Wyner's Common Information under Rényi Divergence Measures

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mon information problem (also coined the distributed source simulation problem). The original common information problem consists in understanding the minimum rate of the common input to independent processors to generate an approximation of a joint distribution when the distance measure used to quantify the discrepancy between the synthesized and target distributions is the normalized relative entropy. Our generalization involves changing the distance measure to the unnormalized and normalized Rényi divergences of order $\alpha = 1 + s \in [0, 2]$. We show that the minimum rate needed to ensure the Rényi divergences between the distribution induced by a code and the target distribution vanishes remains the same as the one in Wyner's setting, except when the order $\alpha = 1 + s = 0$. This implies that Wyner's common information is rather robust to the choice of distance measure employed. As a byproduct of the proofs used to the establish the above results, the exponential strong converse for the common information problem under the total variation distance measure is established.

Abstract-We study a generalized version of Wyner's com-

Index Terms—Wyner's common information, Distributed source simulation, Rényi divergence, Total variation distance, Exponential strong converse

I. INTRODUCTION

How much common randomness is needed to simulate two correlated sources in a distributed fashion? This problem, termed *distributed source simulation*, was first studied by Wyner [1], who used the normalized relative entropy (Kullback-Leibler divergence or KL divergence) to measure the approximation level (discrepancy) between the simulated joint distribution and the joint distribution of the original correlated sources. He defined the minimum rate needed to ensure that the normalized relative entropy vanishes asymptotically as the *common information* between the sources. He also established a single-letter characterization for the common information, i.e., the common information between correlated sources X and Y (with target distribution π_{XY}) is

$$C_{\text{Wyner}}(X;Y) = \min_{P_{XYW}: P_{XY} = \pi_{XY}, X - W - Y} I(XY;W).$$
(1)

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Copyright (c) 2018 IEEE. Personal use of this material is permitted. However, permission to use this material for any other purposes must be obtained from the IEEE by sending a request to pubs-permissions@ieee.org. The common information is also known to be one of many reasonable measures of the dependence between two random variables [2, Section 14.2.2] (other measures include the mutual information and the Gács-Körner-Witsenhausen common information). A related notion is that of the exact common information which was introduced by Kumar, Li, and El Gamal [3]. They assumed variable-length codes and exact generation of the correlated sources (X, Y), instead of block codes and approximate simulation of π_{XY} as assumed by Wyner [1]. The exact common information is not smaller than Wyner's common information. However, it is still not known whether they are equal in general. Furthermore, the common information problem can be also be regarded as a distributed coordination problem. The concept of coordination was first introduced by Cuff, Permuter, and Cover [4], [5], who used the total variation (TV) distance to measure the level of approximation between the simulated and target distributions.

Wyner's common information problem is also closely related to the channel resolvability problem, which was first studied by Han and Verdú [6], and subsequently studied by Hayashi [7], [8], Liu, Cuff, and Verdú [9], and Yu and Tan [10] among others. For the achievability part, both problems rely on so-called soft-covering lemmas [5]. The channel resolvability or common information problems have several interesting applications—including secrecy, channel synthesis, and source coding. For example, in [11] it was used to study the performance of a wiretap channel system under different secrecy measures. In [12] it was used to study the reliability and secrecy exponents of a wiretap channel with cost constraints. In [13] it was used to study the exact secrecy and reliability exponents for a wiretap channel.

A. Main Contributions

Different from Wyner's work, we use (normalized and unnormalized) Rényi divergences of order $1 + s \in [0, 2]$ to measure the level of approximation between the simulated and target distributions. This is motivated in part by our desire to understand the sensitivity of the divergence as approximation measure on Wyner's common information. We prove that for the distributed source simulation problem, the minimum rate needed to guarantee that the (normalized and unnormalized) Rényi divergences vanish asymptotically is equal to Wyner's common information (except for the case when Rényi parameter is equal to 0). This implies that Wyner's common information in (1) is rather robust to the distance measure. For the achievability part, by using the method of types and typicality arguments, we prove that the optimal Rényi divergences vanish (at least) exponentially fast if the code rate is larger than Wyner's common information. However, for the converse part, the proof is not straightforward and we have to first consider an auxiliary problem. We first prove an exponential strong converse for the common information problem under the TV distance measure, i.e., when the code rate is smaller than Wyner's common information, the TV distance between the induced distribution and the target distribution tends to one (at least) exponentially fast. Even though our proof technique mirrors that of Oohama [14] to establish the exponential strong converse for the Wyner-Ziv problem, it differs significantly in some aspects. To wit, some intricate continuity arguments are required to assert that the strong converse exponent is positive for all rates below $C_{Wvner}(X;Y)$ (see part (i) of Lemma 1). Furthermore and interestingly, by leveraging a key relationship between the Rényi divergence and the TV distance [15], this exponential strong converse implies the converse for the normalized Rényi divergence (which in turn also implies the strong converse for the unnormalized Rényi divergence).

It is worth noting that it is quite natural to use various divergences to measure the discrepancy between two distributions. Wyner [1] used the KL divergence to measure the level of approximation in the distributed source synthesis problem; Hayashi [7], [8] and Yu and Tan [10] respectively used the KL divergence and the Rényi divergence to study the channel resolvability problem. The latter also applied their results to study the capacity region for the wiretap channel under these generalized measures. Furthermore, in probability theory, Barron [16] and Bobkov, Chistyakov and Götze [17] respectively used the KL divergence and the Rényi divergence to study the central limit theorem, i.e., they used them to measure the discrepancy between the induced distribution of sum of i.i.d. random variables and the normal distribution with the same mean and variance. Furthermore, special instances of Rényi entropies and divergences-including the KL divergence, the collision entropy (the Rényi divergence of order 2), and min-entropy (the Rényi divergence of order ∞)—were used to study various information-theoretic problems (including security, cryptography, and quantum information) in several works in the recent literature [10], [11], [18]-[22].

B. Notation

We use $P_X(x)$ to denote the probability distribution of a random variable X. This will also be denoted as P(x)(when the random variable X is clear from the context). We also use \tilde{P}_X , \hat{P}_X and Q_X to denote various probability distributions with alphabet \mathcal{X} . All alphabets considered in the sequel are finite. The set of probability measures on \mathcal{X} is denoted as $\mathcal{P}(\mathcal{X})$, and the set of conditional probability measures on \mathcal{Y} given a variable in \mathcal{X} is denoted as $\mathcal{P}(\mathcal{Y}|\mathcal{X}) := \{P_{Y|X} : P_{Y|X}(\cdot|x) \in \mathcal{P}(\mathcal{Y}), x \in \mathcal{X}\}$. Furthermore, the support of a distribution $P \in \mathcal{P}(\mathcal{X})$ is denoted as $\supp(P) = \{x \in \mathcal{X} : P(x) > 0\}$.

We use $T_{x^n}(x) := \frac{1}{n} \sum_{i=1}^n 1\{x_i = x\}$ to denote the type (empirical distribution) of a sequence x^n , T_X and $V_{Y|X}$ to respectively denote a type of sequences in \mathcal{X}^n and a conditional type of sequences in \mathcal{Y}^n (given a sequence $x^n \in \mathcal{X}^n$). For a type T_X , the type class (set of sequences having the same type

 T_X) is denoted by \mathcal{T}_{T_X} . For a conditional type $V_{Y|X}$ and a sequence x^n , the $V_{Y|X}$ -shell of x^n (the set of y^n sequences having the same conditional type $V_{Y|X}$ given x^n) is denoted by $\mathcal{T}_{V_{Y|X}}(x^n)$. For brevity, sometimes we use T(x, y) to denote the joint distributions T(x)V(y|x) or T(y)V(x|y).

The ϵ -typical set of Q_X is denoted as

$$\mathcal{T}^{n}_{\epsilon}(Q_{X}) := \left\{ x^{n} \in \mathcal{X}^{n} : |T_{x^{n}}(x) - Q_{X}(x)| \le \epsilon Q_{X}(x), \forall x \in \mathcal{X} \right\}.$$
(2)

The conditionally ϵ -typical set of Q_{XY} is denoted as

$$\mathcal{T}^n_{\epsilon}(Q_{YX}|x^n) := \{y^n \in \mathcal{Y}^n : (x^n, y^n) \in \mathcal{T}^n_{\epsilon}(Q_{XY})\}.$$
 (3)

For brevity, sometimes we write $\mathcal{T}_{\epsilon}^{n}(Q_{X})$ and $\mathcal{T}_{\epsilon}^{n}(Q_{YX}|x^{n})$ as $\mathcal{T}_{\epsilon}^{n}$ and $\mathcal{T}_{\epsilon}^{n}(x^{n})$ respectively.

The TV distance between two probability mass functions P and Q with a common alphabet \mathcal{X} is defined as

$$|P - Q| := \frac{1}{2} \sum_{x \in \mathcal{X}} |P(x) - Q(x)|.$$
(4)

By the definition of ϵ -typical set, we have that for any $x^n \in \mathcal{T}_{\epsilon}^n(Q_X)$,

$$|T_{x^n} - Q_X| \le \frac{\epsilon}{2}.$$
(5)

Fix distributions $P_X, Q_X \in \mathcal{P}(\mathcal{X})$. The *relative entropy* and the *Rényi divergence of order* 1 + s are respectively defined as

$$D(P_X || Q_X) := \sum_{x \in \text{supp}(P_X)} P_X(x) \log \frac{P_X(x)}{Q_X(x)}$$
(6)

$$D_{1+s}(P_X \| Q_X) := \frac{1}{s} \log \sum_{x \in \text{supp}(P_X)} P_X(x)^{1+s} Q_X(x)^{-s},$$
(7)

and the conditional versions are respectively defined as

$$D(P_{Y|X} \| Q_{Y|X} | P_X) := D(P_X P_{Y|X} \| P_X Q_{Y|X})$$
(8)

$$D_{1+s}(P_{Y|X} \| Q_{Y|X} | P_X) := D_{1+s}(P_X P_{Y|X} \| P_X Q_{Y|X}),$$
(9)

where the summations in (6) and (7) are taken over the elements in $\operatorname{supp}(P_X)$. Throughout, log is to the natural base e and $s \ge -1$. It is known that $\lim_{s\to 0} D_{1+s}(P_X || Q_X) = D(P_X || Q_X)$ so a special case of the Rényi divergence (or the conditional version) is the usual relative entropy (or the conditional version).

Given a number $a \in [0, 1]$, we define $\bar{a} = 1 - a$. We also define $[x]^+ = \max \{x, 0\}$.

C. Problem Formulation

In this paper, we consider the distributed source simulation problem illustrated in Fig. 1. Given a target distribution π_{XY} , we wish to minimize the alphabet size of a random variable M_n that is uniformly distributed over¹ $\mathcal{M}_n := \{1, \ldots, e^{nR}\}$

¹For simplicity, we assume that e^{nR} and similar expressions are integers.



Fig. 1. Distributed source synthesis problem, where the random variable $M_n \in \mathcal{M}_n := \{1, \dots, e^{nR}\}.$

(R is a positive number known as the rate), such that the generated (or synthesized) distribution

$$P_{X^{n}Y^{n}}(x^{n}, y^{n})$$

:= $\frac{1}{|\mathcal{M}_{n}|} \sum_{m \in \mathcal{M}_{n}} P_{X^{n}|M_{n}}(x^{n}|m) P_{Y^{n}|M_{n}}(y^{n}|m)$ (10)

forms a good approximation to the product distribution $\pi_{X^nY^n} := \pi_{XY}^n$. The pair of random mappings $(P_{X^n|M_n}, P_{Y^n|M_n})$ constitutes a *synthesis code*.

Different from Wyner's seminal work on the distributed source simulation problem [1], we employ the unnormalized Rényi divergence

$$D_{1+s}(P_{X^nY^n} \| \pi_{X^nY^n}) \tag{11}$$

and the normalized Rényi divergence

$$\frac{1}{n}D_{1+s}(P_{X^nY^n}\|\pi_{X^nY^n}) \tag{12}$$

to measure the discrepancy between $P_{X^nY^n}$ and $\pi_{X^nY^n}$. The minimum rates required to ensure these two measures vanish asymptotically are respectively termed the *unnormalized and* normalized Rényi common information, and denoted as

$$T_{1+s}(\pi_{XY})$$

:= inf $\left\{ R : \lim_{n \to \infty} D_{1+s}(P_{X^nY^n} \| \pi_{X^nY^n}) = 0 \right\},$ (13)

$$T_{1+s}(\pi_{XY}) = \inf \left\{ R : \lim_{n \to \infty} \frac{1}{n} D_{1+s}(P_{X^nY^n} \| \pi_{X^nY^n}) = 0 \right\}.$$
 (14)

It is clear that

$$\widetilde{T}_{1+s}(\pi_{XY}) \le T_{1+s}(\pi_{XY}).$$
 (15)

We also denote the minimum rate required to ensure the TV distance is bounded above by some constant $\varepsilon \in [0, 1]$ asymptotically as

$$T_{\varepsilon}^{\mathsf{TV}}(\pi_{XY}) = \inf \left\{ R : \limsup_{n \to \infty} |P_{X^n Y^n} - \pi_{X^n Y^n}| \le \varepsilon \right\}.$$
(16)

We say that the *strong converse property* for the common information problem under the TV distance holds if $T_{\varepsilon}^{\mathsf{TV}}(\pi_{XY})$ does not depend on $\varepsilon \in [0, 1)$.

II. MAIN RESULTS

Our main result concerns Wyner's common information problem when the discrepancy measure is the unnormalized or normalized Rényi divergence. It is stated as follows.

Theorem 1 (Rényi Common Informations). *The unnormalized and normalized and Rényi common informations satisfy*

$$T_{1+s}(\pi_{XY}) = \widetilde{T}_{1+s}(\pi_{XY}) \tag{17}$$

$$= \begin{cases} C_{\text{Wyner}}(X;Y) & s \in (-1,1] \\ 0 & s = -1 \end{cases}.$$
 (18)

Furthermore, for $s \in (-1, 1]$, the optimal Rényi divergence $D_{1+s}(P_{X^nY^n} || \pi_{X^nY^n})$ in the definitions of the Rényi common informations decays at least exponentially fast in n when $R > C_{Wyner}(X;Y)$.

Remark 1. For the converse part, $T_{1+s}(\pi_{XY})$ \geq $\geq C_{\mathsf{Wyner}}(X;Y) \text{ for } s \in$ $T_{1+s}(\pi_{XY})$ [0,1] is implied by Wyner's work [1] and the monotonicity of the Rényi divergence. For the achievability part, $\widetilde{T}_{1+s}(\pi_{XY}) \leq C_{\mathsf{Wyner}}(X;Y)$ for $s \in (-1,0]$ is also implied by Wyner's work [1] and the monotonicity of the Rényi divergence. Furthermore, since a channel resolvability code for the memoryless channel $P_{X|W} \times P_{Y|W}$ can be used to form a common information code, the achievability part for the common information problem can be obtained from existing channel resolvability results. Specifically, $T_{1+s}(\pi_{XY}) \leq C_{\mathsf{Wyner}}(X;Y)$ for $s \in (-1,0]$ can be obtained from Hayashi's [7], [8] or Han, Endo, and Sasaki's results [12]. In addition, $T_{1+s}(\pi_{XY}) \leq T_{1+s}(\pi_{XY}) \leq C_{1+s}(X;Y)$ for $s \in (0, 1]$ with

$$C_{1+s}(X;Y) := \min_{P_{XYW}: P_{XY} = \pi_{XY}, X - W - Y} \sum_{w} P_{W}(w) D_{1+s} \left(P_{X|W}(\cdot|w) P_{Y|W}(\cdot|w) \| P_{XY} \right)$$
(19)

can be obtained from the present authors' results [10], but as shown in Theorem 1, this bound is not tight since $C_{1+s}(X;Y) > C_{Wyner}(X;Y)$ in general for $s \in (0,1]$. This is because, on the one hand, for the channel resolvability problem, the discrete memoryless channel is fixed, and, by construction, imposes a product conditional distribution of the output given the input (which is a product distribution), but for the common information problem, the synthesizer has the freedom to choose $P_{X^nY^n|M_n} = P_{X^n|M_n} \times P_{Y^n|M_n}$, so that the Markov chain $X^n - M_n - Y^n$ holds; on the other hand, for the common information problem, in the sequel, we will show that if we utilize a truncated channel (which is not memoryless) as the synthesizer. This results in a smaller achievable rate for the case $s \in (0, 1]$. Therefore, our converse for $s \in [-1, 0)$ and achievability for $s \in (0, 1]$ are new (and also tight).

Remark 2. An exponential achievability result for $s \in (-1, 0]$ can be obtained from Hayashi's [7], [8] and Han, Endo, and Sasaki's results [12], where i.i.d. codes were employed.

For this theorem, the proof of the achievability part for the unnormalized Rényi common information is provided in Appendix A, and the proof of the converse part for the normalized Rényi common information is provided in Section IV. Observe that the unnormalized Rényi divergence is stronger than the normalized one in the sense of (15), hence $\widetilde{T}_{1+s}(\pi_{XY}) \leq T_{1+s}(\pi_{XY})$. This implies, on one hand, the achievability result for the normalized Rényi common information $\widetilde{T}_{1+s}(\pi_{XY})$ can be obtained directly from the achievability result for the unnormalized version $T_{1+s}(\pi_{XY})$, and on the other hand, the converse result for the normalized Rényi common information $\widetilde{T}_{1+s}(\pi_{XY})$ implies the converse result for the unnormalized version $T_{1+s}(\pi_{XY})$.

The Rényi common informations are the same for all $s \in (-1,1]$, and also same as Wyner's common information $C_{Wyner}(X;Y)$ (which corresponds to s = 0 for the normalized case). For the case $s \in (-1, 1]$, to obtain the (unnormalized and normalized) Rényi common informations, we utilize a random code with (W^n, X^n, Y^n) (W is the auxiliary random variable in the definition of $C_{Wyner}(X;Y)$) distributed according to a truncated product distribution, i.e., a product distribution governed by Q_{WXY}^n but whose mass is truncated to the typical set $\mathcal{T}_{\epsilon}^{n}(Q_{WXY})$.² On one hand, the random sequences (W^{n}, X^{n}, Y^{n}) so generated are almost uniformly distributed over the typical set $\mathcal{T}_{\epsilon}^{n}(Q_{WXY})$; and on the other hand, the Rényi common informations can be expressed as some Rényi divergences. Moreover, these Rényi divergences evaluated at the truncated distribution are almost the same regardless of the parameter $s \in (-1, 1]$. Therefore, by using this truncated code, Wyner's common information is achievable for any $s \in (-1, 1]$.

However, the proof of the converse part for the normalized Rényi common information is not straightforward.³ We attempted to use the method of types to prove it, just as in [10] for the Rényi resovability problem, but failed since the code for the common information problem is arbitrary and does not need to be i.i.d. In particular, it is not i.i.d. In the following two sections, we provide an indirect proof using the following strategy: We first prove an exponential strong converse for Wyner's common information problem under the TV distance measure in Section III. Then by using a relationship between the Rényi divergence and the TV distance [15], we show this exponential strong converse implies the converse for normalized Rényi divergence in Section IV.

As an intermediate result, the common information under the TV distance measure is characterized in the following

²Interestingly, a truncated code is not necessary for $s \in (-1, 0]$ case, since an i.i.d. code (without truncation) is optimal as well for this case. This point can be seen from the work of Yu and Tan [10].

³More precisely, the proof of the converse part for $s \in (-1,0)$ case is not easy. The converse part for s = 0 case was proven by Wyner [1], in which the continuity of the normalized entropy under the normalized KL divergence measure was used, i.e., when $\frac{1}{n}D(P_{X^nY^n}||\pi_{XY}^n)$ is small then $\frac{1}{n}H_P(X^nY^n)$ is arbitrarily close to $H_{\pi}(XY)$. But it is not straightforward to apply Wyner's proof to the case $s \in (-1,0)$, since we do not know whether a strong enough continuity condition for the normalized entropy holds under the normalized Rényi divergence measure with order $\alpha = 1 + s \in (0, 1)$. Even if a strong enough continuity condition holds, it is not straightforward to prove. Note that we were not able to directly utilize ideas in Wyner's converse proof to demonstrate this point. The definition of the "relative entropy typical set" $A(n, \epsilon_1) := \{u: \frac{1}{n} \log \frac{p_1(u)}{p_0(u)} \le \epsilon_1\}$ is crucial in Wyner's proof. If we adopt this set with this definition for the Rényi divergence setting (with Rényi parameter < 1), it is not clear to us whether $\mathbb{P}_1(A^c(n, \epsilon_1))$ vanishes (cf. Equation (A.7) in Wyner's paper). theorem.

Theorem 2 (Common Information under the TV Distance Measure). *The following hold:*

(i) The common information under the TV distance measure satisfies

$$T_{\varepsilon}^{\mathsf{TV}}(\pi_{XY}) = \begin{cases} C_{\mathsf{Wyner}}(X;Y) & \varepsilon \in [0,1) \\ 0 & \varepsilon = 1 \end{cases}.$$
 (20)

Hence, the strong converse property for the common information problem under the TV distance holds.

- (ii) Furthermore, there exists a sequence of synthesis codes with rate $R > C_{Wyner}(X;Y)$, such that $|P_{X^nY^n} - \pi_{X^nY^n}|$ tends to zero exponentially fast as n tends to infinity.
- (iii) On the other hand, for any sequence of synthesis codes with rate $R < C_{Wyner}(X;Y)$, we have that $|P_{X^nY^n} \pi_{X^nY^n}|$ tends to one exponentially fast as n tends to infinity.

Part (ii) is an exponential achievability result while the part (iii) is an exponential strong converse result. Combining parts (ii) and (iii) implies part (i). By Pinsker's inequality for Rényi divergences [23], the achievability results (including the exponential achievability result) in Theorem 1 implies the achievability results (including the exponential achievability result) in Theorem 2. Conversely, the exponential strong converse result in part (iii) of Theorem 2 implies the converse results in Theorem 1 for both unnormalized and normalized Rényi divergences. To prove part (iii), we draw on several key ideas from Oohama's work [14] on the exponential strong converse for the Wyner-Ziv problem. However, there are several key differences in our proofs, including the way we establish that the strong converse exponent is positive for all rates larger than $C_{Wyner}(X;Y)$ and the treatment of the cases when various probability mass functions take on the value zero.

We note that conclusion in part (ii) (the exponential achievability result) in Theorem 2 can be also obtained by using the soft-covering lemma by Cuff [5, Lemma IV.1].

The proof of the conclusion in part (iii) is provided in the next section. As mentioned above, the other parts follow directly from Theorem 1.

III. THE PROOF OF PART (III) IN THEOREM 2

In this section, we provide an exponential strong converse theorem for the common information problem under the TV distance measure, which will be used to derive the converse for normalized Rényi divergence in next section.

We define

$$\mathcal{Q} := \Big\{ Q_{XYU} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U}) : \\ |\mathcal{U}| \le |\mathcal{X}| |\mathcal{Y}|, \operatorname{supp}(Q_{XY}) \subseteq \operatorname{supp}(\pi_{XY}) \Big\}.$$
(21)

Given $\alpha \in [0,1]$ and an arbitrary distribution $Q_{XYU} \in \mathcal{Q}$, define the linear combination of the likelihood ratios for $(x, y, u) \in \operatorname{supp}(Q_{XYU}),$

$$\omega_{Q_{XYU}}^{(\alpha)}(x,y|u) := \bar{\alpha} \left(\log \frac{Q_{XY}(x,y)}{\pi_{XY}(x,y)} + \log \frac{Q_{XY|U}(x,y|u)}{Q_{X|U}(x|u)Q_{Y|U}(y|u)} \right) + \alpha \log \frac{Q_{XY|U}(x,y|u)}{\pi_{XY}(x,y)}.$$
(22)

This function is finite for all $(x, y, u) \in \text{supp}(Q_{XYU})$. For $Q_{XYU} \in \mathcal{Q}$ and $\theta \in [0,\infty)$, define the negative cumulant generating functions as

$$\Omega^{(\alpha,\theta)}(Q_{XYU})$$

$$:= -\log \mathbb{E}_{Q_{XYU}} \Big[\exp \Big(-\theta \omega_{Q_{XYU}}^{(\alpha)}(X,Y|U) \Big) \Big], \quad (23)$$

and

$$\Omega^{(\alpha,\theta)} := \min_{Q_{XYU} \in \mathcal{Q}} \Omega^{(\alpha,\theta)}(Q_{XYU}), \qquad (24)$$

where the expectation $\mathbb{E}_{Q_{XYU}}$ is only taken over the set $supp(Q_{XYU})$ (this means we only sum over the elements (x, y, u) such that $Q_{XYU}(x, y, u) > 0$).

Finally, we define the large deviations rate functions

$$F^{(\alpha,\theta)}(R) := \frac{\Omega^{(\alpha,\theta)} - \theta \alpha R}{1 + (5 - 3\alpha)\theta},$$
(25)

$$F(R) := \sup_{(\alpha,\theta) \in [0,1] \times [0,\infty)} F^{(\alpha,\theta)}(R).$$
(26)

In view of the definitions above, we have the following theorem. The proof of this theorem is provided in Appendix B.

Theorem 3. For any synthesis code such that

$$\frac{1}{n}\log|\mathcal{M}_n| \le R,\tag{27}$$

we have

$$P_{X^nY^n} - \pi_{X^nY^n} | \ge 1 - 4 \exp\left(-nF(R)\right).$$
 (28)

If we show F(R) > 0, then Theorem 3 implies the exponential strong converse for TV distance measure. To that end, we need the following lemma.

Lemma 1. The following conclusions hold.

(i) If $R < C_{Wyner}(X; Y)$, then

$$F(R) > 0. \tag{29}$$

(ii) If
$$R \ge C_{Wvner}(X;Y)$$
, then

$$F(R) = 0. \tag{30}$$

The proof of Lemma 1 is provided in Appendix C. We remark that Lemma 1, especially part (i), plays an central role in claiming the exponential strong converse theorem for the common information problem with the TV distance measure. Its proof is completely different from that for the corresponding statement in [14] and requires some intricate continuity arguments (e.g., [24, Lemma 14]). As we have seen in Theorem 3, F(R) in (26) is a lower bound on the exponent

of $1 - |P_{X^nY^n} - \pi_{X^nY^n}|$. This can be regarded as the strong converse exponent.

Combining Lemma 1 and Theorem 3, we conclude that the exponent in the right hand side of (28) is strictly positive if the rate is smaller than $C_{Wyner}(X;Y)$. Hence, we obtain the exponential strong converse result given in the conclusion (iii) of Theorem 2.

IV. CONVERSE PROOF OF THEOREM 1 FOR THE NORMALIZED RÉNYI COMMON INFORMATION

In this section, we provide a proof of the converse part of Theorem 1 for the normalized Rényi common information. To this end, we need the following relationships between the Rényi divergence and the TV distance.

Lemma 2 (Relationship between the Rényi Divergence and the TV Distance (Sason [15])). For any $s \in (-1, +\infty)$,

$$\inf_{\substack{P_X,Q_X:|P_X-Q_X| \ge \epsilon}} D_{1+s}(P_X || Q_X)$$

=
$$\inf_{\substack{P_X,Q_X:|P_X-Q_X| = \epsilon}} D_{1+s}(P_X || Q_X)$$
(31)

$$= \inf_{q \in [0, 1-\epsilon]} d_{1+s}(q+\epsilon ||q),$$
(32)

and for any $s \in (0, 1)$,

$$\inf_{q \in [0, 1-\epsilon]} d_{1-s}(q+\epsilon ||q)$$

$$\geq \left[\min\left\{1, \frac{1-s}{s}\right\} \log \frac{1}{1-\epsilon} - \frac{1}{s} \log 2\right]^+, \quad (33)$$

where

$$d_{1+s}(p||q) := \begin{cases} \frac{1}{s} \log(p^{1+s}q^{-s} + \bar{p}^{1+s}\bar{q}^{-s}), & s \ge -1, s \ne 0\\ p \log \frac{p}{q} + \bar{p} \log \frac{\bar{p}}{\bar{q}}, & s = 0 \end{cases}$$
(34)

denotes the binary Rénvi divergence of order 1 + s.⁴ We also have

$$\inf_{\substack{P_X,Q_X:|P_X-Q_X| \ge \epsilon}} D_0(P_X \| Q_X)$$

=
$$\inf_{\substack{P_X,Q_X:|P_X-Q_X| = \epsilon}} D_0(P_X \| Q_X)$$
(35)
= 0. (36)

3. Pinsker's inequality provides a Remark lower bound for $\inf_{P_X,Q_X:|P_X-Q_X|\geq\epsilon} D_{1+s}(P_X||Q_X)$ or $\inf_{P_X,Q_X:|P_X-Q_X|=\epsilon} D_{1+s}(P_X ||Q_X)$, i.e.,

$$\inf_{P_X, Q_X : |P_X - Q_X| = \epsilon} D_{1+s}(P_X || Q_X) \ge \frac{(1+s)\epsilon^2}{2}.$$
 (37)

Hence $\frac{(1+s)\epsilon^2}{2}$ is also a lower bound of $\inf_{q\in[0,1-\epsilon]} d_{1+s}(q+$ $\epsilon \| q).$

Remark 4. Using (32) and the lower bound in (33), it is easy to obtain the following improved lower bounds. For any $s \in$ (0,1),

$$\inf_{\substack{P_X,Q_X:|P_X-Q_X|\geq\epsilon}} D_{1-s}(P_X \| Q_X)$$

=
$$\inf_{\substack{P_X,Q_X:|P_X-Q_X|\geq\epsilon}} \sup_{t\in[s,1)} D_{1-t}(P_X \| Q_X)$$
(38)

⁴For s = 0 case, the binary Rényi divergence is known as the binary relative entropy, and it is usually denoted as d(p||q).

$$\geq \sup_{t \in [s,1)} \inf_{P_X, Q_X : |P_X - Q_X| \ge \epsilon} D_{1-t}(P_X \| Q_X)$$
(39)

$$\geq \sup_{t \in [s,1)} \inf_{q \in [0,1-\epsilon]} d_{1-t}(q+\epsilon \| q)$$
(40)

$$\geq \sup_{t \in [s,1)} \left[\min\left\{1, \frac{1-t}{t}\right\} \log \frac{1}{1-\epsilon} - \frac{1}{t} \log 2 \right]^+$$
(41)

$$= \begin{cases} \left[\log\frac{1}{4(1-\epsilon)}\right]^{+} & s \in (0,\frac{1}{2}], \\ \left[\frac{1-s}{s}\log\frac{1}{1-\epsilon} - \frac{1}{s}\log2\right]^{+} & s \in (\frac{1}{2},1), \epsilon > \frac{1}{2} \\ 0 & s \in (\frac{1}{2},1), \epsilon \le \frac{1}{2} \end{cases}$$
(42)

and for any $s \in [0, +\infty)$,

$$\inf_{\substack{P_X,Q_X:|P_X-Q_X|\geq\epsilon}} D_{1+s}(P_X \| Q_X) \\
\geq \inf_{\substack{P_X,Q_X:|P_X-Q_X|\geq\epsilon}} \sup_{t\in(0,1)} D_{1-t}(P_X \| Q_X) \tag{43}$$

$$\geq \sup_{t \in (0,1)} \inf_{P_X, Q_X: |P_X - Q_X| \ge \epsilon} D_{1-t}(P_X \| Q_X)$$

$$\tag{44}$$

$$\geq \sup_{t \in (0,1)} \inf_{q \in [0,1-\epsilon]} d_{1-t}(q+\epsilon \| q)$$
(45)

$$\geq \sup_{t \in (0,1)} \left[\min\left\{1, \frac{1-t}{t}\right\} \log \frac{1}{1-\epsilon} - \frac{1}{t} \log 2 \right]^+ \quad (46)$$

$$= \left[\log\frac{1}{4\left(1-\epsilon\right)}\right]^+.$$
(47)

Remark 5. The improved lower bounds (42) and (47) (or combining (32) and the lower bound in (33)) implies if

$$|P_X - Q_X| \to 1,\tag{48}$$

then for any $s \in (-1, +\infty)$,

$$D_{1+s}(P_X \| Q_X) \to \infty. \tag{49}$$

Combining Lemma 2 with Theorem 3, we have the converse part for the normalized Rényi divergence, which implies the strong converse for the unnormalized Rényi divergence.

Theorem 4. For any synthesis codes such that

$$\limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{M}_n| < C_{\mathsf{Wyner}}(X;Y), \tag{50}$$

we have for any s > -1,

$$\liminf_{n \to \infty} \frac{1}{n} D_{1+s}(P_{X^n Y^n} \| \pi_{X^n Y^n}) > 0.$$
(51)

Remark 6. This theorem establishes the converse part of Theorem 1 for the normalized Rényi common information.

Remark 7. Since $\liminf_{n\to\infty} \frac{1}{n} D_{1+s}(P_{X^nY^n} || \pi_{X^nY^n}) > 0$ implies $D_{1+s}(P_{X^nY^n} || \pi_{X^nY^n}) \to \infty$, the theorem above implies the strong converse for the Wyner's common information problem under the unnormalized Rénvi divergence.

Proof: Theorem 3 states if $\frac{1}{n} \log |\mathcal{M}_n| < C_{\mathsf{Wyner}}(X;Y)$, then $|P_{X^nY^n} - \pi_{X^nY^n}| \rightarrow 1$ exponentially fast. In other words,

$$|P_{X^nY^n} - \pi_{X^nY^n}| \ge 1 - e^{-n\delta_n},$$
 (52)

for some sequence $\delta_n > 0$ such that $\liminf_{n \to \infty} \delta_n > 0$. Therefore, using Lemma 2 we have

$$\liminf_{n \to \infty} \frac{1}{n} D_{1+s}(P_{X^n Y^n} \| \pi_{X^n Y^n})$$

$$\geq \liminf_{n \to \infty} \left\{ \min\left\{ 1, \frac{1-s}{s} \right\} \delta_n - \frac{1}{ns} \log 2 \right\}$$
(53)

$$= \min\left\{1, \frac{1-s}{s}\right\} \liminf_{n \to \infty} \delta_n \tag{54}$$
$$> 0. \tag{55}$$

This completes the proof.

V. CONCLUSION AND FUTURE WORK

In this paper, we studied a generalized version of Wyner's common information problem (or the distributed source simulation problem), in which the unnormalized and normalized Rényi divergences were used to measure the level of approximation. We showed the minimum rate needed to ensure that the unnormalized or normalized Rényi divergence vanishes asymptotically remains the same as the one under Wyner's setting where the relative entropy was used.

In the future, we plan to investigate the second-order coding rate for Wyner's common information under the unnormalized Rényi divergence or the TV distance. For the unnormalized Rényi divergence, the one-shot achievability bound given in Lemma 3 can be used to obtain an achievability bound for the second-order coding rate. In fact, it can easily be shown that the optimal second-order coding rate scales as $O(\frac{1}{\sqrt{n}})$. For the TV distance, the one-shot achievability bound given by Cuff [5] can be used to derive an achievability bound. However, the converse parts for both cases are not straightforward. One may leverage the perturbation approach [25] used to prove the second-order coding rate for the Gray-Wyner problem in [26], [27]. This is left as future work.

Furthermore, we are also interested in various closelyrelated problems. Among them, the most interesting one is the distributed channel synthesis problem under the Rényi divergence measure: The coordination problem or distributed channel synthesis problem was studied by Cuff, Permuter, and Cover [4], [5]. In this problem, an observer (encoder) of a source sequence describes the sequence to a distant random number generator (decoder) that produces another sequence. What is the minimum description rate needed to produce achieve a joint distribution that is statistically indistinguishable, under the TV distance, from the distribution induced by a given channel? For this problem, Cuff [5] provided a complete characterization of the minimum rate. We can enhance the level of coordination by replacing the TV measure with the Rényi divergence. For this enhanced version of the problem, we are interested in characterizing the corresponding admissible rate region.

APPENDIX A

ACHIEVABILITY PROOF OF THEOREM 1 FOR THE **UNNORMALIZED RÉNYI COMMON INFORMATION**

A. Achievability

Next we focus on the achievability part. We first consider the case $s \in (0, 1]$. First we introduce the following one-shot achievability bound (i.e., achievability bound for blocklength n equal to 1).

Lemma 3 (One-Shot Achievability Bound). [10] Consider a random mapping $P_{X|W}$ and a random codebook $U = \{W(i)\}_{i \in \mathcal{M}}$ with $W(i) \sim P_W, i \in \mathcal{M}$, where $\mathcal{M} = \{1, \ldots, e^R\}$. We define

$$P_{X|U}(x|\{w(i)\}_{i\in\mathcal{M}}) := \frac{1}{|\mathcal{M}|} \sum_{m\in\mathcal{M}} P_{X|W}(x|w(m)) \quad (56)$$

Then we have for $s \in (0, 1]$,

$$e^{sD_{1+s}(P_{X|U}\|\pi_X|P_U)} \le e^{sD_{1+s}(P_{X|W}\|\pi_X|P_W) - sR} + e^{sD_{1+s}(P_X\|\pi_X)}$$
(57)

$$\leq 2e^{s\Gamma_{1+s}(P_W, P_{X|W}, \pi_X, R)},\tag{58}$$

where

$$\Gamma_{1+s}(P_W, P_{X|W}, \pi_X, R) := \max \left\{ D_{1+s}(P_{X|W} \| \pi_X | P_W) - R, D_{1+s}(P_X \| \pi_X) \right\}.$$
(59)

Remark 8. This lemma provides a one-shot achievability bound for general source synthesis problems, not only for the distributed source synthesis or common information problem as studied in this paper.

By setting π_X , $P_{X|W}$, P_W , and R to $\pi_{X^nY^n}$, $P_{X^nY^n|W^n} = P_{X^n|W^n}P_{Y^n|W^n}$, P_{W^n} , and nR respectively, Lemma 3 can be used to derive an achievability result for the common information problem. Applying Lemma 3 and taking limits appropriately, we obtain if there exists a sequence of distributions $\{P_{W^n}P_{X^n|W^n}P_{Y^n|W^n}\}$ such that $\lim_{n\to\infty} D_{1+s}(P_{X^nY^n}|\pi_{X^nY^n}) \to 0$ and $R > \lim_{n\to\infty} \frac{1}{n}D_{1+s}(P_{X^nY^n|W^n}|\pi_{X^nY^n}|P_{W^n})$, then there exists a sequence of codes such that

$$\begin{split} &\limsup_{n \to \infty} D_{1+s} (P_{X^{n}Y^{n}|U_{n}} \| \pi_{X^{n}Y^{n}} | P_{U_{n}}) \\ &\leq \limsup_{n \to \infty} \frac{1}{s} \log \Big\{ e^{sD_{1+s} (P_{X^{n}Y^{n}|W^{n}} \| \pi_{X^{n}Y^{n}} | P_{W^{n}}) - nsR} \\ &+ e^{sD_{1+s} (P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})} \Big\} \end{split}$$
(60)

$$\leq \frac{1}{s} \log \left\{ \limsup_{n \to \infty} \mathrm{e}^{s \left(D_{1+s} \left(P_{X^n Y^n | W^n} \| \pi_{X^n Y^n} | P_{W^n} \right) - nR \right)} + 1 \right\}$$
(61)

$$\leq \frac{1}{s} \log \left\{ \limsup_{n \to \infty} e^{s(n(R-\epsilon) - nR)} + 1 \right\}$$
(62)

$$=0,$$
 (63)

where (62) follows since

$$R > \limsup_{n \to \infty} \frac{1}{n} D_{1+s} (P_{X^n Y^n | W^n} \| \pi_{X^n Y^n} | P_{W^n})$$

implies there exists a constant $\epsilon > 0$ such that

$$R - \epsilon > \frac{1}{n} D_{1+s}(P_{X^n Y^n | W^n} \| \pi_{X^n Y^n} | P_{W^n})$$

⁵The pair (X^n, Y^n) plays the role of X in Lemma 3.

holds for all sufficiently large n. Therefore, the minimum achievable rate satisfies

$$\inf \left\{ R: D_{1+s}(P_{X^{n}Y^{n}|U_{n}} \| \pi_{X^{n}Y^{n}}|P_{U_{n}}) \to 0 \right\}$$

$$\leq \inf_{\substack{\{P_{W^{n}}, P_{X^{n}|W^{n}}, P_{Y^{n}|W^{n}}\}_{n=1}^{\infty}: \\ D_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}}) \to 0}$$

$$\limsup_{n \to \infty} \frac{1}{n} D_{1+s}(P_{X^{n}Y^{n}|W^{n}} \| \pi_{X^{n}Y^{n}}|P_{W^{n}}). \quad (64)$$

Let Q_{WXY} be a distribution such that $Q_{XY} = \pi_{XY}$ and X - W - Y. For the optimization in (64), to obtain an upper bound, we set the distributions

$$P_{W^{n}}(w^{n}) \propto Q_{W}^{n}(w^{n}) \mathbf{1} \{w^{n} \in \mathcal{T}_{\epsilon'}^{n}(Q_{W})\},\$$

$$P_{X^{n}|W^{n}}(x^{n}|w^{n}) \propto Q_{X|W}^{n}(x^{n}|w^{n}) \mathbf{1} \{x^{n} \in \mathcal{T}_{\epsilon}^{n}(Q_{WX}|w^{n})\},\$$

$$P_{Y^{n}|W^{n}}(x^{n}|w^{n}) \propto Q_{Y|W}^{n}(x^{n}|w^{n}) \mathbf{1} \{y^{n} \in \mathcal{T}_{\epsilon}^{n}(Q_{WY}|w^{n})\},\$$

where $0 < \epsilon' < \epsilon \leq 1$. Then we have

$$P_{X^{n}Y^{n}}(x^{n}, y^{n}) = \sum_{w^{n}} P_{W^{n}}(w^{n}) P_{X^{n}|W^{n}}(x^{n}|w^{n}) P_{Y^{n}|W^{n}}(x^{n}|w^{n}) \quad (65)$$

$$= \sum_{w^{n}} \frac{Q_{W}^{n}(w^{n}) 1 \{w^{n} \in \mathcal{T}_{\epsilon'}^{n}(Q_{W})\}}{Q_{W}^{n}(\mathcal{T}_{\epsilon'}^{n})} \times \frac{Q_{X|W}^{n}(w^{n}|w^{n}) 1 \{x^{n} \in \mathcal{T}_{\epsilon'}^{n}(Q_{WX}|w^{n})\}}{Q_{X|W}^{n}(\mathcal{T}_{\epsilon}^{n}(Q_{WX}|w^{n})|w^{n})} \times \frac{Q_{Y|W}^{n}(x^{n}|w^{n}) 1 \{y^{n} \in \mathcal{T}_{\epsilon}^{n}(Q_{WY}|w^{n})\}}{Q_{Y|W}^{n}(\mathcal{T}_{\epsilon}^{n}(Q_{WY}|w^{n})|w^{n})} \quad (66)$$

$$\leq \frac{\sum_{w^{n}} Q_{WXY}^{n}(w^{n},x^{n},y^{n})}{Q_{W}^{n}(\mathcal{T}_{\epsilon'}^{n})} \times \frac{1}{\min_{w^{n}\in\mathcal{T}_{\epsilon'}^{n}} Q_{X|W}^{n}(\mathcal{T}_{\epsilon}^{n}(Q_{WY}|w^{n})|w^{n})} \times \frac{1}{\min_{w^{n}\in\mathcal{T}_{\epsilon'}^{n}} Q_{Y|W}^{n}(\mathcal{T}_{\epsilon}^{n}(Q_{WY}|w^{n})|w^{n})} \quad (67)$$

$$=\frac{\pi_{X^{n}Y^{n}}(x^{n},y^{n})}{1-\delta_{n}},$$
(68)

where in (68) δ_n is defined as 1 minus the denominator of (67). Here we claim that $\delta_n \to 0$ as $n \to \infty$. This follows since $Q_W^n(\mathcal{T}_{\epsilon'}^n) \to 1$, $\min_{w^n \in \mathcal{T}_{\epsilon'}^n} Q_{X|W}^n(\mathcal{T}_{\epsilon}^n(Q_{WX}|w^n)|w^n) \to 1$, and $\min_{w^n \in \mathcal{T}_{\epsilon'}^n} Q_{Y|W}^n(\mathcal{T}_{\epsilon}^n(Q_{WY}|w^n)|w^n) \to 1$, where the last two limits hold due to the following lemma.

Lemma 4. Assume $0 < \epsilon' < \epsilon \leq 1$, then as $n \to \infty$, $Q_{X|W}^{n}(\mathcal{T}_{\epsilon}^{n}(Q_{WX}|w^{n})|w^{n})$ converges uniformly⁶ to 1 (in $w^{n} \in \mathcal{T}_{\epsilon'}^{n}(Q_{W})$).

$$1 - Q_{X|W}^{n} \left(\mathcal{T}_{\epsilon}^{n}(Q_{WX}|w^{n})|w^{n} \right) \leq |\mathcal{X}| |\mathcal{W}| \left(e^{-\frac{1}{3} \left(\frac{\epsilon - \epsilon'}{1 + \epsilon'} \right)^{2} n Q_{X|W}^{(\min)}} + e^{-\frac{1}{2} \left(\frac{\epsilon - \epsilon'}{1 - \epsilon'} \right)^{2} n Q_{X|W}^{(\min)}} \right), \quad (69)$$

⁶This means that for any $\eta > 0$, there exists an integer $N = N_{\eta}$ such that $\max_{w^n \in \mathcal{T}^n_{\epsilon'}(Q_W)} 1 - Q^n_{X|W}(\mathcal{T}^n_{\epsilon}(Q_{WX}|w^n)|w^n) \leq \eta$ for all $n > N_{\eta}$. Here the notion of "uniform convergence" is a slightly different from the conventional one [28, Definition 7.7]. In the conventional definition, the domain of the functions are fixed but here, the domain $\mathcal{T}^n_{\epsilon'}(Q_W)$ depends on n.

for all $w^n \in \mathcal{T}^n_{\epsilon'}(Q_W)$, where $Q_{X|W}^{(\min)} := \min_{(x,w):Q_X|W}(x|w)>0} Q_{X|W}(x|w)$.

This lemma is a stronger version of the conditional typicality lemma in [2], since here the probability converges uniformly, instead of converging pointwise. However, the proof is merely a refinement of the conditional typicality lemma [2, Appendix 2A] (by applying the Chernoff bound, instead of the law of large numbers), and hence omitted here. Besides, a similar lemma can be found in [29, Lemma 2.12], which is established based on a slightly different definition of strong typicality.

Using this upper bound of $P_{X^nY^n}(x^n, y^n)$ we have

$$D_{1+s}(P_{X^nY^n} \| \pi_{X^nY^n})$$

= $\frac{1}{s} \log \sum_{x^n, y^n} P_{X^nY^n}^{1+s}(x^n, y^n) \pi_{X^nY^n}^{-s}(x^n, y^n)$ (70)

$$\leq \frac{1}{s} \log \sum_{x^{n}, y^{n}} \left(\frac{\pi_{X^{n}Y^{n}}(x^{n}, y^{n})}{1 - \delta_{n}} \right)^{1+s} \pi_{X^{n}Y^{n}}^{-s}(x^{n}, y^{n})$$
(71)

$$=\frac{1}{s}\log\left(\frac{1}{1-\delta_n}\right)^{1+s}\tag{72}$$

$$\rightarrow 0.$$
 (73)

Let $[T_W V_{X|W}]$ denote the joint distribution of X and W induced by the type T_W and conditional type $V_{X|W}$. Now define the sets of tuples of types and conditional types:

$$\mathcal{A} := \left\{ (T_W, V_{X|W}, V_{Y|W}) : \\ \forall w, |T_W(w) - Q_W(w)| \le \epsilon' Q_W(w), \\ \forall (w, x), \left| [T_W V_{X|W}](w, x) - Q_{WX}(w, x) \right| \le \epsilon Q_{WX}(w, x), \\ \forall (w, y), \left| [T_W V_{Y|W}](w, y) - Q_{WY}(w, y) \right| \le \epsilon Q_{WY}(w, y) \right\}$$

$$\tag{74}$$

and

$$\mathcal{B} := \left\{ (T_W, V_{X|W}, V_{Y|W}) : \forall (w, x, y), \\ \frac{(1-\epsilon)^2}{1+\epsilon'} \le \frac{[T_W V_{X|W} V_{Y|W}](w, x, y)}{Q_{WXY}(w, x, y)} \le \frac{(1+\epsilon)^2}{1-\epsilon'} \right\}.$$
(75)

In (75), if $Q_{WXY}(w, x, y) = 0$, this imposes that $[T_W V_{X|W} V_{Y|W}](w, x, y) = 0$. It is easy to verify that $\mathcal{A} \subseteq \mathcal{B}$. Let $\delta_{1,n}$ and $\delta_{2,n}$ be two arbitrary sequences tending to zero as $n \to \infty$. Using these notations, we can write (76)-(84) (shown at the top of the next page), where (79) follows from Lemma 4, (80) follows since $\mathcal{A} \subseteq \mathcal{B}$, (81) follows since

$$\sum_{\substack{(T_X, V_X|_W, V_Y|_W) \in \mathcal{B} \ w^n \in \mathcal{T}_{T_W}, x^n \in \mathcal{T}_{V_X|_W}(w^n), \\ y^n \in \mathcal{T}_{V_Y|_W}(w^n)}} P\left(w^n, x^n, y^n\right) \le 1,$$
(85)

and (83) follows since $Q_{XY} = \pi_{XY}$.

Letting $n \to \infty$ in (84), we have

$$\lim_{n \to \infty} \sup_{n \to \infty} \frac{1}{n} D_{1+s} \left(P_{W^n X^n Y^n} \| P_{W^n} \pi_{X^n Y^n} \right)$$

$$\leq \frac{(1-\epsilon)^2}{1+\epsilon'} I_Q(XY;W) + \frac{4\epsilon}{1-\epsilon'} H_Q(XY).$$
(86)

Combining (86) with (64) and (73), we obtain

$$\inf \left\{ R : D_{1+s}(P_{X^{n}Y^{n}|U_{n}} \| \pi_{X^{n}Y^{n}} | P_{U_{n}}) \to 0 \right\}$$

$$\leq \frac{(1-\epsilon)^{2}}{1+\epsilon'} I_{Q}(XY;W) + \frac{4\epsilon}{1-\epsilon'} H_{Q}(XY).$$
(87)

Since $\epsilon > \epsilon' > 0$ are arbitrary, and $H_Q(XY) = H_{\pi}(XY) \le \log \{|\mathcal{X}||\mathcal{Y}|\}$ is bounded, we have

$$\inf \left\{ R : D_{1+s}(P_{X^n Y^n | U_n} \| \pi_{X^n Y^n} | P_{U_n}) \to 0 \right\}$$

$$\leq I_Q(XY; W).$$
(88)

Since the distribution Q_{WXY} is arbitrary, we can minimize $I_Q(XY;W)$ over all distributions satisfying $Q_{XY} = \pi_{XY}$ and X - W - Y. Hence

$$\inf \left\{ R : D_{1+s}(P_{X^{n}Y^{n}|U_{n}} \| \pi_{X^{n}Y^{n}}|P_{U_{n}}) \to 0 \right\}$$

$$\leq \bigcup_{Q_{XYW}: Q_{XY} = \pi_{XY}, X - W - Y} I_{Q}(XY;W)$$
(89)
(89)

$$= C_{\mathsf{Wyner}}(X;Y). \tag{90}$$

Observe that

$$D_{1+s}(P_{X^{n}Y^{n}|U_{n}} \| \pi_{X^{n}Y^{n}}|P_{U_{n}})$$

$$= \frac{1}{s} \log \mathbb{E}_{U_{n}} \bigg[\sum_{x^{n},y^{n}} P_{X^{n}Y^{n}|U_{n}}(x^{n},y^{n}|U_{n}) \\ \times \bigg(\frac{P_{X^{n}Y^{n}|U_{n}}(x^{n},y^{n}|U_{n})}{\pi_{X^{n}Y^{n}}(x^{n},y^{n})} \bigg)^{s} \bigg], \qquad (91)$$

where \mathbb{E}_{U_n} is the expectation taken with respect to the distribution P_{U_n} . Hence $D_{1+s}(P_{X^nY^n|U_n} \| \pi_{X^nY^n}|P_{U_n}) \to 0$ implies that there must exist at least one sequence of codebooks indexed by $\{u_n\}_{n=1}^{\infty}$ such that $D_{1+s}(P_{X^nY^n|U_n=u_n} \| \pi_{X^nY^n}) \to$ 0. Therefore, the Rényi common information for $s \in (0, 1]$ is not larger than $C_{\text{Wyner}}(X; Y)$. This completes the proof for the case $s \in (0, 1]$.

Now we prove the case $s \in (-1,0)$. Since $D_{1+s}(P_{X^nY^n} || \pi_{X^nY^n})$ is non-decreasing in s, the result for $s \in (0,1]$ implies the achievability result for $s \in (-1,0)$.

B. Exponential Achievability

Since $D_{1+s}(P_{X^nY^nU_n} || \pi_{X^nY^n} \times P_{U_n})$ is non-decreasing in s, to prove the exponential result for $s \in (-1, 1]$, we only need to show the result holds for $s \in (0, 1]$. To this end, we use the random code given in Appendix A-A. For this code, by Lemma 3, we obtain

$$e^{sD_{1+s}(P_{X^{n}Y^{n}U_{n}} \| \pi_{X^{n}Y^{n}} \times P_{U_{n}})} \leq e^{sD_{1+s}(P_{W^{n}X^{n}Y^{n}} \| P_{W^{n}}\pi_{X^{n}Y^{n}}) - nsR} + e^{sD_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})}$$
(92)
$$= e^{sD_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})} (e^{sD_{1+s}(P_{W^{n}X^{n}Y^{n}} \| P_{W^{n}}\pi_{X^{n}Y^{n}}) - nsR} \times e^{-sD_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})} + 1).$$
(93)

$$\begin{split} &\frac{1}{n} D_{1+s} \left(P_{W^n X^n Y^n} \| P_{W^n \pi X^n Y^n} \right) \\ &= \frac{1}{ns} \log \sum_{w^n, x^n, y^n} P\left(w^n\right) \left(P\left(x^n | w^n\right) P\left(y^n | w^n\right) \right)^{1+s} \pi^{-s} \left(x^n, y^n\right) \\ &= \frac{1}{ns} \log \sum_{w^n, x^n, y^n} P\left(w^n, x^n, y^n\right) \\ &\times \left(\frac{Q_{X|W}^n \left(x^n | w^n\right) 1 \left\{x^n \in \mathcal{T}_c^n \left(Q_{WX} | w^n\right) \right\}}{Q_{X|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right)} \frac{Q_{Y|W}^n \left(x^n | w^n\right) 1 \left\{y^n \in \mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right\}}{Q_{Y|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right)} \right)^s \pi_{X^n Y^n}^{-s} \left(x^n, y^n\right) \\ &= \frac{1}{ns} \log \sum_{T_W, V_{X|W}, V_{Y|W}} \sum_{w^n \in \mathcal{T}_{Y_W, w^n}^n \in \mathcal{T}_{Y_{Y|W}}^n \left(w^n\right)} P\left(w^n, x^n, y^n\right) \\ &\times \left(\frac{Q_{X|W}^n \left(x^n | w^n\right) 1 \left\{x^n \in \mathcal{T}_c^n \left(Q_{WX} | w^n\right) \right\}}{Q_{Y|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right)} \frac{Q_{Y|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right)}{Q_{Y|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) \right)} \right)^s \pi_{X^n Y^n}^{-s} \left(x^n, y^n\right) \\ &\times \left(\frac{Q_{X|W}^n \left(\mathcal{T}_c^n \left(Q_{WX} | w^n\right) | w^n - \mathcal{T}_{Y_{Y|W}}^n \left(w^n\right)}{Q_{X|W}^n \left(\mathcal{T}_c^n \left(Q_{WX} | w^n\right) | w^n\right)} \right) \frac{P\left(w^n, x^n, y^n\right)}{Q_{Y|W}^n \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) | w^n\right)} \right)^s \pi_{X^n Y^n}^{-s} \left(x^n, y^n\right) \\ &\leq \frac{1}{ns} \log \left(\sum_{(T_X, V_{X|W}, V_{Y|W}) \in \mathcal{A}} w^n \in \mathcal{T}_{TW}^{-s} e^{\varepsilon \mathcal{T}_{Y,W|W}^n} (w^n)}{1 - \delta_{2,n}} \right) \frac{P\left(w^n, x^n, y^n\right)}{2^{s-r_{Y,W}^n} \left(\mathcal{T}_c^n \left(Q_{WY} | w^n\right) | w^n\right)} \right)^s \pi_{X^n Y^n}^{-s} \left(x^n, y^n\right) \\ &\times \left(\frac{e^n \sum_{(T_X, V_{X|W}, V_{Y|W}) \in \mathcal{B}} w^n \in \mathcal{T}_{TW}^{-s} (T_{W,Y|W}) \log q(y|w)}{1 - \delta_{2,n}} \right)^s e^{-ns \sum_{x,y} \mathcal{T}(x,y) \log \pi(x,y)} \right) \\ &\leq \frac{1}{(T_X, V_{X|W}, V_{Y|W}) \in \mathcal{B}} \left(\sum_{w, x} \mathcal{T}^n \left(w, x\right) \log Q(x|w) + \sum_{w, y} \mathcal{T}^n \left(w, y\right) \log Q(y|w) - \sum_{x, y} \mathcal{T}^n \left(x, y\right) \log \pi(x,y) \right) \\ &\leq \frac{1}{(T_X, V_{X|W}, V_{Y|W}) \in \mathcal{B}} \left(\sum_{w, x} \mathcal{T}^n \left(w, x\right) \log Q(x|w) + \sum_{w, y} \mathcal{T}^n \left(w, y\right) \log Q(y|w) - \sum_{x, y} \mathcal{T}^n \left(x, y\right) \log \pi(x,y) \right) \\ &= \frac{1}{n} \log(1 - \delta_{1,n})(1 - \delta_{2,n}) \end{aligned} \right)$$

$$(81)$$

$$\leq \frac{(1 - e)^2}{1 + e^2} \left(\sum_{w, x} Q(w, x) \log Q(x|w) + \sum_{w, y} Q(w, y) \log Q(y|w) \right) - \frac{(1 + e)^2}{1 - e^2} \sum_{x, y} Q(x, y) \log \pi(x, y) \\ &= \frac{1}{n} \log(1 - \delta_{1,n})(1 - \delta_{2,n}) \end{aligned} \right)$$

$$= -\frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} \left(H_{Q}(X|W) + H_{Q}(Y|W) \right) + \frac{(1+\epsilon)^{2}}{1-\epsilon^{\prime}} H_{Q}(XY) - \frac{1}{n} \log(1-\delta_{1,n})(1-\delta_{2,n})$$

$$(83)$$

$$(1-\epsilon)^{2} H_{Q}(X|W) + \frac{4\epsilon}{1-\epsilon^{\prime}} H_{Q}(X|V) - \frac{1}{1-\epsilon^{\prime}} H_{Q}(XY) - \frac{1}{n} \log(1-\delta_{1,n})(1-\delta_{2,n})$$

$$(83)$$

$$= \frac{(1-\epsilon)^2}{1+\epsilon'} I_Q(XY;W) + \frac{4\epsilon}{1-\epsilon'} H_Q(XY) - \frac{1}{n} \log(1-\delta_{1,n})(1-\delta_{2,n}),$$
(84)

Taking \log 's and normalizing by s,

$$D_{1+s}(P_{X^{n}Y^{n}U_{n}} \| \pi_{X^{n}Y^{n}} \times P_{U_{n}})$$

$$= D_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})$$

$$+ \frac{1}{s} \log \left(e^{sD_{1+s}(P_{W^{n}X^{n}Y^{n}} \| P_{W^{n}}\pi_{X^{n}Y^{n}}) - nsR} \times e^{-sD_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})} + 1 \right)$$
(94)

$$\leq D_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}}) + \frac{1}{s} e^{sD_{1+s}(P_{W^{n}X^{n}Y^{n}} \| P_{W^{n}\pi_{X^{n}Y^{n}}})} \\ \times e^{-nsR - sD_{1+s}(P_{X^{n}Y^{n}} \| \pi_{X^{n}Y^{n}})}.$$
(95)

We first consider the first term of (95). Note that in

(68), δ_n tends to zero exponentially fast as $n \to \infty$, since $Q_W^n(\mathcal{T}_{\epsilon'}^n)$, $\min_{w^n \in \mathcal{T}_{\epsilon'}^n} Q_{X|W}^n(\mathcal{T}_{\epsilon}^n(Q_{WX}|w^n)|w^n)$, and $\min_{w^n \in \mathcal{T}_{\epsilon'}^n} Q_{Y|W}^n(\mathcal{T}_{\epsilon}^n(Q_{WY}|w^n)|w^n)$ all tend to one exponentially fast as $n \to \infty$. Combining this with (73), we obtain that $D_{1+s}(P_{X^nY^n} || \pi_{X^nY^n}) \to 0$ exponentially fast.

Furthermore, by (84) we can write the exponent of the 4) second term of (95) as

$$\liminf_{n \to \infty} sR - \frac{1}{n} sD_{1+s}(P_{W^n X^n Y^n} \| P_{W^n} \pi_{X^n Y^n}) \\ + \frac{1}{n} sD_{1+s}(P_{X^n Y^n} \| \pi_{X^n Y^n})$$

$$= sR - s\left(\frac{(1-\epsilon)^2}{1+\epsilon'}I_Q(XY;W) + \frac{4\epsilon}{1-\epsilon'}H_Q(XY)\right).$$
(96)

Since $H_Q(XY) = H_{\pi}(XY) \le \log \{|\mathcal{X}||\mathcal{Y}|\}\$ is bounded and $R > I_Q(XY; W)$, by choosing sufficiently small $\epsilon > \epsilon' > 0$, we can ensure this exponent is positive.

Combining the two points above, we conclude that the optimal $D_{1+s}(P_{X^nY^nU_n} \| \pi_{X^nY^n} \times P_{U_n})$ tends to zero exponentially fast as long as $R > C_{Wyner}(X;Y)$. On the other hand, by a similar argument in Appendix A-A, $D_{1+s}(P_{X^nY^n|U_n} \| \pi_{X^nY^n} | P_{U_n}) \to 0$ exponentially fast implies that there must exist at least one sequence of codebooks indexed by $\{u_n\}_{n=1}^{\infty}$ such that

$$D_{1+s}(P_{X^nY^n|U_n=u_n} \| \pi_{X^nY^n}) \to 0$$

exponentially fast. Hence the proof is completed.

APPENDIX B PROOF OF THEOREM 3

A. Proof of Theorem 3

In this section, we present the proof of Theorem 3. In the proof, we adapt the information spectrum method proposed by Oohama [14] to first establish a non-asymptotic lower bound on $|P_{X^nY^n} - \pi_{X^nY^n}|$. Invoking the lower bound (cf. Lemma 6) and applying Cramér's bound in the theory of large deviations [30], we can obtain a further lower bound on $|P_{X^nY^n} - \pi_{X^nY^n}|$ leading to (28).

Let $P_{M_n X^n Y^n}$ be the joint distribution of (M_n, X^n, Y^n) , induced by the synthesis code, i.e.,

$$P_{M_n X^n Y^n}(m, x^n, y^n) = \frac{1}{|\mathcal{M}_n|} P_{X^n | M_n}(x^n | m) P_{Y^n | M_n}(y^n | m).$$
(97)

In the following, for brevity sometimes we omit the subscript, and write $P_{M_n X^n Y^n}$ as P.

Let $Q_{X^nY^n}$ and $Q_{X^nY^n|M_n}$ be arbitrary distributions. Given any $\eta > 0$, define the following information-spectrum sets and support sets:

$$\mathcal{A}_{1} := \left\{ (x^{n}, y^{n}) : \frac{1}{n} \log \frac{\pi_{X^{n}Y^{n}}(x^{n}, y^{n})}{Q_{X^{n}Y^{n}}(x^{n}, y^{n})} \ge -\eta \right\} \times \mathcal{M}_{n},$$
(98)

$$\mathcal{A}_{2} := \left\{ (x^{n}, y^{n}, m) : \\ \frac{1}{n} \log \frac{P_{X^{n}|M_{n}}(x^{n}|w)P_{Y^{n}|M_{n}}(y^{n}|m)}{Q_{X^{n}Y^{n}|M_{n}}(x^{n}, y^{n}|m)} \ge -\eta \right\}, \quad (99)$$

$$\mathcal{A}_{3} := \left\{ (x^{n}, y^{n}, m) : \\ \frac{1}{n} \log \frac{Q_{X^{n}Y^{n}|M_{n}}(x^{n}, y^{n}|m)}{\pi_{X^{n}Y^{n}}(x^{n}, y^{n})} \le R + \eta \right\},$$
(100)

$$\widetilde{\mathcal{A}}_1 := \operatorname{supp}(\pi_{X^n Y^n}) \times \mathcal{M}_n, \tag{101}$$

$$\widetilde{\mathcal{A}}_2 := \operatorname{supp}(P_{X^n Y^n M_n}), \tag{102}$$

$$\widetilde{\mathcal{A}} := \widetilde{\mathcal{A}}_1 \cap \widetilde{\mathcal{A}}_2. \tag{103}$$

Choose $U_i = M_n$ and $V_i = (X^{i-1}, Y^{i-1})$. For $i = 1, \ldots, n$, let $Q_{X_i Y_i U_i V_i}$ be any distribution and let

$$\mathcal{B}_{1} := \left\{ (x^{n}, y^{n}, v^{n}) : \\ \frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_{i}Y_{i}|V_{i}}(x_{i}, y_{i}|v_{i})}{\pi_{XY}(x_{i}, y_{i})} \leq \eta \right\} \times \mathcal{M}_{n}^{n}, \quad (104)$$
$$\mathcal{B}_{2} := \left\{ (x^{n}, y^{n}, u^{n}, v^{n}) : \right\}$$

$$\frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_i Y_i | U_i V_i}(x_i, y_i | u_i, v_i)}{P_{X_i | U_i V_i}(x_i | u_i, v_i) P_{Y_i | U_i V_i}(y_i | u_i, v_i)} \le \eta \Big\},$$
(105)

$$\mathcal{B}_{3} := \left\{ (x^{n}, y^{n}, u^{n}, v^{n}) : \frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_{i}Y_{i}|U_{i}V_{i}}(x_{i}, y_{i}|u_{i}, v_{i})}{\pi_{XY}(x_{i}, y_{i})} \le R + \eta \right\}.$$
(106)

We first present a non-asymptotic lower bound on $|P_{X^nY^n} - \pi_{X^nY^n}|$, i.e., a non-asymptotic converse bound for the problem.

Lemma 5. For any synthesis code such that

$$\frac{1}{n}\log|\mathcal{M}_n| \le R,\tag{107}$$

we have

$$|P_{X^nY^n} - \pi_{X^nY^n}| \ge 1 - P\left(\bigcap_{i=1}^3 \mathcal{A}_i \mid \widetilde{\mathcal{A}}\right) - 3\mathrm{e}^{-n\eta}, \quad (108)$$

where $P(\cdot | \widetilde{\mathcal{A}}) = P_{X^n Y^n M_n | \widetilde{\mathcal{A}}}$ denotes the conditional distribution of $(X^n, Y^n, M_n) \sim P_{M_n X^n Y^n}$ given that $(X^n, Y^n, M_n) \in \widetilde{\mathcal{A}}$, with $P_{M_n X^n Y^n}$ denoting the distribution induced by the synthesis code.

The proof of Lemma 5 is given in Appendix B-B.

Invoking Lemma 5 and choosing the distributions $Q_{X^nY^n}$ and $Q_{X^nY^n|M_n}$ as in the paragraph above (104), we obtain the following lemma.

Lemma 6. Given the conditions in Lemma 5, we have

$$|P_{X^nY^n} - \pi_{X^nY^n}| \ge 1 - P\Big(\bigcap_{i=1}^3 \mathcal{B}_i \,\Big|\,\widetilde{\mathcal{A}}\Big) - 3\mathrm{e}^{-n\eta}.$$
 (109)

The proof of Lemma 6 is given in Appendix B-C.

In the following, for simplicity, we will use Q_i to denote $Q_{X_iY_iU_iV_i}$ and use P_i to denote $P_{X_iY_iU_iV_i}$. Let $\alpha \in [0, 1]$. Then we need the following definitions to further lower bound (109). Similar to the definition of $\omega_{Q_{XYU}}^{(\alpha)}(x,y|u)$ in (22), we define

$$\omega_{Q_{i},P_{i}}^{(\alpha)}(x_{i},y_{i}|u_{i},v_{i}) \\
:= \bar{\alpha} \left(\log \frac{Q_{X_{i}Y_{i}|V_{i}}(x_{i},y_{i}|v_{i})}{\pi_{XY}(x_{i},y_{i})} \\
+ \log \frac{Q_{X_{i}Y_{i}|U_{i}V_{i}}(x_{i},y_{i}|u_{i},v_{i})}{P_{X_{i}|U_{i}V_{i}}(x_{i}|u_{i},v_{i})P_{Y_{i}|U_{i}V_{i}}(y_{i}|u_{i},v_{i})} \right) \\
+ \alpha \log \frac{Q_{X_{i}Y_{i}|U_{i}V_{i}}(x_{i},y_{i}|u_{i},v_{i})}{\pi_{XY}(x_{i},y_{i})}. \quad (110)$$

Then, similar to the definition of $\Omega^{(\alpha,\theta)}(Q_{XYU})$ in (23), we define

$$\Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n)$$

$$:= -\log\left(\sum_{x^n, y^n, m} P_{X^n Y^n M_n \mid \widetilde{\mathcal{A}}}(x^n, y^n, m) \times \exp\left(-\lambda \sum_{i=1}^n \omega_{Q_i, P_i}^{(\alpha)}(x_i, y_i \mid u_i, v_i)\right)\right).$$
(111)

where $u_i = m$, $v_i = (x^{i-1}, y^{i-1})$, and $P_{X^nY^nM_n|\widetilde{\mathcal{A}}}$ is the conditional distribution of (X^n, Y^n, M_n) given $(X^n, Y^n, M_n) \in \widetilde{\mathcal{A}}$.

Applying Cramér's bound [30, Section 2.2] and utilizing Lemma 6, we obtain the following lemma. The proof of this lemma is similar to that of [14, Proposition 1], and hence we omit it for the sake of brevity.

Lemma 7. For any $(\alpha, \lambda) \in [0, 1] \times [0, \infty)$, given the condition in Lemma 5, we have

$$|P_{X^{n}Y^{n}} - \pi_{X^{n}Y^{n}}|$$

$$\geq 1 - 4 \exp\left(-n\frac{\frac{1}{n}\Omega^{(\alpha,\lambda)}(\{Q_{i}\}_{i=1}^{n}) - \lambda\alpha R}{1 + (1 + \bar{\alpha})\lambda}\right). \quad (112)$$

Let

$$\underline{\Omega}^{(\alpha,\lambda)} := \inf_{n \ge 1} \inf_{\{Q_i\}_{i=1}^n} \frac{1}{n} \Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n).$$
(113)

Define

$$\theta := \frac{\lambda}{1 - 2\bar{\alpha}\lambda}.\tag{114}$$

Hence, we have

$$\lambda = \frac{\theta}{1 + 2\bar{\alpha}\theta}.$$
(115)

The next lemma is essential in the proof.

Lemma 8. For $\alpha \in [0,1]$ and $\lambda \in [0,\frac{1}{2\bar{\alpha}})$, we have

$$\underline{\Omega}^{(\alpha,\lambda)} \ge \frac{\Omega^{(\alpha,\theta)}}{1+2\bar{\alpha}\theta}.$$
(116)

The proof of Lemma 8 is similar to that of [14, Proposition 2] and given in Appendix B-D. In the proof of Lemma 8, we adopt ideas from [14] and choose appropriate distributions $Q_{X_iY_iU_iV_i}$ via the recursive method.

Combining Lemmas 7 and 8 yields

$$|P_{X^{n}Y^{n}} - \pi_{X^{n}Y^{n}}| \geq 1 - 4 \exp\left(-n\frac{\underline{\Omega}^{(\alpha,\lambda)} - \lambda\alpha R}{1 + (1 + \bar{\alpha})\lambda}\right)$$
(117)

$$\geq 1 - 4 \exp\left(-n\frac{\frac{\Omega^{(\alpha,\theta)}}{1+2\bar{\alpha}\theta} - \frac{\theta\alpha R}{1+2\bar{\alpha}\theta}}{1 + \frac{(1+\bar{\alpha})\theta}{1+2\bar{\alpha}\theta}}\right)$$
(118)

$$= 1 - 4 \exp\left(-n\frac{\Omega^{(\alpha,\theta)} - \theta\alpha R}{1 + (5 - 3\alpha)\theta}\right)$$
(119)

$$\geq 1 - 4 \exp\left(-nF(R)\right),\tag{120}$$

where (120) follows from the definition of F(R) in (26) and the fact that (119) holds for any $(\alpha, \theta) \in [0, 1] \times (0, +\infty)$. The proof of Theorem 3 is now complete.

B. Proof of Lemma 5

Define $\pi_{X^nY^nM_n} := \pi_{X^nY^n}P_{M_n|X^nY^n}$. Then

$$|P_{X^{n}Y^{n}} - \pi_{X^{n}Y^{n}}| = |P_{X^{n}Y^{n}M_{n}} - \pi_{X^{n}Y^{n}M_{n}}|$$
(121)

$$\geq \pi(\widetilde{\mathcal{A}}_1 \cap \mathcal{A}_1 \cap \mathcal{A}_3) - P(\widetilde{\mathcal{A}}_1 \cap \mathcal{A}_1 \cap \mathcal{A}_3)$$
(122)

$$= 1 - \pi(\widetilde{\mathcal{A}}_{1}^{c} \cup \mathcal{A}_{1}^{c} \cup \mathcal{A}_{3}^{c}) - P(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3})$$
(123)

$$= 1 - P\left(\widetilde{\mathcal{A}} \cap \left(\bigcap_{i=1}^{\circ} \mathcal{A}_{i}\right)\right) - P(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3} \cap (\mathcal{A}_{2}^{c} \cup \widetilde{\mathcal{A}}_{2}^{c})) - \pi(\widetilde{\mathcal{A}}_{1}^{c} \cup \mathcal{A}_{1}^{c} \cup \mathcal{A}_{3}^{c}) \geq 1 - P\left(\widetilde{\mathcal{A}} \cap \left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right) - P(\mathcal{A}_{2}^{c}) - P(\widetilde{\mathcal{A}}_{2}^{c})$$
(124)

$$-\pi(\widetilde{\mathcal{A}}_{1}^{c}) - \pi(\mathcal{A}_{1}^{c}) - \pi(\mathcal{A}_{3}^{c})$$
(125)
$$-\pi(\widetilde{\mathcal{A}}_{1}^{c}) - \pi(\widetilde{\mathcal{A}}_{1}^{c}) - \pi(\mathcal{A}_{3}^{c})$$
(125)

$$= 1 - P\left(\widetilde{\mathcal{A}} \cap \left(\bigcap_{i=1}^{c} \mathcal{A}_{i}\right)\right) - P(\mathcal{A}_{2}^{c}) - \pi(\mathcal{A}_{1}^{c}) - \pi(\mathcal{A}_{3}^{c})$$
(126)

The last three terms above can each be bounded above by $e^{-n\eta}$ because

$$P(\mathcal{A}_{2}^{c}) = \sum_{(x^{n}, y^{n}, m) \in \mathcal{A}_{2}^{c}} P(x^{n}, y^{n}, m)$$
(127)

$$\leq \sum_{(x^{n}, y^{n}, m) \in \mathcal{A}_{2}^{c}} P(w)Q(x^{n}, y^{n}|m)e^{-n\eta}$$
(128)

$$\leq e^{-n\eta},\tag{129}$$

and

$$\pi(\mathcal{A}_{3}^{c}) = \sum_{(x^{n}, y^{n}, m) \in \mathcal{A}_{3}^{c}} \pi(x^{n}, y^{n}) P(m | x^{n}, y^{n})$$
(130)

$$\leq \sum_{\substack{(x^n, y^n, m) \in \mathcal{A}_3^c \\ \times e^{-n(R+\eta)} P(m|x^n, y^n)}} Q(x^n, y^n|m)$$
(131)

$$\leq \sum_{(x^n, y^n, m) \in \mathcal{A}_3^c} Q(x^n, y^n | m) \mathrm{e}^{-n(R+\eta)}$$
(132)

$$\leq \mathrm{e}^{-n\eta},\tag{133}$$

and

$$\pi(\mathcal{A}_1^c) = \sum_{(x^n, y^n) \in \mathcal{A}_1^c} \pi(x^n, y^n)$$
(134)

$$\leq \sum_{\substack{(x^n, y^n) \in \mathcal{A}_1^c}} Q(x^n, y^n) \mathrm{e}^{-n\eta}$$
(135)
$$\leq \mathrm{e}^{-n\eta}.$$
(136)

Therefore, we have

$$|P_{X^{n}Y^{n}} - \pi_{X^{n}Y^{n}}| \geq 1 - P\left(\widetilde{\mathcal{A}} \cap \left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right) - 3e^{-n\eta}$$
(137)

$$\geq 1 - P\left(\bigcap_{i=1}^{3} \mathcal{A}_{i} \middle| \widetilde{\mathcal{A}}\right) - 3\mathrm{e}^{-n\eta}.$$
 (138)

C. Proof of Lemma 6

Recall that in Appendix B-A, we choose $U_i = M_n$ and $V_i = (X^{i-1}, Y^{i-1})$. Then $Q_{X^nY^n}$ and $Q_{X^nY^n|M_n}$ can be written as follows:

$$Q_{X^{n}Y^{n}}(x^{n}, y^{n}) = \prod_{i=1}^{n} Q_{X_{i}Y_{i}|X^{i-1}Y^{i-1}}(x_{i}, y_{i}|x^{i-1}, y^{i-1})$$
(139)

$$=\prod_{i=1}^{n} Q_{X_i Y_i | V_i}(x_i, y_i | v_i),$$
(140)

$$=\prod_{i=1}^{n} Q_{X_{i}Y_{i}|M_{n}X^{i-1}Y^{i-1}}(x_{i}, y_{i}|m, x^{i-1}, y^{i-1})$$
(141)

$$=\prod_{i=1}^{n} Q_{X_i Y_i | U_i V_i}(x_i, y_i | u_i, v_i).$$
(142)

Now recall from Appendix B-A that the joint distribution of (X^n, Y^n, M_n) induced by the code is $P_{X^nY^nM_n}$. The marginal distributions of $P_{X^nY^nM_n}$ are as follows:

$$\pi_{X^n Y^n}(x^n, y^n) = \prod_{i=1}^n \pi_{XY}(x_i, y_i),$$
(143)

$$P_{X^{n}|M_{n}}(x^{n}|m) = \prod_{\substack{i=1\\n}}^{n} P_{X_{i}|M_{n}X^{i-1}}(x_{i}|m, x^{i-1})$$
(144)

$$=\prod_{i=1} P_{X_i|M_n X^{i-1} Y^{i-1}}(x_i|m, x^{i-1}, y^{i-1})$$
(145)

$$=\prod_{i=1}^{n} P_{X_i|U_iV_i}(x_i|u_i, v_i),$$
(146)

$$P_{Y^{n}|M_{n}}(y^{n}|m) = \prod_{i=1}^{n} P_{Y_{i}|M_{n}Y^{i-1}}(y_{i}|m, y^{i-1})$$
(147)

$$=\prod_{i=1}^{n} P_{Y_{i}|M_{n}X^{i-1}Y^{i-1}}(y_{i}|m, x^{i-1}, y^{i-1})$$
(148)

$$=\prod_{i=1}^{n} P_{Y_i|U_iV_i}(y_i|u_i, v_i),$$
(149)

where (145) and (148) follow from the Markov chains $X_i - M_n X^{i-1} - Y^{i-1}$ and $Y_i - M_n Y^{i-1} - X^{i-1}$ under distribution $P_{X^n Y^n M_n}$ (these two Markov chains can be easily obtained by observing that $P_{X^i Y^i M_n} = P_{M_n} P_{X_i | M_n X^{i-1}} P_{Y_i | M_n Y^{i-1}}$). Using Lemma 5 and (139)–(149), we obtain

$$|P_{X^nY^n} - \pi_{X^nY^n}| \ge 1 - P\left(\bigcap_{i=1}^3 \mathcal{B}_i \,\middle|\, \widetilde{\mathcal{A}}\right) - 3\mathrm{e}^{-n\eta}.$$
 (150)

D. Proof of Lemma 8

1) Removing Dependence on the Indices: Recall from Appendix B-A that the joint distribution of (X^n, Y^n, M_n) is $P_{X^nY^nM_n}$ and $P_{X_iY_iU_iV_i}$ is induced by $P_{X^nY^nM_n}$. Further, recall that Q_i denotes $Q_{X_iY_iU_iV_i}$ and P_i denotes $P_{X_iY_iU_iV_i}$. Define

$$g_{Q_i,P_i}^{(\alpha,\lambda)}(x_i,y_i|u_i,v_i) := \exp\left(-\lambda\omega_{Q_i,P_i}^{(\alpha)}(x_i,y_i|u_i,v_i)\right),$$
(151)

where $\omega_{Q_i,P_i}^{(\alpha)}(x_i,y_i|u_i,v_i)$ is defined in (110).

Recall the definition of $\Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n)$ in (111), then we obtain that

$$\exp\left(-\Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n)\right)$$
$$=\sum_{x^n,y^n,m} P_{X^nY^nM_n|\tilde{\mathcal{A}}}(x^n,y^n,m) \prod_{i=1}^n g_{Q_i,P_i}^{(\alpha,\lambda)}(x_i,y_i|u_i,v_i),$$
(152)

where $u_i = m$, $v_i = (x^{i-1}, y^{i-1})$, and $P_{X^nY^nM_n|\widetilde{\mathcal{A}}}$ is the conditional distribution of (X^n, Y^n, M_n) given $(X^n, Y^n, M_n) \in \widetilde{\mathcal{A}}$.

For
$$i = 1, ..., n$$
, define

$$\tilde{C}_i := \sum_{x^n, y^n, m} P_{X^n Y^n M_n | \widetilde{\mathcal{A}}}(x^n, y^n, m)$$

$$\times \prod_{j=1}^i g_{Q_j, P_j}^{(\alpha, \lambda)}(x_j, y_j | u_j, v_j), \qquad (153)$$

$$P_{X^nY^nM_n|\widetilde{\mathcal{A}}}^{(\alpha,\lambda)|i}(x^n, y^n, m) := \frac{1}{\widetilde{C}_i} P_{X^nY^nM_n|\widetilde{\mathcal{A}}}(x^n, y^n, m)$$
$$\times \prod_{j=1}^i g_{Q_j, P_j}^{(\alpha,\lambda)}(x_j, y_j|u_j, v_j),$$
(154)

$$\Lambda_{i}^{(\alpha,\lambda)}(\{Q_{j}\}_{j=1}^{i}) := \frac{\tilde{C}_{i}}{\tilde{C}_{i-1}}.$$
(155)

(145) Obviously, $P_{X^nY^nM_n|\widetilde{\mathcal{A}}}^{(\alpha,\lambda)|i}(x^n, y^n, m)$ is a distribution induced by normalizing all the terms of the summation in the definition (146) of \widetilde{C}_i .

Similarly to [14, Lemma 7], we obtain the following lemma, which will be used to simplify $\Lambda_i^{(\alpha,\lambda)}(\{Q_j\}_{j=1}^i)$, defined in (155), in Appendix B-D2.

Lemma 9. For i = 1, ..., n, we have

$$\Lambda_{i}^{(\alpha,\lambda)}(\{Q_{j}\}_{j=1}^{i}) = \sum_{x^{n},y^{n},m} P_{X^{n}Y^{n}M_{n}|\widetilde{\mathcal{A}}}^{(\alpha,\lambda)|i-1}(x^{n},y^{n},m)g_{Q_{i},P_{i}}^{(\alpha,\lambda)}(x_{i},y_{i}|u_{i},v_{i}).$$
(156)

Furthermore, combining (152), (153) and (155) gives us

$$\exp\left(-\Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n)\right) = \prod_{i=1}^n \Lambda_i^{(\alpha,\lambda)}(\{Q_j\}_{j=1}^i). \quad (157)$$

2) Completion of the Proof of Lemma 8: Assume \mathcal{U} and \mathcal{V} are two countable sets. Paralleling (21) to (24), for $(\alpha, \theta) \in (0, 1] \times (0, \infty)$, we define the following quantities:

$$\widetilde{\mathcal{Q}} := \left\{ Q_{XYUV} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U} \times \mathcal{V}) : \\ \operatorname{supp}(Q_{XY}) \subseteq \operatorname{supp}(\pi_{XY}) \right\},$$
(158)

$$\widetilde{\omega}_{Q_{XYUV}}^{(\alpha)}(x,y|u,v) := \bar{\alpha} \left(\log \frac{Q_{XY|V}(x,y|v)}{\pi_{XY}(x,y)} + \log \frac{Q_{XY|UV}(x,y|u,v)}{Q_{X|UV}(x|u,v)Q_{Y|UV}(y|u,v)} \right) + \alpha \log \frac{Q_{XY|UV}(x,y|u,v)}{\pi_{XY}(x,y)},$$
(159)

$$\Omega^{(\alpha,\lambda)}(Q_{XYUV}) = -\log \mathbb{E}_{Q_{XYUV}} \Big[\exp \Big(-\theta \omega_{Q_{XYUV}}^{(\alpha)}(X,Y|U,V) \Big) \Big],$$
(160)
$$\widetilde{\alpha}^{(\alpha,\lambda)} = -\widetilde{\alpha}^{(\alpha,\lambda)}(Q_{XYUV}(X,Y|U,V)) \Big],$$
(160)

$$\Omega^{(\alpha,\lambda)} := \inf_{\substack{Q_{XYUV} \in \widetilde{\mathcal{Q}}}} \Omega^{(\alpha,\lambda)}(Q_{XYUV}), \tag{161}$$

where $\mathbb{E}_{Q_{XYUV}}$ in (160) is only taken over the set $\sup(Q_{XYUV})$.

Recall that $u_i = m$ and $v_i = (x^{i-1}, y^{i-1})$. For each $i = 1, \ldots, n$, define

$$P^{(\alpha,\lambda)}(x_{i}, y_{i}, u_{i}, v_{i}) := \sum_{\substack{x_{i+1}^{n}, y_{i+1}^{n}}} P_{X^{n}Y^{n}M_{n}|\tilde{\mathcal{A}}}^{(\alpha,\lambda)|i-1}(x^{n}, y^{n}, m),$$
(162)

where $P_{X^nY^nM_n}^{(\alpha,\lambda)|i-1}(x^n, y^n, m)$ was defined in (154). Combining Lemma 9 and (162) yields

$$\Lambda_{i}^{(\alpha,\lambda)}(\{Q_{j}\}_{j=1}^{i}) = \sum_{x_{i},y_{i},u_{i},v_{i}} P^{(\alpha,\lambda)}(x_{i},y_{i},u_{i},v_{i}) g_{Q_{i},P_{i}}^{(\alpha,\lambda)}(x_{i},y_{i}|u_{i},v_{i}).$$
(163)

Note that $Q_i = Q_{X_i Y_i U_i V_i}$ can be chosen arbitrarily for all i = 1, ..., n. Here we apply the recursive method. For each i = 1, ..., n, we choose $Q_{X_i Y_i U_i V_i}$ such that

$$Q_{X_i Y_i U_i V_i}(x_i, y_i, u_i, v_i) = P^{(\alpha, \lambda)}(x_i, y_i, u_i, v_i).$$
(164)

Then, let $Q_{X_iY_i|V_i}, Q_{X_iY_i|U_iV_i}$ be induced by $Q_{X_iY_iU_iV_i}$. Define

$$\begin{split} h_{Q_{i}}^{(\alpha,\lambda)}(x_{i},y_{i}|u_{i},v_{i}) &:= g_{Q_{i},P_{i}}^{(\alpha,\lambda)}(x_{i},y_{i}|u_{i},v_{i}) \\ \times \left(\frac{P_{X_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(x_{i}|u_{i},v_{i})P_{Y_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(y_{i}|u_{i},v_{i})}{Q_{X_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(x_{i}|u_{i},v_{i})Q_{Y_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(y_{i}|u_{i},v_{i})} \right)^{-1}, \end{split}$$
(165)

where $g_{Q_i,P_i}^{(\alpha,\lambda)}$ was defined in (151). In the following, for brevity, we drop the subscripts of the distributions. From (163), we obtain

$$\Lambda_i^{(\alpha,\lambda)}(\{Q_j\}_{j=1}^i) = \mathbb{E}_{Q_i}[g_{Q_i,P_i}^{(\alpha,\lambda)}(X_i, Y_i|U_i, V_i)]$$

$$(166)$$

$$= \mathbb{E}_{Q_{i}} \left[h_{Q_{i}}^{\alpha, \alpha'}(X_{i}, Y_{i}|U_{i}, V_{i}) \times \frac{P_{X_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(X_{i}|U_{i}, V_{i})P_{Y_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(Y_{i}|U_{i}, V_{i})}{Q_{X_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(X_{i}|U_{i}, V_{i})Q_{Y_{i}|U_{i}V_{i}}^{\lambda\bar{\alpha}}(Y_{i}|U_{i}, V_{i})} \right]$$

$$\leq \left(\mathbb{E}_{Q_{i}} \left[\left\{ h_{Q_{i}}^{(\alpha, \lambda)}(X_{i}, Y_{i}|U_{i}, V_{i}) \right\}^{\frac{1}{1-2\lambda\bar{\alpha}}} \right] \right)^{1-2\lambda\bar{\alpha}}$$
(167)

$$= \left(\mathbb{E}_{Q_i} \left[\left\{ n_{Q_i}^{P_i} \cap (X_i, Y_i | U_i, V_i) \right\} \right] \right) \\ \times \left(\mathbb{E}_{Q_i} \left[\frac{P_{X_i | U_i V_i}(X_i | U_i, V_i)}{Q_{X_i | U_i V_i}(X_i | U_i, V_i)} \right] \right)^{\lambda \bar{\alpha}} \\ \times \left(\mathbb{E}_{Q_i} \left[\frac{P_{Y_i | U_i V_i}(Y_i | U_i, V_i)}{Q_{Y_i | U_i V_i}(Y_i | U_i, V_i)} \right] \right)^{\lambda \bar{\alpha}}$$
(168)

$$\leq \exp\left(-\left(1-2\lambda\bar{\alpha}\right)\widetilde{\Omega}^{(\alpha,\frac{\lambda}{1-2\lambda\bar{\alpha}})}(Q_i)\right) \tag{169}$$

$$= \exp\left(-\frac{\Omega^{(\alpha,\theta)}(Q_i)}{1+2\bar{\alpha}\theta}\right) \tag{170}$$

$$\leq \exp\left(-\frac{\widetilde{\Omega}^{(\alpha,\theta)}}{1+2\bar{\alpha}\theta}\right) \tag{171}$$

$$= \exp\left(-\frac{\Omega^{(\alpha,\theta)}}{1+2\bar{\alpha}\theta}\right),\tag{172}$$

where (167) follows from (165); (168) follows from Hölder's inequality; (169) follows from the definitions of $\Omega^{(\alpha,\theta)}(\cdot)$ and $h_{Q_i}^{(\alpha,\lambda)}(\cdot)$ in (23) and (165) respectively; (170) follows from (114) and (115); (171) follows since $\Omega^{(\alpha,\theta)}(Q_{XYUV}) \geq \widetilde{\Omega}^{(\alpha,\theta)}$ for any Q_{XYUV} such that $\sup(Q_{XY}) \subseteq \sup(\pi_{XY})$ (The fact that Q_i satisfies this point will be shown in the following paragraph); and (172) follows since by the support lemma [2], the cardinality bounds $|\mathcal{V}| \leq 1$, $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{Y}|$ are sufficient to exhaust $\widetilde{\Omega}^{(\alpha,\theta)}$.

Now we show that according to the choice of $Q_{X_iY_iU_iV_i}$, we have $\operatorname{supp}(Q_{X_iY_i}) \subseteq \operatorname{supp}(\pi_{XY})$, which was used in (171). Note that $P_{X^nY^nM_n}(x^n, y^n, m) > 0$ and $\pi_{X^nY^n}(x^n, y^n) > 0$ for any $(x^n, y^n, m) \in \widetilde{\mathcal{A}}$, and hence the marginal distributions $P_{X_iY_iU_iV_i}, P_{X_i|U_iV_i}$ and $P_{Y_i|U_iV_i}$ when evaluated at any $(x^n, y^n, m) \in \widetilde{\mathcal{A}}$ is positive as well. According to the choice of $Q_{X_iY_iU_iV_i}$ in (164), we have that $Q_{X_iY_iU_iV_i}$ is also positive when evaluated at $(x^n, y^n, m) \in \widetilde{\mathcal{A}}$ (this point can be shown via mathematical induction), i.e.,

$$\sup(Q_{X_iY_iU_iV_i}) \supseteq \{(x, y, u, v) : \exists (x^n, y^n, m) \in \mathcal{A} : \\ x_i = x, y_i = y, m = u, (x^{i-1}, y^{i-1}) = v \}.$$
(173)

On the other hand, also according to the choice of $Q_{X_iY_iU_iV_i}$, we have

$$\sup(Q_{X_iY_iU_iV_i}) \subseteq \{(x, y, u, v) : \exists (x^n, y^n, m) \in \widetilde{\mathcal{A}} : x_i = x, y_i = y, m = u, (x^{i-1}, y^{i-1}) = v\}.$$
(174)

Therefore,

$$\sup (Q_{X_i Y_i U_i V_i}) = \{(x, y, u, v) : \exists (x^n, y^n, m) \in \widetilde{\mathcal{A}} : \\ x_i = x, y_i = y, m = u, (x^{i-1}, y^{i-1}) = v \}.$$
(175)

Further, we have

$$\sup(Q_{X_iY_i}) = \left\{ (x,y) : \exists (x^n, y^n, m) \in \widetilde{\mathcal{A}} : x_i = x, y_i = y \right\}$$
(176)

$$\subseteq \{(x,y): \exists (x,y), m \in \operatorname{supp}(\pi_X \pi_Y \pi) \times \mathcal{M}_n : \\ x_i = x, y_i = y \}$$

$$(177)$$

$$= \operatorname{supp}(\pi_{XY}). \tag{178}$$

Combining (157) and (172), we obtain that

$$\frac{1}{n}\Omega^{(\alpha,\lambda)}(\{Q_i\}_{i=1}^n) = -\frac{1}{n}\sum_{i=1}^n \log \Lambda_i^{(\alpha,\lambda)}(\{Q_j\}_{j=1}^i) \quad (179)$$

$$\geq \frac{\Omega^{(\alpha,b)}}{1+2\bar{\alpha}\theta}.$$
(180)

Finally, combining (113) and (180), we have that

$$\underline{\Omega}^{(\alpha,\lambda)} \ge \frac{\Omega^{(\alpha,\theta)}}{1+2\bar{\alpha}\theta}.$$
(181)

The proof of Lemma 8 is now complete.

APPENDIX C Proof of Lemma 1

Let U be a random variable taking values in a finite alphabet \mathcal{U} . Define a set of joint distributions on $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ as

$$\mathcal{P}^* := \left\{ P_{XYU} : |\mathcal{U}| \le |\mathcal{X}| |\mathcal{Y}|, \ P_{XY} = \pi_{XY}, \ X - U - Y \right\}.$$
(182)

and let

$$R^* := \min_{P_{XYU} \in \mathcal{P}^*} I(XY; U).$$
(183)

A. Preliminary Lemmata for the Proof of Lemma 1

By the support lemma [2, Appendix C], we have the following lemma [1].

Lemma 10. Wyner's common information $C_{Wyner}(X;Y)$ satisfies

$$C_{\mathsf{Wyner}}(X;Y) = R^*. \tag{184}$$

Before proceeding the proof of Lemma 1, we present an alternative expression for Wyner's common information. Recall that given a number $a \in [0, 1]$, we define $\bar{a} = 1 - a$. Then for any $\alpha \in [0, 1]$ and $Q_{XYU} \in \mathcal{Q}$, define

$$R^{(\alpha)}(Q_{XYU}) := \bar{\alpha} \left(D(Q_{XY} \| \pi_{XY}) + D(Q_{XY|U} \| Q_{X|U} Q_{Y|U} | Q_U) \right) + \alpha D(Q_{XY|U} \| \pi_{XY} | Q_U), \quad (185)$$

$$R^{(\alpha)} := \min_{Q_{XYU} \in \mathcal{Q}} R^{(\alpha)}(Q_{XYU}) \tag{186}$$

$$R_{\rm sh} := \sup_{\alpha \in (0,1]} \frac{1}{\alpha} R^{(\alpha)}.$$
 (187)

By observing that both \mathcal{P}^* and \mathcal{Q} are compact, and by utilizing the fact that a continuous function defined on a compact set attains its minimum, we obtain the following.

Fact 1. Both the minima in the definitions of R^* in (183) and $R^{(\alpha)}$ in (186) are attained.

We then have the following lemma.

Lemma 11. The following conclusions hold.

(i) For any $\alpha \in (0, 1]$, we have

$$\frac{1}{\alpha}R^{(\alpha)} \le R^*. \tag{188}$$

Moreover, there exists some decreasing sequence $\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $\lim_{k\to\infty} \alpha_k = 0$ and

$$\frac{1}{\alpha_k} R^{(\alpha_k)} \ge R^* - c(\alpha_k), \tag{189}$$

where $\{c(\alpha_k)\}_{k=1}^{\infty} \subset \mathbb{R}$ is another sequence such that $\lim_{k\to\infty} c(\alpha_k) = 0.$

(ii) We have

$$R_{\rm sh} = R^* = C_{\rm Wyner}(X;Y). \tag{190}$$

Lemma 11 is similar to [14, Property 3], but the proofs are different. Essentially, in both the proof of [14, Property 3] and our proof, an intermediate distribution \tilde{Q}_{XYU} is used to establish the inequality

$$R^* - c(\alpha_k) \le \frac{1}{\alpha_k} R^{(\alpha_k)}(\widetilde{Q}_{XYU}) \le \frac{1}{\alpha_k} R^{(\alpha_k)}.$$
 (191)

However, the construction of such an intermediate distribution is different for these two proofs. The construction in [14] does not apply to our case, since our case does not only require \tilde{Q}_{XYU} to satisfy the Markov chain X - U - Y, but also requires that $\tilde{Q}_{XY} = \pi_{XY}$.

Proof of Lemma 11: It is easy to show (188). Hence, by the definition of $R_{\rm sh}$ in (187),

$$R_{\rm sh} \le R^*. \tag{192}$$

In the following we prove (189). Let $\{\alpha_m\}_{m=1}^{\infty}$ be an arbitrary sequence of decreasing positive real numbers such that $\lim_{m\to\infty} \alpha_m = 0$, and let $Q_{XYU}^{(m)}$ be a minimizing distribution of (186) with $\alpha = \alpha_m$. The existence of this minimizing distribution is guaranteed by Fact 1. Since $\mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U})$ is compact (following from the definition of \mathcal{Q}), there must

exist some sequence of increasing integers $\{m_k\}_{k=1}^{\infty}$ such that $Q_{XYU}^{(m_k)}$ converges to some distribution \widetilde{Q}_{XYU} . Consider,

$$R_{\rm sh} = \sup_{\alpha \in (0,1]} \frac{1}{\alpha} R^{(\alpha)} \tag{193}$$

$$\geq \limsup_{k \to \infty} \frac{1}{\alpha_{m_k}} R^{(\alpha_{m_k})} \tag{194}$$

$$= \limsup_{k \to \infty} \left\{ \frac{\bar{\alpha}_{m_k}}{\alpha_{m_k}} \Big(D(Q_{XY}^{(m_k)} \| \pi_{XY}) \\ + D(Q_{XY|U}^{(m_k)} \| Q_{X|U}^{(m_k)} Q_{Y|U}^{(m_k)} | Q_U^{(m_k)}) \Big) \\ + D(Q_{XY|U}^{(m_k)} \| \pi_{XY} | Q_U^{(m_k)}) \right\}$$
(195)

$$\geq \limsup_{k \to \infty} \left\{ \frac{\bar{\alpha}_{m_k}}{\alpha_{m_k}} \right\} \liminf_{k \to \infty} \left\{ D(Q_{XY}^{(m_k)} \| \pi_{XY}) + D(Q_{XY|U}^{(m_k)} \| Q_{X|U}^{(m_k)} Q_{Y|U}^{(m_k)} | Q_U^{(m_k)}) \right\} + \liminf_{k \to \infty} D(Q_{XY|U}^{(m_k)} \| \pi_{XY} | Q_U^{(m_k)})$$
(196)

$$= \infty \left(D(\widetilde{Q}_{XY} \| \pi_{XY}) + D(\widetilde{Q}_{XY|U} \| \widetilde{Q}_{X|U} \widetilde{Q}_{Y|U} | \widetilde{Q}_{U}) \right) + D(\widetilde{Q}_{XY|U} \| \pi_{XY} | \widetilde{Q}_{U}).$$
(197)

Observe that $R_{\rm sh}$ is finite due to (192). Hence it holds that

$$D(\widetilde{Q}_{XY} \| \pi_{XY}) = 0, \qquad (198)$$

$$D(\widetilde{Q}_{XY|U} \| \widetilde{Q}_{X|U} \widetilde{Q}_{Y|U} | \widetilde{Q}_U) = 0.$$
(199)

That is,

$$\tilde{Q}_{XY} = \pi_{XY},\tag{200}$$

$$\tilde{Q}_{XY|U} = \tilde{Q}_{X|U}\tilde{Q}_{Y|U}.$$
(201)

Therefore, under (200) and (201), we have

$$(197) \ge D(\tilde{Q}_{XY|U} \| \pi_{XY} | \tilde{Q}_U) \tag{202}$$

$$= I(\widetilde{Q}_{XY|U}, \widetilde{Q}_{XY}) \tag{203}$$

$$\geq R^*. \tag{204}$$

Combining (192), (197) and (204) yields us

$$R_{\rm sh} = R^* = \lim_{k \to \infty} \frac{1}{\alpha_{m_k}} R^{(\alpha_{m_k})}.$$
 (205)

Therefore, there exists some sequence $\{c(\alpha_{m_k})\}_{k=1}^{\infty} \subset \mathbb{R}$ (e.g., the sequence $\{R^* - \frac{1}{\alpha_{m_k}}R^{(\alpha_{m_k})}\}_{k=1}^{\infty} \subset \mathbb{R}$) such that $\lim_{k\to\infty} c(\alpha_{m_k}) = 0$ and

$$R^* - c(\alpha_{m_k}) \le \frac{1}{\alpha_{m_k}} R^{(\alpha_{m_k})} \le R^*.$$
 (206)

This concludes the proof.

We also have the following crucial lemma.

Lemma 12. Let $\alpha \in (0,1]$ and $Q_{XYU} \in \mathcal{Q}$. Then we have

$$\lim_{\theta \downarrow 0} \frac{1}{\theta} \Omega^{(\alpha,\theta)} = R^{(\alpha)}, \qquad (207)$$

or equivalently,

$$\frac{1}{\theta}\Omega^{(\alpha,\theta)} = R^{(\alpha)} + \epsilon^{(\alpha,\theta)}, \qquad (208)$$

where $\Omega^{(\alpha,\theta)}$ and $R^{(\alpha)}$ were defined in (24) and (186) respectively, and $\epsilon^{(\alpha,\theta)}$ is a term that vanishes as $\theta \downarrow 0$, the rate being dependent on α .

Proof of Lemma 12: To show this lemma, we first need to show that

$$\widehat{R}^{(\alpha,\theta)}(Q_{XYU}) := \begin{cases} \frac{1}{\theta} \Omega^{(\alpha,\theta)}(Q_{XYU}), & \theta > 0\\ R^{(\alpha)}(Q_{XYU}), & \theta = 0 \end{cases}$$
(209)

is continuous in $(\theta, Q_{XYU}) \in [0, \frac{1}{1+\bar{\alpha}}) \times Q$. It is easy to observe that

$$\Omega^{(\alpha,\theta)}(Q_{XYU}) = -\log \mathbb{E}_{Q_{XYU}} \left[\exp\left(-\theta \omega_{Q_{XYU}}^{(\alpha)}(X,Y|U)\right) \right]$$
(210)
$$= -\log \sum_{x,y,u} Q_{XYU}^{1-\theta(1+\bar{\alpha})}(x,y,u) \left(Q_U(u)\pi_{XY}(x,y)\right)^{\theta} \times \left(Q_{U|XY}(u|x,y)Q_{X|U}(x|u)Q_{Y|U}(y|u)\right)^{\theta\bar{\alpha}}$$
(211)

is jointly continuous in $(\theta, Q_{XYU}) \in [0, \frac{1}{1+\bar{\alpha}}) \times \mathcal{Q}$, hence $\widehat{R}^{(\alpha,\theta)}(Q_{XYU})$ is jointly continuous on $(0, \frac{1}{1+\bar{\alpha}}) \times \mathcal{Q}$. Therefore, to show the continuity of $\widehat{R}^{(\alpha,\theta)}(Q_{XYU})$ in $(\theta, Q_{XYU}) \in [0, \frac{1}{1+\bar{\alpha}}) \times \mathcal{Q}$, it suffices to show it is continuous at any point in $\{0\} \times \mathcal{Q}$, i.e.,

$$\lim_{(\theta,Q_{XYU})\to(0,Q'_{XYU})}\frac{1}{\theta}\Omega^{(\alpha,\theta)}(Q_{XYU}) = R^{(\alpha)}(Q'_{XYU})$$
(212)

for any $Q'_{XYU} \in \mathcal{Q}$. Let

$$Q_{XYU}^{(\alpha,\theta)}(x,y,u) = \frac{Q_{XYU}(x,y,u)\exp\left(-\theta\omega_{Q_{XYU}}^{(\alpha)}(x,y|u)\right)}{\sum_{x,y,u}Q_{XYU}(x,y,u)\exp\left(-\theta\omega_{Q_{XYU}}^{(\alpha)}(x,y|u)\right)}.$$
(213)

Invoking the definition of $\Omega^{(\alpha,\theta)}(Q_{XYU})$ in (23), we obtain

$$\frac{\partial \Omega^{(\alpha,\theta)}(Q_{XYU})}{\partial \theta} = \mathbb{E}_{Q_{XYU}^{(\alpha,\theta)}} \Big[\omega_{Q_{XYU}}^{(\alpha)}(X,Y|U) \Big], \quad (214)$$

and

$$\frac{\partial^2 \Omega^{(\alpha,\theta)}(Q_{XYU})}{\partial \theta^2} = -\operatorname{Var}_{Q_{XYU}^{(\alpha,\theta)}} \left[\omega_{Q_{XYU}}^{(\alpha)}(X,Y|U) \right].$$
(215)

Hence for fixed $Q_{XYU} \in \mathcal{Q}$, we have

$$\frac{\partial \Omega^{(\alpha,\theta)}(Q_{XYU})}{\partial \theta}\bigg|_{\theta=0} = R^{(\alpha)}(Q_{XYU}) > 0, \qquad (216)$$

$$\frac{\partial^2 \Omega^{(\alpha,b)}(Q_{XYU})}{\partial \theta^2} \le 0, \qquad (217)$$

which implies that

$$\theta \frac{\partial \Omega^{(\alpha,\theta)}(Q_{XYU})}{\partial \theta} \le \Omega^{(\alpha,\theta)}(Q_{XYU}) \le \theta R^{(\alpha)}(Q_{XYU}).$$
(218)

Furthermore, observe that $R^{(\alpha)}(Q_{XYU})$ is continuous in $Q_{XYU} \in \mathcal{Q}$, hence

$$\lim_{Q_{XYU}\to Q'_{XYU}} R^{(\alpha)}(Q_{XYU}) = R^{(\alpha)}(Q'_{XYU}).$$
(219)

On the other hand, observe that $\frac{\partial \Omega^{(\alpha,\theta)}(Q_{XYU})}{\partial \theta}$ given in (214) is continuous in $(\theta, Q_{XYU}) \in [0, \frac{1}{1+\tilde{\alpha}}) \times Q$. Hence

$$\lim_{\substack{(\theta, Q_{XYU}) \to (0, Q'_{XYU})}} \frac{\partial \Omega^{(\alpha, \theta)}(Q_{XYU})}{\partial \theta}$$
$$= \sum_{x, y, u} Q'_{XYU}(x, y, u) \omega^{(\alpha)}_{Q'_{XYU}}(X, Y|U) \qquad (220)$$
$$= R^{(\alpha)}(Q'_{XYU}). \qquad (221)$$

Therefore, combining (218), (219), and (221), we observe that the limit
$$\lim_{(\theta,Q_{XYU})\to(0,Q'_{XYU})} \frac{1}{\theta}\Omega^{(\alpha,\theta)}(Q_{XYU})$$
 exists, and moreover, $\lim_{(\theta,Q_{XYU})\to(0,Q'_{XYU})} \frac{1}{\theta}\Omega^{(\alpha,\theta)}(Q_{XYU}) = R^{(\alpha)}(Q'_{XYU})$. Hence, we obtain (212). In other words, $\widehat{R}^{(\alpha,\theta)}(Q_{XYU})$ is jointly continuous in $(\theta,Q_{XYU}) \in [0,\frac{1}{1+\overline{\alpha}}) \times Q$. In addition, observe that Q is a compact set. By using the following lemma we can assert that $\min_{Q_{XYU}\in Q} \widehat{R}^{(\alpha,\theta)}(Q_{XYU})$ is continuous in $\theta \in [0,\frac{1}{1+\overline{\alpha}})$.

Lemma 13 (Lemma 14 in [24]). Let \mathcal{X} and \mathcal{Y} be two metric spaces and let $\mathcal{K} \subset \mathcal{X}$ be a compact set. Let $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a (jointly) continuous real-valued function. Then the function $g : \mathcal{Y} \to \mathbb{R}$, defined as

$$g(y) := \min_{x \in \mathcal{K}} f(x, y), \quad \forall y \in \mathcal{Y},$$
(222)

is continuous on Y.

Considering the point $\theta = 0$, we obtain

$$\lim_{\theta \to 0} \min_{Q_{XYU} \in \mathcal{Q}} \widehat{R}^{(\alpha,\theta)}(Q_{XYU})$$
$$= \min_{Q_{XYU} \in \mathcal{Q}} \widehat{R}^{(\alpha,0)}(Q_{XYU})$$
(223)

$$= \min_{Q_{XYU} \in \mathcal{Q}} R^{(\alpha)}(Q_{XYU}) = R^{(\alpha)}, \qquad (224)$$

where the first equality follows from Lemma 13 which essentially says that the limit and minimum operations can be swapped. On the other hand, observe that

$$\lim_{\theta \to 0} \min_{Q_{XYU} \in \mathcal{Q}} \widehat{R}^{(\alpha,\theta)}(Q_{XYU}) = \lim_{\theta \to 0} \min_{Q_{XYU} \in \mathcal{Q}} \frac{1}{\theta} \Omega^{(\alpha,\theta)}(Q_{XYU})$$
(225)

$$=\lim_{\theta\to 0}\frac{1}{\theta}\Omega^{(\alpha,\theta)}.$$
(226)

Combining (224) and (226), we obtain (207) as desired.

B. Proof of Part (i) in Lemma 1

Using Lemma 10, we obtain that if $R < C_{Wyner}(X;Y)$, then

$$R + \tau \le R^* \tag{227}$$

for some $\tau > 0$. Further, invoking (189) and (227), we obtain that there exists k_0 such that for any $k \ge k_0$,

$$R + \tau \le \frac{1}{\alpha_k} R^{(\alpha_k)} + c(\alpha_k), \qquad (228)$$

and

$$c(\alpha_k) \le \frac{\tau}{2}.\tag{229}$$

Referring to (228) and (229), we obtain that for any $k \ge k_0$,

$$R + \frac{\tau}{2} \le \frac{1}{\alpha_k} R^{(\alpha_k)}.$$
(230)

Therefore, invoking (26), we conclude that for any $k \ge k_0$,

$$F(R) \ge \sup_{\theta \ge 0} F^{(\alpha_k, \theta)}(R)$$
(231)

$$\geq \sup_{\theta \in [0, \frac{1}{1 + \bar{\alpha}_k})} F^{(\alpha_k, \theta)}(R)$$
(232)

$$= \sup_{\theta \in [0, \frac{1}{1+\tilde{\alpha}_k})} \frac{\Omega^{(\alpha_k, \theta)} - \theta \alpha_k R}{1 + (5 - 3\alpha_k)\theta}$$
(233)

$$\geq \sup_{\theta \in [0, \frac{1}{1+\alpha_k})} \frac{1}{1+5\theta} \Big\{ \theta R^{(\alpha_k)} + \theta \epsilon^{(\alpha_k, \theta)} - \theta \alpha_k R \Big\}$$

$$\geq \sup_{\theta \in [0, \frac{1}{1+\alpha_k})} \frac{\theta}{1+5\theta} \left\{ \epsilon^{(\alpha_k, \theta)} + \frac{\alpha_k \tau}{2} \right\}$$
(235)

$$\geq \sup_{\theta \in [0,\tilde{\theta}]} \frac{\alpha_k \tau \theta}{4(1+5\theta)} \tag{236}$$

$$\geq \frac{\alpha_k \tau \widetilde{\theta}}{4(1+5\widetilde{\theta})},\tag{237}$$

where (234) follows from Lemma 12 and the inequality $1 + (5 - 3\alpha_k)\theta \le 1 + 5\theta$, (235) follows from (230), and (236) follows since there exists a sufficiently small $\tilde{\theta} \in (0, \frac{1}{1+\bar{\alpha}_k})$ such that $|\epsilon^{(\alpha_k,\theta)}| \le \frac{1}{4}\alpha_k\tau$ for all $\theta \le \tilde{\theta}$. Since the expression in (237) is positive, we have F(R) > 0 as desired.

C. Proof of Part (ii) in Lemma 1

Because $\exp(\cdot)$ is convex, applying Jensen's inequality, we obtain

$$\Omega^{(\alpha,\theta)}(Q_{XYU}) \le \theta \mathbb{E}_{Q_{XYU}} \left[\omega_{Q_{XYU}}^{(\alpha)}(X,Y|U) \right]$$
(238)

$$=\theta R^{(\alpha)}(Q_{XYU}). \tag{239}$$

Hence we have

$$\Omega^{(\alpha,\theta)} \le \min_{Q_{XYU} \in \mathcal{Q}} \theta R^{(\alpha)}(Q_{XYU})$$
(240)

$$=\theta R^{(\alpha)}.$$
 (241)

Thus, recalling the definition of $F^{(\alpha,\theta)}(R)$ in (25), we obtain that

$$F^{(\alpha,\theta)}(R) = \frac{\Omega^{(\alpha,\theta)} - \theta \alpha R}{1 + (5 - 3\alpha)\theta}$$
(242)

$$\leq \frac{\theta \alpha(\frac{1}{\alpha}R^{(\alpha)} - R)}{1 + (5 - 3\alpha)\theta}$$
(243)

$$\leq \frac{\theta \alpha (R_{\rm sh} - R)}{1 + (5 - 3\alpha)\theta} \tag{244}$$

where (245) follows from the assumption $R \geq C_{\text{Wyner}}(X;Y) = R_{\text{sh}}$. On the other hand, note that

$$\lim_{\theta \to 0} F^{(\alpha,\theta)} = 0.$$
 (246)

Hence, combining (245) and (246), we conclude that

 \leq

$$F = \sup_{(\alpha,\theta) \in [0,1] \times [0,\infty)} F^{(\alpha,\theta)}(R) = 0.$$
(247)

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