# An improvement of the asymptotic Elias bound for non-binary codes.

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#### Abstract

For non-binary codes the Elias bound is a good upper bound for the asymptotic information rate at low relative minimum distance, where as the Plotkin bound is better at high relative minimum distance. In this work, we obtain a hybrid of these bounds which improves both. This in turn is based on the anticode bound which is a hybrid of the Hamming and Singleton bounds and improves both bounds. The question of convexity of the asymptotic rate function is an important open question. We conjecture a much weaker form of the convexity, and we show that our bounds follow immediately if we assume the conjecture.

#### **Index Terms**

information rate, size of a code, anticode.

#### I. INTRODUCTION

Let  $A_a(n, d; \mathcal{L})$  denote the maximum size of a code of length n, minimum distance at least d, and contained in a subset  $\mathcal{L} \subset \mathcal{F}^n$ , where  $\mathcal{F}$  is an alphabet of finite size q. A central problem in coding theory is to obtain good upper and lower bounds for  $A_q(n,d) = A_q(n,d;\mathcal{F}^n)$ . The asymptotic version of this quantity is the asymptotic information rate function:

$$\alpha(x) = \limsup_{n \to \infty} n^{-1} \log_q A_q(n, xn), \ x \in [0, 1].$$

$$\tag{1}$$

The quantities  $A_q(n, d; \mathcal{L})$  and  $A_q(n, d)$  are related by the inequality

$$A_q(n,d) \le q^n A_q(n,d;\mathcal{L})/|\mathcal{L}|,\tag{2}$$

known as the Bassalygo-Elias lemma. Taking  $\mathcal{L}$  to be a Hamming ball of diameter w, and choosing w optimally gives, at the asymptotic level the Hamming and the Elias upper bounds.

$$\alpha_H(x) = 1 - H_q(x/2), \ x \in [0, 1].$$
(3)

$$\alpha_E(x) = \alpha_H(2\theta(1 - \sqrt{1 - x/\theta})), \ x \in [0, \theta].$$
(4)

The bound  $\alpha_E$  is better than  $\alpha_H$  for all x. Here  $H_q(x)$  is the entropy function (9), and  $\theta := 1 - q^{-1}$ . An anticode of diameter w in  $\mathcal{F}^n$  is any subset of  $\mathcal{F}^n$  with Hamming diameter w. Let  $A_q^*(n, w)$  denote the maximum size of an anticode of diameter at most w in  $\mathcal{F}^n$ . In contrast to the situation with  $A_q(n,d)$ , the quantity  $A_a^*(n,d)$  was explicitly determined by Ahlswede and Khachatrian in [1]. From their result, it is easy to determine the asymptotic quantity  $\alpha^*(x) = \lim_{n \to \infty} n^{-1} \log_a A_a^*(n, xn)$ . We actually do not need the results of [1], however it is the main inspiration for this work. Taking  $\mathcal{L}$  to be an  $A_a^*(n, w)$ anticode in (2), and choosing w optimally, we get the following two bounds which improve  $\alpha_H$  and  $\alpha_E$ respectively.

**Theorem 1.** (hybrid Hamming-Singleton bound)

$$\alpha_{HS}(x) = \begin{cases} 1 - H_q(\frac{x}{2}) & \text{if } x \in [0, 2/q] \\ (1 - x)H_q(1) & \text{if } x \in [2/q, 1]. \end{cases}$$

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The bound  $\alpha_{HS}$  improves the Hamming and the Singleton bounds. It is  $\cup$ -convex and continuously differentiable.

**Theorem 2.** (hybrid Elias-Plotkin bound) Let q > 2.

$$\alpha_{EP}(x) = \begin{cases} 1 - H_q(\theta - \sqrt{\theta^2 - x\theta}) & \text{if } x \in [0, \frac{2q-3}{q(q-1)}] \\ (\theta - x)\frac{(q-1)H_q(1)}{q-2} & \text{if } x \in [\frac{2q-3}{q(q-1)}, \theta] \end{cases}$$

The bound  $\alpha_{EP}$  improves the Elias and Plotkin bounds. It is  $\cup$ -convex and continuously differentiable.

It is not known if the function  $\alpha(x)$  itself is  $\cup$ -convex, although it is tempting to believe that it is. We propose a weaker conjecture:

**Conjecture 1.** The function  $\frac{\alpha(x)}{\theta-x}$  is decreasing. In other words

$$\alpha(tx + (1-t)\theta) \le t\alpha(x) + (1-t)\alpha(\theta), \ t \in [0,1].$$

As evidence for this conjecture, we will show that theorems 1 and 2 follow very easily if we admit the truth of the conjecture.

The bound  $\alpha_{EP}$  in Theorem 2 is an elementary and explicit correction to the classical Elias bound. It does not however improve the upper-bounds obtained by the linear programming approach, like the second MRRW bound  $\alpha_{MRRW2}$  (due to Aaltonen [2]) or the further improvement of  $\alpha_{MRRW2}$  due to Ben-Haim and Litsyn [3, Theorem 7]. The reasons for this are as follows: For small  $\delta$  we have  $\alpha_{EP}(\delta) = \alpha_{E}(\delta) \geq \alpha_{MRRW2}(\delta)$ . For large  $\delta$ , the inequality  $\alpha_{EP}(\delta) > \alpha_{MRRW2}(\delta)$  follows from the fact that  $\alpha_{EP}(\delta)$  has a non-zero slope at  $\delta = 1 - 1/q$  where as the actual function  $\alpha(\delta)$  and the bound  $\alpha_{MRRW2}$  have zero slope at  $\delta = 1 - 1/q$ .

The paper is organized as follows. In section II, we collect some results on size of anticodes, which we use in section III to prove Theorems 1 and 2. We discuss Conjecture 1 in section IV.

#### II. SIZE OF ANTICODES

We recall that  $A_q^*(n,d)$  is the maximum size of an anticode of diameter at most d in  $\mathcal{F}^n$ . If we take  $\mathcal{L}$  to be an anticode of size  $A_q^*(n,d-1)$  then clearly  $A_q(n,d;\mathcal{L}) = 1$ . Using this in (2), we get a bound

$$A_q(n,d) \le q^n / A_q^*(n,d-1),$$
 (5)

known as Delsarte's code-anticode bound [4]. Taking d = xn where  $x \in [0, 1]$  we get

$$n^{-1}\log_q A_q(n,xn) \le 1 - n^{-1}\log_q A_q^*(n,xn-1).$$

Taking  $\limsup_{n\to\infty}$  we get:

$$\alpha(x) \le 1 - \alpha^*(x),\tag{6}$$

where

$$\alpha^*(x) = \liminf_{n \to \infty} n^{-1} \log_q A_q^*(n, xn).$$
(7)

This is the the asymptotic form of (5). We use the notation B(r;n) and  $V_q(n,r)$  to denote a Hamming ball of radius r in  $\mathcal{F}^n$  and its volume respectively. The ball B(t;n) where  $t = \lfloor (d-1)/2 \rfloor$  in  $\mathcal{F}^n$  is an anticode of diameter at most d-1. Let  $\mathcal{F}^n = \mathcal{F}^{d-1} \times \mathcal{F}^{n-d+1}$  and let  $v \in \mathcal{F}^{n-d+1}$  be a fixed word. Sets of the form  $\mathcal{F}^{d-1} \times \{v\}$  of size  $q^{d-1}$  are also anticodes of diameter d-1. It follows that:

$$\alpha^*(x) \ge \max\{H_q(x/2), x\}.$$
(8)

Here, we have used the well known formula:

$$\lim_{n \to \infty} n^{-1} \log_q V_q(n, tn) = H_q(t), \ t \in [0, \theta],$$

where,

$$H_q(x) = x \log_q(\frac{q-1}{x}) + (1-x) \log_q(\frac{1}{1-x}), \ x \in [0,1].$$
(9)

While the convexity of  $\alpha(x)$  is an open question, it is quite easy to see that:

#### **Lemma 1.** The function $\alpha^*(x)$ is $\cap$ -convex.

*Proof:* If  $S_1 \subset \mathcal{F}^{n_1}$  and  $S_2 \subset \mathcal{F}^{n_2}$  are anticodes of diameters  $d_1$  and  $d_2$  respectively, then  $S_1 \times S_2 \subset \mathcal{F}^{n_1} \times \mathcal{F}^{n_2}$  is an anticode of diameter  $d_1 + d_2$ . Taking  $S_i$  to be  $A_a^*(n_i, d_i)$  anticodes, we immediately get

$$A_q^*(n_1 + n_2, d_1 + d_2) \ge A_q^*(n_1, d_1)A_q^*(n_2, d_2).$$

Let  $n = n_1 + n_2$  go to infinity with  $n_1/n = t + o(1)$ ,  $d_1/n_1 = x + o(1)$  and  $d_2/n_2 = y + o(1)$ . Applying  $\liminf_{n \to \infty} n^{-1} \log_q$  to this inequality we get:

$$\alpha^*(tx + (1-t)y) \ge t\alpha^*(x) + (1-t)\alpha^*(y), \ t \in [0,1].$$

We note that with codes we have  $d(C_1 \times C_2) = \min\{d(C_1), d(C_2)\}$ , which is why the above proof method does not apply to the question of convexity of  $\alpha(x)$ . From (8) and Lemma 1 we get:

$$\alpha^*(tx + (1-t)y) \ge tH_q(x/2) + (1-t)y, \ t \in [0,1].$$

Let  $\delta = tx + (1 - t)y$ . We can rewrite this as

$$\alpha^*(\delta) \ge f(x, y),$$

where  $f: [0, \delta) \times (\delta, 1]$  is defined by

$$f(x,y) = \frac{y-\delta}{y-x}(H_q(x/2) - x) + \delta.$$
 (10)

We note that

$$\frac{(y-x)^2}{\delta-x}\frac{\partial f}{\partial y}(x,y) = H_q(x/2) - x,$$
$$\frac{x(y-x)^2}{y(y-\delta)}\frac{\partial f}{\partial x}(x,y) = H_q(\frac{x}{2}) - x + (1-\frac{x}{y})\log_q(1-\frac{x}{2}).$$

There is a unique positive number b > 0 satisfying  $H_q(b/2) = b$  (where the Hamming and Singleton bounds intersect). Therefore,  $H_q(x/2) - x$  has the same sign as b - x. Using this in (10), we see that  $f(x, y) \le \delta$  for  $x \ge b$ . Therefore, in order to maximize f(x, y) it suffices to consider x < b. We note that  $\frac{\partial f}{\partial y}(x, y)$  has the same sign as H(x/2) - x and hence that of b - x. Since x < b, we see that for fixed x < b, the function f(x, y) is maximized for y = 1. We are now reduced to maximizing

$$f(x,1) = 1 - (1-\delta)\frac{1-H_q(x/2)}{1-x}, \ x \in [0,\delta].$$

**Lemma 2.** Let  $g(x) = \frac{1-H_q(x/2)}{1-x}$  for  $x \in [0, 1]$ .

$$\operatorname{sign}(g'(x)) = \operatorname{sign}(x - \frac{2}{q}).$$

Proof: We calculate:

$$g'(x) = \frac{1}{2(1-x)^2} \log_q(\frac{q^2 x(2-x)}{4(q-1)}).$$

Therefore  $\operatorname{sign}(g'(x)) = \operatorname{sign}(\frac{q^2x(2-x)}{4(q-1)} - 1)$ . Next, we note that

$$\frac{q^2x(2-x)}{4(q-1)} - 1 = q\left(x - \frac{2}{q}\right)\frac{(q-2)+q(1-x)}{4(q-1)}$$

has the same sign as x - 2/q, as was to be shown. A stronger assertion is that g(x) is in fact  $\cup$ -convex: differentiating once more, we get:

$$\ln(q)(1-x)^3 g''(x) = \ln(\frac{q^2/4}{q-1}) + \left(\frac{1}{2x-x^2} - 1 - \ln(\frac{1}{2x-x^2})\right)$$

We note that  $q^2 \ge 4(q-1)$ , and hence the first term is non-negative. The remaining parenthetical term is non-negative using the inequality

$$t - 1 - \ln(t) \ge 0 \text{ for } t \ge 1,$$
 (11)

and the fact that  $t = 1/(2x - x^2) \ge 1$  for  $x \in (0, 1]$ .

It follows from Lemma 2 that

$$\operatorname{argmin}_{x \in [0,\delta]} \frac{1 - H_q(x/2)}{1 - x} = \min\{\delta, 2/q\}.$$
(12)

Therefore we obtain the bound:

**Theorem 3.**  $\alpha^*(x) \ge \beta(x)$  where

$$\beta(x) = \begin{cases} H_q(x/2) & \text{if } x \in [0, 2/q] \\ 1 - (1 - x)H_q(1) & \text{if } x \in [2/q, 1]. \end{cases}$$
(13)

Moreover,  $\beta(x)$  is continuously differentiable and  $\cap$ -convex.

We have used the relation

$$\frac{1 - H_q(1/q)}{1 - 2/q} = H_q(1) = \log_q(q - 1).$$
(14)

The function  $\beta(x)$  is continuously differentiable because the component for  $x \ge 2/q$  is just the tangent line at x = 2/q to the component for  $x \le 2/q$ , i.e. to  $H_q(x/2)$ . We note that  $\beta'(x)$  equals  $H'_q(x/2)/2$  for  $x \le 2/q$  and  $H'_q(1/q)/2$  for  $x \ge 2/q$ . Since  $\beta'(x)$  is non-increasing, it follows that  $\beta(x)$  is  $\cap$ -convex.

In the next lemma, we show that there is a sequence of anticodes  $S_n \subset \mathcal{F}^n$  of diameter at most  $\delta n$  such that  $\lim_{n\to\infty} n^{-1}\log_q |S_n|$  equals  $\beta(\delta)$ , i.e. the lower bound on  $\alpha^*(\delta)$  given in theorem 3.

**Lemma 3.** Consider the anticodes S(d, n) of diameter d in  $\mathcal{F}^n$  (taken from [1]) given by

$$S(d,n) = B(r_{d,n}; n - d + 2r_{d,n}) \times \mathcal{F}^{d-2r_{d,n}}, \text{ where}$$
$$r_{d,n} = \max\{0, \min\{\lceil \frac{d-1}{2}\rceil, \lceil \frac{n-d-q+1}{q-2}\rceil\}\}.$$

Then  $\lim_{n\to\infty} n^{-1}\log_q |S(\delta n, n)| = \beta(\delta).$ 

*Proof:* We note that

$$\rho = \lim_{n \to \infty} \frac{r_{\delta n, n}}{n} = \begin{cases} \frac{\delta}{2} & \text{if } \delta \in [0, 2/q] \\ \frac{1-\delta}{q-2} & \text{if } \delta \in [2/q, 1] \end{cases}$$

Also  $\lim_{n\to\infty} n^{-1}\log_q |S(\delta n, n)|$  equals

$$(1-\delta+2\rho)H_q(\frac{\rho}{1-\delta+2\rho})+(\delta-2\rho),$$

which simplifies to  $H_q(\delta/2)$  if  $\delta \leq 2/q$  and (on using (14)) to  $1 - (1 - \delta)H_q(1)$  if  $\delta \geq 2/q$ . This is the same as  $\beta(\delta)$ .

We now have all the results we need for proving theorems 1 and 2. However, we will state a remarkable theorem due to Ahlswede and Khachatrian [1], which we will not need. We also obtain an asymptotic version of their result and record it as a corollary, as it does not seem to have appeared in literature. In brief their theorem states that  $A_q^*(n, d)$  equals |S(d, n)|. Moreover any  $A_q^*(n, d)$  anticode is Hamming isometric to the anticode S(d, n) (with some exceptions). At the asymptotic level, the result is again remarkable: The lower bound  $\beta(\delta)$  for  $\alpha^*(\delta)$  given in theorem 3 is actually the exact value of  $\alpha^*(\delta)$ . Moreover  $\alpha^*(\delta)$  need not have been defined using  $\liminf_{n\to\infty} as \lim_{n\to\infty} n^{-1} \log_q A_q^*(n, \delta n)$  already exists.

**Theorem.** [1] Given  $q \ge 2$  and integers  $0 \le d \le n$ , let  $r_{d,n}$  and S(d,n) be as in Lemma 3. Then,

$$A_q^*(n,d) = |S(d,n)|.$$

Moreover, up to a Hamming isometry of  $\mathcal{F}^n$  an anticode S of size  $A_a^*(n,d)$  must be:

- S(d,n)
- or S(d,n) with  $r_{d,n}$  replaced with  $r_{d,n} 1$ . This case is possible only if (n d 1)/(q 2) is a positive integer not exceeding d/2.

#### **Corollary 1.**

$$\alpha^*(x) = \begin{cases} H_q(x/2) & \text{if } 0 \le x \le 2/q \\ 1 - (1 - x)H_q(1) & \text{if } 2/q \le x \le 1 \end{cases}$$

Proof: It follows from the theorem of Ahlswede and Khachatrian, together with Lemma 3 that

$$\lim_{n \to \infty} \frac{\log_q A_q^*(n, \delta n)}{n} = \lim_{n \to \infty} \frac{\log_q |S(\delta n, n)|}{n} = \beta_q(\delta).$$

Therefore

$$\alpha^*(\delta) = \liminf_{n \to \infty} \frac{\log_q A_q^*(n, \delta n)}{n} = \lim_{n \to \infty} \frac{\log_q A_q^*(n, \delta n)}{n} = \beta(\delta).$$

# III. PROOFS OF THEOREMS 1 AND 2

#### A. Proof of Theorem 1

If we use the bound  $\alpha^*(x) \ge \beta(x)$  of Theorem 3 in the inequality  $\alpha(x) \le 1 - \alpha^*(x)$  (see (6)), we obtain the bound

$$\alpha(x) \le 1 - \beta(x) =: \alpha_{HS}(x).$$

Since  $\beta(x)$  is  $\cap$ -convex and continuously differentiable (see Theorem 3), it follows that  $\alpha_{HS}(x)$  is  $\cup$ -convex and continuously-differentiable. To show that  $\alpha_{HS}(x) \leq \alpha_S(x) = 1 - x$ , we note that  $\alpha_S(x)$  being the secant line to  $\alpha_{HS}(x)$  between  $(0, \alpha_{HS}(0))$  and  $(1, \alpha_{HS}(1))$ , lies above the graph of  $\alpha_{HS}(x)$  as the latter is  $\cup$ -convex. To prove that  $\alpha_{HS}(x)$  improves  $\alpha_H(x)$  we note that  $\alpha_{HS}(x)$  coincides with  $\alpha_H(x)$  for  $x \leq 2/q$ , and for  $x \geq 2/q$ , Lemma 2 implies that  $\alpha_H(x) \geq (1 - x)H_q(1) = \alpha_{HS}(x)$ . This finishes the proof of Theorem 1.

It is worth noting that (12) implies the following formula for  $\alpha_{HS}(\delta)$ :

$$\alpha_{HS}(\delta) = \min_{x \in [0,\delta]} \frac{\alpha_H(x)(1-\delta)}{1-x}.$$
(15)

Since  $\frac{\theta-\delta}{\theta-x} \leq \frac{1-\delta}{1-x}$  for  $x \in [0, \delta]$  and  $\delta \leq \theta$ , we get

$$\alpha_{HS}(\delta) \ge \alpha_{HP}(\delta) := \min_{x \in [0,\delta]} \frac{\alpha_H(x)(\theta - \delta)}{\theta - x}.$$
(16)

It can be shown (see subsection III-C) that  $\alpha_{HP}(\delta)$  is an upper bound for  $\alpha(\delta)$  which improves both the Hamming and Plotkin bounds.

### B. Proof of Theorem 2

It will be convenient to identify the alphabet  $\mathcal{F}$  with the abelian group  $\mathbb{Z}/q\mathbb{Z}$ . Given  $0 \leq \delta \leq \omega$ , let  $w_n = \lfloor \omega n \rfloor$  and  $d_n = \lfloor \delta n \rfloor$ . We take  $\mathcal{L}_n \subset \mathcal{F}^n$  to be the anticode (from Lemma 3):

$$\mathcal{L}_n = B(r_n; n - w_n + 2r_n) \times \mathcal{F}^{w_n - 2r_n}, \text{ where}$$

$$r_n = \max\{0, \min\{\lceil \frac{w_n - 1}{2} \rceil, \lceil \frac{n - w_n - q + 1}{q - 2} \rceil\}\}.$$
(17)

We will take the balls B(r; m) to be centered at  $(0, \ldots, 0) \in \mathcal{F}^m$ . As in Lemma 3, we have

$$\rho := \lim_{n \to \infty} \frac{r_n}{n} = \begin{cases} \frac{\omega}{2} & \text{if } \omega \in [0, 2/q] \\ \frac{1-\omega}{q-2} & \text{if } \omega \in [2/q, 1]. \end{cases}$$
(18)

We also note that Lemma 3 gives

$$\lim_{n \to \infty} n^{-1} \log_q |\mathcal{L}_n| = \beta(\omega) = 1 - \alpha_{HS}(\omega).$$
(19)

Let  $A_q(n, d_n; \mathcal{L}_n)$  be the maximum possible size of a code contained in  $\mathcal{L}_n$  and having minimum distance at least  $d_n$ .

**Theorem 4.**  $\lim_{n\to\infty} n^{-1} \log_q A_q(n, d_n; \mathcal{L}_n) = 0$  if

$$\frac{\rho}{\theta(1-\omega+2\rho)} \le 1 - \sqrt{\frac{1-\delta/\theta}{1-\omega+2\rho}}.$$
(20)

*Proof:* Our proof is similar to the standard proof of the analogous result for the Elias bound (which corresponds to taking  $\rho = \omega/2$  instead of the prescription (18)). First let  $C \subset \mathcal{L}$  be a code of size  $M = A_q(n, d; \mathcal{L})$ , where  $\mathcal{L} \subset \mathcal{F}^n$  is the anticode  $\mathcal{L} = B(r; n - w + 2r) \times \mathcal{F}^{w-2r}$  for some  $r \leq w/2$ . Let

$$\gamma_1(\mathcal{C}) = (Mn)^{-1} \sum_{i=1}^n \sum_{a \in \mathcal{F}} m(i, a)^2,$$

where  $m(i, a) = \#\{c \in \mathcal{C} : c_i = a\}$ . We note that  $M = \sum_{a \in \mathcal{F}} m(i, a)$ , and that

$$M(M-1)d \le \sum_{c \in \mathcal{C}} \sum_{c' \in \mathcal{C}} d(c,c') = nM^2(1-\frac{\gamma_1}{M}).$$

We can rewrite this as:

$$M \le \frac{d/n}{\frac{\gamma_1}{M} - (1 - \frac{d}{n})}, \text{ provided } \frac{\gamma_1}{M} > 1 - \frac{d}{n}.$$
(21)

For  $n - w + 2r < i \le n$  we use Cauchy-Schwarz inequality to get  $\sum_{a \in \mathcal{F}} m(i, a)^2 \ge M^2/q$ . In particular

$$\frac{1}{M^2(w-2r)} \sum_{i=n-w+2r+1}^n \sum_{a \in \mathcal{F}} m(i,a)^2 \ge \frac{1}{q}.$$
(22)

Let  $\pi_1$  be the projection of  $\mathcal{F}^n = \mathcal{F}^{n-w+2r} \times \mathcal{F}^{w-2r}$  on to the factor  $\mathcal{F}^{n-w+2r}$ . We note that for  $c \in \mathcal{C}$ , we have wt $(\pi_1(c)) < r$  because  $\pi_1(\mathcal{C}) \subset B(r; n-w+2r)$ . Here wt(v) is the number of nonzero entries of v. Therefore

$$\sum_{i=1}^{-w+2r} \sum_{a \neq 0} m(i,a) \le Mr.$$

Since  $\sum_{i=1}^{n-w+2r} \sum_{a} m(i,a) = M(n-w+2r)$  we get:

$$S = \sum_{i=1}^{n-w+2r} m(i,0) \ge (n-w+r)M.$$

In particular

$$\frac{S}{M(n-w+2r)} - \frac{1}{q} \ge \frac{n-w+r}{n-w+2r} - \frac{1}{q} = \theta - \frac{r}{n-w+2r}.$$
(23)

We note that  $\sum_{a\neq 0} m(i,a) = M - m(i,0)$ . By Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n-w+2r} m(i,0)^2 \ge S^2/(n-w+2r), \ \, \text{and} \ \,$$

$$\sum_{a \neq 0} m(i,a)^2 \ge (M - m(i,0))^2 / (q-1)$$

Since  $\sum_{i=1}^{n-w+2r} \sum_{a \in \mathcal{F}} m(i,a)^2$  equals

$$\sum_{i=1}^{n-w+2r} \left( m(i,0)^2 + \sum_{a \neq 0} m(i,a)^2 \right),\,$$

we get:

$$\sum_{i=1}^{n-w+2r} \sum_{a \in \mathcal{F}} m(i,a)^2 \ge \sum_{i=1}^{n-w+2r} \left(\frac{qm(i,0)^2 + M^2 - 2Mm(i,0)}{q-1}\right)$$

This can be rewritten as:

$$\frac{1}{M^2(n-w+2r)} \sum_{i=1}^{n-w+2r} \sum_{a \in \mathcal{F}} m(i,a)^2 \ge \frac{1}{\theta} \left(\frac{S}{M(n-w+2r)} - \frac{1}{q}\right)^2 + \frac{1}{q}$$

Combining this with (22) we get:

$$\frac{\gamma_1}{M} \ge \frac{n-w+2r}{n\theta} (\frac{S}{M(n-w+2r)} - q^{-1})^2 + \frac{1}{q}.$$

Using (23) this can be written as:

$$\frac{\gamma_1}{M} - \frac{1}{q} \ge \frac{n - w + 2r}{n\theta} \left(\theta - \frac{r}{n - w + 2r}\right)^2.$$

Now let  $C_n \subset \mathcal{L}_n$  be a sequence of codes of size  $M_n = A_q(n, d_n; \mathcal{L}_n)$ . The preceding inequality gives:

$$\frac{\gamma_1(\mathcal{C}_n)}{M_n} - \frac{1}{q} \ge \frac{1 - \omega + 2\rho}{\theta} \left(\theta - \frac{\rho}{1 - \omega + 2\rho}\right)^2 + o(1).$$

Using this in (21), we get:

$$M_n \le \frac{\delta + o(1)}{\frac{1 - \omega + 2\rho}{\theta} \left(\theta - \frac{\rho}{1 - \omega + 2\rho}\right)^2 - \left(\theta - \delta\right) + o(1)};$$

provided the denominator is a positive number. Therefore,  $\lim_{n\to\infty} n^{-1}\log_q M_n = 0$  provided

$$\frac{1-\omega+2\rho}{\theta} \left(\theta - \frac{\rho}{1-\omega+2\rho}\right)^2 \ge \theta - \delta$$

This condition is the same as

$$\frac{\rho}{\theta(1-\omega+2\rho)} \le 1 - \sqrt{\frac{1-\delta/\theta}{1-\omega+2\rho}}$$

Since  $M_n = A_q(n, d_n; \mathcal{L}_n)$  this finishes the proof.

Using (2) we get:

$$\frac{\log_q A_q(n,d_n)}{n} \le 1 - \frac{\log_q |\mathcal{L}_n|}{n} + \frac{\log_q A_q(n,d_n;\mathcal{L}_n)}{n}.$$

Taking  $\limsup as n \to \infty$  and using the result of Theorem 4 and (19) we get:

$$\alpha(\delta) \le \alpha_{HS}(\omega_{\max}(\delta)),\tag{24}$$

where  $\omega_{\max}(\delta)$  is the largest value of  $\omega$  for which the inequality (20) holds. In order to determine  $\omega_{\max}(\delta)$ , we introduce functions  $f_1, f_2$  on  $[0, \theta]$  defined by:

$$f_1(\delta) = 2\theta (1 - \sqrt{1 - \delta/\theta}) \tag{25}$$

$$f_2(\delta) = 1 - (1 - \delta/\theta) \frac{(q-1)^2}{q(q-2)}$$
(26)

**Lemma 4.** *Let* q > 2.

1)  $f_1(\delta) \ge f_2(\delta)$  with equality only at  $\delta = \frac{2q-3}{q(q-1)}$ .

2)  $f_2(\delta)$  is the tangent line to  $f_1(\delta)$  at  $\delta = \frac{2q-3}{q(q-1)}$ . 3)  $sign(f_1(\delta) - 2/q) = sign(f_2(\delta) - 2/q) = sign(\delta - \frac{2q-3}{q(q-1)})$ . *Proof:* Let  $f_3(\delta) := 1 - \frac{q-1}{q-2}\sqrt{1 - \delta/\theta}$ . We observe that

$$\operatorname{sign}(f_3(\delta)) = \operatorname{sign}(\delta - \frac{2q-3}{q(q-1)})$$

The three assertions to be proved follow respectively from the following three relations:

$$f_1(\delta) - f_2(\delta) = (1 - 2/q) f_3(\delta)^2,$$
  

$$f_1'(\delta) - f_2'(\delta) = f_3(\delta)/\sqrt{1 - \delta/\theta},$$
  

$$\frac{f_1(\delta) - 2/q}{2(1 - 2/q)} = \frac{f_2(\delta) - 2/q}{(1 - 2/q) + \theta\sqrt{1 - \delta/\theta}} = f_3(\delta).$$

**Proposition 1.** 

$$\omega_{max}(\delta) = \begin{cases} 2\theta(1 - \sqrt{1 - \delta/\theta}) & \text{if } \delta \in [0, \frac{2q - 3}{q(q - 1)}] \\ 1 - (1 - \delta/\theta)\frac{(q - 1)^2}{q(q - 2)} & \text{if } \delta \in [\frac{2q - 3}{q(q - 1)}, 1]. \end{cases}$$

The function  $\omega_{max}(\delta)$  is increasing, continuously differentiable, and  $\cup$ -convex on  $[0, \theta]$ .

*Proof:* The inequality (20) reduces to

$$\omega \leq \begin{cases} f_1(\delta) & \text{if } \rho = \omega/2\\ f_2(\delta) & \text{if } \rho = (1-\omega)/(2-q), \end{cases}$$

where  $\rho$  is as given in (18). Therefore, for a given  $\delta \in [0, \theta]$ , the quantity  $\omega_{\max}(\delta)$  is the maximum element of the set

$$\{\omega: \delta \le \omega \le \min\{f_1(\delta), 2/q\}\} \cup \{\omega: \max\{\delta, 2/q\} \le \omega \le f_2(\delta)\}.$$

If  $\delta \geq \frac{2q-3}{q(q-1)}$ , then  $f_2(\delta) \geq 2/q$  (by Lemma 4) and hence, the maximum of this set is  $f_2(\delta)$ . If  $\delta \leq \frac{2q-3}{q(q-1)}$ , then  $f_2(\delta) \leq f_1(\delta) \leq 2/q$  (by Lemma 4) and hence, the maximum of this set is  $f_1(\delta)$ . This proves the asserted formula for  $\omega_{\max}(\delta)$ .

We note that the second component of  $\omega_{\max}(\delta)$  is the tangent line to the first component at  $x = \frac{2q-3}{q(q-1)}$ . Therefore  $\omega_{\max}(\delta)$  is continuously differentiable. The derivative of  $\omega_{\max}(x)$  is  $1/\sqrt{1-x/\theta}$  for  $x \leq \frac{2q-3}{q(q-1)}$ , and constant at  $\frac{q-1}{q-2}$  for  $x \geq \frac{2q-3}{q(q-1)}$ . Since the derivative is positive, the function is increasing. Since the derivative is non-decreasing, we see that the function is  $\cup$ -convex.

*Proof of*  $\alpha_{EP}$  *being an upper bound*: We note from lemma 4 that

$$\operatorname{sign}(\omega_{\max}(\delta) - 2/q) = \operatorname{sign}(\delta - \frac{2q-3}{q(q-1)}).$$

Therefore  $\alpha_{HS}(\omega_{\max}(\delta))$  is just the function  $\alpha_{EP}(\delta)$  defined in of theorem 2. The bound  $\alpha(x) \leq \alpha_{EP}(x)$  now follows from (24).

Proof of  $\alpha_{EP}$  being continuously differentiable: The function  $\alpha_{EP}(x) = \alpha_{HS}(\omega_{max}(x))$  being a composition of continuously differentiable functions, is itself continuously differentiable.

Proof of  $\alpha_{EP}$  being  $\cup$ -convex: Both the functions  $\alpha_{HS}$  and  $\omega_{\max}$  are  $\cup$ -convex, but  $\alpha_{HS}$  is decreasing and hence it is not obvious that  $\alpha_{EP}(x) = \alpha_{HS}(\omega_{\max}(x))$  is  $\cup$ -convex. We will show instead that the derivative  $\alpha'_{EP}$  is non-decreasing. Since  $\alpha'_{EP}$  is constant for  $x \ge \frac{2q-3}{q(q-1)}$ , it suffices to show that  $\alpha''_{E}(x) > 0$ for  $x \in (0, \frac{2q-3}{q(q-1)}]$ . This follows from the next lemma.

**Lemma 5.** The Elias bound  $\alpha_E(x)$  is  $\cup$ -convex on  $[0, \delta_E]$  and  $\cap$ -convex on  $[\delta_E, \theta]$  where  $\delta_E$  satisfies:

$$\frac{2q-3}{q(q-1)} < \delta_E < \frac{3}{4} \left( \frac{q-4/3}{q-1} \right)$$

*Proof:* Let  $Z(x) = \theta(1 - \sqrt{1 - x/\theta})$ . A calculation shows that

$$4\theta \ln(q)(1 - \frac{Z(x)}{\theta})^3 \alpha_E''(x) = \varphi(Z(x)), \text{ where}$$
$$\varphi(z) = \int_{\frac{1-\theta}{1-z}}^{\frac{\theta}{z}} (1 - 1/t) dt.$$

To see this, we note:  $\alpha_E(x) = H_q(Z(x))$  and hence

$$\ln(q)\alpha_E''(x) = \frac{Z'}{Z(1-Z)} + Z'' \ln(\frac{(q-1)(1-Z)}{Z}).$$

Since  $Z' = 1/(2(1 - Z/\theta))$  and  $Z'' = Z'/(2\theta(1 - Z/\theta)^2)$ , we get

$$4\theta \ln(q)\left(1 - \frac{Z(x)}{\theta}\right)^3 \alpha_E''(x) = \int_{\frac{1-\theta}{1-Z(x)}}^{\frac{1}{Z(x)}} (1 - 1/t)dt$$

as desired. It follows that  $sign(\alpha''_E(x)) = sign(\varphi(Z(x)))$ . Next we note that Z(x) is increasing on  $[0, \theta]$  and

$$Z(\frac{2q-3}{q(q-1)}) = 1/q, \quad Z(\frac{3}{4}(\frac{q-4/3}{q-1})) = 1/2.$$

It now suffices to show that

$$\operatorname{sign}(\varphi(z)) = \operatorname{sign}(z_E - z), \text{ for some } z_E \in (\frac{1}{q}, \frac{1}{2}).$$

We note that

$$\varphi'(z) = (z - \frac{1}{2}) \frac{2(\theta - z)}{z^2(1 - z)^2}.$$

Thus  $\varphi(z)$  is decreasing on [0, 1/2] and increasing on  $[1/2, \theta]$ . In order to show  $\operatorname{sign}(\varphi(z)) = \operatorname{sign}(z_E - z)$  for some  $z_E \in (1/q, 1/2)$ , it suffices to show that  $\varphi(1/q) > 0$  and  $\varphi(1/2) < 0$ . We calculate

$$\frac{1}{2}\varphi(1/q) = \left(\frac{q-1}{2} - 1 - \ln(\frac{q-1}{2})\right) + \left(\frac{2q-3}{2q-2} - \ln(2)\right).$$

Since  $q \ge 3$ , we have  $\frac{q-1}{2} \ge 1$ . The inequality (11) implies that the first parenthetical term above is non-negative. Again  $q \ge 3$  implies

$$\frac{2q-3}{2q-2} - \ln(2) \ge \frac{3}{4} - \ln(2) > 0$$

and hence the second parenthetical term is positive. Thus  $\varphi(1/q) > 0$ .

Next, we note that  $\varphi(1/2) = 2 - 4/q - \ln(q-1)$ . The function  $a(t) = 2 - 4/t - \ln(t-1)$  satisfies

$$a'(t) = -\frac{(t-2)^2}{t^2(t-1)}$$

and  $a(3) = 2/3 - \ln(2) < 0$ . Therefore a(t) < 0 for  $t \ge 3$ , and hence  $\varphi(1/2) < 0$  for all  $q \ge 3$ .

Proof that  $\alpha_{EP}$  improves the Plotkin bound: We have already shown that  $\alpha_{EP}(x)$  is  $\cup$ -convex, and hence  $\alpha_{EP}(x)$  lies below the secant line between x = 0 and  $x = \theta$ , which is the Plotkin bound.

Proof that  $\alpha_{EP}$  improves the Elias bound: This does not readily follow from our results thus far, and requires more work. The characterization of  $\alpha_{EP}(x)$  given in the next theorem clearly implies  $\alpha_{EP}(x) \leq \alpha_{E}(x)$ .

**Theorem 5.**  $\alpha_{EP}(\delta) = \min_{x \in [0,\delta]} \frac{\alpha_E(x)(\theta - \delta)}{\theta - x}.$ 

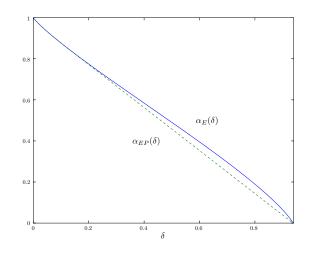


Fig. 1.  $\alpha_E(\delta)$  and  $\alpha_{EP}(\delta)$  for q = 16.

*Proof:* The theorem immediately follows if we show that  $\frac{\alpha_E(x)}{\theta-x}$  is decreasing on  $[0, \frac{2q-3}{q(q-1)}]$  and increasing on  $[\frac{2q-3}{q(q-1)}, \theta]$ . We will use the notation from the proof of Lemma 5. Since  $\alpha_E(x)$  is  $\cap$ -convex for  $x \ge \delta_E$ , it follows that the slope  $\alpha_E(x)/(\theta-x)$  of the secant between x and  $\theta$  is increasing. It remains to show that  $\frac{\alpha_E(x)}{\theta-x}$  is decreasing on  $[0, \frac{2q-3}{q(q-1)}]$  and increasing on  $[\frac{2q-3}{q(q-1)}, \delta_E]$ . Since Z(x) is an increasing function, with  $Z(0) = 0, Z(\frac{2q-3}{q(q-1)}) = 1/q$ , and  $(1 - x/\theta) = (1 - z/\theta)^2$ , it suffices to show that

$$h(z) = \frac{1 - H_q(z)}{(\theta - z)^2}$$

is decreasing on [0, 1/q] and increasing on  $[1/q, z_E]$  where  $z_E = Z(\delta_E)$ . A calculation shows that

$$\ln(q)(\theta - z)^3 h'(z) = \int_{1/q}^z \varphi(t) dt$$

To see this we note that either side of this equation evaluates to  $(\theta + z) \ln(\frac{z}{(1-z)(q-1)}) + 2 \ln(q(1-z))$ . Since  $\varphi(t) > 0$  for  $t \in (0, z_E)$ , we see

$$\operatorname{sign}(h'(z)) = \operatorname{sign}(z - 1/q), \ z \in [0, z_E].$$

Thus we have also shown that h(z) is decreasing for  $z \in [0, 1/q]$  and increasing on  $[1/q, z_E]$  as required.

The bounds  $\alpha_{HS}$ ,  $\alpha_{HP}$  and  $\alpha_{EP}$  are related as

$$\alpha_{EP}(\delta) \le \alpha_{HP}(\delta) \le \alpha_{HS}(\delta)$$

We have already shown  $\alpha_{HP}(\delta) \leq \alpha_{HS}(\delta)$  in (16). Since  $\alpha_E(x) \leq \alpha_H(x)$  for all x, we note that

$$\min_{x \in [0,\delta]} \frac{\alpha_E(x)(\theta - \delta)}{\theta - x} \le \min_{x \in [0,\delta]} \frac{\alpha_H(x)(\theta - \delta)}{\theta - x}.$$

Thus  $\alpha_{EP}(\delta) \leq \alpha_{HP}(\delta)$ . We end this section with a plot comparing  $\alpha_E(x)$  and  $\alpha_{EP}(x)$  for q = 16.

### C. Another proof of Theorem 1

Another proof of  $\alpha_{HS}(x)$  being an upper bound for  $\alpha(x)$  can be given using the following theorem of Laihonen and Litsyn:

**Theorem.** [5] Let  $\delta_1, \delta_2, \mu \in [0, 1]$ .  $\alpha((1 - \mu)\delta_1 + \mu\delta_2) \le (1 - \mu)\alpha_H(\delta_1) + \mu\alpha(\delta_2).$  (27) *Proof:* We give a quick proof. The result follows from the inequality:

$$A_q(n_1 + n_2, d_1 + d_2) \le \frac{q^{n_1}A_q(n_2, d_2)}{V_q(n_1, d_1/2)},$$

by taking  $n_1 + n_2 = n \rightarrow \infty$ , and  $n_1/n, n_2/n, d_1/n_1$  and  $d_2/n_2$  going to  $1 - \mu, \mu, \delta_1$  and  $\delta_2$  respectively. The above inequality in turn comes from the Bassalygo-Elias lemma (2)

$$A_q(n_1 + n_2, d_1 + d_2) \le \frac{q^{n_1 + n_2} A_q(n_1 + n_2, d_1 + d_2; \mathcal{L})}{|\mathcal{L}|},$$

by taking  $\mathcal{L} = B(d_1/2; n_1) \times \mathcal{F}^{n_2}$ , and observing that  $A_q(n_1 + n_2, d_1 + d_2; \mathcal{L}) \leq A_q(n_2, d_2)$ . (If  $\mathcal{C}$  is a  $A_q(n_1 + n_2, d_1 + d_2; \mathcal{L})$  code, and  $\pi_2 : \mathcal{F}^{n_1} \times \mathcal{F}^{n_2} \to \mathcal{F}^{n_2}$  is the projection on the second factor, then the restriction of  $\pi_2$  to  $\mathcal{C}$  is injective, and  $\pi_2(\mathcal{C})$  has minimum distance at least  $d_2$ .)

If we set  $\delta_2 = 1$ ,  $\delta_1 = x$  and  $\mu = (\delta - \delta_1)/(1 - \delta_1)$  in (27), we get:  $\frac{\alpha(y)}{1 - y} \le \frac{\alpha_H(x)}{1 - x} \text{ for } x \le y.$ 

would immediately yield Theorem 5. We believe that such an inequality

 $\frac{1}{1-y} \le \frac{1}{1-x}$  for  $x \le 1$ 

Thus,

$$\alpha(\delta) \le \min_{x \in [0,\delta]} \frac{\alpha_H(x)(1-\delta)}{1-x} = \alpha_{HS}(\delta),$$

where we have used (15).

We now prove that the function  $\alpha_{HP}(\delta)$  defined in (16) is an upper bound for  $\alpha(\delta)$ . Taking  $\delta_2 = \theta$ ,  $\delta_1 = x$ , and  $\mu = (y - \delta_1)/(\theta - \delta_1)$  in (27), we get:

$$\frac{\alpha(y)}{\theta - y} \le \frac{\alpha_H(x)}{\theta - x} \quad \text{for } x \le y$$
  
$$\alpha(\delta) \le \min_{x \in [0, \delta]} \frac{\alpha_H(x)(\theta - \delta)}{\theta - x} = \alpha_{HP}(\delta). \tag{28}$$

Thus

It is not known if the inequality (27) (the theorem of Laihonen-Litsyn) holds if we replace 
$$\alpha_H$$
 by  $\alpha_E$ .  
If such a result were true, then the derivation of the bound  $\alpha_{HP}(x)$  above with  $\alpha_H$  replaced with  $\alpha_E$ 

$$\alpha((1-\mu)\delta_1 + \mu\delta_2) \le (1-\mu)\alpha_E(\delta_1) + \mu\alpha(\delta_2),\tag{29}$$

must be true (it would surely be true if  $\alpha(x)$  is  $\cup$ -convex), but we believe it cannot be obtained just by a simple application of the Bassalygo-Elias lemma (2). If (29) holds, we can obtain an upper bound which improves the Laihonen-Litsyn bound [5]. We recall that the Laihonen-Litsyn bound, which we denote  $\alpha_{HMRRW}$  is a hybrid of the Hamming and MRRW bounds. It coincides with the Hamming bound for  $\delta \in [0, a]$  and with the MRRW bound for  $[b, \theta]$  where a < b are points such that the straight line joining  $(a, \alpha_H(a))$  and  $(b, \alpha_{MRRW}(b))$  is a common tangent to both  $\alpha_H$  at a and  $\alpha_{MRRW}$  at b. Since the Hamming bound is good for small  $\delta$  and the MRRW bound good for large  $\delta$ , the Laihonen-Litsyn bound combines the best features of both bounds in to a single bound. To obtain this bound, we note that (27) implies the inequality

$$\alpha((1-\mu)\delta_1+\mu\delta_2) \le (1-\mu)\alpha_H(\delta_1)+\mu\alpha_{MRRW}(\delta_2)$$

We fix  $\delta = (1 - \mu)\delta_1 + \mu\delta_2$  and choose  $\delta_1$  and  $\delta_2$  optimally in order to minimize the right hand side. This yields the  $\alpha_{HMRRW}$  bound. Since the second MRRW bound  $\alpha_{MRRW2}$  improves the first MRRW bound  $\alpha_{MRRW}$ , a better version  $\alpha_{HMRRW2}$  of the Laihonen-Litsyn bound (see [3, Theorem 2]) can be obtained by using  $\alpha_{MRRW2}$  in place of  $\alpha_{MRRW2}$ . Since the Elias bound  $\alpha_E(\delta)$  is better than the Hamming bound  $\alpha_H(\delta)$  for all  $\delta$ , in case (29) is true, repeating this procedure with  $\alpha_E$  replacing  $\alpha_H$ , would yield the hybrid Elias-MRRW bounds  $\alpha_{EMRRW}(\delta), \alpha_{EMRRW2}(\delta)$  which would improve the respective Laihonen-Litsyn bounds  $\alpha_{HMRRW}, \alpha_{HMRRW2}(\delta)$ . We leave the question of the truth of (29) open.

# IV. On the convexity of $\alpha(x)$

A fundamental open question about the function  $\alpha(x)$  is whether it is  $\cup$ -convex. In other words is it true that

$$\alpha((1-t)x + ty) \le (1-t)\alpha(x) + t\alpha(y), \ t \in [0,1].$$
(30)

It is worth noting that non-convex upper bounds like the Elias bound and the MRRW bound admit corrections to the non-convex part: the bound  $\alpha_{EP}(x)$  for the Elias bound and the Aaltonen straight-line bound (see the theorem below and the Appendix) for the MRRW bound. This may be viewed as some kind of evidence supporting the truth of (30). It is known that (30) holds for x = 0 (for example by taking  $\delta_1 = 0$  in (27)). Another way to state this is that

$$(1 - \alpha(x))/x$$
 is decreasing on [0, 1]

As a consequence, if  $\alpha_u(x)$  is any upper bound for  $\alpha(x)$  we obtain a better upper bound

$$\alpha(\delta) \le \tilde{\alpha}_u(\delta) = 1 - \max_{x \in [\delta, \theta]} \frac{(1 - \alpha_u(x))\delta}{x}$$
(31)

To see this we use:

$$\frac{1-\alpha(\delta)}{\delta} \ge \frac{1-\alpha(x)}{x} \ge \frac{1-\alpha_u(x)}{x}$$
, for  $x \in [\delta, \theta]$ .

Thus  $\frac{1-\alpha(\delta)}{\delta} \ge \max_{x \in [\delta,\theta]} \frac{(1-\alpha_u(x))}{x}$  as desired. If  $(1-\alpha_u(x))/x$  is a decreasing function then the improved bound  $\tilde{\alpha}_u(x)$  coincides with  $\alpha_u(x)$ , but otherwise  $\tilde{\alpha}_u(x)$  improves  $\alpha_u(x)$ . For example let  $\alpha_u(x)$  be the first MRRW bound  $\alpha_{MRRW}(x) =$ 

$$H_q((\sqrt{\theta(1-x)} - \sqrt{x(1-\theta)})^2), x \in [0,\theta].$$

It can be shown that that  $(1 - \alpha_{MRRW}(x))/x$  fails to be decreasing near x = 0, and similarly  $\alpha_{MRRW}(x)$  fails to be  $\cup$ -convex near x = 0. This is immediately rectified by passing to the improved bound  $\tilde{\alpha}_{MRRW}(x)$ , resulting in the following theorem of Aaltonen.

**Theorem.** (*Aaltonen bound*) [6] [7, p.53] Let q > 2.  $\alpha(x) \leq \tilde{\alpha}_{MRRW}(x)$  where

1

$$\tilde{\alpha}_{MRRW}(x) = \begin{cases} 1 - \frac{xH_q(1)}{1 - 2/q} & \text{if } x \in [0, (1 - \frac{2}{q})^2] \\ \alpha_{MRRW}(x) & \text{if } x \in [(1 - \frac{2}{q})^2, \theta] \end{cases}$$
(32)

This bound is  $\cup$ -convex, continuously differentiable, and improves the MRRW bound.

We note that for  $x \leq (1-2/q)^2$  the bound  $\tilde{\alpha}_{MRRW}(x)$  coincides with the tangent line to  $\alpha_{MRRW}(x)$  at  $(1-2/q)^2$ . In particular  $\tilde{\alpha}_{MRRW}(x)$  is continuously differentiable. The assertion that  $\tilde{\alpha}_{MRRW}(x)$  improves  $\alpha_{MRRW}(x)$  follows from the fact that  $\tilde{\alpha}_u(x) \leq \alpha_u(x)$  for any upper bound  $\alpha_u(x)$  for  $\alpha(x)$ . The other assertions are proved in the appendix.

On the other hand, it is not known if the convexity condition (30) holds for  $y = \theta$ , in other words if  $\alpha(x)/(\theta - x)$  is a decreasing function of x. We conjecture that this is true (see Conjecture 1). As evidence for this conjecture, we now show that the bounds  $\alpha_{EP}$ ,  $\alpha_{HP}$  and  $\alpha_{HS}$  can be obtained without doing any work, if we assume the truth of Conjecture 1: if  $\alpha_u(x)$  is any upper bound for  $\alpha(x)$  we obtain a better upper bound

$$\alpha(\delta) \le \alpha_u^{\dagger}(\delta) := \min_{x \in [0,\delta]} \frac{\alpha_u(x)(\theta - \delta)}{\theta - x}$$
(33)

To see this we use:

$$\frac{\alpha(\delta)}{\theta-\delta} \le \frac{\alpha(x)}{\theta-x} \le \frac{\alpha_u(x)}{\theta-x}$$
 for  $x \in [0, \delta]$ .

Thus  $\alpha(\delta) \leq \min_{x \in [0,\delta]} \frac{\alpha_u(x)(\theta-\delta)}{\theta-x}$  as desired. In case  $\frac{\alpha_u(x)}{\theta-x}$  is a decreasing function then the improved bound  $\alpha_u^{\dagger}(x)$  coincides with  $\alpha_u(x)$ , but otherwise  $\alpha_u^{\dagger}(x)$  improves  $\alpha_u(x)$ . Taking  $\alpha_u(x)$  to be the Elias

bound, we get  $\alpha_u^{\dagger}(x)$  to be the bound  $\alpha_{EP}$ . This is the content of Theorem 5. Taking  $\alpha_u(x)$  to be the Hamming bound, we get  $\alpha_u^{\dagger}(x)$  to be the bound  $\alpha_{HP}$ . This is the content of (28). Moreover, if  $\alpha(x)/(\theta-x)$  is decreasing then  $\alpha(x)/(1-x)$  being the product of the non-negative decreasing functions  $\alpha(x)/(\theta-x)$  and  $(\theta-x)/(1-x)$  is itself decreasing. Thus we obtain  $\alpha(\delta) \leq \min_{x \in [0,\delta]} \frac{\alpha_u(x)(1-\delta)}{1-x}$ . Taking  $\alpha_u(x)$  to be the Hamming bound, the bound  $\min_{x \in [0,\delta]} \frac{\alpha_u(x)(1-\delta)}{1-x}$  is  $\alpha_{HS}(\delta)$ . This is the content of (15).

#### APPENDIX A

#### AALTONEN'S STRAIGHT-LINE BOUND

The bound  $\tilde{\alpha}_{MRRW}$  presented above was obtained by Aaltonen in [6, p.156]. The bound follows from (31) and the following result

$$\operatorname{argmax}_{x \in [\delta, \theta]} \frac{1 - \alpha_{MRRW}(x)}{x} = \max\{\delta, (1 - 2/q)^2\}.$$
(34)

The argmax above is not straightforward to obtain, and to quote from [6], was found by a mere chance. The derivation is not presented in [6]. The purpose of this appendix is to i) record a proof of (34), and ii) to prove that  $\tilde{\alpha}_{MRW}(x)$  is  $\cup$ -convex. The author thanks Tero Laihonen for providing a copy of Aaltonen's work [6], which is not easily available.

Let  $\xi : [0,\theta] \to [0,\theta]$  be the function defined by  $\xi(x) = (\sqrt{\theta(1-x)} - \sqrt{x(1-\theta)})^2$ . We note that  $\alpha_{MRRW}(x) = H_q(\xi(x))$ , and that  $\xi(x)$  decreases from  $\theta$  to 0 as x runs from 0 to  $\theta$ . It is easy to check that  $\xi(\xi(x)) = x$  for  $x \in [0,\theta]$ . Therefore we can invert the relation  $y = \xi(x)$  as  $x = \xi(y)$ . We also note that  $\xi((1-2/q)^2) = 1/q$ . Therefore (34) is equivalent to the assertion:

$$\operatorname{argmax}_{y \in [0,t]} \frac{1 - H_q(y)}{\xi(y)} = \min\{t, 1/q\}.$$
(35)

In terms of  $h_A(y) := \frac{1-H_q(y)}{\xi(y)}$  we must show

$$\operatorname{sign}(h'_A(y)) = \operatorname{sign}(1/q - y), \quad y \in (0, \theta).$$

A calculation shows that:

$$h'_A(y)\xi(y)^{3/2}\sqrt{\frac{y(1-y)}{\theta(1-\theta)}}\ln(q) = \sqrt{\frac{y}{1-\theta}}\ln(\frac{y}{\theta}) + \sqrt{\frac{1-y}{\theta}}\ln(\frac{1-y}{1-\theta}) := G(y)$$

Clearly  $\operatorname{sign}(h'_A(y)) = \operatorname{sign}(G(y))$ . Therefore, we must show that  $\operatorname{sign}(G(y)) = \operatorname{sign}(1/q - y)$  for  $y \in (0, \theta)$ . Clearly  $G(1/q) = G(1 - \theta) = 0$ . First we will prove that G(y) > 0 on [0, 1/q). We calculate:

$$-\sqrt{\theta(1-\theta)}G'(y) = \int_{\sqrt{\frac{1-\theta}{1-y}}}^{\sqrt{\frac{\theta}{y}}} \ln(t)dt = \int_{\sqrt{\frac{1-\theta}{1-y}}}^{1} \ln(t)dt + \int_{1}^{\sqrt{\frac{\theta}{y}}} \ln(t)dt.$$

We make the substitution  $t = 1/\tau$  in the first integral to obtain:

$$-\sqrt{\theta(1-\theta)} G'(y) = \int_{1}^{\sqrt{\frac{1-y}{1-\theta}}} \ln(t)(1-\frac{1}{t^2})dt + \int_{\sqrt{\frac{1-y}{1-\theta}}}^{\sqrt{\frac{\theta}{y}}} \ln(t)dt.$$

We note that  $t \ge 1$  in both the integrals, and hence both the integrands are non-negative. Consequently, the first integral is positive, and the second integral is also positive when  $\sqrt{\theta/y} > \sqrt{(1-y)/(1-\theta)}$ . For  $y \in [0, \theta]$ , this inequality is equivalent to  $(\theta - y)(1 - \theta - y) > 0$  which in turn is equivalent to  $y < 1 - \theta$  i.e.  $y \in [0, 1/q)$ . Thus, for  $y \in (0, 1/q)$ , we have shown that G'(y) < 0. Since  $G(0) = \ln(q)\sqrt{q/(q-1)} > 0$  and G(1/q) = 0, the fact that G(y) is strictly decreasing on [0, 1/q] implies G(y) > 0 on [0, 1/q).

Next we prove G(y) < 0 on  $(1/q, \theta)$ . Differentiating the expression for G'(y) we get:

$$4\sqrt{\theta(1-\theta)} G''(y) = \ln(\frac{\theta}{y})\frac{\sqrt{\theta}}{y^{3/2}} + \ln(\frac{1-\theta}{1-y})\frac{\sqrt{1-\theta}}{(1-y)^{3/2}}.$$

Differentiating once more, we get:

$$4\sqrt{\theta(1-\theta)} G'''(y) = \frac{\sqrt{1-\theta}}{(1-y)^{5/2}} \left(1 + \frac{3}{2}\ln(\frac{1-\theta}{1-y})\right) - \frac{\sqrt{\theta}}{y^{5/2}} \left(1 + \frac{3}{2}\ln(\frac{\theta}{y})\right)$$

The second term  $-\frac{\sqrt{\theta}}{y^{5/2}}(1+\frac{3}{2}\ln(\frac{\theta}{y}))$  is negative on  $[1/q, \theta)$  because  $\theta/y > 1$  on this interval. The first term  $\frac{\sqrt{1-\theta}}{(1-y)^{5/2}}(1+\frac{3}{2}\ln(\frac{1-\theta}{1-y}))$  has the same sign as  $y - (1-q^{-1}e^{2/3})$  for  $y \in [1/q, \theta)$ . It follows that G'''(y) < 0 for  $y \in [1/q, 1-q^{-1}e^{2/3}]$ . (We note that the condition for  $1/q < 1-q^{-1}e^{2/3}$ , is  $q \ge 3$ , which is the case here).

For  $y \in (1 - q^{-1}e^{2/3}, \theta)$ , as above  $\frac{\sqrt{1-\theta}}{(1-y)^{5/2}}(1 + \frac{3}{2}\ln(\frac{1-\theta}{1-y}))$  is positive. It is also an increasing function of y, because  $(1-\theta)/(1-y)$  increases with y. For  $y \in [1-q^{-1}e^{2/3}, \theta]$ , we note that  $\theta/y$  decreases with y and  $\theta/y \ge 1$ . Therefore the term  $-\frac{\sqrt{\theta}}{y^{5/2}}(1 + \frac{3}{2}\ln(\frac{\theta}{y}))$  increases with y. Thus G'''(y) is an increasing function of y for  $y \in [1-q^{-1}e^{2/3}, \theta]$ . We note the boundary conditions on G'''(y): we have  $G'''(1/q) < 0 < G'''(\theta)$ . To see this we note that

$$-4(\theta(1-\theta))^3 G'''(1/q) = \frac{3}{2}\ln(q-1)(\theta^3 + (1-\theta)^3) + (\theta^3 - (1-\theta)^3) > 0$$

because q > 2 is equivalent to  $\theta > 1 - \theta$  as well as  $\ln(q - 1) > 0$ . Also

$$G'''(\theta) = \frac{2\theta - 1}{4(\theta(1 - \theta))^{2.5}} > 0.$$

Since  $G'''(1 - q^{-1}e^{2/3}) < 0 < G'''(\theta)$  and G'''(y) is increasing on  $[1 - q^{-1}e^{2/3}, \theta]$ , we conclude that there is a unique  $y_0$  in the interior of this interval such that G'''(y) has the same sign as  $y - y_0$  on this interval. Together with the fact that G'''(y) < 0 on  $[1/q, 1 - q^{-1}e^{2/3}]$ , we obtain:

$$\operatorname{sign}(G'''(y)) = \operatorname{sign}(y - y_0)$$
 on  $[1/q, \theta]$ .

This is illustrated in Figure 2, which shows the graphs of G(y) (dashed plot) and G'''(y) on  $[1/q, \theta]$  for q = 8. The point (y, G'''(y)) for  $y = 1 - q^{-1}e^{2/3}$  is marked. (In this plot, the values of G'''(y) are indicated on the right-vertical axis, and the values of G(y) are indicated on the left-vertical axis). Thus G''(y) is decreasing on  $[1/q, y_0]$  and increasing on  $[y_0, \theta]$ . Since  $G''(\theta) = 0$ , it follows that G''(y) < 0 on  $[y_0, \theta)$ . We note that

$$G''(1/q) = \frac{\ln(q-1)(2\theta-1)}{4(\theta(1-\theta))^2} > 0.$$

Thus  $G''(1/q) > 0 > G''(y_0)$  together with the fact that G''(y) is decreasing on  $[1/q, y_0]$  implies that there is a unique  $y_1$  in the interior of this interval such that G''(y) has the same sign as  $y_1 - y$  on this interval. We have already shown that G''(y) < 0 on  $[y_0, \theta]$ . Thus we conclude

$$\operatorname{sign}(G''(y)) = \operatorname{sign}(y_1 - y)$$
 on  $[1/q, \theta)$ .

This implies G'(y) is increasing on  $[1/q, y_1]$  and decreasing on  $[y_1, \theta]$ . Since  $G'(\theta) = 0$ , we conclude that G'(y) > 0 on  $[y_1, \theta)$ . We note that

$$G'(1/q) = \frac{-1}{\sqrt{\theta(1-\theta)}} \int_{1}^{\sqrt{q-1}} \ln(t)(1-\frac{1}{t^2})dt < 0.$$

Since G'(y) is increasing on  $[1/q, y_1]$  and  $G'(1/q) < 0 < G'(y_1)$ , we conclude that there is a unique  $y_2$  in the interior of the interval  $[1/q, y_1]$  such that G'(y) has the same sign as  $y - y_2$  on this interval. Also G'(y) > 0 on  $[y_1, \theta]$ . Thus we conclude:

$$\operatorname{sign}(G'(y)) = \operatorname{sign}(y - y_2)$$
 on  $[1/q, \theta)$ 

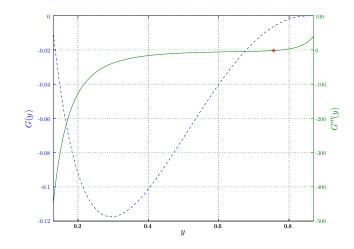


Fig. 2. Graphs of G(y) and G'''(y) on  $[1/q, \theta]$  for q = 8.

This implies that G(y) is decreasing on  $[1/q, y_2]$  and increasing on  $[y_2, \theta]$ . Since  $G(1/q) = G(\theta) = 0$ , we see that G(y) is negative on  $(1/q, y_2]$  as well as  $[y_2, \theta)$ . This finishes the proof of the assertion G(y) < 0 on  $(1/q, \theta)$ , and hence of (35).

Next, we prove the  $\cup$ -convexity of  $\tilde{\alpha}_{MRRW}(x)$ . We must show that the derivative  $\tilde{\alpha}'_{MRRW}(x)$  is nondecreasing. Since the derivative is constant on  $[0, (1 - 2/q)^2]$ , the problem reduces to showing that  $\alpha_{MRRW}(x)$  is  $\cup$ -convex for  $x \in [(1 - 2/q)^2, \theta]$ . This follows from the next lemma:

**Lemma 6.** The first MRRW bound  $\alpha_{MRRW}(\delta)$  is  $\cup$ -convex if q = 2. For q > 2, it is  $\cap$ -convex on  $[0, \delta_{MRRW}]$  and  $\cup$ -convex on  $[\delta_{MRRW}, \theta]$  where  $\delta_{MRRW}$  satisfies:

$$\frac{1}{2} - \frac{\sqrt{q-1}}{q} < \delta_{MRRW} < (1 - \frac{2}{q})^2.$$

*Proof:* Let  $y = \xi(x)$ . Let

$$\chi(y) = 1 - 2y + (2\theta - 1)\sqrt{\frac{y(1-y)}{\theta(1-\theta)}}$$

We calculate:

$$\frac{y'^2}{2y''y(1-y)} = \sqrt{\frac{x(1-x)}{\theta(1-\theta)}} = \chi(y)$$
(36)

Since  $\sqrt{x(1-x)/(\theta(1-\theta))}$  is non-negative, we also make the observation that that  $\chi(y) > 0$  for all  $y \in [0, \theta)$ . Since  $\alpha_{MRRW}(x) = H_q(y)$ , we get:

$$\ln(q)\alpha''_{MRRW}(x) = \frac{-y'^2}{y(1-y)} + y'' \ln \frac{(q-1)(1-y)}{y}$$

Since  $\xi(\xi(x)) = x$ , we get  $y' = \xi'(x) = 1/\xi'(y)$ . Using this we get:

$$\alpha_{MRRW}''(x)(\xi'(y))^2 y(1-y)\ln(q) = -1 + \frac{2y''y(1-y)}{y'^2} \ln \sqrt{\frac{(q-1)(1-y)}{y}}.$$

Using (36), we obtain:

$$\alpha''_{MRRW}(x)(\xi'(y))^2 y(1-y)\ln(q) = -1 + \frac{\ln\sqrt{\frac{(1-y)(q-1)}{y}}}{\chi(y)}$$

Let  $y \in (0, \theta)$ . We recall note  $\chi(y) > 0$  for  $y \in (0, \theta)$ . Thus for  $\alpha''_{MRRW}(x)$  has the same sign as

$$G_2(y) := \ln(\sqrt{\frac{(1-y)(q-1)}{y}}) - \chi(y).$$

We calculate:

$$G'_{2}(y)y(1-y) = \chi(y)(y-\frac{1}{2})$$

Therefore,  $sign(G'_2(y)) = sign(y - 1/2)$ . In other words  $G_2(y)$  is decreasing on [0, 1/2] and increasing on  $[1/2, \theta)$ . We note  $G_2(1/q) = \ln(q-1) - 2(1-\frac{2}{q})$ . The function

$$t \mapsto \ln(t-1) - 2(1-2/t),$$

evaluates to 0 at t = 2, and is an increasing function of t for  $t \ge 2$  (because its derivative  $(1-2/t)^2/(t-1)$  is positive). Thus  $G_2(1/q) > 0$  for q > 2 and  $G_2(1/q) = 0$  for q = 2. Since  $G_2(0) = +\infty$  and  $G_2(y)$  is decreasing on [0, 1/q], we conclude that  $G_2(y) > 0$  on [0, 1/q] if q > 2. If q = 2, then  $G_2(y) \ge 0$  on  $[0, 1/q] = [0, \theta]$ . In particular, for q = 2 the bound  $\alpha_{MRRW}(x)$  is  $\cup$ -convex on  $[0, \theta]$ .

For q > 2, we note that  $G_2(1/2) = \frac{1}{2}(\ln(q-1) - \frac{q-2}{\sqrt{q-1}})$ . The function  $b(t) = \frac{1}{2}(\ln(t-1) - \frac{t-2}{\sqrt{t-1}})$ satisfies  $b(3) = \frac{1}{2}(\ln(2) - \frac{2}{\sqrt{2}}) < 0$  and  $b'(t) = \frac{2\sqrt{t-1}-t}{4(t-1)^{3/2}} < 0$  for  $t \ge 3$ . Thus  $G_2(1/2) < 0$  for all q > 2. Since  $G_2(1/q) > 0$  and  $G_2(1/2) < 0$  and  $G_2(y)$  is decreasing on [1/q, 1/2], we conclude that there is a  $y_{MRRW} \in (1/q, 1/2)$  such that  $\operatorname{sign}(G_2(y)) = \operatorname{sign}(y_{MRRW} - y)$  for  $y \in [0, 1/2]$ . Also  $G_2(\theta) = 0, G_2(1/2) < 0$  and  $G_2(y)$  is increasing on  $[1/2, \theta]$ , which shows that  $G_2(y) < 0$  on  $[1/2, \theta)$ . Thus  $\operatorname{sign}(G_2(y)) = \operatorname{sign}(y_{MRRW} - y)$  for  $y \in (0, \theta)$ . Since  $\alpha''_{MRRW}(x)$  has the same sign as  $G_2(y)$  (where  $y = \xi(x)$ ), we finally obtain  $\operatorname{sign}(\alpha''_{MRRW}(x)) = \operatorname{sign}(x - \delta_{MRWW})$  for  $x \in (0, \theta)$ , where  $\delta_{MRWW} = \xi(y_{MRRW})$  satisfies  $\xi(1/2) < \delta_{MRWW} < \xi(1/q)$ , or in other words:  $\frac{1}{2} - \frac{\sqrt{q-1}}{q} < \delta_{MRRW} < (1 - \frac{2}{q})^2$ . This completes the proof of the lemma.

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