On Gaussian MACs with Variable-Length Feedback and Non-Vanishing Error Probabilities

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***Abstract*—We characterize the fundamental limits of transmission of information over a Gaussian multiple access channel (MAC) with the use of variable-length feedback codes and under a non-vanishing error probability formalism. We develop new achievability and converse techniques to handle the continuous nature of the channel and the presence of expected power constraints. We establish the ε-capacity regions and bounds on the second-order asymptotics of the Gaussian MAC with variable-length feedback with termination (VLFT) codes and stop-feedback codes. We show that the former outperforms the latter significantly. Due to the multi-terminal nature of the channel model, we leverage tools from renewal theory developed by Lai and Siegmund to bound the asymptotic behavior of the maximum of a finite number of stopping times.**

***Index Terms*—Gaussian multiple access channel, Variable-length codes, Variable-length feedback with termination, Stop-feedback, Non-vanishing error probability, Second-order asymptotics, Finite blocklength regime,**

I. Introduction

A. Background and Related Works

Shannon [1] showed that noiseless feedback does not increase the capacity of point-to-point memoryless channels. Despite this seemingly negative result, it is known that feedback significantly simplifies coding schemes and decreases the error probability. For example, Schalkwijk and Kailath (SK) [2] proposed a simple coding scheme for the additive white Gaussian noise (AWGN) channel with fixed-length feedback based on the idea of refining the receiver's knowledge of the noise in each transmission. The sender then iteratively corrects each error in the previous transmission. The error probability for this scheme is known to decay doubly exponentially fast in the blocklength. Burnashev and Yamamoto [3] showed that even with noisy feedback, the reliability function of an AWGN channel improves (over the no feedback case). Ozarow [4] extended SK's coding scheme [2] and showed that the capacity region of the Gaussian MAC is enlarged in the presence of feedback. These ideas are collectively known as *posterior matching* [5]. These ideas have also been extended by Truong, Fong and Tan [6] to the case where the error probability is not required to vanish.

It is also well known that feedback can increase the capacity of channels with memory. Cover and Pombra [7] characterized the feedback capacity of non-stationary additive Gaussian noise channels with memory. Kim [8] found the capacity of the first-order autoregressive moving-average AWGN channel with feedback. For finite alphabet channels with memory and feedback, expressions of feedback capacity have been derived for the trapdoor channel [9] and the Ising channel [10]. It is also known that feedback can increase the second-order coding rates of certain discrete memoryless channels (DMCs) [11].

A greater advantage of feedback can be observed if one allows the length of the feedback signal to vary based on the quality of the channel output. Burnashev [12] demonstrated that the error exponent improves dramatically in this variable-length feedback setting. In fact, the error exponent of a DMC with variable-length feedback is $E(R) = C_1\big(1-\frac{R}{C}\big)$ for all rates $0 \le R \le C$, where C is the capacity of the DMC and C_1 is the maximal relative entropy between the conditional output distributions. Yamamoto and Itoh [13] proposed a simple and conceptually important two-phase coding scheme that attains $E(R)$. While the error exponent results in [12] and [13] are of paramount importance in feedback communications, we focus on the scenario in which the error probability is non-vanishing [14].

For variable-length codes under the *non-vanishing error probability* formalism, Polyanskiy, Poor and Verdú [15] provided non-asymptotic achievability and converse bounds for the coding rates. They also derived asymptotic expansions for the optimal code lengths of DMCs and showed dramatic improvements over the no feedback and the fixed-length feedback settings. In particular the channel dispersion vanishes, and so the backoff from capacity at finite blocklengths is significantly reduced. Trillingsgaard and Popovski [16] generalized the results for DMCs in [15] to the discrete memoryless multiple access channel (DM-MAC). In it, they used ideas contained in Tan and Kosut [17] and MolavianJazi and Laneman [18] to analyze achievable second-order asymptotics for the DM-MAC. However, only achievability results were provided. It was also shown numerically in [16] that variable-length feedback outperforms fixed-length feedback. Achievability and converse bounds under variable-length full-feedback (VLF) and variable-length stop-feedback (VLSF) for the binary erasure channel (BEC) have recently been derived by Devassy *et al.* [19]. In addition, Trillingsgaard *et al.* used ideas related

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to the compound channel [20] to study the 2-user [21] and K-user [22] common-message discrete memoryless broadcast channel with stop-feedback. However, the techniques used in both the achievability and converse parts in [19], [21] and [22] are difficult to extend to Gaussian channels. This is because the authors leveraged the fact that a set of information densities for discrete channels can be bounded. This, together with Hoeffding's inequality, allows the authors to control the expectation of the maximum of a set of stopping times to eventually upper bound the average transmission time. The information density terms for Gaussian channels are not bounded. Hence, to study this important class of channels under variable-length feedback, we develop new techniques. We mention here that while the analysis of variable-length codes for non-vanishing error probabilities has been restricted to the finite alphabet setting, for the *vanishing error probability* formalism, however, general alphabets have been considered both with and without cost constraints in the important works of Burnashev [23] and Nakiboğlu and Gallager [24].

We characterize the information-theoretic limits of the Gaussian MAC when variable-length feedback is available at the encoder and a non-vanishing error probability is permitted. In particular, we circumvent the problem of the continuous nature of the alphabets by deriving new bounds on the moments (e.g., expectation and variance) of the maximum of a set of random variables (e.g., stopping times). These techniques may be of independent interest in other problems.

B. Main Contributions

We propose a variable-length feedback model for Gaussian channels. We carefully define the expected power constraint so that it is analogous to the definition in the fixed-length feedback setting. In the latter setting, the power constraint of a code for a point-to-point channel with (deterministic) blocklength $N \in \mathbb{N}$ is defined to be

$$\mathbb{E}\left[\sum_{n=1}^{N} X_n^2\right] \leq NP, \tag{1}$$

where X_n is the input to the channel at the n-th time slot and $P > 0$ is the admissible power. However, in the variable-length feedback setting, the analogue of N, usually denoted as $\tau \in \mathbb{N}$, is a stopping time (i.e., the random decoding time instant). Hence, one needs to carefully define the analogue of (1) so that we can utilize existing mathematical techniques for analyzing stopping times. We note that the expected power constraint we propose in (6) is analogous to that in [24, Sec. II.A], i.e.,

$$\mathbb{E}\left[\sum_{n=1}^{\tau} X_n^2\right] \leq \mathbb{E}(\tau)P. \tag{2}$$

However, our formulation in (6) is somewhat more convenient to analyze under the non-vanishing error probability formalism.

In our main contribution, we derive achievability and converse bounds for the Gaussian MAC with two forms of variable-length feedback—stop-feedback and variable-length feedback with termination (VLFT). We establish the ε-capacity regions. We show that under the VLFT setting, we can achieve a larger ε-capacity region compared to the stop-feedback setting. We also provide bounds on the second-order terms. Our achievability proof for the Gaussian MAC with stop-feedback uses some non-standard techniques. We find that Doob's optional stopping theorem [25, Thm. 10.10], which was used in [15] for the DMC, is not sufficient to bound the expected blocklength of the code. We develop new results, coupled with work on renewal theory by Gut [26] and Lai and Siegmund [27], to bound the expected blocklength. The converse proof for the Gaussian MAC borrows some ideas from the weak converse proof in Ozarow's analysis for the Gaussian MAC with fixed-length feedback [4]. However, our choice of parameters is different from [4]. This is to account for the variable-length setting that we study.

C. Paper Organization

The rest of this paper is structured as follows: In Section II, we provide a precise problem setting for the Gaussian MAC, state the main results, and provide intuitions for these results. We also explain the novelties of our arguments relative to existing works. The achievability and converse proofs are provided in Sections III and IV respectively. We conclude our discussion and suggest avenues for future work in Section V. Auxiliary technical results that are not essential to the main arguments are relegated to the appendices.

II. Gaussian MAC with Variable-Length Feedback

A. Notation, Channel Model and Definitions

1) Notation: We use $\log x$ to denote the natural logarithm so information units throughout are in nats. We also define $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. The Gaussian capacity and binary entropy functions are respectively defined as $\mathsf{C}(x) := \frac{1}{2}\log(1+x)$ and $h_{\mathrm{b}}(x) := -x\log x - (1-x)\log(1-x)$. The notation for random variables and information-theoretic quantities are standard and mainly follow the text by El Gamal and Kim [28]. We use $\sigma(A)$ to denote the smallest σ-field on which random variable A is measurable. We write $\mathcal{N}(\mu, \nu)$ for the univariate Gaussian distribution with mean μ and variance ν. We also use standard asymptotic notation such as $O(\cdot)$.

2) Channel Model: The channel model is given by

$$Y = X_1 + X_2 + Z, \tag{3}$$

where X_1 and X_2 represent the inputs to the channel, $Z \sim \mathcal{N}(0, 1)$ is additive Gaussian noise with zero mean and unit variance, and Y is the output of the channel. Thus, the channel law from (X_1, X_2) to Y can be written as

$$\mathbb{P}(y|x_1, x_2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - x_1 - x_2)^2\right). \tag{4}$$

3) Basic Definitions: The following definitions generalize [15] to the Gaussian MAC with expected power constraints.

Definition 1. *An $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code for the Gaussian MAC $\mathbb{P}(y|x_1, x_2)$, where N, P_1, P_2 are positive numbers, M_1, M_2 are positive integers, and $0 \leq \varepsilon \leq 1$, is defined by:*

1) *Two spaces* $\mathcal{U}_1,\mathcal{U}_2$ *and probability distributions* P_{U_1}, P_{U_2} *on them, defining independent random variables* $U_j, j = 1, 2$ *each of which is revealed to transmitter* $j = 1, 2$ *and the receiver before the start of transmission; i.e.,* (U_1, U_2) *acts as common randomness.*[1]
2) *Two sequences of encoders* $f_n^{(1)} : \mathcal{U}_1 \times \{1, 2, \ldots, M_1\} \to \mathbb{R}$ *and* $f_n^{(2)} : \mathcal{U}_2 \times \{1, 2, \ldots, M_2\} \to \mathbb{R}$ *(indexed by* $n \in \mathbb{N}$*) defining channel inputs* $X_{jn} = f_n^{(j)}(U_j, W_j)$*. where* W_j *is equiprobable on the message set* $\{1, 2, \ldots, M_j\}$ *for* $j = 1, 2$*.*
3) *A sequence of decoders* $g_n : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathbb{R}^n \to \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}$ *providing estimates* (W_1, W_2) *at times* n*.*
4) *A non-negative integer-valued random variable* τ*, a stopping time of the filtration* $\{\sigma(U_1, U_2, Y^n)\}_{n=1}^{\infty}$*, which satisfies*
$$\mathbb{E}(\tau) \le N. \tag{5}$$
5) *The expected power constraints at the encoders*
$$\sum_{n=1}^{\infty} \mathbb{E}[X_{jn}^2] \le \mathbb{E}(\tau) P_j, \quad j = 1, 2. \tag{6}$$

The final decision $(\hat{W}_1, \hat{W}_2) = g_\tau(U_1, U_2, Y^\tau)$ *is computed at time* τ *and must satisfy*
$$\mathbb{P}[(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)] \le \varepsilon. \tag{7}$$

Definition 2. *An* $(M_1, M_2, N, P_1, P_2, \varepsilon)$ *variable-length feedback with termination code (VLFT) is defined as in Definition 1 except that* τ *is a stopping time of the filtration* $\{\sigma(U_1, U_2, W_1, W_2, Y^n)\}_{n=1}^{\infty}$ *and* $X_{jn} = f_n^{(j)}(U_j, W_j, Y^{n-1})$ *for* $j = 1, 2$*.*

B. Main Results and Discussions

We now state our main results for the Gaussian MAC under various forms of variable-length codes with feedback. The proofs of the achievability parts of Theorems 1 and 2 are provided in Section III. The proofs of the converse parts of Theorems 1 and 2 are provided in Section IV.

Theorem 1. *For the Gaussian MAC* $\mathbb{P}(y|x_1, x_2)$*, there exists a sequence of* $(M_1, M_2, N, P_1, P_2, \varepsilon)$ *stop-feedback codes for any* (M_1, M_2) *satisfying*
$$0 \le \log M_j \le \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \mathsf{C}(P_j) - \log N + O(1), \quad j = 1, 2 \tag{8}$$
$$0 \le \log M_1 M_2 \le \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} \right) \mathsf{C}(P_1 + P_2) - \log N + O(1). \tag{9}$$

where $A \ge 0$ *is a constant given as*
$$A := \min_{(i,j,k) \in \mathrm{perm}[3]} \frac{1}{2}\Big(\sqrt{2(L_i + L_j)} + \sqrt{4L_k}\Big) + \frac{1}{4}\Big(\sqrt{2(L_i + L_k)} + \sqrt{2(L_j + L_k)}\Big), \tag{10}$$
and where $\mathrm{perm}[3]$ *is the set of all permutations of the tuple* $(1, 2, 3)$ *and*
$$L_j := \frac{4P_j}{(1 + P_j)\,[\log(1 + P_j)]^2}, \quad j = 1, 2 \tag{11}$$
$$L_3 := \frac{4(P_1 + P_2)}{(1 + P_1 + P_2)\,[\log(1 + P_1 + P_2)]^2}. \tag{12}$$
Conversely, given any $(M_1, M_2, N, P_1, P_2, \varepsilon)$ *stop-feedback code, the following inequalities hold*
$$0 \le \log M_j \le \frac{N\mathsf{C}(P_j) + h_{\mathrm{b}}(\varepsilon)}{1-\varepsilon}, \quad j = 1, 2 \tag{13}$$
$$0 \le \log M_1 M_2 \le \frac{N\mathsf{C}(P_1 + P_2) + h_{\mathrm{b}}(\varepsilon)}{1-\varepsilon}. \tag{14}$$

Theorem 2. *Given a Gaussian MAC, for any* $\rho \in [0, 1]$*, there exist a sequence of* $(M_1, M_2, N, P_1, P_2, \varepsilon)$ *VLFT-feedback codes for any* M_1, M_2 *satisfying*
$$0 \le \log M_j \le \frac{N\mathsf{C}(P_j(1-\rho^2))}{1-\varepsilon} - \log\log N + O(1), \quad j = 1, 2 \tag{15}$$
$$0 \le \log M_1 M_2 \le \frac{N\mathsf{C}(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})}{1-\varepsilon} - \log\log N + O(1). \tag{16}$$

Conversely, for any $(M_1, M_2, N, P_1, P_2, \varepsilon)$*-VLFT feedback code for the Gaussian MAC, the following inequalities hold for some* $\rho \in [0, 1]$ *and for* $j = 1, 2$*:*
$$0 \le \log M_j \le \frac{1}{1-\varepsilon}\Big[N\mathsf{C}(P_j(1-\rho^2)) + (N+1)h_{\mathrm{b}}\Big(\frac{1}{N+1}\Big) + h_{\mathrm{b}}(\varepsilon) \Big], \tag{17}$$
$$0 \le \log M_1 M_2 \le \frac{1}{1-\varepsilon}\Big[N\mathsf{C}(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) + (N+1)h_{\mathrm{b}}\Big(\frac{1}{N+1}\Big) + h_{\mathrm{b}}(\varepsilon) \Big]. \tag{18}$$

We define the ε-capacity region of a Gaussian MAC under the stop-feedback (resp. VLFT) formalisms $\mathcal{C}_{\mathrm{sf}}(P_1, P_2, \varepsilon)$ (resp. $\mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$) to be the closure of the set of all rate pairs (R_1, R_2) such that there exists a sequence of $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback codes (resp. VLFT codes) such that $\liminf_{N\to\infty} \frac{1}{N}\log M_j \ge R_j$ for $j = 1, 2$. and also that (7) holds. Theorems 1 and 2 immediately imply the following corollary.

Corollary 1. *Let* $0 < \varepsilon < 1$*. The* ε*-capacity region* $\mathcal{C}_{\mathrm{sf}}(P_1, P_2, \varepsilon)$ *is the set of all* $(R_1, R_2) \in \mathbb{R}_+^2$ *satisfying*
$$R_j \le \frac{\mathsf{C}(P_j)}{1-\varepsilon}, \quad j = 1, 2 \tag{19}$$
$$R_1 + R_2 \le \frac{\mathsf{C}(P_1 + P_2)}{1-\varepsilon}. \tag{20}$$

[1] The common randomness is used to initialize the encoders and the decoder before the start of transmission. See the usage of the Bernoulli random variable D in Lemmas 6 and 7. The reader is referred to the analogue of this common randomness and accompanying discussions for the point-to-point case in [15].

Similarly, the ε-capacity region $\mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$ is the set of all $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$R_j \leq \frac{\mathsf{C}(P_j(1-\rho^2))}{1-\varepsilon}, \quad j = 1, 2 \tag{21}$$

$$R_1 + R_2 \leq \frac{\mathsf{C}(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})}{1-\varepsilon} \tag{22}$$

for some $\rho \in [0, 1]$.

Some remarks concerning Theorems 1 and 2 and Corollary 1 are now in order:

1) Trillingsgaard and Popovski [16] generalized the point-to-point variable-length feedback results for the DMC in Polyanskiy, Poor and Verdú [15] to the DM-MAC. In it, they used ideas contained in Tan and Kosut [17] and MolavianJazi and Laneman [18] to analyze achievable second-order asymptotics for the DM-MAC with variable-length feedback. However, Trillingsgaard and Popovski [16] could not analytically bound the expectation of the maximum of several stopping times $\mathbb{E}(\max_k \tau_k)$ and they also could not prove a matching (first-order) converse. Instead, they provided numerical results to show that stop-feedback increases the first-order coding rate compared to the fixed-length feedback setting.
2) The multiplicative gains of $\frac{1}{1-\varepsilon}$ in (19)–(22) are due to the non-vanishing nature of the error probability and the use of variable-length codes with feedback. Note that for the Gaussian MAC without feedback, the strong converse holds in the sense that the ε-capacity is independent of ε [29].
3) The ε-capacity region for VLFT codes is easily seen to be strictly larger than the corresponding region for fixed-length feedback codes recently studied by Truong, Fong and Tan [6]. In that scenario, the ε-capacity region is given by [6]

$$R_j \leq \mathsf{C}\left(\frac{P_j(1-\rho^2)}{1-\varepsilon}\right), \quad j = 1, 2 \tag{23}$$

$$R_1 + R_2 \leq \mathsf{C}\left(\frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{1-\varepsilon}\right), \tag{24}$$

   for some $\rho \in [0, 1]$. The enlargement is due to the following consequence of Jensen's inequality:

$$\mathsf{C}\left(\frac{P}{1-\varepsilon}\right) < \frac{\mathsf{C}(P)}{1-\varepsilon}, \qquad \forall\,(P, \varepsilon) \in (0, \infty) \times (0, 1). \tag{25}$$

   This gain is present as variable-length feedback codes are *adaptive*, i.e., their lengths are adapted to the quality of Y^∞.
4) The ε-capacity region $\mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$ is strictly larger than $\mathcal{C}_{\mathrm{sf}}(P_1, P_2, \varepsilon)$, which clearly illustrates the fact that feedback at encoders can enlarge the ε-capacity region compared to the case where only stop-feedback is available. That $\mathcal{C}_{\mathrm{t}}(P_1, P_2, \varepsilon)$ is strictly larger than $\mathcal{C}_{\mathrm{sf}}(P_1, P_2, \varepsilon)$ is completely analogous to the fact that fixed-length feedback enlarges the capacity region of the Gaussian MAC (cf. Ozarow [4]).
5) In the achievability proofs, we note that Polyanskiy, Poor and Verdú [15] utilize the fact that the relevant information density random variable $i(X;Y)$ (induced by the capacity-achieving input distribution and the channel) is bounded when the channel is a DMC [15, Eqn. (107)]. However, this fact does not hold for the AWGN channel and so our achievability proofs require some novel elements. All previous works on variable-length feedback for systems with non-vanishing error probabilities [15], [19], [21], [22] involve channels with *discrete* alphabets. In addition, we leverage novel bounds (Lemmas 2 and 4) that control the first and second moments of the maximum of a set of stopping times $\max_k \tau_k$ and multi-user information spectrum methods [16]–[18].
6) For the converse of Theorem 1, we make use of Fano-like arguments. Although some of the ideas are inspired by [15], we need to augment the original arguments so that the proof is amenable to Gaussian channels. More specifically, in [15], the authors use the fact that the capacity of the DMC is $\sup\{I(\hat{X};\hat{Y}) : P_{\hat{X}}(\mathsf{T}) = 0\}$, where T is a new symbol appended to the input and output alphabets of the DMC to form $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$ respectively and $\hat{X} \in \hat{\mathcal{X}}$ is the input random variable of the new DMC. However, for Gaussian MAC with the expected power constraints in (6), this does not hold.
7) For the converse proof of Theorem 2, we borrow some ideas from Ozarow's weak converse proof for the Gaussian MAC with fixed-length feedback [4]. However, our parameter settings and the manipulations of the resultant bounds are different from Ozarow. See (157) and (158) in Lemma 9 to follow.
8) Specializing our results for the Gaussian MAC to the point-to-point AWGN channel also yields novel results. In this case, the first-order terms for VLFT code and stop-feedback codes are identical and equal to to $\frac{\mathsf{C}(P)}{1-\varepsilon}$; this can be seen by setting $\rho = 0$ in Theorem 2. However, the achievability result for VLFT codes is better than the corresponding one for stop-feedback codes in the second-order term ($-O(\log\log N)$ compared to $-O(\sqrt{N})$).

III. Achievability Proofs

A. Achievability Proof for Theorem 1

To prove the achievability result for Theorem 1 in (8) and (9), we commence with some technical results in Lemmas 1 to 4. The achievability result for Theorem 1 follows from a combination of Lemmas 5 and 6 to follow.

Definition 3 (Strongly nonlattice [30])**.** *We say that a distribution function F is strongly nonlattice if* $\liminf_{|t|\to\infty} |1 - f(t)| > 0$, *where* $f(t) := \int_{-\infty}^{\infty} e^{itx}\,\mathrm{d}F(x)$ *is the characteristic function of F. This is equivalent to Cramer's condition (C), i.e., that* $\limsup_{|t|\to\infty} |f(t)| < 1$.

Lemma 1 (Asymptotics of Expected Values of Stopping Times)**.** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with positive mean $\mu = \mathbb{E}[X_1]$, finite variance $\sigma^2 = \mathrm{Var}(X_1)$ and*

$\mathbb{E}[X_1^+] < \infty$. *Let* $S_n := X_1 + X_2 + \ldots + X_n$. *For each* $b \geq 0$ *define*

$$\tau = \tau(b) = \inf\{n : S_n > b\}, \tag{26}$$
$$\tau_+ = \tau(0) = \inf\{n : S_n > 0\}. \tag{27}$$

Assume that X_1 *has a distribution function* F_{X_1} *that is strongly nonlattice in the sense of Definition 3. Then as* $b \to \infty$,

$$\mu\mathbb{E}(\tau) = b + \frac{\mathbb{E}(S_{\tau_+}^2)}{2\mathbb{E}(S_{\tau_+})} + o(1). \tag{28}$$

Proof: Follows from Gut [26, Thm. 2.6] and Wald's identity [31, Eqn. (13) in Sec. 12.5]. ■

Lemma 2 (Asymptotics of Variance of Stopping Times [27])**.** *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with positive mean* μ *and finite variance* σ^2 *and* $\mathbb{E}(X_1^+) < \infty$. *Let* $S_n := X_1 + X_2 + \ldots + X_n$. *For each* $b \geq 0$ *define* τ *and* τ_+ *as in* (26) *and* (27)*. If* X_1 *has a distribution function that is strongly nonlattice, then as* $b \to \infty$,

$$\mathrm{Var}(\tau) = \mu^{-3}\sigma^2 b + \mu^{-2}K + o(1), \tag{29}$$

where K *is a constant that does not depend on* b *and is given by*

$$\begin{aligned} K := {} & \frac{\sigma^2 \mathbb{E}S_{\tau_+}^2}{2\mu\mathbb{E}S_{\tau_+}} + \frac{3}{4}\left(\frac{\mathbb{E}S_{\tau_+}^2}{\mathbb{E}S_{\tau_+}}\right)^2 - \frac{2}{3}\left(\frac{\mathbb{E}S_{\tau_+}^3}{\mathbb{E}S_{\tau_+}}\right) \\ & - \left(\frac{\mathbb{E}S_{\tau_+}^2}{\mathbb{E}S_{\tau_+}}\right)\mathbb{E}\Big\{\min_{n\geq 0} S_n\Big\} \\ & - 2\int_0^\infty \mathbb{E}\{S_{\tau(x)} - x\}\mathbb{P}\Big\{\min_{n\geq 0} S_n \leq -x\Big\}\,\mathrm{d}x. \end{aligned} \tag{30}$$

Lemma 3 (Generalization of Wald's equation [32][2])**.** *Let* $\{X_n\}_{n=1}^\infty$ *be an infinite sequence of real-valued random variables and let* τ *be a non-negative integer-valued random variable. Assume that*

- $\{X_n\}_{n=1}^\infty$ *are all integrable (finite-mean) random variables;*
- *for all natural numbers* n, $\mathbb{E}[X_n 1\{\tau \geq n\}] = \mathbb{E}[X_n]\mathbb{P}(\tau \geq n)$*;*
- *the infinite series* $\sum_{n=1}^\infty \mathbb{E}[|X_n| 1\{\tau \geq n\}] < \infty$*;*
- $\{X_n\}_{n=1}^\infty$ *all have the same expectation, and*
- τ *has finite expectation.*

Define $S_\tau := \sum_{n=1}^\tau X_n$*. Then, we have*

$$\mathbb{E}[S_\tau] = \mathbb{E}[\tau]\mathbb{E}[X_1]. \tag{31}$$

Note that this is indeed a generalization of the standard Wald's equation [31], [34] which states that if $\{X_n\}_{n=1}^\infty$ is a sequence of i.i.d. integrable random variables and τ is a finite expectation stopping time with respect to $\{X_n\}_{n=1}^\infty$, then (31) holds. Lemma 3 does not require $\{X_n\}_{n=1}^\infty$ to be i.i.d. The proof of Lemma 3, which can be found in [32], is similar to that of Wald's equation [31], [34].

[2]The proof of Lemma 3 in [32] has been verified correct by the authors and the Associate Editor Prof. A. Tchamkerten [33]. We thank the editor for his kind assistance.

Lemma 4 (Expectation of the Maximum of Random Variables)**.** *Let* $\{(X_{1N}, X_{2N}, X_{3N})\}_{N\geq 1}$ *be three sequences of random variables satisfying*

$$\mathbb{E}[X_{jN}] = N - A\sqrt{N} - G - B_j + o(1), \quad j = 1,2,3 \tag{32}$$

for some constants $B_1, B_2, B_3 \in \mathbb{R}$*, where* A *as given in* (10) *and* G *is defined as follows:*

$$\begin{aligned} G := {} & -\frac{1}{4}\left(B_{i_0} + B_{j_0} + 2B_{k_0}\right) \\ & + \frac{1}{2}\left(\sqrt{2|F_{i_0} + F_{j_0}| + (B_{i_0} - B_{j_0})^2}\right) \\ & + \frac{1}{4}\Bigg(\sqrt{2|F_{i_0} + F_{k_0}| + (B_{i_0} - B_{k_0})^2} \\ & \qquad + \sqrt{2|F_{j_0} + F_{k_0}| + (B_{j_0} - B_{k_0})^2}\Bigg), \end{aligned} \tag{33}$$

where

$$\begin{aligned} (i_0, j_0, k_0) := {} & \operatorname*{arg\,min}_{(i,j,k)\in\mathrm{perm}[3]} \frac{1}{2}\sqrt{2(L_i + L_j)} \\ & + \frac{1}{4}\left(\sqrt{2(L_i + L_k)} + \sqrt{2(L_j + L_k)}\right). \end{aligned} \tag{34}$$

Furthermore assume that

$$\mathrm{Var}(X_{jN}) \leq L_j N + F_j + o(1), \quad j = 1,2,3 \tag{35}$$

for some other constants $L_1 > 0, L_2 > 0, L_3 > 0$ *and* $F_1, F_2, F_3 \in \mathbb{R}$*. Then, we have*

$$\mathbb{E}(\max\{X_{1N}, X_{2N}, X_{3N}\}) \leq N + o(1). \tag{36}$$

Proof: The proof is deferred to Appendix A. ■

Lemma 5. *Consider a standard Gaussian MAC* $\mathbb{P}(y|x_1, x_2)$ *with expected power constraints* P_1, P_2*. For any* $N' > 0$*, and* (M_1, M_2) *satisfying*

$$\begin{aligned} 0 \leq \log M_j \leq {} & (N' - A\sqrt{N'})\mathsf{C}(P_j) \\ & - \log N' + O(1), \quad j = 1,2 \end{aligned} \tag{37}$$
$$\begin{aligned} 0 \leq \log M_1 M_2 \leq {} & (N' - A\sqrt{N'})\mathsf{C}(P_1 + P_2) \\ & - \log N' + O(1), \end{aligned} \tag{38}$$

we can find an $(M_1, M_2, N' + o(1), \frac{1}{N'})$ *stop-feedback code with* A *defined as in* (10).

Proof: Part of the proof is based on [15] and [16] but as mentioned, we need to combine existing ideas with Lemmas 2 and 4 above. First, we show that there exists an $(M_1, M_2, N' + o(1), P_1, P_2, \frac{1}{N'})$ stop-feedback code with stopping time τ^*, where $\mathbb{E}(\tau^*) \leq N' + o(1)$, the sizes of the message sets M_1, M_2 satisfy (37) and (38), and finally, $\mathbb{E}\big[\sum_{n=1}^{\tau^*} X_{jn}^2\big] = \mathbb{E}(\tau^*)P_j$, for $j = 1,2$. To define this code, we define two random variables U_1 and U_2 each with distribution $\mathbb{P}_{U_j} := (\mathbb{P}_{X_j})^\infty \times (\mathbb{P}_{X_j})^\infty \times \ldots \times (\mathbb{P}_{X_j})^\infty$ (M_j times) where $j = 1,2$ and $\mathbb{P}_{X_j} \sim \mathcal{N}(0, P_j)$.

We generate the codebook as follows. For a realization of U_1, we generate M_1 i.i.d. infinite dimensional vectors $\{\mathbf{C}_j^{(1)}\}$ from $\mathbb{P}_{X_1} \sim \mathcal{N}(0, P_1)$. Similarly, for each realization of U_2, we generate M_2 i.i.d. infinite dimensional vectors $\{\mathbf{C}_k^{(2)}\}$ from

$\mathbb{P}_{X_2} \sim \mathcal{N}(0, P_2)$. The encoder and decoder depend on U_1, U_2 implicitly through $\{\mathbf{C}_j^{(1)}\}$ and $\{\mathbf{C}_k^{(2)}\}$.

Encoder $j = 1, 2$ consists of a sequence of encoders $f_n^{(j)}$ that maps message $w_j \in \{1, 2, \ldots, M_j\}$ to an infinite sequence of inputs $\mathbf{C}_{w_j}^{(j)} \in \mathbb{R}^\infty$. The mappings are without regard to feedback, $X_{jn} = f_n^{(j)}(w_j) := \mathbf{C}_{w_j,n}^{(j)}$, where $\mathbf{C}_{w_1,n}^{(1)}$ and $\mathbf{C}_{w_2,n}^{(2)}$ are respectively the n-th coordinates of the infinite vectors $\mathbf{C}_{w_1}^{(1)}$ and $\mathbf{C}_{w_2}^{(2)}$.

Let $\mathbf{C}_j^{(1)}(n) := (\mathbf{C}_{j,1}^{(1)}, \ldots, \mathbf{C}_{j,n}^{(1)})$ and similarly define $\mathbf{C}_k^{(2)}(n)$. At time n, the decoder computes the (conditional) information densities:

$$S_{j,k}^{(1,n)} := i(\mathbf{C}_j^{(1)}(n); Y^n | \mathbf{C}_k^{(2)}(n)), \tag{39}$$
$$S_{j,k}^{(2,n)} := i(\mathbf{C}_k^{(2)}(n); Y^n | \mathbf{C}_j^{(1)}(n)), \tag{40}$$
$$S_{j,k}^{(3,n)} := i(\mathbf{C}_j^{(1)}(n), \mathbf{C}_k^{(2)}(n); Y^n), \tag{41}$$

for all $(j, k) \in \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}$, where

$$\begin{aligned} & i(\mathbf{C}_j^{(1)}(n); Y^n | \mathbf{C}_k^{(2)}(n)) \\ & := \log \frac{\mathrm{d}\mathbb{P}_{X_1^n Y^n | X_2^n}}{\mathrm{d}(\mathbb{P}_{X_1^n | X_2^n} \times \mathbb{P}_{Y^n | X_2^n})} \left(\mathbf{C}_j^{(1)}(n), \mathbf{C}_k^{(2)}(n), Y^n \right), \end{aligned} \tag{42}$$

and similarly for $i(\mathbf{C}_k^{(2)}(n); Y^n | \mathbf{C}_j^{(1)}(n))$ and $i(\mathbf{C}_j^{(1)}(n), \mathbf{C}_k^{(2)}(n); Y^n)$. For a triple of positive real numbers $(\gamma_1, \gamma_2, \gamma_3)$ to be chosen later, the decoder also defines a number of stopping times as follows:

$$\tau_{j,k}^{(1)} := \inf\{n \geq 0 : i(\mathbf{C}_j^{(1)}(n); Y^n | \mathbf{C}_k^{(2)}(n)) > \gamma_1\}, \tag{43}$$
$$\tau_{j,k}^{(2)} := \inf\{n \geq 0 : i(\mathbf{C}_k^{(2)}(n); Y^n | \mathbf{C}_j^{(1)}(n)) > \gamma_2\}, \tag{44}$$
$$\tau_{j,k}^{(3)} := \inf\{n \geq 0 : i(\mathbf{C}_j^{(1)}(n), \mathbf{C}_k^{(2)}(n); Y^n) > \gamma_3\}, \tag{45}$$

and $\tau_{j,k} := \max\{\tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)}\}$. The final decision is made by the decoder at the stopping time

$$\tau^* := \min_{j,k} \tau_{j,k}. \tag{46}$$

The output of the decoder is given by

$$g(Y^{\tau^*}) = \max\{(j, k) : \tau_{j,k} = \tau^*\}, \tag{47}$$

where the maximum is in lexicographic order.

Let $X_1^\infty, X_2^\infty, \bar{X}_1^\infty, \bar{X}_2^\infty, Y^\infty$ be i.i.d. infinite-dimensional vectors with joint distribution

$$\begin{aligned} & \mathbb{P}_{X_1 X_2 Y \bar{X}_1 \bar{X}_2}(x_1, x_2, y, \bar{x}_1, \bar{x}_2) \\ & = \mathbb{P}_{X_1}(x_1) \mathbb{P}_{X_2}(x_2) \mathbb{P}(y | x_1 x_2) \mathbb{P}_{X_1}(\bar{x}_1) \mathbb{P}_{X_2}(\bar{x}_2), \end{aligned} \tag{48}$$

where $\mathbb{P}_{X_1} \sim \mathcal{N}(0, P_1)$, $\mathbb{P}_{X_2} \sim \mathcal{N}(0, P_2)$ and $\mathbb{P}(y | x_1 x_2)$ is the law of the Gaussian MAC.

For each finite n, define three random information density random variables (random walks) $S_n^{(1)} := i(X_1^n; Y^n | X_2^n)$, $S_n^{(2)} := i(X_2^n; Y^n | X_1^n)$, and $S_n^{(3)} := i(X_1^n, X_2^n; Y^n)$ and hitting times

$$\tau^{(1)} := \inf\{n \geq 0 : i(X_1^n; Y^n | X_2^n) > \gamma_1\}, \tag{49}$$
$$\tau_+^{(1)} := \inf\{n \geq 0 : i(X_1^n; Y^n | X_2^n) > 0\}, \tag{50}$$
$$\bar{\tau}^{(1)} := \inf\{n \geq 0 : i(\bar{X}_1^n; Y^n | X_2^n) > \gamma_1\}, \tag{51}$$

Analogously define $\tau^{(2)}$, $\tau_+^{(2)}$, $\bar{\tau}^{(2)}$, $\tau^{(3)}$, $\tau_+^{(3)}$, $\bar{\tau}^{(3)}$ and $\tau' := \max\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\}$.

It follows that the average length of the transmission satisfies

$$\begin{aligned} \mathbb{E}(\tau^*) &= \frac{1}{M_1 M_2} \sum_{j,k} \mathbb{E}(\tau^* | W_1 = j, W_2 = k) && (52) \\ &= \mathbb{E}(\tau^* | W_1 = 1, W_2 = 1) && (53) \\ &\leq \mathbb{E}(\max\{\tau_{1,1}^{(1)}, \tau_{1,1}^{(2)}, \tau_{1,1}^{(3)}\} | W_1 = 1, W_2 = 1) && (54) \\ &= \mathbb{E}(\max\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\}) = \mathbb{E}(\tau'). && (55) \end{aligned}$$

From the analysis of the DM-MAC in Trillingsgaard and Popovski [16], we know that the average probability of error satisfies

$$\begin{aligned} \mathbb{P}(g(Y^{\tau^*}) \neq (W_1, W_2)) \leq (M_1 - 1)(M_2 - 1) \mathbb{P}(\tau' \geq \bar{\tau}^{(3)}) \\ + (M_1 - 1) \mathbb{P}(\tau' \geq \bar{\tau}^{(1)}) \\ + (M_2 - 1) \mathbb{P}(\tau' \geq \bar{\tau}^{(2)}). \end{aligned} \tag{56}$$

Observe that the following statistics are all finite:

$$\mu_1 = \mathbb{E}[i(X_1; Y | X_2)] = I(X_1; Y | X_2) = \mathsf{C}(P_1), \tag{57}$$
$$\sigma_1^2 = \mathrm{Var}\,(i(X_1; Y | X_2)) = \frac{P_1}{1 + P_1}, \tag{58}$$

$$\begin{aligned} \mathbb{E}\left[i(X_1; Y | X_2)^+\right] &\leq \mathbb{E}\left[i(X_1; Y | X_2)^+ + i(X_1; Y | X_2)^-\right] && (59) \\ &= \mathbb{E}\left[|i(X_1; Y | X_2)|\right] && (60) \\ &\leq \sqrt{\mathbb{E}\left[(i(X_1; Y | X_2))^2\right]} && (61) \\ &\leq \sqrt{\mu_1^2 + \sigma_1^2} < \infty. && (62) \end{aligned}$$

Similarly, $\mu_2 = \mathbb{E}[i(X_2; Y | X_1)] = I(X_2; Y | X_1)$, $\mu_3 = \mathbb{E}[i(X_1, X_2; Y)] = I(X_1, X_2; Y)$, $\sigma_2^2 = \mathrm{Var}\,(i(X_2; Y | X_1))$, $\sigma_3^2 = \mathrm{Var}\,(i(X_1, X_2; Y))$, $\mathbb{E}\left[i(X_2; Y | X_1)^+\right]$, and $\mathbb{E}\left[i(X_1, X_2; Y)^+\right]$ are finite. Moreover, by [35, pp. 207], Cramer's condition (C) in Definition 3 is satisfied by those distributions having at least a continuous component in its Lebesgue decomposition. Since $i(X_1, Y | X_2)$, $i(X_2; Y | X_1)$, and $i(X_1, X_2; Y)$ are all continuous random variables, their distribution functions are strongly nonlattice. Hence, it follows from Lemma 1 that

$$I(X_1; Y | X_2) \mathbb{E}(\tau^{(1)}) = \gamma_1 + \xi_1 + o(1), \text{ as } \gamma_1 \to \infty, \tag{63}$$
$$I(X_2; Y | X_1) \mathbb{E}(\tau^{(2)}) = \gamma_2 + \xi_2 + o(1), \text{ as } \gamma_2 \to \infty, \tag{64}$$
$$I(X_1, X_2; Y) \mathbb{E}(\tau^{(3)}) = \gamma_3 + \xi_3 + o(1), \text{ as } \gamma_3 \to \infty, \tag{65}$$

where

$$\xi_j := \frac{\mathbb{E}\left[\left(S_{\tau_+^{(j)}}^{(j)}\right)^2\right]}{2\mathbb{E}\left[S_{\tau_+^{(j)}}^{(j)}\right]}, \quad j = 1, 2, 3. \tag{66}$$

Recall that $S_n^{(1)}$ is the n-letter information density $i(X_1^n; Y^n | X_2^n)$ and $\tau_+^{(1)}$ is defined in (50). Additionally, let

$$\nu_j := \frac{\mathbb{E}\left[\left(S_{\tau_+^{(j)}}^{(j)}\right)^3\right]}{\mathbb{E}\left[S_{\tau_+^{(j)}}^{(j)}\right]}, \quad \text{and} \tag{67}$$

$$\tau^{(j)}(x) := \inf\{n \geq 0 : S_n^{(j)} > x\}, \quad j = 1, 2, 3. \tag{68}$$

From Lemma 2, we have that

$$\mathrm{Var}(\tau^{(j)}) = \mu_j^{-3}\sigma_j^2\gamma_j + \mu_j^{-2}K_j + o(1), \text{ as } \gamma_j \to \infty, \quad (69)$$

where for $j = 1, 2, 3$,

$$K_j := \frac{\sigma_j^2}{\mu_j}\xi_j + 3\xi_j^2 - \frac{2}{3}\nu_j - 2\xi_j\,\mathbb{E}\Big\{\min_{n\geq 0} S_n^{(j)}\Big\} - 2\int_0^\infty \mathbb{E}\{S_{\tau^{(j)}(x)}^{(j)} - x\}\mathbb{P}\Big\{\min_{n\geq 0} S_n^{(j)} \leq -x\Big\}\,\mathrm{d}x. \quad (70)$$

are constants which are not dependent on $\gamma_j, j = 1, 2, 3$, (i.e. $K_1, K_2, K_3 = O(1)$). Now, for any positive real number N', choose

$$\gamma_1 = I(X_1; Y|X_2)(N' - A\sqrt{N'} - G), \quad (71)$$
$$\gamma_2 = I(X_2; Y|X_1)(N' - A\sqrt{N'} - G), \quad (72)$$
$$\gamma_3 = I(X_1, X_2; Y)(N' - A\sqrt{N'} - G), \quad (73)$$

and a pair (M_1, M_2) satisfying

$$0 \leq \log M_j \leq \gamma_j - \log(3N'), \quad j = 1, 2, \quad (74)$$
$$0 \leq \log M_1M_2 \leq \gamma_3 - \log(3N'), \quad (75)$$

for some $A \geq 0, G \geq 0$ to be determined later. These choices of M_1 and M_2 and the fact that $\xi_j = O(1)$ for all $j = 1, 2, 3$ show that (37) and (38) are satisfied.

Combining these choices of γ_j with (63)–(65) we obtain

$$\mathbb{E}[\tau^{(j)}] = N' - A\sqrt{N'} - G - B_j + o(1) \quad j = 1, 2, 3, \quad (76)$$

where

$$B_j := \frac{-2\xi_j}{\log(1 + P_j)}, \quad j = 1, 2 \quad (77)$$
$$B_3 := \frac{-2\xi_3}{\log(1 + P_1 + P_2)}, \quad (78)$$

are constants. By using the facts that $A \geq 0$, $G \geq 0$ and (69), we also have

$$\mathrm{Var}(\tau^{(j)}) = L_j(N' - A\sqrt{N'} - G) + F_j + o(1) \leq L_jN' + F_j + o(1), \quad (79)$$

where the constants L_j and F_j are defined according to Lemma 2. Specifically,

$$L_j := \left(\frac{\sigma_j}{\mu_j}\right)^2 = (11), \quad (80)$$
$$F_j := \mu_j^{-2}K_j, \quad j = 1, 2, 3. \quad (81)$$

It follows from Lemma 4 that

$$\mathbb{E}(\tau^*) \leq \mathbb{E}[\tau'] = \mathbb{E}[\max\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\}] \leq N' + o(1) \quad (82)$$

as $N' \to \infty$. Moreover, from (76) we have $\mathbb{E}(\tau^{(j)}) < \infty$, hence

$$\mathbb{P}(\tau^{(j)} < \infty) = 1, \quad j = 1, 2, 3. \quad (83)$$

Applying a change of measure, we observe that for any measurable function f,

$$\mathbb{E}[f(\bar{X}_1^n, X_2^n, Y^n)] = \mathbb{E}\big[f(X_1^n, X_2^n, Y^n)\exp\big(-S_n^{(1)}\big)\big], \quad (84)$$
$$\mathbb{E}[f(X_1^n, \bar{X}_2^n, Y^n)] = \mathbb{E}\big[f(X_1^n, X_2^n, Y^n)\exp\big(-S_n^{(2)}\big)\big], \quad (85)$$
$$\mathbb{E}[f(\bar{X}_1^n, \bar{X}_2^n, Y^n)] = \mathbb{E}\big[f(X_1^n, X_2^n, Y^n)\exp\big(-S_n^{(3)}\big)\big]. \quad (86)$$

Observe that $1\{\tau^{(j)} \leq n\} \in \sigma(X_1^n, X_2^n, Y^n)$ for $j = 1, 2, 3$, $1\{\tau' \leq n\} \in \sigma(X_1^n, X_2^n, Y^n)$, $1\{\bar{\tau}^{(1)} \leq n\} \in \sigma(\bar{X}_1^n, X_2^n, Y^n)$, $1\{\bar{\tau}^{(2)} \leq n\} \in \sigma(X_1^n, \bar{X}_2^n, Y^n)$, and $1\{\bar{\tau}^{(3)} \leq n\} \in \sigma(\bar{X}_1^n, \bar{X}_2^n, Y^n)$. Following the same arguments as in [15, Eqns. (111)–(118)], we have

$$\mathbb{P}(\bar{\tau}^{(3)} \leq \tau') \leq \mathbb{P}(\bar{\tau}^{(3)} < \infty) \leq \exp(-\gamma_3), \quad (87)$$

Similarly, for $j = 1, 2$,

$$\mathbb{P}(\bar{\tau}^{(j)} \leq \tau') \leq \mathbb{P}(\bar{\tau}^{(j)} < \infty) \leq \exp(-\gamma_j). \quad (88)$$

From the bound on the error probability in (56), the bounds on the individual probabilities in (87) and (88), the choices of M_1 and M_2 in (74) and (75), we see that the average error probability of the stop-feedback code satisfies

$$\varepsilon' \leq \frac{1}{N'}. \quad (89)$$

Observe that

$$\mathbb{E}\Bigg[\sum_{n=1}^{\tau^*} X_{jn}^2\Bigg] = \mathbb{E}\Bigg[\sum_{n=1}^{\tau^*} X_{jn}^2 \,\Bigg|\, W_1 = 1, W_2 = 1\Bigg] \quad (90)$$
$$\leq \mathbb{E}\Bigg[\sum_{n=1}^{\tau_{1,1}} X_{jn}^2 \,\Bigg|\, W_1 = 1, W_2 = 1\Bigg] \quad (91)$$
$$= \mathbb{E}\Bigg[\sum_{n=1}^{\tau'} X_{jn}^2\Bigg], \quad j = 1, 2. \quad (92)$$

To verify that the expected power constraints are satisfied, we now check all the conditions of Lemma 3 (with X_{jn}^2 for $j = 1, 2$ here playing the role of X_n in Lemma 3).

- We have $\mathbb{E}[X_{jn}^2] = P_j$ for $j = 1, 2$ so it follows that X_{1n}^2 and X_{2n}^2 are integrable for all $n \geq 1$.
- Now, we see that $1\{\tau' \geq n\} = 1 - 1\{\tau' \leq n - 1\} \in \sigma(X_1^{n-1}, X_2^{n-1}, Y^{n-1})$. Moreover, since the sequence $\{X_{1n}\}_{n\geq 1}$ as well as the sequence $\{X_{2n}\}_{n\geq 1}$ are i.i.d. generated and the channel is memoryless, we have that $1\{\tau' \geq n\}$ is independent of X_{1n} and X_{2n}. It follows that $\mathbb{E}[X_{jn}^2 1\{\tau' \geq n\}] = \mathbb{E}[X_{jn}^2]\mathbb{E}[1\{\tau' \geq n\}] = \mathbb{E}[X_{jn}^2]\mathbb{P}(\tau' \geq n)$ for $j = 1, 2$;
- For each $j = 1, 2$, the infinite series $\sum_{n=1}^\infty \mathbb{E}[X_{jn}^2 1\{\tau' \geq n\}]$ satisfies

$$\sum_{n=1}^\infty \mathbb{E}[X_{jn}^2 1\{\tau' \geq n\}] = \sum_{n=1}^\infty \mathbb{E}[X_{jn}^2]\mathbb{P}(\tau' \geq n) \quad (93)$$
$$\leq P_j \sum_{n=1}^\infty \mathbb{P}(\tau' \geq n) \quad (94)$$
$$= P_j\mathbb{E}(\tau') \quad (95)$$
$$\leq P_j(N' + o(1)) < \infty, \quad (96)$$

  where (96) follows from (82).
- For each $j = 1, 2$, all random variables $X_{jn}^2, n \geq 1$ have the same expectation P_j.
- $\mathbb{E}(\tau') \leq N' + o(1) < \infty$.

Hence, by (92) and Lemma 3, the expected power constraints at the encoders satisfy

$$\mathbb{E}\left[\sum_{n=1}^{\tau^*} X_{jn}^2\right] \leq \mathbb{E}\left[\sum_{n=1}^{\tau'} X_{jn}^2\right] \tag{97}$$
$$= \mathbb{E}(\tau')\mathbb{E}[X_{j1}^2] \tag{98}$$
$$\leq \mathbb{E}(\tau')P_j, \quad j=1,2. \tag{99}$$

This means that we have shown there exists an $(M_1, M_2, N'+o(1), \frac{1}{N'})$ stop-feedback code with stopping time τ^* such that (99) holds. Since there exists such a code, we can find an $(M_1, M_2, N'+o(1), \frac{1}{N'})$ stop-feedback code with stopping time τ' by increasing the stopping time from τ^* to τ' (using the same decoder at time τ^*). It follows that (99) holds with equality. Moreover, if there exists an $(M_1, M_2, N'+o(1), \frac{1}{N'})$ stop-feedback code with stopping time τ^*, by keeping the same stopping rule and the decoder of the aforementioned code and setting

$$\tilde{X}_{jn} := \begin{cases} X_{jn}, & n \leq \tau^* \\ 0, & n > \tau^* \end{cases}, \quad j=1,2, \tag{100}$$

we have a new $(M_1, M_2, N'+o(1), \frac{1}{N'})$ stop-feedback code satisfying:

$$\sum_{n=1}^{\infty} \mathbb{E}[\tilde{X}_{jn}^2] = \mathbb{E}\left[\sum_{n=1}^{\infty} \tilde{X}_{jn}^2\right] \tag{101}$$
$$= \mathbb{E}\left[\sum_{n=1}^{\tau^*} X_{jn}^2\right] \tag{102}$$
$$= \mathbb{E}(\tau^*)P_j, \quad j=1,2, \tag{103}$$

where (101) follows from Tonelli's theorem [36]. This concludes the proof of Lemma 5. ∎

Lemma 6. *For the Gaussian MAC $\mathbb{P}(y|x_1,x_2)$, there exists an $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code for M_1, M_2 satisfying* (8) *and* (9).

Proof: We propose a stop-feedback coding scheme as follows:

- The decoder chooses numbers N', P_1', P_2' such that
$$\frac{(N')^2(1-\varepsilon)}{N'-1} \leq N, \tag{104}$$
$$P_j' = P_j, \quad j=1,2. \tag{105}$$
- The decoder generates a Bernoulli random variable $D \sim \mathrm{Bern}(p)$, where
$$p := \frac{N'\varepsilon - 1}{N'-1}. \tag{106}$$
- If $D=1$, the decoder sends a stop-feedback (or a NACK) to the encoder via the feedback link. This means that $\tau = 0$.
- If $D=0$, the encoder sends the intended message to the decoder using the stop-feedback $(M_1, M_2, N'+o(1), P_1', P_2', \frac{1}{N'})$ mentioned in Lemma 5 for the Gaussian MAC with expected powers P_1' and P_2' and stops at time τ'. This means that $\tau = \tau'$.

It follows that the error probability of the proposed stop-feedback coding scheme is upper bounded by

$$1\frac{N'\varepsilon - 1}{N'-1} + \left(1 - \frac{N'\varepsilon - 1}{N'-1}\right)\frac{1}{N'} = \varepsilon. \tag{107}$$

In addition, the average length of the proposed stop-feedback coding scheme is less than or equal to

$$\left(1 - \frac{N'\varepsilon - 1}{N'-1}\right)\mathbb{E}(\tau')$$
$$\leq \left(1 - \frac{N'\varepsilon - 1}{N'-1}\right)N' + o(1)\left(1 - \frac{N'\varepsilon - 1}{N'-1}\right) \tag{108}$$
$$= \frac{(N')^2(1-\varepsilon)}{N'-1} + o(1) \tag{109}$$
$$\leq N + o(1). \tag{110}$$

From (104) and (105), the expected powers of the combined scheme satisfy

$$\left(1 - \frac{N'\varepsilon - 1}{N'-1}\right)\mathbb{E}(\tau')P_j' = \mathbb{E}(\tau)P_j' = \mathbb{E}(\tau)P_j, \ j=1,2. \tag{111}$$

Therefore, combining this code construction with Lemma 5, we see that there exists an $(M_1, M_2, N+o(1), P_1, P_2, \varepsilon)$ stop-feedback code where

$$0 \leq \log M_j \leq \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} - G + o(1)\right)\mathsf{C}(P_j)$$
$$- \log\left(\frac{N}{1-\varepsilon}\right) + O(1), \quad j=1,2 \tag{112}$$
$$0 \leq \log M_1M_2 \leq \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}} - G + o(1)\right)\mathsf{C}(P_1+P_2)$$
$$- \log\left(\frac{N}{1-\varepsilon}\right) + O(1). \tag{113}$$

Observe that if there exists an $(M_1, M_2, N+o(1), P_1, P_2, \varepsilon)$ stop-feedback code, then there also exists an $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code by setting the expected length equal to $N - o(1)$. This change of the expected length does not affect the asymptotic approximation of the code rates. This concludes our proof of the achievability part of Theorem 1. ∎

B. Achievability Proof for Theorem 2

Lemma 7. *Given a Gaussian MAC, for any $\rho \in [0,1]$, there exist an $(M_1, M_2, N, P_1, P_2, \varepsilon)$ VLFT-feedback code for any M_1, M_2 satisfying* (15) *and* (16).

Proof: Consider Ozarow's coding scheme (for the Gaussian MAC with fixed-length feedback) [4] with fixed blocklength $N' \in \mathbb{N}$, expected powers bounded by P_1' and P_2', and message sizes M_1 and M_2 satisfying

$$\log M_j = N'\mathsf{C}\left(P_j'(1-\rho^2)\right)$$
$$- \log\log N' + O(1), \quad j=1,2 \tag{114}$$
$$\log M_1M_2 = N'\mathsf{C}\left(P_1 + P_2 + 2\rho\sqrt{P_1P_2}\right)$$
$$- \log\log N' + O(1) \tag{115}$$

where $\rho \in [0,1]$. Then, from [4, Eqn. (13)] and [6, Eqn. (121)], one sees that Ozarow's scheme results in an error probability

$$\varepsilon' \leq \frac{2}{(N')^2} \leq \frac{1}{N'}, \quad \forall N' \geq 2. \tag{116}$$

Therefore, we construct the VLFT coding scheme as follows.

- The decoder chooses the largest natural number N' such that (104) is satisfied. It also chooses positive numbers P_1', P_2' as in (105).
- The decoder generates a Bernoulli random variable $D \sim \mathrm{Bern}(p)$, where p is defined in (106).
- If $D = 1$, the decoder sends a stop-feedback signal (or a NACK) to the encoder via the feedback link. This means that, conditioned on $D = 1$, $\tau = 0$.
- If $D = 0$, the encoder sends the intended message to the decoder using Ozarow's coding scheme with parameters $(M_1, M_2, N', P_1', P_2', \frac{1}{N'})$ with expected powers $P_1' = P_1$ and $P_2' = P_2$ and stops at time τ'. This means that, conditioned on $D = 0$, we have $\tau = N'$.

Similarly to the stop-feedback case, it follows that the error probability of the proposed VLFT coding scheme is upper bounded by ε. The expected powers of the combined scheme are also bounded by $\mathbb{E}(\tau)P_j, j = 1, 2$. Consequently, the achievability part of Theorem 2 is proved. ∎

IV. Converse Proofs

A. Converse Proof for Theorem 1

Lemma 8. *Given a Gaussian MAC $\mathbb{P}(y|x_1,x_2)$, $0 \leq \varepsilon \leq 1 - \max\{\frac{1}{M_1}, \frac{1}{M_2}\}$, any $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code satisfies* (13) *and* (14) *for all $N \in \mathbb{N}$.*

Proof: First, we consider the case $|\mathcal{U}_1| = |\mathcal{U}_2| = 1$. For the stop-feedback formalism, τ is a stopping time of the filtration $\{\sigma(Y^n)\}_{n=0}^\infty$. We note that if there exists a code $(f_n^{(1)}, f_n^{(2)}, g_n, \tau)$, we can construct another code $(\hat{f}_n^{(1)}, \hat{f}_n^{(2)}, \hat{g}_n, \hat{\tau})$ such that $\hat{X}_n = \hat{Y}_n = \mathsf{T}$ for any $n \geq \hat{\tau}$, where $\mathsf{T} \notin \mathbb{R}$ is a special symbol appended to the input and output alphabets to form the common input-output alphabet $\mathbb{R} \cup \{\mathsf{T}\}$ and $\hat{\tau} = \tau + 1 = \inf\{n : \hat{Y}_n = \mathsf{T}\}$. Thus for the converse, it is suffices to consider $(\hat{f}_n^{(1)}, \hat{f}_n^{(2)}, \hat{g}_n, \hat{\tau})$, where the encoders $\hat{f}_n^{(j)}, j = 1, 2$ are defined as in [15, Eqn. (59)] and the decoder $\hat{g}_n$ as in [15, Eqn. (61)].

In addition, using the same arguments as [15, Eqn. (68)] we have

$$(1-\varepsilon)\log M_1 M_2 \leq I(W_1 W_2; \hat{Y}^\infty) + h_{\mathrm{b}}(\varepsilon), \tag{117}$$

$$(1-\varepsilon)\log M_1 \leq I(W_1; \hat{Y}^\infty | W_2 = w_2) + h_{\mathrm{b}}(\varepsilon), \tag{118}$$

$$(1-\varepsilon)\log M_2 \leq I(W_2; \hat{Y}^\infty | W_1 = w_1) + h_{\mathrm{b}}(\varepsilon). \tag{119}$$

By taking expectations of (118) and (119) with respect to P_{W_2} and P_{W_1} respectively, we obtain

$$(1-\varepsilon)\log M_1 \leq I(W_1; \hat{Y}^\infty | W_2) + h_{\mathrm{b}}(\varepsilon), \tag{120}$$

$$(1-\varepsilon)\log M_2 \leq I(W_2; \hat{Y}^\infty | W_1) + h_{\mathrm{b}}(\varepsilon). \tag{121}$$

Define

$$\Psi_n := 1\{\hat{\tau} \leq n-1\} \in \sigma(\hat{Y}^{n-1}). \tag{122}$$

By Lemma 10 in Appendix B, we have

$$I(W_1 W_2; \hat{Y}^\infty) \leq \sum_{n=1}^\infty \frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0]), \tag{123}$$

$$I(W_1; \hat{Y}^\infty | W_2) \leq \sum_{n=1}^\infty \frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[X_{1n}^2 | \Psi_n = 0]), \tag{124}$$

$$I(W_2; \hat{Y}^\infty | W_1) \leq \sum_{n=1}^\infty \frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[X_{2n}^2 | \Psi_n = 0]). \tag{125}$$

We observe that

$$\sum_{n=1}^\infty \mathbb{P}(\Psi_n = 0) = \sum_{n=1}^\infty \mathbb{P}(\tau \geq n) = \mathbb{E}(\tau). \tag{126}$$

It follows that $\{\mathbb{P}(\Psi_n = 0)/\mathbb{E}(\tau)\}_{n=1}^\infty$ is a probability distribution. Moreover, since the function $f(x) = \log(1+x)$ is concave, we have from (117) and (123) that

$$\begin{aligned}
&(1-\varepsilon)\log M_1 M_2 \\
&\leq \frac{1}{2}\mathbb{E}(\tau)\log\left(1 + \sum_{n=1}^\infty \frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)}\mathbb{E}[(X_{1n}+X_{2n})^2 | \Psi_n = 0]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(127)\\
&\leq \frac{N}{2}\log\left(1 + \frac{1}{\mathbb{E}(\tau)}\sum_{n=1}^\infty \mathbb{P}(\Psi_n = 0)\mathbb{E}[(X_{1n}+X_{2n})^2 | \Psi_n = 0]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(128)\\
&\leq \frac{N}{2}\log\left(1 + \frac{1}{\mathbb{E}(\tau)}\sum_{n=1}^\infty \mathbb{E}[(X_{1n}+X_{2n})^2]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(129)\\
&= \frac{N}{2}\log\left(1 + \frac{1}{\mathbb{E}(\tau)}\sum_{n=1}^\infty \mathbb{E}[X_{1n}^2] + \mathbb{E}[X_{2n}^2] + 2\mathbb{E}[X_{1n}X_{2n}]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(130)\\
&\leq \frac{N}{2}\log\left(1 + \frac{P_1\mathbb{E}(\tau) + P_2\mathbb{E}(\tau)}{\mathbb{E}(\tau)}\right) + h_{\mathrm{b}}(\varepsilon) &(131)
\end{aligned}$$

Here, (129) follows from the fact that $\mathbb{E}[(X_{1n} + X_{2n})^2] \geq \mathbb{P}(\Psi_n = 0)\mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0]$. and (131) follows from the power constraints of the stop-feedback code and the fact that $X_{1n} = f_n^{(1)}(W_1)$ is independent of $X_{2n} = f_n^{(2)}(W_2)$.

Similarly, we have from (120) and (124) that

$$\begin{aligned}
&(1-\varepsilon)\log M_1 \\
&\leq \frac{1}{2}\mathbb{E}(\tau)\log\left(1 + \sum_{n=1}^\infty \frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)}\mathbb{E}[X_{1n}^2 | \Psi_n = 0]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(132)\\
&\leq \frac{N}{2}\log\left(1 + \frac{1}{\mathbb{E}(\tau)}\sum_{n=1}^\infty \mathbb{P}(\Psi_n = 0)\mathbb{E}[X_{1n}^2 | \Psi_n = 0]\right) \\
&\quad + h_{\mathrm{b}}(\varepsilon) &(133)
\end{aligned}$$

$$\leq \frac{N}{2}\log\left(1+\frac{1}{\mathbb{E}(\tau)}\sum_{n=1}^{\infty}\mathbb{E}[X_{1n}^2]\right)+h_{\mathrm{b}}(\varepsilon) \tag{134}$$

$$\leq \frac{N}{2}\log\left(1+\frac{P_1\mathbb{E}(\tau)}{\mathbb{E}(\tau)}\right)+h_{\mathrm{b}}(\varepsilon) \tag{135}$$

For the case $|\mathcal{U}_1|\geq 1, |\mathcal{U}_2|\geq 1$, with the above arguments and $\mathcal{F}_n=\sigma(U_1,U_2,\hat{Y}^n)$, the following expressions hold almost surely:

$$\begin{aligned}&(1-\mathbb{P}[(\hat{W}_1,\hat{W}_2)\neq(W_1,W_2)|U_1,U_2])\log M_1M_2\\&\leq\frac{1}{2}\log(1+P_1+P_2)\\&\quad+h_{\mathrm{b}}(\mathbb{P}[(\hat{W}_1,\hat{W}_2)\neq(W_1,W_2)|U_1,U_2]),\end{aligned} \tag{136}$$

$$\begin{aligned}&(1-\mathbb{P}[(\hat{W}_1,\hat{W}_2)\neq(W_1,W_2)|U_1,U_2])\log M_j\\&\leq\frac{1}{2}\log(1+P_j)+h_{\mathrm{b}}(\mathbb{P}[(\hat{W}_1,\hat{W}_2)\neq(W_1,W_2)|U_1,U_2]),\end{aligned} \tag{137}$$

where $j=1,2$. By taking the expectation with respect to (U_1,U_2) on both sides of (136)–(137) and applying Jensen's inequality for the binary entropy terms, we obtain (13)–(14). This concludes the converse proof of Theorem 1. ∎

B. Converse Proof for Theorem 2

Lemma 9. *Given a Gaussian MAC $\mathbb{P}(y|x_1,x_2)$, for any $0\leq\varepsilon\leq 1-\max\{\frac{1}{M_1},\frac{1}{M_2}\}$, any $(M_1,M_2,N,P_1,P_2,\varepsilon)$ VLFT code for any $N\in\mathbb{N}$ satisfies* (17) *and* (18) *for some $\rho\in[0,1]$.*

Proof: Similarly to the converse proof for Gaussian MAC with a stop-feedback code, we first consider the case in which $|\mathcal{U}_1|=|\mathcal{U}_2|=1$. Since the receiver decides on the transmitted messages based only on Y^τ and (W_1,W_2) (not dependent on the channel outputs that are received after time τ), as in [15], we can convert any given code $(f_n^{(1)},f_n^{(2)},g_n,\tau)$ to an equivalent code $(\hat{f}_n^{(1)},\hat{f}_n^{(2)},\hat{g}_n,\tau)$ to remove the dependence of τ on (W_1,W_2). To do so, we append a special symbol $\mathsf{T}\notin\mathbb{R}$ to the input and output alphabets to form the common input-output alphabet $\mathbb{R}\cup\{\mathsf{T}\}$. We also set $\hat{\tau}=\tau+1=\inf\{n:\hat{Y}_n=\mathsf{T}\}$ and

$$\Psi_n:=1\{\hat{\tau}\leq n\}\in\sigma(\hat{Y}^n), \tag{138}$$

which is slightly different from the stop-feedback case (cf. (122)).

Using the same approach as the proof of converse for the Gaussian MAC with a stop-feedback code in Section IV-A, we obtain from the bounds in Appendix B that

$$\begin{aligned}&I(W_1,W_2;\hat{Y}_n|\hat{Y}^{n-1})\leq H(\Psi_n|\hat{Y}^{n-1})\\&+\frac{1}{2}\mathbb{P}(\Psi_n=0)\log(1+\mathbb{E}[(X_{1n}+X_{2n})^2|\Psi_n=0]),\end{aligned} \tag{139}$$

$$\begin{aligned}&I(W_1;\hat{Y}_n|\hat{Y}^{n-1}W_2)\leq H(\Psi_n|\hat{Y}^{n-1})\\&+\mathbb{P}(\Psi_n=0)I(X_{1n};Y_n|\Psi_n=0,Y^{n-1},X_{2n},W_2),\end{aligned} \tag{140}$$

$$\begin{aligned}&I(W_2;\hat{Y}_n|\hat{Y}^{n-1}W_1)\leq H(\Psi_n|\hat{Y}^{n-1})\\&+\mathbb{P}(\Psi_n=0)I(X_{2n};Y_n|\Psi_n=0,Y^{n-1},X_{1n},W_1).\end{aligned} \tag{141}$$

Note that $\{\hat{\tau}\leq n-1\}$ for the stop-feedback case (cf. Lemma 8) is equivalent to $\{\hat{\tau}\leq n\}$ for the VLFT case we consider here. Also compare (122) to (138). Observe that

$$\begin{aligned}&I(X_{1n};Y_n|\Psi_n=0,Y^{n-1},X_{2n},W_2)\\&\leq h(X_{1n}+Z_n|\Psi_n=0,X_{2n})-\frac{1}{2}\log(2\pi e).\end{aligned} \tag{142}$$

From here on, we essentially mimic Ozarow's weak converse proof for the Gaussian MAC with fixed-length feedback [4] but with some changes in the parameter settings. First define

$$\sigma_{jn}^2:=\mathrm{Var}[X_{jn}|\Psi_n=0],\quad j=1,2 \tag{143}$$

$$\lambda_n:=\mathsf{Cov}[X_{1n},X_{2n}|\Psi_n=0]. \tag{144}$$

Using the same approach as in [4], we can show that

$$\begin{aligned}&h(X_{1n}+Z_n|\Psi_n=0,X_{2n})\\&\leq\frac{1}{2}\log\left[2\pi e\sigma_{1n}^2\left(1-\frac{\lambda_n^2}{\sigma_{1n}^2\sigma_{2n}^2}\right)+2\pi e\right].\end{aligned} \tag{145}$$

Therefore, we obtain

$$\begin{aligned}&I(W_1,W_2;\hat{Y}_n|\hat{Y}^{n-1})\leq H(\Psi_n|\hat{Y}^{n-1})\\&\quad+\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{1n}^2+\sigma_{2n}^2+2\lambda_n\right],\end{aligned} \tag{146}$$

$$\begin{aligned}&I(W_1;\hat{Y}_n|\hat{Y}^{n-1}W_2)\leq H(\Psi_n|\hat{Y}^{n-1})\\&\quad+\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{1n}^2\left(1-\frac{\lambda_n^2}{\sigma_{1n}^2\sigma_{2n}^2}\right)\right],\end{aligned} \tag{147}$$

$$\begin{aligned}&I(W_2;\hat{Y}_n|\hat{Y}^{n-1}W_1)\leq H(\Psi_n|\hat{Y}^{n-1})\\&\quad+\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{2n}^2\left(1-\frac{\lambda_n^2}{\sigma_{1n}^2\sigma_{2n}^2}\right)\right].\end{aligned} \tag{148}$$

It follows from (117), (120), and (121) and the above considerations that

$$\begin{aligned}&(1-\varepsilon)\log M_1M_2\leq\sum_{n=1}^{\infty}H(\Psi_n|\hat{Y}^{n-1})\\&+\sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{1n}^2+\sigma_{2n}^2+2\lambda_n\right]+h_{\mathrm{b}}(\varepsilon),\end{aligned} \tag{149}$$

$$\begin{aligned}&(1-\varepsilon)\log M_1\leq\sum_{n=1}^{\infty}H(\Psi_n|\hat{Y}^{n-1})\\&+\sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{1n}^2\left(1-\frac{\lambda_n^2}{\sigma_{1n}^2\sigma_{2n}^2}\right)\right]+h_{\mathrm{b}}(\varepsilon),\end{aligned} \tag{150}$$

$$\begin{aligned}&(1-\varepsilon)\log M_2\leq\sum_{n=1}^{\infty}H(\Psi_n|\hat{Y}^{n-1})\\&+\sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n=0)\log\left[1+\sigma_{2n}^2\left(1-\frac{\lambda_n^2}{\sigma_{1n}^2\sigma_{2n}^2}\right)\right]+h_{\mathrm{b}}(\varepsilon).\end{aligned} \tag{151}$$

Note that by [15, Eqn. (90)], we have

$$\sum_{n=1}^{\infty}H(\Psi_n|\hat{Y}^{n-1})=H(\tau)\leq(N+1)h_{\mathrm{b}}\left(\frac{1}{N+1}\right) \tag{152}$$

$$\leq\log(N+1)+1. \tag{153}$$

Moreover, since we have

$$\sum_{n=1}^{\infty}\mathbb{P}(\Psi_n=0)=\sum_{n=1}^{\infty}\mathbb{P}(\hat{\tau}>n) \tag{154}$$
$$=\sum_{n=1}^{\infty}\mathbb{P}(\tau\ge n) \tag{155}$$
$$=\mathbb{E}(\tau), \tag{156}$$

it follows that $\{\mathbb{P}(\Psi_n=0)/\mathbb{E}(\tau)\}_{n=1}^{\infty}$ is a valid probability distribution. As in Ozarow's weak converse proof for the Gaussian MAC with fixed-length feedback [4], the right-hand-sides of (149), (150), and (151) can be readily shown to be jointly concave in $(\sigma_{1n}^2,\sigma_{2n}^2,\lambda_n)$. Thus, we can use Jensen's inequality to upper bound them.

More specifically, we set

$$G_j^2:=\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n=0)}{\mathbb{E}(\tau)}\sigma_{jn}^2,\quad j=1,2 \tag{157}$$
$$\rho:=\frac{1}{G_1G_2}\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n=0)}{\mathbb{E}(\tau)}\lambda_n. \tag{158}$$

We can bound G_1 as follows:

$$G_1^2=\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n=0)}{\mathbb{E}(\tau)}\sigma_{1n}^2, \tag{159}$$
$$\le\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n=0)}{\mathbb{E}(\tau)}\mathbb{E}(X_{1n}^2|\Psi_n=0) \tag{160}$$
$$\le\sum_{n=1}^{\infty}\frac{\mathbb{E}[X_{1n}^2]}{\mathbb{E}(\tau)}\le P_1. \tag{161}$$

The last step follows from the expected power constraints in (6). Similarly, we have $G_2^2\le P_2$. Moreover, we also have $|\lambda_n|\le\sigma_{1n}\sigma_{2n}$ and so from (158) and the Cauchy-Schwarz inequality,

$$|\rho|^2\le\left(\sum_{n=1}^{\infty}\frac{1}{G_1G_2}\frac{\mathbb{P}(\Psi_n=0)}{\mathbb{E}(\tau)}\sigma_{1n}\sigma_{2n}\right)^2 \tag{162}$$
$$\le\left(\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n\!=\!0)}{\mathbb{E}(\tau)}\frac{\sigma_{1n}^2}{G_1^2}\right)\left(\sum_{n=1}^{\infty}\frac{\mathbb{P}(\Psi_n\!=\!0)}{\mathbb{E}(\tau)}\frac{\sigma_{1n}^2}{G_2^2}\right) \tag{163}$$
$$=1. \tag{164}$$

By applying Jensen's inequality to (149), we obtain

$$(1-\varepsilon)\log M_1M_2$$
$$\le(N\!+\!1)h_{\mathrm{b}}\left(\frac{1}{N\!+\!1}\right)+\frac{\mathbb{E}(\tau)}{2}\log\left[1\!+\!G_1^2\!+\!G_2^2\!+\!2\rho G_1G_2\right] \tag{165}$$
$$\le(N\!+\!1)h_{\mathrm{b}}\left(\frac{1}{N\!+\!1}\right)+\frac{N}{2}\log\left[1\!+\!G_1^2\!+\!G_2^2\!+\!2\rho G_1G_2\right] \tag{166}$$
$$\le(N\!+\!1)h_{\mathrm{b}}\left(\frac{1}{N\!+\!1}\right)+\frac{N}{2}\log\left[1\!+\!P_1\!+\!P_2\!+\!2|\rho|\sqrt{P_1P_2}\right]. \tag{167}$$

Similarly, by applying Jensen's inequality to (150) and (151), we obtain

$$(1-\varepsilon)\log M_j\le(N+1)h_{\mathrm{b}}\left(\frac{1}{N+1}\right)+\frac{N}{2}\log\left[1+P_j^2(1-\rho^2)\right], \tag{168}$$

for $j=1,2$. This completes the proof of Lemma 9 and hence, the converse proof of Theorem 2. ■

V. Conclusion and Future Work

In this paper, we derived bounds on achievable rates of the Gaussian MAC with the use of variable-length codes with feedback and under the non-vanishing error probability formalism. We quantified the gains of VLFT codes over stop-feedback codes. To establish our results, we leveraged some non-standard techniques to deal with the continuous nature of the channel and also to control the overshoot of the barrier (or threshold) of some relevant random walks.

In the future, it would be a fruitful endeavor to improve on the second-order terms in Theorems 1 and 2 as they are likely to be loose. In addition, it would be interesting to check if our newly-developed techniques for systems with variable-length feedback can be extended to other multi-terminal channel models such as the Gaussian broadcast channel.

Appendix A
Proof of Lemma 4

Proof: First, observe that

$$\mathbb{E}[(X_{1N}+X_{2N})^2]+\mathbb{E}[(X_{1N}-X_{2N})^2]$$
$$=2(\mathbb{E}[X_{1N}^2]+\mathbb{E}[X_{2N}^2]) \tag{169}$$
$$=2[\mathrm{Var}(X_{1N})+(\mathbb{E}X_{1N})^2+\mathrm{Var}(X_{2N})+(\mathbb{E}X_{2N})^2] \tag{170}$$
$$\le2[L_1N+F_1+o(1)+(N-A\sqrt{N}-G-B_1+o(1))^2+L_2N+F_2+o(1)+(N-A\sqrt{N}-G-B_2+o(1))^2]. \tag{171}$$

Since, we have

$$\mathbb{E}[(X_{1N}+X_{2N})^2]\ge(\mathbb{E}[X_{1N}+X_{2N}])^2 \tag{172}$$
$$=(N-A\sqrt{N}-G-B_1+o(1)+N-A\sqrt{N}-G-B_2+o(1))^2. \tag{173}$$

It follows from (171) and (173) that

$$\mathbb{E}[(X_{1N}-X_{2N})^2]$$
$$\le2[L_1N+F_1+o(1)+(N-A\sqrt{N}-G-B_1+o(1))^2+L_2N+F_2+o(1)+(N-A\sqrt{N}-G-B_2+o(1))^2]-(N-A\sqrt{N}-G-B_1+o(1)+N-A\sqrt{N}-G-B_2+o(1))^2 \tag{174}$$
$$=2[L_1N+F_1+o(1)+L_2N+F_2+o(1)]+(B_1-B_2+o(1))^2 \tag{175}$$
$$\le2[L_1+L_2]N+2(F_1+F_2)+(B_1-B_2)^2+o(1) \tag{176}$$
$$\le2[L_1+L_2]N+2|F_1+F_2|+(B_1-B_2)^2+o(1). \tag{177}$$

Therefore, we have

$$(\mathbb{E}|X_{1N} - X_{2N}|)^2 \le \mathbb{E}[(X_{1N} - X_{2N})^2] \tag{178}$$
$$\le 2[L_1 + L_2]N + 2|F_1 + F_2| + (B_1 - B_2)^2 + o(1). \tag{179}$$

By using the fact that $(a+b)^{1/2} \le a^{1/2}+b^{1/2}$ for nonnegative a, b, it follows that

$$\mathbb{E}|X_{1N} - X_{2N}| \le \sqrt{2(L_1 + L_2)N} + \sqrt{2|F_1 + F_2| + (B_1 - B_2)^2} + o(1). \tag{180}$$

Similarly, we have

$$\mathbb{E}|X_{iN} - X_{jN}| \le \sqrt{2(L_i + L_j)N} + \sqrt{2|F_i + F_j| + (B_i - B_j)^2} + o(1). \tag{181}$$

for any $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.

Now, we note that

$$\max\{X_{iN}, X_{jN}\} = \frac{1}{2}[X_{iN} + X_{jN} + |X_{iN} - X_{jN}|] \tag{182}$$

for any $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.

Therefore, we have

$$\max\{X_{1N}, X_{2N}, X_{3N}\}$$
$$= \max\{\max\{X_{1N}, X_{2N}\}, X_{3N}\} \tag{183}$$
$$= \frac{1}{2}\max\{X_{1N} + X_{2N} + |X_{1N} - X_{2N}|, 2X_{3N}\} \tag{184}$$
$$= \frac{1}{4}\Big[X_{1N} + X_{2N} + |X_{1N} - X_{2N}| + 2X_{3N} + |(X_{1N} + X_{2N} + |X_{1N} - X_{2N}|) - 2X_{3N}|\Big] \tag{185}$$
$$= \frac{1}{4}\Big[(X_{1N} + X_{2N} + 2X_{3N}) + |X_{1N} - X_{2N}| + |(X_{1N}-X_{3N})+(X_{2N}-X_{3N})+|X_{1N}-X_{2N}||\Big] \tag{186}$$
$$\le \frac{1}{4}\Big[(X_{1N} + X_{2N} + 2X_{3N}) + 2|X_{1N} - X_{2N}| + |X_{1N} - X_{3N}| + |X_{2N} - X_{3N}|\Big]. \tag{187}$$

It follows that

$$\mathbb{E}[\max\{X_{1N}, X_{2N}, X_{3N}\}]$$
$$\le \frac{1}{4}\mathbb{E}[X_{1N} + X_{2N} + 2X_{3N}] + \frac{1}{4}\mathbb{E}\left(2|X_{1N} - X_{2N}| + |X_{1N} - X_{3N}| + |X_{2N} - X_{3N}|\right) \tag{188}$$
$$= \frac{1}{4}\Big(\mathbb{E}[X_{1N}] + \mathbb{E}[X_{2N}] + 2\mathbb{E}[X_{3N}]\Big) + \frac{1}{2}\Big(\mathbb{E}|X_{1N} - X_{2N}|\Big) + \frac{1}{4}\Big(\mathbb{E}|X_{1N} - X_{3N}| + \mathbb{E}|X_{2N} - X_{3N}|\Big) \tag{189}$$
$$\le N - A\sqrt{N} - G - \frac{1}{4}(B_1 + B_2 + 2B_3) + o(1) + \frac{1}{2}\Big[\sqrt{2(L_1{+}L_2)}\sqrt{N}+\sqrt{2|F_1{+}F_2|{+}(B_1{-}B_2)^2}+o(1)\Big] + \frac{1}{4}\Big[\sqrt{2(L_1{+}L_3)}\sqrt{N}+\sqrt{2|F_1{+}F_3|{+}(B_1{-}B_3)^2} + o(1) + \sqrt{2(L_2{+}L_3)}\sqrt{N}+\sqrt{2|F_2{+}F_3|{+}(B_2{-}B_3)^2}+o(1)\Big] \tag{190}$$
$$= N - \sqrt{N}\Big[A - \frac{1}{2}\sqrt{2(L_1 + L_2)} - \frac{1}{4}(\sqrt{2(L_1 + L_3)} + \sqrt{2(L_2 + L_3)})\Big] - G - \frac{1}{4}(B_1 + B_2 + 2B_3) + \frac{1}{2}\left(\sqrt{2|F_1 + F_2| + (B_1 - B_2)^2}\right) + \frac{1}{4}\Big(\sqrt{2|F_1 + F_3| + (B_1 - B_3)^2} + \sqrt{2|F_2 + F_3| + (B_2 - B_3)^2}\Big) + o(1). \tag{191}$$

Now, if we choose

$$A = \frac{1}{2}\sqrt{2(L_1 + L_2)} + \frac{1}{4}\left(\sqrt{2(L_1 + L_3)}+\sqrt{2(L_2 + L_4)}\right) \tag{192}$$

and

$$G = -\frac{1}{4}(B_1{+}B_2{+}2B_3) + \frac{1}{2}\left(\sqrt{2|F_1{+}F_2| + (B_1{-}B_2)^2}\right) + \frac{1}{4}\Big(\sqrt{2|F_1 + F_3| + (B_1 - B_3)^2} + \sqrt{2|F_2 + F_3| + (B_2 - B_3)^2}\Big), \tag{193}$$

from (191), we have

$$\mathbb{E}[\max\{X_{1N}, X_{2N}, X_{3N}\}] \le N + o(1). \tag{194}$$

Notice the symmetry of X_{1N}, X_{2N}, X_{3N} in the expression $\max\{X_{1N}, X_{2N}, X_{3N}\}$. Hence, by the above approximation procedure, the smallest value of A that we can choose is given by (10). The proof of Lemma 4 can now be completed by choosing the order of combination X_{1N}, X_{2N}, X_{3N} in (183) such that A is minimized. ■

Appendix B
Bounds on Mutual Information Quantities for the Gaussian MAC

Lemma 10. *For any stop-feedback code for the Gaussian MAC as in Definition 1 and its equivalent form with the augmented symbol* T *for the case* $|\mathcal{U}_1| = |\mathcal{U}_2| = 1$*, define* $\Psi_n := 1\{\hat{\tau} \le n - 1\} \in \sigma(\hat{Y}^{n-1})$ *(cf. (122)). Then the following bounds hold:*

$$I(W_1W_2; \hat{Y}^\infty) \le \sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[(X_{1n} + X_{2n})^2|\Psi_n = 0]), \tag{195}$$

$$I(W_1; \hat{Y}^\infty|W_2) \le \sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[X_{1n}^2|\Psi_n = 0]), \tag{196}$$

$$I(W_2; \hat{Y}^\infty|W_1) \le \sum_{n=1}^{\infty}\frac{1}{2}\mathbb{P}(\Psi_n = 0)\log(1 + \mathbb{E}[X_{2n}^2|\Psi_n = 0]). \tag{197}$$

Proof: To prove (195), we observe that

$$I(W_1W_2; \hat{Y}^\infty) = \sum_{n=1}^{\infty} I(W_1, W_2; \hat{Y}_n|\hat{Y}^{n-1}). \tag{198}$$

Consider,

$$
\begin{aligned}
&I(W_1, W_2; \hat{Y}_n | \hat{Y}^{n-1}) \\
&= I(W_1, W_2; \hat{Y}_n, \Psi_n | \hat{Y}^{n-1}) && (199)\\
&= I(W_1, W_2; \Psi_n | \hat{Y}^{n-1}) + I(W_1, W_2; \hat{Y}_n | \Psi_n, \hat{Y}^{n-1}) && (200)\\
&\le H(\Psi_n | \hat{Y}^{n-1}) + I(W_1, W_2; \hat{Y}_n | \Psi_n, \hat{Y}^{n-1}) && (201)\\
&= I(W_1, W_2; \hat{Y}_n | \Psi_n, \hat{Y}^{n-1}) && (202)\\
&= \mathbb{P}(\Psi_n = 0) I(W_1, W_2; \hat{Y}_n | \Psi_n = 0, \hat{Y}^{n-1}) && (203)\\
&\le \mathbb{P}(\Psi_n = 0) I(\hat{X}_{1n}, \hat{X}_{2n}; \hat{Y}_n | \Psi_n = 0, \hat{Y}^{n-1}) && (204)\\
&= \mathbb{P}(\Psi_n = 0) I(X_{1n}, X_{2n}; Y_n | \Psi_n = 0, Y^{n-1}) && (205)\\
&= \mathbb{P}(\Psi_n = 0)[h(Y_n | \Psi_n = 0, Y^{n-1}) \\
&\qquad - h(Y_n | X_{1n}, X_{2n}, \Psi_n = 0, Y^{n-1})] && (206)\\
&\le \mathbb{P}(\Psi_n = 0)[h(Y_n | \Psi_n = 0) \\
&\qquad - h(Y_n | X_{1n}, X_{2n}, \Psi_n = 0, Y^{n-1})] && (207)\\
&= \mathbb{P}(\Psi_n = 0)[h(Y_n | \Psi_n = 0) \\
&\qquad - h(Z_n | X_{1n}, X_{2n}, \Psi_n = 0, Y^{n-1})] && (208)\\
&\le \mathbb{P}(\Psi_n = 0)[h(Y_n | \Psi_n = 0) - h(Z_n)] && (209)\\
&\le \mathbb{P}(\Psi_n = 0) \left[\frac{1}{2} \log[2\pi e \mathbb{E}(Y_n^2 | \Psi_n = 0)] - \frac{1}{2} \log[2\pi e] \right] && (210)\\
&= \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[\mathbb{E}(X_{1n} + X_{2n} + Z_n)^2 | \Psi_n = 0] && (211)\\
&= \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[\mathbb{E}((X_{1n} + X_{2n})^2 | \Psi_n = 0) \\
&\qquad + \mathbb{E}(X_{1n} Z_n | \Psi_n = 0) + \mathbb{E}(X_{2n} Z_n | \Psi_n = 0) \\
&\qquad + \mathbb{E}(Z_n^2 | \Psi_n = 0)] && (212)\\
&= \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[1 + \mathbb{E}((X_{1n} + X_{2n})^2 | \Psi_n = 0)], && (213)
\end{aligned}
$$

where (201) follows from the fact that Ψ_n is a binary random variable, (202) follows from the fact that $\Psi_n \in \sigma(\hat{Y}^{n-1})$, (203) follows from the fact that given $\Psi_n = 1$ or $n \ge \hat{\tau} + 1$ we always have $\hat{Y}_n = \mathsf{T}$, (205) follows from the fact that given $\Psi_n = 0$ or $\tau \ge n$ we have $\hat{X}_{1n} = X_{1n}, \hat{X}_{2n} = X_{2n}$, and $\hat{Y}_n = Y_n$, (209) follows from the fact that $\Psi_n = 1\{\hat{\tau} \le n-1\} = 1\{\tau \le n-1\}$ is a function of $\sigma(Y^{n-1}), X_{1n} = f_n^{(1)}(W_1), X_{2n} = f_n^{(1)}(W_2)$ and Z_n is independent of (Y^{n-1}, W_1, W_2), (210) follows from the maximal differential entropy formula, (213) follows from the facts that Ψ_n is a function of Y^{n-1} and Z_n is independent of $(X_{1n}, X_{2n}, Y^{n-1})$.

It follows that

$$
\begin{aligned}
&I(W_1 W_2; \hat{Y}^\infty) \\
&\le \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[1 + \mathbb{E}(X_{1n} + X_{2n})^2 | \Psi_n = 0].
\end{aligned} \tag{214}
$$

The other inequalities can be shown in a completely analogous manner. ■

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