# Extended Product and Integrated Interleaved Codes 

Mario Blaum and Steven Hetzler<br>IBM Research Division<br>Almaden Research Center<br>San Jose, CA 95120, USA<br>Mario.Blaum@ibm.com, hetzler@us.ibm.com


#### Abstract

A new class of codes, Extended Product (EPC) Codes, consisting of a product code with a number of extra parities added, is presented and applications for erasure decoding are discussed. An upper bound on the minimum distance of EPC codes is given, as well as constructions meeting the bound for some relevant cases. A special case of EPC codes, Extended Integrated Interleaved (EII) codes, which naturally unify Integrated Interleaved (II) codes and product codes, is defined and studied in detail. It is shown that EII codes often improve the minimum distance of II codes with the same rate, and they enhance the decoding algorithm by allowing decoding on columns as well as on rows. It is also shown that EII codes allow for encoding II codes with an uniform distribution of the parity symbols.


## Index Terms

Erasure-correcting codes, product codes, Reed-Solomon (RS) codes, generalized concatenated codes, integrated interleaving, MDS codes, PMDS codes, maximally recoverable codes, local and global parities, heavy parities, locally recoverable (LRC) codes.

## I. Introduction

There has been considerable research in recent literature on codes with local and global properties for erasure correction (see for instance [1], [2], [4], [14], [21], [25], [33]-[39], [41] and references within). In general, data symbols are divided into sets and parity symbols (i.e., local parities) are added to each set (often using an MDS code). This way, when a number of erasures not exceeding the number of parity symbols occurs in a set, such erasures are rapidly recovered. In addition to the local parities, a number of global parities (also called heavy parities) are added. Those global parities involve all of the data symbols and may include the local parity symbols as well. The goal of the global parities is to correct situations in which the erasure-correcting power of the local parities has been exceeded.

The interest in erasure correcting codes with local and global properties arises mainly from two applications. One of them is the cloud. A cloud configuration may consist of many storage devices, of which some of them may even be in different geographical locations, and the data is distributed across them. If one or more of those devices fails, it is desirable to recover its contents "locally," that is, using a few parity devices within a set of limited size in order to affect performance as little as possible. However, the local parities may not suffice. Extra protection is needed in case the erasure-correcting capability of a local set is exceeded. To address this situation, some devices consisting of global parities are incorporated, and when the local correction power is exceeded, the global parity devices are invoked and correction is attempted. If such a situation occurs, although there will be an impact on performance, data loss may be averted. It is expected that the cases in which the local parity is exceeded are relatively rare events, so the aforementioned impact on performance does not occur frequently. As an example of this type of application, we refer the reader to the description of the Azure system [22] or to the Xorbas code presented in [37].

A second application occurs in the context of Redundant Arrays of Independent Disks (RAID) architectures [10]. In this case, a RAID architecture protects against one or more storage device failures. For example, RAID 5 adds one extra parity device, allowing for the recovery of the contents of one failed device, while RAID 6 protects against up to two device failures. In particular, if those devices are Solid State Drives (SSDs), like flash memories, their reliability decays with time and with the number of writes and reads [29]. The information in SSDs is generally divided into pages, each page containing its own internal Error-Correction Code (ECC). It may happen that a particular page degrades and its ECC is exceeded. However, the user may not become aware of this situation until the page is accessed (what is known as a silent failure). Assuming an SSD has failed in a RAID 5 scheme, if during reconstruction a silent page failure is encountered in one of the surviving SSDs, then data loss will occur. A method around this situation is using RAID 6 . However, this method is costly, since it requires two whole SSDs as parity. It is more desirable to divide the information in a RAID type of architecture into $m \times n$ stripes: $m$ represents the size of a stripe, and $n$ is the number of SSDs. The RAID architecture may be viewed as consisting of a large number of stripes, each stripe encoded and decoded independently. Certainly, codes like the ones used in cloud applications may be used as well for RAID applications. In practice, the choice of code depends on the statistics of errors and on the
frequency of silent page failures. RAID systems, however, may behave differently than a cloud array of devices, in the sense that each column represents a whole storage device. When a device fails, then the whole column is lost, a correlation that may not occur in cloud applications. For that reason, RAID architectures may benefit from a special class of codes with local and global properties, the so called Sector-Disk (SD) codes, which take into account such correlations [20], [27], [31], [32].

From now on, we will call the entries of the codes considered in the paper "symbols". Such symbols can be whole devices (for example, in the case of cloud applications) or pages (in the case of RAID applications for SSDs). Each symbol may be protected by one local group, but a natural extension is to consider multiple localities [36], [39], [43]. Product codes [28] represent a special case of multiple localities: any symbol is protected by both horizontal and vertical parities.

Product codes by themselves may also be used in RAID-type of architectures: the horizontal parities protect a number of devices from failure. The vertical parities allow for rapid recovery of a page or sector within a device (a first responder type of approach). However, if the number of silent failures exceeds the correcting capability of the vertical code, and the horizontal code is unusable due to device failure, data loss will occur. For that reason, it may be convenient to incorporate a number of extra global parities to the product code. In general, we will simply call extra parities these extra global parities in order to avoid confusion, since in a product code the parities on parities, by affecting all of the symbols, can also be considered as global parities.

In effect, consider a product code consisting of $m \times n$ arrays such that each column has $v$ parity symbols and each row has $h$ parity symbols. If in addition to the horizontal and vertical parities there are $g$ extra parities, we say that the code is an Extended Product (EPC) code and we denote it by $E P(m, v ; n, h ; g)$. Notice that, in particular, $E P(m, v ; n, h ; 0)$ is a regular product code, while $E P(m, 0 ; n, h ; g)$ is a Locally Recoverable (LRC) code [13], [39].

Constructions of LRC codes involve different issues and tradeoffs, like the size of the field and optimality criteria. The same is true for EPC codes, of which, as we have seen above, LRC codes are a special case. In particular, one goal is to keep the size of the required finite field small, since operations over a small field have less complexity than ones over a larger field due to the smaller look-up tables involved. For example, Integrated Interleaved (II) codes [18], [40] over GF $(q)$, where $q>\max \{m, n\}$, were proposed in [2] as LRC codes (II codes are closely related to Generalized Concatenated Codes [6], [45]). Let us mention also the construction in [26] (STAIR codes), which reduces field size when failures are correlated. Similarly, we propose a new family of codes that we call Extended Integrated Interleaved (EII) codes, to be defined in Section III of which both product codes and II codes are special cases. In earlier versions, we called such codes Generalized Product Codes [3]. However, there are several ways of generalizing product codes, and the term Generalized Product Code usually refers to graph theoretic constructions [16], [17]. The new denomination avoids confusion.

As is the case with LRC codes, construction of EPC codes involves optimality issues. For example, LRC codes optimizing the minimum distance were presented in [39]. Except for special cases, II codes are not optimal as LRC codes, but the codes in [39] require a field of size at least $m n$, so there is a tradeoff. The same happens with EII codes: except for special cases to be presented in Section III, they do not optimize the minimum distance.

There are stronger criteria for optimization than the minimum distance in LRC codes. For example, PMDS codes [1], [4], [13], [20], [22] satisfy the Maximally Recoverable (MR) property [13], [15]. The definition of the MR property is extended for EPC codes in [15], but it turns out that EPC codes with the MR property are difficult to obtain. For example, in [15] it was proven that an EPC code $E P(n, 1 ; n, 1 ; 1)$ (i.e., one vertical and one horizontal parity per column and row and one extra parity) with the MR property requires a field whose size is superlinear on $n$. We do not address EPC codes with the MR property in this paper.

Although the constructions we present can be extended to finite fields of any characteristic, for simplicity, we assume that they have characteristic 2 .

The paper is structured as follows: in Section II we present the definition of EII codes and give their properties, like their erasure-correcting capability, their minimum distance and encoding and decoding algorithms. We also show that EII codes effectively enhance the decoding power of regular II codes, by allowing decoding on rows as well as on columns. As another application of EII codes, we show that II codes admit a balanced distribution of parity symbols. In Section III we present an upper bound on the minimum distance of EPC codes. We show that this bound generalizes the known bound on the minimum distance of LRC codes. We also present some constructions of EPC codes optimizing the minimum distance for $E(m, 1 ; n, 1 ; g)$ codes. We end the paper by drawing some conclusions.

## II. Extended Integrated Interleaved (EII) Codes

This section is divided into subsections as follows: in Subsection $I-A$ we give the definition of EII codes and we illustrate it with several examples. In Subsection 【I-B we present the main (erasure) decoding algorithm of EII codes consisting of a triangulation process. In Subsection II-C we give the dimension and the minimum distance of EII codes, as well as an encoding algorithm. In Subsection 【II-D we show that the transpose arrays of the arrays in an EII code also constitute an EII code, and this property allows for an enhancement of the decoding algorithm, since arrays can now be iteratively decoded on rows as well as on columns, a process that generalizes the well known row-column iterative decoding of product codes. As a second application of this property, we show that EII codes allow for an uniform distribution of the parity symbols. In Subsection II-E we show how to extend the erasure decoding algorithm to errors together with erasures.

## A. Definition of Extended Integrated Interleaved (EII) Codes

We start by defining EII codes, which unify product codes and II codes. II codes may be interpreted as $m \times n$ arrays such that each row belongs in a code $\mathcal{C}_{0}$, and certain linear combinations of the rows belong in nested subcodes of $\mathcal{C}_{0}$ [2], [40], [42], [44]. In addition, we assume that each column is also in a (vertical) code, making the arrays a subcode of a product code. We assume that the individual codes are Reed-Solomon [28] (RS) type of codes. Explicitly,
Definition 1. Take $t+1$ integers

$$
0 \leqslant u_{0}<u_{1}<\ldots<u_{t-1}<u_{t}=n
$$

and let $\underline{u}$ be the following vector of length $m=s_{0}+s_{1}+\cdots+s_{t-1}+s_{t}$, where $s_{i} \geqslant 1$ for $0 \leqslant i \leqslant t-1$ and $s_{t} \geqslant 0$ :

$$
\begin{equation*}
\underline{u}=(\overbrace{u_{0}, u_{0}, \ldots, u_{0}}^{s_{0}}, \overbrace{u_{1}, u_{1}, \ldots, u_{1}}^{s_{1}}, \ldots, \overbrace{u_{t-1}, u_{t-1}, \ldots, u_{t-1}}^{s_{t-1}}, \overbrace{u_{t}, u_{t}, \ldots, u_{t}}^{s_{t}}) \tag{1}
\end{equation*}
$$

Consider a set $\left\{\mathcal{C}_{i}\right\}$ of $t$ nested $\left[n, n-u_{i}, u_{i}+1\right], 0 \leqslant i \leqslant t-1$, RS codes with elements in a finite field $G F(q), q>$ $\max \{m, n\}$, such that a parity-check matrix for $\mathcal{C}_{i}$ is given by

$$
H_{u_{i}}=\left(\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1  \tag{2}\\
1 & \alpha & \alpha^{2} & \ldots & \alpha^{n-1} \\
1 & \alpha^{2} & \alpha^{4} & \ldots & \alpha^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{u_{i}-1} & \alpha^{2\left(u_{i}-1\right)} & \ldots & \alpha^{\left(u_{i}-1\right)(n-1)}
\end{array}\right)
$$

where $\alpha$ is an element of order $\mathcal{O}(\alpha) \geqslant \max \{m, n\}$ in $G F(q)$. Assume also that $\mathcal{C}_{t}=\{0\}$.
Let $\mathcal{C}(n, \underline{u})$ be the code consisting of $m \times n$ arrays over $G F(q)$ such that, for each array in the code with rows $\underline{c}_{0}, \underline{c}_{1}, \ldots, \underline{c}_{m-1}$, $\underline{c}_{j} \in \mathcal{C}_{0}$ for $0 \leqslant j \leqslant m-1$ and, if

$$
\begin{equation*}
\hat{s}_{i}=\sum_{j=i}^{t} s_{j} \quad \text { for } \quad 0 \leqslant i \leqslant t \tag{3}
\end{equation*}
$$

then

$$
\begin{equation*}
\bigoplus_{j=0}^{m-1} \alpha^{r j} \underline{c}_{j} \in \mathcal{C}_{t-i} \text { for } 0 \leqslant i \leqslant t-1 \text { and } 0 \leqslant r \leqslant \hat{s}_{t-i}-1 \tag{4}
\end{equation*}
$$

Then we say that $\mathcal{C}(n, \underline{u})$ is a $t$-level Extended Integrated Interleaved (EII) code.

In Definition 11, notice that, if $u_{0}=0$, then the code $\mathcal{C}_{0}$ is an $[n, n, 1]$ code, that is, the whole space, with no erasure-correcting capabilities. Let us mention the recent work in [44] on II codes, in which the definition is modified to improve the locality when only one row has erasures, but the number of erasures in such row exceeds $u_{0}$.

Before giving the properties of $t$-level EII codes, we present some examples.
Example 2. Assume that $s_{t}=0$ in Definition 1] then, in (4), $i \geqslant 1$ and $\mathcal{C}(n, \underline{u})$ is a $t$-level II code [2], [40], [42]. So, $t$-level II codes can be viewed as a special case of $t$-level EII codes.

In [40], [42], when $t>2$, II codes are called Generalized Integrated Interleaved (GII) codes, while II codes refer to the case $t=2$. The reason for this denomination is historical, since the first paper on II codes [18] describes the case $t=2$ only.
Example 3. Assume that $t=1$, then (1) gives $\underline{u}=(\overbrace{u_{0}, u_{0}, \ldots, u_{0}}^{s_{0}}, \overbrace{n, n, \ldots, n}^{s_{1}})$. If $s_{1}>0, \mathcal{C}(n, \underline{u})$ is a regular product code [28] such that each row is in an $\left[n, n-u_{0}\right]$ code and each column in an $\left[m, m-s_{1}\right]$ code. Thus, product codes can be viewed as a special case of $t$-level EII codes.

Example 4. Assume that $t=2$. Then, $\mathcal{C}_{1} \subset \mathcal{C}_{0}$,

$$
\underline{u}=(\overbrace{u_{0}, u_{0}, \ldots, u_{0}}^{s_{0}}, \overbrace{u_{1}, u_{1}, \ldots, u_{1}}^{s_{1}}, \overbrace{n, n, \ldots, n}^{s_{2}})
$$

$s_{0}+s_{1}+s_{2}=m$, and consider the 2-level EII code $\mathcal{C}(n, \underline{u})$. Let $\underline{c}=\left(\underline{c}_{0}, \underline{c}_{1}, \ldots, \underline{c}_{m-1}\right)$ be an $m \times n$ array in $\mathcal{C}(n, \underline{u})$. Then, $\underline{c}_{j} \in \mathcal{C}_{0}$ for each $0 \leqslant j \leqslant m-1$, and, since $\mathcal{C}_{2}=\{0\}$, (4) gives

$$
\begin{align*}
& \bigoplus_{j=0}^{m-1} \alpha^{r j} \underline{\mathcal{c}}_{j}=0 \text { for } 0 \leqslant r \leqslant s_{2}-1  \tag{5}\\
& \bigoplus_{j=0}^{m-1} \alpha^{r j} \underline{\mathcal{C}}_{j} \in \mathcal{C}_{1} \text { for } 0 \leqslant r \leqslant s_{1}+s_{2}-1 \tag{6}
\end{align*}
$$

The 2-level II codes presented in [18] correspond to $s_{2}=0$ in this example, i.e., only equations (6) are taken into account. As a special case of 2-level EII codes, take

$$
\underline{u}=(\overbrace{1,1, \ldots, 1}^{m-2}, 2, n)
$$

(hence, $s_{0}=m-2, s_{1}=s_{2}=1$ ). The rows $\underline{c}_{0}, \underline{c}_{1}, \ldots, \underline{c}_{m-1}$ of $\mathcal{C}(n, \underline{u})$ constitute a 2 -level II code. Each column is in an [ $m, m-1,2$ ] code and each row is in an $[n, n-1,2]$ code (single parity). The $\mathcal{C}_{0}$ code is the $[n, n-1,2]$ code, and the $\mathcal{C}_{1}$ code is the $[n, n-2,3]$ code given, according to (2), by the parity-check matrix

$$
H_{1}=\left(\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1 \\
1 & \alpha & \alpha^{2} & \ldots & \alpha^{n-1}
\end{array}\right) .
$$

Moreover, (5) and (6) give

$$
\begin{gathered}
\bigoplus_{i=0}^{m-1} \underline{c}_{i}=0 \\
\bigoplus_{i=0}^{m-1} \alpha^{i} \underline{\mathcal{c}}_{i} \in \mathcal{C}_{1} .
\end{gathered}
$$

It is not hard to prove directly that this code can correct any 5 erasures, but this will be a consequence of Theorem 15 to be presented below. It consists of a product code (which has minimum distance 4) plus one extra parity. This extra parity brings the minimum distance up from 4 to 6 . For instance, if $m=4$ and $n=5$, erasure patterns like the following (vertices of a rectangle)

|  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
|  | $E$ |  |  | $E$ |
|  |  |  |  |  |
|  | $E$ |  |  | $E$ |

are uncorrectable by the product code but can be corrected by $\mathcal{C}(5,(1,1,2,5))$. An extra erasure in addition to the four depicted above can be corrected by either the horizontal or the vertical code.

Example 5. Assume that $t=3$. Then, $\mathcal{C}_{2} \subset \mathcal{C}_{1} \subset \mathcal{C}_{0}$,

$$
\underline{u}=(\overbrace{u_{0}, u_{0}, \ldots, u_{0}}^{s_{0}}, \overbrace{u_{1}, u_{1}, \ldots, u_{1}}^{s_{1}}, \overbrace{u_{2}, u_{2}, \ldots, u_{2}}^{s_{2}}, \overbrace{n, n, \ldots, n}^{s_{3}}),
$$

$s_{0}+s_{1}+s_{2}+s_{3}=m$, and consider the 3-level EII code $\mathcal{C}(n, \underline{u})$. Let $\underline{c}=\left(\underline{c}_{0}, \underline{c}_{1}, \ldots, \underline{c}_{m-1}\right)$ be an $m \times n$ array in $\mathcal{C}(n, \underline{u})$. Then, $\underline{c}_{j} \in \mathcal{C}_{0}$ for each $0 \leqslant j \leqslant m-1$, and (4) gives

$$
\begin{align*}
& \bigoplus_{j=0}^{m-1} \alpha^{r j} \mathcal{c}_{j}=0 \text { for } 0 \leqslant r \leqslant s_{3}-1  \tag{7}\\
& \bigoplus_{j=0}^{m-1} \alpha^{r j} \underline{c}_{j} \in \mathcal{C}_{2} \text { for } 0 \leqslant r \leqslant s_{2}+s_{3}-1 \\
& \bigoplus_{j=0}^{m-1} \alpha^{r j} \underline{c}_{j} \in \mathcal{C}_{1} \text { for } 0 \leqslant r \leqslant s_{1}+s_{2}+s_{3}-1 \tag{8}
\end{align*}
$$

In Definition 1 of EII codes, we have assumed that the nested codes are RS codes. In [42], the construction of II codes was adapted to include binary BCH codes. In order to replace the component codes by binary BCH codes, the $\underline{\mathcal{c}}_{i}$ s cannot be multiplied by powers of $\alpha$ in (4), since doing so would take us out of the binary field. This problem is overcome in [42] by replacing the powers of $\alpha$ by powers of $x$ modulo the primitive polynomial that defines the finite field $G F(q)$. In future work, we will show how to adapt EII codes to any arbitrary set of nested codes.

Although the nested codes in Definition 1 are RS codes, they can be other types of MDS codes as well, like Extended RS codes [28] or Blaum-Roth (BR) [5] codes. For simplicity, we concentrate on RS codes.

## B. Erasure Decoding of EII Codes

We are now ready to state the main result regarding EII codes.
Theorem 6. Consider an $m \times n$ array corresponding to a $\mathcal{C}(n, \underline{u}) t$-level EII code as given by Definition 1 Then, the code can correct up to $u_{0}$ erasures in any row and up to $u_{i}$ erasures in any $s_{i}$ rows, where $1 \leqslant i \leqslant t$.
Proof: We may assume that the rows with erasures contain more than $u_{0}$ erasures, since each row is in $\mathcal{C}_{0}$, which is an $\left[n, n-u_{0}, u_{0}+1\right]$ code, hence, rows with up to $u_{0}$ erasures can be corrected.

Assume that there are $\ell$ rows with more than $u_{0}$ erasures such that there are up to $u_{i}$ erasures in any up to $s_{i}$ rows, $1 \leqslant i \leqslant t$. We do induction on $\ell$.

If $\ell=0$, there is nothing to prove, so, assume that there are $\ell \geqslant 1$ rows with more than $u_{0}$ erasures each such that there are up to $u_{i}$ erasures in any up to $s_{i}$ rows, $1 \leqslant i \leqslant t$. In particular, $\ell \leqslant s_{1}+s_{2}+\ldots+s_{t}=\hat{s}_{1}$. By induction, up to $\ell-1$ rows with this property are correctable.

Let $i_{0}, i_{1}, \ldots, i_{m-1}$ be an ordering of the rows according to a non-increasing number of erasures such that:

1) Row $i_{j}$ for $0 \leqslant j \leqslant \ell-1$ has $v_{j}$ erasures, where
$n \geqslant v_{0} \geqslant v_{1} \geqslant \ldots \geqslant v_{\ell-1}>u_{0}$.
2) Rows $i_{\ell}, i_{\ell+1}, \ldots, i_{m-1}$ have no erasures.

It suffices to prove that the $v_{\ell-1}$ erasures in row $i_{\ell-1}$ can be corrected. Then we will be left with $\ell-1$ rows with more than $s_{0}$ erasures each such that there are up to $u_{i}$ erasures in any up to $s_{i}$ rows, $1 \leqslant i \leqslant t$, and the result follows by induction.

In effect, define $w, 1 \leqslant w \leqslant t-1$, such that

$$
\begin{equation*}
\hat{s}_{w+1}=\sum_{i=1}^{t-w} s_{w+i}<\ell \leqslant \sum_{i=0}^{t-w} s_{w+i}=\hat{s}_{w} \tag{10}
\end{equation*}
$$

and consider the code $\mathcal{C}_{w}$ from the nested set of codes $\mathcal{C}_{i}$ in Definition 1 which can correct up to $u_{w}$ erasures. Since there are up to $u_{w}$ erasures in any up to $s_{w}$ rows, given that the $v_{j}$ s are non-increasing and $0<\ell-\hat{s}_{w+1} \leqslant s_{w}$, then $v_{j} \leqslant u_{w}$ for $\hat{s}_{w+1} \leqslant j \leqslant \ell-1$. In particular, $v_{\ell-1} \leqslant u_{w}$.

Rearranging the order of the elements of the sums in 44, and since $\mathcal{C}_{t} \subset \mathcal{C}_{t-1} \subset \cdots \subset \mathcal{C}_{w}$, from (4), in particular, we have

$$
\begin{equation*}
\bigoplus_{j=0}^{m-1} \alpha^{r i_{j}} \underline{c}_{i_{j}} \in \mathcal{C}_{w} \text { for } 0 \leqslant r \leqslant \ell-1 \tag{11}
\end{equation*}
$$

Since the $\ell \times m$ matrix corresponding to the coefficients of the $\underline{c}_{i_{j}} \mathrm{~s}$ in 11 is a Vandermonde type of matrix and $\mathcal{O}(\alpha) \geqslant$ $\max \{m, n\}$, this matrix can be triangulated, giving

$$
\begin{equation*}
\underline{c}_{i_{r}} \oplus\left(\bigoplus_{j=r+1}^{m-1} \gamma_{r, j} \underline{c}_{i_{j}}\right) \in \mathcal{C}_{w} \text { for } 0 \leqslant r \leqslant \ell-1 \tag{12}
\end{equation*}
$$

where the coefficients $\gamma_{r, j}$ are a result of the triangulation. In particular, taking $r=\ell-1$ in (12), we obtain

$$
\begin{equation*}
\underline{c}_{i_{\ell-1}} \oplus\left(\bigoplus_{j=\ell}^{m-1} \gamma_{\ell-1, j} \underline{c}_{i_{j}}\right) \in \mathcal{C}_{w} \tag{13}
\end{equation*}
$$

Since $\underline{c}_{i_{\ell-1}}$ has $v_{\ell-1}$ erasures and $\underline{c}_{i_{j}}$ has no erasures for
$\ell \leqslant j \leqslant m-1$, then $\underline{c}_{i_{\ell-1}} \oplus\left(\bigoplus_{j=\ell}^{m-1} \gamma_{\ell-1, j} \underline{c}_{i_{j}}\right)$ has $v_{\ell-1}$ erasures. Since the vector is in $\mathcal{C}_{w}$ and $v_{\ell-1} \leqslant u_{w}$, the erasures can be corrected. Once $\underline{c}_{i_{\ell-1}} \oplus\left(\bigoplus_{j=\ell}^{m-1} \gamma_{\ell-1, j} \underline{c}_{i_{j}}\right)$ is corrected, $\underline{c}_{\ell-1}$ is obtained as

$$
\underline{c}_{i_{\ell-1}}=\left(\underline{c}_{i_{\ell-1}} \oplus\left(\bigoplus_{j=\ell}^{m-1} \gamma_{\ell-1, j} \underline{c}_{i_{j}}\right)\right) \oplus\left(\bigoplus_{j=\ell}^{m-1} \gamma_{\ell-1, j} \underline{c}_{i_{j}}\right)
$$

and the result follows by induction on $\ell$.

Theorem 6 generalizes Theorem 1 in [2]. The proof of Theorem 6 is constructive in the sense that it provides a decoding algorithm. The following example illustrates Theorem 6 and the decoding algorithm.

Example 7. Consider the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ according to Definition 1 and Example 5. We have four codes $\mathcal{C}_{3} \subset \mathcal{C}_{2} \subset \mathcal{C}_{1} \subset \mathcal{C}_{0}$, where $\mathcal{C}_{0}$ is a $[7,6,2]$ code, $\mathcal{C}_{1}$ is a $[7,4,4]$ code, $\mathcal{C}_{2}$ is a $[7,3,5]$ code and $\mathcal{C}_{3}=\{0\}$. We may assume that the entries of these codes are in $G F(8)$ and that $\alpha$ is a primitive element in $G F(8)$.

Consider the following $6 \times 7$ array with erasures denoted by $E$ :

| $\underline{c}_{0}$ |  |  | E |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\underline{c}_{1}$ | $E$ | $E$ | $E$ | $E$ | E | E | $E$ |
| $\underline{c}_{2}$ |  | $E$ | E |  | E |  | $E$ |
| $\underline{c}_{3}$ | E |  |  | $E$ |  | $E$ |  |
| $\underline{C}_{4}$ | E | E | E | E | E | $E$ | $E$ |
| $\underline{C}_{5}$ |  |  |  |  |  | $E$ |  |

The first step is correcting the single erasures in $\underline{c}_{0}$ and in $\underline{c}_{5}$. An ordering of the remaining rows in non-increasing number of erasures is $\left\{i_{0}, i_{1}, i_{2}, i_{3}\right\}=\{1,4,2,3\}(\ell=4)$. In particular, $\underline{c}_{3}$ has three erasures. According to (7), (8) and (9),

$$
\begin{aligned}
& \begin{array}{l}
\underline{c}_{0} \oplus \underline{c}_{1} \oplus \underline{c}_{2} \oplus \underline{c}_{3} \oplus \underline{c}_{4} \oplus \underline{c}_{5}=0 \\
\underline{c}_{0} \oplus \alpha \underline{c}_{1} \oplus \alpha^{2} \underline{c}_{2} \oplus \alpha^{3} \underline{c}_{3} \oplus \alpha^{4} \underline{c}_{4} \oplus \alpha^{5} \underline{c}_{5}=0
\end{array} \\
& \underline{c}_{0} \oplus \alpha^{2} \underline{\underline{c}}_{1} \oplus \alpha^{4} \underline{\underline{c}}_{2} \oplus \alpha^{6} \underline{\underline{c}}_{3} \oplus \alpha^{8} \underline{\underline{c}}_{4} \oplus \alpha^{10} \underline{\mathcal{c}}_{5} \quad \in \quad \mathcal{C}_{2} \\
& \underline{\mathcal{c}}_{0} \oplus \alpha^{3} \underline{\mathcal{c}}_{1} \oplus \alpha^{6} \underline{\mathcal{c}}_{2} \oplus \alpha^{9} \underline{\mathcal{c}}_{3} \oplus \alpha^{12} \underline{\mathcal{c}}_{4} \oplus \alpha^{15} \underline{\mathcal{c}}_{5} \in \mathcal{C}_{1} .
\end{aligned}
$$

Notice that $\mathcal{C}_{1}$ can correct three erasures, i.e., $w=1$ in (10). Rearranging the $\underline{c}_{i}$ s above in non-increasing number of erasures, we obtain

$$
\begin{aligned}
& \alpha^{2} \underline{\mathcal{c}}_{1} \oplus \alpha^{8} \underline{\mathcal{c}}_{4} \oplus \alpha^{4} \underline{\underline{c}}_{2} \oplus \alpha^{6} \underline{\mathcal{c}}_{3} \oplus \underline{\mathcal{c}}_{0} \oplus \alpha^{10} \underline{\mathcal{C}}_{5} \quad \in \mathcal{C}_{2} \\
& \alpha^{3} \underline{\mathcal{c}}_{1} \oplus \alpha^{12} \underline{\mathcal{C}}_{4} \oplus \alpha^{6} \underline{\mathcal{c}}_{2} \oplus \alpha^{9} \underline{\mathcal{c}}_{3} \oplus \underline{\mathcal{c}}_{0} \oplus \alpha^{15} \underline{\mathcal{C}}_{5} \in \mathcal{C}_{1},
\end{aligned}
$$

which corresponds to (11) in the proof of Theorem6. Triangulating this linear system in $G F(8)$, where $1 \oplus \alpha \oplus \alpha^{3}=0$, and since $\mathcal{C}_{2} \subset \mathcal{C}_{1}$, we obtain the following triangulated system:

$$
\begin{aligned}
& \underline{c}_{1} \oplus \underline{c}_{4} \oplus \underline{c}_{2} \oplus \underline{c}_{3} \oplus \underline{c}_{0} \oplus \underline{c}_{5}=0 \\
& \underline{c}_{4} \oplus \alpha^{2} \underline{\mathcal{c}}_{2} \oplus \alpha^{5} \underline{\mathcal{c}}_{3} \oplus \alpha \underline{\mathcal{c}}_{0} \oplus \alpha^{4} \underline{\mathcal{c}}_{5}=0 \\
& \underline{\mathcal{c}}_{2} \oplus \alpha \underline{\underline{c}}_{3} \oplus \alpha^{3} \underline{\underline{c}}_{0} \oplus \alpha \underline{\underline{c}}_{5} \in \mathcal{C}_{2} \\
& \underline{\mathcal{c}}_{3} \oplus \alpha^{3} \underline{\mathcal{c}}_{0} \oplus \alpha^{5} \underline{\mathcal{C}}_{5} \in \mathcal{C}_{1} .
\end{aligned}
$$

Since $\underline{c}_{3}$ has 3 erasures and $\underline{c}_{0}$ and $\underline{c}_{5}$ have no erasures, $\underline{c}_{3} \oplus \alpha^{3} \underline{c}_{0} \oplus \alpha^{5} \underline{c}_{5}$ has 3 erasures, which can be corrected in $\mathcal{C}_{1}$. Then,

$$
\underline{c}_{3}=\left(\underline{c}_{3} \oplus \alpha^{3} \underline{c}_{0} \oplus \alpha^{5} \underline{\underline{c}}_{5}\right) \oplus\left(\alpha^{3}{\underline{c_{0}}}_{0} \oplus \alpha^{5}{\underline{c_{5}}}^{5}\right)
$$

Similarly, $\underline{\mathcal{c}}_{2} \oplus \alpha \underline{\mathcal{C}}_{3} \oplus \alpha^{3} \underline{\mathcal{C}}_{0} \oplus \alpha \underline{\mathcal{C}}_{5}$ has 4 erasures, which can be corrected in $\mathcal{C}_{2}$, and

$$
\underline{c}_{2}=\left(\underline{c}_{2} \oplus \alpha \underline{c}_{3} \oplus \alpha^{3} \underline{c}_{0} \oplus \alpha \underline{c}_{5}\right) \oplus\left(\alpha \underline{c}_{3} \oplus \alpha^{3} \underline{\mathcal{c}}_{0} \oplus \alpha \underline{c}_{5}\right)
$$

Finally, since the first two rows of the triangulated system are equal to zero, we obtain

$$
\begin{aligned}
& \underline{c}_{4}=\alpha^{2} \underline{\mathcal{c}}_{2} \oplus \alpha^{5} \underline{\mathcal{c}}_{3} \oplus \alpha \underline{c}_{0} \oplus \alpha^{4} \underline{c}_{5} \\
& \underline{c}_{1}=\underline{c}_{4} \oplus \underline{\mathcal{c}}_{2} \oplus \underline{\mathcal{c}}_{3} \oplus \underline{c}_{0} \oplus \underline{c}_{5}
\end{aligned}
$$

completing the decoding.
From the proof of Theorem 6 even if the decoding algorithm cannot correct all the erasures, it is often possible to correct a few rows. Specifically, consider a $\mathcal{C}(n, \underline{u}) t$-level EII code as given by Definition 11 and assume that, given a received
array in $\mathcal{C}(n, \underline{u})$, row $i$ of the array has $x_{i}$ erasures. Let $x_{i_{0}} \leqslant x_{i_{1}} \leqslant \cdots \leqslant x_{i_{m-1}}$ and consider $\underline{u}$ as given by (1). Let $\underline{u}=\left(v_{0}, v_{1}, \ldots, v_{m-1}\right)$, where, by (1), $v_{0} \leqslant v_{1} \leqslant \cdots \leqslant v_{m-1}$. Define $y, 0 \leqslant y \leqslant m-1$, such that $x_{i_{j}} \leqslant v_{j}$ for $0 \leqslant j \leqslant y$, and $x_{y+1}>v_{y+1}$. If $y=m-1$, then, by Theorem6, all the erasures in the array are correctable. However, if $y<m-1$, then rows $i_{0}, i_{1}, \ldots, i_{y}$ are still correctable by the triangulation algorithm, but rows $i_{j}$ for $y+1 \leqslant j \leqslant m-1$ are not. We illustrate this fact in the following example.
Example 8. Consider the 4-level EII code $\mathcal{C}(7,(1,2,3,5))$ according to Definition 1 We have four codes $\mathcal{C}_{3} \subset \mathcal{C}_{2} \subset \mathcal{C}_{1} \subset \mathcal{C}_{0}$, where $\mathcal{C}_{0}$ is a $[7,6,2]$ code, $\mathcal{C}_{1}$ is a $[7,5,3]$ code, $\mathcal{C}_{2}$ is a $[7,4,4]$ code and $\mathcal{C}_{3}$ is a $[7,2,6]$ code. As in Example 7 , we assume that the entries of these codes are in $G F(8)$ and that $\alpha$ is a primitive element in $G F(8)$.

Consider the following $4 \times 7$ array with erasures denoted by $E$ :

| $\underline{C}_{0}$ | $E$ |  |  | $E$ | $E$ | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\underline{c}_{1}$ |  | E |  | E |  |  |
| $\underline{c}_{2}$ |  |  | $E$ |  |  |  |
| $\underline{c}_{3}$ | E | E |  |  | E | $E$ |

Then, according to the notation above, $x_{0}=4, x_{1}=2, x_{2}=1$ and $x_{3}=4$. Writing this in non-decreasing order, we have $x_{2} \leqslant x_{1} \leqslant x_{0} \leqslant x_{3}$. Since $x_{2} \leqslant v_{0}, x_{1} \leqslant v_{1}$ and $x_{0}>v_{2}, y=1$. Since $\underline{c}_{2}$ has a single erasure, we may assume that this erasure is corrected in $\mathcal{C}_{0}$, so now $\underline{c}_{2}$ is erasure free. According to Definition 1 we have the following system:

$$
\begin{array}{llccccccc}
\underline{\mathcal{c}}_{0} & \oplus & \underline{\mathcal{c}}_{3} & \oplus & \underline{c}_{1} & \oplus & \underline{c}_{2} & \in & \mathcal{C}_{3} \\
\underline{\boldsymbol{c}}_{0} & \oplus & \alpha^{3} \underline{\underline{c}}_{3} & \oplus & \alpha \underline{\mathcal{c}}_{1} & \oplus & \alpha^{2} \underline{\mathcal{c}}_{2} & \in & \mathcal{C}_{2} \\
\underline{\boldsymbol{c}}_{0} & \oplus & \alpha^{6} \underline{\underline{c}}_{3} & \oplus & \alpha^{2} \underline{c}_{1} & \oplus & \alpha^{4} \underline{\mathcal{c}}_{2} & \in & \mathcal{C}_{1}
\end{array}
$$

Triangulating this system, we obtain

$$
\begin{array}{cccccccccc}
\underline{c}_{0} \oplus & \underline{c}_{3} & \oplus & \underline{c}_{1} & \oplus & \underline{c}_{2} & \in & \mathcal{C}_{3} \\
& \underline{c}_{3} & \oplus & \alpha^{2} \underline{c}_{1} & \oplus & \alpha^{5} \underline{c}_{2} & \in & \mathcal{C}_{2} \\
& & \underline{c}_{1} & \oplus & \alpha \underline{c}_{2} & \in & \mathcal{C}_{1}
\end{array}
$$

Since $\underline{\mathcal{c}}_{1} \oplus \alpha \underline{\mathcal{c}}_{2}$ has two erasures, they can be corrected in $\mathcal{C}_{1}$. Then, $\underline{\mathcal{c}}_{1}$ is obtained as $\underline{\mathcal{c}}_{1}=\left(\underline{c}_{1} \oplus \alpha \underline{\mathcal{c}}_{2}\right) \oplus \alpha \underline{\mathcal{c}}_{2}$. However, with this procedure, $\underline{\mathcal{c}}_{3} \oplus \alpha^{2} \underline{\mathcal{c}}_{1} \oplus \alpha^{5} \underline{\mathcal{c}}_{2}$ cannot be obtained, since 4 erasures are uncorrectable in $\mathcal{C}_{2}$, so after correcting $\underline{\mathcal{c}}_{1}$ and $\underline{\mathcal{c}}_{2}$, we are left with the uncorrectable array


Although the erasure pattern in Example 8 is only partially correctable, we will see after Theorem 18 that it can be fully correctable when expanding the correction to columns.

## C. Dimension, Encoding and Minimum Distance of EII Codes

Before discussing the dimension, the encoding and the minimum distance of $t$-level EII codes, let us state and prove the following lemma.
Lemma 9. Consider the $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 . Then, for each $j$ such that $0 \leqslant j \leqslant t-1$, given $u_{j}+1$ fixed column indices in $\hat{s}_{j+1}+1$ different rows, there is an array in $\mathcal{C}(n, \underline{u})$ that is non-zero in such $\left(\hat{s}_{j+1}+1\right)\left(u_{j}+1\right)$ locations and 0 elsewhere.

Proof: Since $\mathcal{C}_{j}$ is an $\left[n, n-u_{j}, u_{j}+1\right]$ MDS code for each $j$ such that $0 \leqslant j \leqslant t-1$, given $u_{j}+1$ fixed indices in a vector of length $n$, there is a codeword $\underline{w}$ in $\mathcal{C}_{j}$ whose non-zero entries are in such $u_{j}+1$ fixed locations. Assume that the $\hat{s}_{j+1}+1$ rows selected are $i_{0}, i_{1}, \ldots, i_{\hat{s}_{j+1}}$, where

$$
0 \leqslant i_{0}<i_{1}<\ldots<i_{\hat{s}_{j+1}} \leqslant m-1
$$

Let $\underline{v}=\left(v_{0}, v_{1}, \ldots, v_{\hat{s}_{j+1}}\right)$ be a vector of weight $\hat{s}_{j+1}+1$ such that

$$
\bigoplus_{s=0}^{\hat{s}_{j+1}} \alpha^{r i_{s}} v_{s}=0 \text { for } 0 \leqslant r \leqslant \hat{s}_{j+1}-1
$$

Such a vector exists since the coefficients in are in an $\hat{s}_{j+1} \times\left(\hat{s}_{j+1}+1\right)$ Vandermonde matrix (which corresponds to the parity-check matrix of an $\left[\hat{s}_{j+1}+1,1, \hat{s}_{j+1}+1\right]$ RS code $)$. Consider the $m \times n$ array of weight $\left(\hat{s}_{j+1}+1\right)\left(u_{j}+1\right)$ such that row $i_{s}$ equals $v_{s} \underline{w}$ for $0 \leqslant s \leqslant \hat{s}_{j+1}$, and the remaining rows are zero. We will show that this array is in $\mathcal{C}(n, \underline{u})$. Since each row of the array is in $\mathcal{C}_{j}$ by design, in particular, it is in $\mathcal{C}_{0}$. According to 4], we have to show that

$$
\bigoplus_{s=0}^{\hat{s}_{j+1}} \alpha^{r i_{s}}\left(v_{s} \underline{w}\right) \in \mathcal{C}_{t-i} \text { for } 0 \leqslant i \leqslant t-1 \text { and } 0 \leqslant r \leqslant \hat{s}_{t-i}-1
$$

If $0 \leqslant j \leqslant t-i-1$, then, $\hat{s}_{j+1} \geqslant \hat{s}_{t-i}$, and, for $0 \leqslant r \leqslant \hat{s}_{t-i}-1$, by (14),

$$
\bigoplus_{s=0}^{\hat{s}_{j+1}} \alpha^{r i_{s}}\left(v_{s} \underline{w}\right)=\left(\bigoplus_{s=0}^{\hat{s}_{j+1}} \alpha^{r i_{s}} v_{s}\right) \underline{w}=0
$$

so, in particular, (15) follows.
If $t-i \leqslant j \leqslant t-1$, then $\mathcal{C}_{j} \subseteq \mathcal{C}_{t-i}$ and $\underline{w} \in \mathcal{C}_{t-i}$, so (15) also follows in this case.

Example 10. Consider the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Example 7 According to Lemma 9 the locations denoted by $E$ in the following arrays correspond to the non-zero entries of arrays in $\mathcal{C}(7,(1,1,3,4,7,7))$ for $j=0$, 1 and 2 respectively:

|  | $E$ |  | $E$ |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $E$ |  | $E$ | $E$ |  |  | $E$ | $E$ |  | $E$ |
|  |  |  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |  |  |
|  | $E$ | $E$ |  | $E$ |  | $E$ |  |  |  |  |
|  | $E$ |  | $E$ |  |  |  |  |  |  |  |
|  | $E$ | $E$ |  | $E$ |  | $E$ |  |  |  |  |
|  | $E$ |  | $E$ |  |  |  |  |  |  |  |
|  | $E$ |  | $E$ |  |  |  |  |  |  |  |
|  | $E$ | $E$ |  |  | $E$ |  | $E$ |  |  |  |


|  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $E$ |  | $E$ |  | $E$ | $E$ | $E$ |
|  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |
| $E$ |  | $E$ |  | $E$ | $E$ | $E$ |
| $E$ |  | $E$ |  | $E$ | $E$ | $E$ |

The arrays with erasures in locations $E$ above are uncorrectable since, provided the zero array was stored, the decoding cannot decide between the zero array and the arrays with non-zero entries in the locations $E$.

Next we give an auxiliary general lemma.
Lemma 11. Consider an $[n, k]$ linear code, and let $S=\left\{i_{0}, i_{1}, \ldots, i_{s-1}\right\}$, where $0 \leqslant i_{0}<i_{1}<\cdots<i_{S-1} \leqslant n-1$. Assume that, given a codeword with erasures in $S$, the code can correct such erasures, while, for any $i \notin S$, erasures in $S \cup\{i\}$ are not correctable. Then, $n-k=s$.
Proof: Since the erasures in $S$ are correctable, there are at least $s$ linearly independent parity equations, so $n-k \geqslant s$.
Assume that $n-k>s$. Let $H$ be an $(n-k) \times n$ parity-check matrix of the code such that the first $s$ rows of $H$ are used to correct the $s$ erasures in $S$, thus, the $s \times s$ submatrix consisting of those first $s$ rows and columns $i_{0}, i_{1}, \ldots, i_{s-1}$ is invertible.

Consider next the matrix consisting of the first $s+1$ rows in $H$. By row operations, we can make the entries $i_{0}, i_{1}, \ldots, i_{s-1}$ in the $(s+1)$-th row equal to zero. Since the first $s+1$ rows of $H$ have rank $s+1$, then there is a non-zero location $i, i \notin S$, in the $(s+1)$-th row. Thus, columns $S \cup\{i\}$ in the first $s+1$ rows of $H$ are linearly independent and hence erasures in $S \cup\{i\}$ are correctable, a contradiction, so $n-k=s$.

Theorem 12. Consider the $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 Then, $\mathcal{C}(n, \underline{u})$ is an $[m n, k]$ code, where

$$
\begin{equation*}
k=m n-\left(\sum_{i=0}^{t} s_{i} u_{i}\right) \tag{16}
\end{equation*}
$$

Proof: Assume that the zero array is stored, and a received array $W$ has erasures in the last $u_{i}$ entries of rows $m-\hat{s}_{i}$ to $m-\hat{s}_{i+1}-1$ for $0 \leqslant i \leqslant t-1$, and in all the entries of rows $m-s_{t}$ to $m-1$. Thus, $W$ has a total of $\sum_{i=0}^{t} s_{i} u_{i}$ erasures, and by Theorem 6, it will be correctly decoded as the zero codeword.

Consider an array $V$ which coincides with $W$, except in one location in which it has an extra erasure. If we show that any such $V$ is uncorrectable, by Lemma 11 $m n-k=\sum_{i=0}^{t} s_{i} u_{i}$, which is equivalent to (16).

For each $\left(u^{\prime}, v^{\prime}\right)$ that is not in the set of erasures of $W$, define $i^{\prime}, 0 \leqslant i^{\prime} \leqslant t-1$, such that $m-\hat{s}_{i^{\prime}} \leqslant u^{\prime} \leqslant m-\hat{s}_{i^{\prime}+1}-$ 1, and let $Y^{\left(u^{\prime}, v^{\prime}\right)}=\left(y_{a, b}^{\left(u^{\prime}, v^{\prime}\right)}\right)_{\substack{0 \leqslant a \leqslant m-1 \\ 0 \leqslant b \leqslant n-1}}$ be an array in $\mathcal{C}(n, \underline{u})$ whose non-zero coordinates are in the intersection of rows $u^{\prime}, u^{\prime}+1, \ldots, u^{\prime}+\hat{s}_{i^{\prime}+1}$ and columns $v^{\prime}, n-u_{i^{\prime}}, n-u_{i^{\prime}}+1, \ldots, n-1$. Such a non-zero array exists due to Lemma 9

Assume that the extra erasure in $V$ is in location $(u, v)$ and consider $j, 1 \leqslant j \leqslant t$, such that $n-u_{j} \leqslant v \leqslant n-u_{j-1}-1$. Take the arrays $Y^{\left(u^{\prime}, v\right)}$, where $u \leqslant u^{\prime} \leqslant m-\hat{s}_{j}-1$. For each $u^{\prime}, u<u^{\prime} \leqslant m-\hat{s}_{j}-1$, choose constants $c_{u^{\prime}}$ such that

$$
\begin{equation*}
y_{u^{\prime}, v}^{(u, v)} \oplus \bigoplus_{z=u+1}^{u^{\prime}} c_{z} y_{u^{\prime}, v}^{(z, v)}=0 \tag{17}
\end{equation*}
$$

Then, if $Y=\bigoplus_{z=u}^{m-\hat{S}_{j}-1} Y^{(z, v)}$, by 177 , $\Upsilon$ has a non-zero entry in $(u, v)$, while its remaining non-zero entries are contained in the locations of the erasures of $W$. So, array $V$ is uncorrectable, since it can be decoded either as the zero array or as $Y$.

Theorem II. 1 in [40], which corresponds to Corollary 2 in [2], is a special case of Theorem 12,
Example 13. We illustrate the proof of Theorem 12 with the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Examples 7 and 10 , By Theorem 12] this code is a $[42,19]$ code. Following the proof of Theorem 12] denote by $E$ the erased locations in an array $W$ :

$$
W=\begin{array}{|l|l|l|l|l|l|l|}
\hline & & & & & & E \\
\hline & & & & & & E \\
\hline & & & & E & E & E \\
\hline & & & E & E & E & E \\
\hline E & E & E & E & E & E & E \\
\hline E & E & E & E & E & E & E \\
\hline
\end{array}
$$

If the non-erased locations of $W$ are zero, by Theorem 6 the array will be decoded as the zero array. Consider the array $V$ which has an extra erasure in location $(u, v)=(0,1)$ (hence, $i=0$ and $j=3$ in the proof of Theorem 12), rendering


Consider the following arrays $Y^{\left(u^{\prime}, 1\right)}, 0 \leqslant u^{\prime} \leqslant 3$, defined as in Theorem 12, whose non-zero entries are denoted $y_{a, b}^{\left(u^{\prime}, 1\right)}$ below:

$Y^{(0,1)}=$| 0 | $y_{0,1}^{(0,1)}$ | 0 | 0 | 0 | 0 | $y_{0,6}^{(0,1)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $y_{1,1}^{(0,1)}$ | 0 | 0 | 0 | 0 | $y_{1,6}^{(0,1)}$ |
| 0 | $y_{2,1}^{(0,1)}$ | 0 | 0 | 0 | 0 | $y_{2,6}^{(0,1)}$ |
| 0 | $y_{3,1}^{(0,1)}$ | 0 | 0 | 0 | 0 | $y_{3,6}^{(0,1)}$ |
| 0 | $y_{4,1}^{(0,1)}$ | 0 | 0 | 0 | 0 | $y_{4,6}^{(0,1)}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |


$Y^{(1,1)}=$| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $y_{1,1}^{(1,1)}$ | 0 | 0 | 0 | 0 | $y_{1,6}^{(1,1)}$ |
| 0 | $y_{2,1}^{(1,1)}$ | 0 | 0 | 0 | 0 | $y_{2,6}^{(1,1)}$ |
| 0 | $y_{3,1}^{(1,1)}$ | 0 | 0 | 0 | 0 | $y_{3,6}^{(1,1)}$ |
| 0 | $y_{4,1}^{(1,1)}$ | 0 | 0 | 0 | 0 | $y_{4,6}^{(1,1)}$ |
| 0 | $y_{5,1}^{(1,1)}$ | 0 | 0 | 0 | 0 | $y_{5,6}^{(1,1)}$ |


$Y^{(2,1)}=$| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | $y_{2,1}^{(2,1)}$ | 0 | 0 | $y_{2,4}^{(2,1)}$ | $y_{2,5}^{(2,1)}$ | $y_{2,6}^{(2,1)}$ |
| 0 | $y_{3,1}^{(2,1)}$ | 0 | 0 | $y_{3,4}^{(2,1)}$ | $y_{3,5}^{(2,1)}$ | $y_{3,6}^{(2,1)}$ |
| 0 | $y_{4,1}^{(2,1)}$ | 0 | 0 | $y_{4,4}^{(2,1)}$ | $y_{4,5}^{(2,1)}$ | $y_{4,6}^{(2,1)}$ |
| 0 | $y_{5,1}^{(2,1)}$ | 0 | 0 | $y_{5,4}^{(2,1)}$ | $y_{5,5}^{(2,1)}$ | $y_{5,6}^{(2,1)}$ |


$Y^{(3,1)}=$| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | $y_{3,1}^{(3,1)}$ | 0 | $y_{3,2}^{(3,1)}$ | $y_{3,4}^{(3,1)}$ | $y_{3,5}^{(3,1)}$ | $y_{3,6}^{(3,1)}$ |
| 0 | $y_{4,1}^{(3,1)}$ | 0 | $y_{4,2}^{(3,1)}$ | $y_{4,4}^{(3,1)}$ | $y_{4,5}^{(3,1)}$ | $y_{4,6}^{(3,1)}$ |
| 0 | $y_{5,1}^{(3,1)}$ | 0 | $y_{5,2}^{(3,1)}$ | $y_{5,4}^{(3,1)}$ | $y_{5,5}^{(3,1)}$ | $y_{5,6}^{(3,1)}$ |

Such arrays with non-zero entries exist by Lemma 9 (see also Example 10). We choose $c_{1}, c_{2}$ and $c_{3}$ such that

$$
\begin{aligned}
y_{1,1}^{(0,1)} \oplus c_{1} y_{1,1}^{(1,1)} & =0 \\
y_{2,1}^{(0,1)} \oplus c_{1} y_{2,1}^{(1,1)} \oplus c_{2} y_{2,1}^{(2,1)} & =0 \\
y_{3,1}^{(0,1)} \oplus c_{1} y_{3,1}^{(1,1)} \oplus c_{2} y_{3,1}^{(2,1)} \oplus c_{3} y_{3,1}^{(3,1)} & =0
\end{aligned}
$$

Then, defining $Y=Y^{(0,1)} \oplus c_{1} Y^{(1,1)} \oplus c_{2} Y^{(2,1)} \oplus c_{3} Y^{(3,1)}$, we see that

$$
Y=\begin{array}{|c|c|c|c|c|c|c|}
\hline 0 & y_{0,1}^{(0,1)} & 0 & 0 & 0 & 0 & X \\
\hline 0 & 0 & 0 & 0 & 0 & 0 & X \\
\hline 0 & 0 & 0 & 0 & X & X & X \\
\hline 0 & 0 & 0 & X & X & X & X \\
\hline 0 & X & 0 & X & X & X & X \\
\hline 0 & X & 0 & X & X & X & X \\
\hline
\end{array}
$$

where entries denoted by $X$ may take any value. Array $Y$ is non-zero since $y_{0,1}^{(0,1)} \neq 0$. Array $V$ may be decoded as the zero array or as $Y$, so it is uncorrectable. We can make the same argument for any entry $(u, v)$ not contained in the erasures of $W$, so, by Lemma 11 the number of parity symbols is 23 and the dimension of the code is 19 .

The encoding is a special case of the decoding. For example, we may place the parities at the end of the array in increasing order of parities, as shown in Theorem 12 and in Example 13 The parities are considered as erasures and may be obtained using the triangulation method described in Theorem 6. The fact that the locations of the erasures are known allows for a simplification of the decoding algorithm. For example, the triangulated matrix corresponding to the coefficients of (12) may be precomputed. The next example illustrates this encoding process.

Example 14. Take the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Examples 70 and 13 . We have to solve the erasures in array $W$ of Example 13 proceeding by triangulation like in Example 7

The first step is encoding rows $\underline{c}_{0}$ and $\underline{c}_{1}$ (single parity). An ordering of the remaining rows in non-increasing number of erasures is $\left\{i_{0}, i_{1}, i_{2}, i_{3}\right\}=\{5,4,3,2\}$.

According to (7), (8) and (9), and rearranging the $\underline{c}_{i}$ s in non-increasing number of erasures, we obtain

$$
\begin{aligned}
& \begin{array}{l}
\underline{c}_{5} \oplus \underline{c}_{4} \oplus \underline{c}_{3} \oplus \underline{\underline{c}}_{2} \oplus \underline{c}_{1} \oplus \underline{c}_{0}=0 \\
\alpha^{5} \underline{c}_{5} \oplus \alpha^{4} \underline{c}_{4} \oplus \alpha^{3} \underline{c}_{3} \oplus \alpha^{2} \underline{c}_{2} \oplus \underline{c}_{1} \underline{\underline{c}}_{1} \oplus \underline{c}_{0}=0
\end{array} \\
& \alpha^{10} \underline{\mathcal{C}}_{5} \oplus \alpha^{8} \underline{\mathcal{C}}_{4} \oplus \alpha^{6} \underline{\underline{c}}_{3} \oplus \alpha^{4} \underline{\mathcal{C}}_{2} \oplus \alpha^{2} \underline{\underline{c}}_{1} \oplus \underline{\underline{c}}_{0} \in \mathcal{C}_{2} \\
& \alpha^{15} \underline{\mathcal{C}}_{5} \oplus \alpha^{12} \underline{\mathcal{C}}_{4} \oplus \alpha^{9} \underline{\mathcal{C}}_{3} \oplus \alpha^{6} \underline{\mathcal{C}}_{2} \oplus \alpha^{3} \underline{\mathcal{c}}_{1} \oplus \underline{\mathcal{c}}_{0} \in \mathcal{C}_{1}
\end{aligned}
$$

Triangulating this linear system in $G F(8)$, since $\mathcal{C}_{2} \subset \mathcal{C}_{1}$ and $1 \oplus \alpha \oplus \alpha^{3}=0$, we obtain the following triangulated system:

$$
\begin{aligned}
\underline{c}_{5} \oplus \underline{c}_{4} \oplus \underline{c}_{3} \oplus \underline{c}_{2} \oplus \underline{c}_{1} \oplus \underline{c}_{0} & \underline{c}_{0} \\
\underline{c}_{4} \oplus \alpha^{2} \underline{c}_{3} \oplus \alpha^{3} \underline{c}_{2} \oplus \alpha^{6} \underline{c}_{1} \oplus \alpha^{4} \underline{c}_{0} & =0 \\
\underline{c}_{3} \oplus \alpha^{3} \underline{c}_{2} \oplus \underline{c}_{1} \oplus \alpha \underline{c}_{0} & \in \mathcal{C}_{2} \\
& \underline{c}_{2} \oplus \alpha^{6} \underline{c}_{1} \oplus \alpha \underline{c}_{0}
\end{aligned} \in \mathcal{C}_{1} .
$$

This triangulated system is precomputed, so it is not necessary to do Gaussian elimination when encoding.
Then, $\underline{c}_{2} \oplus \alpha^{6} \underline{c}_{1} \oplus \alpha \underline{c}_{0}$ is encoded in $\mathcal{C}_{1}$ and $\underline{c}_{2}$ is obtained as $\underline{c}_{2}=\left(\underline{c}_{2} \oplus \alpha^{6} \underline{c}_{1} \oplus \alpha \underline{c}_{0}\right) \oplus\left(\alpha^{6} \underline{c}_{1} \oplus \alpha \underline{c}_{0}\right)$. Similarly, $\underline{c}_{3} \oplus \alpha^{3} \underline{c}_{2} \oplus \underline{c}_{1} \oplus \alpha \underline{c}_{0}$ is encoded in $\mathcal{C}_{2}$, and $\underline{c}_{3}$ is obtained as $\underline{c}_{3}=\left(\underline{c}_{3} \oplus \alpha^{3} \underline{c}_{2} \oplus \underline{c}_{1} \oplus \alpha \underline{c}_{0}\right) \oplus\left(\alpha^{3} \underline{c}_{2} \oplus \underline{c}_{1} \oplus \alpha \underline{c}_{0}\right)$. Finally, we obtain

$$
\begin{aligned}
& \underline{c}_{4}=\alpha^{2} \underline{c}_{3} \oplus \alpha^{3} \underline{\underline{c}}_{2} \oplus \alpha^{6} \underline{c}_{1} \oplus \alpha^{4} \underline{c}_{0} \\
& \underline{c}_{5}=\underline{c}_{4} \oplus \underline{c}_{3} \oplus \underline{c}_{2} \oplus \underline{c}_{1} \oplus \underline{c}_{0}
\end{aligned}
$$

At every step, we are encoding RS codes.
Another possibility for encoding EII codes is to use an existing encoding algorithm for II codes. In effect, if $\underline{u}$ is given by (11) and $\underline{u}^{\prime}=(\overbrace{u_{0}, \ldots, u_{0}}^{s_{0}}, \overbrace{u_{1}, \ldots, u_{1}}^{s_{1}}, \ldots, \overbrace{u_{t-1}, \ldots, u_{t-1}}^{s_{t-1}+s_{t}})$, then an EII code $\mathcal{C}(n, \underline{u})$, in particular, is contained in an II code $\mathcal{C}\left(n, \underline{u}^{\prime}\right)$, both codes as given by Definition 1 . Then, by again setting the parities in the locations described in Theorem 12 and in Example 13, we obtain the vertical parities in locations $(a, b), m-s_{t} \leqslant a \leqslant m-1,0 \leqslant b \leqslant n-u_{t-1}-1$ using (4) with $i=0$. The remaining parities are computed by encoding the data and the vertical parities obtained in the previous step into the II code $\mathcal{C}\left(n, \underline{u}^{\prime}\right)$. Any encoding algorithm for II codes can be used, like, for example, the one described in 42].

The following theorem extends Theorem II. 2 on $t$-level II codes as stated in [40] and proven as Corollary 3 in [2] (see also [42]). It also generalizes the well known result that the minimum distance of a product code is the product of the minimum distances of the two component codes.

Theorem 15. Consider the $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 Then, the minimum distance of $\mathcal{C}(n, \underline{u})$ is

$$
\begin{equation*}
d=\min \left\{\left(\hat{s}_{i+1}+1\right)\left(u_{i}+1\right), 0 \leqslant i \leqslant t-1\right\} \tag{18}
\end{equation*}
$$

Proof: For each $i$ such that $0 \leqslant i \leqslant t-1$, consider an array in $\mathcal{C}(n, \underline{u})$ that has $\hat{s}_{i+1}$ rows with $u_{i}+1$ erasures each, one row with $u_{i}$ erasures, and all the other entries are zero. By Theorem 6 such arrays will be corrected by the code $\mathcal{C}(n, \underline{u})$ as the zero codeword, thus

$$
d \geqslant \min \left\{\left(\hat{s}_{i+1}+1\right)\left(u_{i}+1\right), 0 \leqslant i \leqslant t-1\right\}
$$

On the other hand, by Lemma 9 , for each $0 \leqslant i \leqslant t-1$, there is an array in $\mathcal{C}(n, \underline{u})$ of weight $\left(\hat{s}_{i+1}+1\right)\left(u_{i}+1\right)$, so

$$
d \leqslant \min \left\{\left(\hat{s}_{i+1}+1\right)\left(u_{i}+1\right), 0 \leqslant i \leqslant t-1\right\}
$$

and 18 follows.

Example 16. Consider the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Examples 710,13 and 14 . According to Theorem 15 since $m=6, u_{0}=1, u_{1}=3, u_{2}=4, s_{0}=2, s_{1}=1, s_{2}=1$ and $s_{3}=2$ (and hence, $\hat{s}_{3}=s_{3}=2, \hat{s}_{2}=s_{2}+s_{3}=3$, $\hat{s}_{1}=s_{1}+s_{2}+s_{3}=4$, by (18), the minimum distance of this code is $d=\min \{(5)(2) ;(4)(4) ;(3)(5)\}=10$.

Although the minimum distance is not the only criterium to determine the correction power of an II code [2] (see also Subsection III-C), given different II codes as $m \times n$ arrays with the same rate, there is one that has the largest minimum distance. A natural question is, if we include EII codes, is there any EII code whose minimum distance is larger than the one of any II code with the same rate? The answer depends on the parameters chosen, but the following example shows that indeed this may be the case.

Example 17. Consider the 4-level EII code $\mathcal{C}(7,(1,3,4,6,7))$. According to Theorem 15, the minimum distance of this code is $d=10$. An II code with the same rate is a code $\mathcal{C}\left(7,\left(v_{0}, v_{1}, v_{2}, v_{3}, v_{4}\right)\right.$ ), where $0 \leqslant v_{0} \leqslant v_{1} \leqslant v_{2} \leqslant v_{3} \leqslant v_{4}<7$ and $v_{0}+v_{1}+v_{2}+v_{3}+v_{4}=21$. Again by Theorem 15 the minimum distance of this code is $d \leqslant v_{4}+1 \leqslant 7$, so the EII code has larger minimum distance than any II code also consisting of $5 \times 7$ arrays and with the same rate.

## D. Transpose Arrays, Iterative Decoding and Uniform Distribution of Parity Symbols

Definition 1 states that a $t$-level EII code $\mathcal{C}(n, \underline{u})$ consists of $m \times n$ arrays such that each row in the array is in a code $\mathcal{C}_{0}$, and that certain linear combinations of the rows belong in nested codes $\mathcal{C}_{i}$. If we take the columns in an array in $\mathcal{C}(n, \underline{u})$, they would be the rows in an $n \times m$ transpose array. A natural question is, are the rows in these transpose arrays also related by certain nested codes?

Before answering this question, we consider the simple example of a product code such that the vertical code is an $\left[m, k_{0}, m-\right.$ $\left.k_{0}+1\right]$ code and the horizontal code is an $\left[n, k_{1}, n-k_{1}+1\right]$ code. We have seen in Example 3 that this product code is a 1-level EII code $\mathcal{C}(n, \underline{u})$ where $\underline{u}$ is the vector $(\overbrace{n-k_{1}, n-k_{1}, \ldots, n-k_{1}}^{k_{0}} \overbrace{n, n, \ldots, n}^{m-k_{0}})$. If we consider the transpose arrays of the product code, the rows of the transpose arrays (that is, the columns of the original arrays) constitute a 1-level EII code $\mathcal{C}\left(m, \underline{u}^{\prime}\right)$, where $\underline{u}^{\prime}$ is the vector $(\overbrace{m-k_{0}, m-k_{0}, \ldots, m-k_{0}}^{k_{1}}, \overbrace{m, m, \ldots, m}^{n-k_{1}})$. The following theorem generalizes this argument for $t$-level EII codes.
Theorem 18. Consider a $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 and take the set of $n \times m$ transpose arrays corresponding to the $m \times n$ arrays in $\mathcal{C}(n, \underline{u})$. Then, this set of $n \times m$ transpose arrays constitute a $t$-level EII code $\mathcal{C}\left(m, \underline{u}^{\prime}\right)$ such that, assuming $u_{-1}=0$,

$$
\begin{equation*}
\underline{u}^{\prime}=(\overbrace{u_{0}^{\prime}, u_{0}^{\prime}, \ldots, u_{0}^{\prime}}^{s_{0}^{\prime}}, \overbrace{u_{1}^{\prime}, u_{1}^{\prime}, \ldots, u_{1}^{\prime}}^{s_{1}^{\prime}}, \ldots, \overbrace{u_{t-1}^{\prime}, u_{t-1}^{\prime}, \ldots, u_{t-1}^{\prime}}^{s_{t-1}^{\prime}} \overbrace{u_{t}^{\prime}, u_{t}^{\prime}, \ldots, u_{t}^{\prime}}^{s_{t}^{\prime}}) \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{t-i}^{\prime}=\hat{s}_{i} \text { and } s_{i}^{\prime}=u_{t-i}-u_{t-i-1} \text { for } 0 \leqslant i \leqslant t \tag{20}
\end{equation*}
$$

Proof: Denote by $\underline{c}_{i}^{(\mathbf{H})}, 0 \leqslant i \leqslant m-1$, the rows of an array in $\mathcal{C}(n, \underline{u})$, and by $\underline{c}_{j}^{(\mathbf{V})}, 0 \leqslant j \leqslant n-1$, the columns (that is, the rows of the $n \times m$ transpose array). Specifically, if the array consists of symbols $\left(c_{i, j}\right)_{\substack{0 \leqslant j \leqslant m-1 \\ 0 \leqslant j \leqslant n-1}}$, then $\underline{c}_{i}^{(\mathbf{H})}=\left(c_{i, 0}, c_{i, 1}, \ldots, c_{i, n-1}\right)$ for $0 \leqslant i \leqslant m-1$ and $\underline{c}_{j}^{(\mathbf{V})}=\left(c_{0, j}, c_{1, j}, \ldots, c_{m-1, j}\right)$ for $0 \leqslant j \leqslant n-1$.

Consider the $t+1$ nested codes (on columns) $\{0\}=\mathcal{C}_{t}^{\prime} \subset \mathcal{C}_{t-1}^{\prime} \subset \mathcal{C}_{t-2}^{\prime} \subset \cdots \subset \mathcal{C}_{0}^{\prime}$, where $\mathcal{C}_{i}^{\prime}$ is an $\left[m, m-u_{i}^{\prime}, u_{i}^{\prime}+1\right]$ code and $u_{i}^{\prime}$ is given by (20). A parity-check matrix of $\mathcal{C}_{i}^{\prime}$ is $H_{u_{i}^{\prime}}$ as given by (2).

In order to prove the theorem, according to (4], we have to prove that each $\underline{c}_{j}^{(\mathbf{V})} \in \mathcal{C}_{0}^{\prime}$ for $0 \leqslant j \leqslant n-1$, and that

$$
\begin{array}{r}
\bigoplus_{j=0}^{n-1} \alpha^{r j} \underline{c}_{j}^{(\mathbf{V})} \in \quad \mathcal{C}_{t-i}^{\prime} \text { for } 0 \leqslant i \leqslant t-1 \\
\quad \text { and } 0 \leqslant r \leqslant \hat{s}_{t-i}^{\prime}-1 \tag{21}
\end{array}
$$

$\mathcal{C}_{0}^{\prime}$ is an $\left[m, m-u_{0}^{\prime}, u_{0}^{\prime}+1\right]$ code and by (20), $u_{0}^{\prime}=s_{t}$, so from (4), taking $i=0, \underline{c}_{j}^{(\mathbf{V})} \in \mathcal{C}_{0}^{\prime}$.
Next we have to prove (21). In effect, (21) holds if and only if, by (2),

$$
\bigoplus_{v=0}^{m-1} \alpha^{u v} \bigoplus_{j=0}^{n-1} \alpha^{r j} c_{v, j}=0
$$

for $0 \leqslant i \leqslant t-1,0 \leqslant u \leqslant u_{t-i}^{\prime}-1$ and $0 \leqslant r \leqslant \hat{s}_{t-i}^{\prime}-1$, if and only if, changing the summation order,

$$
\bigoplus_{j=0}^{n-1} \alpha^{r j} \bigoplus_{v=0}^{m-1} \alpha^{u v} c_{v, j}=0
$$

for $0 \leqslant i \leqslant t-1,0 \leqslant u \leqslant u_{t-i}^{\prime}-1$ and $0 \leqslant r \leqslant \hat{s}_{t-i}^{\prime}-1$, if and only if, since, by (3) and (20), $\hat{s}_{t-i}^{\prime}=\sum_{z=t-i}^{t} s_{z}^{\prime}=$ $\sum_{z=t-i}^{t}\left(u_{t-z}-u_{t-z-1}\right)=u_{i}$ and $u_{t-i}^{\prime}=\hat{s}_{i}$, by (2),

$$
\bigoplus_{v=0}^{m-1} \alpha^{u v}{\underset{\underline{c}}{v}}_{(\mathbf{H})} \in \mathcal{C}_{i} \text { for } 0 \leqslant i \leqslant t-1 \text { and } 0 \leqslant u \leqslant \hat{s}_{i}-1
$$

which is true by (4) and thus is also true.

Theorem 18 is the most important result in this section. One application is an enhancement of the decoding algorithm by extending the iterative decoding algorithm of product codes, in which rows and columns are decoded iteratively until either all the erasures are corrected or an uncorrectable pattern remains. In order to illustrate this process, let us revisit Example 8

Example 19. According to Theorem [18, the transpose $7 \times 4$ arrays of the $4 \times 7$ arrays in the 4 -level II code $\mathcal{C}(7,(1,2,3,5))$ of Example 8 are in a 4-level EII code $\mathcal{C}(4,(0,0,1,1,2,3,4))$. After (partially) decoding the rows of the array with erasures in Example 8, we were left with the uncorrectable array in $\mathcal{C}(7,(1,2,3,5))$


Notice that this array has two columns with no erasures, two columns with one erasure each and three columns with two erasures each. By Theorem 6 the array is correctable in $\mathcal{C}(4,(0,0,1,1,2,3,4))$. Hence, after two iterations the erasures are corrected.
Example 20. Consider the 5-level II code $\mathcal{C}(10,(1,3,6,8,9))$. The transpose arrays of $\mathcal{C}(10,(1,3,6,8,9))$ are in the 5 -level EII code $\mathcal{C}(5,(0,1,2,2,3,3,3,4,4,5))$ by Theorem 18 Assume that the following array is received:

| $E$ |  |  |  | $E$ | $E$ |  | $E$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $E$ | $E$ |  | $E$ | $E$ | $E$ | $E$ |  | $E$ |
|  |  |  |  |  |  |  |  | $E$ |  |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ | $E$ | $E$ |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ |  | $E$ |

Applying the decoding algorithm on rows, only the third row can be corrected, since it has exactly one erasure. After correction of the third row, we have the array

| $E$ |  |  |  | $E$ | $E$ |  | $E$ |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $E$ | $E$ |  | $E$ | $E$ | $E$ | $E$ |  | $E$ |
|  |  |  |  |  |  |  |  |  |  |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ | $E$ | $E$ |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ |  | $E$ |

This array has one column with no erasures, one column with one erasure and one column with two erasures, while the remaining columns contain more than two erasures. The decoding algorithm on columns (i.e., on the code $\mathcal{C}(5,(0,1,2,2,3,3,3,4,4,5))$ ) allows for correction of the column with one erasure and the column with two erasures, giving the array

| $E$ |  |  |  |  | $E$ |  | $E$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $E$ | $E$ |  |  | $E$ | $E$ | $E$ |  | $E$ |
|  |  |  |  |  |  |  |  |  |  |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ |  | $E$ |
| $E$ | $E$ | $E$ |  |  | $E$ | $E$ | $E$ |  | $E$ |

This last array is decodable on rows (i.e., on the code $\mathcal{C}(10,(1,3,6,8,9))$ ), hence, the erasures are corrected after three iterations.

A second application of Theorem 18 is allowing for a balanced distribution of the parity symbols in EII codes. In effect, given an $[m n, k]$ code consisting of $m \times n$ arrays, if $m n-k=q m+r$, where $0 \leqslant r<m$, we say that the code has a balanced distribution of parity symbols if $m-r$ of the rows contain $q$ parity symbols, while the remaining $r$ rows contain $q+1$ parity symbols. Codes somewhat similar to II codes with a balanced distribution of parity symbols were presented in [7]. Actually, in [7] only cases for which $r=0$ are considered, i.e., $m$ divides $m n-k$ and hence each row contains the same number $q$ of parity symbols.

Given a $t$-level EII code $\mathcal{C}(n, \underline{u})$, so far we have placed the parity symbols as in Theorem 12 and in examples 13 and 14 , that is, at the end of each row in non-decreasing order of the $u_{i} \mathrm{~s}$. However, this distribution of symbols in general is not balanced. If it can be shown that there is an uniform distribution of erasures that can be corrected by the code $\mathcal{C}\left(m, \underline{u}^{\prime}\right)$ (i.e., the code on columns as given by Theorem [18), then we can use those erasures as the locations for the parity symbols. The following theorem shows that, using Theorem 18 we can easily obtain a balanced distribution of the parity symbols for a $t$-level EII code $\mathcal{C}(n, \underline{u})$.

Theorem 21. Consider a $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 Then $\mathcal{C}(n, \underline{u})$ admits a balanced distribution of the parity symbols.

Proof: We need to find $s=\sum_{i=0}^{t} s_{i} u_{i}$ erasures such that, if $s=q m+r$ with $0 \leqslant r<m$, then there are $m-r$ rows with $q$ erasures each and $r$ rows with $q+1$ erasures each, and the erasures are correctable by the the $t$-level EII code $\mathcal{C}\left(m, \underline{u}^{\prime}\right)$ on columns as given by Theorem 18 . Then such erasures can be used to place the parity symbols.

In effect, let $v_{0} \geqslant v_{1} \geqslant \ldots \geqslant v_{z-1}$ be the non-zero elements of $\underline{u}^{\prime}$ in non-increasing order. In particular, $s=\sum_{i=0}^{z-1} v_{i}$. We will select the first $z$ columns in an $m \times n$ array such that column $j$ has exactly $v_{j}$ erasures for $0 \leqslant j \leqslant z-1$. Then, by Theorem 18, such erasures are correctable. In addition, we will show that the selection of erasures is balanced. We proceed by induction.

If $z=1$, we have only one column and we place the erasures in the top $v_{0}$ positions of that column. In particular, the distribution is balanced. So assume that $z>1$.

Consider the first $z-1$ columns and let $s^{\prime}=\sum_{i=0}^{z-2} v_{i}$. By induction, if $s^{\prime}=q^{\prime} m+r^{\prime}$, we can place $s_{j}$ erasures in column $j$ for $0 \leqslant j \leqslant z-2$, such that the first $r^{\prime}$ rows contain $q^{\prime}+1$ erasures and the last $m-r^{\prime}$ rows contain $q^{\prime}$ erasures.

If $v_{z-1} \leqslant m-r^{\prime}$, then in column $z-1$ we place the $v_{z-1}$ erasures in locations $r^{\prime}, r^{\prime}+1, \ldots, r^{\prime}+v_{z-1}-1$. Then the first $r^{\prime}+v_{z-1}$ rows contain $q^{\prime}+1$ erasures and the last $m-\left(r^{\prime}+v_{z-1}\right)$ rows contain $q^{\prime}$ erasures, giving a balanced distribution of the erasures.

If $v_{z-1}>m-r^{\prime}$, then in column $z-1$ we place the $v_{z-1}$ erasures in locations

$$
0,1, \ldots, v_{z-1}-\left(m-r^{\prime}\right)-1, r^{\prime}, r^{\prime}+1, \ldots, m-1
$$

Then the first $v_{z-1}-\left(m-r^{\prime}\right)$ rows contain $q^{\prime}+1$ erasures and the remaining rows $q^{\prime}$ erasures, also giving a balanced distribution of the erasures.

Of course the balanced distribution of parity symbols is not unique. We illustrate the method described in Theorem 21 in the next two examples.

Example 22. Consider a product code consisting of $5 \times 7$ arrays such that each row has one parity and each column two parities. We have seen in Example 3 that such a code can be viewed as a 1-level EII code $\mathcal{C}(7,(1,1,1,7,7))$. The distribution of parities given in the proof of Theorem 21 in this case is the following:

| $E$ | $E$ |  | $E$ |  |  | $E$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $E$ | $E$ |  |  | $E$ |  | $E$ |
| $E$ |  | $E$ |  | $E$ |  |  |
| $E$ |  | $E$ |  |  | $E$ |  |
| $E$ |  |  | $E$ |  | $E$ |  |

Certainly it is not necessary to invoke Theorem 21 to obtain a balanced distribution of the parities in a product code. The next example is more representative.

Example 23. Consider the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Examples $10,13,14$ and 16 According to Theorem 18 the $7 \times 6$ transpose arrays of this code constitute a 3-level EII code $\mathcal{C}(6,(2,2,2,3,4,4,6))$. The balanced distribution of parities given by Theorem 21 is

| $E$ | $E$ | $E$ |  | $E$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $E$ | $E$ | $E$ |  |  | $E$ |  |
| $E$ | $E$ |  | $E$ |  | $E$ |  |
| $E$ | $E$ |  | $E$ |  |  | $E$ |
| $E$ |  | $E$ | $E$ |  |  | $E$ |
| $E$ |  | $E$ |  | $E$ |  |  |

For encoding using a balanced distribution of parities, we apply the decoding algorithm by triangulation on the $t$-level EII code $\mathcal{C}\left(m, \underline{u}^{\prime}\right)$ on columns given by Theorem 18 Again, the fact that the erasures are known a priori allows for precomputing the coefficients arising from the triangulation. Obviously, any $t$-level EII code $\mathcal{C}(n, \underline{u})$ also admits a balanced distribution of symbols on columns.

## E. Error and Erasure Decoding of EII Codes

Although in this paper we concentrate on the erasure model, the decoding algorithm can be adapted to handle errors together with erasures. Specifically:
Algorithm 24. Consider a $t$-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 and assume that a received $m \times n$ array contains both errors and erasures. Then proceed as follows:

1) Attempt to correct in $\mathcal{C}_{0}$ rows with up to $i$ errors together with up to $j$ erasures, where $2 i+j \leqslant u_{0}$.
2) Consider the $\ell$ rows for which the decoding has failed. If $\ell=0$, then correction has been successful and exit the algorithm.
3) If $\ell>\hat{s}_{1}$, then declare that the algorithm has failed. Otherwise, as in Theorem 6 let $i_{0}, i_{1}, \ldots, i_{m-1}$ be an ordering of the rows according to a non-increasing number of erasures such that rows $i_{0}, i_{1}, \ldots, i_{\ell-1}$ correspond to the $\ell$ rows for which the decoding in $\mathcal{C}_{0}$ has failed.
4) Define $w$ as in (10), i.e., $\hat{s}_{w+1}<\ell \leqslant \hat{s}_{w}$ and consider the code $\mathcal{C}_{w}$ from the nested set of codes in Definition 1 which can correct up to $i$ errors together with up to $j$ erasures for $2 i+j \leqslant u_{w}$.
5) Proceeding as in Theorem 6, after triangulation, obtain (13). Then attempt to correct up to $i$ errors together with up to $j$ erasures, where $2 i+j \leqslant u_{w}$. If the decoding is successful, continue by induction with the remaining $\ell-1$ rows. If the decoding is unsuccessful, change the order of the $\ell$ uncorrected rows (for example, by rotating them) and repeat the procedure until $\underline{c}_{i_{\ell-1}}$ is decoded successfully and then proceed by induction. If none of the $\ell$ rows is decoded successfully after this procedure, declare failure.
If the algorithm fails, then correction is attempted on columns.

Contrary to the case of erasures only, there is now a probability of miscorrection each time individual decoding or errors together with erasures is attempted: if the error-erasure correcting power of the codes is exceeded, the decoder may miscorrect and give the wrong codeword. However, if the finite field is fairly large and the codes can correct a substantial number of errors, such probability is small [8], [9], [30] and we assume that miscorrection does not occur (otherwise, the decoding algorithm gets more complicated). A similar assumption was made in [42]. Certainly, also the decoding algorithms of [40] and [42] can be adapted for $t$-level EII codes.

We illustrate Algorithm 24 with an example.
Example 25. Consider $\mathcal{C}(15,(3,3,5,8,8,15))$ as a 3-level EII code over the field $G F(16)$, hence, $\mathcal{C}_{0}$ is a $[15,12,4]$ code, $\mathcal{C}_{1}$ is a $[15,10,6]$ code, $\mathcal{C}_{2}$ is a $[15,7,9]$ code and $\mathcal{C}_{3}=\{0\}$. Denoting errors by $X$ and erasures by $E$, assume that the following array has been received:


Following Algorithm 24 correction of up to one error together with one erasure is attempted using code $\mathcal{C}_{0}$. This correction succeeds for rows $\underline{c}_{1}$ and $\underline{c}_{4}$, and fails in the remaining 4 rows (i.e., $\ell=4$ ). Ordering the rows in non-decreasing number of erasures and according to (7), (8) and (9) as in Example 7 we obtain (now rows $\underline{c}_{1}$ and $\underline{c}_{4}$ are error-free)

Triangulating this linear system in $G F(16)$ and assuming that $1 \oplus \alpha \oplus \alpha^{4}=0$, we obtain

$$
\begin{array}{rllllllllll}
\underline{\mathcal{c}}_{5} \oplus \underline{\mathcal{c}}_{0} & \oplus & \underline{c}_{2} & \oplus & \underline{c}_{3} & \oplus & \underline{c}_{1} & \oplus & \underline{c}_{4} & = & 0 \\
\underline{c}_{0} \oplus \alpha^{6} \underline{\mathcal{c}}_{2} & \oplus & \alpha_{3} & \oplus & \alpha^{7} \underline{c}_{1} & \oplus & \alpha^{13} \underline{c}_{4} & \in & \mathcal{C}_{2} \\
& \underline{c}_{2} & \oplus & \alpha \underline{c}_{3} & \oplus & \alpha^{12} \underline{c}_{1} & \oplus & \underline{c}_{4} & \in & \mathcal{C}_{2}
\end{array}
$$

Since $\underline{c}_{3} \oplus \alpha^{10} \underline{\mathcal{c}}_{1} \oplus \alpha^{3} \underline{\mathcal{c}}_{4}$ has two errors, they can be corrected in $\mathcal{C}_{1}$. So, we get

$$
\underline{c}_{3}=\left(\underline{c}_{3} \oplus \alpha^{10} \underline{c}_{1} \oplus \alpha^{3} \underline{c}_{4}\right) \oplus\left(\alpha^{10} \underline{c}_{1} \oplus \alpha^{3} \underline{c}_{4}\right)
$$

Next we attempt to decode $\underline{c}_{2} \oplus \alpha \underline{\mathcal{c}}_{3} \oplus \alpha^{12} \underline{\underline{c}}_{1} \oplus \underline{\mathcal{c}}_{4}$ in $\mathcal{C}_{2}$. But this vector has five errors, which are uncorrectable in $\mathcal{C}_{2}$. Making a rotation of the 3 uncorrected vectors, we have

Triangulating this linear system, we obtain

$$
\begin{aligned}
\underline{c}_{2} \oplus \underline{c}_{5} \oplus \underline{\mathcal{c}}_{0} \oplus \underline{\mathcal{c}}_{3} \oplus \underline{c}_{1} \oplus \underline{c}_{4} & =0 \\
\underline{c}_{5} \oplus \alpha^{7} \underline{c}_{0} \oplus \alpha^{5} \underline{c}_{3} \oplus \alpha^{4} \underline{c}_{1} \oplus \alpha^{9} \underline{c}_{4} & \in \mathcal{C}_{2} \\
\underline{\underline{c}}_{0} \oplus \alpha^{14} \underline{\underline{c}}_{3} \oplus \alpha^{4} \underline{c}_{1} \oplus \underline{c}_{4} & \in \mathcal{C}_{2} .
\end{aligned}
$$

Now, $\underline{c}_{0} \oplus \alpha^{14} \underline{\underline{c}}_{3} \oplus \alpha^{4} \underline{\underline{c}}_{1} \oplus \underline{c}_{4}$ has four errors, that are correctable in $\mathcal{C}_{2}$, so $\underline{c}_{0}$ is obtained as

$$
\underline{c}_{0}=\left(\underline{c}_{0} \oplus \alpha^{14} \underline{c}_{3} \oplus \alpha^{4} \underline{c}_{1} \oplus \underline{c}_{4}\right) \oplus\left(\alpha^{14} \underline{c}_{3} \oplus \alpha^{4} \underline{c}_{1} \oplus \underline{c}_{4}\right) .
$$

Next, $\underline{c}_{5} \oplus \alpha^{7} \underline{\mathcal{c}}_{0} \oplus \alpha^{5} \underline{\mathcal{c}}_{3} \oplus \alpha^{4} \underline{\mathcal{c}}_{1} \oplus \alpha^{9} \underline{c}_{4}$ has two errors and four erasures, which are also correctable in $\mathcal{C}_{2}$, so

$$
\underline{c}_{5}=\left(\underline{c}_{5} \oplus \alpha^{7} \underline{\underline{c}}_{0} \oplus \alpha^{5} \underline{\underline{c}}_{3} \oplus \alpha^{4} \underline{\underline{c}}_{1} \oplus \alpha^{9} \underline{\underline{c}}_{4}\right) \oplus\left(\alpha^{7}{\underline{c_{0}}}_{0} \oplus \alpha^{5} \underline{\underline{c}}_{3} \oplus \alpha^{4} \underline{\underline{c}}_{1} \oplus \alpha^{9} \underline{\underline{c}}_{4}\right) .
$$

Finally,

$$
\underline{\mathcal{c}}_{2}=\left(\underline{c}_{2} \oplus \underline{c}_{5} \oplus \underline{c}_{0} \oplus \underline{c}_{3} \oplus \underline{c}_{1} \oplus \underline{c}_{4}\right) \oplus\left(\underline{c}_{5} \oplus \underline{c}_{0} \oplus \underline{c}_{3} \oplus \underline{c}_{1} \oplus \underline{c}_{4}\right),
$$

completing the decoding.

## III. Extended Product Codes and Optimality Issues

This section is structured as follows: in Subsection III-A we present an upper bound on the minimum distance of EPC codes, we illustrate it with examples and we show that other bounds, like the bound on LRC codes, are special cases of this bound. In Subsection III-B we present some constructions of codes meeting this upper bound for important special cases. In Subsection III-C, we briefly discuss tradeoffs between different codes by giving Monte Carlo simulations for some specific parameters.

## A. Upper Bound on the Minimum Distance of Extended Product Codes

The $t$-level EII codes $\mathcal{C}(n, \underline{u})$ described in Section $\Pi$ are a special case of product codes with some extra parities defined in Section where we called these codes extended product (EPC) codes and we denoted them by $E P(m, v ; n, h ; g)$, $v$ the number of vertical parities in each column, $h$ the number of horizontal parities in each row, and $g$ the number of extra parities. From Definition 1 it is easy to determine $v, h$ and $g$ for $t$-level EII codes $\mathcal{C}(n, \underline{u})$. In effect, since $v=s_{t}$ and $h=u_{0}$, the extra parities consist of all the remaining parities, i.e., $g=\left(\sum_{i=0}^{t} u_{i} s_{i}\right)-u_{0} m-s_{t}\left(n-u_{0}\right)$. For example, the 3-level EII code $\mathcal{C}(7,(1,1,3,4,7,7))$ of Examples 710,13 and 23 is an $E P(6,2 ; 7,1 ; 5)$ code.

The next theorem gives an upper bound on the minimum distance of an $E P(m, v ; n, h ; g)$ code.
Theorem 26. Let $d(m, v ; n, h ; g)$ be the minimum distance of an $E P(m, v ; n, h ; g)$ code. Given $a$ such that $1 \leqslant a \leqslant g+1$, $b=\lfloor(g+1) / a\rfloor$ and $r=g+1-a b$, let

$$
\begin{align*}
& d(v, h, g ; a)=(v+b)(h+a) \quad \text { if } \quad r=0  \tag{22}\\
& d(v, h, g ; a)=(v+b)(h+a)+h+r \text { if } r \neq 0 \tag{23}
\end{align*}
$$

Then,

$$
\begin{equation*}
d(m, v ; n, h ; g) \leqslant \min \{d(v, h, g ; a):\lceil(g+1) /(m-v)\rceil \leqslant a \leqslant \min \{g+1, n-h\}\} \tag{24}
\end{equation*}
$$

Proof: Assume first that $r=0$, the zero array is stored, and the received array has the $(v+b)(h+a)$ locations $(i, j)$ erased, where $0 \leqslant i \leqslant v+b-1$ and $0 \leqslant j \leqslant h+a-1$. Notice that, since $\lceil(g+1) /(m-v)\rceil \leqslant a \leqslant \min \{g+1, n-h\}$,

$$
0 \leqslant i \leqslant v+b-1 \leqslant v+\frac{g+1}{(g+1) /(m-v)}-1=m-1
$$

and $0 \leqslant j \leqslant h+a-1 \leqslant h+(n-h)-1=n-1$, so all the erasures are within the array. The erasures are covered by $h(v+b)$ horizontal parities, $v(h+a)$ vertical parities and $g$ extra parities, but $h v$ of such parities are dependent. Since $a b=g+1$, there are only $(v+b)(h+a)-1$ independent parities covering the $(v+b)(h+a)$ erasures, insufficient to correct them.

Similarly, assume that $r \neq 0$, the zero array is stored, and the received array has the $(v+b)(h+a)+h+r$ locations $(i, j)$ erased, where either $0 \leqslant i \leqslant v+b-1$ and $0 \leqslant j \leqslant h+a-1 \leqslant n-1$, or $i=v+b$ and $0 \leqslant j \leqslant h+r-1$. Observe that all the erasures are within the array. In effect, since $a$ does not divide $g+1$,

$$
v+b=v+\left\lfloor\frac{g+1}{a}\right\rfloor<v+\frac{g+1}{a} \leqslant v+\frac{g+1}{\lceil(g+1) /(m-v)\rceil} \leqslant v+(m-v)=m
$$

The erasures are covered by $h(v+b+1)$ horizontal parities, $v(h+a)$ vertical parities and $g$ extra parities. Since $h v$ of such parities are dependent and $g+1=a b+r$, this gives a total of $(v+b)(h+a)+h+r-1$ independent parities covering the $(v+b)(h+a)+h+r$ erasures, insufficient to correct them.

Example 27. Consider an $E P(5,2 ; 8,3 ; 3)$ code and let $d(5,2 ; 8,3 ; 3)$ be its minimum distance. According to (24), we have $d(5,2 ; 8,3 ; 3) \leqslant \min \{d(2,3,3 ; a): 2 \leqslant a \leqslant 4\}$, where, by 22) and 23), $d(2,3,3 ; 2)=20, d(2,3,3 ; 3)=22$ and $d(2,3,3 ; 4)=21$, so $d(5,2 ; 8,3 ; 3) \leqslant 20$.

Example 28. Consider $E P(m, 1 ; n, 1 ; g)$ codes such that $g+1<\min \{m, n\}$. The following table gives the upper bound, according to Theorem [26, for $d(m, 1 ; n, 1 ; g)$, where $0 \leqslant g \leqslant 13$ :

| $g$ | $d(m, 1 ; n, 1 ; g) \leqslant$ |
| :---: | :---: |
| 0 | 4 |
| 1 | 6 |
| 2 | 8 |
| 3 | 9 |
| 4 | 11 |
| 5 | 12 |
| 6 | 14 |


| $g$ | $d(m, 1 ; n, 1 ; g) \leqslant$ |
| :---: | :---: |
| 7 | 15 |
| 8 | 16 |
| 9 | 18 |
| 10 | 19 |
| 11 | 20 |
| 12 | 22 |
| 13 | 23 |

The special case $v=0$ in Theorem 26 corresponds to LRC codes [14], [34], [39], i.e., there is no vertical code. Let us state explicitly the result for this case.

Corollary 29 Consider an $E P(m, 0 ; n, h ; g)$ code. Then,

$$
\begin{equation*}
d(m, 0 ; n, h ; g) \leqslant\left\lceil\frac{g+1}{n-h}\right\rceil h+g+1 \tag{25}
\end{equation*}
$$

Proof: Taking $a=\min \{g+1, n-h\}$ in gives for this case

$$
\begin{equation*}
d(m, 0 ; n, h ; g) \leqslant d(0, h, g ; \min \{g+1, n-h\}) \tag{26}
\end{equation*}
$$

If $g+1<n-h$, then $b=1$, so (22) gives $d(0, h, g ; g+1)=h+g+1$ and (25) follows from (26).
If $g+1 \geqslant n-h$ and $n-h$ divides $g+1$, by (22),

$$
d(0, h, g ; n-h)=\left(\frac{g+1}{n-h}\right)(h+n-h)=\left(\frac{g+1}{n-h}\right) h+g+1
$$

and (25) follows from 26). If $g+1>n-h$ and $n-h$ does not divide $g+1$, by (23),

$$
d(0, h, g ; n-h)=\left\lfloor\frac{g+1}{n-h}\right\rfloor n+h+g+1-(n-h)\left\lfloor\frac{g+1}{n-h}\right\rfloor=\left(\left\lfloor\frac{g+1}{n-h}\right\rfloor+1\right) h+g+1=\left\lceil\frac{g+1}{n-h}\right\rceil h+g+1
$$

and also in this case, (25) follows from (26).

Bound (25) is well known, albeit it is usually given in a slightly different form [14], [34] as a function of the dimension of the code, while bound (25) is given as a function of the redundancy. It was also shown that bound (25) can be achieved with efficient constructions [39] over a field $G F(q)$, where $q \geqslant m n$ and in general $q$ is minimal with this property.

## B. Some Optimal Extended Product Codes

We say that an $E P(m, v ; n, h ; g)$ code is optimal if it meets bound with equality. We believe that there are optimal $E P(m, v ; n, h ; g)$ codes for any choice of parameters, but the subject requires further research.

The next theorem shows that there is a range of parameters for which 2-level EII codes are optimal extended product codes.
Theorem 30. Consider the 2-level EII code $\mathcal{C}(n, \underline{u})$ as given by Definition 1 where

$$
\begin{equation*}
\underline{u}=(\overbrace{h, h, \ldots, h}^{m-v-1}, h+g, \overbrace{n, n, \ldots, n}^{v}) \tag{27}
\end{equation*}
$$

$0 \leqslant v<m-1, v \leqslant h, h+g<n$ and $1 \leqslant g \leqslant\left\lceil\frac{h-v+1}{v+1}\right\rceil$. Then, $\mathcal{C}(n, \underline{u})$ is an optimal $E P(m, v ; n, h ; g)$ code with minimum distance

$$
\begin{equation*}
d=(h+g+1)(v+1) \tag{28}
\end{equation*}
$$

Proof: Applying Theorem 15 to (27), the minimum distance of $\mathcal{C}(n, \underline{u})$ is $d=\min \{(h+g+1)(v+1),(h+1)(v+2)\}$. In fact, we will show that $(h+g+1)(v+1) \leqslant(h+1)(v+2)$ and hence 28 follows.

In effect, $(h+g+1)(v+1) \leqslant(h+1)(v+2)$ if and only if $g \leqslant(h+1) /(v+1)$. By the conditions on $g$, it suffices to prove that

$$
\begin{equation*}
\left\lceil\frac{h-v+1}{v+1}\right\rceil \leqslant \frac{h+1}{v+1} . \tag{29}
\end{equation*}
$$

If $\left\lceil\frac{h-v+1}{v+1}\right\rceil=\frac{h-v+1}{v+1}$, (29) is immediate, so assume that $\left\lceil\frac{h-v+1}{v+1}\right\rceil>\frac{h-v+1}{v+1}$. Then, there is a $j, 1 \leqslant j \leqslant v$, such that $\left\lceil\frac{h-v+1}{v+1}\right\rceil=\frac{h-v+j+1}{v+1}$. But $\frac{h-v+j+1}{v+1} \leqslant \frac{h+1}{v+1}$ since $j \leqslant v$, so (29) follows also in this case.

Next we have to prove that this minimum distance $d$ given by (28) meets bound (24) of Theorem 26, It suffices to observe that $d=d(v, h, g ; g+1)$ by (22).

Notice that, in particular, if $v=0$, we have an LRC code, as in Corollary 29 In this case, Theorem 30 asserts that when $1 \leqslant g \leqslant h+1$, then $\mathcal{C}(n, \underline{u})$ is an optimal LRC code with minimum distance $d=h+g+1$. This result was also observed in [2], Corollary 2.3.

Let us examine now the case of $E P(m, 1 ; n, 1 ; 2)$ codes, where $m, n \geqslant 3$. In this case, bound gives $d(m, 1 ; n, 1 ; 2) \leqslant 8$.
Consider for example a 2-level EII code $\mathcal{C}(n,(\overbrace{1,1, \ldots, 1}^{m-2}, 3, n))$ or a 2-level EII code $\mathcal{C}(n,(\overbrace{1,1, \ldots, 1}^{m-3}, 2, n))$. These are the only cases of EII codes that are $E P(m, 1 ; n, 1 ; 2)$ codes. In both cases, according to Theorem 15 the minimum distance is 6 , so bound (24) is not met. We present next an optimal $E P(m, 1 ; n, 1 ; 2)$ code. The construction is related to the PMDS constructions in [4]. The tradeoff is that the finite field is larger than the required one for EII codes.

Let $G F\left(2^{b}\right)$ be a finite field and $\alpha$ an element in $G F\left(2^{b}\right)$ such that $m n \leqslant \mathcal{O}(\alpha)$ (remember, $\mathcal{O}(\alpha)$ denotes the order of $\alpha$ ). Consider the parity-check matrix $\mathcal{H}_{2}$ given by

$$
\mathcal{H}_{2}=\left(\begin{array}{c}
I_{m} \otimes(\overbrace{(1,1, \ldots, 1}^{m})  \tag{30}\\
\frac{(\overbrace{1,1, \ldots, 1}^{m}) \otimes I_{n}}{n} \begin{array}{cccc}
1 & \alpha & \alpha^{2} & \ldots
\end{array} \alpha^{m n-1} \\
1
\end{array} \alpha^{-1} \begin{array}{ccc}
\alpha^{-2} & \ldots & \alpha^{-(m n-1)}
\end{array}\right),
$$

where $I_{m}$ denotes the $m \times m$ identity matrix and $\otimes$ the Kronecker product [28] of two matrices. Notice that the first $m+n$ rows of $\mathcal{H}_{2}$ in (30) correspond to the parity-check matrix of the product code with single parity in rows and columns. We denote the matrix in (30) $\mathcal{H}_{2}$ to indicate that two extra parities are added to the product code.

The following theorem shows that the code whose parity-check matrix is $\mathcal{H}_{2}$ is an optimal $E(m, 1 ; n, 1 ; 2)$ code.
Theorem 31. Consider the $E P(m, 1 ; n, 1 ; 2)$ code whose parity-check matrix $\mathcal{H}_{2}$ is given by $\sqrt{30}, m, n \geqslant 3$ and $m n \leqslant \mathcal{O}(\alpha)$. Then, the code has minimum distance 8 .

Proof: We have to prove that any 7 erasures can be corrected.
First assume that there are six erasures in locations $\left(i_{0}, j_{0}\right),\left(i_{0}, j_{1}\right),\left(i_{0}, j_{2}\right),\left(i_{1}, j_{0}\right),\left(i_{1}, j_{1}\right)$ and $\left(i_{1}, j_{2}\right)$, where $0 \leqslant i_{0}<i_{1} \leqslant$ $m-1$ and $0 \leqslant j_{0}<j_{1}<j_{2} \leqslant n-1$ or $\left(i_{0}, j_{0}\right),\left(i_{0}, j_{1}\right),\left(i_{1}, j_{0}\right),\left(i_{1}, j_{1}\right),\left(i_{2}, j_{0}\right)$ and $\left(i_{2}, j_{1}\right)$, where $0 \leqslant i_{0}<i_{1}<i_{2} \leqslant m-1$
and $0 \leqslant j_{0}<j_{1} \leqslant n-1$, and a seventh erasure in any other location. This seventh erasure can be corrected using either horizontal or vertical parities, thus, it is enough to prove that the two situations of six erasures described above are correctable.

Consider the first case. It suffices to prove, using the parity-check matrix as given by (30), that the $6 \times 6$ matrix

$$
\left(\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
\alpha^{i_{0} n+j_{0}} & \alpha^{i_{0} n+j_{1}} & \alpha^{i_{0} n+j_{2}} & \alpha^{i_{1} n+j_{0}} & \alpha^{i_{1} n+j_{1}} & \alpha^{i_{1} n+j_{2}} \\
\alpha^{-i_{0} n-j_{0}} & \alpha^{-i_{0} n-j_{1}} & \alpha^{-i_{0} n-j_{2}} & \alpha^{-i_{1} n-j_{0}} & \alpha^{-i_{1} n-j_{1}} & \alpha^{-i_{1} n-j_{2}}
\end{array}\right)
$$

is invertible. Redefining $i \leftarrow i_{1}-i_{0}, j_{1} \leftarrow j_{1}-j_{0}$ and $j_{2} \leftarrow j_{2}-j_{0}$, where now $1 \leqslant i \leqslant m-1$ and $1 \leqslant j_{1}<j_{2} \leqslant n-1$, this matrix is invertible if and only if matrix

$$
\left(\begin{array}{cccccc}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & \alpha^{j_{1}} & \alpha^{j_{2}} & \alpha^{i n} & \alpha^{i n+j_{1}} & \alpha^{i n+j_{2}} \\
1 & \alpha^{-j_{1}} & \alpha^{-j_{2}} & \alpha^{-i n} & \alpha^{-i n-j_{1}} & \alpha^{-i n-j_{2}}
\end{array}\right)
$$

is invertible. By Gaussian elimination, this $6 \times 6$ matrix is invertible if and only if the $2 \times 2$ matrix

$$
\left(\begin{array}{cc}
\left(1 \oplus \alpha^{j_{1}}\right)\left(1 \oplus \alpha^{i n}\right) & \left(1 \oplus \alpha^{j_{2}}\right)\left(1 \oplus \alpha^{i n}\right) \\
\left(1 \oplus \alpha^{-j_{1}}\right)\left(1 \oplus \alpha^{-i n}\right) & \left(1 \oplus \alpha^{-j_{2}}\right)\left(1 \oplus \alpha^{-i n}\right)
\end{array}\right)
$$

is invertible, if and only if, since $1 \oplus \alpha^{i n} \neq 0$,

$$
\left(\begin{array}{cc}
1 \oplus \alpha^{j_{1}} & 1 \oplus \alpha^{j_{2}} \\
1 \oplus \alpha^{-j_{1}} & 1 \oplus \alpha^{-j_{2}}
\end{array}\right)=\left(\begin{array}{cc}
1 \oplus \alpha^{j_{1}} & 1 \oplus \alpha^{j_{2}} \\
\alpha^{-j_{1}}\left(1 \oplus \alpha^{j_{1}}\right) & \alpha^{-j_{2}}\left(1 \oplus \alpha^{j_{2}}\right)
\end{array}\right)
$$

is invertible, if and only if, since $1 \oplus \alpha^{j_{1}} \neq 0$ and $1 \oplus \alpha^{j_{2}} \neq 0, \alpha^{j_{1}} \neq \alpha^{j_{2}}$, which is the case since $1 \leqslant j_{1}<j_{2} \leqslant n-1<\mathcal{O}(\alpha)$. The second case is proven similarly.

Next, assume that there are seven erasures, such that each row and column has at least two erasures. This can only happen if one row (column) has three erasures and two rows (columns) have two erasures.

Let $i_{0}$ be the row with three erasures, and $j_{0}$ the column with three erasures, while $j_{1}<j_{2}$ and $i_{1}$ is such that erasures are in $\left(i_{1}, j_{0}\right)$ and $\left(i_{1}, j_{1}\right)$ so the remaining two erasures are in $\left(i_{2}, j_{0}\right)$ and $\left(i_{2}, j_{2}\right)$. Redefining $i_{1} \leftarrow i_{1}-i_{0}, i_{2} \leftarrow i_{2}-i_{0}$, $j_{1} \leftarrow j_{1}-j_{0}$ and $j_{2} \leftarrow j_{2}-j_{0}$, it suffices to prove, using the parity-check matrix $\mathcal{H}_{2}$ as given by $\sqrt{30}$, that the $7 \times 7$ matrix

$$
\left(\begin{array}{ccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & \alpha^{j_{1}} & \alpha^{j_{2}} & \alpha^{i_{1} n} & \alpha^{i_{1} n+j_{1}} & \alpha^{i_{2} n} & \alpha^{i_{2} n+j_{2}} \\
1 & \alpha^{-j_{1}} & \alpha^{-j_{2}} & \alpha^{-i_{1} n} & \alpha^{-i_{1} n-j_{1}} & \alpha^{-i_{2} n} & \alpha^{-i_{2} n-j_{2}}
\end{array}\right)
$$

is invertible, if and only if, doing Gaussian elimination like in the other two cases, the $2 \times 2$ matrix

$$
\left(\begin{array}{cc}
\left(1 \oplus \alpha^{j_{1}}\right)\left(1 \oplus \alpha^{i_{1} n}\right) & \left(1 \oplus \alpha^{j_{2}}\right)\left(1 \oplus \alpha^{i_{2} n}\right) \\
\left(1 \oplus \alpha^{-j_{1}}\right)\left(1 \oplus \alpha^{-i_{1} n}\right) & \left(1 \oplus \alpha^{-j_{2}}\right)\left(1 \oplus \alpha^{-i_{2} n}\right)
\end{array}\right)=\left(\begin{array}{cc}
\left(1 \oplus \alpha^{j_{1}}\right)\left(1 \oplus \alpha^{i_{1} n}\right) & \left(1 \oplus \alpha^{j_{2}}\right)\left(1 \oplus \alpha^{i_{2} n}\right) \\
\alpha^{-i_{1} n-j_{1}}\left(1 \oplus \alpha^{j_{1}}\right)\left(1 \oplus \alpha^{i_{1} n}\right) & \alpha^{-i_{2} n-j_{2}}\left(1 \oplus \alpha^{j_{2}}\right)\left(1 \oplus \alpha^{i_{2} n}\right)
\end{array}\right)
$$

is invertible, if and only if, since $1 \oplus \alpha^{j_{1}}, 1 \oplus \alpha^{i_{1} n}, 1 \oplus \alpha^{j_{2}}$ and $1 \oplus \alpha^{i_{2} n}$ are non-zero, $\alpha^{i_{1} n+j_{1}} \neq \alpha^{i_{2} n+j_{2}}$ which is the case since $m n \leqslant \mathcal{O}(\alpha)$, thus $\left(i_{2}-i_{1}\right) n+j_{2}-j_{1} \not \equiv 0(\bmod \mathcal{O}(\alpha))$. For complete details, see [3].

Consider next the 3-level EII code $\mathcal{C}(n, \underline{u})$, where $\underline{u}=(\overbrace{1,1, \ldots, 1}^{m-3}, 2,3, n)$. This is an $E P(m, 1 ; n, 1 ; 3)$ code. According to Theorem 15, $\mathcal{C}(n, \underline{u})$ has minimum distance 8 , the same as the code given by parity-check matrix $\mathcal{H}_{2}$, at the cost of an extra parity. However, there is a tradeoff: the size of the field required by $\mathcal{C}(n, \underline{u})$ is greater than $\max \{m ; n\}$, while the field required by the code whose parity-check matrix is $\mathcal{H}_{2}$ must have size greater than $m n$. Also, by Theorem 6 $\mathcal{C}(n, \underline{u})$ can correct any 8 erasures involving two rows with 3 erasures and one row with two erasures, like for example a pattern with
erasures in locations $\left(i_{0}, j_{0}\right),\left(i_{0}, j_{1}\right),\left(i_{0}, j_{2}\right),\left(i_{1}, j_{0}\right),\left(i_{1}, j_{1}\right),\left(i_{1}, j_{2}\right),\left(i_{2}, j_{0}\right)$ and $\left(i_{2}, j_{1}\right)$. The code generated by $\mathcal{H}_{2}$ is unable to correct such pattern since it does not have enough parities; so, even if both codes have the same minimum distance, $\mathcal{C}(n, \underline{u})$ can correct more erasure patterns. These tradeoffs need to be evaluated when implementation is considered.

We end this section with a construction of $E P(m, 1 ; n, 1 ; g)$ codes. Let $f(x)$ be a binary irreducible polynomial of degree $b$, $G F\left(2^{b}\right)$ the field of polynomials modulo $f(x)$ and $\alpha$ an element in $G F\left(2^{b}\right)$ such that $f(\alpha)=0$. Consider the following paritycheck matrix of a code consisting of $m \times n$ arrays:

$$
\mathcal{H}(m, n ; g)=\left(\begin{array}{ccccc} 
 \tag{31}\\
I_{m} \otimes(\overbrace{1,1, \ldots, 1}) \\
n \\
\overbrace{1,1, \ldots, 1}^{m}) & I_{n} \\
\hline 1 & \alpha & \alpha^{2} & \ldots & \alpha^{m n-1} \\
1 & \alpha^{2} & \alpha^{4} & \ldots & \alpha^{2(m n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{g} & \alpha^{2 g} & \ldots & \alpha^{g(m n-1)}
\end{array}\right)
$$

Notice that the first $m+n$ rows of $\mathcal{H}(m, n ; g)$ correspond to the parity-check matrix of a product code with single horizontal and vertical parities, while the last $g$ rows correspond to the parity-check matrix of an $[m n, m n-g, g+1]$ RS code over $G F\left(2^{b}\right)$. Assume that $b \geqslant \sum_{i=0}^{g-1}(i+1)(m n-g+i)$. We will show that the code $\mathcal{C}(m, n ; g)$ whose parity-check matrix is $\mathcal{H}(m, n ; g)$ as given by 31 is an optimal $E(m, 1 ; n, 1 ; g)$ code. Before proving this result, we need the following lemma:

Lemma 32. Let

$$
\Delta_{j_{0}, j_{1}, \ldots, j_{g-1}}(x)=\left(\begin{array}{ccccc}
x^{j_{0}} & x^{j_{1}} & x^{j_{2}} & \ldots & x^{j_{g-1}}  \tag{32}\\
x^{2 j_{0}} & x^{2 j_{1}} & x^{2 j_{2}} & \ldots & x^{2 j_{g-1}} \\
x^{3 j_{0}} & x^{3 j_{1}} & x^{3 j_{2}} & \ldots & x^{3 j_{g-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x^{g j_{0}} & x^{g j_{1}} & x^{g j_{2}} & \ldots & x^{g j_{g-1}}
\end{array}\right)
$$

be a $g \times g$ matrix of powers of $x$, where the $j_{i}$ s are integers such that $0 \leqslant j_{0}<j_{1}<\cdots<j_{g-1}$. Then the binary polynomial $\operatorname{det}\left(\Delta_{j_{0}, j_{1}, \ldots, j_{g-1}}(x)\right)$ has degree $\sum_{i=0}^{g-1}(i+1) j_{i}$.

Proof: By properties of Vandermonde determinants,

$$
\operatorname{det}\left(\Delta_{j_{0}, j_{1}, \ldots, j_{g-1}}(x)\right)=\left(x^{\sum_{i=0}^{g-1} j_{i}}\right) \prod_{0 \leqslant u<v \leqslant g-1}\left(x^{j_{u}} \oplus x^{j_{v}}\right)
$$

The degree of this polynomial is $\sum_{i=0}^{g-1}(i+1) j_{i}$ by induction on $g$.

Theorem 33. Consider the code $\mathcal{C}(m, n ; g)$ whose parity-check matrix is $\mathcal{H}(m, n ; g)$ as given by 31), where $G F\left(2^{b}\right)$ is the field of polynomials modulo the binary irreducible polynomial $f(x), f(\alpha)=0$ and $b \geqslant \sum_{i=0}^{g-1}(i+1)(m n-g+i)$. Then, $\mathcal{C}(m, n ; g)$ is an optimal $E(m, 1 ; n, 1 ; g)$ code.

Proof: Assume that $d$ is the upper bound on the minimum distance of a code $E(m, 1 ; n, 1 ; g)$ given by Theorem 26 . We will prove that any $d-1$ erasures can be corrected by code $\mathcal{C}(m, n ; g)$. So, assume that we have $d-1$ erasures, say, in locations $0 \leqslant j_{0}<j_{1}<\cdots<j_{d-2} \leqslant m n-1$. From Theorem 26, there are $d-1-g$ horizontal and vertical parities covering the erasures that are linearly independent (otherwise, we would be violating the bound). Consider the $(d-1) \times(d-1)$ submatrix of $\mathcal{H}(m, n ; g)$ whose entries are given by the intersection of columns $j_{0}, j_{1}, \ldots, j_{d-2}$ with the rows corresponding to the aforementioned $d-1-g$ linearly independent horizontal and vertical parities, followed by the last $g$ rows of $\mathcal{H}(m, n ; g)$. We have to prove that this matrix is invertible. The matrix looks as follows:

$$
H_{d-1}=\left(\frac{V}{W}\right)
$$

where $V$ is a $(d-1-g) \times(d-1)$ matrix of rank $d-1-g$ whose entries are 0 s and 1 s and

$$
W=\left(\begin{array}{cccc}
\alpha^{j_{0}} & \alpha^{j_{1}} & \ldots & \alpha^{j_{d-2}} \\
\alpha^{2 j_{0}} & \alpha^{2 j_{1}} & \ldots & \alpha^{2 j_{d-2}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^{g j_{0}} & \alpha^{g j_{1}} & \ldots & \alpha^{g j_{d-2}}
\end{array}\right)
$$

If $U \subseteq\{0,1, \ldots, d-2\}$ and $\bar{U}=\{0,1, \ldots, d-2\}-U$, denote by $V[U]$ the columns of $V$ in locations $U$ and by $W[\bar{U}]$ the columns of $W$ in $\bar{U}$. By properties of determinants and since $V$ is a binary matrix,

$$
\begin{equation*}
\operatorname{det} H_{d-1}=\bigoplus_{\substack{U \subseteq\{0,1, \ldots, d-2\} \\|U|=d-1-g}}(\operatorname{det} V[U])(\operatorname{det} W[\bar{U}])=\bigoplus_{\substack{U \subseteq\{0,1, \ldots, d-2\} \\|U|=d-1-g, \operatorname{det} V[U]=1}} \operatorname{det} W[\bar{U}] . \tag{33}
\end{equation*}
$$

Since $V$ has rank $d-1-g$, let $U_{0}$ be the first subset of $\{0,1, \ldots, d-2\}$ in alphabetical order such that $\left|U_{0}\right|=d-1-g$ and $\operatorname{det} V\left[U_{0}\right]=1$. Then, $\bar{U}_{0}$ is the last subset in alphabetical order such that $\operatorname{det} V\left[U_{0}\right]=1$, i.e., if $U \neq U_{0}$, $\operatorname{det} V[U]=1$, $\bar{U}=\left\{u_{0}, u_{1}, \ldots, u_{g-1}\right\}$ and $\bar{U}_{0}=\left\{w_{0}, w_{1}, \ldots, w_{g-1}\right\}$, where $u_{0}<u_{1}<\cdots<u_{g-1}$ and $w_{0}<w_{1}<\cdots<w_{g-1}$, then $u_{i} \leqslant w_{i}$ for $0 \leqslant i \leqslant g-1$. By Lemma 32] $\operatorname{det} W[\bar{U}]$ has degree (as a polynomial in $\left.\alpha\right) \sum_{i=0}^{g-1}(i+1) j_{u_{i}}<\sum_{i=0}^{g-1}(i+1) j_{w_{i}}$, which is the degree of $\operatorname{det} W\left[\bar{U}_{0}\right]$. This means, $\alpha^{\sum_{i=0}^{g-1}(i+1) j_{w_{i}}}$ cannot be canceled by any other power of $\alpha$ in (33) and det $H_{d-1}$ is a polynomial in $\alpha$ of degree $\sum_{i=0}^{g-1}(i+1) j_{w_{i}}$. Since $\sum_{i=0}^{g-1}(i+1) j_{w_{i}}<\sum_{i=0}^{g-1}(i+1)(m n-g+i) \leqslant b, \operatorname{det} H_{d-1} \neq 0$ and $H_{d-1}$ is invertible.

We illustrate the proof of Theorem 33 in the following example:
Example 34. Consider the code $\mathcal{C}(4,9 ; 3)$ whose parity-check matrix is $\mathcal{H}(4,6 ; 3)$ as given by (31), and assume that $b \geqslant 33+$ $(2)(34)+(3)(35)=206$. For instance, we may take $\mathcal{C}(4,9 ; 3)$ over the field $G F\left(2^{206}\right)$ with $f(x)$ an irreducible polynomial of degree 206 and $f(\alpha)=0$. According to Theorems 26 and 33, the minimum distance of this code is $d=9$, i.e., any 8 erasures can be corrected. In effect, assume that we have the following array with 8 erasures:

|  | $E$ |  |  | $E$ |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |  |  |  |  |
| $E$ | $E$ |  |  | $E$ | $E$ |  |  |  |
| $E$ |  |  |  |  | $E$ |  |  |  |

Following the proof of Theorem 33, erasures have occurred in locations

$$
\left\{j_{0}, j_{1}, j_{2}, j_{3}, j_{4}, j_{5}, j_{6}, j_{7}\right\}=\{1,4,18,19,22,23,27,32\}
$$

Using the parity-check matrix $\mathcal{H}(4,9 ; 3)$ given by 31), it suffices to prove that the following $8 \times 8$ determinant in $\alpha$ is non-zero:

$$
g(\alpha)=\operatorname{det}\left(\begin{array}{cc|cccc|cc}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0  \tag{34}\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline \alpha & \alpha^{4} & \alpha^{18} & \alpha^{19} & \alpha^{22} & \alpha^{23} & \alpha^{27} & \alpha^{32} \\
\alpha^{2} & \alpha^{8} & \alpha^{36} & \alpha^{38} & \alpha^{44} & \alpha^{46} & \alpha^{54} & \alpha^{64} \\
\alpha^{3} & \alpha^{12} & \alpha^{54} & \alpha^{57} & \alpha^{66} & \alpha^{69} & \alpha^{81} & \alpha^{96}
\end{array}\right)
$$

The first invertible $5 \times 5$ submatrix of the first 5 rows is

$$
\left(\begin{array}{lllll}
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\hline 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{array}\right)
$$

which corresponds to the subset of columns $U_{0}=\{0,1,2,3,6\}$. The complement of this set of columns is $\bar{U}_{0}=\{4,5,7\}$, so, since $\left\{j_{4}, j_{5}, j_{7}\right\}=\{22,23,32\}$, the degree of $g(\alpha)$ in (34) corresponds to the degree of $\Delta_{22,23,32}(\alpha)$ which, by Lemma 32, is $22+(2)(23)+(3)(32)=164$. Since $b=206, g(\alpha)$ is non-zero.

Theorem 33 provides an infinite family of $E(m, 1 ; n, 1 ; g)$ codes. It is sufficient to use a code $\mathcal{C}(m, n ; g)$ over a field $G F\left(2^{b}\right)$ with $b \geqslant \sum_{i=0}^{g-1}(i+1)(m n-g+i)$. However, from a practical point of view, this process requires a very large finite field with the corresponding increase in complexity even for relatively small values of $m, n$ and $g$, as we have seen in Example 34 ,

A way to overcome this problem and make the codes practical for implementation is to use the field $G F\left(2^{p-1}\right), p$ a prime number, such that $G F\left(2^{p-1}\right)$ is generated by the irreducible polynomial $M_{p}(x)=1+x+x^{2}+\cdots+x^{p-1}$. This field was often used in array codes requiring symbols of large size [5]. The polynomial $M_{p}(x)$ is not irreducible for every prime number $p$. For example, $M_{5}(x)$ is irreducible but $M_{7}(x)=\left(1+x+x^{3}\right)\left(1+x^{2}+x^{3}\right)$. For an irreducible $M_{p}(x)$, if $M_{p}(\alpha)=0$, then $\alpha^{p}=1$. If we choose $M_{p}(x)$ and $p$ is a prime number large enough, then we can apply Theorem 33 and the code will be optimal. We state this result as a corollary.
Corollary 35 Consider the $E P(m, 1 ; n, 1 ; g)$ code whose parity-check matrix is given by with $\alpha$ in 31) a zero of $M_{p}(x)$, $p$ a prime number, $M_{p}(x)$ irreducible and $\sum_{i=0}^{g-1}(i+1)(m n-g+i) \leqslant p-1$. Then the code is an optimal $E P(m, 1 ; n, 1 ; g)$ code.

Although the field of polynomials modulo $M_{p}(x)$ has size $2^{p-1}$ (possibly a very large number), no look-up tables are necessary in implementation, since most operations reduce to XORs and rotations [5], but we omit the details here. Strictly speaking, it is not proven that the number of primes such that $M_{p}(x)$ is irreducible is infinite, but from a practical point of view, it is always possible to find such a large enough prime number.

## C. Some Performance Considerations

It is known that in product codes, the row-column iterative decoding algorithm does not necessarily correct all the correctable erasure patterns [15], [23]. Similarly, there may be correctable erasure patterns in a $t$-level EII code that cannot be corrected by the row-column iterative decoding algorithm. In effect, if after applying the iterative decoding algorithm there are still erasures left, sometimes such erasures may be corrected by solving a linear system using the parity-check matrix of the code. However, we do not deal here with this residual erasure correcting capability. A natural question is, how much does the row-column iterative decoding algorithm enhance the individual row or column decoding algorithms? The answer depends on the particular parameters considered. For example, take a 4 -level II code $\mathcal{C}(7,(1,2,3,6,6))$. By Theorem 15 the minimum distance of this code is $d=7$. Assume that the erasures, which may correspond to failures of whole storage devices, occur randomly, one after the other. An important parameter is the average number of erasures that will produce an uncorrectable pattern [2]. If we decode by rows only, a Monte Carlo simulation for this example gives that this average number is 14.1. The column code, by Theorem [18, is a 4 -level EII code $\mathcal{C}(5,(0,2,2,2,3,4,5))$. Decoding by columns, the Monte Carlo simulation gives that the average number of erasures producing an uncorrectable pattern is 13.3. The iterative row-column decoding algorithm gives an average of 15.3 , better than the other two algorithms taken separately. Another way of looking at the performance of the three algorithms is as follows: assume that a (random) number of erasures greater than the minimum distance has occurred. What is the probability that each of the algorithms will correct such pattern? For example, taking the same code $\mathcal{C}(7,(1,2,3,6,6))$, assume that 13 erasures have occurred. Again by Monte Carlo simulation, we found out that the row decoding algorithm corrects $64 \%$ of such patterns, the column decoding algorithm corrects $49 \%$, while the iterative row-column decoding algorithm corrects $84 \%$ of them.

The average number of erasures causing an uncorrectable erasure pattern in a $t$-level EII code is very related to the mean time to data loss (MTTDL) [10]-[12] in RAID type of architectures, specially when failures occur following a Poisson model [12], and to birthday surprise type of problems [24]. For example, assume that we have a RAID 5 type of architecture, where each row of an $m \times n$ array is in an $[n, n-1,2]$ code. Using the notation of $t$-level EII codes, this scheme corresponds to a 1-level II code $\mathcal{C}(n,(\overbrace{1,1, \ldots, 1}^{m}))$. When erasures start occurring, there will be an uncorrectable pattern when one row has two erasures. What is the average number of erasures causing this uncorrectable pattern? This question is equivalent to the birthday surprise problem: assuming that people arrive at random in a planet whose year has $m$ days, what is the average number of people that arrive until two of them have the same birthday? An exact formula for this number is well known, mainly, it is $m \int_{0}^{\infty} e^{-m x}(1+x)^{m} d x$ [24]. On Earth, $m=365$ and this average gives 24.6 , the birthday surprise number. It is possible to obtain exact formulae for the average of general $t$-level EII codes, but such formulae become too complicated. The Monte Carlo simulations provide good approximations though.

We end this subsection with an example comparing an EII code with other types of codes. Consider a code of length 64 and rate $1 / 2$. Certainly, an MDS code has minimum distance 33, but an (extended) RS code requires the code to be at least over the field $G F(64)$. If we want a smaller field like $G F(16)$, we can use, for instance, Algebraic Geometry (AG) codes. In [19], page 23 , a $[64,32,27]$ AG code over $G F(16)$ is presented. Consider a 5 -level II code $\mathcal{C}(8,(2,3,3,4,4,5,5,6))$, also over $G F(16)$. This code, by Theorem 15, is a $[64,32,7]$ code, so its minimum distance is considerably smaller than the one of the AG code. However, the average number of erasures that are uncorrectable, by Monte Carlo simulation, is 30.1 . If the AG does not correct erasures beyond its minimum distance, the average number of uncorrectable erasures is precisely the minimum distance 27 , meaning that on average the AG code corrects less erasures than $\mathcal{C}(8,(2,3,3,4,4,5,5,6))$. Also, the

AG code, as well as the RS code, have no locality properties. In particular, $\mathcal{C}(8,(2,3,3,4,4,5,5,6))$ is an LRC code of length 64 and locality 6 . In addition to the 16 local parities, there are 16 extra (global) parities. By (25), an upper bound on the minimum distance of such a code is 23 . Assuming a code meeting the bound (i.e., optimal) is used and there is no correction beyond the minimum distance once the local erasures have been corrected, a Monte Carlo simulation gives that the average number of uncorrectable erasures is 27 , which is below 30.1 as given by the EII code. Another way of looking at the problem is the following: assuming that a number of erasures have occurred, what are the probabilities that an optimal LRC code or the row-column decoding algorithm of $\mathcal{C}(8,(2,3,3,4,4,5,5,6))$ will decode such a pattern? For example, assuming that 27 erasures have occurred, a Monte Carlo simulation gives that an optimal LRC code can correct roughly $50 \%$ of such patterns, while $\mathcal{C}(8,(2,3,3,4,4,5,5,6))$ can correct $88 \%$ of them. The best constructions of optimal LRC codes would require a field of size at least the length of the code [39], 64 in this case.

The examples given above show that there are tradeoffs to be considered when choosing an EII or an optimal LRC code in applications.

## IV. Conclusions

We have studied extended product (EPC) codes, which consist of a product code with some extra parities added in order to increase the minimum distance. We presented an upper bound on the minimum distance of EPC codes and we gave constructions of codes achieving this upper bound for the case in which the product code consists of single parity on rows and columns. We also studied in detail a special case of EPC codes: Extended Integrated Interleaved (EII) codes, which in general do not meet the bound on the minimum distance, but require a small finite field and allow for a large variety of possible parameters, making them an attractive alternative for implementation in practical cases. We showed that EII codes naturally unify product codes and Integrated Interleaved (II) codes. We provided the distance, the dimension and encoding and (erasure) decoding algorithms for any EII code. We showed that EII codes often have better minimum distance than II codes with the same rate, they allow for decoding on columns as well as on rows (enhancing the correction capability of the decoding algorithm) and they permit an uniform distribution of the parity in the array.

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