

On the Tanner Graph Cycle Distribution of Random LDPC, Random Protograph-Based LDPC, and Random Quasi-Cyclic LDPC Code Ensembles

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Abstract

In this paper, we study the cycle distribution of random low-density parity-check (LDPC) codes, randomly constructed protograph-based LDPC codes, and random quasi-cyclic (QC) LDPC codes. We prove that for a random bipartite graph, with a given (irregular) degree distribution, the distributions of cycles of different length tend to independent Poisson distributions, as the size of the graph tends to infinity. We derive asymptotic upper and lower bounds on the expected values of the Poisson distributions that are independent of the size of the graph, and only depend on the degree distribution and the cycle length. For a random lift of a bi-regular protograph, we prove that the asymptotic cycle distributions are essentially the same as those of random bipartite graphs as long as the degree distributions are identical. For random QC-LDPC codes, however, we show that the cycle distribution can be quite different from the other two categories. In particular, depending on the protograph and the value of c , the expected number of cycles of length c , in this case, can be either $\Theta(N)$ or $\Theta(1)$, where N is the lifting degree (code length). We also provide numerical results that match our theoretical derivations. Our results provide a theoretical foundation for empirical results that were reported in the literature but were not well-justified. They can also be used for the analysis and design of LDPC codes and associated algorithms that are based on cycles.

Index Terms: Low-density parity-check (LDPC) codes, random LDPC codes, quasi cyclic (QC) LDPC codes, protograph-based LDPC codes, cycle distribution of LDPC codes, lifting, cyclic lifting.

I. INTRODUCTION

The performance of low-density parity-check (LDPC) codes under iterative message-passing algorithms is highly dependent on the structure of the code's Tanner graph, in general, and the distribution of short cycles, in particular, see, e.g., [1], [2], [3], [4]. The cycles play a particularly important role in the error floor performance of LDPC codes, where they form the main substructure of the trapping sets [5], [6], [7], [8], [9].

Counting and enumerating (finding) cycles of a given length in a general graph is known to be NP-hard [10]. (For a rather comprehensive literature review on algorithms to count and enumerate cycles in different types of graphs, including bipartite graphs, and their complexity, the reader is referred to [11].) It is thus of interest to have simple approximations for the number of cycles of a given length in a given graph. Related to this, it is also interesting to obtain the distribution of cycles of a given length in an ensemble of Tanner graphs (LDPC codes). The knowledge of such a distribution, including the expected value and variance, can help in the analysis and in guiding the design of LDPC codes. The expected value can also be used as an approximation for the number of cycles of a given length in a given graph in the ensemble, with the variance providing a measure of accuracy of the approximation.

In [12], Bollobás showed that, for a given random graph with an arbitrary degree distribution and a fixed c , as the size of the graph tends to infinity, the multiplicities of cycles of lengths $3, 4, 5, \dots, c$, tend to independent Poisson random variables. He also derived the expected values of the random variables. Later, in [13], the authors considered random bipartite graphs, in which all the nodes have the same degree d , and c can grow as a function of the number nodes in the graph, and proved that as the size of the graph tends to infinity, the distributions of cycles of different length c tend to independent Poisson distributions with expected values $\mu = (d-1)^c/c$.

In this work, we consider the case of random bipartite graphs with arbitrary degree distributions $\{d_i\}$ and $\{d'_i\}$ on the two parts of the graph, respectively, and prove that the multiplicities of cycles of different length c , as the size of the graph tends to infinity, tend to independent Poisson random variables with the following expected values:

$$\mu \approx \frac{\left(\frac{2}{|E|} \sum_{i=1}^n \binom{d_i}{2}\right) \left(\frac{2}{|E|} \sum_{i=1}^m \binom{d'_i}{2}\right)^{c/2}}{c}, \quad (1)$$

where n and m are the number of nodes in the two parts of the graph, and $|E|$ is the number of edges of the graph. The notation “ \approx ” in (1) is used to mean “approximately equal,” and the

approximation is within some fixed multiplicative factor of the exact value. Unlike the bipartite graphs studied in [13], the graphs studied in this work are those representing (irregular and bi-regular) LDPC codes.

For the special case of bi-regular LDPC codes, Equation (1) reduces to

$$\mu \sim \frac{\left((d_u - 1)(d_w - 1)\right)^{c/2}}{c}, \quad (2)$$

in which d_u and d_w denote the degrees of nodes in the two parts of the graph. The notation “ \sim ” in (2) is used to mean “asymptotically equal.” Equation (2) implies that, at sufficiently large block lengths, the average number of cycles, as well as the variances, do not depend on the block length of the code. This matches the observation made in [11] through numerical results.

The construction of LDPC codes by lifting a small bipartite graph, called *base graph* or *protograph*, was first appeared in [14]. Since then, there has been a flurry of research activity on the analysis and design of protograph-based LDPC codes, see, e.g., [15], [16], and the references therein. A particularly popular category of protograph-based LDPC codes are those constructed by *cyclic liftings* [17], [18], [19], [20], [21], [22], [6]. Such codes are quasi cyclic (QC), and are of most interest in practice, as they lend themselves to simpler implementation of encoding and decoding algorithms. For that reason, they have also been adopted in a number of standards [23], [24].

It was shown by Fortin and Rudinsky in [25] that for a random lift of a protograph, the distributions of cycles of different length tend to independent Poisson distributions as the size of the graph tends to infinity. They also showed that the expected value of the number of cycles of length c is equal to $T(G, c)$, where $T(G, c)$ is the number of tailless backtrackless closed walks of length c in the protograph G . In this work, we calculate $T(G, c)$ for bi-regular protographs, in general, and fully-connected bipartite protographs, in particular. Using these results, we show that the cycle distributions of random bi-regular graphs and those of random lifts of a bi-regular protograph with a similar degree distribution are essentially identical in the asymptotic regime, where the graph size tends to infinity.

In [20], an efficient algorithm for counting short cycles in the Tanner graph of a QC-LDPC code is proposed. Using numerical results, it was shown in [20], that randomly constructed QC-LDPC codes have a much better girth distribution compared to their counterparts that lack the QC structure. In this work, by viewing the Tanner graphs of QC-LDPC codes as cyclic lifts of protographs, we study their cycle distribution. We demonstrate that the cycle distributions for

random cyclic lifts of a bipartite protograph can be quite different from those of random bipartite graphs and random lifts of bipartite protographs of similar degree distributions. In particular, we show that depending on the protograph and the cycle length c , the expected value of the number of cycles of length c in random cyclic lifts can increase linearly with the size of the graph. This is while for random bipartite graphs and random lifts of bipartite protographs, the expected number of cycles of length c remains constant with increase in the graph size, regardless of the value of c or the choice of protograph or degree distribution. These results explain the differences observed in [20] regarding the cycle distributions of QC-LDPC codes versus LDPC codes that lack the QC structure.

In addition to providing theoretical justification for empirical results in the literature, the results presented here can be used for the analysis and design of LDPC codes and associated algorithms that are based on cycles. As an example, it was shown very recently [26] that among trapping set structures with cycles, only those that contain a single (chordless) cycle have non-zero multiplicity asymptotically. This asymptotic multiplicity has been estimated in [26] using the results of this work. (More details are provided in Subsection III-B.) As another example, the *dpl* characterization and search algorithm of [9], [27] is known to be the most efficient in exhaustively finding the elementary trapping sets of LDPC codes. The starting point of *dpl* search is chordless cycles in a graph. These cycles are then recursively expanded using three simple expansion techniques. Our theoretical results on the average number of cycles can be used to establish theoretical bounds on the average complexity of *dpl* search. In the absence of such theoretical results, complexity discussions in [9], related to the number of cycles, relied on empirical results provided in [11].

The organization of the rest of the paper is as follows: In Section II, we present some definitions and notations. This is followed in Section III by our results on the cycle distribution of random LDPC codes. In this section, we also present an application of our results to estimate the asymptotic multiplicity of trapping sets. In Section IV, we discuss the cycle distribution of random lifts of a protograph, and calculate $T(G, c)$ for bi-regular protographs. The results on the expected value and the variance of the number of cycles for QC-LDPC codes are presented in Section V. Section VI is devoted to numerical results. The paper is concluded with some remarks in Section VII.

II. DEFINITIONS AND NOTATIONS

An undirected graph $G = (V, E)$ is defined as a set of vertices or nodes V and a set of edges E , where E is a subset of the pairs $\{\{u, v\} : u, v \in V, u \neq v\}$. In this work, we consider graphs with no loop or parallel edges. A graph is called *complete* if every node is connected to all the other nodes. We use the notation K_a for a complete graph with a nodes. A *walk* of length k in the graph G is a sequence of nodes v_1, v_2, \dots, v_{k+1} in V such that $\{v_i, v_{i+1}\} \in E$, for all $i \in \{1, \dots, k\}$. Equivalently, a walk of length k can be described by the corresponding sequence of k edges. A walk is a *path* if all the nodes v_1, v_2, \dots, v_k are distinct. A walk is called a *closed walk* if the two end nodes are identical, i.e., if $v_1 = v_{k+1}$. Under the same condition, a path is called a *cycle*. We call a cycle *chordless* if no two nodes of the cycle are connected by an edge that does not itself belong to the cycle. Otherwise, such an edge is called a *chord* of the cycle. We denote cycles of length k , also referred to as k -cycles, by C_k . We use N_k for $|C_k|$. The length of the shortest cycle in a graph is called *girth*.

Consider a walk \mathcal{W} of length k represented by the sequence of edges $e_{i_1}, e_{i_2}, \dots, e_{i_k}$. The walk \mathcal{W} is *backtrackless*, if $e_{i_s} \neq e_{i_{s+1}}$, for any $s \in \{1, \dots, k-1\}$. Also, the walk \mathcal{W} is *tailless*, if $e_{i_1} \neq e_{i_k}$. In this paper, we use the term *TBC walk* to refer to a tailless backtrackless closed walk.

The *adjacency matrix* of a graph G is the matrix $A = [a_{ij}]$, where a_{ij} is the number of edges connecting the node i to the node j for all $i, j \in V$. Matrix A is symmetric and since we have assumed that G has no parallel edges or loops, $a_{ij} \in \{0, 1\}$ for all $i, j \in V$, and $a_{ii} = 0$ for all $i \in V$. One important property of the adjacency matrix that we will use for our results is that the number of walks between any two nodes of the graph can be determined using the powers of this matrix. More precisely, the entry in the i^{th} row and the j^{th} column of A^k , $[A^k]_{ij}$, is the number of walks of length k between nodes i and j . In particular, $[A^k]_{ii}$ is the number of closed walks of length k containing node i .

A graph $G = (V, E)$ is called *bipartite*, if the node set V can be partitioned into two disjoint subsets U and W , i.e., $V = U \cup W$ and $U \cap W = \emptyset$, such that every edge in E connects a node from U to a node from W . Tanner graphs of LDPC codes are bipartite graphs, in which U and W are referred to as *variable nodes* and *check nodes*, respectively. Parameters n and m in this case are used to denote $|U|$ and $|W|$, respectively. Parameter n is the code's block length and the code rate R satisfies $R \geq 1 - (m/n)$.

The number of edges connected to a node v is called the *degree* of the node v , and is denoted by d_v (or $\deg(v)$). We call a bipartite graph $G = (U \cup W, E)$ *bi-regular*, if all the nodes on the same side of the given bipartition have the same degree, i.e., if all the nodes in U have the same degree d_u and all the nodes in W have the same degree d_w . Note that, for a bi-regular graph, $|U|d_u = |W|d_w = |E|$. A bipartite graph that is not bi-regular is called *irregular*. We call a bipartite graph *fully-connected* or *complete*, if it is bi-regular and if $d_u = |W|$ and $d_w = |U|$.

Let $G(V = U \cup W, E)$ be a bipartite graph with $|U| = n'$ and $|W| = m'$, and consider an assignment of a permutation $\pi^e \in S_N$ to each edge e in E , where S_N is the symmetric group over $\mathbb{Z}_N = \{0, 1, 2, \dots, N-1\}$. Consider the following construction of the graph $\tilde{G}(\tilde{V}, \tilde{E})$ from $G(V, E)$: We make N copies of G such that for each node $v \in V$, we have a set of nodes $\tilde{v} = \{v^0, \dots, v^{N-1}\}$ in \tilde{V} . Similarly, for each edge $e = \{u, w\} \in E$, we have a set of edges $\tilde{e} = \{e^0, \dots, e^{N-1}\}$ in \tilde{E} such that $\{u^i, w^j\}$ belongs to \tilde{E} if and only if $\pi^e(i) = j$. In this construction, graph \tilde{G} is called an *N-lifting* of G . Graph G is called the *base graph* or *protograph*, and the parameter N is referred to as the *lifting degree*. The lifted graph \tilde{G} can be considered as the Tanner graph of an LDPC code \tilde{C} , i.e., the parity-check matrix \tilde{H} of \tilde{C} is defined to be the incidence matrix of \tilde{G} . The code \tilde{C} , in this case, is called the *lifted code*, and the incidence matrix H of G is called the *base matrix*. The $m'N \times n'N$ parity-check matrix \tilde{H} of \tilde{C} consists of $m' \times n'$ submatrices $[\tilde{H}]_{ij}$, $0 \leq i \leq m' - 1$, $0 \leq j \leq n' - 1$, where each submatrix is a permutation matrix of size $N \times N$, if the entry $[H]_{ij} \neq 0$; otherwise, $[\tilde{H}]_{ij}$ is the all-zero matrix. The LDPC codes constructed by the lifting process, just explained, are referred to as *protograph-based LDPC codes*. In the lifting process, if the permutations are selected randomly from S_N , the constructed codes are called *random lifts*.

Consider the subgroup C_N of symmetric group S_N over \mathbb{Z}_N , where C_N contains all circulant permutations π_p . The index p of the permutation π_p corresponds to p cyclic shifts to the left. If the permutations in the lifting process are cyclic, i.e., if they are selected from C_N , then the resulting graph \tilde{G} is called a *cyclic lift* of G , and the associated code is quasi-cyclic (QC). In this case, the non-zero submatrices of \tilde{H} are circulant permutation matrices (CPM). In particular, when the entry $[H]_{ij} \neq 0$, then $[\tilde{H}]_{ij} = I^{p_{ij}}$, $p_{ij} \in \mathbb{Z}_N$, where $I^{p_{ij}}$ is a CPM whose rows are obtained by cyclically shifting the rows of the identity matrix to the left by p_{ij} . We also take $I^{+\infty}$ to represent the all-zero matrix. We refer to the $m' \times n'$ matrix $P = [p_{ij}]; 0 \leq i \leq m' - 1, 0 \leq j \leq n' - 1$, as the *permutation shift matrix* or the *exponent matrix* corresponding to the lifted code \tilde{C} or to the lifted graph \tilde{G} . Clearly, there is a one-to-one correspondence between P and \tilde{H} .

Consider a QC-LDPC code \tilde{C} corresponding to an exponent matrix P . It is well-known that a necessary condition for the existence of a cycle of length $2k$ in the Tanner graph of \tilde{C} , corresponding to \tilde{H} , is

$$\sum_{i=0}^{k-1} (p_{m_i, n_i} - p_{m_i, n_{i+1}}) = 0 \pmod{N}, \quad (3)$$

where $n_k = n_0$, $m_i \neq m_{i+1}$, $n_i \neq n_{i+1}$, and none of the permutation shifts in (3) is $+\infty$ [17]. The sequence of permutation shifts in (3) corresponds to a TBC walk in the base graph, i.e., cycles of the lifted graph are the inverse images of TBC walks with zero permutation shift in the base graph [28], [22]. In fact, an additional requirement for the sequence of permutation shifts in (3) to correspond to a cycle in the lifted graph is that no subsequence of the permutation shifts should correspond to a TBC walk of permutation shift zero in the base graph [28], [22]. For a TBC walk w in G , we refer to the summation in (3) as the permutation shift corresponding to w and denote it by $\mathcal{P}(w)$. It is clear that depending on the starting index n_0 of w , or the direction of travel along w , the sign of $\mathcal{P}(w)$ may change. As we are only concerned about the value of $\mathcal{P}(w)$ being zero or non-zero, in the context of this work, the two values $\pm\mathcal{P}(w)$ are considered equivalent.

It is well-known that there are cycles in cyclic lifts of a base graph that are independent of the lifting degree N or the choice of the exponent matrix P [17], [18]. Such cycles, referred to as *inevitable cycles*, occur if there exists a TBC walk w in the base graph, in which, each edge is traversed in both directions equal number of times. In this case, $\mathcal{P}(w) = 0 \pmod{N}$, regardless of the value of N , or the choice of P . Such TBC walks are referred to as *zero-permutation (ZP) TBC walks* in this paper. We also use the terminology *prime ZP TBC walk* for a ZP TBC walk that does not contain any ZP TBC subwalk. In fact, inevitable cycles in the lifted graph are the inverse images of prime ZP TBC walks in the base graph. Clearly, inevitable cycles (prime ZP TBC walks) only depend on the structure of the base graph.

III. RANDOM IRREGULAR AND BI-REGULAR GRAPHS

A. Main Result

In the following, we prove our result on the cycle distribution of random irregular bipartite graphs with arbitrary degree distributions.

Theorem 1. *Let Δ_u , Δ_w , δ_u and δ_w be fixed natural numbers satisfying $\Delta_u = d_1 \geq d_2 \geq \dots \geq d_n = \delta_u > 1$, and $\Delta_w = d'_1 \geq d'_2 \geq \dots \geq d'_m = \delta_w > 1$, where $\sum_{i=1}^n d_i = \sum_{i=1}^m d'_i = \eta$.*

Consider the probability space \mathcal{G} of all bipartite graphs with node set (U, W) , where $U = \{u_1, u_2, \dots, u_n\}$, $W = \{w_1, w_2, \dots, w_m\}$, and in which the degree of node u_i is d_i and the degree of node w_i is d'_i . Suppose that the graphs in \mathcal{G} are selected uniformly at random. For $G \in \mathcal{G}$, denote by $N_i(G)$ the number of cycles of length i in G . Then, as $n, m \rightarrow \infty$, for any fixed even value of $k \geq 4$, the random variables N_4, N_6, \dots, N_k , are asymptotically independent Poisson random variables with N_c having the expected value

$$E(N_c) \approx \frac{\left(\left(\frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2}\right)\left(\frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2}\right)\right)^{c/2}}{c},$$

where the approximation is an asymptotic upper bound within the fixed multiplicative factor of $[S(h_u) \times S(h_w)]^{-c/2}$ from the exact value, with Specht's ratio $S(h)$ defined by $S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h}$ for $h \neq 1$, and $S(1) = 1$ (e is Euler's constant), and $h_u = \frac{\Delta_u(\Delta_u-1)}{\delta_u(\delta_u-1)}$, $h_w = \frac{\Delta_w(\Delta_w-1)}{\delta_w(\delta_w-1)}$.

To prove the result of Theorem 1, we need a series of intermediate results as discussed below.

We first construct the ensemble \mathcal{G} of random bipartite graphs, indicated in Theorem 1, in two steps. In the first step, for each node z , we consider a bin that contains $\deg(z)$ cells. We then consider random perfect matchings to pair the cells on the U side of the graph to the cells on the W side. The set of all such matchings is denoted by Φ , and we have $|\Phi| = \eta!$, where η is the number of edges in the graph. Corresponding to each matching, there is a so-called *configuration*, in which the matched cells on the two sides of the graph are connected by an edge. In the rest of the paper, we assume that configurations are selected uniformly at random. Corresponding to each matching (configuration), we construct a bipartite graph such that if there is an edge between two cells, then we place an edge between the corresponding nodes (bins) in the bipartite graph. The bipartite graphs are thus represented as images of the configurations. We denote the ensemble of bipartite graphs so constructed by \mathcal{G}^* . We note that \mathcal{G}^* contains bipartite graphs with parallel edges,¹ and that a uniform distribution over the configurations induces a non-uniform distribution over the ensemble \mathcal{G}^* . The second step in the construction of \mathcal{G} is to remove all the bipartite graphs with parallel edges from \mathcal{G}^* . It is now straightforward to see that, with the condition that the bipartite graphs constructed from random configurations have

¹In the rest of the paper, due to the possibility of parallel edges existing in the bipartite graphs in \mathcal{G}^* , we use the term *multigraph* to refer to such graphs. A multigraph is called *simple* if it has no parallel edges.

no parallel edges, the distribution of bipartite graphs (those in \mathcal{G}) is uniform. This is because corresponding to each graph in \mathcal{G} , we have the same number $d_1! \times \cdots \times d_n! \times d'_1! \times \cdots \times d'_m!$ of configurations.

We now prove a result on the cycle distribution of \mathcal{G}^* (Theorem 3) as an intermediate step to prove Theorem 1. To prove the result of Theorem 3, we first recall the joint version of Poisson approximation theorem as follows (see, e.g., [29], p. 145).

Theorem 2. (*Joint version of Poisson approximation theorem*) For each $i \in \{1, 2, \dots, m\}$, consider the sequence of random variables $X_{i,1}, X_{i,2}, \dots$, each taking values in $\mathbb{N} \cup \{0\}$. Suppose there exist $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}_{\geq 0}$, such that for any fixed $r_1, r_2, \dots, r_m \in \mathbb{N} \cup \{0\}$, we have

$$E[(X_{1,n})_{r_1} (X_{2,n})_{r_2} \cdots (X_{m,n})_{r_m}] \rightarrow \prod_{i=1}^m \lambda_i^{r_i} \quad \text{as } n \rightarrow \infty,$$

where $(X)_r = X(X-1)\cdots(X-r+1)$, for $r \in \mathbb{N}$, and $(X)_0 = 1$. Then as $n \rightarrow \infty$, the random vector $(X_{1,n}, X_{2,n}, \dots, X_{m,n})$ converges to (Y_1, Y_2, \dots, Y_m) in distribution, where the random variables Y_i are independent Poisson random variables with $E[Y_i] = \lambda_i$ (i.e., for each i , Y_i is $\text{Poisson}(\lambda_i)$).

Theorem 3. Let $\Delta_u, \Delta_w, \delta_u$ and δ_w be fixed natural numbers satisfying $\Delta_u = d_1 \geq d_2 \geq \dots \geq d_n = \delta_u > 1$, and $\Delta_w = d'_1 \geq d'_2 \geq \dots \geq d'_m = \delta_w > 1$, where $\sum_{i=1}^n d_i = \sum_{i=1}^m d'_i = \eta$. Consider the probability space \mathcal{G}^* of all bipartite multigraphs with node set (U, W) , where $U = \{u_1, u_2, \dots, u_n\}$, $W = \{w_1, w_2, \dots, w_m\}$, and in which the degree of node u_i is d_i and the degree of node w_i is d'_i . (The probability distribution of graphs in \mathcal{G}^* is assumed to be induced by the uniform distribution over configurations.) For $G \in \mathcal{G}^*$, denote by $N_i(G)$ the number of cycles of length i in G . Then, as $n, m \rightarrow \infty$, for any fixed even value of $k \geq 2$, the random variables N_2, N_4, \dots, N_k , are asymptotically independent Poisson random variables with N_c having the expected value

$$E(N_c) \approx \frac{\left(\left(\frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2} \right) \left(\frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2} \right) \right)^{c/2}}{c},$$

where the approximation is an asymptotic upper bound within a fixed multiplicative factor from the exact value as described in Theorem 1.

Proof. We start by computing the expectation of N_c , also denoted by λ_c in the course of the proof. We then apply Theorem 2 to prove that cycle multiplicities are independent Poisson

random variables.

Calculation of $E(N_c)$. To simplify the calculation of $E(N_c)$, rather than working in the non-uniform probability space of \mathcal{G}^* , we perform the calculations in the space of configurations (with uniform distribution). For a configuration, we define a cycle of length k to be a set of k edges, like $\{e_1, e_2, \dots, e_k\}$, that connect k distinct bins, like D_{i_1}, \dots, D_{i_k} . The connections are such that for each $j \in \{1, \dots, k\}$, the edge e_j , connects a cell in bin D_{i_j} to a cell in bin $D_{i_{j+1}}$, where $D_{i_{k+1}} = D_{i_1}$, and the two cells in each bin D_{i_j} , connected to the two edges e_j and e_{j-1} , are distinct ($e_0 = e_k$). We now compute the number of k -cycles, \mathcal{C}_k , in a configuration. To form a k -cycle, one needs to choose $k/2$ bins from U and $k/2$ bins from W . Next, from each bin, one needs to choose two cells (the order of the two cells is important). Suppose that bin i contains d_i cells. We thus have $(d_i)(d_i - 1)$ choices for the two cells of bin i . Hence, in order to choose all the cells on both sides of the graph, we have

$$\left(\sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \quad (4)$$

choices. To count the number of k -cycles in a configuration, we also need to consider different orderings of the $k/2$ bins on each side of the graph. This results in

$$\mathcal{C}_k = \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left(\frac{(\frac{k}{2})!(\frac{k}{2})!}{k} \right), \quad (5)$$

where the division by k is for counting each cycle in the above process k times.

We note that given a set of ℓ fixed edges, there are $(\eta - \ell)!$ configurations containing those edges. We then have

$$\begin{aligned} E(N_c) &= \frac{\mathcal{C}_c \times (\eta - c)!}{\eta!} \\ &= \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=c/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left(\frac{(\frac{c}{2})!(\frac{c}{2})!}{c} \right) \times \frac{(\eta - c)!}{\eta!} \\ &\sim \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=c/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left(\frac{(\frac{c}{2})!(\frac{c}{2})!}{c} \right) \times \frac{1}{\eta^c}. \end{aligned} \quad (6)$$

In the following, we derive asymptotic upper and lower bounds on (6) that differ only in a constant multiplicative factor. For this, we first use Maclaurin's inequality (see [30], pp 117-119), as described below. Let a_1, a_2, \dots, a_n be positive real numbers, and for $k = 1, 2, \dots, n$, define the averages S_k as follows:

$$S_k = \frac{\sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k}}{\binom{n}{k}},$$

where the summation is over all distinct sets of k indices. Maclaurin's inequality then states:

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \dots \geq \sqrt[n]{S_n}.$$

Using Maclaurin's inequality, we thus have:

$$S_1 \geq \sqrt[c/2]{S_{\frac{c}{2}}} \geq \sqrt[n]{S_n},$$

or equivalently,

$$\sqrt[n]{\prod_{j=1}^n a_j} \leq \sqrt[c/2]{S_{\frac{c}{2}}} \leq \frac{\sum_{j=1}^n a_j}{n}. \quad (7)$$

On the other hand, we have [31]:

$$\frac{1}{S(h)} \times \frac{\sum_{j=1}^n a_j}{n} \leq \sqrt[n]{\prod_{j=1}^n a_j}, \quad (8)$$

where $h = \frac{M}{m} (\geq 1)$ with M and m equal to the maximum and minimum values of numbers a_1, a_2, \dots, a_n , and Specht's ratio $S(h)$ is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \text{ for } h \neq 1, \text{ and } S(1) = 1, \quad (9)$$

in which e is Euler's number.

Combining (7) and (8), we have

$$S(h)^{-c/2} \binom{n}{c/2} \left(\frac{\sum_{j=1}^n a_j}{n} \right)^{c/2} \leq \sum_{1 \leq i_1 < \dots < i_{c/2} \leq n} a_{i_1} a_{i_2} \cdots a_{i_{c/2}} \leq \binom{n}{c/2} \left(\frac{\sum_{j=1}^n a_j}{n} \right)^{c/2}. \quad (10)$$

Now, let $a_{i_k} = d_k(d_k - 1)$. Focusing on the upper bound in (10), we then have

$$\sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \leq \binom{n}{c/2} \left(\frac{\sum_{u_i \in U} (d_i)(d_i - 1)}{n} \right)^{c/2} \quad (11)$$

$$\begin{aligned} &\sim \frac{n^{c/2}}{(c/2)!} \left(\frac{\sum_{u_i \in U} (d_i)(d_i - 1)}{n} \right)^{c/2} \\ &= \frac{1}{(c/2)!} \left(2 \sum_{u_i \in U} \binom{d_i}{2} \right)^{c/2}. \end{aligned} \quad (12)$$

Similarly, we can establish the following asymptotic lower bound:

$$\frac{S(h_u)^{-c/2}}{(c/2)!} \left(2 \sum_{u_i \in U} \binom{d_i}{2} \right)^{c/2} \leq \sum_{\substack{\sigma \subset U \\ |\sigma|=c/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1). \quad (13)$$

Now, by applying (12) and (13) to (6) for both sides of the graph, we obtain the asymptotic upper and lower bounds on $E(N_c)$. In particular, the asymptotic value of the upper bound, used as the approximate value of $E(N_c)$, is calculated as follows:

$$\begin{aligned} E(N_c) &\approx \frac{1}{(c/2)!} \left(2 \sum_{u_i \in U} \binom{d_i}{2} \right)^{c/2} \frac{1}{(c/2)!} \left(2 \sum_{w_i \in W} \binom{d'_i}{2} \right)^{c/2} \left(\frac{(\frac{c}{2})! (\frac{c}{2})!}{c} \right) \times \frac{1}{\eta^c} \\ &= \frac{\left(\left(\frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2} \right) \left(\frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2} \right) \right)^{c/2}}{c}. \end{aligned} \quad (14)$$

The asymptotic lower bound on $E(N_c)$ is equal to the upper bound of (14) multiplied by $[S(h_u) \times S(h_w)]^{-c/2}$, proving the claim about the accuracy of the approximation. This completes the calculation of $E(N_c)$.

To continue the proof of Theorem 3, we need the following lemma, whose proof is provided in Appendix I.

Lemma 1. *Consider the ensemble of multigraphs \mathcal{G}^* in Theorem 3, and a fixed multigraph H with more edges than nodes. Then, the expected number of copies of H in a multigraph in \mathcal{G}^* is $\mathcal{O}(\frac{1}{n})$.²*

We now proceed with the calculation of joint factorial moments $E[(N_2)_{r_2} (N_4)_{r_4} \cdots (N_k)_{r_k}]$, where r_{2i} , $i = 1, \dots, k/2$, are arbitrary non-negative integers, constant with respect to n .

Calculation of joint factorial moments. We begin with calculating $E[(N_c)_2]$, and show that $E[(N_c)_2] \sim \lambda_c^2$. This will then be generalized to the asymptotic expression for the joint factorial moments.

We note that $(N_c)_2$ is the number of ordered pairs of two distinct c -cycles in \mathcal{G}^* . The two c -cycles may or may not intersect. We thus write $(N_c)_2 = N' + N''$, where N' is the number of ordered pairs of node disjoint c -cycles, and N'' is the number of ordered pairs of distinct

²The notation $f(x) = \mathcal{O}(g(x))$ is used, if for sufficiently large values of x , we have $|f(x)| \leq a|g(x)|$, for some positive value a .

c -cycles that have at least one node in common. Based on Lemma 1, we have $E(N'') = \mathcal{O}(\frac{1}{n})$. In the following, we prove $E[N'] \sim \lambda_c^2$.

We first count the number \mathcal{C}_{kk} of ordered pairs of node-disjoint k -cycles in a configuration. Similar to the derivation of (5), we have

$$\mathcal{C}_{kk} = \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=k}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=k}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left(\frac{k!k!}{k^2} \right). \quad (15)$$

In order to simplify (15), we use the following lemma, whose proof is provided in Appendix I.

Lemma 2. *Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of variables such that for each i , $0 < s \leq x_i \leq t$, where both s and t are constant numbers. Also, let k be a constant number and let n tend to infinity. We then have*

$$\left(\sum_{\substack{\sigma \subset X \\ |\sigma|=k}} \prod_{x_i \in \sigma} x_i \right) \binom{k}{k/2} \sim \left(\sum_{\substack{\sigma \subset X \\ |\sigma|=k/2}} \prod_{x_i \in \sigma} x_i \right)^2. \quad (16)$$

By the application of Lemma 2 to (15), we obtain the following:

$$\mathcal{C}_{kk} \sim \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right)^2 \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right)^2 \left(\frac{(k/2)!(k/2)!}{k} \right)^2. \quad (17)$$

This together with (5) result in

$$\mathcal{C}_{kk} \sim \mathcal{C}_k^2. \quad (18)$$

We thus have

$$\begin{aligned} E[N'] &= \frac{\mathcal{C}_{cc} \times (\eta - 2c)!}{\eta!} \\ &\sim \mathcal{C}_c^2 \times \frac{1}{\eta^{2c}} \\ &\sim \lambda_c^2, \end{aligned}$$

where in the last step, we have used (6).

We note that the joint factorial moment under consideration is the expected value of the product of the number of ordered r_i distinct i -cycles, for even values $i = 2, \dots, k$. This can be interpreted as the expected number of sequences of $\beta = r_2 + r_4 + \dots + r_k$ distinct cycles such

that the first r_2 have length 2, the next r_4 have length 4, and so on. In the following, we call such sequences β -sequences. Similar to the approach for two cycles, the number of β -sequences can be written as $N' + N''$, where N' counts the sequences of node-disjoint cycles and N'' counts the sequences of distinct cycles such that in each sequence there are at least two cycles that share at least one node. For N'' , by Lemma 1, we have $E(N'') = \mathcal{O}(\frac{1}{n})$.

Now, we study N' . We use the notation \mathcal{C}' to denote the number of possible β -sequences with node-disjoint cycles in a configuration. Similar to (15), we have

$$\mathcal{C}' = \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=\alpha}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right) \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=\alpha}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right) \left(\frac{\alpha! \alpha!}{2^{r_2} 4^{r_4} \dots k^{r_k}} \right), \quad (19)$$

where $\alpha = r_2 + 2r_4 + \dots + (k/2)r_k$.

Lemma 2 can be extended to the following asymptotic equality:

$$\left(\sum_{\substack{\sigma \subset X \\ |\sigma|=\alpha}} \prod_{x_i \in \sigma} x_i \right) \left(\frac{\alpha}{2}, \dots, \frac{\alpha}{2}, \dots, \frac{\alpha}{2} \right) \sim \left(\sum_{\substack{\sigma \subset X \\ |\sigma|=2/2}} \prod_{x_i \in \sigma} x_i \right)^{r_2} \left(\sum_{\substack{\sigma \subset X \\ |\sigma|=4/2}} \prod_{x_i \in \sigma} x_i \right)^{r_4} \dots \left(\sum_{\substack{\sigma \subset X \\ |\sigma|=k/2}} \prod_{x_i \in \sigma} x_i \right)^{r_k} \quad (20)$$

Using (20) in (19), we obtain

$$\begin{aligned} \mathcal{C}' &\sim \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=2/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right)^{r_2} \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=4/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right)^{r_4} \dots \left(\sum_{\substack{\sigma \subset U \\ |\sigma|=k/2}} \prod_{u_i \in \sigma} (d_i)(d_i - 1) \right)^{r_k} \\ &\times \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=2/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right)^{r_2} \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=4/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right)^{r_4} \dots \left(\sum_{\substack{\sigma \subset W \\ |\sigma|=k/2}} \prod_{w_i \in \sigma} (d'_i)(d'_i - 1) \right)^{r_k} \\ &\times \frac{\left(((2/2)!)^{r_2} ((4/2)!)^{r_4} \dots ((k/2)!)^{r_k} \right)^2}{2^{r_2} 4^{r_4} \dots k^{r_k}}. \end{aligned}$$

By the application of (5) and (6) to the expected value of the above equation, we then have $E[N'] \sim \lambda_2^{r_2} \dots \lambda_k^{r_k}$. Hence,

$$E[(N_2)_{r_2} (N_4)_{r_4} \dots (N_k)_{r_k}] \rightarrow \prod_{i=1}^{k/2} \lambda_{2i}^{r_{2i}} \quad \text{as } n \rightarrow \infty.$$

Thus, by Theorem 2, for any fixed even value of $k \geq 2$, the random variables N_2, \dots, N_k , are asymptotically independent Poisson random variables with N_c having the expected value

$$E(N_c) \approx \frac{\left(\left(\frac{2}{\eta} \sum_{i=1}^n \binom{d_i}{2} \right) \left(\frac{2}{\eta} \sum_{i=1}^m \binom{d'_i}{2} \right) \right)^{c/2}}{c}.$$

■

Proof of Theorem 1. We note that multigraphs in \mathcal{G}^* are simple if and only if $N_2 = 0$, and also that \mathcal{G}^* conditioned on $N_2 = 0$ yields \mathcal{G} . Let \mathcal{S} denote the event that the multigraphs in \mathcal{G}^* are simple. By Theorem 3, we have $\Pr(\mathcal{S}) \sim e^{-\lambda_2}$, and thus, $\Pr(\mathcal{S}) > 0$. We now show that any property P that holds true asymptotically almost surely (a.a.s.) for \mathcal{G}^* (including that of Theorem 3 on cycle distributions), also holds true a.a.s. for \mathcal{G} . Let $\bar{\mathcal{P}}^*$ and $\bar{\mathcal{P}}$ denote the events that multigraphs in \mathcal{G}^* and bipartite graphs in \mathcal{G} have Property P , respectively. We then have

$$\Pr(\bar{\mathcal{P}}) = \Pr(\bar{\mathcal{P}}^* | \mathcal{S}) = \frac{\Pr(\bar{\mathcal{P}}^* \cap \mathcal{S})}{\Pr(\mathcal{S})} \leq \frac{\Pr(\bar{\mathcal{P}}^*)}{\Pr(\mathcal{S})} \rightarrow 0, \quad (21)$$

where the last part follows from the fact that Property P holds true a.a.s. on \mathcal{G}^* . ■

Remark 1. For bi-regular graphs, where $\Delta_u = \delta_u$ and $\Delta_w = \delta_w$, we have $S(h_u) = S(h_w) = 1$, and thus the asymptotic upper and lower bounds on $E(N_c)$, derived in Theorem 1, coincide. In this case, the asymptotic approximation provided for $E(N_c)$ in Theorem 1 turns into an asymptotic equality.

For irregular graphs, we have $h_u > 1$ or $h_w > 1$, and thus $[S(h_u) \times S(h_w)]^{-c/2} < 1$. Specht's ratio $S(h)$ is a monotone increasing function on $(1, \infty)$, and thus the asymptotic lower bound on $E(N_c)$ decreases monotonically with increase in h_u or h_w . The numerical results in Section VI, however, show that the asymptotic upper bound presented in Theorem 1 is often much tighter than the asymptotic lower bound presented in this theorem.

Corollary 1. Let $G = (U \cup W, E)$ be a random bi-regular graph in which all the nodes in U have the same degree d_u and all the nodes in W have the same degree d_w . Consider the ensemble of such graphs as the number of nodes tends to infinity. In this case, for a fixed even value k , random variables $N_4(G), \dots, N_k(G)$, are independent with Poisson distribution, where the expected value of N_c is given by

$$E(N_c) \sim \frac{\left((d_u - 1)(d_w - 1) \right)^{c/2}}{c}. \quad (22)$$

B. An Application

It is well-known that the performance of LDPC codes in the error floor region is determined by certain substructures of the code's Tanner graph, referred to as trapping sets [32]. The error floor performance of an LDPC code is not only a function of the Tanner graph of the code but also depends on the channel model, quantization scheme and the iterative algorithm used for

the decoding. Depending on the scenario, different categories (types) of trapping sets may prove to be relevant. Such categories include elementary trapping sets (ETS), leafless ETSs (LETS), absorbing sets, and stopping sets. For example, while stopping sets are known to be the culprit in belief propagation decoding of LDPC codes over the binary erasure channel (BEC) [33], LETSs are the relevant structures in the context of iterative decoding of LDPC codes over the additive white Gaussian noise (AWGN) channel [9], [27].

Very recently, it was shown in [26] that, regardless of the type of a trapping set structure, its asymptotic average multiplicity in a random ensemble of Tanner graphs depends only on the trapping set's constituent cycles. In particular, a structure with no cycle, with only one cycle, and with more than one cycle has an asymptotic average multiplicity of infinity, a non-zero constant, and zero, respectively. For the non-trivial case where the structure has only a single (chordless) cycle, the asymptotic average multiplicity can be estimated using Theorem 1.

Example 1. *Consider the random ensemble of bi-regular LDPC codes with $d_u = 3$ and $d_w = 6$. The error floor of this ensemble over the AWGN channel is determined by the distribution of LETS structures of the codes. It is proved in [26] that among all (a, b) classes³ of LETS structures for this ensemble, only those with $b = a$ have a non-zero asymptotic multiplicity. Such structures correspond to chordless cycles of length $2a$. We can thus use Corollary 1 and estimate the average number of such structures by $10^a / (2a)$. One should note that while Theorem 1 and Corollary 1 consider both chordless cycles and cycles with chords, the multiplicity of cycles with chords tends to zero asymptotically [26], and thus the results given here provide good estimates for the asymptotic multiplicity of chordless cycles.*

To demonstrate the accuracy of this estimate at finite block lengths, we have randomly constructed five LDPC codes with $d_u = 3$ and $d_w = 6$, and block lengths $n = 816, 1008, 4000, 20000$, and 50000 . All the codes have girth 6. The multiplicity of (a, a) LETS structures of these codes for $a = 3, 4, 5$, are listed in Table I, along with the estimate of Corollary 1. As can be seen from Table I, the estimates for different values of a are rather accurate even for relatively short block lengths.

³A trapping set is often identified by the number of its variable nodes a , and the number of unsatisfied check nodes b in its induced subgraph. Such a trapping set is said to belong to the class of (a, b) trapping sets.

TABLE I

MULTIPLICITIES OF (3, 3), (4, 4), AND (5, 5) LETSS FOR RANDOMLY CONSTRUCTED BI-REGULAR LDPC CODES WITH $d_u = 3$ AND $d_w = 6$, IN COMPARISON WITH THE ASYMPTOTIC EXPECTED VALUES OF COROLLARY 1

(a, a) Class	Block Length					Expected value by Corollary 1
	816	1008	4000	20000	50000	
(3,3)	132	165	171	161	178	166
(4,4)	1491	1252	1219	1260	1268	1250
(5,5)	9169	10019	9935	10046	10231	10000

IV. RANDOM LIFTS OF AN ARBITRARY BIPARTITE BASE GRAPH

In this section, we study the cycle distribution of protograph-based LDPC codes that are random lifts of a base graph with no parallel edges. The following result shows that, similar to random bipartite graphs, for random lifts also, the cycles of different length have independent Poisson distributions.

Theorem 4. [25] *For a random N -lift of a protograph G , as N tends to infinity, the distributions of cycles of different length c tend to independent Poisson distributions with the expected value equal to $T(G, c)$, where $T(G, c)$ is the number of TBC walks of length c in G .*

In the following, we calculate $T(G, c)$ for two special cases of base graphs commonly used in the construction of protograph-based LDPC codes: fully-connected and bi-regular. Although, fully-connected graphs are themselves a special case of bi-regular graphs, in the following, we first consider the case of fully-connected graphs, since for this case, we can in fact, derive an exact expression for $T(G, c)$. For the more general case of bi-regular base graphs, our approximation is in the form of an upper bound.

A. Calculation of $T(G, c)$ for fully-connected base graphs

Theorem 5. *Let $G = (U \cup W)$ be a fully-connected bipartite graph with $|U| = a$ and $|W| = b$. For any even value $c \geq 4$, we have*

$$T(G, c) = \frac{(a-1)(b-1)}{c} \left((-1)^{c/2} + (a-1)^{c/2-1} \right) \left((-1)^{c/2} + (b-1)^{c/2-1} \right).$$

Proof. To calculate $T(G, c)$, we consider the number of TBC walks of length c , $R_{c,e}$, that go through a specific edge e in the base graph G . Due to the symmetry of G , this number is

independent of e . In the rest of the proof, we thus use the notation R_c for this number. Since there are ab edges in G , we have

$$T(G, c) = \frac{ab \times R_c}{c}, \quad (23)$$

where the division by c is because each TBC walk is accounted for c times through its c edges.

To calculate R_c , we note that any TBC walk of length c in the fully-connected based graph can be uniquely described by two interleaving sequences of variable and check nodes, where each sequence corresponds to a closed walk of length $c/2$ in the complete graph K_a and K_b , respectively. Suppose that the number of closed walks of length $c/2$ starting from a specific node in K_a is denoted by $\mathcal{W}_{c/2}^a$. We thus have

$$R_c = \mathcal{W}_{c/2}^a \times \mathcal{W}_{c/2}^b. \quad (24)$$

To obtain \mathcal{W}_k^a , we need to calculate a diagonal element of A_a^k , where A_a is the $a \times a$ adjacency matrix of K_a . It is easy to see that the k -th power of A_a has the following general form

$$A_a^k = \begin{pmatrix} \alpha_k & \beta_k & \cdots & \beta_k \\ \beta_k & \alpha_k & \cdots & \beta_k \\ \vdots & \vdots & \ddots & \vdots \\ \beta_k & \beta_k & \cdots & \alpha_k \end{pmatrix},$$

where

$$\alpha_1 = 0, \quad \alpha_{k+1} = (a-1)\beta_k$$

$$\beta_1 = 1, \quad \beta_{k+1} = \alpha_k + (a-2)\beta_k = (a-2)\beta_k + (a-1)\beta_{k-1}.$$

To solve the recursion $\beta_{k+1} = (a-2)\beta_k + (a-1)\beta_{k-1}$, we solve the corresponding quadratic equation $x^2 - (a-2)x - (a-1) = 0$. The roots of this equation are -1 and $a-1$. Thus, $\beta_k = \gamma(-1)^k + \gamma'(a-1)^k$. Using $\beta_1 = 1$ and $\beta_2 = a-2$, we obtain $\gamma' = -\gamma = \frac{1}{a}$. Hence, $\beta_k = \frac{-1}{a}(-1)^k + \frac{1}{a}(a-1)^k$. We thus have

$$\mathcal{W}_k^a = \alpha_k = (a-1)\beta_{k-1} = \frac{a-1}{a}(-1)^k + \frac{1}{a}(a-1)^k. \quad (25)$$

Combining (25) with (24) and (23) completes the proof. ■

Corollary 2. *Let $G = (U \cup W)$ be a fully-connected bipartite graph with $|U| = a$ and $|W| = b$. For any even value $c \geq 4$, we have*

$$T(G, c) \approx \frac{\left((a-1)(b-1)\right)^{c/2}}{c}.$$

Remark 2. *Combination of Theorem 4 and Corollary 2, and the comparison with the result of Corollary 1 show that, in the asymptotic regime, where the size of the graph tends to infinity, the cycle distributions of random lifts of a fully-connected base graph are identical to those of random bi-regular graphs with the same variable and check node degrees.*

B. Calculation of $T(G, c)$ for general bi-regular graphs

In this part, we consider the graphs that are bi-regular but not necessarily fully-connected.

Theorem 6. *Let $G = (U \cup W)$ be a bi-regular graph. Then, for any even value $c \geq 4$, we have*

$$T(G, c) \leq \frac{|U|d_u}{c} \left((d_u - 1)(d_w - 1)\right)^{c/2-1}.$$

Proof. By counting the TBC walks in G from the viewpoint of the edges, we have

$$T(G, c) \leq \frac{|U|d_u}{c} K_c, \tag{26}$$

where K_c is defined as the maximum number of TBC walks of length c to go through a specific edge in G (the maximum is taken over all the edges in G). Now, for a given edge e in G , consider a potential TBC walk in G that starts from e . There are $[(d_u - 1)(d_w - 1)]^{c/2-1}$ possibilities for selecting the following $c - 2$ edges of such a potential TBC walk. For the last edge of the TBC walk, there would be only one choice e' that can connect the end node of the last edge to the beginning node of e . This is if such an edge $e' \neq e$ exists. We thus have $K_c \leq [(d_u - 1)(d_w - 1)]^{c/2-1}$. This together with (26) completes the proof. ■

Remark 3. *Note that for a fully-connected base graph, the upper bound of Theorem 6 is approximately equal to the value given in Corollary 2.*

V. RANDOM CYCLIC LIFTS OF AN ARBITRARY BIPARTITE BASE GRAPH

In this section, we focus on random cyclic liftings of degree N of a given bipartite base graph G . The randomness is with respect to the exponent matrix P , where each non-infinity element of P is selected in an independent and identically distributed (i.i.d.) fashion from a uniform

distribution over \mathbb{Z}_N . In the following, we first derive upper and lower bounds on the expected value of the number of c -cycles, followed by an upper bound on the variance.

A. Calculation of $E(N_c)$

We use the notation $\mathcal{T}(G, c)$ to denote the set of all TBC walks of length c in a base graph G . This set has size $T(G, c)$. To derive our results, we need to partition $\mathcal{T}(G, c)$ into three subsets $\mathcal{T}_1(G, c)$, $\mathcal{T}_2(G, c)$, and $\mathcal{T}_3(G, c)$. The partition $\mathcal{T}_1(G, c)$ is the set of all prime ZP TBC walks of length c in G , while $\mathcal{T}_2(G, c)$ consists of all TBC walks w of length c in G such that w contains at least a ZP TBC subwalk. The partition $\mathcal{T}_3(G, c)$ covers the rest of the TBC walks of length c in G , i.e., $\mathcal{T}_3(G, c) = \mathcal{T}(G, c) \setminus (\mathcal{T}_1(G, c) \cup \mathcal{T}_2(G, c))$. In the following, for simplicity of notations, we use \mathcal{T} for $\mathcal{T}(G, c)$, and \mathcal{T}_i for $\mathcal{T}_i(G, c)$.

Consider an edge e involved in a TBC walk w in \mathcal{T} . Assume that e is traversed i times in one direction and j times in the opposite direction. The contribution of e in $\mathcal{P}(w)$ is thus $(i - j)p_e$, where p_e is the permutation shift of e . In this case, we say edge e is of *multiplicity* $|i - j|$ in w . We now organize the contribution of different edges of w in $\mathcal{P}(w)$ in accordance with their multiplicity, as follows:

$$\mathcal{P}(w) = \sum_{e \in E_1} p_e + 2 \times \sum_{e \in E_2} p_e + \cdots + k \times \sum_{e \in E_k} p_e, \quad (27)$$

where E_i is the set of edges of multiplicity i , and k is the largest multiplicity of edges in w . In (27), with a slight abuse of notation, we have used p_e to denote either p_e or $-p_e$ depending on the sign of $i - j$. In relation to (27), we say TBC walk w is of *degree* k . Assuming that ℓ summations (out of k) in (27) are non-zero, we refer to w as a TBC walk of *weight* ℓ . Clearly, ZP TBC walks have both degree zero and weight zero.

Lemma 3. *Consider a random cyclic N -lift of a base bipartite graph G with no parallel edges, and consider a TBC walk w of length c and weight $\ell \geq 1$ in G . We then have*

$$\frac{1}{N^\ell} \leq \Pr(\mathcal{P}(w) = 0) \leq \frac{c}{4N}. \quad (28)$$

Proof. We first note that the degree k of a TBC walk of length c is at most $c/4$. This can be easily seen by noting that passing through an edge e , k times, requires passing through k closed

walks, each containing e . Since graph G is assumed to have no parallel edges and is bipartite, the length of each such closed walk is at least 4.

For each non-empty set E_i , the corresponding summation in (27), denoted by X_i , takes one of the N values in \mathbb{Z}_N with equal probability. Also, different summations in (27) are independent, since they share no permutation shifts. The relationship (27) is then a linear integer combination of i.i.d. random variables X_i 's, and we are interested in evaluating the probability that this linear combination is equal to zero modulo N . Considering that the weight of w is ℓ , we are thus interested in the probability that the following equation is satisfied:

$$j_1 X_{j_1} + \cdots + j_\ell X_{j_\ell} = 0 \pmod{N}, \quad (29)$$

where $j_i, i = 1, \dots, \ell$, are the indices corresponding to non-zero random variables. The lower bound of (28) immediately follows by noticing that setting all the random variables equal to zero satisfies (29).

For the upper bound, consider (29), in which all the random variables except X_{j_i} are fixed. The number of solutions to this equation (considering X_{j_i} as the variable) is then at most $\gcd(j_i, N)$, where $\gcd(\cdot, \cdot)$ denotes the greatest common divisor. This implies that the probability of (29) being satisfied is upper bounded by $\gcd(j_i, N)/N$, and thus by $\min\{\gcd(j_1, N)/N, \dots, \gcd(j_\ell, N)/N\}$. Now, the upper bound in (28) follows from $\gcd(j_i, N) \leq j_i \leq k \leq c/4$, for any j_i . ■

Lemma 4. *Consider a random cyclic N -lift of a base bipartite graph G with no parallel edges, and consider a TBC walk w in \mathcal{T}_1 . We then have*

$$\Pr(A_w) \geq 1 - \frac{c^3}{4N}, \quad (30)$$

where A_w is the event that none of the subsequences of permutation shifts for w corresponds to a TBC walk with zero permutation shift.

Proof. Denote by \bar{A}_w , the complement event of A_w . It is easy to see that the number of subwalks of w is upper bounded by c^2 . Each such subwalk, based on Lemma 3, is a TBC walk of permutation zero with probability at most $c/(4N)$. We thus have $\Pr(\bar{A}_w) \leq c^3/(4N)$. This together with $\Pr(A_w) = 1 - \Pr(\bar{A}_w)$, completes the proof. ■

Theorem 7. *Let \tilde{G} be a random cyclic N -lift of a base bipartite graph G with no parallel edges. For any even value $c \geq 4$, we have*

$$\left(N - \frac{c^3}{4}\right) \times T_1 \leq E[N_c(\tilde{G})] \leq N \times T_1 + \frac{c}{4} \times T_3,$$

where T_i is the size of the set \mathcal{T}_i .

Proof. Consider the base graph G , and the ensemble of random cyclic N -lifts \tilde{G} . The number of cycles of length c in \tilde{G} is then given by the following random variable:

$$N_c(\tilde{G}) = N \sum_{w \in \mathcal{T}} I(\{\mathcal{P}(w) = 0\} \cap A_w), \quad (31)$$

where $I(\cdot)$ is the indicator function, and A_w is the event as defined in Lemma 4. By (31), and using the definition of conditional probability, we have

$$E[N_c(\tilde{G})] = N \sum_{w \in \mathcal{T}} \Pr(\{\mathcal{P}(w) = 0\} \cap A_w) = N \sum_{w \in \mathcal{T}} \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0). \quad (32)$$

Consider breaking down the summation over the set \mathcal{T} in (32) to summations over the three partitions of \mathcal{T} , i.e., over the sets \mathcal{T}_1 , \mathcal{T}_2 , and \mathcal{T}_3 . In the following, we evaluate the probability $\Pr(\{\mathcal{P}(w) = 0\} \cap A_w)$, for TBC walks w in the three sets, respectively.

For each TBC walk w in \mathcal{T}_1 , by the definition of \mathcal{T}_1 , we have: $\Pr(\{\mathcal{P}(w) = 0\}) = 1$. Combining this with Lemma 4, we have

$$1 - \frac{c^3}{4N} \leq \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0) \leq 1. \quad (33)$$

For each TBC walk w in \mathcal{T}_2 , by the definition of \mathcal{T}_2 , there is a TBC subwalk w' of w such that $\mathcal{P}(w') = 0$. So $\Pr(A_w) = 0$, and thus

$$\Pr(\{\mathcal{P}(w) = 0\} \cap A_w) = 0. \quad (34)$$

For each TBC walk w in \mathcal{T}_3 , using (28) and $0 \leq \Pr(A_w | \mathcal{P}(w) = 0) \leq 1$, we have

$$0 \leq \Pr(\mathcal{P}(w) = 0) \times \Pr(A_w | \mathcal{P}(w) = 0) \leq \frac{c}{4N}. \quad (35)$$

Replacing (33), (34), and (35) in (32) completes the proof. ■

We note that, for a given base graph, the values T_1 and T_3 are fixed with respect to the lifting degree N . We thus have the following corollary, which demonstrates that the growth of the expected number of c -cycles with N can follow two very different trajectories depending on the value of c and whether the lifted graph has any inevitable cycle of length c or not.

Corollary 3. *Let \tilde{G} be a random cyclic N -lift of a base bipartite graph G . If \tilde{G} contains inevitable cycles of length c (i.e., graph G contains at least one prime ZP TBC walk of length c), then, as N tends to infinity, the expected number of cycles of length c in \tilde{G} will be dominated*

by that of inevitable cycles and grows as $\Theta(N)$.⁴ On the other hand, if \tilde{G} contains no inevitable cycles of length c (i.e., graph G contains no prime ZP TBC walk of length c), then, as N tends to infinity, the expected number of cycles of length c in \tilde{G} is $\Theta(1)$ (is asymptotically constant with respect to N).

Remark 4. It was shown in [19] that cyclic lifts \tilde{G} of a base graph with girth g and no parallel edges have no inevitable cycles of length smaller than $3g$. Thus, based on Corollary 3, for $c < 3g$, the expected number of cycles of length c in \tilde{G} is $\Theta(1)$.

Remark 5. It is important to note the difference between the expected number of c -cycles of random lifts, discussed in Section IV, and that of cyclic lifts, discussed in this section. While for random lifts, the expected value is $\Theta(1)$ with respect to lifting degree N , regardless of the value of c or the base graph, for cyclic lifts, it can be $\Theta(N)$, depending on the value of c and the base graph, as explained in Corollary 3.

B. Calculation of $\text{Var}(N_c)$

In the following, we prove that the variance of the number of cycles of length c in a random cyclic N -lift increases at most linearly with N .

Theorem 8. Let \tilde{G} be a random cyclic N -lift of a base bipartite graph G with no parallel edges. As N tends to infinity, for any fixed even value $c \geq 4$, we have

$$\text{Var}[N_c(\tilde{G})] \leq \left(\frac{c^3}{2}T_1^2 + \frac{c}{4}T_3^2 + \frac{c}{2}T_1T_3\right) \times N + \mathcal{O}(1). \quad (36)$$

Proof. Following the same notations as in Theorem 7, the number of cycles of length c in \tilde{G} is given by the following random variable:

$$N_c(\tilde{G}) = N \sum_{w \in \mathcal{T}} I(\{\mathcal{P}(w) = 0\} \cap A_w).$$

⁴We use the notation $f(x) = \Theta(g(x))$, if for sufficiently large values of x , we have $a \times g(x) \leq f(x) \leq b \times g(x)$, for some positive a and b values.

We have $\text{Var}[N_c(\tilde{G})] = E[N_c^2(\tilde{G})] - E^2[N_c(\tilde{G})]$. In the following, we derive an upper bound on $E[N_c^2(\tilde{G})]$. This together with the lower bound on $E^2[N_c(\tilde{G})]$, derived in Theorem 7, will prove the theorem. We have

$$\begin{aligned} E[N_c^2(\tilde{G})] &= N^2 \sum_{w \in \mathcal{T}} \sum_{w' \in \mathcal{T}} E[I(\mathcal{P}(w) = 0 \cap A_w) I(\mathcal{P}(w') = 0 \cap A_{w'})] \\ &= N^2 \sum_{w \in \mathcal{T}} \sum_{w' \in \mathcal{T}} \Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}). \end{aligned} \quad (37)$$

To obtain an upper bound on $E[N_c^2(\tilde{G})]$, we break each of the two summations in (37) into three, each on one of the three partitions $\mathcal{T}_1, \mathcal{T}_2$, and \mathcal{T}_3 of \mathcal{T} .

Consider the case where $w \in \mathcal{T}_1$ and $w' \in \mathcal{T}_1$. In this case, we simply use the upper bound of one on $\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'})$. This contributes $T_1^2 \times N^2$ to the upper bound on the variance.

Now consider the case where $w \in \mathcal{T}_1$ and $w' \in \mathcal{T}_3$. In this case, we have

$$\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}) \leq \Pr(\mathcal{P}(w') = 0) \leq \frac{c}{4N}, \quad (38)$$

where the last inequality is from (28). Based on (38), the contribution of this scenario plus the case where $w \in \mathcal{T}_3$ and $w' \in \mathcal{T}_1$ in the upper bound is $c/2 \times T_1 \times T_3 \times N$. Similarly, based on (38), the contribution of cases where $w \in \mathcal{T}_3$ and $w' \in \mathcal{T}_3$ is upper bounded by $c/4 \times T_3^2 \times N$.

For all the cases where either w or w' is in \mathcal{T}_2 , we have $\Pr(\mathcal{P}(w) = 0 \cap A_w \cap \mathcal{P}(w') = 0 \cap A_{w'}) = 0$, and thus no contribution to the upper bound.

Adding up all the contributions of different cases, as discussed above, we obtain the following upper bound on $E[N_c^2(\tilde{G})]$:

$$E[N_c^2(\tilde{G})] \leq T_1^2 \times N^2 + \left(\frac{c}{4}T_3^2 + \frac{c}{2}T_1T_3\right) \times N.$$

This combined with the lower bound of Theorem 7 on $E^2[N_c(\tilde{G})]$ complete the proof. \blacksquare

VI. NUMERICAL RESULTS

A. Random regular and irregular bipartite graphs

In [11], the authors generated random codes from different bi-regular ensembles of LDPC codes, and empirically studied the distribution of cycles of different length in such codes as a function of code's degree distribution and block length. The conclusion of [11] was that the cycle distribution highly depends on the degree distribution but does not change much with the block

length n . In Corollary 1, we reached a similar conclusion through our theoretical analysis. In fact, we proved that, in the asymptotic regime of $n \rightarrow \infty$, the cycle distributions are independent of n , and that the expected values of the number of c -cycles increase polynomially with the node degrees and exponentially with the cycle length c .

In the following, we demonstrate through some examples that the expected values that we derived in Theorem 1 and Corollary 1, match the numerical results. We start by the same examples considered in Table IV of [11]. The multiplicities of cycles of different lengths for rate-1/2 bi-regular codes of different degree distributions and lengths are reproduced in Table II here, and compared with the result of Corollary 1. As can be seen, the expected values of Corollary 1 are very close to the cycle multiplicities of random realizations of the graphs for different block lengths, ranging from 200 all the way to 20000.

TABLE II
MULTIPLICITIES OF SHORT CYCLES IN THE TANNER GRAPHS OF RATE-1/2 RANDOM BI-REGULAR LDPC CODES WITH DIFFERENT DEGREE DISTRIBUTIONS AND DIFFERENT BLOCK LENGTHS

Degree Distribution	Short Cycle Distribution	Block Length						$E[N_c]$ Corollary 1
		200	500	1000	5000	10000	20000	
(3,6)	N_6	171	167	181	156	166	148	167
	N_8	1265	1239	1226	1235	1253	1285	1250
	N_{10}	10069	10110	9939	9982	9858	9974	10000
(4, 8)	N_6	1636	1611	1584	1562	1537	1572	1544
	N_8	25005	24419	24379	24363	24529	24557	24310
	N_{10}	409335	409373	408595	407958	408246	409051	408410
(5, 10)	N_6	8626	8064	8055	7978	7858	7926	7776
	N_8	213639	212484	210767	210153	209614	210159	209952
	N_{10}	6052158	6054661	6049148	6043400	6049583	6043704	6046617

As the next example, we consider two irregular degree distributions, and construct random codes of different block lengths with those degree distributions. The first degree distribution is selected as $\lambda_I(x) = 0.4286x^2 + 0.5714x^3$, and $\rho_I(x) = x^6$, where the coefficients λ_i and ρ_i represent the fraction of edges connected to variable and check nodes of degree $i+1$, respectively. This degree distribution, which is mildly irregular, corresponds to an LDPC code with rate 0.5. We thus have $n = 2m$. The second degree distribution is selected from Table I of [34]. It is more irregular than the first degree distribution and is as follows: $\lambda_{II}(x) = 0.2690x + 0.2603x^2 +$

$0.0451x^4 + 0.4256x^9$, and $\rho_{II}(x) = 0.6398x^6 + 0.3602x^7$. The code rate corresponding to this degree distribution is 0.4998 [34], and thus $n \simeq 2m$. In Table III, we have provided the cycle multiplicities of the random realizations of the two degree distributions at block lengths 200, 500, 1000, 5000, 10000 and 20000, along with the approximation of expected values obtained based on the asymptotic upper bound of Theorem 1. Comparison of the results of Table III with those of Table II shows a larger discrepancy between the approximations of expected values and the cycle multiplicities in random realizations for irregular graphs vs. regular ones. This can be, at least in part, explained by Remark 1. Moreover, comparison of the results for the two irregular degree distributions, particularly for the largest block length of 20000, shows that the approximations provided for $E(N_c)$ by the asymptotic upper bound of Theorem 1 are more accurate for the less irregular ensemble. We also note that the asymptotic lower bounds for $E(N_c)$, $c = 4, 6, 8, 10$, corresponding to the irregular ensembles I and II are 52, 512, 5577, 64840, and 10, 44, 210, 1077, respectively. One can clearly see that the asymptotic upper bound provides a much more accurate estimate for the number of cycles of different length in comparison with the asymptotic lower bound derived in Theorem 1. This is particularly the case for the more irregular degree distribution.

TABLE III
MULTIPLICITIES OF SHORT CYCLES IN THE TANNER GRAPHS OF IRREGULAR LDPC CODES WITH DIFFERENT DEGREE DISTRIBUTIONS AND DIFFERENT BLOCK LENGTHS

Degree Distribution	Short Cycle Distribution	Block Length						$E[N_c]$
		200	500	1000	5000	10000	20000	Theorem 1
$\lambda_I(x), \rho_I(x)$	N_4	56	62	61	52	61	59	59
	N_6	599	602	587	590	597	602	611
	N_8	6653	6814	6742	6881	7011	7158	7067
	N_{10}	85244	87260	84846	86436	87046	87311	87181
$\lambda_{II}(x), \rho_{II}(x)$	N_4	230	222	244	236	243	196	225
	N_6	4871	4759	4057	4571	4562	4769	4500
	N_8	109017	107523	104599	106620	105685	107479	101250
	N_{10}	2610260	2557357	2212847	2585699	2548117	2605595	2430000

B. Random lifts of a base graph

We consider random lifts of the 3×5 fully-connected base graph with lifting degrees 400, 1000 and 2000. The cycle multiplicities of the random lifts for cycles of length 4 all the way to 16 are shown in Table IV, and compared with the expected value obtained from Theorem 5. As can be seen, for different lifting degrees, the expected value provides a good approximation for the multiplicities of cycles of different length in random realizations.

TABLE IV
MULTIPLICITIES OF SHORT CYCLES OF DIFFERENT LENGTH FOR RANDOM LIFTS OF DIFFERENT DEGREES OF THE 3×5
FULLY-CONNECTED BASE GRAPH

Cycle Length	Lifting Degree			$E[N_c]$
	$N = 400$	$N = 1000$	$N = 2000$	Theorem 5
4	31	27	29	30
6	64	62	66	60
8	590	588	515	585
10	2994	3111	3083	3060
12	22730	22636	22919	22550
14	147395	148141	147894	147420
16	1058149	1061667	1052401	1056832

C. Random QC bipartite graphs

In [20], the authors studied the cycle distribution of random cyclic lifts of the 3×5 fully-connected base graph for different lifting degrees (block lengths), and observed that such graphs have generally larger girth compared to random bi-regular codes with the same degree distribution and block length. The example also showed that the girth of QC codes was improved by the increase in the lifting degree N . The above results reported in Table I of [20] are reproduced here in Table V.

We note that the 3×5 fully-connected base graph has girth 4, and thus, based on Remark 4, for $c \leq 10$, cyclic random lifts of this base graph have no inevitable cycles of length c . This means that for $c \leq 10$, the expected value of the number of cycles of length c does not increase with the lifting degree N . On the other hand, one can find prime ZP TBC walks of length 12, 14 and 16 in the base graph: let $G = (U, W)$ be the 3×5 fully-connected base graph with

$U = \{1, 2, 3\}$ and $W = \{4, 5, 6, 7, 8\}$. It is then easy to verify that the following TBC walks in G are prime with zero permutation shifts: $w_{12} = 5243514253415$, $w_{14} = 342536143524163$ and $w_{16} = 25362714263524172$. This means that the random cyclic lifts of the base graph will have inevitable cycles with these lengths and that, based on Corollary 3, the expected value of cycles with these lengths increases linearly with N for sufficiently large N values. These theoretical predictions are consistent with the numerical results reported in Table V, for these cycle lengths. For cycles of length 18, however, there is no prime ZP TBC walk in the 3×5 fully-connected base graph, and thus the expected number of such cycles remains constant with respect to N . This is also consistent with the results of Table V.

For comparison, we have also included, in the last column of Table V, the expected value of the number of cycles in random lifts of the 3×5 fully-connected base graph, obtained based on Theorem 5. One can see the large difference between these values and the corresponding values for random cyclic lifts for cases of $c = 12, 14$, and 16 , where the cyclic lifts have inevitable cycles.

TABLE V
MULTIPLICITIES OF CYCLES OF DIFFERENT LENGTH FOR RANDOM CYCLIC LIFTS OF DIFFERENT DEGREES OF THE 3×5
FULLY-CONNECTED BASE GRAPH

Cycle Length	Lifting Degree			$E[N_c]$
	$N = 400$	$N = 1000$	$N = 2000$	Theorem 5
6	0	0	0	60
8	0	0	0	585
10	2000	1000	0	3060
12	33200	54000	98000	22550
14	193200	275000	478000	147420
16	1022200	1169000	1490000	1056832
18	7143600	7251000	8282000	7427300

VII. CONCLUSION

In this paper, we studied the cycle distribution of different ensembles of LDPC codes, often used in the literature, in the asymptotic regime where the block length tends to infinity (but the degree distribution is fixed). These ensembles were random irregular and bi-regular, random lifts of protographs, and random cyclic lifts of protographs. We demonstrated that for the first

ensemble, the multiplicities of cycles of different lengths have independent Poisson distributions. We derived asymptotic upper and lower bounds on the expected values of the distributions. These bounds are only a function of cycle length and degree distributions, and independent of the block length. We also showed that for the second ensemble, the asymptotic cycle distributions have the same behavior as those of the first ensemble as long as the degree distributions are identical. For the third ensemble, we proved that the cycle distributions can be significantly different than those of the first two ensembles. In particular, we showed that for some values of c , and depending on the protograph, the expected number of c -cycles can increase linearly with the block length. We also derived an upper bound, linearly increasing with the block length, on the variance of the number of c -cycles.

Using numerical results, we demonstrated that our asymptotic results provide good approximations for the number of cycles in realizations of finite-length LDPC codes, even when the block length is as short as a few hundred bits. Moreover, our results provided theoretical justification for some of the observations made empirically in the literature about cycle distributions of LDPC codes.

The results presented in this paper can be used in the analysis and design of LDPC codes in cases where such processes depend on the knowledge of the cycle distributions. As a particular example, we showed how the asymptotic average number of trapping sets can be estimated using the results presented in this work.

Finally, our numerical results show that for irregular graphs, the asymptotic upper bound provided in Theorem 1 on the expected value of cycle multiplicities is much more accurate than the asymptotic lower bound in estimating the cycle multiplicities. This suggests that it may be possible to tighten the asymptotic lower bound derived in Theorem 1.

VIII. ACKNOWLEDGMENT

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IX. APPENDIX I

Proof of Lemma 1. Let $|V(H)|$ and $|E(H)|$ be the number of nodes and the number of edges of H , respectively, and let \mathcal{C}_H be the number of structures in a random configuration whose projections in \mathcal{G}^* are copies of H . There are at most $\mathcal{O}\left(\binom{\Delta+1}{|V(H)|}^n\right)$ choices for the node set

of the copy of H . Thus, we have $\mathcal{C}_H = \mathcal{O}(n^{|V(H)|})$. On the other hand, the probability of each given edge set of size $|E(H)|$ is $\frac{(\eta - |E(H)|)!}{\eta!} = \mathcal{O}(n^{-|E(H)|})$. Thus, the expected number of copies of H in \mathcal{G}^* is $\frac{\mathcal{C}_H \times (\eta - |E(H)|)!}{\eta!} = \mathcal{O}(n^{|V(H)| - |E(H)|}) = \mathcal{O}(\frac{1}{n})$. ■

Proof of Lemma 2. Consider the left hand side of (16). There are $\binom{n}{k}$ terms added together, each being a product of k distinct variables from the set X , and each with the multiplicative coefficient $\binom{k}{k/2}$. This implies that on the left side, we have $\binom{n}{k} \times \binom{k}{k/2} = \Theta(n^k)$ terms, each a product of k distinct variables from the set X , added together. Now, consider the right hand side of (16). It is the product of two identical expressions, each a sum of $\binom{n}{k/2}$ terms, where each such term is a product of $k/2$ distinct variables from the set X . If we expand the product of the two identical expressions, we have the sum of $\binom{n}{k/2} \times \binom{n}{k/2}$ product terms, where each product involves k variables from the set X . We can partition such product terms into two categories: (1) those with all k variables being distinct, and (2) those with at least one variable repeated at least once. In the following, we show that the first category consists of exactly the same product terms as in the left hand side of (16), and that the second category contains $\mathcal{O}(n^{k-1})$ terms. This will then prove the asymptotic equality of (16).

On the right hand side of (16), the number of product terms in Category 1 is equal to $\binom{n}{k/2} \times \binom{n-k/2}{k/2}$. The term $\binom{n}{k/2}$ is the number of product terms of size $k/2$ in the first expression, and the term $\binom{n-k/2}{k/2}$ is the number of product terms in the second expression that have no common variable with the selected product term from the first expression. It is now easy to see that considering all the possible $\binom{n}{k}$ product terms with k distinct variables, each is repeated $\binom{k}{k/2}$ times in the product terms of Category 1. In fact, we have $\binom{n}{k/2} \times \binom{n-k/2}{k/2} = \binom{n}{k} \times \binom{k}{k/2}$. Now, the number of terms in Category 2 is equal to $\binom{n}{k/2} \times \binom{n}{k/2} - \binom{n}{k/2} \times \binom{n-k/2}{k/2}$, which is $\mathcal{O}(n^{k-1})$. ■

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