# Wyner's Common Information under Rényi Divergence Measures 

Lei Yu and Vincent Y. F. Tan, Senior Member, IEEE


#### Abstract

We study a generalized version of Wyner's common information problem (also coined the distributed source simulation problem). The original common information problem consists in understanding the minimum rate of the common input to independent processors to generate an approximation of a joint distribution when the distance measure used to quantify the discrepancy between the synthesized and target distributions is the normalized relative entropy. Our generalization involves changing the distance measure to the unnormalized and normalized Rényi divergences of order $\alpha=1+s \in[0,2]$. We show that the minimum rate needed to ensure the Rényi divergences between the distribution induced by a code and the target distribution vanishes remains the same as the one in Wyner's setting, except when the order $\alpha=1+s=0$. This implies that Wyner's common information is rather robust to the choice of distance measure employed. As a byproduct of the proofs used to the establish the above results, the exponential strong converse for the common information problem under the total variation distance measure is established.


Index Terms-Wyner's common information, Distributed source simulation, Rényi divergence, Total variation distance, Exponential strong converse

## I. Introduction

How much common randomness is needed to simulate two correlated sources in a distributed fashion? This problem, termed distributed source simulation, was first studied by Wyner [1], who used the normalized relative entropy (Kullback-Leibler divergence or KL divergence) to measure the approximation level (discrepancy) between the simulated joint distribution and the joint distribution of the original correlated sources. He defined the minimum rate needed to ensure that the normalized relative entropy vanishes asymptotically as the common information between the sources. He also established a single-letter characterization for the common information, i.e., the common information between correlated sources $X$ and $Y$ (with target distribution $\pi_{X Y}$ ) is

$$
\begin{equation*}
C_{\mathrm{Wynner}}(X ; Y)=\min _{P_{X Y W}: P_{X Y}=\pi_{X Y}, X-W-Y} I(X Y ; W) . \tag{1}
\end{equation*}
$$

[^0]The common information is also known to be one of many reasonable measures of the dependence between two random variables [2] Section 14.2.2] (other measures include the mutual information and the Gács-Körner-Witsenhausen common information). A related notion is that of the exact common information which was introduced by Kumar, Li, and El Gamal [3]. They assumed variable-length codes and exact generation of the correlated sources $(X, Y)$, instead of block codes and approximate simulation of $\pi_{X Y}$ as assumed by Wyner [1]. The exact common information is not smaller than Wyner's common information. However, it is still not known whether they are equal in general. Furthermore, the common information problem can be also be regarded as a distributed coordination problem. The concept of coordination was first introduced by Cuff, Permuter, and Cover [4], [5], who used the total variation (TV) distance to measure the level of approximation between the simulated and target distributions.
Wyner's common information problem is also closely related to the channel resolvability problem, which was first studied by Han and Verdú [6], and subsequently studied by Hayashi [7], [8], Liu, Cuff, and Verdú [9], and Yu and Tan [10] among others. For the achievability part, both problems rely on so-called soft-covering lemmas [5]. The channel resolvability or common information problems have several interesting applications-including secrecy, channel synthesis, and source coding. For example, in [11] it was used to study the performance of a wiretap channel system under different secrecy measures. In [12] it was used to study the reliability and secrecy exponents of a wiretap channel with cost constraints. In [13] it was used to study the exact secrecy and reliability exponents for a wiretap channel.

## A. Main Contributions

Different from Wyner's work, we use (normalized and unnormalized) Rényi divergences of order $1+s \in[0,2]$ to measure the level of approximation between the simulated and target distributions. This is motivated in part by our desire to understand the sensitivity of the divergence as approximation measure on Wyner's common information. We prove that for the distributed source simulation problem, the minimum rate needed to guarantee that the (normalized and unnormalized) Rényi divergences vanish asymptotically is equal to Wyner's common information (except for the case when Rényi parameter is equal to 0 ). This implies that Wyner's common information in (1) is rather robust to the distance measure. For the achievability part, by using the method of types and typicality arguments, we prove that the optimal Rényi divergences vanish (at least) exponentially fast if the code rate is larger
than Wyner's common information. However, for the converse part, the proof is not straightforward and we have to first consider an auxiliary problem. We first prove an exponential strong converse for the common information problem under the TV distance measure, i.e., when the code rate is smaller than Wyner's common information, the TV distance between the induced distribution and the target distribution tends to one (at least) exponentially fast. Even though our proof technique mirrors that of Oohama [14] to establish the exponential strong converse for the Wyner-Ziv problem, it differs significantly in some aspects. To wit, some intricate continuity arguments are required to assert that the strong converse exponent is positive for all rates below $C_{\text {Wyner }}(X ; Y)$ (see part (i) of Lemma 11. Furthermore and interestingly, by leveraging a key relationship between the Rényi divergence and the TV distance [15], this exponential strong converse implies the converse for the normalized Rényi divergence (which in turn also implies the strong converse for the unnormalized Rényi divergence).

It is worth noting that it is quite natural to use various divergences to measure the discrepancy between two distributions. Wyner [1] used the KL divergence to measure the level of approximation in the distributed source synthesis problem; Hayashi [7], [8] and Yu and Tan [10] respectively used the KL divergence and the Rényi divergence to study the channel resolvability problem. The latter also applied their results to study the capacity region for the wiretap channel under these generalized measures. Furthermore, in probability theory, Barron [16] and Bobkov, Chistyakov and Götze [17] respectively used the KL divergence and the Rényi divergence to study the central limit theorem, i.e., they used them to measure the discrepancy between the induced distribution of sum of i.i.d. random variables and the normal distribution with the same mean and variance. Furthermore, special instances of Rényi entropies and divergences-including the KL divergence, the collision entropy (the Rényi divergence of order 2), and min-entropy (the Rényi divergence of order $\infty$ )-were used to study various information-theoretic problems (including security, cryptography, and quantum information) in several works in the recent literature [10], [11], [18]-[22].

## B. Notation

We use $P_{X}(x)$ to denote the probability distribution of a random variable $X$. This will also be denoted as $P(x)$ (when the random variable $X$ is clear from the context). We also use $\widetilde{P}_{X}, \widehat{P}_{X}$ and $Q_{X}$ to denote various probability distributions with alphabet $\mathcal{X}$. All alphabets considered in the sequel are finite. The set of probability measures on $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{X})$, and the set of conditional probability measures on $\mathcal{Y}$ given a variable in $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{Y} \mid \mathcal{X}):=\left\{P_{Y \mid X}: P_{Y \mid X}(\cdot \mid x) \in \mathcal{P}(\mathcal{Y}), x \in \mathcal{X}\right\}$. Furthermore, the support of a distribution $P \in \mathcal{P}(\mathcal{X})$ is denoted as $\operatorname{supp}(P)=\{x \in \mathcal{X}: P(x)>0\}$.

We use $T_{x^{n}}(x):=\frac{1}{n} \sum_{i=1}^{n} 1\left\{x_{i}=x\right\}$ to denote the type (empirical distribution) of a sequence $x^{n}, T_{X}$ and $V_{Y \mid X}$ to respectively denote a type of sequences in $\mathcal{X}^{n}$ and a conditional type of sequences in $\mathcal{Y}^{n}$ (given a sequence $x^{n} \in \mathcal{X}^{n}$ ). For a type $T_{X}$, the type class (set of sequences having the same type
$T_{X}$ ) is denoted by $\mathcal{T}_{T_{X}}$. For a conditional type $V_{Y \mid X}$ and a sequence $x^{n}$, the $V_{Y \mid X}$-shell of $x^{n}$ (the set of $y^{n}$ sequences having the same conditional type $V_{Y \mid X}$ given $x^{n}$ ) is denoted by $\mathcal{T}_{V_{Y \mid X}}\left(x^{n}\right)$. For brevity, sometimes we use $T(x, y)$ to denote the joint distributions $T(x) V(y \mid x)$ or $T(y) V(x \mid y)$.

The $\epsilon$-typical set of $Q_{X}$ is denoted as

$$
\begin{align*}
\mathcal{T}_{\epsilon}^{n}\left(Q_{X}\right): & :=\left\{x^{n} \in \mathcal{X}^{n}:\right. \\
& \left.\left|T_{x^{n}}(x)-Q_{X}(x)\right| \leq \epsilon Q_{X}(x), \forall x \in \mathcal{X}\right\} . \tag{2}
\end{align*}
$$

The conditionally $\epsilon$-typical set of $Q_{X Y}$ is denoted as

$$
\begin{equation*}
\mathcal{T}_{\epsilon}^{n}\left(Q_{Y X} \mid x^{n}\right):=\left\{y^{n} \in \mathcal{Y}^{n}:\left(x^{n}, y^{n}\right) \in \mathcal{T}_{\epsilon}^{n}\left(Q_{X Y}\right)\right\} \tag{3}
\end{equation*}
$$

For brevity, sometimes we write $\mathcal{T}_{\epsilon}^{n}\left(Q_{X}\right)$ and $\mathcal{T}_{\epsilon}^{n}\left(Q_{Y X} \mid x^{n}\right)$ as $\mathcal{T}_{\epsilon}^{n}$ and $\mathcal{T}_{\epsilon}^{n}\left(x^{n}\right)$ respectively.

The TV distance between two probability mass functions $P$ and $Q$ with a common alphabet $\mathcal{X}$ is defined as

$$
\begin{equation*}
|P-Q|:=\frac{1}{2} \sum_{x \in \mathcal{X}}|P(x)-Q(x)| . \tag{4}
\end{equation*}
$$

By the definition of $\epsilon$-typical set, we have that for any $x^{n} \in$ $\mathcal{T}_{\epsilon}^{n}\left(Q_{X}\right)$,

$$
\begin{equation*}
\left|T_{x^{n}}-Q_{X}\right| \leq \frac{\epsilon}{2} \tag{5}
\end{equation*}
$$

Fix distributions $P_{X}, Q_{X} \in \mathcal{P}(\mathcal{X})$. The relative entropy and the Rényi divergence of order $1+s$ are respectively defined as

$$
\begin{align*}
D\left(P_{X} \| Q_{X}\right) & :=\sum_{x \in \operatorname{supp}\left(P_{X}\right)} P_{X}(x) \log \frac{P_{X}(x)}{Q_{X}(x)}  \tag{6}\\
D_{1+s}\left(P_{X} \| Q_{X}\right) & :=\frac{1}{s} \log \sum_{x \in \operatorname{supp}\left(P_{X}\right)} P_{X}(x)^{1+s} Q_{X}(x)^{-s} \tag{7}
\end{align*}
$$

and the conditional versions are respectively defined as

$$
\begin{align*}
D\left(P_{Y \mid X} \| Q_{Y \mid X} \mid P_{X}\right) & :=D\left(P_{X} P_{Y \mid X} \| P_{X} Q_{Y \mid X}\right)  \tag{8}\\
D_{1+s}\left(P_{Y \mid X} \| Q_{Y \mid X} \mid P_{X}\right) & :=D_{1+s}\left(P_{X} P_{Y \mid X} \| P_{X} Q_{Y \mid X}\right) \tag{9}
\end{align*}
$$

where the summations in (6) and (7) are taken over the elements in $\operatorname{supp}\left(P_{X}\right)$. Throughout, $\log$ is to the natural base e and $s \geq-1$. It is known that $\lim _{s \rightarrow 0} D_{1+s}\left(P_{X} \| Q_{X}\right)=$ $D\left(P_{X} \| Q_{X}\right)$ so a special case of the Rényi divergence (or the conditional version) is the usual relative entropy (or the conditional version).

Given a number $a \in[0,1]$, we define $\bar{a}=1-a$. We also define $[x]^{+}=\max \{x, 0\}$.

## C. Problem Formulation

In this paper, we consider the distributed source simulation problem illustrated in Fig. 1 Given a target distribution $\pi_{X Y}$, we wish to minimize the alphabet size of a random variable $M_{n}$ that is uniformly distributed overl $\mathcal{M}_{n}:=\left\{1, \ldots, \mathrm{e}^{n R}\right\}$

[^1]

Fig. 1. Distributed source synthesis problem, where the random variable $M_{n} \in \mathcal{M}_{n}:=\left\{1, \ldots, \mathrm{e}^{n R}\right\}$.
( $R$ is a positive number known as the rate), such that the generated (or synthesized) distribution

$$
\begin{align*}
& P_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right) \\
& \quad:=\frac{1}{\left|\mathcal{M}_{n}\right|} \sum_{m \in \mathcal{M}_{n}} P_{X^{n} \mid M_{n}}\left(x^{n} \mid m\right) P_{Y^{n} \mid M_{n}}\left(y^{n} \mid m\right) \tag{10}
\end{align*}
$$

forms a good approximation to the product distribution $\pi_{X^{n} Y^{n}}:=\pi_{X Y}^{n}$. The pair of random mappings $\left(P_{X^{n} \mid M_{n}}, P_{Y^{n} \mid M_{n}}\right)$ constitutes a synthesis code.

Different from Wyner's seminal work on the distributed source simulation problem [1], we employ the unnormalized Rényi divergence

$$
\begin{equation*}
D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \tag{11}
\end{equation*}
$$

and the normalized Rényi divergence

$$
\begin{equation*}
\frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \tag{12}
\end{equation*}
$$

to measure the discrepancy between $P_{X^{n} Y^{n}}$ and $\pi_{X^{n} Y^{n}}$. The minimum rates required to ensure these two measures vanish asymptotically are respectively termed the unnormalized and normalized Rényi common information, and denoted as

$$
\begin{align*}
& T_{1+s}\left(\pi_{X Y}\right) \\
& \quad:=\inf \left\{R: \lim _{n \rightarrow \infty} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)=0\right\}  \tag{13}\\
& \widetilde{T}_{1+s}\left(\pi_{X Y}\right) \\
& \quad:=\inf \left\{R: \lim _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)=0\right\} . \tag{14}
\end{align*}
$$

It is clear that

$$
\begin{equation*}
\widetilde{T}_{1+s}\left(\pi_{X Y}\right) \leq T_{1+s}\left(\pi_{X Y}\right) \tag{15}
\end{equation*}
$$

We also denote the minimum rate required to ensure the TV distance is bounded above by some constant $\varepsilon \in[0,1]$ asymptotically as

$$
\begin{align*}
& T_{\varepsilon}^{\mathrm{TV}}\left(\pi_{X Y}\right) \\
& \quad:=\inf \left\{R: \limsup _{n \rightarrow \infty}\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \leq \varepsilon\right\} . \tag{16}
\end{align*}
$$

We say that the strong converse property for the common information problem under the TV distance holds if $T_{\varepsilon}^{\mathrm{TV}}\left(\pi_{X Y}\right)$ does not depend on $\varepsilon \in[0,1)$.

## II. Main Results

Our main result concerns Wyner's common information problem when the discrepancy measure is the unnormalized or normalized Rényi divergence. It is stated as follows.

Theorem 1 (Rényi Common Informations). The unnormalized and normalized and Rényi common informations satisfy

$$
\begin{align*}
T_{1+s}\left(\pi_{X Y}\right) & =\widetilde{T}_{1+s}\left(\pi_{X Y}\right)  \tag{17}\\
& = \begin{cases}C_{\mathrm{Wyner}}(X ; Y) & s \in(-1,1] \\
0 & s=-1\end{cases} \tag{18}
\end{align*}
$$

Furthermore, for $s \in(-1,1]$, the optimal Rényi divergence $D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)$ in the definitions of the Rényi common informations decays at least exponentially fast in $n$ when $R>$ $C_{\text {Wyner }}(X ; Y)$.
Remark 1. For the converse part, $T_{1+s}\left(\pi_{X Y}\right) \geq$ $\widetilde{T}_{1+s}\left(\pi_{X Y}\right) \geq C_{\mathrm{Wyner}}(X ; Y)$ for $s \in[0,1]$ is implied by Wyner's work [1] and the monotonicity of the Rényi divergence. For the achievability part, $\widetilde{T}_{1+s}\left(\pi_{X Y}\right) \leq C_{\text {Wyner }}(X ; Y)$ for $s \in(-1,0]$ is also implied by Wyner's work [1] and the monotonicity of the Rényi divergence. Furthermore, since a channel resolvability code for the memoryless channel $P_{X \mid W} \times P_{Y \mid W}$ can be used to form a common information code, the achievability part for the common information problem can be obtained from existing channel resolvability results. Specifically, $T_{1+s}\left(\pi_{X Y}\right) \leq C_{\text {Wyner }}(X ; Y)$ for $s \in(-1,0]$ can be obtained from Hayashi's [7], [8] or Han, Endo, and Sasaki's results [12]. In addition, $\widetilde{T}_{1+s}\left(\pi_{X Y}\right) \leq T_{1+s}\left(\pi_{X Y}\right) \leq C_{1+s}(X ; Y)$ for $s \in(0,1]$ with

$$
\begin{align*}
& C_{1+s}(X ; Y):=\min _{P_{X Y W}: P_{X Y}=\pi_{X Y}, X-W-Y} \\
& \quad \sum_{w} P_{W}(w) D_{1+s}\left(P_{X \mid W}(\cdot \mid w) P_{Y \mid W}(\cdot \mid w) \| P_{X Y}\right) \tag{19}
\end{align*}
$$

can be obtained from the present authors' results [10], but as shown in Theorem 1 this bound is not tight since $C_{1+s}(X ; Y)>C_{\mathrm{Wyner}}(X ; Y)$ in general for $s \in(0,1]$. This is because, on the one hand, for the channel resolvability problem, the discrete memoryless channel is fixed, and, by construction, imposes a product conditional distribution of the output given the input (which is a product distribution), but for the common information problem, the synthesizer has the freedom to choose $P_{X^{n} Y^{n} \mid M_{n}}=P_{X^{n} \mid M_{n}} \times P_{Y^{n} \mid M_{n}}$, so that the Markov chain $X^{n}-M_{n}-Y^{n}$ holds; on the other hand, for the common information problem, in the sequel, we will show that if we utilize a truncated channel (which is not memoryless) as the synthesizer. This results in a smaller achievable rate for the case $s \in(0,1]$. Therefore, our converse for $s \in[-1,0)$ and achievability for $s \in(0,1]$ are new (and also tight).
Remark 2. An exponential achievability result for $s \in(-1,0]$ can be obtained from Hayashi's [7], [8] and Han, Endo, and Sasaki's results [12], where i.i.d. codes were employed.

For this theorem, the proof of the achievability part for the unnormalized Rényi common information is provided in Appendix A and the proof of the converse part for
the normalized Rényi common information is provided in Section IV] Observe that the unnormalized Rényi divergence is stronger than the normalized one in the sense of (15), hence $\widetilde{T}_{1+s}\left(\pi_{X Y}\right) \leq T_{1+s}\left(\pi_{X Y}\right)$. This implies, on one hand, the achievability result for the normalized Rényi common information $\widetilde{T}_{1+s}\left(\pi_{X Y}\right)$ can be obtained directly from the achievability result for the unnormalized version $T_{1+s}\left(\pi_{X Y}\right)$, and on the other hand, the converse result for the normalized Rényi common information $\widetilde{T}_{1+s}\left(\pi_{X Y}\right)$ implies the converse result for the unnormalized version $T_{1+s}\left(\pi_{X Y}\right)$.

The Rényi common informations are the same for all $s \in(-1,1]$, and also same as Wyner's common information $C_{\text {Wyner }}(X ; Y)$ (which corresponds to $s=0$ for the normalized case). For the case $s \in(-1,1]$, to obtain the (unnormalized and normalized) Rényi common informations, we utilize a random code with $\left(W^{n}, X^{n}, Y^{n}\right)$ ( $W$ is the auxiliary random variable in the definition of $C_{\text {Wyner }}(X ; Y)$ ) distributed according to a truncated product distribution, i.e., a product distribution governed by $Q_{W X Y}^{n}$ but whose mass is truncated to the typical set $\mathcal{T}_{\epsilon}^{n}\left(Q_{W X Y}\right)^{2}$ On one hand, the random sequences $\left(W^{n}, X^{n}, Y^{n}\right)$ so generated are almost uniformly distributed over the typical set $\mathcal{T}_{\epsilon}^{n}\left(Q_{W X Y}\right)$; and on the other hand, the Rényi common informations can be expressed as some Rényi divergences. Moreover, these Rényi divergences evaluated at the truncated distribution are almost the same regardless of the parameter $s \in(-1,1]$. Therefore, by using this truncated code, Wyner's common information is achievable for any $s \in(-1,1]$.

However, the proof of the converse part for the normalized Rényi common information is not straightforward $\sqrt[3]{ }$ We attempted to use the method of types to prove it, just as in [10] for the Rényi resovability problem, but failed since the code for the common information problem is arbitrary and does not need to be i.i.d. In particular, it is not i.i.d. In the following two sections, we provide an indirect proof using the following strategy: We first prove an exponential strong converse for Wyner's common information problem under the TV distance measure in Section IIII Then by using a relationship between the Rényi divergence and the TV distance [15], we show this exponential strong converse implies the converse for normalized Rényi divergence in Section IV.

As an intermediate result, the common information under the TV distance measure is characterized in the following

[^2]theorem.
Theorem 2 (Common Information under the TV Distance Measure). The following hold:
(i) The common information under the TV distance measure satisfies
\[

T_{\varepsilon}^{\mathrm{TV}}\left(\pi_{X Y}\right)= $$
\begin{cases}C_{\mathrm{Wyner}}(X ; Y) & \varepsilon \in[0,1)  \tag{20}\\ 0 & \varepsilon=1\end{cases}
$$
\]

Hence, the strong converse property for the common information problem under the TV distance holds.
(ii) Furthermore, there exists a sequence of synthesis codes with rate $R>C_{\mathrm{Wyner}}(X ; Y)$, such that $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$ tends to zero exponentially fast as $n$ tends to infinity.
(iii) On the other hand, for any sequence of synthesis codes with rate $R<C_{\mathrm{Wyner}}(X ; Y)$, we have that $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$ tends to one exponentially fast as $n$ tends to infinity.

Part (ii) is an exponential achievability result while the part (iii) is an exponential strong converse result. Combining parts (ii) and (iii) implies part (i). By Pinsker's inequality for Rényi divergences [23], the achievability results (including the exponential achievability result) in Theorem 1 implies the achievability results (including the exponential achievability result) in Theorem 2 Conversely, the exponential strong converse result in part (iii) of Theorem 2 implies the converse results in Theorem 1 for both unnormalized and normalized Rényi divergences. To prove part (iii), we draw on several key ideas from Oohama's work [14] on the exponential strong converse for the Wyner-Ziv problem. However, there are several key differences in our proofs, including the way we establish that the strong converse exponent is positive for all rates larger than $C_{\text {Wyner }}(X ; Y)$ and the treatment of the cases when various probability mass functions take on the value zero.

We note that conclusion in part (ii) (the exponential achievability result) in Theorem 2 can be also obtained by using the soft-covering lemma by Cuff [5, Lemma IV.1].

The proof of the conclusion in part (iii) is provided in the next section. As mentioned above, the other parts follow directly from Theorem 1

## III. The Proof of Part (iil) in Theorem 2

In this section, we provide an exponential strong converse theorem for the common information problem under the TV distance measure, which will be used to derive the converse for normalized Rényi divergence in next section.

We define

$$
\begin{align*}
\mathcal{Q}:= & \left\{Q_{X Y U} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U}):\right. \\
& \left.|\mathcal{U}| \leq|\mathcal{X}||\mathcal{Y}|, \operatorname{supp}\left(Q_{X Y}\right) \subseteq \operatorname{supp}\left(\pi_{X Y}\right)\right\} \tag{21}
\end{align*}
$$

Given $\alpha \in[0,1]$ and an arbitrary distribution $Q_{X Y U} \in \mathcal{Q}$, define the linear combination of the likelihood ratios for $(x, y, u) \in \operatorname{supp}\left(Q_{X Y U}\right)$,

$$
\begin{align*}
& \omega_{Q_{X Y U}}^{(\alpha)}(x, y \mid u):=\bar{\alpha}\left(\log \frac{Q_{X Y}(x, y)}{\pi_{X Y}(x, y)}\right. \\
& \left.+\log \frac{Q_{X Y \mid U}(x, y \mid u)}{Q_{X \mid U}(x \mid u) Q_{Y \mid U}(y \mid u)}\right)+\alpha \log \frac{Q_{X Y \mid U}(x, y \mid u)}{\pi_{X Y}(x, y)} \tag{22}
\end{align*}
$$

This function is finite for all $(x, y, u) \in \operatorname{supp}\left(Q_{X Y U}\right)$. For $Q_{X Y U} \in \mathcal{Q}$ and $\theta \in[0, \infty)$, define the negative cumulant generating functions as

$$
\begin{align*}
& \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right) \\
& \quad:=-\log \mathbb{E}_{Q_{X Y U}}\left[\exp \left(-\theta \omega_{Q_{X Y U}}^{(\alpha)}(X, Y \mid U)\right)\right], \tag{23}
\end{align*}
$$

and

$$
\begin{equation*}
\Omega^{(\alpha, \theta)}:=\min _{Q_{X Y U} \in \mathcal{Q}} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right) \tag{24}
\end{equation*}
$$

where the expectation $\mathbb{E}_{Q_{X Y U}}$ is only taken over the set $\operatorname{supp}\left(Q_{X Y U}\right)$ (this means we only sum over the elements $(x, y, u)$ such that $\left.Q_{X Y U}(x, y, u)>0\right)$.

Finally, we define the large deviations rate functions

$$
\begin{align*}
F^{(\alpha, \theta)}(R) & :=\frac{\Omega^{(\alpha, \theta)}-\theta \alpha R}{1+(5-3 \alpha) \theta}  \tag{25}\\
F(R) & :=\sup _{(\alpha, \theta) \in[0,1] \times[0, \infty)} F^{(\alpha, \theta)}(R) . \tag{26}
\end{align*}
$$

In view of the definitions above, we have the following theorem. The proof of this theorem is provided in Appendix $B$
Theorem 3. For any synthesis code such that

$$
\begin{equation*}
\frac{1}{n} \log \left|\mathcal{M}_{n}\right| \leq R \tag{27}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \geq 1-4 \exp (-n F(R)) \tag{28}
\end{equation*}
$$

If we show $F(R)>0$, then Theorem 3 implies the exponential strong converse for TV distance measure. To that end, we need the following lemma.

Lemma 1. The following conclusions hold.
(i) If $R<C_{\mathrm{Wyner}}(X ; Y)$, then

$$
\begin{equation*}
F(R)>0 \tag{29}
\end{equation*}
$$

(ii) If $R \geq C_{\mathrm{Wyner}}(X ; Y)$, then

$$
\begin{equation*}
F(R)=0 \tag{30}
\end{equation*}
$$

The proof of Lemma 1 is provided in Appendix C We remark that Lemma 1) especially part (i), plays an central role in claiming the exponential strong converse theorem for the common information problem with the TV distance measure. Its proof is completely different from that for the corresponding statement in [14] and requires some intricate continuity arguments (e.g., [24, Lemma 14]). As we have seen in Theorem 3, $F(R)$ in 26 is a lower bound on the exponent
of $1-\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$. This can be regarded as the strong converse exponent.

Combining Lemma 1 and Theorem 3 we conclude that the exponent in the right hand side of (28) is strictly positive if the rate is smaller than $C_{\text {Wyner }}(X ; Y)$. Hence, we obtain the exponential strong converse result given in the conclusion (iii) of Theorem 2

## IV. Converse Proof of Theorem 1 for the Normalized Rényi Common Information

In this section, we provide a proof of the converse part of Theorem 1 for the normalized Rényi common information. To this end, we need the following relationships between the Rényi divergence and the TV distance.

Lemma 2 (Relationship between the Rényi Divergence and the TV Distance (Sason 【[15])). For any $s \in(-1,+\infty)$,

$$
\begin{align*}
& \inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right) \\
& =\inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right|=\epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right)  \tag{31}\\
& =\inf _{q \in[0,1-\epsilon]} d_{1+s}(q+\epsilon \| q), \tag{32}
\end{align*}
$$

and for any $s \in(0,1)$,

$$
\begin{align*}
& \inf _{q \in[0,1-\epsilon]} d_{1-s}(q+\epsilon \| q) \\
\geq & {\left[\min \left\{1, \frac{1-s}{s}\right\} \log \frac{1}{1-\epsilon}-\frac{1}{s} \log 2\right]^{+} } \tag{33}
\end{align*}
$$

where
$d_{1+s}(p \| q):= \begin{cases}\frac{1}{s} \log \left(p^{1+s} q^{-s}+\bar{p}^{1+s} \bar{q}^{-s}\right), & s \geq-1, s \neq 0 \\ p \log \frac{p}{q}+\bar{p} \log \frac{\bar{p}}{\bar{q}}, & s=0\end{cases}$
denotes the binary Rényi divergence of order $1+s, 4$ We also have

$$
\begin{align*}
& \inf _{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon \\
& =\inf _{0}\left(P_{X} \| Q_{X}\right)  \tag{35}\\
& =P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right|=\epsilon  \tag{36}\\
& =0 .
\end{align*}
$$

Remark 3. Pinsker's inequality provides a lower bound for $\inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right) \quad$ or $\inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right|=\epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right)$, i.e.,

$$
\begin{equation*}
\inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right|=\epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right) \geq \frac{(1+s) \epsilon^{2}}{2} \tag{37}
\end{equation*}
$$

Hence $\frac{(1+s) \epsilon^{2}}{2}$ is also a lower bound of $\inf _{q \in[0,1-\epsilon]} d_{1+s}(q+$ $\epsilon \| q)$.
Remark 4. Using (32) and the lower bound in (33), it is easy to obtain the following improved lower bounds. For any $s \in$ $(0,1)$,

$$
\begin{align*}
& P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon \\
& =D_{1-s}\left(P_{X} \| Q_{X}\right)  \tag{38}\\
& =\inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} \sup _{t \in[s, 1)} D_{1-t}\left(P_{X} \| Q_{X}\right)
\end{align*}
$$

[^3]\[

$$
\begin{align*}
& \geq \sup _{t \in[s, 1)} \inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} D_{1-t}\left(P_{X} \| Q_{X}\right)  \tag{39}\\
& \geq \sup _{t \in[s, 1)} \inf _{q \in[0,1-\epsilon]} d_{1-t}(q+\epsilon \| q)  \tag{40}\\
& \geq \sup _{t \in[s, 1)}\left[\min \left\{1, \frac{1-t}{t}\right\} \log \frac{1}{1-\epsilon}-\frac{1}{t} \log 2\right]^{+}  \tag{41}\\
& = \begin{cases}{\left[\log \frac{1}{4(1-\epsilon)}\right]^{+}} & s \in\left(0, \frac{1}{2}\right], \\
{\left[\frac{1-s}{s} \log \frac{1}{1-\epsilon}-\frac{1}{s} \log 2\right]^{+}} & s \in\left(\frac{1}{2}, 1\right), \epsilon>\frac{1}{2} \\
0 & s \in\left(\frac{1}{2}, 1\right), \epsilon \leq \frac{1}{2}\end{cases} \tag{42}
\end{align*}
$$
\]

and for any $s \in[0,+\infty)$,

$$
\begin{align*}
& \inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} D_{1+s}\left(P_{X} \| Q_{X}\right) \\
& \geq \inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} \sup _{t \in(0,1)} D_{1-t}\left(P_{X} \| Q_{X}\right)  \tag{43}\\
& \geq \sup _{t \in(0,1)} \inf _{P_{X}, Q_{X}:\left|P_{X}-Q_{X}\right| \geq \epsilon} D_{1-t}\left(P_{X} \| Q_{X}\right)  \tag{44}\\
& \geq \sup _{t \in(0,1)} \inf _{q \in[0,1-\epsilon]} d_{1-t}(q+\epsilon \| q)  \tag{45}\\
& \geq \sup _{t \in(0,1)}\left[\min \left\{1, \frac{1-t}{t}\right\} \log \frac{1}{1-\epsilon}-\frac{1}{t} \log 2\right]^{+}  \tag{46}\\
& =\left[\log \frac{1}{4(1-\epsilon)}\right]^{+} . \tag{47}
\end{align*}
$$

Remark 5. The improved lower bounds (42) and 47) (or combining (32) and the lower bound in (33) implies if

$$
\begin{equation*}
\left|P_{X}-Q_{X}\right| \rightarrow 1 \tag{48}
\end{equation*}
$$

then for any $s \in(-1,+\infty)$,

$$
\begin{equation*}
D_{1+s}\left(P_{X} \| Q_{X}\right) \rightarrow \infty \tag{49}
\end{equation*}
$$

Combining Lemma 2 with Theorem 3 we have the converse part for the normalized Rényi divergence, which implies the strong converse for the unnormalized Rényi divergence.

Theorem 4. For any synthesis codes such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{1}{n} \log \left|\mathcal{M}_{n}\right|<C_{\mathrm{Wyner}}(X ; Y) \tag{50}
\end{equation*}
$$

we have for any $s>-1$,

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)>0 \tag{51}
\end{equation*}
$$

Remark 6. This theorem establishes the converse part of Theorem 1 for the normalized Rényi common information.
Remark 7. Since $\liminf _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)>0$ implies $D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \rightarrow \infty$, the theorem above implies the strong converse for the Wyner's common information problem under the unnormalized Rényi divergence.

Proof: Theorem 3 states if $\frac{1}{n} \log \left|\mathcal{M}_{n}\right|<C_{\text {Wyner }}(X ; Y)$, then $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \rightarrow 1$ exponentially fast. In other words,

$$
\begin{equation*}
\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \geq 1-\mathrm{e}^{-n \delta_{n}} \tag{52}
\end{equation*}
$$

for some sequence $\delta_{n}>0$ such that $\liminf _{n \rightarrow \infty} \delta_{n}>0$. Therefore, using Lemma 2 we have

$$
\begin{align*}
& \liminf _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \\
& \geq \liminf _{n \rightarrow \infty}\left\{\min \left\{1, \frac{1-s}{s}\right\} \delta_{n}-\frac{1}{n s} \log 2\right\}  \tag{53}\\
& =\min \left\{1, \frac{1-s}{s}\right\} \liminf _{n \rightarrow \infty} \delta_{n}  \tag{54}\\
& >0 \tag{55}
\end{align*}
$$

This completes the proof.

## V. Conclusion and Future Work

In this paper, we studied a generalized version of Wyner's common information problem (or the distributed source simulation problem), in which the unnormalized and normalized Rényi divergences were used to measure the level of approximation. We showed the minimum rate needed to ensure that the unnormalized or normalized Rényi divergence vanishes asymptotically remains the same as the one under Wyner's setting where the relative entropy was used.

In the future, we plan to investigate the second-order coding rate for Wyner's common information under the unnormalized Rényi divergence or the TV distance. For the unnormalized Rényi divergence, the one-shot achievability bound given in Lemma 3 can be used to obtain an achievability bound for the second-order coding rate. In fact, it can easily be shown that the optimal second-order coding rate scales as $O\left(\frac{1}{\sqrt{n}}\right)$. For the TV distance, the one-shot achievability bound given by Cuff [5] can be used to derive an achievability bound. However, the converse parts for both cases are not straightforward. One may leverage the perturbation approach [25] used to prove the second-order coding rate for the Gray-Wyner problem in [26], [27]. This is left as future work.

Furthermore, we are also interested in various closelyrelated problems. Among them, the most interesting one is the distributed channel synthesis problem under the Rényi divergence measure: The coordination problem or distributed channel synthesis problem was studied by Cuff, Permuter, and Cover [4], [5]. In this problem, an observer (encoder) of a source sequence describes the sequence to a distant random number generator (decoder) that produces another sequence. What is the minimum description rate needed to produce achieve a joint distribution that is statistically indistinguishable, under the TV distance, from the distribution induced by a given channel? For this problem, Cuff [5] provided a complete characterization of the minimum rate. We can enhance the level of coordination by replacing the TV measure with the Rényi divergence. For this enhanced version of the problem, we are interested in characterizing the corresponding admissible rate region.

## Appendix A

Achievability Proof of Theorem 1 for the Unnormalized Rényi Common Information

## A. Achievability

Next we focus on the achievability part. We first consider the case $s \in(0,1]$. First we introduce the following one-shot
achievability bound (i.e., achievability bound for blocklength $n$ equal to 1).

Lemma 3 (One-Shot Achievability Bound). [10] Consider a random mapping $P_{X \mid W}$ and a random codebook $U=$ $\{W(i)\}_{i \in \mathcal{M}}$ with $W(i) \sim P_{W}, i \in \mathcal{M}$, where $\mathcal{M}=$ $\left\{1, \ldots, \mathrm{e}^{R}\right\}$. We define

$$
\begin{equation*}
P_{X \mid U}\left(x \mid\{w(i)\}_{i \in \mathcal{M}}\right):=\frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} P_{X \mid W}(x \mid w(m)) \tag{56}
\end{equation*}
$$

Then we have for $s \in(0,1]$,

$$
\begin{align*}
& \mathrm{e}^{s D_{1+s}\left(P_{X \mid U} \| \pi_{X} \mid P_{U}\right)} \\
& \leq \mathrm{e}^{s D_{1+s}\left(P_{X \mid W} \| \pi_{X} \mid P_{W}\right)-s R}+\mathrm{e}^{s D_{1+s}\left(P_{X} \| \pi_{X}\right)}  \tag{57}\\
& \leq 2 \mathrm{e}^{s \Gamma_{1+s}\left(P_{W}, P_{X \mid W}, \pi_{X}, R\right)} \tag{58}
\end{align*}
$$

where

$$
\begin{align*}
& \Gamma_{1+s}\left(P_{W}, P_{X \mid W}, \pi_{X}, R\right) \\
& :=\max \left\{D_{1+s}\left(P_{X \mid W} \| \pi_{X} \mid P_{W}\right)-R, D_{1+s}\left(P_{X} \| \pi_{X}\right)\right\} \tag{59}
\end{align*}
$$

Remark 8. This lemma provides a one-shot achievability bound for general source synthesis problems, not only for the distributed source synthesis or common information problem as studied in this paper.

By setting $\pi_{X}, P_{X \mid W}, P_{W}$, and $R$ to $\pi_{X^{n} Y^{n}}, P_{X^{n} Y^{n} \mid W^{n}}=$ $P_{X^{n} \mid W^{n}} P_{Y^{n} \mid W^{n}}{ }^{5} P_{W^{n}}$, and $n R$ respectively, Lemma 3 can be used to derive an achievability result for the common information problem. Applying Lemma 3 and taking limits appropriately, we obtain if there exists a sequence of distributions $\left\{P_{W^{n}} P_{X^{n} \mid W^{n}} P_{Y^{n} \mid W^{n}}\right\}$ such that $\lim _{n \rightarrow \infty} D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \quad \rightarrow \quad 0 \quad$ and $\quad R \quad>$ $\limsup _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n} \mid W^{n}} \| \pi_{X^{n} Y^{n}} \mid P_{W^{n}}\right)$, then there exists a sequence of codes such that

$$
\begin{align*}
& \limsup _{n \rightarrow \infty} D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \\
& \leq \limsup _{n \rightarrow \infty} \frac{1}{s} \log \left\{\mathrm{e}^{s D_{1+s}\left(P_{X^{n} Y^{n} \mid W^{n}} \| \pi_{\left.X^{n} Y^{n} \mid P_{W^{n}}\right)-n s R}{ }^{2}\right)}\right. \\
& \left.+\mathrm{e}^{s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{\left.X^{n} Y^{n}\right)}\right.}\right\} \tag{60}
\end{align*}
$$

$$
\begin{align*}
& \leq \frac{1}{s} \log \left\{\limsup _{n \rightarrow \infty} \mathrm{e}^{s(n(R-\epsilon)-n R)}+1\right\}  \tag{61}\\
& =0, \tag{62}
\end{align*}
$$

where (62) follows since

$$
R>\limsup _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n} \mid W^{n}} \| \pi_{X^{n} Y^{n}} \mid P_{W^{n}}\right)
$$

implies there exists a constant $\epsilon>0$ such that

$$
R-\epsilon>\frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n} \mid W^{n}} \| \pi_{X^{n} Y^{n}} \mid P_{W^{n}}\right)
$$

[^4]holds for all sufficiently large $n$. Therefore, the minimum achievable rate satisfies
\[

$$
\begin{align*}
& \inf \left\{R: D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0\right\} \\
& \leq \inf _{\substack{\left\{P_{W^{n}}, P_{X^{n}} \mid W^{n}, P_{Y^{n} \mid W^{n}}\right\}_{n=1}^{\infty} \\
D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n}} Y^{n}\right) \rightarrow 0}}: \\
& \limsup _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{X^{n} Y^{n} \mid W^{n}} \| \pi_{X^{n} Y^{n}} \mid P_{W^{n}}\right) . \tag{64}
\end{align*}
$$
\]

Let $Q_{W X Y}$ be a distribution such that $Q_{X Y}=\pi_{X Y}$ and $X-W-Y$. For the optimization in (64), to obtain an upper bound, we set the distributions

$$
\begin{aligned}
P_{W^{n}}\left(w^{n}\right) & \propto Q_{W}^{n}\left(w^{n}\right) 1\left\{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)\right\}, \\
P_{X^{n} \mid W^{n}}\left(x^{n} \mid w^{n}\right) & \propto Q_{X \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{x^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right)\right\}, \\
P_{Y^{n} \mid W^{n}}\left(x^{n} \mid w^{n}\right) & \propto Q_{Y \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{y^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right)\right\},
\end{aligned}
$$

where $0<\epsilon^{\prime}<\epsilon \leq 1$. Then we have

$$
\begin{align*}
& P_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right) \\
& =\sum_{w^{n}} P_{W^{n}}\left(w^{n}\right) P_{X^{n} \mid W^{n}}\left(x^{n} \mid w^{n}\right) P_{Y^{n} \mid W^{n}}\left(x^{n} \mid w^{n}\right)  \tag{65}\\
& =\sum_{w^{n}} \frac{Q_{W}^{n}\left(w^{n}\right) 1\left\{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)\right\}}{Q_{W}^{n}\left(\mathcal{T}_{\epsilon^{\prime}}^{n}\right)} \\
& \quad \times \frac{Q_{X \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{x^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right)\right\}}{Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)} \\
& \quad \times \frac{Q_{Y \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{y^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right)\right\}}{Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right)}  \tag{66}\\
& \leq \frac{\sum_{w^{n}} Q_{W X Y}^{n}\left(w^{n}, x^{n}, y^{n}\right)}{Q_{W}^{n}\left(\mathcal{T}_{\epsilon^{\prime}}^{n}\right)} \\
& \quad \times \frac{1}{\min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)} \\
& \quad \times \frac{1}{\min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right)}  \tag{67}\\
& =\frac{\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)}{1-\delta_{n}}, \tag{68}
\end{align*}
$$

where in (68) $\delta_{n}$ is defined as 1 minus the denominator of 67). Here we claim that $\delta_{n} \rightarrow 0$ as $n \rightarrow \infty$. This follows since $Q_{W}^{n}\left(\mathcal{T}_{\epsilon^{\prime}}^{n}\right) \rightarrow 1, \min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right) \rightarrow 1$, and $\min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right) \rightarrow 1$, where the last two limits hold due to the following lemma.
Lemma 4. Assume $0<\epsilon^{\prime}<\epsilon \leq 1$, then as $n \rightarrow \infty$, $Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)$ converges uniformly to 1 (in $\left.w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)\right)$.

$$
\begin{align*}
& 1-Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right) \leq \\
& |\mathcal{X}||\mathcal{W}|\left(\mathrm{e}^{-\frac{1}{3}\left(\frac{\epsilon-\epsilon^{\prime}}{1+\epsilon^{\prime}}\right)^{2} n Q_{X \mid W}^{(\min )}}+\mathrm{e}^{-\frac{1}{2}\left(\frac{\epsilon-\epsilon^{\prime}}{1-\epsilon^{\prime}}\right)^{2} n Q_{X \mid W}^{(\min )}}\right) \tag{69}
\end{align*}
$$

[^5]for all $w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)$, where $Q_{X \mid W}^{(\min )} \quad:=$ $\min _{(x, w): Q_{X \mid W}(x \mid w)>0} Q_{X \mid W}(x \mid w)$.

This lemma is a stronger version of the conditional typicality lemma in [2], since here the probability converges uniformly, instead of converging pointwise. However, the proof is merely a refinement of the conditional typicality lemma [2, Appendix 2A] (by applying the Chernoff bound, instead of the law of large numbers), and hence omitted here. Besides, a similar lemma can be found in [29, Lemma 2.12], which is established based on a slightly different definition of strong typicality.

Using this upper bound of $P_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)$ we have

$$
\begin{align*}
& D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \\
& =\frac{1}{s} \log \sum_{x^{n}, y^{n}} P_{X^{n} Y^{n}}^{1+s}\left(x^{n}, y^{n}\right) \pi_{X^{n} Y^{n}}^{-s}\left(x^{n}, y^{n}\right)  \tag{70}\\
& \leq \frac{1}{s} \log \sum_{x^{n}, y^{n}}\left(\frac{\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)}{1-\delta_{n}}\right)^{1+s} \pi_{X^{n} Y^{n}}^{-s}\left(x^{n}, y^{n}\right)  \tag{71}\\
& =\frac{1}{s} \log \left(\frac{1}{1-\delta_{n}}\right)^{1+s}  \tag{72}\\
& \rightarrow 0 \tag{73}
\end{align*}
$$

Let $\left[T_{W} V_{X \mid W}\right]$ denote the joint distribution of $X$ and $W$ induced by the type $T_{W}$ and conditional type $V_{X \mid W}$. Now define the sets of tuples of types and conditional types:

$$
\begin{align*}
& \mathcal{A}:=\left\{\left(T_{W}, V_{X \mid W}, V_{Y \mid W}\right):\right. \\
& \forall w,\left|T_{W}(w)-Q_{W}(w)\right| \leq \epsilon^{\prime} Q_{W}(w) \\
& \forall(w, x),\left|\left[T_{W} V_{X \mid W}\right](w, x)-Q_{W X}(w, x)\right| \leq \epsilon Q_{W X}(w, x) \\
& \left.\forall(w, y),\left|\left[T_{W} V_{Y \mid W}\right](w, y)-Q_{W Y}(w, y)\right| \leq \epsilon Q_{W Y}(w, y)\right\} \tag{74}
\end{align*}
$$

and

$$
\begin{align*}
& \mathcal{B}:=\left\{\left(T_{W}, V_{X \mid W}, V_{Y \mid W}\right): \forall(w, x, y)\right. \\
& \left.\frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} \leq \frac{\left[T_{W} V_{X \mid W} V_{Y \mid W}\right](w, x, y)}{Q_{W X Y}(w, x, y)} \leq \frac{(1+\epsilon)^{2}}{1-\epsilon^{\prime}}\right\} . \tag{75}
\end{align*}
$$

In (75), if $Q_{W X Y}(w, x, y)=0$, this imposes that $\left[T_{W} V_{X \mid W} V_{Y \mid W}\right](w, x, y)=0$. It is easy to verify that $\mathcal{A} \subseteq \mathcal{B}$. Let $\delta_{1, n}$ and $\delta_{2, n}$ be two arbitrary sequences tending to zero as $n \rightarrow \infty$. Using these notations, we can write (76)(84) (shown at the top of the next page), where (79) follows from Lemma 4 follows since $\mathcal{A} \subseteq \mathcal{B}$, follows since

$$
\sum_{\substack{\left(T_{X}, V_{X \mid W}, V_{Y \mid W}\right) \in \mathcal{B}}} \sum_{w^{n} \in \mathcal{T}_{T_{W}, x^{n} \in \mathcal{T}_{V_{X \mid W}}\left(w^{n}\right),}} P\left(w^{n}, x^{n}, y^{n}\right) \leq 1,
$$

and (83) follows since $Q_{X Y}=\pi_{X Y}$.
Letting $n \rightarrow \infty$ in (84), we have

$$
\begin{align*}
& \limsup _{n \rightarrow \infty} \frac{1}{n} D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right) \\
& \leq \frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} I_{Q}(X Y ; W)+\frac{4 \epsilon}{1-\epsilon^{\prime}} H_{Q}(X Y) \tag{86}
\end{align*}
$$

Combining (86) with (64) and (73), we obtain

$$
\begin{align*}
& \inf \left\{R: D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0\right\} \\
& \leq \frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} I_{Q}(X Y ; W)+\frac{4 \epsilon}{1-\epsilon^{\prime}} H_{Q}(X Y) . \tag{87}
\end{align*}
$$

Since $\epsilon>\epsilon^{\prime}>0$ are arbitrary, and $H_{Q}(X Y)=H_{\pi}(X Y) \leq$ $\log \{|\mathcal{X}||\mathcal{Y}|\}$ is bounded, we have

$$
\begin{align*}
& \inf \left\{R: D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0\right\} \\
& \leq I_{Q}(X Y ; W) . \tag{88}
\end{align*}
$$

Since the distribution $Q_{W X Y}$ is arbitrary, we can minimize $I_{Q}(X Y ; W)$ over all distributions satisfying $Q_{X Y}=\pi_{X Y}$ and $X-W-Y$. Hence

$$
\begin{align*}
& \inf \left\{R: D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0\right\} \\
& \leq \min _{Q_{X Y W}: Q_{X Y}=\pi_{X Y}, X-W-Y} I_{Q}(X Y ; W)  \tag{89}\\
& =C_{\text {Wyner }}(X ; Y) . \tag{90}
\end{align*}
$$

Observe that

$$
\begin{align*}
& D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \\
& =\frac{1}{s} \log \mathbb{E}_{U_{n}}\left[\sum_{x^{n}, y^{n}} P_{X^{n} Y^{n} \mid U_{n}}\left(x^{n}, y^{n} \mid U_{n}\right)\right. \\
& \left.\quad \times\left(\frac{P_{X^{n} Y^{n} \mid U_{n}}\left(x^{n}, y^{n} \mid U_{n}\right)}{\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)}\right)^{s}\right], \tag{91}
\end{align*}
$$

where $\mathbb{E}_{U_{n}}$ is the expectation taken with respect to the distribution $P_{U_{n}}$. Hence $D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0$ implies that there must exist at least one sequence of codebooks indexed by $\left\{u_{n}\right\}_{n=1}^{\infty}$ such that $D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}=u_{n}} \| \pi_{X^{n} Y^{n}}\right) \rightarrow$ 0 . Therefore, the Rényi common information for $s \in(0,1]$ is not larger than $C_{\text {Wyner }}(X ; Y)$. This completes the proof for the case $s \in(0,1]$.

Now we prove the case $s \in(-1,0)$. Since $D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)$ is non-decreasing in $s$, the result for $s \in(0,1]$ implies the achievability result for $s \in(-1,0)$.

## B. Exponential Achievability

Since $D_{1+s}\left(P_{X^{n} Y^{n} U_{n}} \| \pi_{X^{n} Y^{n}} \times P_{U_{n}}\right)$ is non-decreasing in $s$, to prove the exponential result for $s \in(-1,1]$, we only need to show the result holds for $s \in(0,1]$. To this end, we use the random code given in Appendix $\mathrm{A}-\mathrm{A}$. For this code, by Lemma 3 we obtain

$$
\begin{align*}
& \mathrm{e}^{s D_{1+s}\left(P_{X^{n}} Y^{n} U_{n} \| \pi_{\left.X^{n} Y^{n} \times P_{U_{n}}\right)}\right.} \\
& \leq \mathrm{e}^{s D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right)-n s R} \\
& \quad+\mathrm{e}^{s D_{1+s}\left(P_{X^{n}} Y^{n} \| \pi_{X^{n}} Y^{n}\right)}  \tag{92}\\
& =\mathrm{e}^{s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)}\left(\mathrm{e}^{s D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right)-n s R}\right. \\
& \left.\quad \times \mathrm{e}^{-s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)}+1\right) . \tag{93}
\end{align*}
$$

$$
\begin{align*}
& \frac{1}{n} D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right) \\
& =\frac{1}{n s} \log \sum_{w^{n}, x^{n}, y^{n}} P\left(w^{n}\right)\left(P\left(x^{n} \mid w^{n}\right) P\left(y^{n} \mid w^{n}\right)\right)^{1+s} \pi^{-s}\left(x^{n}, y^{n}\right)  \tag{76}\\
& =\frac{1}{n s} \log \sum_{w^{n}, x^{n}, y^{n}} P\left(w^{n}, x^{n}, y^{n}\right) \\
& \times\left(\frac{Q_{X \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{x^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right)\right\}}{Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)} \frac{Q_{Y \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{y^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right)\right\}}{Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right)}\right)^{s} \pi_{X^{n} Y^{n}}^{-s}\left(x^{n}, y^{n}\right)  \tag{77}\\
& =\frac{1}{n s} \log \sum_{T_{W}, V_{X \mid W}, V_{Y \mid W}} \sum_{w^{n} \in \mathcal{T}_{T_{W}}, x^{n} \in \mathcal{T}_{V_{X \mid W}}\left(w^{n}\right),} P\left(w^{n}, x^{n}, y^{n}\right) \\
& y^{n} \in \mathcal{T}_{V_{Y \mid W}}\left(w^{n}\right) \\
& \times\left(\frac{Q_{X \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{x^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right)\right\}}{Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)} \frac{Q_{Y \mid W}^{n}\left(x^{n} \mid w^{n}\right) 1\left\{y^{n} \in \mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right)\right\}}{Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right)}\right)^{s} \pi_{X^{n} Y^{n}}^{-s}\left(x^{n}, y^{n}\right)  \tag{78}\\
& \leq \frac{1}{n s} \log \sum_{\left(T_{X}, V_{X \mid W}, V_{Y \mid W}\right) \in \mathcal{A}} \sum_{\substack{n \\
w^{n} \in \mathcal{T}_{T_{W}}, x^{n} \in \mathcal{T}_{V_{X \mid W}}\left(w^{n}\right), y^{n} \in \mathcal{T}_{V_{Y \mid W}}\left(w^{n}\right)}} P\left(w^{n}, x^{n}, y^{n}\right) \\
& \times\left(\frac{\mathrm{e}^{n \sum_{w, x} T(w, x) \log Q(x \mid w)}}{1-\delta_{1, n}} \frac{\mathrm{e}^{n \sum_{w, y} T(w, y) \log Q(y \mid w)}}{1-\delta_{2, n}}\right)^{s} e^{-n s \sum_{x, y} T(x, y) \log \pi(x, y)}  \tag{79}\\
& \leq-\frac{1}{n} \log \left(1-\delta_{1, n}\right)\left(1-\delta_{2, n}\right)+\frac{1}{n s} \log \sum_{\left(T_{X}, V_{X \mid W}, V_{Y \mid W}\right) \in \mathcal{B}} \sum_{w^{n} \in \mathcal{T}_{T_{W},}, x^{n} \in \mathcal{T}_{V_{X \mid W}}\left(w^{n}\right),} P\left(w^{n}, x^{n}, y^{n}\right) \\
& y^{n} \in \mathcal{T}_{V_{Y \mid W}}\left(w^{n}\right) \\
& \times \max _{\left(T_{X}, V_{X \mid W}, V_{Y \mid W}\right) \in \mathcal{B}} \mathrm{e}^{s n \sum_{w, x} T(w, x) \log Q(x \mid w)+s n \sum_{w, y} T(w, y) \log Q(y \mid w)-n s \sum_{x, y} T(x, y) \log \pi(x, y)}  \tag{80}\\
& \leq \max _{\left(T_{X}, V_{X \mid W}, V_{Y \mid W}\right) \in \mathcal{B}}\left(\sum_{w, x} T(w, x) \log Q(x \mid w)+\sum_{w, y} T(w, y) \log Q(y \mid w)-\sum_{x, y} T(x, y) \log \pi(x, y)\right) \\
& -\frac{1}{n} \log \left(1-\delta_{1, n}\right)\left(1-\delta_{2, n}\right)  \tag{81}\\
& \leq \frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}}\left(\sum_{w, x} Q(w, x) \log Q(x \mid w)+\sum_{w, y} Q(w, y) \log Q(y \mid w)\right)-\frac{(1+\epsilon)^{2}}{1-\epsilon^{\prime}} \sum_{x, y} Q(x, y) \log \pi(x, y) \\
& -\frac{1}{n} \log \left(1-\delta_{1, n}\right)\left(1-\delta_{2, n}\right)  \tag{82}\\
& =-\frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}}\left(H_{Q}(X \mid W)+H_{Q}(Y \mid W)\right)+\frac{(1+\epsilon)^{2}}{1-\epsilon^{\prime}} H_{Q}(X Y)-\frac{1}{n} \log \left(1-\delta_{1, n}\right)\left(1-\delta_{2, n}\right)  \tag{83}\\
& =\frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} I_{Q}(X Y ; W)+\frac{4 \epsilon}{1-\epsilon^{\prime}} H_{Q}(X Y)-\frac{1}{n} \log \left(1-\delta_{1, n}\right)\left(1-\delta_{2, n}\right), \tag{84}
\end{align*}
$$

Taking log's and normalizing by $s$,

$$
\begin{align*}
& D_{1+s}\left(P_{X^{n} Y^{n} U_{n}} \| \pi_{X^{n} Y^{n}} \times P_{U_{n}}\right) \\
& =D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \\
& \quad+\frac{1}{s} \log \left(\mathrm{e}^{s D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right)-n s R}\right. \\
& \left.\quad \times \mathrm{e}^{-s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)}+1\right) \\
& \leq D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)+\frac{1}{s} \mathrm{e}^{s D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{\left.X^{n} Y^{n}\right)}\right.} \\
& \quad \times \mathrm{e}^{-n s R-s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{\left.X^{n} Y^{n}\right)}\right.} . \tag{95}
\end{align*}
$$

We first consider the first term of 95). Note that in
(68), $\delta_{n}$ tends to zero exponentially fast as $n \rightarrow \infty$, since $Q_{W}^{n}\left(\mathcal{T}_{\epsilon^{\prime}}^{n}\right), \min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right)$, and $\min _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}} Q_{Y \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W Y} \mid w^{n}\right) \mid w^{n}\right)$ all tend to one exponentially fast as $n \rightarrow \infty$. Combining this with (73), we obtain that $D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right) \rightarrow 0$ exponentially fast.

Furthermore, by (84) we can write the exponent of the second term of (95) as

$$
\begin{aligned}
& \liminf _{n \rightarrow \infty} s R-\frac{1}{n} s D_{1+s}\left(P_{W^{n} X^{n} Y^{n}} \| P_{W^{n}} \pi_{X^{n} Y^{n}}\right) \\
& \quad+\frac{1}{n} s D_{1+s}\left(P_{X^{n} Y^{n}} \| \pi_{X^{n} Y^{n}}\right)
\end{aligned}
$$

$$
\begin{equation*}
=s R-s\left(\frac{(1-\epsilon)^{2}}{1+\epsilon^{\prime}} I_{Q}(X Y ; W)+\frac{4 \epsilon}{1-\epsilon^{\prime}} H_{Q}(X Y)\right) . \tag{96}
\end{equation*}
$$

Since $H_{Q}(X Y)=H_{\pi}(X Y) \leq \log \{|\mathcal{X}||\mathcal{Y}|\}$ is bounded and $R>I_{Q}(X Y ; W)$, by choosing sufficiently small $\epsilon>\epsilon^{\prime}>0$, we can ensure this exponent is positive.

Combining the two points above, we conclude that the optimal $D_{1+s}\left(P_{X^{n} Y^{n} U_{n}} \| \pi_{X^{n} Y^{n}} \times P_{U_{n}}\right)$ tends to zero exponentially fast as long as $R>C_{\mathrm{Wyner}}(X ; Y)$. On the other hand, by a similar argument in Appendix A-A $D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}} \| \pi_{X^{n} Y^{n}} \mid P_{U_{n}}\right) \rightarrow 0$ exponentially fast implies that there must exist at least one sequence of codebooks indexed by $\left\{u_{n}\right\}_{n=1}^{\infty}$ such that

$$
D_{1+s}\left(P_{X^{n} Y^{n} \mid U_{n}=u_{n}} \| \pi_{X^{n} Y^{n}}\right) \rightarrow 0
$$

exponentially fast. Hence the proof is completed.

## Appendix B

## Proof of Theorem 3

## A. Proof of Theorem 3

In this section, we present the proof of Theorem 3. In the proof, we adapt the information spectrum method proposed by Oohama [14] to first establish a non-asymptotic lower bound on $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$. Invoking the lower bound (cf. Lemma 6) and applying Cramér's bound in the theory of large deviations [30], we can obtain a further lower bound on $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$ leading to (28).

Let $P_{M_{n} X^{n} Y^{n}}$ be the joint distribution of $\left(M_{n}, X^{n}, Y^{n}\right)$, induced by the synthesis code, i.e.,

$$
\begin{align*}
& P_{M_{n} X^{n} Y^{n}}\left(m, x^{n}, y^{n}\right) \\
& =\frac{1}{\left|\mathcal{M}_{n}\right|} P_{X^{n} \mid M_{n}}\left(x^{n} \mid m\right) P_{Y^{n} \mid M_{n}}\left(y^{n} \mid m\right) \tag{97}
\end{align*}
$$

In the following, for brevity sometimes we omit the subscript, and write $P_{M_{n} X^{n} Y^{n}}$ as $P$.

Let $Q_{X^{n} Y^{n}}$ and $Q_{X^{n} Y^{n} \mid M_{n}}$ be arbitrary distributions. Given any $\eta>0$, define the following information-spectrum sets and support sets:

$$
\begin{align*}
\mathcal{A}_{1}:= & \left\{\left(x^{n}, y^{n}\right):\right. \\
& \left.\frac{1}{n} \log \frac{\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)}{Q_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)} \geq-\eta\right\} \times \mathcal{M}_{n},  \tag{98}\\
\mathcal{A}_{2}:= & \left\{\left(x^{n}, y^{n}, m\right):\right. \\
& \left.\frac{1}{n} \log \frac{P_{X^{n} \mid M_{n}}\left(x^{n} \mid w\right) P_{Y^{n} \mid M_{n}}\left(y^{n} \mid m\right)}{Q_{X^{n} Y^{n} \mid M_{n}}\left(x^{n}, y^{n} \mid m\right)} \geq-\eta\right\},  \tag{99}\\
\mathcal{A}_{3}:= & \left\{\left(x^{n}, y^{n}, m\right):\right. \\
& \left.\frac{1}{n} \log \frac{Q_{X^{n} Y^{n} \mid M_{n}}\left(x^{n}, y^{n} \mid m\right)}{\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)} \leq R+\eta\right\},  \tag{100}\\
\widetilde{\mathcal{A}}_{1}:= & \operatorname{supp}\left(\pi_{X^{n} Y^{n}}\right) \times \mathcal{M}_{n},  \tag{101}\\
\widetilde{\mathcal{A}}_{2}:= & \operatorname{supp}\left(P_{X^{n} Y^{n} M_{n}}\right),  \tag{102}\\
\widetilde{\mathcal{A}}:= & \widetilde{\mathcal{A}}_{1} \cap \widetilde{\mathcal{A}}_{2} . \tag{103}
\end{align*}
$$

Choose $U_{i}=M_{n}$ and $V_{i}=\left(X^{i-1}, Y^{i-1}\right)$. For $i=1, \ldots, n$, let $Q_{X_{i} Y_{i} U_{i} V_{i}}$ be any distribution and let
$Q_{X^{n} Y^{n}}=\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid X^{i-1} Y^{i-1}}=\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid V_{i}}$ and $Q_{X^{n} Y^{n} \mid M_{n}}=\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid M_{n} X^{i-1} Y^{i-1}}=\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid U_{i} V_{i}}$, where $Q_{X_{i} Y_{i} \mid V_{i}}$ and $Q_{X_{i} Y_{i} \mid U_{i} V_{i}}$ are conditional distributions induced by $Q_{X_{i} Y_{i} U_{i} V_{i}}$. Paralleling (98) to 100, given any $\eta>0$, we define the following memoryless version of information-spectrum sets:

$$
\begin{align*}
\mathcal{B}_{1}:= & \left\{\left(x^{n}, y^{n}, v^{n}\right):\right. \\
& \left.\frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_{i} Y_{i} \mid V_{i}}\left(x_{i}, y_{i} \mid v_{i}\right)}{\pi_{X Y}\left(x_{i}, y_{i}\right)} \leq \eta\right\} \times \mathcal{M}_{n}^{n},  \tag{104}\\
\mathcal{B}_{2}:= & \left\{\left(x^{n}, y^{n}, u^{n}, v^{n}\right):\right. \\
& \left.\frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_{i} Y_{i} \mid U_{i} V_{i}}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)}{P_{X_{i} \mid U_{i} V_{i}}\left(x_{i} \mid u_{i}, v_{i}\right) P_{Y_{i} \mid U_{i} V_{i}}\left(y_{i} \mid u_{i}, v_{i}\right)} \leq \eta\right\}, \tag{105}
\end{align*}
$$

$$
\begin{align*}
\mathcal{B}_{3}:= & \left\{\left(x^{n}, y^{n}, u^{n}, v^{n}\right):\right. \\
& \left.\frac{1}{n} \sum_{i=1}^{n} \log \frac{Q_{X_{i} Y_{i} \mid U_{i} V_{i}}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)}{\pi_{X Y}\left(x_{i}, y_{i}\right)} \leq R+\eta\right\} . \tag{106}
\end{align*}
$$

We first present a non-asymptotic lower bound on $\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right|$, i.e., a non-asymptotic converse bound for the problem.

Lemma 5. For any synthesis code such that

$$
\begin{equation*}
\frac{1}{n} \log \left|\mathcal{M}_{n}\right| \leq R \tag{107}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \geq 1-P\left(\bigcap_{i=1}^{3} \mathcal{A}_{i} \mid \widetilde{\mathcal{A}}\right)-3 \mathrm{e}^{-n \eta} \tag{108}
\end{equation*}
$$

where $P(\cdot \mid \widetilde{\mathcal{A}})=P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}$ denotes the conditional distribution of $\left({\underset{\sim}{X}}^{n}, Y^{n}, M_{n}\right) \sim P_{M_{n} X^{n} Y^{n}}$ given that $\left(X^{n}, Y^{n}, M_{n}\right) \in \widetilde{\mathcal{A}}$, with $P_{M_{n} X^{n} Y^{n}}$ denoting the distribution induced by the synthesis code.

The proof of Lemma 5 is given in Appendix B-B
Invoking Lemma 5 and choosing the distributions $Q_{X^{n} Y^{n}}$ and $Q_{X^{n} Y^{n} \mid M_{n}}$ as in the paragraph above (104), we obtain the following lemma.

Lemma 6. Given the conditions in Lemma 5 we have

$$
\begin{equation*}
\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \geq 1-P\left(\bigcap_{i=1}^{3} \mathcal{B}_{i} \mid \widetilde{\mathcal{A}}\right)-3 \mathrm{e}^{-n \eta} \tag{109}
\end{equation*}
$$

The proof of Lemma 6 is given in Appendix $\mathrm{B}-\mathrm{C}$.
In the following, for simplicity, we will use $Q_{i}$ to denote $Q_{X_{i} Y_{i} U_{i} V_{i}}$ and use $P_{i}$ to denote $P_{X_{i} Y_{i} U_{i} V_{i}}$. Let $\alpha \in[0,1]$. Then we need the following definitions to further lower bound
(109). Similar to the definition of $\omega_{Q_{X Y U}}^{(\alpha)}(x, y \mid u)$ in (22), we define

$$
\begin{align*}
& \omega_{Q_{i}, P_{i}}^{(\alpha)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) \\
&:= \bar{\alpha}\left(\log \frac{Q_{X_{i} Y_{i} \mid V_{i}}\left(x_{i}, y_{i} \mid v_{i}\right)}{\pi_{X Y}\left(x_{i}, y_{i}\right)}\right. \\
&\left.+\log \frac{Q_{X_{i} Y_{i} \mid U_{i} V_{i}}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)}{P_{X_{i} \mid U_{i} V_{i}}\left(x_{i} \mid u_{i}, v_{i}\right) P_{Y_{i} \mid U_{i} V_{i}}\left(y_{i} \mid u_{i}, v_{i}\right)}\right) \\
&+\alpha \log \frac{Q_{X_{i} Y_{i} \mid U_{i} V_{i}}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)}{\pi_{X Y}\left(x_{i}, y_{i}\right)} \tag{110}
\end{align*}
$$

Then, similar to the definition of $\Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ in (23), we define

$$
\begin{align*}
& \Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right) \\
& :=-\log \left(\sum_{x^{n}, y^{n}, m} P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}\left(x^{n}, y^{n}, m\right)\right. \\
& \left.\quad \times \exp \left(-\lambda \sum_{i=1}^{n} \omega_{Q_{i}, P_{i}}^{(\alpha)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)\right)\right) . \tag{111}
\end{align*}
$$

where $u_{i}=m, v_{i}=\left(x^{i-1}, y^{i-1}\right)$, and $P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}$ is the conditional distribution of $\left(X^{n}, Y^{n}, M_{n}\right)$ given $\left(X^{n}, Y^{n}, M_{n}\right) \in$ $\widetilde{\mathcal{A}}$.

Applying Cramér's bound [30, Section 2.2] and utilizing Lemma 6, we obtain the following lemma. The proof of this lemma is similar to that of [14, Proposition 1], and hence we omit it for the sake of brevity.
Lemma 7. For any $(\alpha, \lambda) \in[0,1] \times[0, \infty)$, given the condition in Lemma [5] we have

$$
\begin{align*}
& \left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \\
& \geq 1-4 \exp \left(-n \frac{\frac{1}{n} \Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right)-\lambda \alpha R}{1+(1+\bar{\alpha}) \lambda}\right) \tag{112}
\end{align*}
$$

Let

$$
\begin{equation*}
\underline{\Omega}^{(\alpha, \lambda)}:=\inf _{n \geq 1} \inf _{\left\{Q_{i}\right\}_{i=1}^{n}} \frac{1}{n} \Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right) \tag{113}
\end{equation*}
$$

Define

$$
\begin{equation*}
\theta:=\frac{\lambda}{1-2 \bar{\alpha} \lambda} \tag{114}
\end{equation*}
$$

Hence, we have

$$
\begin{equation*}
\lambda=\frac{\theta}{1+2 \bar{\alpha} \theta} \tag{115}
\end{equation*}
$$

The next lemma is essential in the proof.
Lemma 8. For $\alpha \in[0,1]$ and $\lambda \in\left[0, \frac{1}{2 \bar{\alpha}}\right)$, we have

$$
\begin{equation*}
\underline{\Omega}^{(\alpha, \lambda)} \geq \frac{\Omega^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta} \tag{116}
\end{equation*}
$$

The proof of Lemma 8 is similar to that of [14] Proposition 2] and given in Appendix B-D In the proof of Lemma 8 , we adopt ideas from [14] and choose appropriate distributions $Q_{X_{i} Y_{i} U_{i} V_{i}}$ via the recursive method.

Combining Lemmas 7 and 8 yields

$$
\begin{align*}
& \left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \\
& \geq 1-4 \exp \left(-n \frac{\Omega^{(\alpha, \lambda)}-\lambda \alpha R}{1+(1+\bar{\alpha}) \lambda}\right)  \tag{117}\\
& \geq 1-4 \exp \left(-n \frac{\frac{\Omega^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta}-\frac{\theta \alpha R}{1+2 \bar{\alpha} \theta}}{1+\frac{(1+\bar{\alpha}) \theta}{1+2 \bar{\alpha} \theta}}\right)  \tag{118}\\
& =1-4 \exp \left(-n \frac{\Omega^{(\alpha, \theta)}-\theta \alpha R}{1+(5-3 \alpha) \theta}\right)  \tag{119}\\
& \geq 1-4 \exp (-n F(R)) \tag{120}
\end{align*}
$$

where (120) follows from the definition of $F(R)$ in and the fact that 119 holds for any $(\alpha, \theta) \in[0,1] \times(0,+\infty)$. The proof of Theorem 3 is now complete.

## B. Proof of Lemma 5

Define $\pi_{X^{n} Y^{n} M_{n}}:=\pi_{X^{n} Y^{n}} P_{M_{n} \mid X^{n} Y^{n}}$. Then

$$
\begin{align*}
& \left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \\
& =\left\lvert\, P_{X^{n} Y^{n} M_{n}-\pi_{X^{n} Y^{n} M_{n}} \mid}^{\geq} \begin{aligned}
& \geq\left(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3}\right)-P\left(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3}\right) \\
&= 1-\pi\left(\widetilde{\mathcal{A}}_{1}^{c} \cup \mathcal{A}_{1}^{c} \cup \mathcal{A}_{3}^{c}\right)-P\left(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3}\right) \\
&=1-P\left(\widetilde{\mathcal{A}} \cap\left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right) \\
& \quad-P\left(\widetilde{\mathcal{A}}_{1} \cap \mathcal{A}_{1} \cap \mathcal{A}_{3} \cap\left(\mathcal{A}_{2}^{c} \cup \widetilde{\mathcal{A}}_{2}^{c}\right)\right) \\
& \quad-\pi\left(\widetilde{\mathcal{A}}_{1}^{c} \cup \mathcal{A}_{1}^{c} \cup \mathcal{A}_{3}^{c}\right) \\
& \geq 1-P\left(\widetilde{\mathcal{A}} \cap\left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right)-P\left(\mathcal{A}_{2}^{c}\right)-P\left(\widetilde{\mathcal{A}}_{2}^{c}\right) \\
& \quad-\pi\left(\widetilde{\mathcal{A}}_{1}^{c}\right)-\pi\left(\mathcal{A}_{1}^{c}\right)-\pi\left(\mathcal{A}_{3}^{c}\right) \\
&=1-P\left(\widetilde{\mathcal{A}} \cap\left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right)-P\left(\mathcal{A}_{2}^{c}\right)-\pi\left(\mathcal{A}_{1}^{c}\right)-\pi\left(\mathcal{A}_{3}^{c}\right)
\end{aligned}\right. \tag{121}
\end{align*}
$$

The last three terms above can each be bounded above by $\mathrm{e}^{-n \eta}$ because

$$
\begin{align*}
P\left(\mathcal{A}_{2}^{c}\right) & =\sum_{\left(x^{n}, y^{n}, m\right) \in \mathcal{A}_{2}^{c}} P\left(x^{n}, y^{n}, m\right)  \tag{127}\\
& \leq \sum_{\left(x^{n}, y^{n}, m\right) \in \mathcal{A}_{2}^{c}} P(w) Q\left(x^{n}, y^{n} \mid m\right) \mathrm{e}^{-n \eta}  \tag{128}\\
& \leq \mathrm{e}^{-n \eta} \tag{129}
\end{align*}
$$

and

$$
\begin{align*}
\pi\left(\mathcal{A}_{3}^{c}\right)= & \sum_{\left(x^{n}, y^{n}, m\right) \in \mathcal{A}_{3}^{c}} \pi\left(x^{n}, y^{n}\right) P\left(m \mid x^{n}, y^{n}\right)  \tag{130}\\
\leq & \sum_{\left(x^{n}, y^{n}, m\right) \in \mathcal{A}_{3}^{c}} Q\left(x^{n}, y^{n} \mid m\right) \\
& \times \mathrm{e}^{-n(R+\eta)} P\left(m \mid x^{n}, y^{n}\right)  \tag{131}\\
\leq & \sum_{\left(x^{n}, y^{n}, m\right) \in \mathcal{A}_{3}^{c}} Q\left(x^{n}, y^{n} \mid m\right) \mathrm{e}^{-n(R+\eta)}  \tag{132}\\
\leq & \mathrm{e}^{-n \eta}, \tag{133}
\end{align*}
$$

and

$$
\begin{align*}
\pi\left(\mathcal{A}_{1}^{c}\right) & =\sum_{\left(x^{n}, y^{n}\right) \in \mathcal{A}_{1}^{c}} \pi\left(x^{n}, y^{n}\right)  \tag{134}\\
& \leq \sum_{\left(x^{n}, y^{n}\right) \in \mathcal{A}_{1}^{c}} Q\left(x^{n}, y^{n}\right) \mathrm{e}^{-n \eta}  \tag{135}\\
& \leq \mathrm{e}^{-n \eta} . \tag{136}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& \left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \\
& \geq 1-P\left(\widetilde{\mathcal{A}} \cap\left(\bigcap_{i=1}^{3} \mathcal{A}_{i}\right)\right)-3 \mathrm{e}^{-n \eta}  \tag{137}\\
& \geq 1-P\left(\bigcap_{i=1}^{3} \mathcal{A}_{i} \mid \widetilde{\mathcal{A}}\right)-3 \mathrm{e}^{-n \eta} \tag{138}
\end{align*}
$$

## C. Proof of Lemma 6

Recall that in Appendix B-A, we choose $U_{i}=M_{n}$ and $V_{i}=\left(X^{i-1}, Y^{i-1}\right)$. Then $Q_{X^{n} Y^{n}}$ and $Q_{X^{n} Y^{n} \mid M_{n}}$ can be written as follows:

$$
\begin{align*}
& Q_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right) \\
& =\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid X^{i-1} Y^{i-1}}\left(x_{i}, y_{i} \mid x^{i-1}, y^{i-1}\right)  \tag{139}\\
& =\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid V_{i}}\left(x_{i}, y_{i} \mid v_{i}\right)  \tag{140}\\
& Q_{X^{n} Y^{n} \mid M_{n}}\left(x^{n}, y^{n} \mid m\right) \\
& =\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid M_{n} X^{i-1} Y^{i-1}}\left(x_{i}, y_{i} \mid m, x^{i-1}, y^{i-1}\right)  \tag{141}\\
& =\prod_{i=1}^{n} Q_{X_{i} Y_{i} \mid U_{i} V_{i}}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) \tag{142}
\end{align*}
$$

Now recall from Appendix B-A that the joint distribution of ( $X^{n}, Y^{n}, M_{n}$ ) induced by the code is $P_{X^{n} Y^{n} M_{n}}$. The marginal distributions of $P_{X^{n} Y^{n}} M_{n}$ are as follows:

$$
\begin{align*}
\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right) & =\prod_{i=1}^{n} \pi_{X Y}\left(x_{i}, y_{i}\right)  \tag{143}\\
P_{X^{n} \mid M_{n}}\left(x^{n} \mid m\right) & =\prod_{i=1}^{n} P_{X_{i} \mid M_{n} X^{i-1}}\left(x_{i} \mid m, x^{i-1}\right)  \tag{144}\\
& =\prod_{i=1}^{n} P_{X_{i} \mid M_{n} X^{i-1} Y^{i-1}}\left(x_{i} \mid m, x^{i-1}, y^{i-1}\right)  \tag{145}\\
& =\prod_{i=1}^{n} P_{X_{i} \mid U_{i} V_{i}}\left(x_{i} \mid u_{i}, v_{i}\right)  \tag{146}\\
P_{Y^{n} \mid M_{n}}\left(y^{n} \mid m\right) & =\prod_{i=1}^{n} P_{Y_{i} \mid M_{n} Y^{i-1}}\left(y_{i} \mid m, y^{i-1}\right) \tag{147}
\end{align*}
$$

$$
\begin{align*}
& =\prod_{i=1}^{n} P_{Y_{i} \mid M_{n} X^{i-1} Y^{i-1}}\left(y_{i} \mid m, x^{i-1}, y^{i-1}\right) \\
& =\prod_{i=1}^{n} P_{Y_{i} \mid U_{i} V_{i}}\left(y_{i} \mid u_{i}, v_{i}\right) \tag{148}
\end{align*}
$$

where (145) and (148) follow from the Markov chains $X_{i}-$ $M_{n} X^{i-1}-Y^{i-1}$ and $Y_{i}-M_{n} Y^{i-1}-X^{i-1}$ under distribution $P_{X^{n} Y^{n} M_{n}}$ (these two Markov chains can be easily obtained by observing that $\left.P_{X^{i} Y^{i} M_{n}}=P_{M_{n}} P_{X_{i} \mid M_{n} X^{i-1}} P_{Y_{i} \mid M_{n} Y^{i-1}}\right)$.

Using Lemma 5 and 139-149, we obtain

$$
\begin{equation*}
\left|P_{X^{n} Y^{n}}-\pi_{X^{n} Y^{n}}\right| \geq 1-P\left(\bigcap_{i=1}^{3} \mathcal{B}_{i} \mid \widetilde{\mathcal{A}}\right)-3 \mathrm{e}^{-n \eta} \tag{150}
\end{equation*}
$$

## D. Proof of Lemma 8

1) Removing Dependence on the Indices: Recall from Appendix B-A that the joint distribution of $\left(X^{n}, Y^{n}, M_{n}\right)$ is $P_{X^{n} Y^{n} M_{n}}$ and $P_{X_{i} Y_{i} U_{i} V_{i}}$ is induced by $P_{X^{n} Y^{n} M_{n}}$. Further, recall that $Q_{i}$ denotes $Q_{X i Y_{i} U_{i} V_{i}}$ and $P_{i}$ denotes $P_{X_{i} Y_{i} U_{i} V_{i}}$. Define

$$
\begin{equation*}
g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right):=\exp \left(-\lambda \omega_{Q_{i}, P_{i}}^{(\alpha)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)\right) \tag{151}
\end{equation*}
$$

where $\omega_{Q_{i}, P_{i}}^{(\alpha)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right)$ is defined in 110).
Recall the definition of $\Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right)$ in 111), then we obtain that

$$
\begin{align*}
& \exp \left(-\Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right)\right) \\
& =\sum_{x^{n}, y^{n}, m} P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}\left(x^{n}, y^{n}, m\right) \prod_{i=1}^{n} g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) \tag{152}
\end{align*}
$$

where $u_{i}=m, v_{i}=\left(x^{i-1}, y^{i-1}\right)$, and $P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}$ is the conditional distribution of $\left(X^{n}, Y^{n}, M_{n}\right)$ given $\left(X^{n}, Y^{n}, M_{n}\right) \in$ A.

For $i=1, \ldots, n$, define

$$
\begin{align*}
& \tilde{C}_{i}:= \sum_{x^{n}, y^{n}, m} P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}\left(x^{n}, y^{n}, m\right) \\
& \times \prod_{j=1}^{i} g_{Q_{j}, P_{j}}^{(\alpha, \lambda)}\left(x_{j}, y_{j} \mid u_{j}, v_{j}\right)  \tag{153}\\
& P_{X^{n} Y^{n} M_{n} \mid \tilde{\mathcal{A}}}^{(\alpha, \lambda) \mid i}\left(x^{n}, y^{n}, m\right):=\frac{1}{\tilde{C}_{i}} P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}\left(x^{n}, y^{n}, m\right) \\
& \times \prod_{j=1}^{i} g_{Q_{j}, P_{j}}^{(\alpha, \lambda)}\left(x_{j}, y_{j} \mid u_{j}, v_{j}\right)  \tag{154}\\
& \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right):=\frac{\tilde{C}_{i}}{\tilde{C}_{i-1}} . \tag{155}
\end{align*}
$$

Obviously, $P_{X^{n} Y^{n} M_{n} \mid \widetilde{\mathcal{A}}}^{(\alpha, \lambda) \mid i}\left(x^{n}, y^{n}, m\right)$ is a distribution induced by normalizing all the terms of the summation in the definition of $\tilde{C}_{i}$.

Similarly to [14, Lemma 7], we obtain the following lemma, which will be used to simplify $\Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right)$, defined in (155), in Appendix B-D2

Lemma 9. For $i=1, \ldots, n$, we have

$$
\begin{align*}
& \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1)}^{i}\right) \\
& \left.=\sum_{x^{n}, y^{n}, m} P_{X^{n} Y^{n} M_{n} \mid \tilde{\mathcal{A}}}^{(\alpha, \lambda) \mid x^{n}}, y^{n}, m\right) g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) . \tag{156}
\end{align*}
$$

Furthermore, combining (152), (153) and (155) gives us

$$
\begin{equation*}
\exp \left(-\Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right)\right)=\prod_{i=1}^{n} \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right) . \tag{157}
\end{equation*}
$$

2) Completion of the Proof of Lemma 8 . Assume $\mathcal{U}$ and $\mathcal{V}$ are two countable sets. Paralleling (21) to (24), for $(\alpha, \theta) \in$ $(0,1] \times(0, \infty)$, we define the following quantities:

$$
\begin{align*}
& \widetilde{\mathcal{Q}}:=\left\{Q_{X Y U V} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U} \times \mathcal{V}):\right. \\
&\left.\operatorname{supp}\left(Q_{X Y}\right) \subseteq \operatorname{supp}\left(\pi_{X Y}\right)\right\},  \tag{158}\\
& \widetilde{\omega}_{Q_{X Y U V}}^{(\alpha)}(x, y \mid u, v):=\bar{\alpha}\left(\log \frac{Q_{X Y \mid V}(x, y \mid v)}{\pi_{X Y}(x, y)}\right. \\
&\left.+\log \frac{Q_{X Y \mid U V}(x, y \mid u, v)}{Q_{X \mid U V}(x \mid u, v) Q_{Y \mid U V}(y \mid u, v)}\right) \\
&+\alpha \log \frac{Q_{X Y \mid U V}(x, y \mid u, v)}{\pi_{X Y}(x, y)},  \tag{159}\\
& \widetilde{\Omega}^{(\alpha, \lambda)}\left(Q_{X Y U V}\right) \\
&:=-\log \mathbb{E}_{Q_{X Y U V}}\left[\exp \left(-\theta \omega_{Q_{X Y U V}}^{(\alpha)}(X, Y \mid U, V)\right)\right], \tag{160}
\end{align*}
$$

$$
\begin{equation*}
\left.\widetilde{\Omega}^{(\alpha, \lambda)}:=\inf _{Q_{X Y U V} \in \widetilde{\mathcal{Q}}^{(\alpha, \lambda)}} \widetilde{Q}_{X Y U V}\right), \tag{161}
\end{equation*}
$$

where $\mathbb{E}_{Q_{\text {XYUV }}}$ in 160 is only taken over the set $\operatorname{supp}\left(Q_{\text {XYUV }}\right)$.
Recall that $u_{i}=m$ and $v_{i}=\left(x^{i-1}, y^{i-1}\right)$. For each $i=$ $1, \ldots, n$, define

$$
\begin{equation*}
P^{(\alpha, \lambda)}\left(x_{i}, y_{i}, u_{i}, v_{i}\right):=\sum_{x_{i+1}^{n}, y_{i+1}^{n}} P_{X^{n} Y^{n} M_{n} \mid \tilde{\mathcal{A}}}^{(\alpha, \lambda) \mid i-1}\left(x^{n}, y^{n}, m\right), \tag{162}
\end{equation*}
$$

where $P_{X^{n} Y^{n} M_{n}}^{(\alpha, \lambda) \mid i-1}\left(x^{n}, y^{n}, m\right)$ was defined in 1544 .
Combining Lemma 9 and (162) yields

$$
\begin{align*}
& \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right) \\
& =\sum_{x_{i}, y_{i}, u_{i}, v_{i}} P^{(\alpha, \lambda)}\left(x_{i}, y_{i}, u_{i}, v_{i}\right) g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) . \tag{163}
\end{align*}
$$

Note that $Q_{i}=Q_{X_{i} Y_{i} U_{i} V_{i}}$ can be chosen arbitrarily for all $i=1, \ldots, n$. Here we apply the recursive method. For each $i=1, \ldots, n$, we choose $Q_{X_{i} Y_{i} U_{i} V_{i}}$ such that

$$
\begin{equation*}
Q_{X_{i} Y_{i} U_{i} V_{i}}\left(x_{i}, y_{i}, u_{i}, v_{i}\right)=P^{(\alpha, \lambda)}\left(x_{i}, y_{i}, u_{i}, v_{i}\right) . \tag{164}
\end{equation*}
$$

Then, let $Q_{X_{i} Y_{i} \mid V_{i}}, Q_{X_{i} Y_{i} \mid U_{i} V_{i}}$ be induced by $Q_{X_{i} Y_{i} U_{i} V_{i}}$. Define

$$
\begin{align*}
& h_{Q_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right):=g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(x_{i}, y_{i} \mid u_{i}, v_{i}\right) \\
& \times\left(\frac{P_{X,}^{\lambda \bar{\alpha}} \mid U_{i} V_{i}}{}\left(x_{i} \mid u_{i}, v_{i}\right) P_{Y_{i}}^{\lambda \bar{\alpha}} U_{i} V_{i}\left(y_{i} \mid u_{i}, v_{i}\right)\right)^{-1}, \tag{165}
\end{align*}
$$

where $g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}$ was defined in (151). In the following, for brevity, we drop the subscripts of the distributions. From (163), we obtain

$$
\begin{align*}
& \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right) \\
& =\mathbb{E}_{Q_{i}}\left[g_{Q_{i}, P_{i}}^{(\alpha, \lambda)}\left(X_{i}, Y_{i} \mid U_{i}, V_{i}\right)\right]  \tag{166}\\
& =\mathbb{E}_{Q_{i}}\left[h_{Q_{i}}^{(\alpha, \lambda)}\left(X_{i}, Y_{i} \mid U_{i}, V_{i}\right)\right. \\
& \left.\times \frac{P_{X_{i l}}^{\lambda \bar{\alpha}} U_{U_{i}} V_{i}}{}\left(X_{i} \mid U_{i}, V_{i}\right) P_{Y_{i} \mid U_{i} V_{i}}^{\lambda \bar{\alpha}}\left(Y_{i} \mid U_{i}, V_{i}\right)\right]  \tag{167}\\
& \leq\left(\mathbb{E}_{Q_{i}}\left[\left\{h_{Q_{i}}^{(\alpha, \lambda)}\left(X_{i}, Y_{i} \mid U_{i}, V_{i}\right)\right\}^{\frac{1}{1-2 \lambda \alpha}}\right]\right)^{1-2 \lambda \bar{\alpha}} \\
& \times\left(\mathbb{E}_{Q_{i}}\left[\frac{P_{X_{i} \mid U_{U} V_{i}}\left(X_{i} \mid U_{i}, V_{i}\right)}{Q_{X_{i} \mid U_{i} V_{i}}\left(X_{i} \mid U_{i}, V_{i}\right)}\right]\right)^{\lambda \bar{\alpha}} \\
& \times\left(\mathbb{E}_{Q_{i}}\left[\frac{P_{Y_{i} \mid U_{i} V_{i}}\left(Y_{i} \mid U_{i}, V_{i}\right)}{Q_{Y_{i} \mid U_{i} V_{i}}\left(Y_{i} \mid U_{i}, V_{i}\right)}\right]\right)^{\lambda \bar{\alpha}}  \tag{168}\\
& \leq \exp \left(-(1-2 \lambda \bar{\alpha}) \widetilde{\Omega}^{\left(\alpha, \frac{\lambda}{1-2 \lambda \alpha}\right)}\left(Q_{i}\right)\right)  \tag{169}\\
& =\exp \left(-\frac{\widetilde{\Omega}^{(\alpha, \theta)}\left(Q_{i}\right)}{1+2 \bar{\alpha} \theta}\right)  \tag{170}\\
& \leq \exp \left(-\frac{\widetilde{\Omega}^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta}\right)  \tag{171}\\
& =\exp \left(-\frac{\Omega^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta}\right), \tag{172}
\end{align*}
$$

where (167) follows from (165); (168) follows from Hölder's inequality; (169) follows from the definitions of $\Omega^{(\alpha, \theta)}(\cdot)$ and $h_{Q_{i}}^{(\alpha, \lambda)}(\cdot)$ in (23) and (165) respectively; (170) follows from (114) and (115); (171) follows since $\Omega^{(\alpha, \theta)}\left(Q_{X Y U V}\right) \geq$ $\widetilde{\Omega}^{(\alpha, \theta)}$ for any $Q_{X Y U V}$ such that $\operatorname{supp}\left(Q_{X Y}\right) \subseteq \operatorname{supp}\left(\pi_{X Y}\right)$ (The fact that $Q_{i}$ satisfies this point will be shown in the following paragraph); and (172) follows since by the support lemma [2], the cardinality bounds $|\mathcal{V}| \leq 1,|\mathcal{U}| \leq|\mathcal{X}||\mathcal{Y}|$ are sufficient to exhaust $\widetilde{\Omega}^{(\alpha, \theta)}$.

Now we show that according to the choice of $Q_{X_{i} Y_{i} U_{i} V_{i}}$, we have $\operatorname{supp}\left(Q_{X_{i} Y_{i}}\right) \subseteq \operatorname{supp}\left(\pi_{X Y}\right)$, which was used in (1711. Note that $P_{X^{n} Y^{n} M_{n}}\left(x^{n}, y^{n}, m\right)>0$ and $\pi_{X^{n} Y^{n}}\left(x^{n}, y^{n}\right)>0$ for any $\left(x^{n}, y^{n}, m\right) \in \widetilde{\mathcal{A}}$, and hence the marginal distributions $P_{X_{i} Y_{i} U_{i} V_{i}}, P_{X_{i} \mid U_{i} V_{i}}$ and $P_{Y_{i} \mid U_{i} V_{i}}$ when evaluated at any $\left(x^{n}, y^{n}, m\right) \in \mathcal{A}$ is positive as well. According to the choice of $Q_{X_{i} Y_{i} U_{i} V_{i}}$ in (164), we have that $Q_{X_{i} Y_{i} U_{i} V_{i}}$ is also positive when evaluated at $\left(x^{n}, y^{n}, m\right) \in \mathcal{A}$ (this point can be shown via mathematical induction), i.e.,

$$
\begin{align*}
\operatorname{supp}\left(Q_{X_{i} Y_{i} U_{i} V_{i}}\right) \supseteq & \left\{(x, y, u, v): \exists\left(x^{n}, y^{n}, m\right) \in \widetilde{\mathcal{A}}:\right. \\
& \left.x_{i}=x, y_{i}=y, m=u,\left(x^{i-1}, y^{i-1}\right)=v\right\} . \tag{173}
\end{align*}
$$

On the other hand, also according to the choice of $Q_{X_{i} Y_{i} U_{i} V_{i}}$, we have

$$
\left.\begin{array}{rl}
\operatorname{supp}\left(Q_{X_{i} Y_{i} U_{i} V_{i}}\right) \subseteq & \left\{(x, y, u, v): \exists\left(x^{n}, y^{n}, m\right) \in \widetilde{\mathcal{A}}:\right. \\
& x_{i} \tag{174}
\end{array}=x, y_{i}=y, m=u,\left(x^{i-1}, y^{i-1}\right)=v\right\} .
$$

Therefore,

$$
\begin{align*}
\operatorname{supp}\left(Q_{X_{i} Y_{i} U_{i} V_{i}}\right)= & \left\{(x, y, u, v): \exists\left(x^{n}, y^{n}, m\right) \in \widetilde{\mathcal{A}}:\right. \\
& \left.x_{i}=x, y_{i}=y, m=u,\left(x^{i-1}, y^{i-1}\right)=v\right\} \tag{175}
\end{align*}
$$

Further, we have

$$
\begin{align*}
& \operatorname{supp}\left(Q_{X_{i} Y_{i}}\right) \\
& =\left\{(x, y): \exists\left(x^{n}, y^{n}, m\right) \in \widetilde{\mathcal{A}}: x_{i}=x, y_{i}=y\right\}  \tag{176}\\
& \subseteq\left\{(x, y): \exists\left(x^{n}, y^{n}, m\right) \in \operatorname{supp}\left(\pi_{X^{n} Y^{n}}\right) \times \mathcal{M}_{n}:\right. \\
& \left.\quad x_{i}=x, y_{i}=y\right\}  \tag{177}\\
& =\operatorname{supp}\left(\pi_{X Y}\right) \tag{178}
\end{align*}
$$

Combining 157) and (172), we obtain that

$$
\begin{align*}
\frac{1}{n} \Omega^{(\alpha, \lambda)}\left(\left\{Q_{i}\right\}_{i=1}^{n}\right) & =-\frac{1}{n} \sum_{i=1}^{n} \log \Lambda_{i}^{(\alpha, \lambda)}\left(\left\{Q_{j}\right\}_{j=1}^{i}\right)  \tag{179}\\
& \geq \frac{\Omega^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta} \tag{180}
\end{align*}
$$

Finally, combining (113) and (180), we have that

$$
\begin{equation*}
\underline{\Omega}^{(\alpha, \lambda)} \geq \frac{\Omega^{(\alpha, \theta)}}{1+2 \bar{\alpha} \theta} \tag{181}
\end{equation*}
$$

The proof of Lemma 8 is now complete.

## Appendix C

Proof of Lemma 1
Let $U$ be a random variable taking values in a finite alphabet $\mathcal{U}$. Define a set of joint distributions on $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ as
$\mathcal{P}^{*}:=\left\{P_{X Y U}:|\mathcal{U}| \leq|\mathcal{X}||\mathcal{Y}|, P_{X Y}=\pi_{X Y}, X-U-Y\right\}$.
and let

$$
\begin{equation*}
R^{*}:=\min _{P_{X Y U} \in \mathcal{P}^{*}} I(X Y ; U) \tag{183}
\end{equation*}
$$

## A. Preliminary Lemmata for the Proof of Lemma 1

By the support lemma [2, Appendix C], we have the following lemma [1].
Lemma 10. Wyner's common information $C_{\mathrm{Wyner}}(X ; Y)$ satisfies

$$
\begin{equation*}
C_{\mathrm{Wyner}}(X ; Y)=R^{*} . \tag{184}
\end{equation*}
$$

Before proceeding the proof of Lemma 1 we present an alternative expression for Wyner's common information. Recall that given a number $a \in[0,1]$, we define $\bar{a}=1-a$. Then for any $\alpha \in[0,1]$ and $Q_{X Y U} \in \mathcal{Q}$, define

$$
\begin{align*}
R^{(\alpha)}\left(Q_{X Y U}\right):= & \bar{\alpha}\left(D\left(Q_{X Y} \| \pi_{X Y}\right)\right. \\
& \left.+D\left(Q_{X Y \mid U} \| Q_{X \mid U} Q_{Y \mid U} \mid Q_{U}\right)\right) \\
& +\alpha D\left(Q_{X Y \mid U} \| \pi_{X Y} \mid Q_{U}\right)  \tag{185}\\
R^{(\alpha)}:= & \min _{Q_{X Y U} \in \mathcal{Q}} R^{(\alpha)}\left(Q_{X Y U}\right)  \tag{186}\\
R_{\mathrm{sh}}:= & \sup _{\alpha \in(0,1]} \frac{1}{\alpha} R^{(\alpha)} \tag{187}
\end{align*}
$$

By observing that both $\mathcal{P}^{*}$ and $\mathcal{Q}$ are compact, and by utilizing the fact that a continuous function defined on a compact set attains its minimum, we obtain the following.

Fact 1. Both the minima in the definitions of $R^{*}$ in (183) and $R^{(\alpha)}$ in (186) are attained.

We then have the following lemma.
Lemma 11. The following conclusions hold.
(i) For any $\alpha \in(0,1]$, we have

$$
\begin{equation*}
\frac{1}{\alpha} R^{(\alpha)} \leq R^{*} \tag{188}
\end{equation*}
$$

Moreover, there exists some decreasing sequence $\left\{\alpha_{k}\right\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $\lim _{k \rightarrow \infty} \alpha_{k}=0$ and

$$
\begin{equation*}
\frac{1}{\alpha_{k}} R^{\left(\alpha_{k}\right)} \geq R^{*}-c\left(\alpha_{k}\right) \tag{189}
\end{equation*}
$$

where $\left\{c\left(\alpha_{k}\right)\right\}_{k=1}^{\infty} \subset \mathbb{R}$ is another sequence such that $\lim _{k \rightarrow \infty} c\left(\alpha_{k}\right)=0$.
(ii) We have

$$
\begin{equation*}
R_{\mathrm{sh}}=R^{*}=C_{\mathrm{Wyner}}(X ; Y) \tag{190}
\end{equation*}
$$

Lemma 11 is similar to [14, Property 3], but the proofs are different. Essentially, in both the proof of [14, Property 3] and our proof, an intermediate distribution $\widetilde{Q}_{X Y U}$ is used to establish the inequality

$$
\begin{equation*}
R^{*}-c\left(\alpha_{k}\right) \leq \frac{1}{\alpha_{k}} R^{\left(\alpha_{k}\right)}\left(\widetilde{Q}_{X Y U}\right) \leq \frac{1}{\alpha_{k}} R^{\left(\alpha_{k}\right)} \tag{191}
\end{equation*}
$$

However, the construction of such an intermediate distribution is different for these two proofs. The construction in [14] does not apply to our case, since our case does not only require $\widetilde{Q}_{X Y U}$ to satisfy the Markov chain $X-U-Y$, but also requires that $\widetilde{Q}_{X Y}=\pi_{X Y}$.

Proof of Lemma 11. It is easy to show 188. Hence, by the definition of $R_{\mathrm{sh}}$ in 187),

$$
\begin{equation*}
R_{\mathrm{sh}} \leq R^{*} \tag{192}
\end{equation*}
$$

In the following we prove (189). Let $\left\{\alpha_{m}\right\}_{m=1}^{\infty}$ be an arbitrary sequence of decreasing positive real numbers such that $\lim _{m \rightarrow \infty} \alpha_{m}=0$, and let $Q_{X Y U}^{(m)}$ be a minimizing distribution of (186) with $\alpha=\alpha_{m}$. The existence of this minimizing distribution is guaranteed by Fact 1 Since $\mathcal{P}(\mathcal{X} \times \mathcal{Y} \times \mathcal{U})$ is compact (following from the definition of $\mathcal{Q}$ ), there must
exist some sequence of increasing integers $\left\{m_{k}\right\}_{k=1}^{\infty}$ such that $Q_{X Y U}^{\left(m_{k}\right)}$ converges to some distribution $\widetilde{Q}_{X Y U}$. Consider,

$$
\begin{align*}
& R_{\mathrm{sh}}= \sup _{\alpha \in(0,1]} \frac{1}{\alpha} R^{(\alpha)}  \tag{193}\\
& \geq \limsup _{k \rightarrow \infty} \frac{1}{\alpha_{m_{k}}} R^{\left(\alpha_{m_{k}}\right)}  \tag{194}\\
&=\limsup _{k \rightarrow \infty}\left\{\frac { \overline { \alpha } _ { m _ { k } } } { \alpha _ { m _ { k } } } \left(D\left(Q_{X Y}^{\left(m_{k}\right)} \| \pi_{X Y}\right)\right.\right. \\
&\left.+D\left(Q_{X Y \mid U}^{\left(m_{k}\right)} \| Q_{X \mid U}^{\left(m_{k}\right)} Q_{Y \mid U}^{\left(m_{k}\right)} \mid Q_{U}^{\left(m_{k}\right)}\right)\right) \\
&\left.+D\left(Q_{X Y \mid U}^{\left(m_{k}\right)} \| \pi_{X Y} \mid Q_{U}^{\left(m_{k}\right)}\right)\right\}  \tag{195}\\
& \geq \limsup _{k \rightarrow \infty}\left\{\frac{\bar{\alpha}_{m_{k}}}{\alpha_{m_{k}}}\right\} \liminf _{k \rightarrow \infty}\left\{D\left(Q_{X Y}^{\left(m_{k}\right)} \| \pi_{X Y}\right)\right. \\
&\left.+D\left(Q_{X Y \mid U}^{\left(m_{k}\right)} \| Q_{X \mid U}^{\left(m_{k}\right)} Q_{Y \mid U}^{\left(m_{k}\right)} \mid Q_{U}^{\left(m_{k}\right)}\right)\right\} \\
&+\liminf _{k \rightarrow \infty} D\left(Q_{X Y \mid U}^{\left(m_{k}\right)} \| \pi_{X Y} \mid Q_{U}^{\left(m_{k}\right)}\right)  \tag{196}\\
&=\infty\left(D\left(\widetilde{Q}_{X Y} \| \pi_{X Y}\right)+D\left(\widetilde{Q}_{X Y \mid U} \| \widetilde{Q}_{X \mid U} \widetilde{Q}_{Y \mid U} \mid \widetilde{Q}_{U}\right)\right) \\
&+D\left(\widetilde{Q}_{X Y \mid U} \| \pi_{X Y} \mid \widetilde{Q}_{U}\right) . \tag{197}
\end{align*}
$$

Observe that $R_{\text {sh }}$ is finite due to (192). Hence it holds that

$$
\begin{align*}
D\left(\widetilde{Q}_{X Y} \| \pi_{X Y}\right) & =0  \tag{198}\\
D\left(\widetilde{Q}_{X Y \mid U} \| \widetilde{Q}_{X \mid U} \widetilde{Q}_{Y \mid U} \mid \widetilde{Q}_{U}\right) & =0 \tag{199}
\end{align*}
$$

That is,

$$
\begin{align*}
\widetilde{Q}_{X Y} & =\pi_{X Y}  \tag{200}\\
\widetilde{Q}_{X Y \mid U} & =\widetilde{Q}_{X \mid U} \widetilde{Q}_{Y \mid U} \tag{201}
\end{align*}
$$

Therefore, under (200) and (201), we have

$$
\begin{align*}
\text { (197) } & \geq D\left(\widetilde{Q}_{X Y \mid U} \| \pi_{X Y} \mid \widetilde{Q}_{U}\right)  \tag{202}\\
& =I\left(\widetilde{Q}_{X Y \mid U}, \widetilde{Q}_{X Y}\right)  \tag{203}\\
& \geq R^{*} \tag{204}
\end{align*}
$$

Combining (192), 197) and 204b yields us

$$
\begin{equation*}
R_{\mathrm{sh}}=R^{*}=\lim _{k \rightarrow \infty} \frac{1}{\alpha_{m_{k}}} R^{\left(\alpha_{m_{k}}\right)} \tag{205}
\end{equation*}
$$

Therefore, there exists some sequence $\left\{c\left(\alpha_{m_{k}}\right)\right\}_{k=1}^{\infty} \subset \mathbb{R}$ (e.g., the sequence $\left\{R^{*}-\frac{1}{\alpha_{m_{k}}} R^{\left(\alpha_{m_{k}}\right)}\right\}_{k=1}^{\infty} \subset \mathbb{R}$ ) such that $\lim _{k \rightarrow \infty} c\left(\alpha_{m_{k}}\right)=0$ and

$$
\begin{equation*}
R^{*}-c\left(\alpha_{m_{k}}\right) \leq \frac{1}{\alpha_{m_{k}}} R^{\left(\alpha_{m_{k}}\right)} \leq R^{*} \tag{206}
\end{equation*}
$$

This concludes the proof.
We also have the following crucial lemma.
Lemma 12. Let $\alpha \in(0,1]$ and $Q_{X Y U} \in \mathcal{Q}$. Then we have

$$
\begin{equation*}
\lim _{\theta \downarrow 0} \frac{1}{\theta} \Omega^{(\alpha, \theta)}=R^{(\alpha)} \tag{207}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
\frac{1}{\theta} \Omega^{(\alpha, \theta)}=R^{(\alpha)}+\epsilon^{(\alpha, \theta)} \tag{208}
\end{equation*}
$$

where $\Omega^{(\alpha, \theta)}$ and $R^{(\alpha)}$ were defined in (24) and (186) respectively, and $\epsilon^{(\alpha, \theta)}$ is a term that vanishes as $\theta \downarrow 0$, the rate being dependent on $\alpha$.

Proof of Lemma 12. To show this lemma, we first need to show that

$$
\widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right):= \begin{cases}\frac{1}{\theta} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right), & \theta>0  \tag{209}\\ R^{(\alpha)}\left(Q_{X Y U}\right), & \theta=0\end{cases}
$$

is continuous in $\left(\theta, Q_{X Y U}\right) \in\left[0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$. It is easy to observe that

$$
\begin{align*}
& \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right) \\
& =-\log \mathbb{E}_{Q_{X Y U}}\left[\exp \left(-\theta \omega_{Q_{X Y U}}^{(\alpha)}(X, Y \mid U)\right)\right]  \tag{210}\\
& =-\log \sum_{x, y, u} Q_{X Y U}^{1-\theta(1+\bar{\alpha})}(x, y, u)\left(Q_{U}(u) \pi_{X Y}(x, y)\right)^{\theta} \\
& \quad \times\left(Q_{U \mid X Y}(u \mid x, y) Q_{X \mid U}(x \mid u) Q_{Y \mid U}(y \mid u)\right)^{\theta \bar{\alpha}} \tag{211}
\end{align*}
$$

is jointly continuous in $\left(\theta, Q_{X Y U}\right) \in\left[0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$, hence $\widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ is jointly continuous on $\left(0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$. Therefore, to show the continuity of $\widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ in $\left(\theta, Q_{X Y U}\right) \in$ $\left[0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$, it suffices to show it is continuous at any point in $\{0\} \times \mathcal{Q}$, i.e.,

$$
\begin{equation*}
\lim _{\left(\theta, Q_{X Y U}\right) \rightarrow\left(0, Q_{X Y U}^{\prime}\right)} \frac{1}{\theta} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)=R^{(\alpha)}\left(Q_{X Y U}^{\prime}\right) \tag{212}
\end{equation*}
$$

for any $Q_{X Y U}^{\prime} \in \mathcal{Q}$.
Let

$$
\begin{align*}
& Q_{X Y U}^{(\alpha, \theta)}(x, y, u) \\
& :=\frac{Q_{X Y U}(x, y, u) \exp \left(-\theta \omega_{Q X Y U}^{(\alpha)}(x, y \mid u)\right)}{\sum_{x, y, u} Q_{X Y U}(x, y, u) \exp \left(-\theta \omega_{Q X Y U}^{(\alpha)}(x, y \mid u)\right)} . \tag{213}
\end{align*}
$$

Invoking the definition of $\Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ in (23), we obtain

$$
\begin{equation*}
\frac{\partial \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta}=\mathbb{E}_{Q_{X Y U}^{(\alpha, \theta)}}\left[\omega_{Q_{X Y U}}^{(\alpha)}(X, Y \mid U)\right] \tag{214}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta^{2}}=-\operatorname{Var}_{Q_{X Y U}^{(\alpha, \theta)}}\left[\omega_{Q_{X Y U}}^{(\alpha)}(X, Y \mid U)\right] \tag{215}
\end{equation*}
$$

Hence for fixed $Q_{X Y U} \in \mathcal{Q}$, we have

$$
\begin{align*}
&\left.\frac{\partial \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta}\right|_{\theta=0}=R^{(\alpha)}\left(Q_{X Y U}\right)>0  \tag{216}\\
& \frac{\partial^{2} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta^{2}} \leq 0 \tag{217}
\end{align*}
$$

which implies that

$$
\begin{equation*}
\theta \frac{\partial \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta} \leq \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right) \leq \theta R^{(\alpha)}\left(Q_{X Y U}\right) \tag{218}
\end{equation*}
$$

Furthermore, observe that $R^{(\alpha)}\left(Q_{X Y U}\right)$ is continuous in $Q_{X Y U} \in \mathcal{Q}$, hence

$$
\begin{equation*}
\lim _{Q_{X Y U} \rightarrow Q_{X Y U}^{\prime}} R^{(\alpha)}\left(Q_{X Y U}\right)=R^{(\alpha)}\left(Q_{X Y U}^{\prime}\right) \tag{219}
\end{equation*}
$$

On the other hand, observe that $\frac{\partial \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta}$ given in (214) is continuous in $\left(\theta, Q_{X Y U}\right) \in\left[0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$. Hence

$$
\begin{align*}
& \quad \lim _{\left(\theta, Q_{X Y U}\right) \rightarrow\left(0, Q_{X Y U}^{\prime}\right)} \frac{\partial \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)}{\partial \theta} \\
& =\sum_{x, y, u} Q_{X Y U}^{\prime}(x, y, u) \omega_{Q_{X Y U}^{\prime}}^{(\alpha)}(X, Y \mid U)  \tag{220}\\
& =R^{(\alpha)}\left(Q_{X Y U}^{\prime}\right) \tag{221}
\end{align*}
$$

Therefore, combining (218), 219), and (221, we observe that the limit $\lim _{\left(\theta, Q_{X Y U}\right) \rightarrow\left(0, Q_{X Y U}^{\prime}\right)} \frac{1}{\theta} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ exists, and moreover, $\lim _{\left(\theta, Q_{X Y U}\right) \rightarrow\left(0, Q_{X Y U}^{\prime}\right)} \frac{1}{\theta} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)=$ $R^{(\alpha)}\left(Q_{X Y U}^{\prime}\right)$. Hence, we obtain (212). In other words, $\widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ is jointly continuous in $\left(\theta, Q_{X Y U}\right) \in$ $\left[0, \frac{1}{1+\bar{\alpha}}\right) \times \mathcal{Q}$. In addition, observe that $\mathcal{Q}$ is a compact set. By using the following lemma we can assert that $\min _{Q_{X Y U} \in \mathcal{Q}} \widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right)$ is continuous in $\theta \in\left[0, \frac{1}{1+\bar{\alpha}}\right)$.
Lemma 13 (Lemma 14 in [24]). Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and let $\mathcal{K} \subset \mathcal{X}$ be a compact set. Let $f: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ be a (jointly) continuous real-valued function. Then the function $g: \mathcal{Y} \rightarrow \mathbb{R}$, defined as

$$
\begin{equation*}
g(y):=\min _{x \in \mathcal{K}} f(x, y), \quad \forall y \in \mathcal{Y} \tag{222}
\end{equation*}
$$

is continuous on $\mathcal{Y}$.
Considering the point $\theta=0$, we obtain

$$
\begin{align*}
& \lim _{\theta \rightarrow 0} \min _{Q_{X Y U} \in \mathcal{Q}} \widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right) \\
& =\min _{Q_{X Y U} \in \mathcal{Q}} \widehat{R}^{(\alpha, 0)}\left(Q_{X Y U}\right)  \tag{223}\\
& =\min _{Q_{X Y U} \in \mathcal{Q}} R^{(\alpha)}\left(Q_{X Y U}\right)=R^{(\alpha)} \tag{224}
\end{align*}
$$

where the first equality follows from Lemma 13 which essentially says that the limit and minimum operations can be swapped. On the other hand, observe that

$$
\begin{align*}
& \lim _{\theta \rightarrow 0} \min _{Q X Y U} \widehat{R}^{(\alpha, \theta)}\left(Q_{X Y U}\right) \\
& =\lim _{\theta \rightarrow 0} \min _{Q_{X Y U} \in \mathcal{Q}} \frac{1}{\theta} \Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right)  \tag{225}\\
& =\lim _{\theta \rightarrow 0} \frac{1}{\theta} \Omega^{(\alpha, \theta)} \tag{226}
\end{align*}
$$

Combining (224) and 226, we obtain (207) as desired.

## B. Proof of Part (i) in Lemma 1

Using Lemma 10, we obtain that if $R<C_{\text {Wyner }}(X ; Y)$, then

$$
\begin{equation*}
R+\tau \leq R^{*} \tag{227}
\end{equation*}
$$

for some $\tau>0$. Further, invoking (189) and (227), we obtain that there exists $k_{0}$ such that for any $k \geq k_{0}$,

$$
\begin{equation*}
R+\tau \leq \frac{1}{\alpha_{k}} R^{\left(\alpha_{k}\right)}+c\left(\alpha_{k}\right) \tag{228}
\end{equation*}
$$

and

$$
\begin{equation*}
c\left(\alpha_{k}\right) \leq \frac{\tau}{2} \tag{229}
\end{equation*}
$$

Referring to (228) and (229), we obtain that for any $k \geq k_{0}$,

$$
\begin{equation*}
R+\frac{\tau}{2} \leq \frac{1}{\alpha_{k}} R^{\left(\alpha_{k}\right)} \tag{230}
\end{equation*}
$$

Therefore, invoking (26), we conclude that for any $k \geq k_{0}$,

$$
\begin{align*}
F(R) & \geq \sup _{\theta \geq 0} F^{\left(\alpha_{k}, \theta\right)}(R)  \tag{231}\\
& \geq \sup _{\theta \in\left[0, \frac{1}{1+\bar{\alpha}_{k}}\right)} F^{\left(\alpha_{k}, \theta\right)}(R)  \tag{232}\\
& =\sup _{\theta \in\left[0, \frac{1}{1+\bar{\alpha}_{k}}\right)} \frac{\Omega^{\left(\alpha_{k}, \theta\right)}-\theta \alpha_{k} R}{1+\left(5-3 \alpha_{k}\right) \theta}  \tag{233}\\
& \geq \sup _{\theta \in\left[0, \frac{1}{1+\bar{\alpha}_{k}}\right)} \frac{1}{1+5 \theta}\left\{\theta R^{\left(\alpha_{k}\right)}+\theta \epsilon^{\left(\alpha_{k}, \theta\right)}-\theta \alpha_{k} R\right\}  \tag{234}\\
& \geq \sup _{\theta \in\left[0, \frac{1}{1+\bar{\alpha}_{k}}\right)} \frac{\theta}{1+5 \theta}\left\{\epsilon^{\left(\alpha_{k}, \theta\right)}+\frac{\alpha_{k} \tau}{2}\right\}  \tag{235}\\
& \geq \sup _{\theta \in[0, \widetilde{\theta}]} \frac{\alpha_{k} \tau \theta}{4(1+5 \theta)}  \tag{236}\\
& \geq \frac{\alpha_{k} \tau \widetilde{\theta}}{4(1+5 \widetilde{\theta})} \tag{237}
\end{align*}
$$

where (234) follows from Lemma 12 and the inequality $1+$ $\left(5-3 \alpha_{k}\right) \theta \leq 1+5 \theta$, 235) follows from 230), and 236) follows since there exists a sufficiently small $\widetilde{\theta} \in\left(0, \frac{1}{1+\bar{\alpha}_{k}}\right)$ such that $\left|\epsilon^{\left(\alpha_{k}, \theta\right)}\right| \leq \frac{1}{4} \alpha_{k} \tau$ for all $\theta \leq \widetilde{\theta}$. Since the expression in 237) is positive, we have $F(R)>0$ as desired.

## C. Proof of Part (ii) in Lemma 1

Because $\exp (\cdot)$ is convex, applying Jensen's inequality, we obtain

$$
\begin{align*}
\Omega^{(\alpha, \theta)}\left(Q_{X Y U}\right) & \leq \theta \mathbb{E}_{Q_{X Y U}}\left[\omega_{Q_{X Y U}}^{(\alpha)}(X, Y \mid U)\right]  \tag{238}\\
& =\theta R^{(\alpha)}\left(Q_{X Y U}\right) \tag{239}
\end{align*}
$$

Hence we have

$$
\begin{align*}
\Omega^{(\alpha, \theta)} & \leq \min _{Q_{X Y U} \in \mathcal{Q}} \theta R^{(\alpha)}\left(Q_{X Y U}\right)  \tag{240}\\
& =\theta R^{(\alpha)} \tag{241}
\end{align*}
$$

Thus, recalling the definition of $F^{(\alpha, \theta)}(R)$ in (25), we obtain that

$$
\begin{align*}
F^{(\alpha, \theta)}(R) & =\frac{\Omega^{(\alpha, \theta)}-\theta \alpha R}{1+(5-3 \alpha) \theta}  \tag{242}\\
& \leq \frac{\theta \alpha\left(\frac{1}{\alpha} R^{(\alpha)}-R\right)}{1+(5-3 \alpha) \theta}  \tag{243}\\
& \leq \frac{\theta \alpha\left(R_{\mathrm{sh}}-R\right)}{1+(5-3 \alpha) \theta}  \tag{244}\\
& \leq 0 \tag{245}
\end{align*}
$$

where (245) follows from the assumption $R \geq$ $C_{\text {Wyner }}(X ; Y)=R_{\mathrm{sh}}$. On the other hand, note that

$$
\begin{equation*}
\lim _{\theta \rightarrow 0} F^{(\alpha, \theta)}=0 \tag{246}
\end{equation*}
$$

Hence, combining (245) and (246), we conclude that

$$
\begin{equation*}
F=\sup _{(\alpha, \theta) \in[0,1] \times[0, \infty)} F^{(\alpha, \theta)}(R)=0 \tag{247}
\end{equation*}
$$

## Acknowledgements

The authors thank the reviewers and the editor for their suggestions to enhance the quality of the paper.

## REFERENCES

[1] A. Wyner. The common information of two dependent random variables. IEEE Trans. on Inform. Theory, 21(2):163-179, 1975.
[2] A. El Gamal and Y.-H. Kim. Network Information Theory. Cambridge university press, 2011.
[3] G. R. Kumar, C. T. Li, and A. El Gamal. Exact common information. In IEEE International Symposium on Information Theory (ISIT), pages 161-165. IEEE, 2014.
[4] P. Cuff, H. Permuter, and T. Cover. Coordination capacity. IEEE Trans. on Inform. Theory, 56(9):4181-4206, 2010.
[5] P. Cuff. Distributed channel synthesis. IEEE Trans. on Inform. Theory, 59(11):7071-7096, 2013.
[6] T. Han and S. Verdú. Approximation theory of output statistics. IEEE Trans. on Inform. Theory, 39(3):752-772, 1993.
[7] M. Hayashi. General nonasymptotic and asymptotic formulas in channel resolvability and identification capacity and their application to the wiretap channel. IEEE Trans. on Inform. Theory, 52(4):1562-1575, 2006.
[8] M. Hayashi. Exponential decreasing rate of leaked information in universal random privacy amplification. IEEE Trans. on Inform. Theory, 57(6):3989-4001, 2011.
[9] J. Liu, P. Cuff, and S. Verdú. E ${ }_{\gamma}$-resolvability. IEEE Trans. on Inform. Theory, 63(5):2629-2658, 2017.
[10] L. Yu and V. Y. F. Tan. Rényi resolvability and its applications to the wiretap channel. arXiv preprint 1707.00810, 2017.
[11] M. R. Bloch and J. N. Laneman. Strong secrecy from channel resolvability. IEEE Trans. on Inform. Theory, 59(12):8077-8098, 2013.
[12] T. S. Han, H. Endo, and M. Sasaki. Reliability and secrecy functions of the wiretap channel under cost constraint. IEEE Trans. on Inform. Theory, 60(11):6819-6843, 2014.
[13] M. B. Parizi, E. Telatar, and N. Merhav. Exact random coding secrecy exponents for the wiretap channel. IEEE Trans. on Inform. Theory, 63(1):509-531, 2017.
[14] Y. Oohama. Exponent function for source coding with side information at the decoder at rates below the rate distortion function. arXiv preprint arXiv:1601.05650, 2016.
[15] I. Sason. On the Rényi divergence, joint range of relative entropies, and a channel coding theorem. IEEE Trans. on Inform. Theory, 62(1):23-34, 2016.
[16] A. R. Barron. Entropy and the central limit theorem. The Annals of Probability, pages 336-342, 1986.
[17] S. G. Bobkov, G. P. Chistyakov, and F. Götze. Rényi divergence and the central limit theorem. arXiv preprint arXiv:1608.01805, 2016.
[18] J. Hou and G. Kramer. Effective secrecy: Reliability, confusion and stealth. In IEEE International Symposium on Information Theory (ISIT), pages 601-605. IEEE, 2014.
[19] S. Beigi and A. Gohari. Quantum achievability proof via collision relative entropy. IEEE Trans. on Inform. Theory, 60(12):7980-7986, 2014.
[20] Y. Dodis and Y. Yu. Overcoming weak expectations. In Theory of Cryptography, pages 1-22. Springer, 2013.
[21] M. Hayashi and V. Y. F. Tan. Equivocations, exponents, and secondorder coding rates under various Rényi information measures. IEEE Trans. on Inform. Theory, 63(2):975-1005, 2017.
[22] V. Y. F. Tan and M. Hayashi. Analysis of remaining uncertainties and exponents under various conditional Rényi entropies. IEEE Trans. on Inform. Theory, to be published. DOI: 10.1109/TIT.2018.2792495., 2018.
[23] T. Van Erven and P. Harremoës. Rényi divergence and Kullback-Leibler divergence. IEEE Trans. on Inform. Theory, 60(7):3797-3820, 2014.
[24] V. Y. F. Tan, A. Anandkumar, L. Tong, and A. S. Willsky. A largedeviation analysis of the maximum-likelihood learning of Markov tree structures. IEEE Trans. on Inform. Theory, 57(3):1714-1735, 2011.
[25] W.-H. Gu and M. Effros. A strong converse for a collection of network source coding problems. In IEEE International Symposium on Information Theory (ISIT), pages 2316-2320. IEEE, 2009.
[26] S. Watanabe. Second-order region for Gray-Wyner network. IEEE Trans. on Inform. Theory, 63(2):1006-1018, 2017.
[27] L. Zhou, V. Y. F. Tan, and M. Motani. Discrete lossy Gray-Wyner revisited: Second-order asymptotics, large and moderate deviations. IEEE Trans. on Inform. Theory, 63(3):1766-1791, 2017.
[28] W. Rudin. Principles of Mathematical Analysis. McGraw-Hill, 1976.
[29] I. Csiszar and J. Körner. Information Theory: Coding Theorems for Discrete Memoryless Systems. Cambridge University Press, 2011.
[30] A. Dembo and O. Zeitouni. Large Deviations Techniques and Applications. Springer-Verlag, 2nd edition, 1998.

Lei Yu received the B.E. and Ph.D. degrees, both in electronic engineering, from University of Science and Technology of China (USTC) in 2010 and 2015, respectively. From 2015 to 2017, he was a postdoctoral researcher at the Department of Electronic Engineering and Information Science (EEIS), USTC. Currently, he is a research fellow at the Department of Electrical and Computer Engineering, National University of Singapore. His research interests include information theory, probability theory, and security.

Vincent Y. F. Tan (S'07-M'11-SM'15) was born in Singapore in 1981. He is currently an Associate Professor in the Department of Electrical and Computer Engineering and the Department of Mathematics at the National University of Singapore (NUS). He received the B.A. and M.Eng. degrees in Electrical and Information Sciences from Cambridge University in 2005 and the Ph.D. degree in Electrical Engineering and Computer Science (EECS) from the Massachusetts Institute of Technology in 2011. His research interests include information theory and machine learning.

Dr. Tan received the MIT EECS Jin-Au Kong outstanding doctoral thesis prize in 2011, the NUS Young Investigator Award in 2014, the NUS Engineering Young Researcher Award in 2018, and the Singapore National Research Foundation (NRF) Fellowship (Class of 2018). He has authored a research monograph on "Asymptotic Estimates in Information Theory with Non-Vanishing Error Probabilities" in the Foundations and Trends in Communications and Information Theory Series (NOW Publishers). He is currently an Editor of the IEEE Transactions on Communications and a Guest Editor for the IEEE Journal of Selected Topics in Signal Processing.


[^0]:    Manuscript received September 11, 2017; revised December 07, 2017, January 25, 2018; accepted January 25, 2018. This work was supported by the Singapore National Research Foundation (NRF) National Cybersecurity R\&D Grant under Grants R-263-000-C74-281 and NRF2015NCR-NCR003-006.
    L. Yu is with the Department of Electrical and Computer Engineering, National University of Singapore (NUS), Singapore 117583 (e-mail: leiyu@nus.edu.sg). V. Y. F. Tan is with the with the Department of Electrical and Computer Engineering and the Department of Mathematics, NUS, Singapore 119076 (e-mail: vtan@nus.edu.sg).

    Communicated by A. Khisti, Associate Editor for Shannon Theory.
    Copyright (c) 2018 IEEE. Personal use of this material is permitted. However, permission to use this material for any other purposes must be obtained from the IEEE by sending a request to pubs-permissions@ieee.org.

[^1]:    ${ }^{1}$ For simplicity, we assume that $\mathrm{e}^{n R}$ and similar expressions are integers.

[^2]:    ${ }^{2}$ Interestingly, a truncated code is not necessary for $s \in(-1,0]$ case, since an i.i.d. code (without truncation) is optimal as well for this case. This point can be seen from the work of Yu and Tan [10].
    ${ }^{3}$ More precisely, the proof of the converse part for $s \in(-1,0)$ case is not easy. The converse part for $s=0$ case was proven by Wyner [1], in which the continuity of the normalized entropy under the normalized KL divergence measure was used, i.e., when $\frac{1}{n} D\left(P_{X^{n} Y^{n}} \| \pi_{X Y}^{n}\right)$ is small then $\frac{1}{n} H_{P}\left(X^{n} Y^{n}\right)$ is arbitrarily close to $H_{\pi}(X Y)$. But it is not straightforward to apply Wyner's proof to the case $s \in(-1,0)$, since we do not know whether a strong enough continuity condition for the normalized entropy holds under the normalized Rényi divergence measure with order $\alpha=1+s \in(0,1)$. Even if a strong enough continuity condition holds, it is not straightforward to prove. Note that we were not able to directly utilize ideas in Wyner's converse proof to demonstrate this point. The definition of the "relative entropy typical set" $A\left(n, \epsilon_{1}\right):=\left\{\boldsymbol{u}: \frac{1}{n} \log \frac{p_{1}(\boldsymbol{u})}{p_{0}(\boldsymbol{u})} \leq \epsilon_{1}\right\}$ is crucial in Wyner's proof. If we adopt this set with this definition for the Rényi divergence setting (with Rényi parameter $<1$ ), it is not clear to us whether $\mathbb{P}_{1}\left(A^{c}\left(n, \epsilon_{1}\right)\right)$ vanishes (cf. Equation (A.7) in Wyner's paper).

[^3]:    ${ }^{4}$ For $s=0$ case, the binary Rényi divergence is known as the binary relative entropy, and it is usually denoted as $d(p \| q)$.

[^4]:    ${ }^{5}$ The pair $\left(X^{n}, Y^{n}\right)$ plays the role of $X$ in Lemma 3

[^5]:    ${ }^{6}$ This means that for any $\eta>0$, there exists an integer $N=N_{\eta}$ such that $\max _{w^{n} \in \mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)} 1-Q_{X \mid W}^{n}\left(\mathcal{T}_{\epsilon}^{n}\left(Q_{W X} \mid w^{n}\right) \mid w^{n}\right) \leq \eta$ for all $n>$ $N_{\eta}$. Here the notion of "uniform convergence" is a slightly different from the conventional one [28 Definition 7.7]. In the conventional definition, the domain of the functions are fixed but here, the domain $\mathcal{T}_{\epsilon^{\prime}}^{n}\left(Q_{W}\right)$ depends on $n$.

