

FROM RATE DISTORTION THEORY TO METRIC MEAN DIMENSION: VARIATIONAL PRINCIPLE

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ABSTRACT. The purpose of this paper is to point out a new connection between information theory and dynamical systems. In the information theory side, we consider rate distortion theory, which studies lossy data compression of stochastic processes under distortion constraints. In the dynamical systems side, we consider mean dimension theory, which studies how many parameters per second we need to describe a dynamical system. The main results are new variational principles connecting rate distortion function to metric mean dimension.

1. INTRODUCTION

1.1. Main results. There is a long tradition in the study of dynamical systems to consider the interplay between ergodic theory and topological dynamics (see e.g. Glasner–Weiss [GW] for an in depth discussion). An important manifestation of this interplay is the *variational principle* relating measure theoretic and topological entropy (Goodwyn [Goodw], Dinaburg [Din] and Goodman [Goodm]). Let (\mathcal{X}, T) be a dynamical system, i.e. \mathcal{X} is a compact metric space and T is a continuous map from \mathcal{X} to \mathcal{X} . We denote by $\mathcal{M}^T(\mathcal{X})$ the set of all invariant probability measures on \mathcal{X} . The variational principle connects the measure theoretic entropy $h_\mu(T)$ to the topological entropy $h_{\text{top}}(T)$ by

$$(1.1) \quad h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}^T(\mathcal{X})} h_\mu(T).$$

In the end of the last century Gromov [Gro] proposed a new topological invariant of dynamical systems called *mean dimension*. The mean

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dimension of a dynamical system (\mathcal{X}, T) is denoted by $\text{mdim}(\mathcal{X}, T)$. This invariant counts the average number of parameters needed per iteration for describing a point in \mathcal{X} , and gives a non-degenerate numerical invariant for dynamical systems of *infinite dimensional* and *infinite entropy*.

For example, consider the infinite product of the unit interval

$$[0, 1]^{\mathbb{Z}} = \cdots \times [0, 1] \times [0, 1] \times [0, 1] \times \cdots$$

and let σ be the shift map on this space. The system $([0, 1]^{\mathbb{Z}}, \sigma)$ is obviously infinite dimensional and has infinite topological entropy, but its mean dimension is one. Intuitively this means that to describe an orbit in $([0, 1]^{\mathbb{Z}}, \sigma)$ one needs one parameter per iterate. This is analogous to the fact that the symbolic shift $\{1, 2, \dots, n\}^{\mathbb{Z}}$ has the topological entropy $\log n$.

Mean dimension has applications to topological dynamics, which cannot be touched within the framework of topological entropy. Here we briefly explain an application to a natural *embedding problem*, raised many years before the definition of mean dimension:

When can we embed a dynamical system (\mathcal{X}, T) in the shift $([0, 1]^{\mathbb{Z}}, \sigma)$?

Mean dimension provides a necessary condition: If (\mathcal{X}, T) is embeddable in $([0, 1]^{\mathbb{Z}}, \sigma)$ then $\text{mdim}(\mathcal{X}, T) \leq 1$. A deeper result [GT, Theorem 1.4] states that a minimal system (\mathcal{X}, T) of mean dimension less than $1/2$ can be embedded into $([0, 1]^{\mathbb{Z}}, \sigma)$ (this strengthened [Lin, Theorem 5.1], which proved a similar result but with a non-optimal constant). The result [GT, Theorem 1.4] is optimal in the sense that there exists a minimal system of mean dimension $1/2$ which cannot be embedded in $([0, 1]^{\mathbb{Z}}, \sigma)$ ([LT, Theorem 1.3]). These results show that mean dimension is certainly a reasonable measure of the “size” of dynamical systems. The theory of mean dimension turns out to have connection to problems in several different mathematical fields, e.g. topological dynamics ([Lin, LW, Li, Gut1, Gut2, Gut3]), geometric analysis ([Cos, MT]), and operator algebra ([LL, EN]).

Motivated by the success of the variational principle (1.1), one might want to define a *measure theoretic mean dimension* and try to prove a corresponding variational principle, but any naïve attempt to carry out this idea is doomed to failure. The reason can be easily seen by using the Jewett–Krieger theorem ([Jew, Kri]): Every ergodic measurable dynamical system has a uniquely ergodic model. Consider an arbitrary ergodic measurable dynamical system \mathcal{X} . Suppose we want to define its “measure theoretic mean dimension”. There exists a topological

system (\mathcal{X}, T) such that it is uniquely ergodic (i.e. $\mathcal{M}^T(\mathcal{X})$ consists of a single measure, say μ) and (\mathcal{X}, μ, T) is measurably isomorphic to \mathcal{X} . It is known that uniquely ergodic systems always have zero topological mean dimension ([LW, Theorem 5.4]). Then if we have a “variational principle”, the only possibility is that the “measure theoretic mean dimension” of \mathcal{X} is zero.

It turned out that *rate distortion theory* and *metric mean dimension* provide a much better framework to study this interplay. Rate distortion theory is a standard concept in information theory originally introduced by the monumental paper of Shannon [Sha]. Its primary object is data compression of *continuous* random variables and their processes. Continuous random variables always have infinite entropy, so it is impossible to describe them perfectly with only finitely many bits. Instead rate distortion theory studies a *lossy* data compression method achieving some *distortion* constraints. A friendly introduction can be found in Cover–Thomas [CT, Chapter 10]. Metric mean dimension is a metric space version of mean dimension introduced by Weiss and the first named author [LW] that is related to mean dimension in a way that is very analogous to how Minkowski or Hausdorff dimensions are related to the topological dimension. Both rate distortion theory and metric mean dimension use *distance* as a crucial ingredient. This metric structure enables us to give a meaningful variational principle.

First we explain rate distortion theory. For a couple (X, Y) of random variables X and Y we denote its mutual information by $I(X; Y)$. We review the definition and basic properties of $I(X; Y)$ in Section 2. Intuitively it is the amount of information which X and Y share. Let (\mathcal{X}, T) be a dynamical system with a distance d on \mathcal{X} . Take an invariant probability measure $\mu \in \mathcal{M}^T(\mathcal{X})$. For a positive number ε we define the **rate distortion function** $R_\mu(\varepsilon)$ as the infimum of

$$(1.2) \quad \frac{I(X; Y)}{n},$$

where n runs over all natural numbers, and X and $Y = (Y_0, \dots, Y_{n-1})$ are random variables defined on some probability space (Ω, \mathbb{P}) such that

- X takes values in \mathcal{X} , and its law is given by μ .
- Each Y_k takes values in \mathcal{X} , and Y approximates the process $(X, TX, \dots, T^{n-1}X)$ in the sense that

$$(1.3) \quad \mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Y_k) \right) < \varepsilon.$$

Here $\mathbb{E}(\cdot)$ is the expectation with respect to the probability measure \mathbb{P} . Note that $R_\mu(\varepsilon)$ depends on the distance d although it is not explicitly written in the notation.

Roughly speaking, $R_\mu(\varepsilon)$ is the minimum rate of quantizations of the process $\{T^k X\}_{k=0}^\infty$ under the distortion constraint (1.3). More precisely, a main theorem of rate distortion theory [Ber, Chapter 7] states that if the invariant measure μ is ergodic then there exists a sequence of maps $f_n = (f_{n,0}, \dots, f_{n,n-1}) : \mathcal{X} \rightarrow \mathcal{X}^n$ ($n \geq 1$) satisfying

$$\lim_{n \rightarrow \infty} \frac{\log |f_n(\mathcal{X})|}{n} = R_\mu(\varepsilon), \quad \mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, f_{n,k}(X)) \right) < \varepsilon,$$

where X is a random variable obeying μ (and $|f_n(\mathcal{X})|$ denotes the cardinality of the set $f_n(\mathcal{X})$). Then we can represent the process $\{T^k X\}_{k=0}^\infty$ by the quantization

$$f_{n,0}(X), \dots, f_{n,n-1}(X), f_{n,0}(T^n X), \dots, \\ f_{n,n-1}(T^n X), f_{n,0}(T^{2n} X), \dots, f_{n,n-1}(T^{2n} X), \dots$$

This approximates $\{T^k X\}_{k=0}^\infty$ by ε in average, and (if n is sufficiently large) we need

$$\frac{\log |f_n(\mathcal{X})|}{n} \approx R_\mu(\varepsilon) \text{ nats per second}$$

for describing the sequence¹. There also exists a similar theorem for non-ergodic μ , but the statement is a bit more involved. See [ECG, LDN] for the details.

Next we explain metric mean dimension. Let (\mathcal{X}, T) be a dynamical system with a distance d as above. For a positive number ε we define $\#(\mathcal{X}, d, \varepsilon)$ as the minimum cardinality N of the open covering $\{U_1, \dots, U_N\}$ of \mathcal{X} such that all U_n have diameter smaller than ε . For a natural number n we define a distance d_n on \mathcal{X} by

$$(1.4) \quad d_n(x, y) = \max_{0 \leq k < n} d(T^k x, T^k y).$$

We set

$$S(\mathcal{X}, T, d, \varepsilon) = \lim_{n \rightarrow \infty} \frac{\log \#(\mathcal{X}, d_n, \varepsilon)}{n}.$$

This limit always exists because $\log \#(\mathcal{X}, d_n, \varepsilon)$ is a subadditive function of n . The topological entropy $h_{\text{top}}(T)$ is the limit of $S(\mathcal{X}, T, d, \varepsilon)$ as $\varepsilon \rightarrow 0$. When the topological entropy is infinite, we are interested

¹“nats” means “natural unit of information”. Here the base of the logarithm is e not 2.

in the growth of $S(\mathcal{X}, T, d, \varepsilon)$. This motivates the definition of upper and lower **metric mean dimension**:

$$\overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \limsup_{\varepsilon \rightarrow 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|},$$

$$\underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \liminf_{\varepsilon \rightarrow 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|}.$$

If the limit supremum and infimum agree, we denote the common value by $\text{mdim}_{\text{M}}(\mathcal{X}, T, d)$.

By [LW, Theorem 4.2] the metric mean dimensions always dominate the topological mean dimension:

$$(1.5) \quad \text{mdim}(\mathcal{X}, T) \leq \underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) \leq \overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d).$$

It is also known ([Lin, Theorem 4.3]) that if (\mathcal{X}, T) is minimal then there exists a distance d on \mathcal{X} satisfying

$$\underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \text{mdim}(\mathcal{X}, T).$$

It is conjectured that such a distance exists for every system.

Metric mean dimension is not just a theoretical object. It is an important tool for computing topological mean dimension. At least in our experience, it is generally difficult to prove upper bounds on topological mean dimension. The most powerful method (known to the authors) is to use metric mean dimension. If we obtain an upper bound on metric mean dimension, then we can also bound topological mean dimension by the inequality (1.5). The papers [Tsu1, Tsu2] employ this method to compute the topological mean dimensions of certain dynamical systems in geometric analysis and complex geometry.

The main purpose of this paper is to establish a variational principle connecting rate distortion function to metric mean dimension. Before going further, we look at an example:

Example 1.1. Let $\mathcal{X} = [0, 1]^{\mathbb{Z}}$ be the infinite product of the unit interval, and let $T : \mathcal{X} \rightarrow \mathcal{X}$ be the shift: $T((x_m)_{m \in \mathbb{Z}}) = (x_{m+1})_{m \in \mathbb{Z}}$. We define a distance d on \mathcal{X} by

$$(1.6) \quad d(x, y) = \sum_{m \in \mathbb{Z}} 2^{-|m|} |x_m - y_m|, \quad (x = (x_m)_{m \in \mathbb{Z}}, y = (y_m)_{m \in \mathbb{Z}}).$$

First we calculate the metric mean dimension. Let $\varepsilon > 0$ and set $l = \lceil \log_2(4/\varepsilon) \rceil$. Then $\sum_{|n| > l} 2^{-|n|} \leq \varepsilon/2$. We consider an open covering of $[0, 1]$ by

$$I_k = \left(\frac{(k-1)\varepsilon}{12}, \frac{(k+1)\varepsilon}{12} \right), \quad 0 \leq k \leq \lfloor 12/\varepsilon \rfloor.$$

I_k has length $\varepsilon/6$. For $n \geq 1$, consider

$$[0, 1]^{\mathbb{Z}} = \bigcup_{0 \leq k_{-l}, \dots, k_{n+l} \leq \lfloor 12/\varepsilon \rfloor} \{x \mid x_{-l} \in I_{k_{-l}}, x_{-l+1} \in I_{k_{-l+1}}, \dots, x_{n+l} \in I_{k_{n+l}}\}.$$

Each open set in the right-hand side has diameter less than ε with respect to the distance d_n . Hence

$$(1.7) \quad \#([0, 1]^{\mathbb{Z}}, d_n, \varepsilon) \leq (1 + \lfloor 12/\varepsilon \rfloor)^{n+2l+1} = (1 + \lfloor 12/\varepsilon \rfloor)^{n+2\lceil \log_2(4/\varepsilon) \rceil + 1}.$$

On the other hand, any two distinct points in the sets

$$\{x \in [0, 1]^{\mathbb{Z}} \mid x_m \in \{0, \varepsilon, 2\varepsilon, \dots, \lfloor 1/\varepsilon \rfloor \varepsilon\} \text{ for all } 0 \leq m < n\}$$

have distance $\geq \varepsilon$ with respect to d_n . It follows $\#(\mathcal{X}, d_n, \varepsilon) \geq (1 + \lfloor 1/\varepsilon \rfloor)^n$. Therefore

$$S(\mathcal{X}, T, d, \varepsilon) = \lim_{n \rightarrow \infty} \frac{\log \#(\mathcal{X}, d_n, \varepsilon)}{n} \sim |\log \varepsilon| \quad (\varepsilon \rightarrow 0).$$

Thus $\text{mdim}_M(\mathcal{X}, T, d) = 1$.

Next we consider the rate distortion function for the measure $\mu = (\text{Lebesgue measure})^{\otimes \mathbb{Z}}$. The calculation of $R_\mu(\varepsilon)$ requires some familiarity with mutual information, so we postpone it to Example 2.11 in Section 2, and here we state only the result:

$$(1.8) \quad R_\mu(\varepsilon) \sim |\log \varepsilon| \quad (\varepsilon \rightarrow 0).$$

Therefore

$$\lim_{\varepsilon \rightarrow 0} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|} = 1 = \text{mdim}_M(\mathcal{X}, T, d).$$

The purpose of this paper is to generalize this phenomena to arbitrary dynamical systems.

For some of our results, we need to introduce a certain regularity condition on the underlying metric space.

Condition 1.2. Let (\mathcal{X}, d) be a compact metric space. It is said to have *tame growth of covering numbers* if for every $\delta > 0$ we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) = 0.$$

Note that this is purely a condition on metric spaces and does not involve the dynamics.

For example, if \mathcal{X} is a compact subset of the Euclidean space \mathbb{R}^n , then

$$\#(\mathcal{X}, \text{Euclidean distance}, \varepsilon) = O((1/\varepsilon)^n),$$

and so \mathcal{X} satisfies Condition 1.2. Indeed the tame growth of covering numbers condition is a fairly mild condition:

Lemma 1.3. *Every compact metrizable space admits a distance satisfying Condition 1.2.*

Proof. Every compact metrizable space can be topologically embedded into the infinite dimensional cube $[0, 1]^{\mathbb{Z}}$, so it is enough to prove the statement for $[0, 1]^{\mathbb{Z}}$. Let d be the distance introduced in (1.6). By (1.7)

$$\#([0, 1]^{\mathbb{Z}}, d, \varepsilon) \leq (1 + \lfloor 12/\varepsilon \rfloor)^{2\lceil \log_2(4/\varepsilon) \rceil + 2}.$$

It follows

$$\log \#([0, 1]^{\mathbb{Z}}, d, \varepsilon) = O(|\log \varepsilon|^2).$$

This satisfies the tame growth of covering numbers condition. \square

Remark 1.4. It is easy to check that if (A, d) is a compact metric space satisfying Condition 1.2 then the distance d' on the shift $A^{\mathbb{Z}}$ defined by

$$d'(x, y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} d(x_n, y_n)$$

also satisfies Condition 1.2.

Our first main result is:

Theorem 1.5. *Let (\mathcal{X}, T) be a dynamical system with a distance d . Suppose d satisfies Condition 1.2. Then*

$$(1.9) \quad \begin{aligned} \overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_{\mu}(\varepsilon)}{|\log \varepsilon|}, \\ \underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \liminf_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_{\mu}(\varepsilon)}{|\log \varepsilon|}. \end{aligned}$$

Therefore we can say that metric mean dimension is a topological dynamics counterpart of rate distortion theory.

Remark 1.6. Our formulation of the variational principle (1.9) is strongly influenced by the work of Kawabata–Dembo [KD]. For a metric space A , they studied connections between the fractal dimensions of A and the rate distortion functions of i.i.d. processes taking values in A . Theorem 1.5 can be seen as a generalization of [KD, Proposition 3.1] from the case of $(\mathcal{X}, T) = (A^{\mathbb{Z}}, \text{shift})$ to arbitrary dynamical systems.

Although Condition 1.2 is a mild condition, it might still look technical and one might want to remove it. But indeed the equalities (1.9) do *not* hold in general without an additional assumption:

Proposition 1.7. *There exists a dynamical system (\mathcal{X}, T) with a distance d such that*

$$\text{mdim}_M(\mathcal{X}, T, d) = \infty, \quad \lim_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_\mu(\varepsilon)}{|\log \varepsilon|} = 0.$$

Remark 1.8. In the proof of Theorem 1.5, we use Condition 1.2 to compare the two distances

$$(1.10) \quad \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^k y) \quad \text{and} \quad \max_{0 \leq k < n} d(T^k x, T^k y).$$

The former is closely related to the distortion condition (1.3) in the definition of rate distortion function. The latter is used in the definition of metric mean dimension. Under Condition 1.2, these two distances behave quite similarly. A rough idea of the proof of Proposition 1.7 is to construct a system (\mathcal{X}, T) where the two distances (1.10) show radically different behaviors.

The above definition of the rate distortion function $R_\mu(\varepsilon)$, or the similar L^2 -rate distortion function defined in §1.2, seems to be the most widely used one. It has from our point of view the disadvantage that in this case we need to assume Condition 1.2 for establishing the variational principle (1.9). Next we propose another version of rate distortion function and establish a corresponding variational principle without any additional condition.

Let (\mathcal{X}, T) be a dynamical system with a distance d and an invariant probability measure μ . For positive numbers ε and α we define the L^∞ -**rate distortion function** $\tilde{R}_\mu(\varepsilon, \alpha)$ as the infimum of

$$\frac{I(X; Y)}{n},$$

where n runs over all natural numbers, and X and $Y = (Y_0, \dots, Y_{n-1})$ are random variables defined on some probability space (Ω, \mathbb{P}) such that

- X takes values in \mathcal{X} , and its law is given by μ .
- Each Y_k takes values in \mathcal{X} , and they satisfy the following *modified distortion condition*:

$$(1.11) \quad \mathbb{E}(\text{the number of } k \in [0, n-1] \text{ satisfying } d(T^k X, Y_k) \geq \varepsilon) < \alpha n.$$

In other words, we define $\tilde{R}_\mu(\varepsilon, \alpha)$ by replacing the distortion condition (1.3) in the definition of $R_\mu(\varepsilon)$ with (1.11). We set

$$\tilde{R}_\mu(\varepsilon) = \lim_{\alpha \rightarrow 0} \tilde{R}_\mu(\varepsilon, \alpha).$$

The reason for our use of terminology “ L^∞ -rate distortion function” will (hopefully) become clearer to the reader in the next subsection.

Our second main result is:

Theorem 1.9. *For any dynamical system (\mathcal{X}, T) with a distance d , we have*

$$(1.12) \quad \begin{aligned} \overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \tilde{R}_\mu(\varepsilon)}{|\log \varepsilon|}, \\ \underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \liminf_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} \tilde{R}_\mu(\varepsilon)}{|\log \varepsilon|}. \end{aligned}$$

We emphasize that we do not need any additional condition for establishing (1.12) in this case.

1.2. L^p -variants. We can also consider L^p -versions of the variational principle. The L^2 -case might be of special interest because it is related to the least squares method. Let (\mathcal{X}, T) be a dynamical system with a distance d . For $1 \leq p < \infty$, $\varepsilon > 0$ and $\mu \in \mathcal{M}^T(\mathcal{X})$ we define the **L^p -rate distortion function** $R_{\mu,p}(\varepsilon)$ by replacing the distortion condition (1.3) in the definition of $R_\mu(\varepsilon)$ with

$$(1.13) \quad \mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Y_k)^p \right) < \varepsilon^p.$$

By the Hölder inequality, this is stronger than (1.3), hence $R_\mu(\varepsilon) \leq R_{\mu,p}(\varepsilon)$. On the other hand, the condition (1.13) is essentially weaker than (1.11) in the definition of $\tilde{R}_\mu(\varepsilon)$. Indeed

$$\begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Y_k)^p &\leq \\ \varepsilon^p + (\text{diam}(\mathcal{X}, d))^p &\cdot \frac{1}{n} \cdot |\{k \in [0, n-1] \mid d(T^k X, Y_k) \geq \varepsilon\}|. \end{aligned}$$

So the condition (1.11) implies

$$\mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Y_k)^p \right) < \varepsilon^p + \alpha (\text{diam}(\mathcal{X}, d))^p.$$

This leads to $R_{\mu,p}(\varepsilon') \leq \tilde{R}_\mu(\varepsilon)$ for any $\varepsilon' > \varepsilon$. Thus we get

$$R_\mu(\varepsilon') \leq R_{\mu,p}(\varepsilon') \leq \tilde{R}_\mu(\varepsilon) \quad \text{for any } \varepsilon' > \varepsilon > 0.$$

Therefore Theorems 1.5 and 1.9 imply

Corollary 1.10. *If the distance d satisfies Condition 1.2, then for any $p \geq 1$*

$$\overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_{\mu,p}(\varepsilon)}{|\log \varepsilon|},$$

$$\underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \liminf_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_{\mu,p}(\varepsilon)}{|\log \varepsilon|}.$$

1.3. Comments on the proofs and the organization of the paper. The uniform distribution on the set $\{1, 2, \dots, n\}$ has entropy $\log n$, and this is the maximal entropy measure among all probability distributions on it. There exists a similar result about mutual information $I(X; Y)$: Roughly speaking, if X is uniformly distributed over an ε -separated set S of a compact metric space \mathcal{X} , and if $\varepsilon^{-1}\mathbb{E}(d(X, Y))$ is sufficiently small, then $I(X; Y)$ is almost equal to $\log |S|$ (for precise statements, see Corollary 2.5 and Lemma 2.6 below). This observation is key to the proofs of Theorems 1.5 and 1.9. Starting from this, we will follow a line of ideas analogous to Misiurewicz’s proof [Mis] of the variational principle (1.1). Misiurewicz’s argument adapts quite naturally (perhaps even suprisingly so) to the setting of rate distortion theory.

Organization of the paper is as follows: We recall some basics of mutual information in Section 2. Theorems 1.5 and 1.9 are proved in Sections 3 and 4 respectively. We prove Proposition 1.7 in Section 5. We recall some elementary results on optimal transport (which are used in Sections 3 and 4) in the Appendix.

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2. MUTUAL INFORMATION

In this section we recall some basic properties of mutual information. A good reference is Cover–Thomas [CT, Chapter 2].

Throughout this section (Ω, \mathbb{P}) is a probability space. Let \mathcal{X} and \mathcal{Y} be measurable spaces, and $X : \Omega \rightarrow \mathcal{X}$ and $Y : \Omega \rightarrow \mathcal{Y}$ measurable maps. We define the **mutual information** $I(X; Y)$ as the supremum

of

$$(2.1) \quad \sum_{m=1}^M \sum_{n=1}^N \mathbb{P}((X, Y) \in P_m \times Q_n) \log \frac{\mathbb{P}((X, Y) \in P_m \times Q_n)}{\mathbb{P}(X \in P_m) \mathbb{P}(Y \in Q_n)},$$

where $\{P_1, \dots, P_M\}$ and $\{Q_1, \dots, Q_N\}$ are partitions of \mathcal{X} and \mathcal{Y} respectively, with the convention that $0 \log(0/a) = 0$ for all $a \geq 0$. The mutual information $I(X; Y)$ is nonnegative and symmetric: $I(X; Y) = I(Y; X) \geq 0$.

If \mathcal{X} and \mathcal{Y} are finite sets, then

$$(2.2) \quad \begin{aligned} I(X; Y) &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathbb{P}(X = x, Y = y) \log \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(X = x) \mathbb{P}(Y = y)} \\ &= H(X) - H(X|Y) = H(X) + H(Y) - H(X, Y), \end{aligned}$$

where $H(X|Y)$ is the conditional entropy of X given Y . The formula $I(X; Y) = H(X) - H(X|Y)$ shows an intuitive meaning of mutual information; it is the amount of information which the random variables X and Y share.

The following two lemmas are trivial but important in the proofs of the main theorems.

Lemma 2.1. *Suppose \mathcal{X} and \mathcal{Y} are finite sets. Let $(X_n, Y_n) : \Omega \rightarrow \mathcal{X} \times \mathcal{Y}$ ($n \geq 1$) be a sequence of measurable maps converging to $(X, Y) : \Omega \rightarrow \mathcal{X} \times \mathcal{Y}$ in law. Then $I(X_n; Y_n)$ converges to $I(X; Y)$.*

Proof. This follows from the first equation of (2.2). \square

Lemma 2.2 (Data-processing inequality). *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be measurable spaces, and $X : \Omega \rightarrow \mathcal{X}$ and $Y : \Omega \rightarrow \mathcal{Y}$ measurable maps. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a measurable map. Then²*

$$I(X; f(Y)) \leq I(X; Y).$$

Proof. This immediately follows from the definition of $I(X; Y)$. \square

Remark 2.3. Lemma 2.2 implies that, in the definition (1.2) of the rate distortion function $R_\mu(\varepsilon)$, we can assume that the random variable Y there takes only finitely many values, namely that its distribution is supported on a finite set. Indeed, let X and Y be as in (1.2) and (1.3). Take a sufficiently fine partition \mathcal{P} of \mathcal{X} and for each atom A of \mathcal{P} choose one point $p_A \in A$. Define $f : \mathcal{X} \rightarrow \mathcal{X}$ by $f(A) = \{p_A\}$, and set

²Indeed data-processing inequality is a more general statement; see [CT, Section 2.8]. But we need only this statement here.

$Z = (Z_0, \dots, Z_{n-1}) = (f(Y_0), \dots, f(Y_{n-1}))$. Then

$$\begin{aligned} \mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Z_k) \right) &\leq \max_{A \in \mathcal{P}} \text{diam}(A) + \mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, Y_k) \right) \\ &< \varepsilon \end{aligned}$$

if \mathcal{P} is sufficiently fine. Hence Z satisfies the distortion condition (1.3). Lemma 2.2 implies

$$I(X; Z) \leq I(X; Y).$$

The random variable Z obviously takes only finitely many values.

Similarly we can also assume that Y takes only finitely many values in the definition of $\tilde{R}_\mu(\varepsilon, \alpha)$: Suppose Y satisfies the modified distortion condition (1.11). Then we can find $0 < \varepsilon' < \varepsilon$ satisfying

$$\mathbb{E} (\text{number of } 0 \leq k \leq n-1 \text{ satisfying } d(T^k X, Y_k) \geq \varepsilon') < \alpha n.$$

If the partition \mathcal{P} is sufficiently fine, then for Z_k as above

$$\begin{aligned} \mathbb{E} (\text{number of } 0 \leq k \leq n-1 \text{ satisfying } d(T^k X, Z_k) \geq \varepsilon) \\ \leq \mathbb{E} (\text{number of } 0 \leq k \leq n-1 \text{ satisfying } d(T^k X, Y_k) \geq \varepsilon') \\ < \alpha n. \end{aligned}$$

For real numbers $0 \leq p \leq 1$ we set $H(p) = -p \log p - (1-p) \log(1-p)$ (with $H(0) = H(1) = 0$).

Lemma 2.4 (Fano's inequality). *Suppose \mathcal{X} , \mathcal{Y} and \mathcal{Z} are finite sets. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a map, and let $X : \Omega \rightarrow \mathcal{X}$ and $Y : \Omega \rightarrow \mathcal{Y}$ be measurable maps. Set $P_e = \mathbb{P}(X \neq f(Y))$ (the probability of error). Then³*

$$H(X|Y) \leq H(P_e) + P_e \log |\mathcal{X}|.$$

Proof. We briefly explain the proof given by [CT, Section 2.10] for the convenience of readers. We define a random variable E by

$$E = 0 \text{ if } X = f(Y), \quad E = 1 \text{ if } X \neq f(Y).$$

We expand $H(X, E|Y)$ in two ways:

$$\begin{aligned} H(X, E|Y) &= H(X|Y) + H(E|X, Y) \\ &= H(E|Y) + H(X|E, Y). \end{aligned}$$

We have $H(E|X, Y) = 0$ because E is determined by X and Y . Thus

$$\begin{aligned} H(X|Y) &= H(E|Y) + H(X|E, Y) \\ &\leq H(E) + \mathbb{P}(E = 0)H(X|E = 0, Y) + \mathbb{P}(E = 1)H(X|E = 1, Y). \end{aligned}$$

³As in the case of data-processing inequality, Fano's inequality is more general than this statement; see [CT, Section 2.10].

It follows from the definition of E that $H(E) = H(P_e)$ and $H(X|E = 0, Y) = 0$ (because $E = 0$ means that X is determined by Y). Since X takes at most $|\mathcal{X}|$ values, $H(X|E = 1, Y) \leq H(X) \leq \log |\mathcal{X}|$. Thus

$$H(X|Y) \leq H(P_e) + P_e \cdot H(X|E = 1, Y) \leq H(P_e) + P_e \log |\mathcal{X}|.$$

□

The next corollary is essentially contained in [KD, Corollary A.1]. This is the basis of the proof of Theorem 1.5.

Corollary 2.5. *Let (\mathcal{X}, d) be a compact metric space. Let $\varepsilon > 0$ and $D > 2$. Suppose $S \subset \mathcal{X}$ is a $(2D\varepsilon)$ -separated set (i.e. any two distinct points in S have distance $\geq 2D\varepsilon$). Let X and Y be measurable maps from Ω to \mathcal{X} such that X is uniformly distributed over S and*

$$\mathbb{E}(d(X, Y)) < \varepsilon.$$

Then

$$I(X; Y) \geq \left(1 - \frac{1}{D}\right) \log |S| - H(1/D).$$

Proof. Since S is a finite set, X takes only finitely many values. We can assume that Y also takes only finitely many values as in Remark 2.3. Define $f : \mathcal{X} \rightarrow \mathcal{X}$ by

$$f(x) = \begin{cases} a & \text{if } x \in B_{D\varepsilon}(a) \text{ for some } a \in S, \\ x & \text{otherwise,} \end{cases}$$

with $B_r(x)$ denoting the open ball of radius r around a point $x \in \mathcal{X}$. Set $P_e = \mathbb{P}(X \neq f(Y))$. Since $\{X \neq f(Y)\}$ is contained in $\{d(X, Y) \geq D\varepsilon\}$,

$$P_e \leq \mathbb{P}(d(X, Y) \geq D\varepsilon) \leq \frac{1}{D\varepsilon} \mathbb{E}(d(X, Y)) < \frac{1}{D} < \frac{1}{2}.$$

By Lemma 2.4,

$$H(X|Y) \leq H(P_e) + P_e \log |S| \leq H(1/D) + (1/D) \log |S|.$$

Since X is uniformly distributed over S , its entropy is $\log |S|$. Thus

$$\begin{aligned} I(X; Y) &= H(X) - H(X|Y) \\ &= \log |S| - H(X|Y) \geq \left(1 - \frac{1}{D}\right) \log |S| - H(1/D). \end{aligned}$$

□

The next lemma is used in the proof of Theorem 1.9.

Lemma 2.6. *Let (\mathcal{X}, d) be a compact metric space with a finite subset A . Let n be a natural number and ε, α positive numbers with $\alpha \leq 1/2$. Suppose $S \subset A^n$ is a 2ε -separated set with respect to the distance*

$$d_n((x_0, \dots, x_{n-1}), (y_0, \dots, y_{n-1})) = \max_{0 \leq k \leq n-1} d(x_k, y_k).$$

Let $X = (X_0, \dots, X_{n-1})$ and $Y = (Y_0, \dots, Y_{n-1})$ be measurable maps from Ω to \mathcal{X}^n such that X is uniformly distributed over S and

$$(2.3) \quad \mathbb{E}(\text{number of } k \in [0, n-1] \text{ satisfying } d(X_k, Y_k) \geq \varepsilon) < \alpha n.$$

Then

$$I(X; Y) \geq \log |S| - nH(\alpha) - \alpha n \log |A|.$$

Proof. The argument is similar to the proof of Fano's inequality. We can assume that Y takes only finitely many values as in Remark 2.3. We define a random variable Z by

$$Z = \{k \in [0, n-1] \mid d(X_k, Y_k) \geq \varepsilon\} \subset \{0, 1, \dots, n-1\}.$$

Note that by assumption (2.3) we have that $\mathbb{E}|Z| < \alpha n$.

Claim 2.7.

$$H(Z) \leq nH(\alpha).$$

Proof. We define Z_k ($0 \leq k \leq n-1$) by

$$Z_k = 0 \text{ if } k \notin Z, \quad Z_k = 1 \text{ if } k \in Z.$$

We have $|Z| = Z_0 + \dots + Z_{n-1}$ and $H(Z) = H(Z_0, \dots, Z_{n-1}) \leq H(Z_0) + \dots + H(Z_{n-1})$. Set $\alpha_k = \mathbb{P}(Z_k = 1)$. From the concavity of $H(p) = -p \log p - (1-p) \log(1-p)$,

$$H(Z) \leq \sum_{k=0}^{n-1} H(\alpha_k) \leq nH\left(\frac{1}{n} \sum_{k=0}^{n-1} \alpha_k\right) \leq nH(\alpha),$$

where we used $\sum \alpha_k = \mathbb{E}|Z| < \alpha n$ and $\alpha \leq 1/2$. □

Expanding $H(X, Z|Y)$ in two ways:

$$\begin{aligned} H(X, Z|Y) &= H(X|Y) + H(Z|X, Y) \\ &= H(Z|Y) + H(X|Y, Z). \end{aligned}$$

We have $H(Z|X, Y) = 0$ because Z is determined by X and Y . Hence by Claim 2.7

$$(2.4) \quad H(X|Y) = H(X|Y, Z) + H(Z|Y) \leq H(X|Y, Z) + nH(\alpha).$$

Take a subset $E \subset \{0, 1, \dots, n-1\}$. (We write $E^c = \{0, 1, \dots, n-1\} \setminus E$.) We estimate the conditional entropy $H(X|Y, Z = E)$. Under

the condition $Z = E$, we have $\max_{k \in E^c} d(X_k, Y_k) < \varepsilon$. Since S is 2ε -separated with respect to d_n , for each $a \in \mathcal{X}^n$ the number of $x \in S$ satisfying

$$\max_{k \in E^c} d(x_k, a_k) < \varepsilon$$

is at most $|A|^{|E|}$. Therefore the number of possible outcomes of X (given Y and $Z = E$) is at most $|A|^{|E|}$. Thus

$$H(X|Y, Z = E) \leq |E| \log |A|.$$

It follows that

$$\begin{aligned} H(X|Y, Z) &= \sum_E \mathbb{P}(Z = E) H(X|Y, Z = E) \\ &\leq \log |A| \sum_E |E| \cdot \mathbb{P}(Z = E) \\ &= \log |A| \cdot \mathbb{E}|Z| \\ &\leq \alpha n \log |A| \quad (\text{by the assumption } \mathbb{E}|Z| < \alpha n). \end{aligned}$$

Combining (2.4)

$$I(X; Y) = H(X) - H(X|Y) \geq \log |S| - nH(\alpha) - \alpha n \log |A|.$$

Here we used $H(X) = \log |S|$ since X is uniformly distributed over S . \square

In the rest of this section we assume for simplicity that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are finites sets.

Lemma 2.8 (Subadditivity of mutual information). *Let X, Y, Z be measurable maps from Ω to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively. Suppose X and Z are conditionally independent given Y , namely for every $y \in \mathcal{Y}$ with $\mathbb{P}(Y = y) \neq 0$ we have*

$$(2.5) \quad \mathbb{P}(X = x, Z = z|Y = y) = \mathbb{P}(X = x|Y = y)\mathbb{P}(Z = z|Y = y)$$

for every $x \in \mathcal{X}$ and $z \in \mathcal{Z}$. Then

$$I(Y; X, Z) \leq I(Y; X) + I(Y; Z).$$

Proof. From the conditional independence,

$$(2.6) \quad H(X, Z|Y) = H(X|Y) + H(Z|Y).$$

Indeed $H(X, Z|Y)$ is equal to

$$- \sum_y \mathbb{P}(Y = y) \left(\sum_{x, z} \mathbb{P}(X = x, Z = z|Y = y) \log \mathbb{P}(X = x, Z = z|Y = y) \right).$$

By using (2.5) we can easily check (2.6). Then

$$\begin{aligned}
I(Y; X, Z) &= H(X, Z) - H(X, Z|Y) \\
&= H(X, Z) - H(X|Y) - H(Z|Y) \\
&\leq H(X) + H(Z) - H(X|Y) - H(Z|Y) \\
&= I(X; Y) + I(Z; Y).
\end{aligned}$$

In the passage from the second line to the third, we used $H(X, Z) \leq H(X) + H(Z)$. \square

On the other hand, we have:

Lemma 2.9 (Superadditivity of mutual information). *Let X, Y, Z be measurable maps from Ω to $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ respectively. Suppose X and Z are independent. Then*

$$I(Y; X, Z) \geq I(Y; X) + I(Y; Z).$$

Proof. Since X, Z are independent, $H(X, Z) = H(X) + H(Z)$ hence

$$\begin{aligned}
I(Y; X, Z) &= H(X, Z) - H(X, Z|Y) = H(X) + H(Z) - H(X, Z|Y) \\
&\geq H(X) + H(Z) - H(X|Y) - H(Z|Y) \\
&= I(Y; X) + I(Y; Z).
\end{aligned}$$

\square

Let $X : \Omega \rightarrow \mathcal{X}$ and $Y : \Omega \rightarrow \mathcal{Y}$ be measurable maps. We define a probability mass function $\mu(x)$ and a conditional probability mass function $\nu(y|x)$ by

$$\mu(x) = \mathbb{P}(X = x), \quad \nu(y|x) = \mathbb{P}(Y = y|X = x).$$

Notice that $\nu(y|x)$ is defined only for $x \in \mathcal{X}$ with $\mathbb{P}(X = x) \neq 0$. The distribution of (X, Y) is given by $\mu(x)\nu(y|x)$ and it determines the mutual information $I(X; Y)$, hence we sometimes write $I(X; Y) = I(\mu, \nu)$.

Lemma 2.10 (Concavity/convexity of mutual information). *$I(\mu, \nu)$ is a concave function of $\mu(x)$ for fixed $\nu(y|x)$ and a convex function of $\nu(y|x)$ for fixed $\mu(x)$. More precisely,*

- (1) *Suppose that for each $x \in \mathcal{X}$ we are given a probability mass function $\nu(\cdot|x)$ on \mathcal{Y} . Let μ_1 and μ_2 be two probability mass functions on \mathcal{X} . Then*

$$I((1-t)\mu_1 + t\mu_2, \nu) \geq (1-t)I(\mu_1, \nu) + tI(\mu_2, \nu) \quad (0 \leq t \leq 1).$$

Here the left-hand side is the mutual information of the joint distribution $(1-t)\mu_1(x)\nu(y|x) + t\mu_2(x)\nu(y|x)$.

- (2) Suppose that for each $x \in \mathcal{X}$ we are given two probability mass functions $\nu_1(\cdot|x)$ and $\nu_2(\cdot|x)$ on \mathcal{Y} . Let μ be a probability mass function on \mathcal{X} . Then

$$I(\mu, (1-t)\nu_1 + t\nu_2) \leq (1-t)I(\mu, \nu_1) + tI(\mu, \nu_2) \quad (0 \leq t \leq 1).$$

Here the left-hand side is the mutual information of the joint distribution $(1-t)\mu(x)\nu_1(y|x) + t\mu(x)\nu_2(y|x)$.

Proof. See [CT, Theorem 2.7.4] for the detailed proof. Here we sketch the outline. First we explain (1).

$$I(\mu, \nu) = I(X; Y) = H(Y) - H(Y|X).$$

If $\nu(y|x)$ is fixed, $H(Y)$ is a concave function of $\mu(x)$ and $H(Y|X)$ is a linear function of $\mu(x)$. The difference $I(\mu, \nu)$ is a concave function of $\mu(x)$.

Next we explain (2). The function $\phi(t) = t \log t$ is convex. So

$$\phi\left(\frac{a+a'}{b+b'}\right) \leq \frac{b}{b+b'}\phi\left(\frac{a}{b}\right) + \frac{b'}{b+b'}\phi\left(\frac{a'}{b'}\right)$$

for positive numbers a, a', b, b' . This leads to

$$(2.7) \quad (a+a') \log \frac{a+a'}{b+b'} \leq a \log \frac{a}{b} + a' \log \frac{a'}{b'}.$$

Set $\sigma_i(y) = \sum_{x \in \mathcal{X}} \mu(x)\nu_i(y|x)$ for $i = 1, 2$. Then $I(\mu, (1-t)\nu_1 + t\nu_2)$ is given by

$$\sum_{x,y} \{(1-t)\mu(x)\nu_1(y|x) + t\mu(x)\nu_2(y|x)\} \log \frac{(1-t)\mu(x)\nu_1(y|x) + t\mu(x)\nu_2(y|x)}{(1-t)\mu(x)\sigma_1(y) + t\mu(x)\sigma_2(y)}.$$

Applying the inequality (2.7) to each summand, $I(\mu, (1-t)\nu_1 + t\nu_2)$ is bounded by

$$\sum_{x,y} (1-t)\mu(x)\nu_1(y|x) \log \frac{\mu(x)\nu_1(y|x)}{\mu(x)\sigma_1(y)} + \sum_{x,y} t\mu(x)\nu_2(y|x) \log \frac{\mu(x)\nu_2(y|x)}{\mu(x)\sigma_2(y)}.$$

This is equal to $(1-t)I(\mu, \nu_1) + tI(\mu, \nu_2)$. \square

Example 2.11 (Continuation of Example 1.1). Here we sketch the proof of the estimate (1.8) in Example 1.1. Note that this is not used for the proofs of Theorems 1.5 and 1.9. We use the notations in Example 1.1. It is easy to prove

$$\limsup_{\varepsilon \rightarrow 0} \frac{R_\mu(\varepsilon)}{|\log \varepsilon|} \leq 1.$$

See Lemma 3.1 below for the details. The main issue is a lower bound on $R_\mu(\varepsilon)$. Let X and $Y = (Y_0, \dots, Y_{n-1})$ be random variables defined

on some probability space such that X has distribution μ and Y_k take values in $[0, 1]^{\mathbb{Z}}$ satisfying the distortion condition (1.3). We write $X = (X_m)_{m \in \mathbb{Z}}$ and $Y_k = (Y_{k,m})_{m \in \mathbb{Z}}$.

(2.8)

$$\begin{aligned}
I(X; Y) &\geq I((X_0, \dots, X_{n-1}); (Y_{0,0}, Y_{1,0}, \dots, Y_{n-1,0})) \\
&\quad \text{(by data-processing inequality; see Lemma 2.2)} \\
&\geq \sum_{m=0}^{n-1} I(X_m; (Y_{0,0}, Y_{1,0}, \dots, Y_{n-1,0})) \\
&\quad \text{(since } X_0, \dots, X_{n-1} \text{ are independent; see Lemma 2.9)} \\
&\geq \sum_{m=0}^{n-1} I(X_m; Y_{m,0}) \quad \text{(by data-processing inequality).}
\end{aligned}$$

It follows from the distortion condition (1.3) that

$$(2.9) \quad \frac{1}{n} \sum_{m=0}^{n-1} \mathbb{E}|X_m - Y_{m,0}| \leq \frac{1}{n} \mathbb{E} \left(\sum_{m=0}^{n-1} d(T^m X, Y_m) \right) < \varepsilon.$$

We denote by $r(\varepsilon)$ the infimum of the mutual information $I(U; V)$ such that U and V are random variables (defined on some probability space) taking values in $[0, 1]$ satisfying

- U obeys the Lebesgue measure.
- V satisfies $\mathbb{E}|U - V| \leq \varepsilon$.

The convexity/concavity properties of mutual information, specifically Lemma 2.10.(2), imply that $r(\varepsilon)$ is a convex function in ε (c.f. [CT, Lemma 10.4.1].) Thus it follows from (2.8) and (2.9) that

$$\frac{I(X; Y)}{n} \geq \frac{1}{n} \sum_{m=0}^{n-1} r(\mathbb{E}|X_m - Y_{m,0}|) \geq r \left(\frac{1}{n} \sum_{m=0}^{n-1} \mathbb{E}|X_m - Y_{m,0}| \right) \geq r(\varepsilon),$$

and hence $R_\mu(\varepsilon) \geq r(\varepsilon)$. Then $R_\mu(\varepsilon) \sim |\log \varepsilon|$ follows from the next claim.

Claim 2.12.

$$r(\varepsilon) \sim |\log \varepsilon| \quad (\varepsilon \rightarrow 0).$$

Proof. It is again easy to prove $\limsup_{\varepsilon \rightarrow 0} r(\varepsilon)/|\log \varepsilon| \leq 1$. So we prove a lower bound on $r(\varepsilon)$. Let U and V be random variables in the above definition of $r(\varepsilon)$. Fix $D > 1$ and set $l = \lfloor 1/(D\varepsilon) \rfloor$. We define a partition \mathcal{P} of $[0, 1]$ by

$$\mathcal{P} = \{[0, D\varepsilon), [D\varepsilon, 2D\varepsilon), [2D\varepsilon, 3D\varepsilon), \dots, [lD\varepsilon, 1]\}.$$

For $u \in [0, 1]$ we denote by $\mathcal{P}(u)$ the atom of \mathcal{P} containing u . It follows from $\mathbb{E}|U - V| \leq \varepsilon$ that

$$\mathbb{P}(|U - V| \geq D\varepsilon) \leq \frac{\mathbb{E}|U - V|}{D\varepsilon} \leq \frac{1}{D}.$$

By the data-processing inequality

$$I(U; V) \geq I(\mathcal{P}(U); V) = H(\mathcal{P}(U)) - H(\mathcal{P}(U)|V).$$

Under the condition $|U - V| < D\varepsilon$, if we know V then the number of possibilities of $\mathcal{P}(U)$ is at most three. This implies

$$H(\mathcal{P}(U)|V) \leq \log 3 + \mathbb{P}(|U - V| \geq D\varepsilon) \log(l + 1) \leq \log 3 + \frac{\log(l + 1)}{D}.$$

Since U obeys the Lebesgue measure, $H(\mathcal{P}(U))$ is bounded from below by

$$l(D\varepsilon) \log(1/D\varepsilon) \geq (1 - D\varepsilon) \log(1/D\varepsilon).$$

Thus

$$r(\varepsilon) \geq (1 - D\varepsilon) \log(1/D\varepsilon) - \frac{\log(1 + \lfloor 1/(D\varepsilon) \rfloor)}{D} - \log 3.$$

It follows

$$\liminf_{\varepsilon \rightarrow 0} \frac{r(\varepsilon)}{|\log \varepsilon|} \geq 1 - \frac{1}{D}.$$

Letting $D \rightarrow \infty$ we get $\liminf_{\varepsilon \rightarrow 0} r(\varepsilon)/|\log \varepsilon| \geq 1$. \square

3. PROOF OF THEOREM 1.5

In this section we prove Theorem 1.5. Throughout this section (\mathcal{X}, T) is a dynamical system, and d a metric on \mathcal{X} . Recall that for $n \geq 1$ we defined the distance d_n on \mathcal{X} by

$$d_n(x, y) = \max_{0 \leq k < n} d(T^k x, T^k y).$$

We define another distance \bar{d}_n on \mathcal{X} by

$$\bar{d}_n(x, y) = \frac{1}{n} \sum_{k=0}^{n-1} d(T^k x, T^k y).$$

Obviously $\bar{d}_n(x, y) \leq d_n(x, y)$. For $\varepsilon > 0$ we set

$$\tilde{S}(\mathcal{X}, T, d, \varepsilon) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \#(\mathcal{X}, \bar{d}_n, \varepsilon).$$

This limit exists because $\log \#(\mathcal{X}, \bar{d}_n, \varepsilon)$ is a subadditive function of n . We have

$$(3.1) \quad \tilde{S}(\mathcal{X}, T, d, \varepsilon) \leq S(\mathcal{X}, T, d, \varepsilon) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \#(\mathcal{X}, d_n, \varepsilon).$$

3.1. Metric mean dimension dominates rate distortion functions.

Lemma 3.1. *For $\varepsilon > 0$ and every invariant probability measure μ on \mathcal{X} we have*

$$R_\mu(\varepsilon) \leq \tilde{S}(\mathcal{X}, T, d, \varepsilon) \leq S(\mathcal{X}, T, d, \varepsilon).$$

Proof. Let $n > 0$, and let $\{U_1, \dots, U_K\}$ be an open covering of \mathcal{X} such that every U_k has diameter smaller than ε with respect to the distance \bar{d}_n . We choose a point $p_k \in U_k$ for each k . We define a map $f : \mathcal{X} \rightarrow \{p_1, \dots, p_K\}$ by setting $f(x) = p_k$ where k is the smallest number satisfying $x \in U_k$. Obviously $\bar{d}_n(x, f(x)) < \varepsilon$. Let X be a random variable obeying μ . We set $Y = (f(X), Tf(X), \dots, T^{n-1}f(X))$. This satisfies the distortion condition (1.3):

$$\mathbb{E} \left(\frac{1}{n} \sum_{k=0}^{n-1} d(T^k X, T^k f(X)) \right) = \mathbb{E} \bar{d}_n(X, f(X)) < \varepsilon.$$

The mutual information $I(X; Y)$ is bounded by

$$I(X; Y) \leq H(Y) \leq \log K,$$

where the second inequality holds because Y takes at most K values. This shows $R_\mu(\varepsilon) \leq \tilde{S}(\mathcal{X}, T, d, \varepsilon)$. \square

Lemma 3.1 immediately implies one direction of Theorem 1.5:

$$(3.2) \quad \overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) \geq \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_\mu(\varepsilon)}{|\log \varepsilon|}.$$

The case of $\underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d)$ is the same. Notice that we have not used Condition 1.2 so far.

3.2. Condition 1.2 implies that d_n and \bar{d}_n look the same. This subsection is the only place where Condition 1.2 plays a role. We set $[n] = \{0, 1, 2, \dots, n-1\}$. For a finite subset $A \subset \mathbb{Z}$ we define $d_A(x, y) = \max_{a \in A} d(T^a x, T^a y)$ for $x, y \in \mathcal{X}$. In particular $d_n = d_{[n]}$.

Lemma 3.2. *For any natural number n and any real numbers $\varepsilon > 0$ and $L > 1$ we have*

$$\frac{1}{n} \log \#(\mathcal{X}, d_n, 2L\varepsilon) \leq \log 2 + \frac{1}{L} \log \#(\mathcal{X}, d, \varepsilon) + \frac{1}{n} \log \#(\mathcal{X}, \bar{d}_n, \varepsilon).$$

Proof. Let $X = W_1 \cup \dots \cup W_M$ be an open covering such that $\text{diam}(W_m, d) < \varepsilon$ for all $1 \leq m \leq M$ and $M = \#(\mathcal{X}, d, \varepsilon)$. We also take an open covering $X = U_1 \cup \dots \cup U_N$ such that $\text{diam}(U_i, \bar{d}_n) < \varepsilon$ for all $1 \leq i \leq N$ and $N = \#(\mathcal{X}, \bar{d}_n, \varepsilon)$.

We choose a point $p_i \in U_i$ for each $1 \leq i \leq N$. Every point $x \in U_i$ satisfies $\bar{d}_n(x, p_i) < \varepsilon$, and hence

$$|\{0 \leq k \leq n-1 \mid d(T^k x, T^k p_i) \geq L\varepsilon\}| < \frac{n}{L}.$$

It follows that U_i is contained in the union of the open balls

$$B_{L\varepsilon}(p_i, d_{[n] \setminus A}),$$

where A runs over subsets of $[n] = \{0, 1, 2, \dots, n-1\}$ satisfying $|A| < n/L$. For $A = \{k_1, \dots, k_a\} \subset [n]$ with $a < n/L$, the ball $B_{L\varepsilon}(p_i, d_{[n] \setminus A})$ is equal to the union of

(3.3)

$$B_{L\varepsilon}(p_i, d_{[n] \setminus A}) \cap T^{-k_1} W_{m_1} \cap \dots \cap T^{-k_a} W_{m_a}, \quad (1 \leq m_1, \dots, m_a \leq M).$$

The sets (3.3) have diameter less than $2L\varepsilon$ with respect to the distance d_n . Hence

$$\#(B_{L\varepsilon}(p_i, d_{[n] \setminus A}), d_n, 2L\varepsilon) \leq M^a \leq M^{n/L}.$$

There are N choices of U_i and 2^n choices of $A \subset [n]$. Thus

$$\#(X, d_n, 2L\varepsilon) \leq 2^n M^{n/L} N.$$

This proves the statement. \square

Lemma 3.3. *Under Condition 1.2,*

$$\begin{aligned} \overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \limsup_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|}, \\ \underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) &= \liminf_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|}. \end{aligned}$$

Proof. We prove the equality for $\overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d)$. The case of $\underline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d)$ is the same. From $S(\mathcal{X}, T, d, \varepsilon) \geq \tilde{S}(\mathcal{X}, T, d, \varepsilon)$, the inequality

$$\overline{\text{mdim}}_{\text{M}}(\mathcal{X}, T, d) = \limsup_{\varepsilon \rightarrow 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} \geq \limsup_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|}$$

is obvious. Take $0 < \delta < 1$ and apply Lemma 3.2 with $L = (1/\varepsilon)^\delta$. Then we get

$$\frac{1}{n} \log \#(\mathcal{X}, d_n, 2\varepsilon^{1-\delta}) \leq \log 2 + \frac{\log \#(\mathcal{X}, d, \varepsilon)}{(1/\varepsilon)^\delta} + \frac{1}{n} \log \#(\mathcal{X}, \bar{d}_n, \varepsilon).$$

Letting $n \rightarrow \infty$

$$S(\mathcal{X}, T, d, 2\varepsilon^{1-\delta}) \leq \log 2 + \varepsilon^\delta \log \#(\mathcal{X}, d, \varepsilon) + \tilde{S}(\mathcal{X}, T, d, \varepsilon).$$

By Condition 1.2, the second term in the right-hand side goes to zero as $\varepsilon \rightarrow 0$ (this is the only place where we use Condition 1.2). It follows that

$$(1 - \delta) \cdot \overline{\text{mdim}}_{\mathbf{M}}(\mathcal{X}, T, d) \leq \limsup_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|}.$$

Letting $\delta \rightarrow 0$, we get the statement. \square

3.3. Completion of the proof of Theorem 1.5. For $n \geq 1$ we define a distance \bar{d}_n on \mathcal{X}^n by

$$\bar{d}_n((x_0, \dots, x_{n-1}), (y_0, \dots, y_{n-1})) = \frac{1}{n} \sum_{k=0}^{n-1} d(x_k, y_k).$$

In particular

$$\bar{d}_n(x, y) = \bar{d}_n((x, Tx, \dots, T^{n-1}x), (y, Ty, \dots, T^{n-1}y)) \quad (x, y \in \mathcal{X}).$$

Proposition 3.4. *For any real numbers $\varepsilon > 0$ and $D > 2$ there exists an invariant probability measure μ on \mathcal{X} satisfying*

$$R_\mu(\varepsilon) \geq \left(1 - \frac{1}{D}\right) \tilde{S}(\mathcal{X}, T, d, (12D + 4)\varepsilon).$$

Proof. For each $n \geq 1$ we choose $S_n \subset \mathcal{X}$ a maximal $(6D+2)\varepsilon$ -separated set with respect to the distance \bar{d}_n . It follows

$$(3.4) \quad |S_n| \geq \#(\mathcal{X}, \bar{d}_n, (12D + 4)\varepsilon).$$

Let ν_n be the uniform distribution over S_n :

$$\nu_n = \frac{1}{|S_n|} \sum_{p \in S_n} \delta_p.$$

Set

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \nu_n.$$

We can choose a subsequence $\{\mu_{n_i}\}_{i=1}^\infty$ converging to an invariant probability measure μ in the weak* topology. We prove that this μ satisfies the statement.

We choose a partition $\mathcal{P} = \{P_1, \dots, P_K\}$ of \mathcal{X} such that

- Every P_k has diameter smaller than ε with respect to the distance d .
- $\mu(\partial P_k) = 0$ for all $1 \leq k \leq K$.

We choose a point $p_k \in P_k$ for each $1 \leq k \leq K$. Set $A = \{p_1, \dots, p_K\}$. We define a map $\mathcal{P} : \mathcal{X} \rightarrow A$ by $\mathcal{P}(x) = p_k$ for $x \in P_k$. It follows that

$$(3.5) \quad d(x, \mathcal{P}(x)) < \varepsilon.$$

For $n \geq 1$ we set

$$\mathcal{P}^n(x) = (\mathcal{P}(x), \mathcal{P}(Tx), \dots, \mathcal{P}(T^{n-1}x)).$$

Claim 3.5. (1) *The set*

$$\mathcal{P}^n(S_n) = \{\mathcal{P}^n(x) \mid x \in S_n\}$$

is a $6D\varepsilon$ -separated set with respect to the distance \bar{d}_n .

(2) *The push-forward measure $\mathcal{P}_*^n \nu_n$ is the uniform distribution over $\mathcal{P}^n(S_n)$. Moreover $|\mathcal{P}^n(S_n)| = |S_n|$.*

Proof. By (3.5) we have $\bar{d}_n((x, Tx, \dots, T^{n-1}x), \mathcal{P}^n(x)) < \varepsilon$. For any two distinct points x, y in S_n , the distance $\bar{d}_n(\mathcal{P}^n(x), \mathcal{P}^n(y))$ is bounded from below by

$$\begin{aligned} & \bar{d}_n(x, y) - \bar{d}_n((x, Tx, \dots, T^{n-1}x), \mathcal{P}^n(x)) - \bar{d}_n((y, Ty, \dots, T^{n-1}y), \mathcal{P}^n(y)) \\ & \geq (6D + 2)\varepsilon - 2\varepsilon = 6D\varepsilon. \end{aligned}$$

This proves part (1) of the claim. Moreover it shows that the map

$$S_n \ni x \mapsto \mathcal{P}^n(x) \in \mathcal{P}^n(S_n)$$

is bijective. Since ν_n is uniformly distributed over S_n , the measure $\mathcal{P}_*^n \nu_n$ is uniformly distributed over $\mathcal{P}^n(S_n)$. This establishes part (2). \square

Consider random variables X and $Y = (Y_0, \dots, Y_{m-1})$ defined on a probability space (Ω, \mathbb{P}) such that $\text{Law}(X) = \mu$ and Y_i take values in \mathcal{X} with

$$(3.6) \quad \mathbb{E} \left(\frac{1}{m} \sum_{i=0}^{m-1} d(T^i X, Y_i) \right) < \varepsilon.$$

We estimate the mutual information $I(X; Y)$ from below. As in Remark 2.3, we can assume that the distribution of Y is supported on a finite set $\mathcal{Y} \subset \mathcal{X}^m$. From the data-processing inequality (Lemma 2.2)

$$I(X; Y) \geq I(\mathcal{P}^m(X); Y).$$

So it is enough to estimate $I(\mathcal{P}^m(X); Y)$ from below. Let $\tau = \text{Law}(\mathcal{P}^m(X), Y)$ be the law of $(\mathcal{P}^m(X), Y)$, which is a probability measure on $A^m \times \mathcal{Y}$.

It follows that

$$(3.7) \quad \begin{aligned} \int_{A^m \times \mathcal{Y}} \bar{d}_m(x, y) d\tau(x, y) &= \mathbb{E} \left(\frac{1}{m} \sum_{i=0}^{m-1} d(\mathcal{P}(T^i X), Y_i) \right) \\ &\leq \varepsilon + \mathbb{E} \left(\frac{1}{m} \sum_{i=0}^{m-1} d(T^i X, Y_i) \right) < 2\varepsilon. \end{aligned}$$

Here we used $d(\mathcal{P}(T^i X), T^i X) < \varepsilon$ and (3.6).

For each $n \geq 1$ we choose a probability measure π_n on $A^m \times A^m$ such that

- π_n is a coupling of $(\mathcal{P}_*^m \mu_n, \mathcal{P}_*^m \mu)$, namely its first and second marginals are $\mathcal{P}_*^m \mu_n$ and $\mathcal{P}_*^m \mu$ respectively.
- π_n minimizes the integral

$$\int_{A^m \times A^m} \bar{d}_m(x, y) d\pi(x, y)$$

among all couplings π of $(\mathcal{P}_*^m \mu_n, \mathcal{P}_*^m \mu)$.

(These two conditions means that π_n is an optimal transference plan in the language of Optimal Transport.)

Claim 3.6. *The sequence π_{n_i} converges to $(\mathcal{P}^m \times \mathcal{P}^m)_* \mu$ in the weak* topology.*

Proof. Since $\mu(\partial P_k) = 0$, the sequence $\mathcal{P}_*^m \mu_{n_i}$ converges to $\mathcal{P}_*^m \mu$. Then the statement becomes a very special case of a theorem of optimal transport [Vil, Theorem 5.20]. As all the measures here are supported on finite sets, our situation is simpler than the general setting in [Vil], and we provide a self-contained elementary proof in Lemma A.2 in the Appendix. \square

Both the second marginal of π_n and the first marginal of τ are equal to the measure $\mathcal{P}_*^m \mu$. So we can compose them and produce a coupling τ_n of $(\mathcal{P}_*^m \mu_n, \text{Law}(Y))$. Namely

$$\tau_n(x, y) = \sum_{x' \in A^m} \pi_n(x, x') \mathbb{P}(Y = y | \mathcal{P}^m(X) = x'), \quad (x \in A^m, y \in \mathcal{Y}).$$

Here we identify probability measures with their probability mass functions. From Claim 3.6 the measures τ_{n_i} converge to τ in the weak* topology. In particular, it follows from (3.7) that

$$(3.8) \quad \mathbb{E}_{\tau_{n_i}}(\bar{d}_m(x, y)) := \int_{A^m \times \mathcal{Y}} \bar{d}_m(x, y) d\tau_{n_i}(x, y) < 2\varepsilon$$

for all sufficiently large n_i .

We define a conditional probability mass function $\tau_n(y|x)$ by

$$\tau_n(y|x) = \frac{\tau_n(x, y)}{\mathcal{P}_*^m \mu_n(x)}.$$

This is defined for

$$x \in \bigcup_{k=0}^{n-1} \mathcal{P}^m(T^k S_n), \quad y \in \mathcal{X}^m.$$

Take $n \geq 2m$ and let $n = qm + r$ with $m \leq r \leq 2m - 1$. Fix a point $a \in \mathcal{X}$. We denote by $\delta_a(\cdot)$ the delta probability measure at a on \mathcal{X} . For $x = (x_1, \dots, x_n) \in \mathcal{P}^n(S_n)$ we let x_k^l denote the $(l - k + 1)$ -tuple $x_k^l = (x_k, \dots, x_l)$ for $0 \leq k \leq l < n$. For such an x we define probability mass functions $\sigma_{n,0}(\cdot|x), \dots, \sigma_{n,m-1}(\cdot|x)$ on \mathcal{X}^n as follows:

$$(3.9) \quad \begin{aligned} \sigma_{n,0}(y|x) &= \prod_{j=0}^{q-1} \tau_n(y_{jm}^{jm+m-1} | x_{jm}^{jm+m-1}) \cdot \prod_{k=n-r}^{n-1} \delta_a(y_k), \\ \sigma_{n,1}(y|x) &= \delta_a(y_0) \cdot \prod_{j=0}^{q-1} \tau_n(y_{jm+1}^{jm+m} | x_{jm+1}^{jm+m}) \cdot \prod_{k=n-r+1}^{n-1} \delta_a(y_k), \\ &\dots \\ \sigma_{n,m-1}(y|x) &= \prod_{k=0}^{m-2} \delta_a(y_k) \cdot \prod_{j=0}^{q-1} \tau_n(y_{jm+m-1}^{jm+2m-2} | x_{jm+m-1}^{jm+2m-2}) \cdot \prod_{k=n-r+m-1}^{n-1} \delta_a(y_k). \end{aligned}$$

See Figure 3.1. Finally we set

$$\sigma_n(y|x) = \frac{\sigma_{n,0}(y|x) + \sigma_{n,1}(y|x) + \dots + \sigma_{n,m-1}(y|x)}{m}.$$

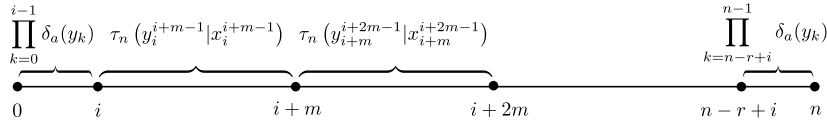


FIGURE 3.1. Definition of $\sigma_{n,i}(y|x)$

Claim 3.7.

$$\frac{1}{m} I(\mathcal{P}_*^m \mu_n, \tau_n) \geq \frac{1}{n} I(\mathcal{P}_*^n \nu_n, \sigma_n).$$

Here $I(\mathcal{P}_*^m \mu_n, \tau_n)$ and $I(\mathcal{P}_*^n \nu_n, \sigma_n)$ are the mutual informations of the probability distributions

$$\mathcal{P}_*^m \mu_n(x) \tau_n(y|x), \quad \mathcal{P}_*^n \nu_n(x) \sigma_n(y|x)$$

respectively.

Proof. We use the concavity/convexity of mutual information (Lemma 2.10). From the convexity

$$I(\mathcal{P}_*^n \nu_n, \sigma_n) \leq \frac{1}{m} \sum_{i=0}^{m-1} I(\mathcal{P}_*^n \nu_n, \sigma_{n,i}).$$

From (3.9) and the subadditivity of mutual information (Lemma 2.8)

$$I(\mathcal{P}_*^n \nu_n, \sigma_{n,i}) \leq \sum_{j=0}^{q-1} I(\mathcal{P}_*^m(T_*^{i+jm} \nu_n), \tau_n).$$

Therefore

$$\begin{aligned} I(\mathcal{P}_*^n \nu_n, \sigma_n) &\leq \frac{1}{m} \sum_{i=0}^{m-1} \sum_{j=0}^{q-1} I(\mathcal{P}_*^m(T_*^{i+jm} \nu_n), \tau_n) \\ &= \frac{1}{m} \sum_{k=0}^{qm-1} I(\mathcal{P}_*^m(T_*^k \nu_n), \tau_n) \\ &\leq \frac{1}{m} \sum_{k=0}^{n-1} I(\mathcal{P}_*^m(T_*^k \nu_n), \tau_n) \\ &\leq \frac{n}{m} I\left(\mathcal{P}_*^m\left(\frac{1}{n} \sum_{k=0}^{n-1} T_*^k \nu_n\right), \tau_n\right) \\ &\quad \text{(by the concavity in Lemma 2.10 (1))} \\ &= \frac{n}{m} I(\mathcal{P}_*^m \mu_n, \tau_n) \quad \text{(by } \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \nu_n \text{).} \end{aligned}$$

We would like to remark that the above calculation is quite analogous to Misiurewicz's proof [Mis] of the standard variational principle. \square

Claim 3.8. We denote by $\mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n}(\bar{d}_n(x, y))$ the expected value of $\bar{d}_n(x, y)$ ($x, y \in \mathcal{X}^n$) with respect to the probability measure

$$\mathcal{P}_*^n \nu_n(x) \sigma_n(y|x).$$

Then $\mathbb{E}_{\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}} (\bar{d}_{n_i}(x, y)) < 3\varepsilon$ for sufficiently large n_i . Moreover

$$(3.10) \quad I(\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}) \geq \left(1 - \frac{1}{D}\right) \log |S_{n_i}| - H(1/D) \quad \text{for sufficiently large } n_i.$$

Proof.

$$\mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} (\bar{d}_n(x, y)) = \frac{1}{m} \sum_{i=0}^{m-1} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_{n,i}} (\bar{d}_n(x, y)).$$

By (3.9)

$$\frac{1}{m} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_{n,i}} (\bar{d}_n(x, y)) \leq \frac{1}{n} \sum_{j=0}^{q-1} \mathbb{E}_{\mathcal{P}_*^m (T^{i+jm})_* \nu_n, \tau_n} (\bar{d}_m(x', y')) + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{mn}.$$

Here x, y are random points in \mathcal{X}^n , whereas x', y' are in \mathcal{X}^m . Therefore

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} (\bar{d}_n(x, y)) &\leq \frac{1}{n} \sum_{i=0}^{m-1} \sum_{j=0}^{q-1} \mathbb{E}_{\mathcal{P}_*^m (T^{i+jm})_* \nu_n, \tau_n} (\bar{d}_m(x', y')) + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n} \\ &= \frac{1}{n} \sum_{k=0}^{qm-1} \mathbb{E}_{\mathcal{P}_*^m (T_*^k \nu_n), \tau_n} (\bar{d}_m(x', y')) + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n} \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}_{\mathcal{P}_*^m (T_*^k \nu_n), \tau_n} (\bar{d}_m(x', y')) + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n} \\ &= \mathbb{E}_{\mathcal{P}_*^m \mu_n, \tau_n} (\bar{d}_m(x', y')) + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n}. \end{aligned}$$

In the last line we used $\mu_n = (1/n) \sum_{k=0}^{n-1} T_*^k \nu_n$. As a conclusion,

$$\mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} (\bar{d}_n(x, y)) \leq \int_{A^m \times \mathcal{Y}} \bar{d}_m(x', y') d\tau_n(x', y') + \frac{r \cdot \text{diam}(\mathcal{X}, d)}{n}.$$

By (3.8) and $r \leq 2m - 1$, this is bounded by 3ε for sufficiently large $n = n_i$.

By Claim 3.5, $\mathcal{P}_*^n \nu_n$ is uniformly distributed over $\mathcal{P}^n(S_n)$, which is a $(6D\varepsilon)$ -separated set of cardinality $|S_n|$. Then (3.10) follows from Corollary 2.5. \square

We conclude that for sufficiently large n_i

$$\begin{aligned}
\frac{1}{m} I(\mathcal{P}_*^m \mu_{n_i}, \tau_{n_i}) &\geq \frac{1}{n_i} I(\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}) \quad (\text{by Claim 3.7}) \\
&\geq \left(1 - \frac{1}{D}\right) \frac{\log |S_{n_i}|}{n_i} - \frac{H(1/D)}{n_i} \quad (\text{by Claim 3.8}) \\
&\geq \left(1 - \frac{1}{D}\right) \frac{\log \#(\mathcal{X}, \bar{d}_{n_i}, (12D+4)\varepsilon)}{n_i} - \frac{H(1/D)}{n_i} \\
&\quad (\text{by (3.4)}).
\end{aligned}$$

The probability measures $\tau_{n_i}(x, y)$ converge to $\tau = \text{Law}(\mathcal{P}^m(X), Y)$ in the weak* topology. Therefore it follows from Lemma 2.1 that

$$\frac{1}{m} I(\mathcal{P}^m(X); Y) \geq \left(1 - \frac{1}{D}\right) \tilde{S}(\mathcal{X}, T, d, (12D+4)\varepsilon).$$

From the data-processing inequality (Lemma 2.2)

$$\frac{1}{m} I(X; Y) \geq \left(1 - \frac{1}{D}\right) \tilde{S}(\mathcal{X}, T, d, (12D+4)\varepsilon).$$

This proves the statement. \square

Lemma 3.1 and Proposition 3.4 immediately imply:

Corollary 3.9.

$$\begin{aligned}
\limsup_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} &= \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_\mu(\varepsilon)}{|\log \varepsilon|} \\
\liminf_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} &= \liminf_{\varepsilon \rightarrow 0} \frac{\sup_{\mu \in \mathcal{M}^T(\mathcal{X})} R_\mu(\varepsilon)}{|\log \varepsilon|}.
\end{aligned}$$

Theorem 1.5 follows from Lemma 3.3 and Corollary 3.9.

4. PROOF OF THEOREM 1.9

Here we prove Theorem 1.9. The proof is very close to that of Theorem 1.5, and in view of this our explanation is more concise. Throughout this section, (\mathcal{X}, T) is a dynamical system with a distance d . For $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$ in \mathcal{X}^n we set

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(x_i, y_i).$$

Lemma 4.1. *For every $\varepsilon > 0$ and every invariant probability measure μ on \mathcal{X} we have*

$$\tilde{R}_\mu(\varepsilon) \leq S(\mathcal{X}, T, d, \varepsilon).$$

Proof. Let $n > 0$ and choose an open covering $\{U_1, \dots, U_K\}$ of \mathcal{X} such that every U_k has diameter less than ε with respect to d_n . Choose a point $p_k \in U_k$ for each k . We define $f : \mathcal{X} \rightarrow \{p_1, \dots, p_K\}$ by $f(x) = p_k$ where k is the smallest integer satisfying $x \in U_k$. Then $d_n(x, f(x)) < \varepsilon$. Let X be a random variable obeying μ , and set $Y = (f(X), Tf(X), \dots, T^{n-1}f(X))$. We have $d_n(X, f(X)) < \varepsilon$ almost surely. It follows that

$$\mathbb{E}(\text{the number of } i \in [0, n-1] \text{ with } d(T^i X, T^i f(X)) \geq \varepsilon) = 0.$$

Thus (X, Y) satisfies the distortion condition (1.11) for any $\alpha > 0$. Since Y takes at most K values

$$I(X; Y) \leq H(Y) \leq \log K.$$

This proves the statement. \square

Proposition 4.2. *For any positive number ε there exists an invariant probability measure μ on \mathcal{X} satisfying*

$$\tilde{R}_\mu(\varepsilon) \geq S(\mathcal{X}, T, d, 12\varepsilon).$$

Proof. For each $n \geq 1$ we take a maximal 6ε -separated set $S_n \subset \mathcal{X}$ with respect to the distance d_n . It follows $|S_n| \geq \#(\mathcal{X}, d_n, 12\varepsilon)$. Let ν_n be the uniform distribution over S_n and set

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \nu_n.$$

Choose a subsequence $\{n_i\}$ so that μ_{n_i} converges to $\mu \in \mathcal{M}^T(\mathcal{X})$ in the weak* topology. We prove that this μ satisfies the statement. For $n \geq 1$, $x = (x_0, \dots, x_{n-1})$ and $y = (y_0, \dots, y_{n-1})$ in \mathcal{X}^n we set

$$f_n(x, y) = \text{the number of } k \in [0, n-1] \text{ satisfying } d(x_k, y_k) \geq 2\varepsilon.$$

Here we chose “ 2ε ” for the later convenience.

We take a partition $\mathcal{P} = \{P_1, \dots, P_K\}$ such that $\text{diam}(P_k, d) < \varepsilon$ and $\mu(\partial P_k) = 0$ for all $1 \leq k \leq K$. Choose a point $p_k \in P_k$ for each k and set $A = \{p_1, \dots, p_K\}$. We define a map $\mathcal{P} : \mathcal{X} \rightarrow A$ by $\mathcal{P}(P_k) = \{p_k\}$. We have $d(x, \mathcal{P}(x)) < \varepsilon$ for all $x \in \mathcal{X}$. For $n \geq 1$ we set $\mathcal{P}^n(x) = (\mathcal{P}(x), \mathcal{P}(Tx), \dots, \mathcal{P}(T^{n-1}x))$.

Claim 4.3. (1) *The set $\mathcal{P}^n(S_n)$ is 4ε -separated with respect to the distance d_n .*

(2) *The measure $\mathcal{P}_*^n \nu_n$ is uniformly distributed over $\mathcal{P}^n(S_n)$ and $|\mathcal{P}^n(S_n)| = |S_n|$.*

Proof. See Claim 3.5. \square

Let $0 < \alpha < 1/4$. Let X and $Y = (Y_0, \dots, Y_{m-1})$ be random variables such that $\text{Law}(X) = \mu$, and Y_i take values in \mathcal{X} and satisfy

$$\mathbb{E} \left(\text{the number of } 0 \leq i \leq m-1 \text{ satisfying } d(T^i X, Y_i) \geq \varepsilon \right) < \alpha m.$$

We estimate $I(X; Y) \geq I(\mathcal{P}^m(X); Y)$ from below. As in Remark 2.3, we can assume that the distribution of Y is supported on a finite set $\mathcal{Y} \subset \mathcal{X}^m$. Set $\tau = \text{Law}(\mathcal{P}^m(X), Y)$, which is a probability measure on $A^m \times \mathcal{Y}$. Since $d(T^i X, \mathcal{P}(T^i X)) < \varepsilon$, it follows that

$$\{0 \leq i \leq m-1 \mid d(\mathcal{P}(T^i X), Y_i) \geq 2\varepsilon\} \subset \{0 \leq i \leq m-1 \mid d(T^i X, Y_i) \geq \varepsilon\}.$$

Thus

$$\begin{aligned} \mathbb{E}_\tau f_m(x, y) &:= \int_{A^m \times \mathcal{Y}} f_m(x, y) d\tau(x, y) \\ &= \mathbb{E} \left(\text{the number of } 0 \leq i \leq m-1 \text{ s.t. } d(\mathcal{P}(T^i X), Y_i) \geq 2\varepsilon \right) \\ &< \alpha m. \end{aligned}$$

For each $n \geq 1$ we take a coupling π_n of $(\mathcal{P}_*^m \mu_n, \mathcal{P}_*^m \mu)$ which minimizes

$$\int_{A^m \times A^m} d_m(x, y) d\pi(x, y)$$

among all couplings π of $(\mathcal{P}_*^m \mu_n, \mathcal{P}_*^m \mu)$. As in Claim 3.6 in Section 3, it follows from $\mu(\partial P_k) = 0$ and Lemma A.2 in Appendix that the measures π_{n_i} converge to $(\mathcal{P}^m \times \mathcal{P}^m)_* \mu$ in the weak* topology. We define a coupling τ_n of $(\mathcal{P}_*^m \mu_n, \text{Law}(Y))$ by composing π_n and τ :

$$\tau_n(x, y) = \sum_{x' \in A^m} \pi_n(x, x') \mathbb{P}(Y = y \mid \mathcal{P}^m(X) = x'), \quad (x \in A^m, y \in \mathcal{Y}).$$

τ_{n_i} converges to τ in the weak* topology. In particular

$$(4.1) \quad \mathbb{E}_{\tau_{n_i}} f_m(x, y) = \int_{A^m \times \mathcal{Y}} f_m(x, y) d\tau_{n_i}(x, y) < \alpha m \quad \text{for sufficiently large } n_i.$$

(Here notice that $f_m(x, y)$ is a *continuous* function on $A^m \times \mathcal{Y}$ because $A^m \times \mathcal{Y}$ is a finite set.) We define a conditional probability mass function $\tau_n(y|x)$ by

$$\tau_n(y|x) = \frac{\tau_n(x, y)}{\mathcal{P}_*^m \mu_n(x)},$$

which is defined for

$$x \in \bigcup_{k=0}^{n-1} \mathcal{P}^m(T^k S_n), \quad y \in \mathcal{X}^m.$$

Fix a point $a \in \mathcal{X}$. For $n \geq 2m$, let $n = mq + r$ with $m \leq r \leq 2m - 1$. For $x \in \mathcal{P}^n(S_n)$ we define probability mass functions $\sigma_{n,i}(\cdot|x)$ ($0 \leq i \leq m - 1$) on \mathcal{X}^n as in (3.9):

$$\sigma_{n,i}(y|x) = \prod_{j=0}^{q-1} \tau_n(y_{i+jm}^{i+jm+m-1} | x_{i+jm}^{i+jm+m-1}) \cdot \prod_{k \in [0,i) \cup [n-r+i,n)} \delta_a(y_k).$$

We set

$$\sigma_n(y|x) = \frac{1}{m} \sum_{i=0}^{m-1} \sigma_{n,i}(y|x).$$

Exactly as in Claim 3.7

$$(4.2) \quad \frac{1}{m} I(\mathcal{P}_*^m \mu_n, \tau_n) \geq \frac{1}{n} I(\mathcal{P}_*^n \nu_n, \sigma_n).$$

Claim 4.4. *We denote by $\mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} f_n(x, y)$ the expected value of the function $f_n(x, y)$ (i.e. the number of $k \in [0, n-1]$ satisfying $d(x_k, y_k) \geq 2\varepsilon$) with respect to the measure*

$$\mathcal{P}_*^n \nu_n(x) \sigma_n(y|x).$$

Then for sufficiently large n_i

$$\mathbb{E}_{\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}} f_{n_i}(x, y) < 2\alpha n_i.$$

Proof.

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} f_n(x, y) &= \frac{1}{m} \sum_{i=0}^{m-1} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_{n,i}} f_n(x, y). \\ \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_{n,i}} f_n(x, y) &\leq r + \sum_{j=0}^{q-1} \mathbb{E}_{\mathcal{P}_*^m T_*^{i+jm} \nu_n, \tau_n} f_m(x, y). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}_{\mathcal{P}_*^n \nu_n, \sigma_n} f_n(x, y) &\leq r + \frac{1}{m} \sum_{i=0}^{m-1} \sum_{j=0}^{q-1} \mathbb{E}_{\mathcal{P}_*^m T_*^{i+jm} \nu_n, \tau_n} f_m(x, y) \\ &\leq r + \frac{1}{m} \sum_{k=0}^{n-1} \mathbb{E}_{\mathcal{P}_*^m T_*^k \nu_n, \tau_n} f_m(x, y) \\ &= r + \frac{n}{m} \mathbb{E}_{\mathcal{P}_*^m \mu_n, \tau_n} f_m(x, y) \quad (\text{by } \mu_n = \frac{1}{n} \sum_{k=0}^{n-1} T_*^k \nu_n) \\ &= r + \frac{n}{m} \int_{A^m \times \mathcal{Y}} f_m(x, y) d\tau_n(x, y) = r + \frac{n}{m} \mathbb{E}_{\tau_n} f_m(x, y). \end{aligned}$$

We have $\mathbb{E}_{\tau_n} f_m(x, y) < \alpha m$ for sufficiently large $n = n_i$ by (4.1). Thus (by $r \leq 2m - 1$)

$$\mathbb{E}_{\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}} f_{n_i}(x, y) < \alpha n_i + r < 2\alpha n_i$$

for sufficiently large n_i . \square

In view of Claim 4.3, Claim 4.4 and Lemma 2.6 imply that for sufficiently large n_i ,

$$(4.3) \quad \frac{1}{n_i} I(\mathcal{P}_*^{n_i} \nu_{n_i}, \sigma_{n_i}) \geq \frac{1}{n_i} \log |S_{n_i}| - 2\alpha \log K - H(2\alpha).$$

It follows from $|S_n| \geq \#(\mathcal{X}, d_n, 12\varepsilon)$ and the inequalities (4.2) and (4.3) that

$$\frac{1}{m} I(\mathcal{P}_*^m \mu_{n_i}, \tau_{n_i}) \geq \frac{1}{n_i} \log \#(\mathcal{X}, d_{n_i}, 12\varepsilon) - 2\alpha \log K - H(2\alpha)$$

for sufficiently large n_i . Recall that the measures $\tau_{n_i}(x, y)$ converge to $\tau = \text{Law}(\mathcal{P}^m(X), Y)$. By letting $n_i \rightarrow \infty$ we obtain that

$$\frac{1}{m} I(\mathcal{P}^m(X); Y) \geq S(\mathcal{X}, T, d, 12\varepsilon) - 2\alpha \log K - H(2\alpha).$$

Thus we conclude

$$\tilde{R}_\mu(\varepsilon, \alpha) \geq S(\mathcal{X}, T, d, 12\varepsilon) - 2\alpha \log K - H(2\alpha).$$

Now notice that K depends only on ε and independent of α . By letting $\alpha \rightarrow 0$ we get

$$\tilde{R}_\mu(\varepsilon) \geq S(\mathcal{X}, T, d, 12\varepsilon).$$

\square

Theorem 1.9 follows from Lemma 4.1 and Proposition 4.2.

5. PROOF OF PROPOSITION 1.7

In this section we construct a dynamical system (\mathcal{X}, T) with a distance d satisfying

$$(5.1) \quad \lim_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} = 0, \quad \text{mdim}_M(\mathcal{X}, T, d) = \infty.$$

This proves Proposition 1.7 because $R_\mu(\varepsilon) \leq \tilde{S}(\mathcal{X}, T, d, \varepsilon)$ by Lemma 3.1.

Let V be an infinite dimensional Hilbert space. We denote its norm by $\|\cdot\|$. We can take $A_1, A_2, \dots \subset V$ such that

- $0 \in A_n$ for every n .
- For every n and any two distinct points $a, b \in A_n$ we have $\|a - b\| = 1/n$.

- $\log |A_n| = \Theta(2^n(\log n)^2)$, namely there exists $C > 1$ independent of n satisfying

$$C^{-1}2^n(\log n)^2 \leq \log |A_n| \leq C2^n(\log n)^2.$$

Set $B = \bigcup_{n \geq 1} A_n$. This is a compact subset of V and its diameter is bounded by 2. For each $n \geq 1$ we define $\mathcal{X}_n \subset A_n^{\mathbb{Z}}$ as the set of $(x_k)_{k \in \mathbb{Z}}$ such that

$$\exists l \in \mathbb{Z} : x_k = 0 \text{ for all } k \in \mathbb{Z} \setminus (l + 2^n \mathbb{Z}).$$

Set $\mathcal{X} = \bigcup_{n \geq 1} \mathcal{X}_n \subset B^{\mathbb{Z}}$. This is compact with respect to the distance

$$d(x, y) = \sum_{k \in \mathbb{Z}} 2^{-|k|} \|x_k - y_k\|.$$

Let $T : \mathcal{X} \rightarrow \mathcal{X}$ be the shift. We show that (\mathcal{X}, T, d) satisfies the property (5.1).

Claim 5.1.

$$\text{mdim}_{\text{M}}(\mathcal{X}, T, d) = \infty.$$

Proof. Let N be a multiple of 2^n . For $0 < \varepsilon \leq 1/n$

$$\#(\mathcal{X}_n, d_N, \varepsilon) \geq |A_n|^{N/2^n}.$$

Thus

$$S(\mathcal{X}_n, T, d, \varepsilon) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \#(\mathcal{X}_n, d_N, \varepsilon) \geq 2^{-n} \log |A_n| = \Theta((\log n)^2).$$

For any $0 < \varepsilon < 1$

$$S(\mathcal{X}, T, d, \varepsilon) \geq S(\mathcal{X}_{\lfloor 1/\varepsilon \rfloor}, T, d, \varepsilon) \geq \Theta((\log \lfloor 1/\varepsilon \rfloor)^2).$$

It follows

$$\text{mdim}_{\text{M}}(\mathcal{X}, T, d) = \lim_{\varepsilon \rightarrow 0} \frac{S(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} = \infty.$$

□

Let $\varepsilon > 0$ and set $L = L(\varepsilon) = \lceil \log_2(8/\varepsilon) \rceil$. It follows $\sum_{|n| > L} 2^{-|n|} \leq \varepsilon/4$.

Claim 5.2. *If $N \geq 2L + 2^n$ and $n > \log_2(1/\varepsilon) + \log_2(48L + 24)$ then every $x \in X_n$ satisfies $\bar{d}_N(x, 0) < \varepsilon/2$. Here $0 = (\dots, 0, 0, 0, \dots) \in X$.*

Proof. Let $x \in X_n$. There exists an integer l such that $x_k = 0$ for all $k \in \mathbb{Z} \setminus (l + 2^n \mathbb{Z})$. Then $d(T^i x, 0) \leq \varepsilon/4$ for any i outside of $[l - L, l + L] + 2^n \mathbb{Z}$.

We count how many $i \in [0, N]$ fall in $[l - L, l + L] + 2^n \mathbb{Z}$:

$$\begin{aligned} \frac{1}{N} |([l - L, l + L] + 2^n \mathbb{Z}) \cap [0, N]| &\leq \frac{1}{N} \left(1 + \frac{N + 2L}{2^n}\right) (2L + 1) \\ &= \left(1 + \frac{2L + 2^n}{N}\right) \frac{2L + 1}{2^n} \\ &\leq \frac{4L + 2}{2^n} \quad (\text{by } N \geq 2L + 2^n). \end{aligned}$$

Therefore

$$\bar{d}_N(x, 0) \leq \frac{\varepsilon}{4} + \frac{3(4L + 2)}{2^n} < \frac{\varepsilon}{2} \quad (\text{by } n > \log_2(1/\varepsilon) + \log_2(48L + 24)).$$

□

We take $\varepsilon_0 > 0$ so that all $0 < \varepsilon < \varepsilon_0$ satisfy $\log_2(1/\varepsilon) > \log_2(48L(\varepsilon) + 24)$. We set $N_0(\varepsilon) = 2L(\varepsilon) + 2^{\lfloor 6/\varepsilon \rfloor}$. In the rest of this section we always assume

$$0 < \varepsilon < \varepsilon_0, \quad N \geq N_0(\varepsilon).$$

Claim 5.3.

$$\bigcup_{n \geq 2 \log_2(1/\varepsilon)} \mathcal{X}_n \subset B_{\varepsilon/2}(0, \bar{d}_N),$$

where the right-hand side is the open $\varepsilon/2$ -ball around 0 with respect to the distance \bar{d}_N . Therefore

$$\# \left(\bigcup_{n \geq 2 \log_2(1/\varepsilon)} \mathcal{X}_n, \bar{d}_N, \varepsilon \right) = 1.$$

Proof. For $n > 6/\varepsilon$ every $x \in \mathcal{X}_n$ satisfies $d(x, 0) \leq 3/n < \varepsilon/2$. Thus $\mathcal{X}_n \subset B_{\varepsilon/2}(0, \bar{d}_N)$. For $2 \log_2(1/\varepsilon) \leq n \leq 6/\varepsilon$ it also follows that $\mathcal{X}_n \subset B_{\varepsilon/2}(0, \bar{d}_N)$ by Claim 5.2 because the assumptions imply $N \geq 2L + 2^n$ and $n > \log_2(1/\varepsilon) + \log_2(48L + 24)$. □

From Claim 5.3 and an elementary inequality

$$\log(a_1 + a_2 + \cdots + a_K) \leq \log K + \max_{1 \leq i \leq K} \log a_i, \quad (a_1, \dots, a_K > 0),$$

it follows that

$$\log \#(\mathcal{X}, \bar{d}_N, \varepsilon) \leq \log(1 + 2 \log_2(1/\varepsilon)) + \max_{1 \leq n < 2 \log_2(1/\varepsilon)} \log \#(\mathcal{X}_n, \bar{d}_N, \varepsilon).$$

The term $\log \#(\mathcal{X}_n, \bar{d}_N, \varepsilon)$ can be easily estimated:

$$\#(\mathcal{X}_n, \bar{d}_N, \varepsilon) \leq \#(\mathcal{X}_n, d_N, \varepsilon) \leq 2^n |A_n|^{1+2^{-n}(N+2L)} \quad (\text{by } \sum_{|n|>L} 2^{-|n|} < \varepsilon/4).$$

$$\begin{aligned}\log \#(\mathcal{X}_n, \bar{d}_N, \varepsilon) &\leq n \log 2 + \{1 + 2^{-n}(N + 2L)\} \log |A_n| \\ &\leq n \log 2 + (2^n + N + 2L)O((\log n)^2).\end{aligned}$$

Hence $\log \#(\mathcal{X}, \bar{d}_N, \varepsilon)$ is bounded by

$$(2 \log 2) \log_2(1/\varepsilon) + \log(1 + \log_2(1/\varepsilon)) + ((1/\varepsilon)^2 + N + 2L)O((\log \log 1/\varepsilon)^2).$$

Thus

$$\tilde{S}(\mathcal{X}, T, d, \varepsilon) = \lim_{N \rightarrow \infty} \frac{1}{N} \log \#(\mathcal{X}, \bar{d}_N, \varepsilon) \leq O((\log \log 1/\varepsilon)^2).$$

So we conclude

$$\lim_{\varepsilon \rightarrow 0} \frac{\tilde{S}(\mathcal{X}, T, d, \varepsilon)}{|\log \varepsilon|} = 0.$$

APPENDIX A. ELEMENTARY LEMMAS ON OPTIMAL TRANSPORT

The purpose of this appendix is to prove lemmas on optimal transport which are used in the proofs of Theorems 1.5 and 1.9. Our argument here is completely elementary. Much more general and systematic treatments can be found in [AGS] and [Vil]. In this appendix we identify probability measures with their probability mass functions.

Let A be a finite set with a distance d . For two probability measures μ and ν on A we denote by $\mathcal{M}(\mu, \nu)$ the set of probability measures π on $A \times A$ whose first and second marginals are μ and ν respectively. We define the L^1 -Wasserstein distance $W(\mu, \nu)$ by

$$W(\mu, \nu) = \min_{\pi \in \mathcal{M}(\mu, \nu)} \int_{A \times A} d(x, y) d\pi(x, y).$$

A measure $\pi \in \mathcal{M}(\mu, \nu)$ attaining this minimum is called an optimal transference plan between μ and ν .

Lemma A.1. *Let $\{\mu_n\}_{n \geq 1}$ be a sequence of probability measures on A converging to μ in the weak* topology. Then*

$$\lim_{n \rightarrow \infty} W(\mu_n, \mu) = 0.$$

Proof. This is a consequence of the general fact that the Wasserstein distance metrizes the weak* topology ([Vil, Theorem 6.9]). Here we prove it directly. For the notational convenience we identify A with some cyclic group $\mathbb{Z}/K\mathbb{Z}$.

We define $\pi_n \in \mathcal{M}(\mu_n, \mu)$ as follows. First we set

$$\begin{aligned}\pi_n(0, 0) &= \min(\mu_n(0), \mu(0)), \\ \pi_n(0, y) &= \min\left(\mu_n(0) - \sum_{k=0}^{y-1} \pi_n(0, k), \mu(y)\right) \quad (1 \leq y \leq K-1).\end{aligned}$$

Here we defined $\pi_n(0, y)$ inductively with respect to y . Next we set

$$\pi_n(1, 1) = \min(\mu_n(1), \mu(1) - \pi_n(0, 1)),$$

and for $2 \leq y \leq K$

$$\pi_n(1, y) = \min\left(\mu_n(1) - \sum_{k=1}^{y-1} \pi_n(1, k), \mu(y) - \pi_n(0, y)\right).$$

Note that $y = K$ is the same as $y = 0$ in $\mathbb{Z}/K\mathbb{Z}$. In general we set

$$\pi_n(x, x) = \min\left(\mu_n(x), \mu(x) - \sum_{k=0}^{x-1} \pi_n(k, x)\right),$$

and for $x + 1 \leq y \leq K + x - 1$

$$\pi_n(x, y) = \min\left(\mu_n(x) - \sum_{k=x}^{y-1} \pi_n(x, k), \mu(y) - \sum_{k=0}^{x-1} \pi_n(k, y)\right).$$

The assumed convergence $\mu_n \rightarrow \mu$ in the weak* topology means that $\mu_n(x) \rightarrow \mu(x)$ for every x . Then it is easy to check that

$$\pi_n(x, x) \rightarrow \mu(x), \quad \pi_n(x, y) \rightarrow 0 \quad (x \neq y).$$

This implies

$$W(\mu_n, \mu) \leq \int_{A \times A} d(x, y) d\pi_n(x, y) \rightarrow 0.$$

□

Lemma A.2. *Let $\{\mu_n\}_{n \geq 1}$ be a sequence of probability measures on A converging to μ in the weak* topology. Let π_n be an optimal transference plan between μ_n and μ . Then the sequence π_n converges to $(\text{Id} \times \text{Id})_* \mu$.*

Proof. For any $a \neq b$ in A

$$\pi_n(a, b) \leq \frac{1}{d(a, b)} \int_{A \times A} d(x, y) d\pi_n(x, y) = \frac{W(\mu_n, \mu)}{d(a, b)}.$$

The right-hand side converges to zero by Lemma A.1. In the diagonal

$$\pi_n(b, b) = \mu(b) - \sum_{a \neq b} \pi_n(a, b) \rightarrow \mu(b).$$

□

REFERENCES

- [AGS] L. Ambrosio, N. Gigli, G. Savaré, Gradient flows in metric spaces and in the space of probability measures, Birkhäuser Verlag, Basel, 2005.
- [Ber] T. Berger, Rate distortion theory: A mathematical basis for data compression, Englewood Cliffs, NJ: Princeton-Hall, 1971.
- [Cos] B. F. P. Da Costa, Deux exemples sur la dimension moyenne dun espace de courbes de Brody, *Ann. Inst. Fourier* **63** (2013) 2223-2237.
- [CT] T.M. Cover, J.A. Thomas, Elements of information theory, second edition, Wiley, New York, 2006.
- [Din] E. I. Dinaburg, A correlation between topological entropy and metric entropy, *Dokl. Akad. Nauk SSSR* **190** (1970) 19-22.
- [ECG] M. Effros, P. A. Chou, G. M. Gray, Variable-rate source coding theorems for stationary nonergodic sources, *IEEE Trans. Inf. Theory* **40** (1994) 1920-1925.
- [EN] G. A. Elliot, Z. Niu, The C^* -algebra of a minimal homeomorphism of zero mean dimension, [arXiv:1406.2382](https://arxiv.org/abs/1406.2382).
- [GW] E. Glasner, B. Weiss, On the interplay between measurable and topological dynamics, *Handbook of dynamical systems*, Vol. 1B, Elsevier B.V. Amsterdam, 2006.
- [Goodm] T. N. T. Goodman, Relating topological entropy and measure entropy, *Bull. London Math. Soc.* **3** (1971) 176-180.
- [Goodw] L. W. Goodwyn, Topological entropy bounds measure-theoretic entropy, *Proc. Amer. Math. Soc.* **23** (1969) 679-688.
- [Gra] R.M. Gray, Entropy and information theory, New York, Springer-Verlag, 1990.
- [Gro] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, *Math. Phys. Anal. Geom.* **2** (1999) 323-415.
- [Gut1] Y. Gutman, Embedding \mathbb{Z}^k -actions in cubical shifts and \mathbb{Z}^k -symbolic extensions, *Ergodic Theory Dynam. Systems* **31** (2011) 383-403.
- [Gut2] Y. Gutman, Mean dimension & Jaworski-type theorems, *Proc. London Math. Soc.* **111** (2015) 831-850.
- [Gut3] Y. Gutman, Embedding topological dynamical systems with periodic points in cubical shifts. To appear in *Ergodic Theory Dynam. Systems*.
- [GT] Y. Gutman, M. Tsukamoto, Embedding minimal dynamical systems into Hilbert cubes, preprint, [arXiv:1511.01802](https://arxiv.org/abs/1511.01802).
- [Jew] R. I. Jewett, The prevalence of uniquely ergodic systems, *J. Math. Mech.* **19** (1970) 717-729.
- [KD] T. Kawabata, A. Dembo, The rate distortion dimension of sets and measures, *IEEE Trans. Inf. Theory* **40** (1994) 1564-1572.
- [Kri] W. Krieger, On unique ergodicity, *Proc. sixth Berkeley symposium, Math. Statist. Probab. Univ. of California Press*, 1970, 327-346.
- [LDN] A. Leon-Garcia, L. D. Davisson, D. L. Neuhoff, New results on coding of stationary nonergodic sources, *IEEE Trans. Inform. Theory* **25** (1979) 137-144.
- [Li] H. Li, Sofic mean dimension, *Adv. Math.* **244** (2013) 570-604.
- [LL] H. Li, B. Liang, Mean dimension, mean rank and von Neumann–Lück rank, *J. Reine Angew. Math. ISSN (Online) 1435-5345, ISSN (Print) 0075-4102*, DOI: 10.1515/crelle-2015-0046, September 2015.
- [Lin] E. Lindenstrauss, Mean dimension, small entropy factors and an embedding theorem, *Inst. Hautes Études Sci. Publ. Math.* **89** (1999) 227-262.

- [LT] E. Lindenstrauss, M. Tsukamoto, Mean dimension and an embedding problem: an example, *Israel J. Math.* **199** (2014) 573-584.
- [LW] E. Lindenstrauss, B. Weiss, Mean topological dimension, *Israel J. Math.* **115** (2000) 1-24.
- [MT] S. Matsuo, M. Tsukamoto, Brody curves and mean dimension, *J. Amer. Math. Soc.* **28** (2015) 159-182.
- [Mis] M. Misiurewicz, A short proof of the variational principle for \mathbb{Z}_+^N actions on a compact space, *International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975)*, *Astérisque* **40** (1976) 145-157, Soc. Math. France, Paris.
- [Sha] C.E. Shannon, A mathematical theory of communication, *Bell Syst. Tech. J.* **27** (1948) 379-423, 623-656.
- [Tsu1] M. Tsukamoto, Large dynamics of Yang–Mills theory: mean dimension formula, arXiv:1407.2058, to appear in *J. Anal. Math.*
- [Tsu2] M. Tsukamoto, Mean dimension of the dynamical system of Brody curves, preprint, arXiv:1410.1143.
- [Vil] C. Villani, *Optimal transport old and new*, Springer-Verlag, Berlin, 2009.

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