# On Independence And Capacity of Multidimensional Semiconstrained Systems 

Ohad Elishco, Student Member, IEEE, Tom Meyerovitch, Moshe Schwartz, Senior Member, IEEE


#### Abstract

We find a new formula for the limit of the capacity of certain sequences of multidimensional semiconstrained systems as the dimension tends to infinity. We do so by generalizing the notion of independence entropy, originally studied in the context of constrained systems, to the study of semiconstrained systems. Using the independence entropy, we obtain new lower bounds on the capacity of multidimensional semiconstrained systems in general, and $d$-dimensional axial-product systems in particular. In the case of the latter, we prove our bound is asymptotically tight, giving the exact limiting capacity in terms of the independence entropy. We show the new bound improves upon the best-known bound in a case study of $(0, k, p)$-RLL.


## Index Terms

Semiconstrained systems, capacity, independence entropy, bounds

## I. Introduction

ERROR-correcting codes and constrained codes may be considered as two extreme ways of coping with a noisy channel. The former are usually data independent, and assume errors are a statistical phenomenon, reducing data-transmission rate to protect against such errors. Constrained codes, however, assume certain patterns in the data stream are responsible for the occurrence of errors. Thus, constrained codes eliminate all undesirable patterns, at the cost of reduced data-transmission rate.

Recently in [9], [10], semiconstrained systems (SCSs) were suggested as a generalization to constrained systems (which we emphasize by calling fully constrained systems). In SCSs we do not eliminate the undesirable patterns entirely but rather we allow them to appear with a restriction on their frequency. To illustrate, consider a binary channel in which the appearance of $k$-consecutive 1's is forbidden. The set of allowed words is the well known inverted ( $0, k$ )-RLL. However, if $k$-consecutive 1 's are not forbidden entirely, but instead are allowed to appear in at most a fraction $p$ of places, then the set of allowed words forms a SCS called the $(0, k, p)$-RLL system. Informally, a SCS is defined by a set $\Gamma$ of probability measures over $k$-tuples. The allowed words in the SCS are those in which the empirical distribution of $k$-tuples belongs to $\Gamma$. This may be viewed as a generalization of fully constrained systems since taking $\Gamma$ to be a subset with a 0 -frequency restriction on some $k$-tuples yields a fully constrained system.

SCSs not only generalize fully constrained systems, but also subsume a range of other settings, which were mainly dealt with in an ad-hoc fashion. Among these we can find DC-free RLL coding [17], constant-weight ICI coding for flash memories [5], [6], [15], [27], coding to mitigate the appearance of ghost pulses in optical communication [30], [31], and the more general, channel with cost constraints [13], [16].

In the one-dimensional case, the capacity of a SCS is given by a relatively explicit expression as the solution to a certain optimization problem on a finite dimensional space, e.g., [22]. A probabilistic encoder for SCSs was constructed in [10], and constant-bit-rate to constant-bit-rate encoders are possible by approximating a SCS with a fully constrained system, as described in [9].

A natural extension, and the goal of this work, is to study multidimensional SCSs. This is an extremely challenging problem, considering the fact that even for fully constrained systems in complete generality it is provably impossible to find an exact solution. The capacity of multidimensional fully constrained systems is known exactly only in a handful of cases [1], [18], [20], [28]. In the absence of a general method for computing the capacity, various bounds and approximations were studied, e.g., [3], [11], [12], [14], [24], [25], [29], [32]-[34]. It should be emphasized that apart from its independent intellectual merit, studying multidimensional systems is of practical importance since most storage media are two- or three-dimensional, including magnetic recording devices such as hard drives, optical recording devices such as CDs and DVDs, and flash memories.

The approach we take in this work is bounding the capacity by studying the independence entropy of SCSs, thus extending the works [19], [23]. The independence entropy appeared in previous works on $d$-dimensional shifts of finite type. Although this notion was first defined in [19], the idea stemmed from tradeoff functions studied in [26]. It was defined in a combinatorial

[^0]fashion, where in this work we redefine it in a probabilistic fashion. We show that the two definitions are equal for the special case of fully constrained systems.

The motivation for the use of independence entropy is the fact that it is more easily computable, since we only need to consider independent probability measures which satisfy the constraints. We also focus on the class of $d$-dimensional axialproduct constraints, which form a significant proportion of multidimensional fully constrained systems studied thus far. For this class, our approach has an additional major advantage in that instead of calculating the independence entropy for a $d$ dimensional axial product SCS, we may calculate it directly from the one-dimensional system. This dimensionality reduction offers further simplification of the calculations.

There are new features and difficulties that come up when adapting the results from fully constrained systems. In an abstract sense, a very useful property of fully constrained systems is the following: If a measure $\mu$ is contained in some fully constrained system, and $\mu$ is a convex combination of measures, then each of them is contained in the same fully constrained system. This property does not hold for general semiconstrained systems. This is manifested for instance in the fact that any subword of an admissible word in a fully constrained system is also admissible, leading to sub-additivity of the sequence of the amount of admissible words. This, in turn, allows the use of Fekete's Lemma.

The main contributions of this paper are a formulation of the independence entropy for SCSs, and its study in relation to the capacity of SCSs. As a result, we obtain a new lower bound on the capacity of multidimensional SCSs, generalizing the results of [19], [23], and in an example test case, improving upon the best known bounds on the capacity of multidimensional $(0,1, p)$-RLL SCSs given in [10].

In this work we also establish an equality of the limiting capacity and independence entropy for the $d$-axial-product SCSs. As the independence entropy is a lower bound on the entropy of a given SCS in every dimension, the capacity approaches the independence entropy as the dimension grows.

This paper is organized as follows. In Section II we describe the notation and give the required definitions used throughout the paper. In Section III we define the independence entropy and provide results characterizing the independence entropy. In Section IV we show that the capacity is lower bounded by the independence entropy. In Section $V$ we show that the limiting capacity of the $d$-axial-product SCS is equal to the independence entropy. We conclude in Section VI by describing a short case study, and comparing it with previous results. The appendices provide proofs that the generalized notions we define in this paper indeed contain fully constrained systems as a special case, thus providing a generalization for them.

## II. Preliminaries

Let $\mathbb{N}$ denote the set of natural numbers. We use $\mathbf{e}_{i}$ to denote the unit vector of direction $i, \mathbf{0}$ to denote the all-zero vector, and 1 to the denote the all-one vector, where in all cases, the dimension of the vectors is implied by the context. For $n \in \mathbb{N}$ we define

$$
[n] \triangleq\{0,1, \ldots, n-1\}
$$

We shall often use $[n] \mathbf{e}_{i}$ to denote the set $\left\{0 \cdot \mathbf{e}_{i}, 1 \cdot \mathbf{e}_{i}, \ldots,(n-1) \cdot \mathbf{e}_{i}\right\}$. For $d, n \in \mathbb{N}$, denote by $F_{n}^{d}$ the $d$-dimensional cube of length $n$, i.e., the set $F_{n}^{d} \triangleq[n]^{d}$. Obviously $\left|F_{n}^{d}\right|=n^{d}$. Additionally, for $\left(n_{0}, \ldots, n_{d-1}\right) \in \mathbb{N}^{d}$ we conveniently denote

$$
\left[\left(n_{0}, \ldots, n_{d-1}\right)\right] \triangleq\left[n_{0}\right] \times\left[n_{1}\right] \times \cdots \times\left[n_{d-1}\right]
$$

Throughout the paper, $\Sigma$ will be used to denote a finite alphabet. A word (or block) $w$ of length $n$ is a sequence of $n$ letters from $\Sigma$, denoted $w=a_{0} a_{1} \ldots a_{n-1}$, with $a_{i} \in \Sigma$. We let $|w|$ denote the length of the word $w$. We can also consider infinite-sized words by mapping letters from $\Sigma$ to positions on the integer grid $\mathbb{Z}^{d}$. Such a word will be denoted by $x \in \Sigma^{\mathbb{Z}^{d}}$, and the letter in the $\mathbf{v} \in \mathbb{Z}^{d}$ position will be denoted by $x_{\mathbf{v}}$ (sometimes referred to as the restriction of $x$ to $\mathbf{v}$ ). More generally, given any subset of the integer grid, $S \subseteq \mathbb{Z}^{d}$, a word $x \in \Sigma^{S}$ is a mapping of letters from $\Sigma$ to positions indexed by elements of $S$.

We require a notation for sets of probability measures and their marginals. For a set $W$ we denote by $\mathcal{P}(W)$ the set of all probability measures over $W$.
Definition 1. Let $(X, \mathcal{B})$ be a measurable space. For every $\mu, v \in \mathcal{P}(X)$, the total variation distance is defined as

$$
\|\mu-v\|_{T V} \triangleq \sup _{W \in \mathcal{B}}|\mu(W)-v(W)|
$$

Given a compact topological space $X$, the space $\mathcal{P}(X)$ is itself a compact topological space with respect to the weak $*$ topology. In particular, when $X$ is a finite set with the discrete topology, the topology on $\mathcal{P}(X)$ is given by the total variation distance which also satisfies $\|\mu-v\|_{T V}=\frac{1}{2} \sum_{x \in X}|\mu(x)-v(x)|$.

Given a continuous map $f: X \rightarrow Y$ between topological spaces, and $\mu \in \mathcal{P}(X)$, let $f(\mu) \in \mathcal{P}(Y)$ be given by

$$
f(\mu)(W) \triangleq \mu\left(f^{-1}(W)\right), W \subseteq Y
$$

Definition 2. For $d \in \mathbb{N}, S \subseteq \tilde{S} \subseteq \mathbb{Z}^{d}$, and $x \in \Sigma^{\tilde{S}}$, let $x_{S}$ denote the restriction of $x$ to the coordinates in $S$. Let $\pi_{S}^{\tilde{S}}: \Sigma^{\tilde{S}} \rightarrow \Sigma^{S}$ denote the restriction map given by

$$
\pi_{S}^{\tilde{S}}(x) \triangleq x_{S}
$$

When $\tilde{S}$ is clear from the context, we shall write $\pi_{S}$ instead of $\pi_{S}^{\tilde{S}}$.
While having the notation $\pi_{S}(x)$ in addition to the equivalent notation $x_{S}$, seems superfluous, we shall require the former to simplify our presentation. As a consequence of the previous definition, for $\mu \in \mathcal{P}\left(\Sigma^{\tilde{S}}\right)$ and $S \subseteq \tilde{S}$, we note that $\pi_{S}(\mu) \in \mathcal{P}\left(\Sigma^{S}\right)$ is the $S$-marginal of $\mu$.
Definition 3. For $d \in \mathbb{N}, \mathbf{v} \in \mathbb{Z}^{d}$, let $\sigma_{\mathbf{v}}: \Sigma^{\mathbb{Z}^{d}} \rightarrow \Sigma^{\mathbb{Z}^{d}}$ be the shift by the vector $\mathbf{v}$, given by

$$
\left(\sigma_{\mathbf{v}}(x)\right)_{\mathbf{u}} \triangleq x_{\mathbf{u}+\mathbf{v}}, \mathbf{u} \in \mathbb{Z}^{d}, x \in \Sigma^{\mathbb{Z}^{d}}
$$

We denote by $\mathcal{P}_{\mathrm{si}}\left(\Sigma^{\mathbb{Z}^{d}}\right)$ the space of shift-invariant probability measures on $\Sigma^{\mathbb{Z}^{d}}$, namely,

$$
\mathcal{P}_{\mathrm{si}}\left(\Sigma^{\mathbb{Z}^{d}}\right) \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{\mathbb{Z}^{d}}\right): \sigma_{\mathbf{v}}(\mu)=\mu \text { for all } \mathbf{v} \in \mathbb{Z}^{d}\right\}
$$

For $k \in \mathbb{N}$ we say that $\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$ is shift invariant if it is the projection of some shift-invariant measure on $\Sigma^{\mathbb{Z}^{d}}$, i.e., if there exists $\tilde{\mu} \in \mathcal{P}_{\mathrm{si}}\left(\Sigma^{\mathbb{Z}^{d}}\right)$ such that $\mu=\pi_{F_{k}^{d}} \tilde{\mu}$. We denote by $\mathcal{P}_{\mathrm{si}}\left(\Sigma^{F_{k}^{d}}\right)$ the space of shift-invariant probability measures on $\Sigma^{F_{k}^{d}}$, namely,

$$
\mathcal{P}_{\mathrm{si}}\left(\Sigma^{F_{k}^{d}}\right) \triangleq \pi_{F_{k}^{d}}\left(\mathcal{P}_{\mathrm{si}}\left(\Sigma^{\mathbb{Z}^{d}}\right)\right) \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)
$$

In the one-dimensional case, $d=1$, it is rather easy to check whether a given probability measure $\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{1}}\right)$ is shift invariant. Indeed, $\mu \in \mathcal{P}_{\mathrm{si}}\left(\Sigma^{F_{k}^{1}}\right)$ if and only if it satisfies the following finite system of linear equations,

$$
\sum_{a \in \Sigma} \mu\left(a, a_{1}, \ldots, a_{k-1}\right)=\sum_{a \in \Sigma} \mu\left(a_{1}, \ldots, a_{k-1}, a\right)
$$

for all $a_{1}, \ldots, a_{k-1} \in \Sigma$.
When $d \geqslant 2$ the space of finite marginals of shift invariant measures becomes much more complicated. It is still not difficult to formulate an analogous system of linear equations that are satisfied for every $\mu \in \mathcal{P}_{\mathrm{si}}\left(\Sigma^{F_{k}^{d}}\right)$. However, these linear conditions are no longer sufficient conditions for shift invariance. In fact, the problem of checking whether a given $\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$ is shift invariant, is undecidable (assuming some computable representation of $\mu$ ). See for instance [4], and references within, for a related discussion.

We are interested in defining empirical distributions of words. To that end, we give some more general definitions that we then specialize to our specific needs. Given $x \in \Sigma^{\mathbb{Z}^{d}}$, the delta measure at $x$, denoted by $\delta_{x} \in \mathcal{P}\left(\Sigma^{\mathbb{Z}^{d}}\right)$, is defined by $\delta_{x}(\{x\})=1$. Additionally, given $n \in \mathbb{N}$, the empirical measure associated with $x$ and $n$, denoted $\operatorname{fr}_{x, n} \in \mathcal{P}\left(\Sigma^{\mathbb{Z}^{d}}\right)$, is given by

$$
\mathrm{fr}_{x, n} \triangleq \frac{1}{n^{d}} \sum_{\mathbf{v} \in F_{n}^{d}} \delta_{\sigma_{\mathbf{v}}(x)}
$$

For $S \subseteq \mathbb{Z}^{d}$ we can take the $S$-marginal, and define $\mathrm{fr}_{x, n}^{S} \in \mathcal{P}\left(\Sigma^{S}\right)$ by

$$
\mathrm{fr}_{x, n}^{S} \triangleq \pi_{S}\left(\mathrm{fr}_{x, n}\right)
$$

Any word $w \in \Sigma^{F_{n}^{d}}$ may be extended periodically to the entire integer grid $\hat{w} \in \Sigma^{\mathbb{Z}^{d}}$ by defining

$$
\hat{w}_{\mathbf{v}} \triangleq w_{\mathbf{v} \bmod n}
$$

for all $\mathbf{v} \in \mathbb{Z}^{d}$, and where the modulo is taken entry-wise. The empirical distribution we shall be requiring may now be defined.

Definition 4. Let $d, n \in \mathbb{N}, w \in \Sigma^{F_{n}^{d}}$, and $S \subseteq \mathbb{Z}^{d}$. The empirical distribution of $w$ with respect to $S$, denoted $\mathrm{fr}_{w}^{S}$, is defined by

$$
\mathrm{fr}_{w}^{S} \triangleq \mathrm{fr}_{\hat{w}, n}^{S}
$$

Combinatorially speaking, the empirical distribution $\mathrm{fr}_{w}^{S}$ is obtained by cyclically scanning $w$ with an $S$-shaped window and recording the frequency of the $S$-tuples in $w$. Thus, for instance, given a word $w=w_{0} \ldots w_{n-1} \in \Sigma^{n}, w_{i} \in \Sigma$, and $a \in \Sigma^{k}$ we have

$$
\operatorname{fr}_{w}^{[k]}(a)=\frac{1}{|w|} \sum_{i=0}^{|w|-1} \mathbb{1}_{a}\left(w_{i} \ldots w_{i+k-1}\right)
$$

where all coordinate indices are taken modulo $|w|$, and $\mathbb{1}_{a}: \Sigma^{k} \rightarrow\{0,1\}$ is the indicator function of the singleton $\{a\}$.
Example 5. Let $\Sigma=\{0,1\}$ and let $w=0010111001 \in \Sigma^{F_{10}^{1}}$. We have that $\left|F_{10}^{1}\right|=10$ and

$$
\begin{aligned}
\operatorname{fr}_{w}^{[3]}(110) & =\frac{1}{10} \sum_{i=0}^{9} \mathbb{1}_{110}\left(w_{i} w_{i+1} w_{i+2}\right)=\frac{1}{10} \\
\operatorname{fr}_{w}^{[2]}(10) & =\frac{1}{10} \sum_{i=0}^{9} \mathbb{1}_{10}\left(w_{i} w_{i+1}\right)=\frac{3}{10}
\end{aligned}
$$

Example 6. Let $\Sigma=\{0,1\}$ and consider

$$
w=\left[\begin{array}{cccc}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{array}\right] \in \Sigma^{F_{4}^{2},} \quad a=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \in \Sigma^{F_{2}^{2}}
$$

Then $\mathrm{fr}_{w}^{F_{2}^{2}}(a)=\frac{2}{16}$ since, of the sixteen $2 \times 2$ windows, exactly two contain $a$, shown in bold in the following:

$$
\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & \mathbf{0} & \mathbf{1} \\
1 & 0 & \mathbf{1} & \mathbf{0}
\end{array}\right], \quad\left[\begin{array}{llll}
\mathbf{0} & 1 & 1 & \mathbf{1} \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
\mathbf{1} & 0 & 1 & \mathbf{0}
\end{array}\right]
$$

Lemma 7. Suppose $d, n \in \mathbb{N}, w \in \Sigma^{F_{n}^{d}}$, and $S \subseteq \tilde{S} \subseteq \mathbb{Z}^{d}$. Then

$$
\pi_{S}^{\tilde{S}}\left(\mathrm{fr}_{w}^{\tilde{S}}\right)=\mathrm{fr}_{w}^{S}
$$

Proof: Let us denote $\mu \triangleq \operatorname{fr}_{\hat{w}, n} \in \mathcal{P}\left(\Sigma^{\mathbb{Z}^{d}}\right)$. By definition, for the right-hand side of the claim, for every $W \subseteq \Sigma^{S}$,

$$
\operatorname{fr}_{w}^{S}(W)=\pi_{S}^{\mathbb{Z}^{d}}(\mu)(W)=\mu\left(\left(\pi_{S}^{\mathbb{Z}^{d}}\right)^{-1}(W)\right)
$$

Similarly, for the left-hand side,

$$
\pi_{S}^{\tilde{S}}\left(\operatorname{fr}_{w}^{\tilde{S}}\right)(W)=\pi_{S}^{\tilde{S}}\left(\pi_{\tilde{S}}^{Z^{d}}(\mu)\right)(W)=\pi_{\tilde{S}}^{Z^{d}}(\mu)\left(\left(\pi_{S}^{\tilde{S}}\right)^{-1}(W)\right)=\mu\left(\left(\pi_{\tilde{S}}^{Z^{d}}\right)^{-1}\left(\left(\pi_{S}^{\tilde{S}}\right)^{-1}(W)\right)\right)
$$

But clearly for all $A \subseteq \Sigma^{S}$,

$$
\left(\pi_{S}^{Z^{d}}\right)^{-1}(W)=\left(\pi_{\tilde{S}}^{Z^{d}}\right)^{-1}\left(\left(\pi_{S}^{\tilde{S}}\right)^{-1}(W)\right)
$$

Lemma 7 implies that the empirical frequency of $S$-tuples in $w$ can be calculated by first calculating the empirical frequency of $\tilde{S}$-tuples, and then taking the $S$-marginal.

Example 8. Let $\Sigma=\{0,1\}$ and consider

$$
w=\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{array}\right] \in \Sigma^{F_{4}^{2}}
$$

Take $S=[1]^{2}=\{(0,0)\}$ and $\tilde{S}=[(2,1)]=\{(0,0),(1,0)\}$. Then

$$
\begin{array}{ll}
\operatorname{fr}_{w}^{\tilde{S}}(00)=\frac{2}{16}, & \operatorname{fr}_{w}^{\tilde{S}}(01)=\frac{5}{16} \\
\operatorname{fr}_{w}^{\tilde{S}}(10)=\frac{5}{16}, & \operatorname{fr}_{w}^{\tilde{S}}(11)=\frac{4}{16}
\end{array}
$$

Moreover, we have that

$$
\operatorname{fr}_{w}^{S}(0)=\frac{7}{16}, \quad \operatorname{fr}_{w}^{S}(1)=\frac{9}{16}
$$

We can verify now that

$$
\pi_{S}^{\tilde{S}}\left(\operatorname{fr}_{w}^{\tilde{S}}\right)(0)=\operatorname{fr}_{w}^{\tilde{S}}\left(\left(\pi_{S}^{\tilde{S}}\right)^{-1}(0)\right)=\operatorname{fr}_{w}^{\tilde{S}}(\{00,01\})=\frac{7}{16}=\operatorname{fr}_{w}^{S}(0)
$$

We are now ready to define multidimensional semiconstrained systems.
Definition 9. For $d \in \mathbb{N}$, a $\mathbb{Z}^{d}$-semiconstrained system (SCS) is a set $\Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$ for some finite set $S \subseteq \mathbb{Z}^{d}$. For $n \in \mathbb{N}$, the admissible $n$-blocks of $\Gamma$ are

$$
\mathcal{B}_{n}(\Gamma) \triangleq\left\{w \in \Sigma^{F_{n}^{d}}: \operatorname{fr}_{w}^{S} \in \Gamma\right\}
$$

Since all SCSs we study in this paper are $\mathbb{Z}^{d}$-SCSs, we shall abbreviate and call them just SCSs, where the dimension, $d$, will be clear from the context.

Note that SCSs generalize $d$-dimensional fully constrained systems. Recall that fully constrained systems are defined by a set of "forbidden patterns", $A \subseteq \Sigma^{F_{k}^{d}}$, such that a word $w \in \Sigma^{\mathbb{Z}^{d}}$ is admissible if and only if none of the elements of $A$ appear as an $F_{k}^{d}$-tuple of $w$. Thus, fully constrained systems correspond to subshifts of finite type in symbolic dynamics. In our notation, we therefore have the following.
Definition 10. For $d, k \in \mathbb{N}$, we say that $\Gamma \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$ is fully constrained if there exists some $L \subseteq \Sigma^{F_{k}^{d}}$ such that

$$
\Gamma=\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right): \mu(L)=1\right\}
$$

Example 11. Let $\Sigma=\{0,1\}$, take

$$
L=\Sigma^{F_{2}^{2}} \backslash\left\{\left[\begin{array}{ll}
0 & 0 \\
1 & 1
\end{array}\right],\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right],\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right],\left[\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right],\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right]\right\}
$$

and consider the fully constrained system, $\Gamma$, defined by

$$
\Gamma=\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{2}^{2}}\right): \mu(L)=1\right\}
$$

Note that $\mathcal{B}_{n}(\Gamma)$ is the set of all $n \times n$ two-dimensional binary arrays such that none of the six patterns above appears within a $2 \times 2$ window in them. It is simple to verify that in fact, no two horizontally adjacent 1 's may appear, and no two vertically adjacent 1 's may appear, in any admissible word. Thus, the $n \times n$ arrays in $\mathcal{B}_{n}(\Gamma)$ are the admissible words of the (cyclical) $(1, \infty)$-RLL fully constrained system.

An important figure of merit we associate with any set of words, and in particular, with SCSs, is the capacity, which we now define.
Definition 12. Let $d \in \mathbb{N}$, and let $S \subseteq \mathbb{Z}^{d}$ be a finite subset. For any $S C S, \Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$, and for $\epsilon>0$, let

$$
\mathbb{B}_{\epsilon}(\Gamma) \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{S}\right): \inf _{v \in \Gamma}\|\mu-v\|_{T V} \leqslant \epsilon\right\}
$$

The capacity of $\Gamma$ is defined as,

$$
\operatorname{cap}(\Gamma) \triangleq \lim _{\epsilon \rightarrow 0^{+}} \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \log _{2}\left(\left|\mathcal{B}_{n}(\Gamma)\right|\right)
$$

First, we mention that $\lim _{\epsilon \rightarrow 0^{+}}$in the definition of the capacity exists due to monotonicity, since $\left|\mathcal{B}_{n}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right|$ is nonincreasing in $\epsilon$.

To avoid certain pathological scenarios, [9], [10] defined sets of weakly-admissible words and their capacity. We contend that the capacity definition provided here is the proper multidimensional generalization of these definitions. Intuitively, the capacity measures the exponential growth rate of the number of words that "almost" satisfy the semiconstraints given by $\Gamma$. Additionally, it has the nice property that the capacity of a set $\Gamma$ is equal to the capacity of the closure of $\Gamma$.

At first glance this definition of capacity may seem odd. A naive definition, which we call the internal capacity, might be as follows.

Definition 13. Let $d \in \mathbb{N}, S \subseteq \mathbb{Z}^{d}$ finite, and $\Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$ be a $S C S$. The internal capacity of $\Gamma$ is defined as

$$
\widehat{\operatorname{cap}}(\Gamma) \triangleq \underset{n \rightarrow \infty}{\limsup } \frac{1}{n^{d}} \log _{2}\left(\left|\mathcal{B}_{n}(\Gamma)\right|\right)
$$

By definition we have

$$
\operatorname{cap}(\Gamma)=\lim _{\epsilon \rightarrow 0^{+}} \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)
$$

which means that

$$
\begin{equation*}
\widehat{\operatorname{cap}}(\Gamma) \leqslant \operatorname{cap}(\Gamma) . \tag{1}
\end{equation*}
$$

We also observe that for some "nice" SCSs $\Gamma, \widehat{\operatorname{cap}}(\Gamma)=\operatorname{cap}(\Gamma)$. For instance, we have the following result for one-dimensional SCSs.

Theorem 14. 9. Section 2] Let $k \in \mathbb{N}$, and $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be convex and equal to the closure of its relative interior in $\mathcal{P}_{\text {si }}\left(\Sigma^{k}\right)$. Then

$$
\operatorname{cap}(\Gamma)=\widehat{\operatorname{cap}}(\Gamma)=\log _{2}|\Sigma|-\inf _{\eta \in \Gamma \cap \mathcal{P}_{\mathrm{si}}\left(\Sigma^{k}\right)} H(\eta \mid \mu)
$$

where $H(\cdot \mid \cdot)$ is the relative entropy function, and $\mu$ is defined by $\mu(\phi a) \triangleq \frac{1}{|\Sigma|} \sum_{a^{\prime} \in \Sigma} \eta\left(\phi a^{\prime}\right)$ for all $\phi \in \Sigma^{k-1}$ and $a \in \Sigma$.
Remark 15. Consider the (compact) space $M=\mathcal{P}\left(\Sigma^{S}\right)$ and let $C(M)$ be the set of all closed (hence, compact) subsets of $M$. Thus, $C(M)$ is a compact topological space (under the Hausdorff metric). Since $\widehat{\operatorname{cap}}(\Gamma)$ is monotone, the set of $\Gamma$ s for which $\widehat{\operatorname{cap}}(\Gamma) \neq \operatorname{cap}(\Gamma)$ is meager. In other words, if we consider $\widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)$ as a function of $\epsilon, f(\epsilon) \triangleq \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)$, then $\operatorname{cap}\left(\mathbb{B}_{\delta}(\Gamma)\right)=\widehat{\operatorname{cap}}\left(\mathbb{B}_{\delta}(\Gamma)\right)$ whenever $f$ is continuous in $\delta$. Since $f$ is a monotone function, it is discontinuous on a countable number of places. In practice, it means that if for a specific $\Gamma, \operatorname{cap}(\Gamma) \neq \widehat{\operatorname{cap}}(\Gamma)$ an arbitrary small change in $\Gamma$ will give an equality.
Remark 16. For a fully constrained system, $\Gamma \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$, non-emptyness of $\mathcal{B}_{n}(\Gamma)$ for all $n>0$ is equivalent to the fact that the subshift of finite type

$$
\left\{w \in \Sigma^{\mathbb{Z}^{d}}: \forall \mathbf{v} \in \mathbb{Z}^{d},\left(\sigma_{\mathbf{v}}(w)\right)_{F_{k}^{d}} \in L\right\}
$$

is not empty. Berger's Theorem [2] implies that it is undecidable whether a subshift of finite type is empty given L. Because (under reasonable assumptions on the representation) it is undecidable if a given multidimensional SCS is non-empty, it is difficult to understand what a SCS really looks like.

At this point we pause to ponder the following: Note that the definition of empirical frequency is cyclic (in the sense that coordinates are taken modulo $n$ ) while in traditional fully constrained systems it is not. This seems at odds with our claim of SCSs generalizing fully constrained systems. The necessity of the modulo in the definition of SCSs stems from working with the space of shift-invariant measures and their associated admissible words. Shift-invariant measures are defined over $\mathbb{Z}^{d}$, hence, it is necessary to complete a word $w \in \Sigma^{F_{n}^{d}}$ to a word from $\Sigma^{\mathbb{Z}^{d}}$. We choose to do this completion periodically using the modulo notion, extending $w$ to $\hat{w}$. This choice simplifies the analysis which follows. We contend that with respect to this issue, the capacity is more natural than the internal capacity, since it is equal to the non-cyclic capacity of fully constrained systems. To avoid a lengthy detour, the full details are provided in Appendix A

Finally, we raise the question: what multidimensional SCSs are of interest? If we examine the extensive literature for fully constrained systems, a significant proportion of multidimensional fully constrained systems are defined as an axial product of one-dimensional fully constrained systems. Intuitively speaking, if we have a set of "forbidden patterns" defining a onedimensional fully constrained system, we can define its $d$-dimensional axial product by forbidding these patterns along each dimension. We now formally define this for the case of $d$-dimensional SCSs with slightly more generality. This definition generalizes the $d$-dimensional axial product defined in [19].

Definition 17. Consider $S_{0}, \ldots, S_{d-1} \subseteq \mathbb{N}$, with $0 \in S_{i}$ for all $i \in[d]$, and $S C S s \Gamma_{i} \subseteq \mathcal{P}\left(\Sigma^{S_{i}}\right)$. Denote $S \triangleq \bigcup_{i \in[d]} S_{i} \mathbf{e}_{i} \subseteq \mathbb{Z}^{d}$. The $d$-axial-product SCS, denoted $\otimes_{i \in[d]} \Gamma_{i}$, is defined by

$$
\otimes_{i \in[d]} \Gamma_{i} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{S}\right): \forall i \in[d], \pi_{S_{i} \mathbf{e}_{i}}(\mu) \in \Gamma_{i}\right\}
$$

It follows from the above definition, that for every $n \in \mathbb{N}$ we have

$$
\mathcal{B}_{n}\left(\otimes_{i \in[d]} \Gamma_{i}\right)=\left\{w \in F_{n}^{d}: \forall i \in[d], \operatorname{fr}_{w}^{S_{i} \mathbf{e}_{i}} \in \Gamma_{i}\right\}
$$

with coordinates taken modulo $n$. Intuitively, the arrays of a $d$-axial-product SCS satisfy that along the $i$ th direction, the empirical distribution of $S_{i}$-tuples is in $\Gamma_{i}$. Note that $\otimes_{i \in[d]} \Gamma_{i}$ induces a set of measures over $\Sigma^{F_{k}^{d}}$ where $k=\max _{i}\left\{k_{i}: k_{i} \in S_{i}\right\}$. Hence, we sometimes consider a $d$-axial-product $\operatorname{SCS} \otimes_{i \in[d]} \Gamma_{i}$ as a subset of $\mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$.
Example 18. Let $\Sigma=\{0,1\}$. Consider two real constants $0 \leqslant p_{0}, p_{1} \leqslant 1$, and the one-dimensional SCSs, $\Gamma_{0}$ and $\Gamma_{1}$, given by

$$
\begin{aligned}
& \Gamma_{0}=\left\{\mu \in \mathcal{P}\left(\Sigma^{2}\right): \mu(11) \leqslant p_{0}\right\}, \\
& \Gamma_{1}=\left\{\mu \in \mathcal{P}\left(\Sigma^{2}\right): \mu(11) \leqslant p_{1}\right\} .
\end{aligned}
$$

Here we are taking $S_{0}=S_{1}=\{0,1\}$. The admissible words in the 2-axial-product $S C S, \Gamma_{0} \otimes \Gamma_{1}$, are all two-dimensional words in which the empirical frequency of two horizontally adjacent 1 s is at most $p_{0}$, and the empirical frequency of two vertically adjacent $1 s$ is at most $p_{1}$, i.e., all the words $w \in \Sigma^{F_{n}^{2}}$ such that

$$
\begin{aligned}
& \operatorname{fr}_{w}^{\{(0,0),(1,0)\}}(11) \leqslant p_{0} \\
& \operatorname{fr}_{w}^{\{(0,0),(0,1)\}}(11) \leqslant p_{1} .
\end{aligned}
$$

We may also consider $\Gamma_{0} \otimes \Gamma_{1}$ as a subset of $\mathcal{P}\left(\Sigma^{F_{2}^{2}}\right)$

$$
\Gamma_{0} \otimes \Gamma_{1}=\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{2}^{2}}\right): \pi_{\{(0,0),(0,1)\}}(\mu)(11) \leqslant p_{0}, \pi_{\{(0,0),(1,0)\}}(\mu)(11) \leqslant p_{1}\right\} .
$$

Note that

$$
\begin{aligned}
& \pi_{\{(0,0),(1,0)\}}(\mu)(11)=\mu\left(\left[\begin{array}{ll}
0 & 0 \\
1 & 1
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]\right), \\
& \pi_{\{(0,0),(0,1)\}}(\mu)(11)=\mu\left(\left[\begin{array}{ll}
1 & 0 \\
1 & 0
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
1 & 1 \\
1 & 0
\end{array}\right]\right)+\mu\left(\left[\begin{array}{ll}
1 & 1 \\
1 & 1
\end{array}\right]\right) .
\end{aligned}
$$

In this paper we are interested in the capacity and the internal capacity of multidimensional SCSs. Although the capacity is easier to work with, as we will see later on, the task of computing it is still daunting. Thus, there is a necessity for more easily computable bounds on the capacity. To this end, we define the independence entropy of a $d$-dimensional SCS, which is the basis of the main results of this paper.

## III. Independence Entropy

In this section we define the independence entropy of multidimensional SCSs and present some of its properties. It will be used to bound the capacity. The independence entropy is not a new notion, and has appeared previously in [19] in relation to the capacity of fully constrained systems. However, the formulation of the independence entropy was combinatorial and therefore less suitable for our purposes. Thus, we modify the definition of independence entropy and formulate it as a statistical notion.

The admissible words of SCSs (see Definition 9) have their empirical $S$-tuple distribution from $\Gamma$. Finding such words inexorably involves intricate dependencies between coordinates. This affects not only the task of generating such words, but also the very basic problem of calculating or bounding the capacity of the SCS - the problem that is the focus of this paper.

In an attempt to simplify this problem, we study the independence-entropy approach. We eliminate all dependencies by considering only product measures, i.e., where the symbol in each coordinate of the word is chosen independently of other coordinates. Accordingly, we only require the average of $S$-marginals to be in $\Gamma$. We then ask what is the entropy of such a system. Intuitively, we are seeking the maximum rate of transmission in a system where word coordinates are transmitted independently and in parallel, designed such that the average $S$-marginals are in $\Gamma$. The following model provides a rough interpretation of the independence entropy: Suppose each bit of the output is transmitted by a different agent, and the number of agents is very large. The agents are allowed to coordinate a protocol in advance, but are unable to communicate once they receive the messages to be transmitted. In addition, the statistics of the output should roughly satisfy the constraints given by $\Gamma$, with high probability (as a function of the number of agents). In this case under suitable assumptions, the maximal transmission rate would coincide with the independence entropy. We proceed with formal definitions, starting with a product measure.
Definition 19. Let $d \in \mathbb{N}$, and let $S \subseteq \mathbb{Z}^{d}$ be a finite set. We say that $\mu \in \mathcal{P}\left(\Sigma^{S}\right)$ is an independent probability measure or a product measure if $\mu(w)=\prod_{\mathbf{v} \in S} \pi_{\{\mathbf{v}\}}(\mu)(w)$. For $S \subseteq \mathbb{Z}^{d}$ that is possibly infinite, $\mu \in \mathcal{P}\left(\Sigma^{S}\right)$ is a product measure whenever $\pi_{S^{\prime}}(\mu)$ is a product measure for every finite $S^{\prime} \subseteq S$.

In other words, we say that $\mu$ is independent if there exists $\left\{p_{\mathbf{v}} \in \mathcal{P}(\Sigma): \mathbf{v} \in S\right\}$ such that $\mu=\prod_{\mathbf{v} \in S} p_{\mathbf{v}}$. We naturally identify the set of product measures in $\mathcal{P}\left(\Sigma^{S}\right)$ with $(\mathcal{P}(\Sigma))^{S}$.

Next, we define the average of a marginal.
Definition 20. Given $d, n \in \mathbb{N}, \mu \in \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right)$, and $S \subseteq F_{n}^{d}$, let $\bar{\pi}_{S}(\mu) \in \mathcal{P}\left(\Sigma^{S}\right)$ be the average of the $S$-marginals over translates of $\mu$ :

$$
\bar{\pi}_{S}(\mu) \triangleq \frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{S+\mathbf{v}}(\mu)
$$

where the coordinates $S+\mathbf{v}$ are taken modulo $n$.
Let $S \subseteq F_{k}^{d}$ and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$ be a SCS. For $n \geqslant k$ we define

$$
\overline{\mathcal{P}}_{n}(\Gamma) \triangleq\left\{\mu \in(\mathcal{P}(\Sigma))^{F_{n}^{d}}: \bar{\pi}_{S}(\mu) \in \Gamma\right\}
$$

Thus, $\overline{\mathcal{P}}_{n}(\Gamma)$ consists of product measures on $\Sigma^{F_{n}^{d}}$ such that the average of the $S$-marginals is in $\Gamma$. We can now define the independence entropy of a SCS.
Definition 21. Let $d, k \in \mathbb{N}, S \subseteq F_{k}^{d}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$ be a $d$-dimensional SCS. The internal independence entropy of $\Gamma$ is defined by

$$
\widehat{h_{\mathrm{ind}}}(\Gamma) \triangleq \limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n}(\Gamma)} \frac{1}{n^{d}} H(\mu)
$$

where $H(\mu) \triangleq-\sum_{w \in \Sigma^{d}} \mu(w) \log _{2} \mu(w)$ is the entropy of $\mu$. The independence entropy of $\Gamma$ is defined by

$$
h_{\mathrm{ind}}(\Gamma) \triangleq \lim _{\epsilon \rightarrow 0^{+}} \widehat{h_{\mathrm{ind}}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)
$$

Again, it is clear by definition that

$$
\begin{equation*}
\widehat{h_{\mathrm{ind}}}(\Gamma) \leqslant h_{\mathrm{ind}}(\Gamma) \tag{2}
\end{equation*}
$$

The notion of independence entropy which appears here is a generalization of the combinatorial notion for fully constrained systems that appears in [19].
Theorem 22. Let $d, k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$ be a fully constrained system. Then

$$
h_{\mathrm{ind}}(\Gamma)=h_{\mathrm{ind}}^{\mathrm{com}}(\Gamma)
$$

where $h_{\text {ind }}^{\text {com }}$ is the combinatorial independence entropy from 19$]$.
To avoid a significant diversion from the main discussion, the proof of Theorem 22, together with the required definitions from [19], are given in Appendix B]

We now show properties of $\widehat{h_{\text {ind }}}$ and $h_{\text {ind }}$ which make them easier to analyze by reducing the multidimensional case to the one-dimensional case. We start with an inequality given in the following lemma. The proof follows the same argument that was used in [19] to show the inequality for fully constrained systems. However, the equality for fully constrained systems holds in an easier and stronger sense.
Lemma 23. Let $k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a one-dimensional SCS. Then for all $d \in \mathbb{N}$,

$$
\widehat{h_{\mathrm{ind}}}(\Gamma) \leqslant \widehat{h_{\mathrm{ind}}}\left(\Gamma^{\otimes d}\right)
$$

Proof: Take $\hat{\mu} \in \overline{\mathcal{P}}_{n}(\Gamma)$. Since $\hat{\mu}$ is a product measure, it can be written as $\hat{\mu}=\prod_{i=0}^{n-1} \pi_{\{i\}}(\hat{\mu})$. We now construct a measure $\mu \in \overline{\mathcal{P}}_{n}\left(\Gamma^{\otimes d}\right)$ using $\hat{\mu}$. For every $\mathbf{v} \in F_{n}^{d}$ set

$$
\pi_{\{\mathbf{v}\}}(\mu) \triangleq \pi_{\{\ell(\mathbf{v})\}}(\hat{\mu})
$$

where $\ell(\mathbf{v}) \triangleq\left(\sum_{i=0}^{d-1} v_{i}\right) \bmod n$ is the modulo $n$ of the sum of the coordinates of $\mathbf{v}$.
Observe that $\mu$ is such that in every row in every direction, i.e., a set of coordinates of the form $\mathbf{v}+[n] \mathbf{e}_{i}$, we obtain some cyclic rotation of $\hat{\mu}$ by $t$ positions, denoted $\sigma_{t}(\hat{\mu})$. However, $\hat{\mu} \in \overline{\mathcal{P}}_{n}(\Gamma)$ implies $\sigma_{t}(\hat{\mu}) \in \overline{\mathcal{P}}_{n}(\Gamma)$. Thus, we obtain that $\mu \in \overline{\mathcal{P}}_{n}\left(\Gamma^{\otimes d}\right)$ and

$$
\frac{1}{n} H(\hat{\mu})=\frac{1}{n^{d}} H(\mu)
$$

Since we are taking the supremum over all measures $\hat{\mu}$, we have $\widehat{h_{\text {ind }}}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\Gamma^{\otimes d}\right)$.
Theorem 24. Let $k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a one-dimensional SCS. Then for all $d \in \mathbb{N}$,

$$
h_{\mathrm{ind}}\left(\Gamma^{\otimes d}\right)=h_{\mathrm{ind}}(\Gamma) .
$$

Proof: We first show that $h_{\text {ind }}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}(\Gamma)$. Fix $\delta>0$ and take $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right)\right)$. Recall that

$$
\overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right)\right) \subseteq \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right)
$$

Let $\left(\mathbf{v}_{i}\right)_{i \in\left[n^{d-1}\right]}$ be an enumeration of $\{0\} \times F_{n}^{d-1}$, i.e.,

$$
\left\{\mathbf{v}_{0}, \ldots, \mathbf{v}_{n^{d-1}-1}\right\}=\{0\} \times F_{n}^{d-1}
$$

For $i \in\left[n^{d-1}\right]$, define $\mu_{i} \in \mathcal{P}\left(\Sigma^{n}\right)$ by $\mu_{i} \triangleq \pi_{[n] \mathbf{e}_{0}+\mathbf{v}_{i}}(\mu)$. Now let $\hat{\mu} \in \mathcal{P}\left(\Sigma^{n^{d}}\right)$ be the product measure that is the product of all the $\mu_{i}$ 's. This means that for a word $a=a_{0} \ldots a_{n^{d}-1} \in \Sigma^{n^{d}}$,

$$
\hat{\mu}(a) \triangleq \mu_{0}\left(a_{0} \ldots a_{n-1}\right) \mu_{1}\left(a_{n} \ldots a_{2 n-1}\right) \cdots \mu_{n^{d}-1}\left(a_{n\left(n^{d-1}-1\right)} \ldots a_{n^{d}-1}\right)
$$

Since each of the $\mu_{i}$ 's is already a product measure, $\hat{\mu} \in \mathcal{P}\left(\Sigma^{n^{d}}\right)$ is also a product measure. We have

$$
\begin{aligned}
\bar{\pi}_{[k]}(\hat{\mu}) & =\frac{1}{n^{d}} \sum_{j=0}^{n^{d}-1} \pi_{j+[k]}(\hat{\mu}) \\
& =\frac{1}{n^{d}}\left(\sum_{i=0}^{n^{d-1}-1} \sum_{j=i n}^{(i+1) n-1} \pi_{j+[k]}(\hat{\mu})\right) \\
& =\frac{1}{n^{d}}\left(\sum_{i=0}^{n^{d-1}-1}\left(\sum_{j=i n}^{(i+1) n-k} \pi_{j+[k]}(\hat{\mu})+\sum_{j=(i+1) n-k+1}^{(i+1) n-1} \pi_{j+[k]}(\hat{\mu})\right)\right) \\
& \stackrel{(a)}{=} \frac{1}{n^{d}}\left(\sum_{i=0}^{n^{d-1}-1} \sum_{j=i n}^{(i+1) n-k} \pi_{(j-i n)+[k]}\left(\mu_{i}\right)+\sum_{i=0}^{n^{d-1}-1} \sum_{j=(i+1) n-k+1}^{(i+1) n-1} \pi_{j+[k]}(\hat{\mu})\right) \\
& =\frac{1}{n^{d}}\left(\sum_{i=0}^{n^{d-1}-1} \sum_{j=i n}^{(i+1) n-1} \pi_{(j-i n)+[k]}\left(\mu_{i}\right)-\sum_{i=0}^{n^{d-1}-1} \sum_{j=(i+1) n-1} \sum_{(j-i n)+[k]}\left(\mu_{i}\right)+\sum_{i=0}^{n^{d-1}-1} \sum_{(i+1) n-1}^{j=(i+1) n-k+1} \pi_{j+[k]}(\hat{\mu})\right) \\
& \left.=\frac{1}{n^{d-1}}\left(\sum_{i=0}^{n^{d-1}-1} \bar{\pi}_{[k]}\left(\mu_{i}\right)-\frac{1}{n} \sum_{i=0}^{n^{d-1}-1} \sum_{j=(i+1) n-k+1}^{(i+1) n-1} \sum_{(j-i n)+[k]}\left(\mu_{i}\right)-\pi_{j+[k]}(\hat{\mu})\right)\right) \\
& =\bar{\pi}_{[k] \mathbf{e}_{0}}(\mu)-\frac{1}{n^{d}} \sum_{i=0}^{n^{d-1}-1} \sum_{j=(i+1) n-k+1}^{(i+1) n-1}\left(\pi_{(j-i n)+[k]}\left(\mu_{i}\right)-\pi_{j+[k]}(\hat{\mu})\right)
\end{aligned}
$$

where $(a)$ follows from the definition of $\hat{\mu}$ and since the coordinates are taken modulo $n$ when calculating $\pi_{[k]}\left(\mu_{i}\right)$. Each $\left(\pi_{(j-i n)+[k]}\left(\mu_{i}\right)-\pi_{j+[k]}(\hat{\mu})\right)$ is a signed measure of total variation norm at most 1 . Therefore,

$$
\left\|\bar{\pi}_{[k]}(\hat{\mu})-\bar{\pi}_{[k] \mathbf{e}_{0}}(\mu)\right\|_{T V} \leqslant \frac{k}{n}
$$

This means that $\bar{\pi}_{[k]}(\hat{\mu}) \in \mathbb{B}_{\frac{k}{n}+\delta}(\Gamma)$. We obtained that for every $\epsilon>\delta>0$, and every $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right)\right)$, we can find $n_{0} \in \mathbb{N}$ such that for every $n>n_{0}, \hat{\mu} \in \overline{\mathcal{P}}_{n^{d}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)$. Since $\mu$ and $\hat{\mu}$ are both product measures we have

$$
\begin{aligned}
H(\mu) & =\sum_{\mathbf{v} \in F_{n}^{d}} H\left(\pi_{\{\mathbf{v}\}}(\mu)\right) \\
& =\sum_{i \in\left[n^{d}\right]} H\left(\pi_{\{i\}}(\hat{\mu})\right) \\
& =H(\hat{\mu})
\end{aligned}
$$

This implies that for every $\epsilon>\delta>0$,

$$
\limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right)\right)} \frac{1}{n^{d}} H(\mu) \leqslant \limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n^{d}}\left(\mathbb{B}_{e}(\Gamma)\right)} \frac{1}{n^{d}} H(\mu)
$$

We therefore obtain $h_{\text {ind }}\left(\Gamma^{\otimes d}\right) \leqslant \widehat{h_{\text {ind }}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)$ for every $\epsilon>0$. Taking the limit as $\epsilon \rightarrow 0^{+}$, by the definition of $h_{\text {ind }}(\Gamma)$ we have

$$
h_{\text {ind }}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}(\Gamma)
$$

We now show the other direction. By Lemma 23, For every $\delta>0$ we have

$$
\widehat{h_{\mathrm{ind}}}\left(\mathbb{B}_{\delta}(\Gamma)\right) \leqslant \widehat{h_{\mathrm{ind}}}\left(\mathbb{B}_{\delta}(\Gamma)^{\otimes d}\right)
$$

By monotonicity of $\widehat{h_{\text {ind }}}$ it thus follows that for every $\delta>0$,

$$
h_{\text {ind }}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\mathbb{B}_{\delta}(\Gamma)^{\otimes d}\right)
$$

Now observe that for every $\epsilon>0$ there exists $\delta>0$ so that

$$
\mathbb{B}_{\delta}(\Gamma)^{\otimes d} \subseteq \mathbb{B}_{\epsilon}\left(\Gamma^{\otimes d}\right)
$$

It follows that for every $\epsilon>0$

$$
h_{\text {ind }}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\mathbb{B}_{\epsilon}\left(\Gamma^{\otimes d}\right)\right)
$$

Thus, by taking the limit $\epsilon \rightarrow 0^{+}$,

$$
h_{\mathrm{ind}}(\Gamma) \leqslant h_{\mathrm{ind}}\left(\Gamma^{\otimes d}\right)
$$

We conclude this section by noting that Lemma 23 and Theorem 24 show that for $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$,

$$
\begin{equation*}
\widehat{h_{\mathrm{ind}}}(\Gamma) \leqslant \widehat{h_{\mathrm{ind}}}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\Gamma^{\otimes d}\right)=h_{\text {ind }}(\Gamma) \tag{3}
\end{equation*}
$$

## IV. Independence Entropy Lower Bounds The Capacity

This section and the next explore the relationship between the independence entropy and the capacity. In this section we show that the capacity of any $d$-dimensional SCS (not necessarily an axial product) is lower bounded by the independence entropy.

Before proceeding we require a simple lemma.
Lemma 25. Let $d, n \in \mathbb{N}$, and $S \subseteq F_{n}^{d}$, then $\pi_{S}$ and $\bar{\pi}_{S}$ are contractions with respect to the total-variation distance, i.e., for all $\mu, v \in \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right)$,

$$
\begin{aligned}
& \left\|\pi_{S}(\mu)-\pi_{S}(v)\right\|_{T V} \leqslant\|\mu-v\|_{T V}, \\
& \left\|\bar{\pi}_{S}(\mu)-\bar{\pi}_{S}(v)\right\|_{T V} \leqslant\|\mu-v\|_{T V} .
\end{aligned}
$$

Proof: For every $W \subseteq \Sigma^{S}$ we have

$$
\left|\pi_{S}(\mu)(W)-\pi_{S}(v)(W)\right|=\left|\mu\left(\pi_{S}^{-1}(W)\right)-v\left(\pi_{S}^{-1}(W)\right)\right| \leqslant \sup _{A^{\prime} \subset \Sigma^{S}}\left|\mu\left(W^{\prime}\right)-v\left(W^{\prime}\right)\right|=\|\mu-v\|_{T V}
$$

Hence the function $\pi_{S+\mathbf{v}}$ is a contraction for every $\mathbf{v} \in F_{n}^{d}$. Then $\bar{\pi}_{S}$, being an average of contractions, is itself a contraction.
We are now ready to state and prove the main result of this section - a lower bound on the capacity. The corresponding result for fully constrained systems was obtained in [19].
Theorem 26. Let $d \in \mathbb{N}, S \subseteq \mathbb{Z}^{d}$ be a finite set, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{S}\right)$ be a SCS. Then $h_{\text {ind }}(\Gamma) \leqslant \operatorname{cap}(\Gamma)$.
Proof: Fix $\delta>0, n \in \mathbb{N}$ such that $S \subseteq F_{n}^{d}$, and let $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)\right)$. For $m \in \mathbb{N}$, we have a natural identification isomorphism $\Sigma^{F_{n m}^{d}} \cong\left(\Sigma_{n}^{F_{n}^{d}} F_{m}^{d}\right.$ that identifies $\mathbf{v} \in F_{n m}^{d}$ with the unique pair $\mathbf{r} \in F_{n}^{d}$ and $\mathbf{q} \in F_{m}^{d}$ such that $\mathbf{v}=n \mathbf{q}+\mathbf{r}$. Consider the product measure $\mu^{m} \in \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right)^{F_{m}^{d}} \subseteq \mathcal{P}\left(\Sigma^{F_{n m}^{d}}\right)$ satisfying

$$
\mu^{m}(\{x\})=\prod_{\mathbf{v} \in F_{m}^{d}} \mu\left(\pi_{F_{n}^{d}}\left(\sigma_{n \mathbf{v}}(x)\right)\right)
$$

Note that since $\mu$ is a product measure, $\mu^{m}$ is also a product measure.
For a word $w \in \Sigma^{F_{n m}^{d}}$, denote by $\hat{\mathrm{fr}}_{w}^{F_{n}^{d}}$ the empirical distribution of non-overlapping $F_{n}^{d}$-tuples, i.e.,

$$
\hat{\mathrm{fr}}_{w}^{F_{n}^{d}} \triangleq \frac{1}{\left|F_{m}^{d}\right|} \sum_{\mathbf{u} \in F_{m}^{d}} \delta_{\pi_{n}^{d}\left(\sigma_{n \mathbf{u}}(\hat{w})\right)}
$$

Additionally, observe that

$$
\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \hat{\mathrm{fr}}_{\sigma_{\mathbf{v}}(w)}^{F_{n}^{d}}=\mathrm{fr}_{w}^{F_{n}^{d}}
$$

Also, because $\pi_{S}^{F_{n}^{d}}$ is an affine map, it follows that

$$
\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{S}^{F_{n}^{d}} \hat{\mathrm{fr}}_{\sigma_{\mathbf{v}}(w)}^{F_{n}^{d}}=\pi_{S}\left(\mathrm{fr}_{w}^{F_{n}^{d}}\right)
$$

By Lemma 7. $\pi_{S}\left(\mathrm{fr}_{w}^{F_{n}^{d}}\right)=\mathrm{fr}_{w}^{S}$.
Note that by the construction of $\mu^{m}$ we have $\bar{\pi}_{S}(\mu)=\bar{\pi}_{S}\left(\mu^{m}\right)$, and we obtain,

$$
\begin{aligned}
\left\|\mathrm{fr}_{w}^{S}-\bar{\pi}_{S}\left(\mu^{m}\right)\right\|_{T V} & =\left\|\mathrm{fr}_{w}^{S}-\bar{\pi}_{S}(\mu)\right\|_{T V} \\
& =\left\|\frac{1}{\left|F_{n m}^{d}\right|} \sum_{\mathbf{u} \in F_{n m}^{d}} \pi_{S}\left(\delta_{\sigma_{\mathbf{u}}(w)}\right)-\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& =\left\|\frac{1}{\left|F_{n}^{d}\right|\left|F_{m}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{\mathbf{u} \in F_{m}^{d}} \pi_{S}\left(\delta_{\sigma_{n \mathbf{u}+\mathbf{v}}(w)}\right)-\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& \stackrel{(a)}{=}\left\|\frac{1}{\left|F_{n}^{d}\right|\left|F_{m}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{\mathbf{u} \in F_{m}^{d}} \pi_{S+\mathbf{v}}\left(\delta_{\sigma_{n \mathbf{u}}(w)}\right)-\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& \stackrel{(b)}{\leqslant} \frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}}\left\|\frac{1}{\left|F_{m}^{d}\right|} \sum_{\mathbf{u} \in F_{m}^{d}} \pi_{S+\mathbf{v}}\left(\delta_{\sigma_{n \mathbf{u}}(w)}\right)-\pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& \stackrel{(c)}{=} \frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}}\left\|\pi_{S+\mathbf{v}}\left(\frac{1}{\left|F_{m}^{d}\right|} \sum_{\mathbf{u} \in F_{m}^{d}} \delta_{\sigma_{n \mathbf{u}}(w)}\right)-\pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& =\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}}\left\|\pi_{S+\mathbf{v}}\left(\hat{f r}_{w}^{F_{n}^{d}}\right)-\pi_{S+\mathbf{v}}(\mu)\right\|_{T V} \\
& \stackrel{(d)}{\leqslant} \frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}}\left\|\hat{f r}_{w}^{F_{n}^{d}}-\mu\right\|_{T V} \\
& =\left\|\hat{\mathrm{fr}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V}
\end{aligned}
$$

where:

- (a) follows since $\pi_{S}\left(\delta_{\sigma_{\mathbf{v}}(w)}\right)=\pi_{S+\mathbf{v}}\left(\delta_{w}\right)$.
- (b) follows by the triangle inequality.
- (c) follows since $\pi_{S}$ is an affine map.
- (d) follows by Lemma 25

Thus, for $\epsilon>\delta$, if $\left\|\hat{\mathrm{fr}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V}<\epsilon-\delta$ then $\left\|\mathrm{fr}_{w}^{S}-\bar{\pi}_{S}\left(\mu^{m}\right)\right\|_{T V}<\epsilon-\delta$. Therefore,

$$
\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\operatorname{fr}_{w}^{S}-\bar{\pi}_{S}(\mu)\right\|_{T V} \geqslant \epsilon-\delta\right\} \subseteq\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\hat{\operatorname{fr}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V} \geqslant \epsilon-\delta\right\}
$$

Using the fact that $\bar{\pi}_{S}(\mu) \in \mathbb{B}_{\delta}(\Gamma)$, it follows that

$$
\begin{equation*}
\left\{w \in \Sigma^{F_{n m}^{d}}: \operatorname{fr}_{w}^{S} \notin \operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right\} \subseteq\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\hat{\mathrm{fr}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V} \geqslant \epsilon-\delta\right\} \tag{4}
\end{equation*}
$$

where $\operatorname{int}(\cdot)$ denotes the interior of a set, i.e., $\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)=\left\{v \in \mathcal{P}\left(\Sigma^{S}\right): \inf _{\mu \in \Gamma}\|v-\mu\|_{T V}<\epsilon\right\}$.
If $w \in \Sigma^{F_{n m}^{d}}$ was randomly drawn according to $\mu^{m}$, the non-overlapping $F_{n}^{d}$-tuples are distributed i.i.d. according to $\mu$. Apply Cramer's Theorem (as in [7, Theorem 2.2.3 remark c]) to deduce that for $\epsilon>\delta$ and for every $m$,

$$
\mu^{m}\left(\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\hat{\mathrm{fr}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V} \geqslant \epsilon-\delta\right\}\right) \leqslant 2 \exp \left(-m \inf _{v \in \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right):\|v-\mu\|_{T V} \geqslant \epsilon-\delta} H(v \mid \mu)\right)
$$

Note that the function $v \times \mu \mapsto H(v \mid \mu)$ is continuous and strictly positive off the diagonal. Thus, for every $\epsilon>\delta$ we have

$$
c_{\mu}(\epsilon) \triangleq \inf _{v \in \mathcal{P}\left(\Sigma^{F_{n}^{d}}\right):\|v-\mu\|_{T V} \geqslant \epsilon-\delta} H(v \mid \mu)>0
$$

Hence

$$
\begin{equation*}
\mu^{m}\left(\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\hat{\mathrm{f}}_{w}^{d_{n}^{d}}-\mu\right\|_{T V} \geqslant \epsilon-\delta\right\}\right) \leqslant 2 \exp \left(-m c_{\mu}(\epsilon)\right) \tag{5}
\end{equation*}
$$

Recall that $\mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)=\left\{w \in \Sigma^{F_{n m}^{d}}: \operatorname{fr}_{w}^{S} \in \operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right\}$. By (4), we have

$$
\begin{align*}
& \mu^{m}\left(\Sigma^{F_{n m}^{d}} \backslash \mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right)  \tag{6}\\
& =\mu^{m}\left(\left\{w \in \Sigma^{F_{n m}^{d}}: \operatorname{fr}_{w}^{S} \notin \operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right\}\right) \\
& \leqslant \mu^{m}\left(\left\{w \in \Sigma^{F_{n m}^{d}}:\left\|\hat{\mathrm{f}}_{w}^{F_{n}^{d}}-\mu\right\|_{T V} \geqslant \epsilon-\delta\right\}\right) .
\end{align*}
$$

Combining (5) and (6) we have,

$$
\xi \triangleq \mu^{m}\left(\Sigma^{F_{n m}^{d}} \backslash \mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right) \leqslant 2 \exp \left(-m c_{\mu}(\epsilon)\right)
$$

It now follows that,

$$
\begin{aligned}
\frac{1}{n^{d}} H(\mu)= & \frac{1}{(n m)^{d}} H\left(\mu^{m}\right) \\
= & -\frac{1}{(n m)^{d}} \sum_{w \in \Sigma^{I_{n m}^{d}}} \mu^{m}(w) \log _{2} \mu^{m}(w) \\
= & -\frac{1}{(n m)^{d}} \sum_{w \in \mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)} \mu^{m}(w) \log _{2} \mu^{m}(w) \\
& -\frac{1}{(n m)^{d}} \sum_{w \notin \mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)} \mu^{m}(w) \log _{2} \mu^{m}(w) \\
\stackrel{(a)}{\leqslant} & (1-\xi) \cdot \frac{\log _{2}\left|\mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right|}{(n m)^{d}}+\xi \cdot \frac{\log _{2}\left|\Sigma^{F_{n m}^{d}} \backslash \mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right|}{(n m)^{d}}+H_{2}(\xi) \\
\leqslant & \frac{\log _{2}\left|\mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right|}{(n m)^{d}}+2 e^{-m c_{\mu}(\epsilon)} \frac{\log _{2}|\Sigma|^{(n m)^{d}}}{(n m)^{d}}+H_{2}(\xi) \\
\leqslant & \frac{1}{(n m)^{d}} \log _{2}\left|\mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right|+2 e^{-m c_{\mu}(\epsilon)} \frac{1}{(n m)^{d}} \log _{2}|\Sigma|^{(n m)^{d}}+H_{2}(\xi)
\end{aligned}
$$

where $(a)$ follows from standard maximization of entropy arguments, and where $H_{2}(\xi) \triangleq-\xi \log _{2} \xi-(1-\xi) \log _{2}(1-\xi)$ is the binary entropy function. This implies

$$
\begin{aligned}
\frac{1}{n^{d}} H(\mu) & =\limsup _{m \rightarrow \infty} \frac{1}{n^{d}} H(\mu) \\
& \leqslant \limsup _{m \rightarrow \infty} \frac{1}{(n m)^{d}} \log _{2}\left|\mathcal{B}_{n m}\left(\operatorname{int}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right)\right| \\
& \leqslant \limsup _{m \rightarrow \infty} \frac{1}{(n m)^{d}} \log _{2}\left|\mathcal{B}_{n m}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)\right| \\
& \leqslant \underset{\operatorname{cap}}{ }\left(\mathbb{B}_{\epsilon}(\Gamma)\right)
\end{aligned}
$$

This is true for every $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)\right)$ and hence

$$
\sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)\right)} \frac{1}{n^{d}} H(\mu) \leqslant \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right) .
$$

Since this holds for every $n$ we have that for every $\epsilon>\delta>0$,

$$
\widehat{h_{\mathrm{ind}}}\left(\mathbb{B}_{\delta}(\Gamma)\right) \leqslant \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)
$$

Taking the limit as $\delta \rightarrow 0$, this implies that for every $\epsilon>0$,

$$
h_{\text {ind }}(\Gamma) \leqslant \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)
$$

Finally, taking the limit as $\epsilon \rightarrow 0$, it follows that

$$
h_{\text {ind }}(\Gamma) \leqslant \operatorname{cap}(\Gamma)
$$

We summarize our results thus far by noting that for a $\operatorname{SCS} \Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$, since $\widehat{h_{\text {ind }}}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\Gamma^{\otimes d}\right)$, Theorem 26 together with (3) show that

$$
\begin{align*}
& \widehat{h_{\text {ind }}}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\Gamma^{\otimes d}\right) \leqslant \operatorname{cap}\left(\Gamma^{\otimes d}\right),  \tag{7}\\
& \widehat{h_{\text {ind }}}(\Gamma) \leqslant \widehat{h_{\text {ind }}}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\Gamma^{\otimes d}\right)=h_{\text {ind }}(\Gamma) \leqslant \operatorname{cap}(\Gamma) . \tag{8}
\end{align*}
$$

## V. Upper Bound on Limiting Capacity

In this section we prove that if $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ is a convex one-dimensional SCS and $\Gamma^{\otimes d}$ its $d$-axial product, then

$$
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\Gamma^{\otimes d}\right)
$$

The main idea is to show that for any $\epsilon>0$ we are able to find $d$ large enough for which the independence entropy is $\epsilon$-close to $\operatorname{cap}\left(\Gamma^{\otimes d}\right)$. This is the main result of [23] and the proof here is an adaptation of it.

Before going into details we introduce a different form of $d$-axial product which we call the weak $d$-axial product. For a one dimensional SCS, $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$, define

$$
\Gamma^{\boxtimes d} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right): \frac{1}{d} \sum_{i \in[d]} \pi_{[k] \mathbf{e}_{i}}(\mu) \in \Gamma\right\}
$$

and thus

$$
\mathcal{B}_{n}\left(\Gamma^{\boxtimes d}\right)=\left\{w \in F_{n}^{d}: \frac{1}{d} \sum_{i \in[d]} \operatorname{fr}_{w}^{[k] \mathbf{e}_{i}} \in \Gamma\right\}
$$

For the weak $d$-axial product we define,

$$
\overline{\mathcal{P}}_{n}\left(\Gamma^{\boxtimes d}\right) \triangleq\left\{\mu \in(\mathcal{P}(\Sigma))^{F_{n}^{d}} \quad: \frac{1}{d} \sum_{i \in[d]} \bar{\pi}_{[k] \mathbf{e}_{i}}(\mu) \in \Gamma\right\}
$$

This last definition is a relaxed version of $\Gamma^{\otimes d}$, since $\overline{\mathcal{P}}_{n}\left(\Gamma^{\otimes d}\right)$ is the set of all independent measures for which the average of the $k$-marginals in each direction (separately) belongs to $\Gamma$, whereas $\overline{\mathcal{P}}_{n}\left(\Gamma^{\boxtimes d}\right)$ is the set of all independent measures for which the average of $k$-marginals (over all directions) belongs to $\Gamma$.

Correspondingly, we have,

$$
h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right) \triangleq \lim _{\epsilon \rightarrow 0^{+}} \limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}(\Gamma)^{\boxtimes d}\right)} \frac{1}{n^{d}} H(\mu)
$$

where $H(\mu) \triangleq-\sum_{w \in \Sigma_{n}^{d}} \mu(w) \log _{2} \mu(w)$ is the entropy of $\mu$.
As will become clearer later on, it will be somewhat easier to use $h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right)$ than $h_{\text {ind }}\left(\Gamma^{\otimes d}\right)$ in this section. First, the following lemma shows that the relaxation leading to $h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right)$ does not affect the independence entropy.

Lemma 27. Let $k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a convex one-dimensional SCS, then

$$
h_{\mathrm{ind}}(\Gamma)=h_{\mathrm{ind}}\left(\Gamma^{\otimes d}\right)=h_{\mathrm{ind}}\left(\Gamma^{\boxtimes d}\right)
$$

Proof: By Theorem 24 we already know that $h_{\text {ind }}\left(\Gamma^{\otimes d}\right)=h_{\text {ind }}(\Gamma)$. Thus, we are left with proving the last equality. Since $\Gamma$ is convex, for every $\delta>0$,

$$
\overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right)\right) \subseteq \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\otimes d}\right) \subseteq \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\boxtimes d}\right)
$$

Hence,

$$
h_{\text {ind }}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right) .
$$

The other direction follows essentially by using the same method as in the proof of Theorem 24, as we now describe. Let $\left(\mathbf{v}_{i}^{j}\right)_{i \in\left[n^{d-1}\right]}$ be an enumeration of $F_{n}^{j-1} \times\{0\} \times F_{n}^{d-j}$, i.e.,

$$
\left\{\mathbf{v}_{0}^{j}, \ldots, \mathbf{v}_{n^{d-1}-1}^{j}\right\}=F_{n}^{j-1} \times\{0\} \times F_{n}^{d-j}
$$

Fix $\delta>0$ and $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\boxtimes d}\right)$. For $i \in\left[n^{d-1}\right]$ and $j \in[d]$, define $\mu_{i}^{j} \in \mathcal{P}\left(\Sigma^{n}\right)$ by $\mu_{i}^{j} \triangleq \pi_{[n] \mathbf{e}_{j}+\mathbf{v}_{i}^{j}}(\mu)$. Now let $\hat{\mu} \in \mathcal{P}\left(\Sigma^{d n^{d}}\right)$ be the product measure satisfying

$$
\hat{\mu}(\{a\})=\prod_{j \in[d]} \prod_{i \in\left[n^{d-1}\right]} \mu_{i}^{j}\left(a_{i n+j n^{d}} \ldots a_{(i+1) n+j n^{d}-1}\right)
$$

for every word $a=a_{0} \ldots a_{d n^{d}-1} \in \Sigma^{d n^{d}}$. It is clear that $\hat{\mu}$ is indeed a product measure, because every $\mu_{i}^{j}$ is also a product measure. Now,

$$
\begin{aligned}
\bar{\pi}_{[k]}(\hat{\mu}) & =\frac{1}{d n^{d}} \sum_{i \in\left[d n^{d}\right]} \pi_{i+[k]}(\hat{\mu}) \\
& =\frac{1}{d n^{d}} \sum_{j \in[d]} \sum_{i \in\left[n^{d-1}\right]} \sum_{\ell \in[n]} \pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu}) \\
& =\frac{1}{d n^{d}} \sum_{j \in[d]} \sum_{i \in\left[n^{d-1}\right]}\left(\sum_{\ell \in[n-k]} \pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu})+\sum_{\ell=n-k}^{n-1} \pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu})\right) \\
& =\frac{1}{d n^{d}} \sum_{j \in[d]} \sum_{i \in\left[n^{d-1}\right]}\left(\sum_{\ell \in[n-k]} \pi_{[k]+\ell}\left(\mu_{i}^{j}\right)+\sum_{\ell=n-k}^{n-1} \pi_{[k]+i n+\ell+j n^{d}(\hat{\mu})}\right) \\
& =\frac{1}{d n^{d}} \sum_{j \in[d]} \sum_{i \in\left[n^{d-1}\right]}\left(\sum_{\ell \in[n]} \pi_{[k]+\ell}\left(\mu_{i}^{j}\right)-\sum_{\ell=n-k}^{n-1} \pi_{[k]+\ell}\left(\mu_{i}^{j}\right)+\sum_{\ell=n-k}^{n-1} \pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu})\right) \\
& \stackrel{(a)}{=} \frac{1}{d} \sum_{j \in[d]} \frac{\bar{\pi}_{[k] \mathbf{e}_{j}}}{}(\mu)-\frac{1}{d n^{d}} \sum_{j \in[d]} \sum_{i \in\left[n^{d-1}\right]} \sum_{\ell=n-k}^{n-1}\left(\pi_{[k]+\ell}\left(\mu_{i}^{j}\right)-\pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu})\right) .
\end{aligned}
$$

Recall that from the definition of $\overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\boxtimes d}\right)$, we have

$$
\frac{1}{d} \sum_{j \in[d]} \bar{\pi}_{[k] \mathbf{e}_{j}}(\mu) \in \mathbb{B}_{\delta}(\Gamma)
$$

Since $\left(\pi_{[k]+\ell}\left(\mu_{i}^{j}\right)-\pi_{[k]+i n+\ell+j n^{d}}(\hat{\mu})\right)$ is a signed measure of total variation norm at most 2 , it follows that $\bar{\pi}_{[k]}(\hat{\mu}) \in$ $\mathbb{B}_{\frac{2 k}{n}+\delta}(\Gamma)$, so $\hat{\mu} \in \overline{\mathcal{P}}_{d n^{d}}\left(\mathbb{B}_{\frac{2 k}{n}+\delta}(\Gamma)\right)$. Hence, for every $\epsilon>\delta>0$, and every $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\boxtimes d}\right)$, we can find $n_{0} \in \mathbb{N}$ such that for every $n>n_{0}, \hat{\mu} \in \frac{n}{\mathcal{P}} d n^{d}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)$, and therefore,

$$
\limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\boxtimes d}\right)} \frac{1}{n^{d}} H(\mu) \leqslant \limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{d n^{d}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)} \frac{1}{d n^{d}} H(\mu)
$$

Thus, we obtain $h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right) \leqslant \widehat{h_{\text {ind }}}\left(\mathbb{B}_{\epsilon}(\Gamma)\right)$ for every $\epsilon>0$, and by definition it follows that

$$
h_{\text {ind }}\left(\Gamma^{\boxtimes d}\right) \leqslant h_{\text {ind }}(\Gamma) .
$$

Given a probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$, denote by $L^{2}\left(\mathcal{X}, \mathcal{F}, \mathbb{P}, \mathbb{C}^{n}\right)$ the Hilbert space of $\mathcal{F}$-measurable functions $f: \mathcal{X} \rightarrow \mathbb{C}^{n}$ satisfying

$$
\|f\|_{L^{2}}^{2} \triangleq \int\langle f, f\rangle \mathrm{d} \mathbb{P}<\infty
$$

where $\langle\cdot, \cdot\rangle$ is the standard inner product on $\mathbb{C}^{n}$.
The following lemma is based on Dirichlet's "pigeon hole principle" and different versions of it are used in many de-Finetti type proofs (see, for example, [8] [21, Lemma 4.1]).

Lemma 28. Let $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{F}_{0} \subseteq \mathcal{F}_{1} \subseteq \cdots \subseteq \mathcal{F}_{m} \subseteq \mathcal{F}$ be a sequence of sub- $\sigma$-algebras. Let $f \in L^{2}\left(\mathcal{X}, \mathcal{F}, \mathbb{P}, \mathbb{C}^{n}\right)$, and denote $f_{j} \triangleq E\left[f \mid \mathcal{F}_{j}\right]$, the conditional expectation of $f$ with respect to the sub- $\sigma$-algebra $\mathcal{F}_{j}$. Then, there exists $t \in[m]$ such that

$$
\left\|f_{t+1}-f_{t}\right\|_{L^{2}}^{2} \leqslant \frac{1}{m}\|f\|_{L^{2}}^{2}
$$

Proof: For every $\ell$, let $V_{\ell} \triangleq L^{2}\left(\mathcal{X}, \mathcal{F}_{\ell}, \mathbb{P}, \mathbb{C}^{n}\right)$ denote the corresponding sub-space of the Hilbert space $V \triangleq L^{2}\left(\mathcal{X}, \mathcal{F}, \mathbb{P}, \mathbb{C}^{n}\right)$. Then $f_{\ell}$ is an orthogonal projection of $f$ onto $V_{\ell}$. Thus, $\left\langle f-f_{\ell}, g\right\rangle_{L^{2}}=0$ for every $g \in V$. Therefore,

$$
\left\|f_{m}\right\|_{L^{2}}^{2}=\sum_{\ell \in[m]}\left\|f_{\ell+1}-f_{\ell}\right\|_{L^{2}}^{2}+\left\|f_{0}\right\|_{L^{2}}^{2}
$$

Additionally, $0 \leqslant\left\|f_{m}\right\|_{L^{2}}^{2} \leqslant\|f\|_{L^{2}}^{2}$. The result follows by noticing that if $m$ non-negative real numbers sum to at most $\|f\|_{L^{2}}^{2}$ then the value of at least one element is at most $\frac{1}{m}\|f\|_{L^{2}}^{2}$.

Before stating the lemmas, we need the following notation. Recall that for $k \in \mathbb{N}$, we defined $[k] \triangleq\{0, \ldots, k-1\}$. We now define $[-k] \triangleq\{-1, \ldots,-k\}$.

Lemma 29. For every $\epsilon>0$, and any $m \in \mathbb{N}$, there exists $d_{0} \in \mathbb{N}$ such that for every $d \geqslant d_{0}$, and every $n, j \in \mathbb{N}, n \geqslant j+2$, there exists a sequence of $m+1$ random subsets $X_{0}, X_{1}, \ldots, X_{m} \subseteq F_{n}^{d}$, and random variables $I_{t, \mathbf{v}} \in[d]$, for all $t \in[m], \mathbf{v} \in F_{n}^{d}$, all defined on an appropriate probability space $\left(\mathcal{X}, 2^{\mathcal{X}}, \mathbb{P}\right)$, such that all the following hold:

1) $\mathbb{P}\left(X_{i} \subseteq X_{i+1}\right)=1$ for all $i \in[m]$.
2) $\mathbb{P}\left(\left|X_{m}\right| \leqslant \epsilon\left|F_{n}^{d}\right|\right) \geqslant 1-\epsilon$.
3) For all $\mathbf{v} \in F_{n}^{d}$ and $t \in[m], I_{t, \mathbf{v}}$ is distributed uniformly on $[d]$ and is independent of $X_{t}$. Furthermore, for every value of $X_{t}$,

$$
\mathbb{P}\left(X_{t} \cup\left([-(j+1)] \mathbf{e}_{I_{t, \mathbf{v}}}+\mathbf{v}\right) \subseteq X_{t+1} \mid X_{t}\right) \geqslant 1-\epsilon
$$

Proof: Choose $0<p<1$ small enough so that $1-(1-p)^{m+1} \leqslant \frac{\epsilon}{2}$, and conveniently denote $p_{i} \triangleq 1-(1-p)^{i+1}$. For all $i \in[m+1]$, consider random subsets $A_{i} \subseteq F_{n}^{d}$ whose coordinates are chosen i.i.d. Bernoulli $(p)$, i.e., $\mathbb{P}\left(\mathbf{v} \in A_{i}\right)=p$ for all $\mathbf{v} \in F_{n}^{d}$, independently of $F_{n}^{d} \backslash\{\mathbf{v}\}$. Define $X_{-1} \triangleq \varnothing$, and for all $i \in[m+1]$, define

$$
X_{i} \triangleq X_{i-i} \cup A_{i}
$$

Thus, $\mathbb{P}\left(\mathbf{v} \in X_{i}\right)=p_{i}$ for all $\mathbf{v} \in F_{n}^{d}$, independently of $F_{n}^{d} \backslash\{\mathbf{v}\}$. We contend that for large enough $d$, the claims hold.
First, it is clear that $\mathbb{P}\left(X_{i} \subseteq X_{i+1}\right)=1$ for $i \in[m+1]$ by construction. Second, we have

$$
\mathbb{P}\left(\left|X_{m}\right| \leqslant \epsilon\left|F_{n}^{d}\right|\right) \geqslant \mathbb{P}\left(\left|X_{m}\right|<2 p_{m}\left|F_{n}^{d}\right|\right) \geqslant 1-e^{-2 p_{m}^{2} n^{d}}
$$

where the last inequality follows from Hoeffding's inequality. Since the right-hand side approaches 1 when $n \geqslant 2$ and $d \rightarrow \infty$, claim 2 holds for large enough $d$.

We now address claim 3. Fix $t \in[m]$ and consider $A_{t+1}$. For a coordinate $\mathbf{v} \in F_{n}^{d}$, denote by $D(t, \mathbf{v})$ the set

$$
D(t, \mathbf{v}) \triangleq\left\{i \in[d]: \mathbf{v}+[-(j+1)] \mathbf{e}_{i} \subseteq A_{t+1}\right\}
$$

If $D(t, \mathbf{v}) \neq \varnothing$ then draw $I_{t, \mathbf{v}}$ uniformly from $D(t, \mathbf{v})$. Otherwise, draw $I_{t, \mathbf{v}}$ uniformly from [d]. Note that $I_{t, \mathbf{v}}$ is distributed uniformly on $[d]$ since the distribution of $A_{t+1}$ is invariant under coordinate permutation. Since the coordinates in $A_{t+1}$ are chosen independently of $A_{t}, A_{t-1}, \ldots, A_{0}$ we obtain that $I_{t, \text { v }}$ is independent of $X_{t}$. Finally, we have

$$
\mathbb{P}\left(X_{t} \cup\left([-(j+1)] \mathbf{e}_{I_{t, \mathbf{v}}}+\mathbf{v}\right) \subseteq X_{t+1} \mid X_{t}\right) \geqslant \mathbb{P}(D(t, \mathbf{v}) \neq \varnothing)=1-\left(1-p^{j+1}\right)^{d}
$$

Since the right-hand side approaches 1 as $d \rightarrow \infty$, claim 3 holds for large enough $d$.
If $X$ is a random variable over some probability space, we use $\mathbb{P}_{X}$ to denote its distribution. Let $X_{0}, \ldots, X_{k-1}$ be random variables over the same probability space $\left(\mathcal{X}, 2^{\mathcal{X}}, \mathbb{P}\right)$. We denote by $\left(X_{0}, \ldots, X_{k-1}\right)$ the vector distributed according to their joint probability, $\mathbb{P}_{X_{0}, \ldots, X_{k-1}}$, and denote by $\left(X_{0} \times \cdots \times X_{k-1}\right)$ the vector distributed according to their product probability, i.e., $\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}} \triangleq \prod_{i \in[k]} \mathbb{P}_{X_{i}}$.

Lemma 30. Let $\mathcal{X}$ be a finite set, and $X_{0}, \ldots, X_{k-1}$ be $k$ random variables defined over the same probability space $\left(\mathcal{X}, 2^{\mathcal{X}}, \mathbb{P}\right)$. Then

$$
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V} \leqslant \sum_{i=0}^{k-2} E_{X_{0}, \ldots, X_{i}}\left[\left\|\mathbb{P}_{X_{i+1} \mid X_{0}, \ldots, X_{i}}-\mathbb{P}_{X_{i+1}}\right\|_{T V}\right]
$$

Proof: We prove this by induction on $k$. The case of $k=1$ is trivially true. In the base case of $k=2$ we have,

$$
\begin{align*}
\left\|\mathbb{P}_{X_{0}, X_{1}}-\mathbb{P}_{X_{0} \times X_{1}}\right\|_{T V} & =\frac{1}{2} \sum_{x_{0}, x_{1}}\left|\mathbb{P}_{X_{0}, X_{1}}\left(x_{0}, x_{1}\right)-\mathbb{P}_{X_{0}}\left(x_{0}\right) \mathbb{P}_{X_{1}}\left(x_{1}\right)\right|  \tag{9}\\
& =\frac{1}{2} \sum_{x_{0}, x_{1}}\left|\mathbb{P}_{X_{0}}\left(x_{0}\right) \mathbb{P}_{X_{1} \mid X_{0}}\left(x_{1} \mid x_{0}\right)-\mathbb{P}_{X_{0}}\left(x_{0}\right) \mathbb{P}_{X_{1}}\left(x_{1}\right)\right|
\end{align*}
$$

where the sum of $x_{0}$ and $x_{1}$ is over the support of $X_{0}$ and $X_{1}$, respectively. Since $\mathbb{P}_{X_{0}}\left(x_{0}\right) \geqslant 0$ we have

$$
\begin{equation*}
\frac{1}{2} \sum_{x_{0}, x_{1} \in \mathcal{X}}\left|\mathbb{P}_{X_{0}}\left(x_{0}\right) \mathbb{P}_{X_{1} \mid X_{0}}\left(x_{1} \mid x_{0}\right)-\mathbb{P}_{X_{0}}\left(x_{0}\right) \mathbb{P}_{X_{1}}\left(x_{1}\right)\right|=\sum_{x_{0} \in \mathcal{X}} \mathbb{P}_{X_{0}}\left(x_{0}\right)\left(\frac{1}{2} \sum_{x_{1} \in \mathcal{X}}\left|\mathbb{P}_{X_{1} \mid X_{0}}\left(x_{1} \mid x_{0}\right)-\mathbb{P}_{X_{1}}\left(x_{1}\right)\right|\right) \tag{10}
\end{equation*}
$$

Combining (9) and (10) and using the total variation distance definition we obtain

$$
\left\|\mathbb{P}_{X_{0}, X_{1}}-\mathbb{P}_{X_{0} \times X_{1}}\right\|_{T V}=E_{X_{0}}\left[\left\|\mathbb{P}_{X_{1} \mid X_{0}}-\mathbb{P}_{X_{1}}\right\|_{T V}\right]
$$

Now assume the statement is correct for $k-1$ random variables and we show it is correct for $k$ random variables. We write

$$
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V}=\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}+\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V}
$$

By applying the triangle inequality we obtain

$$
\begin{equation*}
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V} \leqslant\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}\right\|_{T V}+\left\|\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V} \tag{11}
\end{equation*}
$$

Considering $Y=\left(X_{0}, \ldots, X_{k-2}\right)$ as a tuple-valued radom variable, and applying the case $k=2$ on the pair of random variables ( $Y, X_{k-1}$ ) we have:

$$
\begin{equation*}
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}-\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}\right\|_{T V} \leqslant E_{X_{0}, \ldots, X_{k-2}}\left[\left\|\mathbb{P}_{X_{k-1 \mid\left(X_{0}, \ldots, X_{k-2}\right)}}-\mathbb{P}_{X_{k-1}}\right\|_{T V}\right] \tag{12}
\end{equation*}
$$

It is easy to check that

$$
\begin{equation*}
\left\|\mathbb{P}_{\left(X_{0}, \ldots, X_{k-2}\right) \times X_{k-1}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}\right\|_{T V}=\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-2}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-2}}\right\|_{T V} \tag{13}
\end{equation*}
$$

By the induction hypothesis we have

$$
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-2}}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-2}}\right\|_{T V} \leqslant \sum_{i=0}^{k-3} E_{X_{0}, \ldots, X_{i}}\left[\left\|\mathbb{P}_{X_{i+1} \mid X_{0}, \ldots, X_{i}}-\mathbb{P}_{X_{i+1}}\right\|_{T V}\right]
$$

Combining this with (11), (12) and (13) completes the proof.
For $A \subseteq F_{n}^{d}$, let $\mathcal{F}_{A} \subseteq 2^{\Sigma^{F_{n}^{d}}}$ denote the $\sigma$-algebra generated by the coordinates in $A$, namely,

$$
\mathcal{F}_{A} \triangleq\left\{\left\{x \in \Sigma^{F_{n}^{d}}: \pi_{A}(x) \in W\right\}: W \subseteq \Sigma^{A}\right\}
$$

Definition 31. Let $d, k, n \in \mathbb{N}, A \subseteq F_{n}^{d}$, and let $y \in \Sigma^{F_{n}^{d}}$. For a one-dimensional $S C S, \Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$, and its $d$-axial-product $S C S$, $\Gamma^{\otimes d}$, we define the following:

$$
\begin{aligned}
& \mu^{n, d} \text { is the uniform measure over } \mathcal{B}_{n}\left(\Gamma^{\otimes d}\right) \\
& \mu_{y, A} \triangleq \mu^{n, d}\left(\cdot \mid \mathcal{F}_{A}\right)(y) \\
& \eta_{y, A} \triangleq \prod_{\mathbf{v} \in F_{n}^{d}} \pi_{\{\mathbf{v}\}}\left(\mu_{y, A}\right)
\end{aligned}
$$

In other words, $\mu_{y, A}$ is the uniform distribution on $\mathcal{B}_{n}\left(\Gamma^{\otimes d}\right)$ given whose positions in $A$ agree with $y_{A}$. Moreover, $\eta_{y, A}$ is the independent version of $\mu_{y, A}$. The following statement is a particular application of Lemma 30 above.
Lemma 32. For every $d, n \in \mathbb{N}, i \in[d]$, and $A \subseteq F_{n}^{d}$, we have

$$
\sum_{\mathbf{v} \in F_{n}^{d}} E\left[\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)\right\|_{T V}\right] \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{j \in[k]} E\left[\left\|\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}\right)-\pi_{\{\mathbf{v}\}}\left(\mu_{y,\left(A \cup\left([-j] \mathbf{e}_{i}+\mathbf{v}\right)\right.}\right)\right\|_{T V}\right]
$$

Proof: First note that if $k=1$ the result is immediate since all the summands on the left-hand side are 0 . We now examine the case of $k \geqslant 2$. For the time being, let us fix $\mathbf{v} \in F_{n}^{d}$ and $y \in \Sigma^{F_{n}^{d}}$. We define the random variables $X_{j}, j \in[k]$,
where $X_{0}, \ldots, X_{k-1}$ is distributed according to $\mathbb{P}_{X_{0}, \ldots, X_{k-1}}^{y} \triangleq \pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)$. In particular, each $X_{j}$ is distributed according to $\mathbb{P}_{X_{j}}^{y} \triangleq \pi_{\mathbf{j}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)=\pi_{\mathbf{j}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)$. Additionally, $\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}^{y}=\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)$. We use the superscript $y$ to emphasize that these distributions depend $y$. Also for $z \in \Sigma^{F_{n}^{d}}$ such that $z_{A}=y_{A}$, the conditional probability $\mathbb{P}_{X_{j+1} \mid X_{0}, \ldots, X_{j}}^{y}$ evaluated at $z$ is equal to the measure $\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{z, A \cup\left([j+1] \mathbf{e}_{i}+\mathbf{v}\right)}\right)$. By Lemma 30, we have

$$
\begin{equation*}
\left\|\mathbb{P}_{X_{0}, \ldots, X_{k-1}}^{y}-\mathbb{P}_{X_{0} \times \cdots \times X_{k-1}}^{y}\right\|_{T V} \leqslant \sum_{j=0}^{k-2} E\left[\left\|\mathbb{P}_{X_{j+1} \mid X_{0}, \ldots, X_{j}}^{y}-\mathbb{P}_{X_{i+1}}^{y}\right\|_{T V}\right] \tag{14}
\end{equation*}
$$

The expectations in the right-hand side are with respect to the conditioning on the random variables $X_{0}, \ldots, X_{j}$. We can rewrite the above equation as follows:

$$
\begin{equation*}
\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V} \leqslant \sum_{j=0}^{k-2} \int\left\|\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{z, A \cup\left([j+1] \mathbf{e}_{i}+\mathbf{v}\right)}\right)-\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{z, A}\right)\right\|_{T V} \mathrm{~d} \mu_{y, A}(z) \tag{15}
\end{equation*}
$$

Integrating the above inequality over $y$ with respect to $\mu^{n, d}$ we have:

$$
\begin{aligned}
& \int\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V} \mathrm{~d} \mu^{n, d}(y) \\
& \quad \leqslant \sum_{j=0}^{k-2} \iint\left\|\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{\left.z, A \cup(j+1] \mathbf{e}_{i}+\mathbf{v}\right)}\right)-\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{z, A}\right)\right\|_{T V} \mathrm{~d} \mu_{y, A}(z) \mathrm{d} \mu^{n, d}(y) .
\end{aligned}
$$

By definition of $\mu_{y, A}$ as the conditional measure, for every $f: \Sigma^{F_{n}^{d}} \rightarrow \mathbb{R}$ we have

$$
\iint f(z) d \mu_{y, A}(z) \mathrm{d} \mu^{n, d}(y)=\int f(y) \mathrm{d} \mu^{n, d}(y) .
$$

Writing the integeral with repect to $\mu^{n, d}$ as $E[\cdot]$, we thus have

$$
E\left[\left\|\pi_{\left[k \mathbf{e}_{i}+\mathbf{v}\right.}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right] \leqslant \sum_{j=0}^{k-2} E\left[\left\|\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A \cup\left([j+1] \mathbf{e}_{i}+\mathbf{v}\right)}\right)-\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right]
$$

Summing over all $\mathbf{v} \in F_{n}^{d}$ we obtain

$$
\begin{equation*}
\sum_{\mathbf{v} \in F_{n}^{d}} E\left[\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right] \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{j=0}^{k-2} E\left[\left\|\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A \cup\left([j+1] \mathbf{e}_{i}+\mathbf{v}\right)}\right)-\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right] \tag{16}
\end{equation*}
$$

Recall that $[-j] \triangleq\{-1, \ldots,-j\}$, hence

$$
[j+1] \mathbf{e}_{i}=(j+1) \mathbf{e}_{i}+[-(j+1)] \mathbf{e}_{i} .
$$

Thus, (16) can be written as

$$
\begin{align*}
& \sum_{\mathbf{v} \in F_{n}^{d}} E\left[\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right]  \tag{17}\\
& \quad \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{j=0}^{k-2} E\left[\left\|\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A \cup\left((j+1) \mathbf{e}_{i}+\mathbf{v}+\left[-(j+1) \mathbf{e}_{i}\right)\right.}\right)-\pi_{(j+1) \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right]
\end{align*}
$$

Since we are summing over all $\mathbf{v} \in F_{n}^{d}$, and since coordinates are taken modulo $n$, we may write 17 as follows,

$$
\begin{equation*}
\sum_{\mathbf{v} \in F_{n}^{d}} E\left[\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)\right\|_{T V}\right] \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{j=0}^{k-2} E\left[\left\|\pi_{\{\mathbf{v}\}}\left(\mu_{y, A \cup\left(\mathbf{v}+[-(j+1)] \mathbf{e}_{i}\right)}\right)-\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}\right)\right\|_{T V}\right] \tag{18}
\end{equation*}
$$

Since the total variation distance is non-negative, (18) implies the lemma.
The following proposition, which is used to prove the main result of this section, considers the following scenario. Assume $y \in \Sigma^{F_{n}^{d}}$ is randomly drawn using the measure $\mu^{n, d}$, i.e., it is drawn uniformly at random from the set of admissible words $\mathcal{B}_{n}\left(\Gamma^{\otimes d}\right)$. We then study the random variable $\eta_{y, A}$ (a measure in itself), and ask what is the probability that it resides within the set of measures $\overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}(\Gamma)\right)^{\boxtimes d}\right)$. For convex SCSs, we prove this probability is $\epsilon$-close to 1 , assuming $d$ is sufficiently large.

Proposition 33. Let $k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a convex SCS. For any $\epsilon>0$, there exists $d_{0} \in \mathbb{N}$, such that for all $d \in \mathbb{N}$, $d \geqslant d_{0}, n \in \mathbb{N}, n \geqslant k+2$, there exists $A \subseteq F_{n}^{d},|A| \leqslant \epsilon n^{d}$, such that for $y \in \Sigma^{F_{n}^{d}}$ drawn randomly using the measure $\mu^{n, d}$,

$$
\mu^{n, d}\left(\eta_{y, A} \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}(\Gamma)\right)^{\boxtimes d}\right)\right) \geqslant 1-\epsilon
$$

Proof: Recall that by Definition 31, $\eta_{y, A}$ is a product measure, while $\mu_{y, A}$ is not necessarily so. Additionally, we contend that $\bar{\pi}_{[k] \mathbf{e}_{i}}\left(\mu_{y, A}\right) \in \Gamma$ for all $y \in \mathcal{B}_{n}\left(\bar{\Gamma}^{\otimes d}\right), A \subseteq F_{n}^{d}$ and $i \in[d]$. Indeed,

$$
\begin{aligned}
\bar{\pi}_{[k] \mathbf{e}_{i}}\left(\mu_{y, A}\right) & =\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right) \\
& =\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \frac{1}{\left|\pi_{A}^{-1}\left(y_{A}\right)\right|} \sum_{x \in \pi_{A}^{-1}\left(y_{A}\right)} \pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\delta_{\hat{x}}\right) \\
& =\frac{1}{\left|\pi_{A}^{-1}\left(y_{A}\right)\right|} \sum_{x \in \pi_{A}^{-1}\left(y_{A}\right)} \frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\delta_{\hat{x}}\right) \\
& =\frac{1}{\left|\pi_{A}^{-1}\left(y_{A}\right)\right|} \sum_{x \in \pi_{A}^{-1}\left(y_{A}\right)} \pi_{[k] \mathbf{e}_{i}}\left(\frac{1}{\left|F_{n}^{d}\right|} \sum_{\mathbf{v} \in F_{n}^{d}} \delta_{\sigma_{\mathbf{v}}(\hat{x})}\right) \\
& =\frac{1}{\left|\pi_{A}^{-1}\left(y_{A}\right)\right|} \sum_{x \in \pi_{A}^{-1}\left(y_{A}\right)} \mathrm{fr}_{x}^{[k] \mathbf{e}_{i}},
\end{aligned}
$$

where we recall that $\pi_{A}^{-1}\left(y_{A}\right)=\left\{x \in \mathcal{B}_{n}\left(\Gamma^{\otimes d}\right): x_{A}=y_{A}\right\}$. Since $\operatorname{fr}_{x}^{[k] e_{i}} \in \Gamma$ for every $x \in \pi_{A}^{-1}\left(y_{A}\right)$ and since $\Gamma$ is convex the contention is proved. Additionally, by the convexity of $\Gamma, \bar{\pi}_{[k] \mathbf{e}_{i}}\left(\mu_{y, A}\right) \in \Gamma$ implies

$$
\frac{1}{d} \sum_{i \in[d]} \bar{\pi}_{[k] \mathbf{e}_{i}}\left(\mu_{y, A}\right) \in \Gamma
$$

Draw $y \in \Sigma^{F_{n}^{d}}$ randomly using the measure $\mu^{n, d}$. For any $A \subseteq F_{n}^{d}$, let us denote

$$
D_{A, y} \triangleq\left\|\frac{1}{d} \sum_{i \in[d]} \bar{\pi}_{[k] \mathbf{e}_{i}}\left(\eta_{y, A}\right)-\frac{1}{d} \sum_{i \in[d]} \bar{\pi}_{[k] \mathbf{e}_{i}}\left(\mu_{y, A}\right)\right\|_{T V}
$$

We will use $E_{y}[\cdot]$ to denote expectation with respect to the random variable $y$ which is randomly drawn using the measure $\mu^{n, d}$. Denote

$$
D_{A} \triangleq E_{y}\left[D_{A, y}\right]
$$

To prove the theorem, it suffices to show that for any $\epsilon>0$, if $d$ is large enough there exists $A \subseteq F_{n}^{d},|A| \leqslant \epsilon n^{d}$, and with probability at least $1-\epsilon$ (with respect to $\mu^{n, d}$ ) we have $D_{A, y} \leqslant \epsilon$. By a standard application of the Markov inequality, it is sufficient to show that (under the above conditions) $D_{A} \leqslant \epsilon^{2}$.

By definition, for any $A \subseteq F_{n}^{d}$ and any $y \in \Sigma^{F_{n}^{d}}$ we have

$$
D_{A, y}=\left\|\frac{1}{d\left|F_{n}^{d}\right|} \sum_{i \in[d]} \sum_{\mathbf{v} \in F_{n}^{d}}\left(\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)\right)\right\|_{T V}
$$

Applying the triangle inequality we obtain

$$
D_{A, y} \leqslant \frac{1}{d\left|F_{n}^{d}\right|} \sum_{i \in[d]} \sum_{\mathbf{v} \in F_{n}^{d}}\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)\right\|_{T V}
$$

Taking the expectation, $E_{y}$, on both sides and using its linearity we get

$$
D_{A} \leqslant \frac{1}{d\left|F_{n}^{d}\right|} \sum_{i \in[d]} \sum_{\mathbf{v} \in F_{n}^{d}} E_{y}\left[\left\|\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\eta_{y, A}\right)-\pi_{[k] \mathbf{e}_{i}+\mathbf{v}}\left(\mu_{y, A}\right)\right\|_{T V}\right]
$$

By Lemma 32 and the linearity of the expectation we obtain

$$
D_{A} \leqslant \sum_{j \in[k]} \frac{1}{d\left|F_{n}^{d}\right|} \sum_{i \in[d]} \sum_{\mathbf{v} \in F_{n}^{d}} E_{y}\left[\left\|\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}\right)-\pi_{\{\mathbf{v}\}}\left(\mu_{y, A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right)\right\|_{T V}\right]
$$

Consider another random variable $x \in \Sigma^{F_{n}^{d}}$, also randomly drawn using the measure $\mu^{n, d}$. Now define $f: \Sigma^{F_{n}^{d}} \rightarrow\{0,1\}_{n}^{F_{n}^{d} \times \Sigma}$ by

$$
f(x)_{(\mathbf{v}, a)} \triangleq \begin{cases}1 & x_{\mathbf{v}}=a \\ 0 & \text { otherwise }\end{cases}
$$

for all $\mathbf{v} \in F_{n}^{d}$ and $a \in \Sigma$. Thus, by definition we have that

$$
\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}\right)(a)=\pi_{\{\mathbf{v}\}}\left(\mu_{y, A}\right)(a)=E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)
$$

Since $\Sigma$ is finite we can write the total variation distance as a sum, and then apply the triangle inequality, which results in

$$
\begin{equation*}
D_{A} \leqslant \frac{1}{2} \sum_{j \in[k]} \frac{1}{d\left|F_{n}^{d}\right|} \sum_{i \in[d]} \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right] . \tag{19}
\end{equation*}
$$

For any $j \in[k]$, viewing the expression

$$
\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right]
$$

as an inner product of a vector in $\mathbb{R}^{F_{n}^{d} \times \Sigma}$ whose $(\mathbf{v}, a)^{\prime}$ th coordinate is equal to

$$
E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right]
$$

and 1, we may apply Cauchy-Schwarz (C.S) inequality and obtain

$$
\begin{align*}
& \sum_{(\mathbf{v}, a) \in F_{n}^{d} \times \Sigma} E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right] \cdot 1  \tag{20}\\
& \stackrel{\text { C.S }}{\leqslant} \sqrt{\left(\sum_{(\mathbf{v}, a) \in F_{n}^{d} \times \Sigma}\left(E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right]\right)^{2}\right)\left(\sum_{(\mathbf{v}, a) \in F_{n}^{d} \times \Sigma} 1^{2}\right)} \\
&= \sqrt{\left(\sum_{(\mathbf{v}, a) \in F_{n}^{d} \times \Sigma}\left(E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right]\right)^{2}\right) \cdot\left|F_{n}^{d}\right| \cdot|\Sigma|}
\end{align*}
$$

Thus, combining (19) and we have

$$
D_{A} \leqslant \frac{\sqrt{|\Sigma|}}{2 d \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \sum_{i \in[d]} \sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma}\left(E_{y}\left[\left|E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right|\right]\right)^{2}}
$$

Using the fact that $(E[|X|])^{2} \leqslant E\left[X^{2}\right]$ (again, by C.S), we have

$$
\begin{equation*}
D_{A} \leqslant \frac{\sqrt{|\Sigma|}}{2 d \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \sum_{i \in[d]} \sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma}\left(E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{A \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right)^{2}\right]\right)} \tag{21}
\end{equation*}
$$

Choose $m$ large enough such that $\frac{1}{\sqrt{m}} \leqslant \frac{\epsilon^{2}}{k \sqrt{|\Sigma|}}$ and denote $\epsilon_{0}=\frac{\epsilon^{2}}{k|\Sigma|}$. Now let $\mathbb{P}, I_{t, \mathbf{v}}, X_{0}, X_{1}, \ldots, X_{m}$ be as given by Lemma 29 with $n>k+2$ and with $\epsilon_{0}$ and obtain $d_{0}$. From here on, assume $d \geqslant d_{0}$. Let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. First, from 21] we may bound $D_{X_{t}}$, for any $t \in[m+1]$, by

$$
\begin{equation*}
D_{X_{t}} \leqslant \frac{\sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \frac{1}{d} \sum_{i \in[d]} \sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t} \cup\left([-(j+1)] \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right)^{2}\right]} \tag{22}
\end{equation*}
$$

By the properties of $X_{t}$ and $X_{t+1}$ given in Lemma 29 for every $\mathbf{v} \in F_{n}^{d}$ there is a random variable $I_{t, \mathbf{v}}$ independent of $X_{t}$ and distributed uniformly on $[d]$ so that $\mathbb{P}\left(X_{t} \cup\left([-(j+1)] \mathbf{e}_{I_{t, \mathbf{v}}}+\mathbf{v}\right) \subseteq X_{t+1} \mid X_{t}\right) \geqslant 1-\epsilon_{0}$. Denote

$$
X_{t, \mathbf{v}} \triangleq X_{t} \cup\left([-(j+1)] \mathbf{e}_{I_{t, \mathbf{v}}}+\mathbf{v}\right)
$$

Since $I_{t, \mathrm{v}}$ is independent of $X_{t}$ we have

$$
\begin{align*}
& \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t, \mathbf{v}}}\right](y)\right)^{2}\right]} \mid X_{t}\right]  \tag{23}\\
& =\frac{1}{d} \sum_{i \in[d]} \sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t} \cup\left([-(j+1)) \mathbf{e}_{i}+\mathbf{v}\right)}\right](y)\right)^{2}\right]} .
\end{align*}
$$

From (22) and (23) we obtain

$$
\begin{equation*}
\left.D_{X_{t}} \leqslant \frac{\sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a) \mid} \mathcal{F}_{X_{t, \mathbf{v}}}\right](y)\right)^{2}\right]}\right] X_{t}\right] \tag{24}
\end{equation*}
$$

Since we may view $E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right]$ as the orthogonal projection of $f(x)_{(\mathbf{v}, a)}$ on $L^{2}\left(F_{n}^{d}, \mathcal{F}_{X_{t}}, \mu^{n, d}, \mathbb{R}\right)$, if $X_{t, \mathbf{v}} \subseteq X_{t+1}$ we have

$$
\begin{aligned}
& E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t, \mathbf{v}}}\right](y)\right)^{2}\right] \\
& \leqslant E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t+1}}\right](y)\right)^{2}\right]
\end{aligned}
$$

Otherwise, if $X_{t, \mathbf{v}} \nsubseteq X_{t+1}$, then

$$
E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t, \mathbf{v}}}\right](y)\right)^{2}\right] \leqslant 1
$$

By the properties of $X_{t}$ given in Lemma 29, we have that

$$
\mathbb{P}\left(X_{t, \mathbf{v}} \subseteq X_{t+1} \mid X_{t}\right) \geqslant 1-\epsilon_{0}
$$

Thus,

$$
\begin{align*}
& \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{\left.X_{t, \mathbf{v}}\right]}(y)\right)^{2}\right]\right.} \mid X_{t}\right]  \tag{25}\\
& \leqslant\left(1-\epsilon_{0}\right) \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{\left.X_{t+1}\right]}\right](y)\right)^{2}\right]} \mid X_{t}\right]+\epsilon_{0} \sqrt{|\Sigma|\left|F_{n}^{d}\right|} \\
& \left.\leqslant \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{\left.X_{t+1}\right]}\right](y)\right)^{2}\right]}\right] X_{t}\right]+\epsilon_{0} \sqrt{|\Sigma|\left|F_{n}^{d}\right|}
\end{align*}
$$

From (24) and (25) we obtain

$$
\begin{align*}
D_{X_{t}} \leqslant & \left.\frac{\sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t+1}}\right](y)\right)^{2}\right]}\right] X_{t}\right]  \tag{26}\\
& +\frac{\sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \sum_{j \in[k]} \epsilon_{0} \sqrt{|\Sigma| \mid F_{n}^{d \mid}} \\
= & \frac{k \sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \mathbb{E}\left[\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t+1}}\right](y)\right)^{2}\right]} \mid X_{t}\right]+\frac{k|\Sigma| \epsilon_{0}}{2} .
\end{align*}
$$

Observe that viewing $f$ as a random variable with respect to $\mu^{n, d}$ we have $\|f\|_{2}=\sqrt{\left|F_{n}^{d}\right|}$. Note also that

$$
\begin{equation*}
\left\|E\left[f \mid \mathcal{F}_{X_{t}}\right]-E\left[f \mid \mathcal{F}_{X_{t+1}}\right]\right\|_{2}=\sqrt{\sum_{\mathbf{v} \in F_{n}^{d}} \sum_{a \in \Sigma} E_{y}\left[\left(E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t}}\right](y)-E_{x}\left[f(x)_{(\mathbf{v}, a)} \mid \mathcal{F}_{X_{t+1}}\right](y)\right)^{2}\right]} \tag{27}
\end{equation*}
$$

From 26) and 27] we obtain that for every $t \in[m]$,

$$
\begin{equation*}
D_{X_{t}} \leqslant \frac{k \sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \mathbb{E}\left[\left\|E\left[f \mid \mathcal{F}_{X_{t}}\right]-E\left[f \mid \mathcal{F}_{X_{t+1}}\right]\right\|_{2} \mid X_{t}\right]+\frac{k|\Sigma| \epsilon_{0}}{2} \tag{28}
\end{equation*}
$$

Note that the probability that a random variable is greater or equal to its expectation is always strictly positive. Because $X_{t}$ takes only finitely many values, this means that for every $t \in[m]$, for every realization of $X_{t}$, denoted as $\chi_{t}$, there exists a realization of $X_{t+1}$, denoted as $\chi_{t+1}=\chi_{t+1}\left(\chi_{t}\right)$ such that $\mathbb{P}\left(X_{t+1}=\chi_{t+1} \mid X_{t}\right)>0$ and

$$
\mathbb{E}\left[\left\|E\left[f \mid \mathcal{F}_{X_{t}}\right]-E\left[f \mid \mathcal{F}_{X_{t+1}}\right]\right\|_{2} \mid X_{t}\right] \leqslant\left\|E\left[f \mid \mathcal{F}_{\chi_{t}}\right]-E\left[f \mid \mathcal{F}_{\chi_{t+1}}\right]\right\|_{2}
$$

Together with 28 we obtain

$$
\begin{equation*}
D_{\chi_{t}} \leqslant \frac{k \sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}}\left\|E\left[f \mid \mathcal{F}_{\chi_{t}}\right]-E\left[f \mid \mathcal{F}_{\chi_{t+1}}\right]\right\|_{2}+\frac{k|\Sigma| \epsilon_{0}}{2} \tag{29}
\end{equation*}
$$

Since [29] holds for every $t$, we obtain that there exists a sequence $\left(\chi_{t}\right)_{t \in[m+1]}$ of realizations of $\left(X_{t}\right)_{t \in[m+1]}$ with positive probabilities, such that for every $t \in[m]$,

$$
\begin{equation*}
D_{\chi_{t}} \leqslant \frac{k \sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}}\left\|E\left[f \mid \mathcal{F}_{\chi_{t}}\right]-E\left[f \mid \mathcal{F}_{\chi_{t+1}}\right]\right\|_{2}+\frac{k|\Sigma| \epsilon_{0}}{2} \tag{30}
\end{equation*}
$$

From Lemma 28, there exists $t \in[m]$ such that

$$
\begin{equation*}
\left\|E\left[f \mid \mathcal{F}_{\chi_{t}}\right]-E\left[f \mid \mathcal{F}_{\chi_{t+1}}\right]\right\|_{2}^{2} \leqslant \frac{1}{m}\|f\|_{2}^{2} \tag{31}
\end{equation*}
$$

Combining 31 with we obtain that there exists $t \in[m]$ such that

$$
\begin{aligned}
D_{\chi_{t}} & \leqslant \frac{k \sqrt{|\Sigma|}}{2 \sqrt{\left|F_{n}^{d}\right|}} \frac{1}{\sqrt{m}}\|f\|_{2}+\frac{k|\Sigma| \epsilon_{0}}{2} \\
& =\frac{k \sqrt{|\Sigma|}}{2 \sqrt{m}}+\frac{k|\Sigma| \epsilon_{0}}{2}
\end{aligned}
$$

Taking $A=\chi_{t}$, and recalling our choice of $\epsilon_{0}=\frac{\epsilon^{2}}{k|\Sigma|}$ and $\frac{1}{\sqrt{m}} \leqslant \frac{\epsilon^{2}}{k \sqrt{|\Sigma|}}$, we obtain that

$$
D_{A} \leqslant \frac{k \sqrt{|\Sigma|}}{2 \sqrt{m}}+\frac{k|\Sigma| \epsilon_{0}}{2} \leqslant \epsilon^{2}
$$

which completes the proof.
We have reached the main result of this section. We show that capacity of a convex $d$-axial product is arbitrarily close to the independence entropy, as the dimension grows.
Theorem 34. Let $k \in \mathbb{N}$, and let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a convex one-dimensional SCS. Then

$$
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right)=h_{\text {ind }}(\Gamma)
$$

Proof: First note that $\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right) \geqslant h_{\text {ind }}(\Gamma)$ by applying Theorem 26 to $\Gamma^{\otimes d}$ for every $d$ and taking $d \rightarrow \infty$ on both sides. For the other direction, fix $\epsilon_{0}>0$ and choose

$$
0<\epsilon<\min \left\{\frac{\epsilon_{0}}{2 \log _{2}|\Sigma|}, 1\right\}, \quad 0<\delta<\frac{\epsilon}{2}
$$

Replace $\Gamma$ by $\mathbb{B}_{\delta}(\Gamma)$ in Definition 31 and denote the resulting measures by $\mu_{\delta}^{n, d}, \mu_{y, A}^{\delta}$, and $\eta_{y, A}^{\delta}$.
Recall that for a measure $\mu$ and a $\sigma$-algebra $\mathcal{F}$,

$$
\begin{equation*}
H(\mu \mid \mathcal{F}) \triangleq E[H(\mu(\cdot \mid \mathcal{F}))]=\int H(\mu(\cdot \mid \mathcal{F}))(x) \mathrm{d} \mu(x) \tag{32}
\end{equation*}
$$

In other words, $H(\mu \mid \mathcal{F})$ is the expected entropy of the conditional measure $\mu(\cdot \mid \mathcal{F})$. Also recall that for $A \subseteq F_{n}^{d}, \pi_{A}\left(\mu_{\delta}^{n, d}\right)$ denotes the $A$-marginal of $\mu_{\delta}^{n, d}$, and that $\mathcal{F}_{A}$ denotes the $\sigma$-algebra generated by the coordinates in $A$. We have that

$$
H\left(\mu_{\delta}^{n, d}\right)=H\left(\pi_{A}\left(\mu_{\delta}^{n, d}\right)\right)+H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right)
$$

By Proposition 33, for any $n \in \mathbb{N}, n \geqslant k+2$, there exists $d_{0} \in \mathbb{N}$, such that for every $d \geqslant d_{0}$, there exists $A \subseteq \Sigma^{F_{n}^{d}}$, $|A| \leqslant \epsilon n^{d}$, such that,

$$
\mu_{\delta}^{n, d}\left(\eta_{y, A}^{\delta} \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\boxtimes d}\right)\right) \geqslant 1-\epsilon>0 .
$$

In particular, there exists a word $y \in \Sigma^{F_{n}^{d}}$ such that $\eta_{y, A} \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\boxtimes d}\right)$. Since clearly

$$
H\left(\pi_{A}\left(\mu_{\delta}^{n, d}\right)\right) \leqslant \log _{2}\left|\Sigma^{A}\right|,
$$

by combining the above we have

$$
\begin{equation*}
H\left(\mu_{\delta}^{n, d}\right) \leqslant H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right)+\epsilon n^{d} \log _{2}|\Sigma| . \tag{33}
\end{equation*}
$$

Because the joint entropy of a finite set of random variables is bounded from above by the sum of their entropies (and the same statement holds for conditional entropy), we have:

$$
H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right) \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} H\left(\pi_{\{\mathbf{v}\}}\left(\mu_{\delta}^{n, d}\right) \mid \mathcal{F}_{A}\right) .
$$

By definition of the random measure $\eta_{y, A}^{\delta}$ and from 32], we have

$$
H\left(\pi_{\{\mathbf{v}\}}\left(\mu_{\delta}^{n, d}\right) \mid \mathcal{F}_{A}\right)=\sum_{y \in \Sigma^{d_{n}^{d}}} H\left(\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}^{\delta}\right)\right) \mu_{\delta}^{n, d}(y)
$$

Thus,

$$
H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right) \leqslant \sum_{\mathbf{v} \in F_{n}^{d}} \sum_{y \in \Sigma^{E_{n}^{d}}} H\left(\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}^{\delta}\right)\right) \mu_{\delta}^{n, d}(y) .
$$

Now, since $\eta_{y, A}$ is a product measure, we have

$$
H\left(\eta_{y, A}^{\delta}\right)=\sum_{\mathbf{v} \in F_{n}^{d}} H\left(\pi_{\{\mathbf{v}\}}\left(\eta_{y, A}^{\delta}\right)\right) .
$$

It follows that,

$$
\begin{equation*}
H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right) \leqslant \sum_{y \in \Sigma^{F_{n}^{d}}} H\left(\eta_{y, A}^{\delta}\right) \mu_{\delta}^{n, d}(y) . \tag{34}
\end{equation*}
$$

Let us conveniently use $p$ to denote the value

$$
p \triangleq \mu_{\delta}^{n, d}\left(\eta_{y, A}^{\delta} \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\boxtimes d}\right)\right),
$$

and recall that $p \geqslant 1-\epsilon>0$. Then

$$
\begin{equation*}
\sum_{y \in \Sigma^{E_{n}^{d}}} H\left(\eta_{y, A}^{\delta}\right) \mu_{\delta}^{n, d}(y) \leqslant p \cdot \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{e}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\otimes d}\right)} H(\eta)+(1-p) \cdot \log _{2}\left|\Sigma^{F_{n}^{d}}\right| . \tag{35}
\end{equation*}
$$

Using the fact that $p \geqslant 1-\epsilon>0$ combined with (34) and (35), it follows that

$$
\begin{align*}
H\left(\mu_{\delta}^{n, d} \mid \mathcal{F}_{A}\right) & \leqslant p \cdot \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{e}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\otimes d}\right)} H(\eta)+(1-p) n^{d} \cdot \log _{2}|\Sigma|  \tag{3}\\
& \leqslant \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{e}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\mathbb{X d}}\right)} H(\eta)+\epsilon n^{d} \log _{2}|\Sigma| .
\end{align*}
$$

Combining (36) with (33) we obtain

$$
\frac{1}{n^{d}} H\left(\mu_{\delta}^{n, d}\right) \leqslant \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\mathbb{} d}\right)} H(\eta)+2 \epsilon \log _{2}|\Sigma| .
$$

By our choice of $\epsilon$, we have $\epsilon+\delta \leqslant \epsilon_{0}$, hence $\left(\mathbb{B}_{\epsilon}\left(\mathbb{B}_{\delta}(\Gamma)\right)\right)^{\otimes d} \subseteq\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\otimes d}$, as well as

$$
\begin{equation*}
\frac{1}{n^{d}} H\left(\mu_{\delta}^{n, d}\right) \leqslant \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\varepsilon_{0}}(\Gamma)\right)^{\mathbb{} d}\right)} H(\eta)+\epsilon_{0} . \tag{37}
\end{equation*}
$$

Since $\mu_{\delta}^{n, d}$ is the uniform measure on $\mathcal{B}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\otimes d}\right)$,

$$
H\left(\mu_{\delta}^{n, d}\right)=\log _{2}\left|\mathcal{B}_{n}\left(\mathbb{B}_{\delta}(\Gamma)^{\otimes d}\right)\right| .
$$

Thus,

$$
\frac{1}{n^{d}} \log _{2}\left|\mathcal{B}_{n}\left(\left(\mathbb{B}_{\delta}(\Gamma)\right)^{\otimes d}\right)\right| \leqslant \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\otimes d}\right)} H(\eta)+\epsilon_{0}
$$

Taking $\lim \sup _{n \rightarrow \infty}$ we obtain

$$
\widehat{\operatorname{cap}}\left(\left(\mathbb{B}_{\delta}(\Gamma)\right)^{\otimes d}\right) \leqslant \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\otimes d}\right)} H(\eta)+\epsilon_{0}
$$

Since

$$
\mathbb{B}_{\delta}\left(\Gamma^{\otimes d}\right) \subseteq\left(\mathbb{B}_{\delta}(\Gamma)\right)^{\otimes d}
$$

we have

$$
\widehat{\operatorname{cap}}\left(\mathbb{B}_{\delta}\left((\Gamma)^{\otimes d}\right)\right) \leqslant \widehat{\operatorname{cap}}\left(\left(\mathbb{B}_{\delta}(\Gamma)\right)^{\otimes d}\right) \leqslant \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\otimes d}\right)} H(\eta)+\epsilon_{0}
$$

Taking $\lim _{\delta \rightarrow 0^{+}}$, we get

$$
\begin{equation*}
\operatorname{cap}\left(\Gamma^{\otimes d}\right) \leqslant \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\boxtimes d}\right)} H(\eta)+\epsilon_{0} . \tag{38}
\end{equation*}
$$

At this point we take a slight detour. For $\xi>0, \mathbb{B}_{\epsilon_{0}}(\Gamma) \subseteq \mathbb{B}_{\xi}\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)$ and hence we have

$$
\begin{aligned}
\limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\boxtimes d}\right)} H(\eta)+\epsilon_{0} & \leqslant \limsup _{\xi \rightarrow 0^{+}} \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\xi}\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)\right)^{\boxtimes d}\right)} H(\eta)+\epsilon_{0} \\
& \stackrel{(a)}{=} h_{\text {ind }}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)^{\boxtimes d}\right)+\epsilon_{0} \\
& \stackrel{(b)}{=} h_{\text {ind }}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)\right)+\epsilon_{0}
\end{aligned}
$$

where (a) follows by definition, and (b) follows by Lemma 27. Substituting this in 38 and taking $d \rightarrow \infty$ we obtain

$$
\begin{equation*}
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right) \leqslant h_{\text {ind }}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)\right)+\epsilon_{0} \tag{39}
\end{equation*}
$$

Note that since $\Gamma$ is convex we have that for $\epsilon_{1}>0, \mathbb{B}_{\epsilon_{1}}\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)=\mathbb{B}_{\epsilon_{1}+\epsilon_{0}}(\Gamma)$. Therefore, by the definition of limit we have

$$
\limsup _{\epsilon_{0} \rightarrow 0^{+}} \lim \operatorname{\epsilon }_{1} \rightarrow 0^{+} \sup _{n \rightarrow \infty} \operatorname{simsup}_{n \rightarrow \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\varepsilon_{1}}\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)\right)\right)} H(\eta)=\limsup _{\epsilon_{0} \rightarrow 0^{+}} \limsup _{n \rightarrow \infty} \frac{1}{n} \sup _{\eta \in \overline{\mathcal{P}}_{n}\left(\left(\mathbb{B}_{\epsilon_{0}}(\Gamma)\right)\right)} H(\eta)
$$

Therefore, taking the limit as $\epsilon_{0} \rightarrow 0$ in 39 we obtain

$$
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right) \leqslant h_{\mathrm{ind}}(\Gamma) .
$$

## VI. Discussion

Our initial motivation behind this work is to approximate the capacity of multidimensional SCSs using "meaningful" expressions. The main challenges were defining exactly what is the capacity of multidimensional SCSs, and obtaining the connections between the capacity and the independence entropy. Our approach, which uses the independence entropy, extends previous combinatorial works [19], [23], [26], which apply only to fully constrained systems. At the core of our results, for $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ and its axial product $\Gamma^{\otimes d}$, by Theorem 24 and Theorem 26 that

$$
h_{\mathrm{ind}}(\Gamma) \leqslant \operatorname{cap}\left(\Gamma^{\otimes d}\right)
$$

Thus, the problem of bounding the capacity of a $d$-dimensional axial-product SCS is simplified by having to consider only product measures, which are much easier to handle. Moreover, any number of dimensions $d$, may be reduced via this bound to the one-dimensional case. This bound is asymptotically tight, as together with Theorem 34 for convex $\Gamma$,

$$
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right)=h_{\text {ind }}(\Gamma)
$$

It also appears that the capacity cap, and independence entropy $h_{\text {ind }}$, are robust generalizations of their one-dimensional combinatorial counterparts.


Figure 1: The Hasse diagram for a general $d$-dimensional $\operatorname{SCS} \Gamma \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$, where (a) follows from (1), (b) follows from Theorem 26 and (c) follows from (2).


Figure 2: The Hasse diagram for a convex one-dimensional $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ and its $d$-axial-product $\operatorname{SCS} \Gamma^{\otimes d}$, where (a) and (d) follow from (1), (b) and (c) follow from Theorem 26, (e) follows from (2), and (f) follows from Lemma 23

The paper contains many connections between the various capacities and entropies. Figure 1 shows the Hasse diagram for the bounds pertaining to general $d$-dimensional $\Gamma \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$. In the case of a convex one-dimensional $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ and its $d$-axial-product SCS $\Gamma^{\otimes d}$, a more elaborate Hasse diagram emerges, which is shown in Figure 2
We note here that following the same arguments used in the proof of Theorem 24 would show that $\operatorname{cap}\left(\Gamma^{\otimes d}\right) \geqslant \operatorname{cap}\left(\Gamma^{\otimes d+1}\right)$ which means that in limsup $\sup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma^{\otimes d}\right)$ the limit actually exists and equals to $\inf _{d} \operatorname{cap}\left(\Gamma^{\otimes d}\right)$.

We would also like to compare our results, as they apply to a specific case study described in $[10]$. Let $\Gamma \subseteq \mathcal{P}\left(\Sigma^{k}\right)$ be a convex one-dimensional SCS, and recall that the axial product $\Gamma^{\otimes d}$ is defined as

$$
\Gamma^{\otimes d} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right): \forall i \in[d], \pi_{[k] \mathbf{e}_{i}}(\mu) \in \Gamma\right\},
$$

and thus

$$
\mathcal{B}_{n}\left(\Gamma^{\otimes d}\right)=\left\{w \in F_{n}^{d}: \forall i \in[d], \operatorname{fr}_{w}^{[k] e_{i}} \in \Gamma\right\} .
$$

The SCSs studied in [10] were an averaged version of the axial product, namely,

$$
\Gamma^{\boxtimes d} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right): \frac{1}{d} \sum_{i \in[d]} \pi_{[k] \mathbf{e}_{i}}(\mu) \in \Gamma\right\},
$$

and thus

$$
\mathcal{B}_{n}\left(\Gamma^{\boxtimes d}\right)=\left\{w \in F_{n}^{d}: \frac{1}{d} \sum_{i \in[d]} \operatorname{fr}_{w}^{[k] \mathrm{e}_{i}} \in \Gamma\right\} .
$$

By convexity, it easily follows that

$$
\mathcal{B}_{n}\left(\Gamma^{\otimes d}\right) \subseteq \mathcal{B}_{n}\left(\Gamma^{\boxtimes d}\right),
$$

and thus

$$
\operatorname{cap}\left(\Gamma^{\otimes d}\right) \leqslant \operatorname{cap}\left(\Gamma^{\boxtimes d}\right) .
$$

We now focus on the simple example known as the $(0, k, p)$-RLL SCS over the binary alphabet $\Sigma=\{0,1\}$, which was the case study of [10]. The one-dimensional $(0, k, p)$-RLL SCS, $0 \leqslant p \leqslant 1$, is defined by

$$
\begin{equation*}
\Gamma_{k, p} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{k+1}\right): \mu\left(1^{k+1}\right) \leqslant p\right\}, \tag{40}
\end{equation*}
$$

where $1^{k+1}$ denotes the all-ones string of length $k+1$. This example is a generalization of the well known inverted $(0, k)$-RLL fully constrained system, since if we take $p=0$ we obtain the inverted $(0, k)$-RLL. In [10], the authors found lower and upper bounds on the internal capacity of $\Gamma_{k, p}^{\boxtimes d}$. We recall the relevant lower bound here.
Theorem 35. [10. Th. 20] Let $\Gamma_{k, p}$ denote the one-dimensional ( $0, k, p$ )-RLL SCS given in (40). Then, for all $0 \leqslant p \leqslant \frac{1}{2^{k+1}}$,

$$
\widehat{\operatorname{cap}}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant 1+d(\widehat{\operatorname{cap}}(\Gamma)-1),
$$

whereas for all $\frac{1}{2^{k+1}} \leqslant p \leqslant 1, \widehat{\operatorname{cap}}\left(\Gamma_{k, p}^{\boxtimes d}\right)=1$.
We first note that this theorem implies a lower bound on $\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right)$,

$$
\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant \widehat{\operatorname{cap}}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant 1+d(\widehat{\operatorname{cap}}(\Gamma)-1) .
$$

The lower bound of [10] eventually becomes negative, as the dimension $d$ grows, and therefore, degenerate. However, using the results of this paper,

$$
\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant \widehat{h_{\text {ind }}}\left(\Gamma_{k, p}\right),
$$

and this bound does not depend on the dimension, and therefore, does not degenerate. We provide an explicit numerical example:
Example 36. Let us take $k=2$, and $p=0.05$, meaning that we restrict the frequency of the pattern 111 to be at most 0.05 . Fix $d=3$. The lower bound on $\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right)$ from $\lceil 10]$ uses $\widehat{\operatorname{cap}}\left(\Gamma_{k, p}\right)$. The latter can be calculated by solving an optimization problem using a computer. We obtain that $\overline{c a p}\left(\Gamma_{k, p}\right) \approx 0.976$ which means that

$$
\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant 1+3 \cdot(0.976-1) \approx 0.928 .
$$

Using the results of this paper, we use $\widehat{h_{\text {ind }}}\left(\Gamma_{k, p}\right)$ as a lower bound to cap $\left(\Gamma_{k, p}^{\boxtimes d}\right)$. Finding the supremum involved in the definition of $\widehat{h_{\text {ind }}}\left(\Gamma_{k, p}\right)$ is also not easy, and we lower bound it by guessing a specific measure. We take each coordinate to be i.i.d. Bernoulli $\sqrt[3]{0.05}$, and we get

$$
\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant \widehat{h_{\text {ind }}}\left(\Gamma_{k, p}\right) \geqslant H_{2}(\sqrt[3]{0.05}) \approx 0.949,
$$

which is a better lower bound than that of 10$]$. Note that the upper bound gives $\widehat{\operatorname{cap}}\left(\Gamma^{\prime}\right) \leqslant 0.983$. We further mention that the lower bound of [10] gets increasingly worse as the dimension grows. For example, when $d=10$ we obtain by Theorem 35 that $\operatorname{cap}\left(\Gamma_{k, p}^{\boxtimes d}\right) \geqslant 0.76$ whereas using the independence entropy, the bound stays the same, i.e., $\operatorname{cap}\left(\Gamma^{\boxtimes d}\right) \geqslant 0.949$. Finally, for all $d \geqslant 42$, the lower bound of [10] becomes degenerate.

We present another example for $(0,1, p)$ with a more elaborate lower bound.
Example 37. Take $k=1$ and consider $\Gamma_{k, p}$. From the results of this paper,

$$
\limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma_{1, p}^{\boxtimes d}\right) \geqslant \limsup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma_{1, p}^{\otimes d}\right)=h_{\text {ind }}\left(\Gamma_{1, p}\right) \geqslant \widehat{h_{\text {ind }}}\left(\Gamma_{1, p}\right) .
$$

We lower bound $\widehat{h_{\text {ind }}}\left(\Gamma_{1, p}\right)$ by devising a product measure $\mu_{2 n} \in(\mathcal{P}(\Sigma))^{2 n}$, for all $n \in \mathbb{N}$. The measures use two parameters $0 \leqslant x, y \leqslant 1$, using a Bernoulli $(x)$ distribution for positions with odd indices, and a Bernoulli $(y)$ for positions with even indices. Thus,

$$
\widehat{h_{\text {ind }}}\left(\Gamma_{1, p}\right) \geqslant \max _{x, y} \frac{1}{2 n} H\left(\mu_{2 n}\right)=\max \left\{\frac{1}{2}\left(H_{2}(x)+H_{2}(y)\right): 0 \leqslant x, y \leqslant 1, x y \leqslant p\right\} .
$$

Due to monotonicity, the maximization problem always has a solution on the curve $x y=p$, which in the high range is unique $x=$ $y=\sqrt{p}$, and in the lower range has two symmetric solutions. For example, for $p=0.2$ the optimal solution is $x=y=\sqrt{0.2}$. However, for $p=0.01$, the first optimal solution is $x \approx 0.454, y \approx 0.022$, and the symmetric solution is $x \approx 0.022, y \approx 0.454$. This is depicted in Figure 3 .

We note that this bound agrees with the solution for the fully constrained case, $\lim _{\sup _{d \rightarrow \infty}} \operatorname{cap}\left(\Gamma_{k, 0}^{\boxtimes d}\right)=\frac{1}{2}$ which was solved in [23]. We conjecture that Figure 3(a) indeed shows the exact limiting capacity.

(a)

(b)

Figure 3: A lower bound on $\lim \sup _{d \rightarrow \infty} \operatorname{cap}\left(\Gamma_{1, p}^{\boxtimes d}\right)$ is shown in (a), where (b) shows a contour plot of $\frac{1}{2}\left(H_{2}(x)+H_{2}(y)\right)$ as well as the curves $x y=p$ for $p=0.01,0.05,0.1,0.2$.

## Appendix A

## CyClic and Non-CyClic Capacities

The goal of this appendix is to show that the capacity, as we defined it cyclically, equals the (traditionally non-cyclic) capacity in the case of fully constrained systems.
Definition 38. Let $d, k \in \mathbb{N}$. A (traditional) fully constrained system is a set $\Phi \subseteq \Sigma^{F_{k}^{d}}$ of $d$-dimensional words, called forbidden patterns. The set of all admissible words in $\Sigma^{F_{n}^{d}}$ is defined as

$$
\mathcal{B}_{n}^{\mathrm{com}}(\Phi) \triangleq\left\{x \in \Sigma^{F_{n}^{d}}: \forall \mathbf{v} \in F_{n-k}^{d}, x_{\mathbf{v}+F_{k}^{d}} \notin \Phi\right\}
$$

The (combinatorial) capacity of $\Phi$ is defined by

$$
\operatorname{cap}^{\operatorname{com}}(\Phi) \triangleq \limsup _{n \rightarrow \infty} \frac{1}{\left|F_{n}^{d}\right|} \log _{2}\left|\mathcal{B}_{n}^{\mathrm{com}}(\Phi)\right|
$$

Intuitively, a traditional fully constrained system is a set of words that do not contain any forbidden pattern non-cyclically. Given a (traditional) fully constrained system $\Phi \subseteq \Sigma^{F_{k}^{d}}$, we can construct a set of measures $\Gamma_{\Phi}$ defined as follows,

$$
\begin{equation*}
\Gamma_{\Phi} \triangleq\left\{\mu \in \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right): \mu\left(\Sigma^{F_{k}^{d}} \backslash \Phi\right)=1\right\} \tag{41}
\end{equation*}
$$

Thus, $\Gamma_{\Phi}$ is a SCS which is fully constrained in the sense of Definition 10 Since Definition 10 is more restrictive, by requiring forbidden patterns to not appear in admissible words cyclically, we immediately have

$$
\mathcal{B}_{n}\left(\Gamma_{\Phi}\right) \subseteq \mathcal{B}_{n}^{\text {com }}(\Phi)
$$

implying also

$$
\widehat{\operatorname{cap}}\left(\Gamma_{\Phi}\right) \leqslant \operatorname{cap}^{\operatorname{com}}(\Phi)
$$

However, we now prove that the capacity of $\Gamma_{\Phi}$ does equal the (combinatorial) capacity of $\Phi$.
Proposition 39 Let $d, k \in \mathbb{N}$. Let $\Phi \subseteq \Sigma^{F_{k}^{d}}$ be a fully constrained system as in Definition 38, and let $\Gamma_{\Phi} \subseteq \mathcal{P}\left(\Sigma^{F_{k}^{d}}\right)$ be its corresponding fully constrained system as in Definition 10. If $\mathcal{B}_{n}^{\text {com }}(\Phi) \neq \varnothing$ for all large enough $n \in \mathbb{N}$, then

$$
\operatorname{cap}\left(\Gamma_{\Phi}\right)=\operatorname{cap}^{\operatorname{com}}(\Phi)
$$

Proof: We first show that $\operatorname{cap}^{\text {com }}(\Phi) \leqslant \operatorname{cap}\left(\Gamma_{\Phi}\right)$. Fix $\epsilon>0$, and for $n \in \mathbb{N}, n \geqslant k$, consider the $k$-boundary of $F_{n}^{d}$ which is defined as $F_{n}^{d} \backslash F_{n-k}^{d}$. Note that $\left|F_{n}^{d} \backslash F_{n-k}^{d}\right|=n^{d}-(n-k)^{d}$. Let $w \in \mathcal{B}_{n}^{\text {com }}(\Phi)$. While $w$ does not contain any forbidden pattern when considering the coordinates non-cyclically, it may contain some when considering the coordinates cyclically. The
number of occurrences of forbidden patterns (cyclically) in $w$ is at most $\left|F_{n}^{d} \backslash F_{n-k}^{d}\right|=n^{d}-(n-k)^{d}$. For all large enough $n$ we have $\frac{n^{d}-(n-k)^{d}}{n^{d}} \leqslant \epsilon$, hence

$$
\mathcal{B}_{n}^{\mathrm{com}}(\Phi) \subseteq \mathcal{B}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)
$$

Thus, for every $\epsilon>0$,

$$
\operatorname{cap}^{\mathrm{com}}(\Phi) \leqslant \widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)
$$

Taking the limit as $\epsilon \rightarrow 0$ we obtain

$$
\operatorname{cap}^{\operatorname{com}}(\Phi) \leqslant \operatorname{cap}\left(\Gamma_{\Phi}\right)
$$

In the other direction, we now show that $\operatorname{cap}\left(\Gamma_{\Phi}\right) \leqslant \operatorname{cap}^{\operatorname{com}}(\Phi)$. Let $\delta_{0}>0$ and take $n_{0} \in \mathbb{N}$ large enough such that

$$
\frac{1}{n_{0}^{d}} \log _{2}\left|\mathcal{B}_{n_{0}}^{\text {com }}(\Phi)\right| \leqslant \operatorname{cap}^{\operatorname{com}}(\Phi)+\frac{1}{3} \delta_{0}
$$

Denote the number of forbidden patterns by $t \triangleq|\Phi|$. Take $\delta>0$ small enough such that both

$$
\frac{t(1+\delta)}{n_{0}^{d}} H_{2}\left(\frac{\delta}{1+\delta}\right) \leqslant \frac{1}{3} \delta_{0}, \quad \text { and } \quad t \delta \log _{2}|\Sigma| \leqslant \frac{1}{3} \delta_{0}
$$

where $H_{2}(\cdot)$ is the binary entropy function. Finally, for every $n \geqslant n_{0}$, denote $m \triangleq\left\lfloor n / n_{0}\right\rfloor$, and choose any $0<\epsilon \leqslant \delta / n_{0}^{d}$.
Consider a word $w \in \mathcal{B}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)$. We say $w$ is made up a concatenation of $m^{d} F_{n_{0}}^{d}$-blocks, namely, a block is a set of positions $n_{0} \mathbf{v}+F_{n_{0}}^{d}$, where $\mathbf{v} \in F_{m}^{d}$, as well a boundary, namely, the set of positions $F_{n}^{d} \backslash F_{m n_{0}}^{d}$. By our choice of parameters, the number of occurrences (perhaps cyclically) of any forbidden pattern from $\Phi$ is at most

$$
\epsilon\left|F_{n}^{d}\right| \leqslant \epsilon(m+1)^{d} n_{0}^{d} \leqslant \delta(m+1)^{d}
$$

This serves also as an upper bound on the number of blocks fully containing (non-cyclically) this forbidden pattern. Since there are $t$ forbidden patterns, the number of blocks that are devoid (non-cyclically) of any forbidden pattern, is at least $m^{d}-t \delta(m+1)^{d}$. Such blocks are in fact words from $\mathcal{B}_{n_{0}}^{\text {com }}(\Phi)$.

Fixing a specific type of forbidden pattern, and considering each occurrence of it as a ball, we have at most $\delta(m+1)^{d}$ balls, which we throw into $m^{d}+1$ bins ( $m^{d}$ blocks, and another "virtual" bin for patterns that are not fully contained within a single block). The total number of ways to throw these ball into bins is exactly $\binom{m^{d}+1+\delta(m+1)^{d}}{\delta(m+1)^{d}}$. Raising this to the power of $t$ gives an upper bound on the number of ways the $t$ forbidden patterns are dispersed among the blocks. In total we have,

$$
\left|\mathcal{B}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)\right| \leqslant\binom{ m^{d}+1+\delta(m+1)^{d}}{\delta(m+1)^{d}}^{t}\left|\mathcal{B}_{n_{0}}^{\mathrm{com}}(\Phi)\right|^{m^{d}-t(m+1)^{d} \delta}|\Sigma|^{t \delta(m+1)^{d} n_{0}^{d}}|\Sigma|^{n^{d}-\left(m n_{0}\right)^{d}}
$$

where the binomial coefficient follows from upper bounding the way forbidden patterns are dispersed among blocks, the following term counts the number of ways to fill blocks that do not contain (non-cyclically) any forbidden word, and the last term counts the ways to arbitrarily fill in the rest of the positions. Thus,

$$
\begin{aligned}
\widehat{\operatorname{cap}}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right) & =\limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \log _{2}\left|\mathcal{B}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)\right| \\
& \leqslant \frac{t(1+\delta)}{n_{0}^{d}} H_{2}\left(\frac{\delta}{1+\delta}\right)+\frac{1}{n_{0}^{d}} \log _{2}\left|\mathcal{B}_{n_{0}}^{\text {com }}(\Phi)\right|+t \delta \log _{2}|\Sigma| \\
& \leqslant \delta_{0}+\operatorname{cap}^{\text {com }}(\Phi)
\end{aligned}
$$

Taking the limit as $\epsilon \rightarrow 0$, we get

$$
\operatorname{cap}\left(\Gamma_{\Phi}\right) \leqslant \delta_{0}+\operatorname{cap}^{\operatorname{com}}(\Phi)
$$

Finally, since this holds for any $\delta_{0}>0$, we get the desired result,

$$
\operatorname{cap}\left(\Gamma_{\Phi}\right) \leqslant \operatorname{cap}^{\operatorname{com}}(\Phi)
$$

## Appendix B

## Independence Entropy for Fully Constrained Systems

Here we Prove Theorem 22. We begin by recalling relevant definitions from [19]. A $\mathbb{Z}^{d}$ shift space $X$, is a subset $X \subseteq \Sigma^{Z^{d}}$ that is closed under shifts, i.e., for all $\mathbf{v} \in \mathbb{Z}^{d}$, and all $x \in X, \sigma_{\mathbf{v}}(x) \in X$.

Definition 40. Let $d, k \in \mathbb{N}$. Given a set of forbidden words $\Phi \subseteq \Sigma^{F_{k}^{d}}$, the $\mathbb{Z}^{d}$ shift space over $\Sigma$ defined by $\Phi$ is

$$
X_{\Phi} \triangleq\left\{x \in \Sigma^{\mathbb{Z}^{d}}: \forall \mathbf{v} \in \mathbb{Z}^{d}, x_{\mathbf{v}+F_{k}^{d}} \notin \Phi\right\}
$$

Given a finite alphabet $\Sigma$, let $\tilde{\Sigma}$ denote the set of all non-empty subset of $\Sigma$, i.e.,

$$
\tilde{\Sigma} \triangleq\{A \subseteq \Sigma: A \neq \varnothing\}
$$

Definition 41. Let $d \in \mathbb{N}, S \subseteq \mathbb{Z}^{d}$, and let $\tilde{x}$ be a configuration on $S$ over $\tilde{\Sigma}$, i.e., $\tilde{x} \in \tilde{\Sigma}^{S}$. Denote by $\varphi(\tilde{x})$ the set of fillings of $\tilde{x}$,

$$
\varphi(\tilde{x}) \triangleq\left\{x \in \Sigma^{S}: \forall \mathbf{v} \in S, x_{\{\mathbf{v}\}} \in \tilde{x}_{\{\mathbf{v}\}}\right\}
$$

Definition 42. Let $d \in \mathbb{N}$, and let $X$ be a $\mathbb{Z}^{d}$ shift space over $\Sigma$. We denote by $\tilde{X}$ the multi-choice shift space corresponding to $X$,

$$
\tilde{X} \triangleq\left\{\tilde{x} \in \tilde{\Sigma}^{\mathbb{Z}^{d}}: \varphi(\tilde{x}) \subseteq X\right\}
$$

We also denote by $\mathcal{B}_{n}(\tilde{X})$ the set of all eligible configurations on $F_{n}^{d}$ in $\tilde{X}$, i.e.,

$$
\mathcal{B}_{n}(\tilde{X}) \triangleq\left\{\tilde{x}_{F_{n}^{d}}: \tilde{x} \in \tilde{X}\right\} .
$$

Definition 43. Let $d \in \mathbb{N}$, and let $X$ be a $\mathbb{Z}^{d}$ shift space. We define the combinatorial independence entropy of $X$, denoted as $h_{\text {ind }}^{\text {com }}(X)$, by

$$
h_{\mathrm{ind}}^{\mathrm{com}}(X) \triangleq \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \max \left\{\log _{2}|\varphi(\tilde{w})|: \tilde{w} \in \mathcal{B}_{n}(\tilde{X})\right\}
$$

Note that in [19] the definition of combinatorial independence entropy is slightly more general and defined over all shapes and not only on the shapes $F_{n}^{d}$. Finally, given a fully constrained system $\Phi \subseteq \Sigma^{F_{k}^{n}}$ (see Definition 38), its representation as a SCS is given by $\Gamma_{\Phi}$ in 41 . We are now ready to prove Theorem 22

Proof of Theorem 22. Let $d, k \in \mathbb{N}$, and let $\Phi \subseteq \Sigma^{F_{k}^{d}}$ be a fully constrained system, with its SCS representation $\Gamma_{\Phi}$ from (41). The claim we want to prove is that

$$
h_{\mathrm{ind}}^{\mathrm{com}}\left(X_{\Phi}\right)=h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right) .
$$

First, we show that $h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right) \leqslant h_{\text {ind }}\left(\Gamma_{\Phi}\right)$. For every $n \in \mathbb{N}$ choose $\tilde{w}_{n} \in \mathcal{B}_{n}\left(\tilde{X}_{\Phi}\right)$ which maximizes $\left|\varphi\left(\tilde{w}_{n}\right)\right|$. Now consider the independent measures $\mu_{n}$ such that $\pi_{\{\mathbf{v}\}}\left(\mu_{n}\right)$ is the uniform distribution over $\left(\tilde{w}_{n}\right)_{\{\mathbf{v}\}}$. Note that in $\mathcal{B}_{n}(\tilde{X})$, the forbidden patterns are considered without modulo while in $\mathcal{B}_{n}\left(\Gamma_{\Phi}\right)$ the calculation of the marginals' average uses modulo $n$. Therefore, if a filling $\varphi\left(\tilde{w}_{n}\right)$ belongs to $X_{\Phi}$, in $\mu_{n}$ there is perhaps a positive probability to see a forbidden pattern only in the boundaries. In $F_{n}^{d}$, the $k$-boundary is the set $F_{n}^{d} \backslash F_{n-k}^{d}$ of size $n^{d}-(n-k)^{d}$. Since $\left(n^{d}-(n-k)^{d}\right) / n^{d} \rightarrow 0$ as $n \rightarrow \infty$, we obtain that for every $\epsilon>0$, for every $n \in \mathbb{N}$ such that $\left(n^{d}-(n-k)^{d}\right) / n^{d} \leqslant \epsilon$, we have that $\mu_{n} \in \mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)$. Thus,

$$
\begin{aligned}
\widehat{h_{\mathrm{ind}}}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right) & =\limsup _{n \rightarrow \infty} \sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)} \frac{1}{n^{d}} H(\mu) \\
& \geqslant \limsup _{n \rightarrow \infty} \frac{1}{n^{d}} H\left(\mu_{n}\right) \\
& =\limsup _{n \rightarrow \infty} \frac{1}{n^{d}} \log _{2}\left|\varphi\left(\tilde{w}_{n}\right)\right| \\
& =h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right) .
\end{aligned}
$$

Taking $\epsilon \rightarrow 0$ we obtain

$$
h_{\text {ind }}\left(\Gamma_{\Phi}\right) \geqslant h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right)
$$

We now show that $h_{\text {ind }}\left(\Gamma_{\Phi}\right) \leqslant h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right)$. Fix $\delta>0$ and take $\delta_{1}>0$ small enough such that $\delta_{1}<\frac{1}{3} \delta$. Take $n_{0} \in \mathbb{N}$ large enough such that for all $n \geqslant n_{0}$,

$$
\begin{equation*}
\frac{1}{n^{d}} \max _{\tilde{w} \in \mathcal{B}_{n}\left(\tilde{X}_{\Phi}\right)}\left\{\log _{2}|\varphi(\tilde{w})|\right\} \leqslant h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right)+\frac{1}{3} \delta \tag{42}
\end{equation*}
$$

We now take $\epsilon>0$ small enough such that all the following hold,

$$
\begin{align*}
&-|\Sigma| \sqrt[k^{d}]{n_{0}^{d} \epsilon^{\frac{1}{4}}} \log _{2} \sqrt[k^{d}]{n_{0}^{d} \epsilon^{\frac{1}{4}}}<\frac{1}{3} \delta  \tag{43}\\
& 2^{d} \epsilon^{\frac{3}{4}} \log _{2}|\Sigma|<\frac{1}{2} \delta_{1}  \tag{44}\\
&\left|\widehat{h_{\text {ind }}}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)-h_{\text {ind }}\left(\Gamma_{\Phi}\right)\right| \leqslant \frac{1}{16} \delta_{1}
\end{align*}
$$

By the definition of $\widehat{h_{\text {ind }}}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)$ we may find $n \geqslant n_{0}$ large enough such that all the following hold,

$$
\begin{align*}
2\left(1-\left(1-\frac{n_{0}}{n}\right)^{d}\right) \log _{2}|\Sigma| & \leqslant \frac{1}{4} \delta_{1},  \tag{45}\\
\left.\sup _{\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)} \frac{1}{n^{d}} H(\mu)-h_{\text {ind }}\left(\Gamma_{\Phi}\right) \right\rvert\, & \leqslant \frac{1}{8} \delta_{1}
\end{align*}
$$

and there exists $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)$ for which

$$
\left|\frac{1}{n^{d}} H(\mu)-h_{\text {ind }}\left(\Gamma_{\Phi}\right)\right| \leqslant \frac{1}{4} \delta_{1} .
$$

Since $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)$, we have

$$
\frac{1}{n^{d}} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{F_{k}^{d}+\mathbf{v}}(\mu)(\Phi) \leqslant \epsilon
$$

Denote $m \triangleq\left\lfloor n / n_{0}\right\rfloor$. We now partition $F_{n}^{d}$ into $m^{d}$ blocks of shape $F_{n_{0}}^{d}$ in the natural way, $\left\{n_{0} \mathbf{v}+F_{n_{0}}^{d}: \mathbf{v} \in F_{m}^{d}\right\}$, as well as a boundary $F_{n}^{d} \backslash F_{m n_{0}}^{d}$. Note that

$$
\mu \cong \bigotimes_{\mathbf{v} \in F_{m}^{d}} \pi_{n_{0} \mathbf{v}+F_{n_{0}}^{d}}(\mu) \otimes \pi_{F_{n}^{d} \backslash F_{m n_{0}}^{d}}(\mu)
$$

Since $\mu$ is independent we obtain

$$
\begin{align*}
\left|\frac{1}{\left(m n_{0}\right)^{d}} H\left(\pi_{F_{m n_{0}}^{d}}(\mu)\right)-h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)\right| & \leqslant\left|\frac{1}{\left(m n_{0}\right)^{d}} H\left(\pi_{F_{m n_{0}}^{d}}(\mu)\right)-\frac{1}{n^{d}} H(\mu)\right|+\left|\frac{1}{n^{d}} H(\mu)-h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)\right| \\
& \leqslant\left(m n_{0}\right)^{d}\left(\frac{1}{\left(m n_{0}\right)^{d}}-\frac{1}{n^{d}}\right) \log _{2}|\Sigma|+\frac{n^{d}-\left(m n_{0}\right)^{d}}{n^{d}} \log _{2}|\Sigma|+\frac{1}{4} \delta_{1} \\
& \leqslant 2 \frac{n^{d}-\left(m n_{0}\right)^{d}}{n^{d}} \log _{2}|\Sigma|+\frac{1}{4} \delta_{1} \\
& \leqslant 2\left(1-\left(1-\frac{n_{0}}{n}\right)^{d}\right) \log _{2}|\Sigma|+\frac{1}{4} \delta_{1} \\
& \leqslant \frac{1}{2} \delta_{1} \tag{46}
\end{align*}
$$

where the last inequality holds due to 45). Let $Z: F_{m}^{d} \rightarrow \mathbb{R}$ be a function defined by

$$
Z(\mathbf{v}) \triangleq \frac{1}{n_{0}^{d}} \sum_{\mathbf{u} \in F_{n_{0}}^{d}} \pi_{F_{k}^{d}+n_{0} \mathbf{v}+\mathbf{u}}(\mu)(\Phi)
$$

(with coordinates taken modulo $n$ ). Note that since $\mu \in \overline{\mathcal{P}}_{n}\left(\mathbb{B}_{\epsilon}\left(\Gamma_{\Phi}\right)\right)$, we have

$$
\frac{1}{n^{d}} \sum_{\mathbf{v} \in F_{n}^{d}} \pi_{F_{k}^{d}+\mathbf{v}}(\mu)(\Phi) \leqslant \epsilon
$$

If we now take $\mathbf{v}$ to be random uniformly distributed in $F_{m}^{d}$, then

$$
E[Z(\mathbf{v})]=\frac{1}{m^{d}} \sum_{\mathbf{v} \in F_{m}^{d}} \frac{1}{n_{0}^{d}} \sum_{\mathbf{u} \in F_{n_{0}}^{d}} \pi_{F_{k}^{d}+n_{0} \mathbf{v}+\mathbf{u}}(\mu)(\Phi) \leqslant\left(1+\frac{1}{m}\right)^{d} \epsilon
$$

By Markov’s inequality we have

$$
\begin{equation*}
\operatorname{Pr}\left(Z(\mathbf{v}) \geqslant \epsilon^{\frac{1}{4}}\right) \leqslant \epsilon^{-\frac{1}{4}} E[Z(\mathbf{v})] \leqslant\left(1+\frac{1}{m}\right)^{d} \epsilon^{\frac{3}{4}} \tag{47}
\end{equation*}
$$

Recall that each $\mathbf{v} \in F_{m}^{d}$ may be identified with the $F_{n_{0}}^{d}$ block of $F_{n}^{d}$ in coordinates $n_{0} \mathbf{v}+F_{n_{0}}^{d}$. Define,

$$
\mathcal{L} \triangleq\left\{\mathbf{v} \in F_{m}^{d}: Z(\mathbf{v}) \geqslant \epsilon^{\frac{1}{4}}\right\}
$$

Since $\mathbf{v}$ was distributed uniformly in $F_{m}^{d}$, by 47) we have,

$$
\begin{equation*}
|\mathcal{L}| \leqslant(m+1)^{d} \epsilon^{\frac{3}{4}} \tag{48}
\end{equation*}
$$

It now follows that

$$
\begin{aligned}
\frac{1}{\left(m n_{0}\right)^{d}} \sum_{\mathbf{v} \in F_{m}^{d} \backslash \mathcal{L}} H\left(\pi_{n_{0} \mathbf{v}+F_{n_{0}}^{d}}(\mu)\right) & =\frac{1}{\left(m n_{0}\right)^{d}} H\left(\pi_{F_{m n_{0}}^{d}}(\mu)\right)-\frac{1}{\left(m n_{0}\right)^{d}} \sum_{\mathbf{v} \in \mathcal{L}} H\left(\pi_{n_{0} \mathbf{v}+F_{n_{0}}^{d}}(\mu)\right) \\
& \stackrel{(a)}{\geqslant} h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)-\frac{1}{2} \delta_{1}-\frac{1}{\left(m n_{0}\right)^{d}} \sum_{\mathbf{v} \in \mathcal{L}} H\left(\pi_{n_{0} \mathbf{v}+F_{n_{0}}^{d}}(\mu)\right) \\
& \stackrel{(b)}{\geqslant} h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)-\frac{1}{2} \delta_{1}-\frac{(m+1)^{d} \epsilon^{\frac{3}{4}} n_{0}^{d}}{\left(m n_{0}\right)^{d}} \log _{2}|\Sigma| \\
& \geqslant h_{\text {ind }}\left(\Gamma_{\Phi}\right)-\frac{1}{2} \delta_{1}-2^{d} \epsilon^{\frac{3}{4}} \log _{2}|\Sigma| \\
& \stackrel{(c)}{>} h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)-\delta_{1} \\
& >h_{\mathrm{ind}}\left(\Gamma_{\Phi}\right)-\frac{1}{3} \delta
\end{aligned}
$$

where (a) follows from (46), (b) follows from (48), and (c) follows from (44). Since there are at most $m^{d}$ summands on the left-hand side, there exists $\mathbf{v}_{0} \in F_{m}^{d} \backslash \mathcal{L}$ such that

$$
\begin{equation*}
\frac{1}{n_{0}^{d}} H\left(\pi_{n_{0} \mathbf{v}_{0}+F_{n_{0}}^{d}}(\mu)\right) \geqslant h_{\text {ind }}\left(\Gamma_{\Phi}\right)-\frac{1}{3} \delta \tag{49}
\end{equation*}
$$

We denote by $v$ the independent measure $v \triangleq \pi_{F_{n_{0}}^{d}+n_{0} \mathbf{v}_{0}}(\mu)$.
Note that if we consider $v$ in a non-cyclic manner, we obtain that

$$
\frac{1}{\left(n_{0}-k+1\right)^{d}} \sum_{\mathbf{u} \in F_{n_{0}-k+1}^{d}} \pi_{\mathbf{u}+F_{k}^{d}}(v)(\Phi) \leqslant \frac{n_{0}^{d}}{\left(n_{0}-k+1\right)^{d}} \epsilon^{\frac{1}{4}}
$$

and in particular, for every coordinate $\mathbf{u} \in F_{n_{0}-k+1}^{d}$, we have that $\pi_{\mathbf{u}+F_{k}^{d}}(v)(\Phi) \leqslant n_{0}^{d} \epsilon^{\frac{1}{4}}$. Let us define

$$
p \triangleq \sqrt[k^{d}]{n_{0}^{d} \epsilon^{\frac{1}{4}}}
$$

Hence, since $v$ is an independent measure, if $a \in \Phi$ then there must be a coordinate $\mathbf{t} \in F_{k}^{d}$ for which $\pi_{\mathbf{u}+\mathfrak{t}}(v)\left(a_{\mathbf{t}}\right) \leqslant p$.
We now construct a configuration $\tilde{w} \in \tilde{\Sigma}^{F_{n_{0}}^{d}}$. For every coordinate $\mathbf{u} \in F_{n_{0}}^{d}$ we take

$$
\tilde{w}_{\mathbf{u}}=\left\{a \in \Sigma: \pi_{\{\mathbf{u}\}}(v)(a)>p\right\}
$$

By our previous observation, $\tilde{w} \in \mathcal{B}_{n_{0}}\left(\tilde{X}_{\Phi}\right)$ since any filling of $\tilde{w}$ cannot contain a forbidden word from $\Phi$ as it requires at least one position $\mathbf{u}$ such that $\pi_{\{\mathbf{u}\}}(v) \leqslant p$. Moreover,

$$
\begin{aligned}
\log _{2}\left|\tilde{w}_{\mathbf{u}}\right| & \geqslant-\sum_{a \in \tilde{w}_{\mathbf{u}}} \pi_{\{\mathbf{u}\}}(v)(a) \log _{2}\left(\pi_{\{\mathbf{u}\}}(v)(a)\right) \\
& =H\left(\pi_{\{\mathbf{u}\}}(v)\right)+\sum_{a \in \Sigma \backslash \tilde{w}_{\mathbf{u}}} \pi_{\{\mathbf{u}\}}(v)(a) \log _{2}\left(\pi_{\{\mathbf{u}\}}(v)(a)\right) \\
& \geqslant H\left(\pi_{\{\mathbf{u}\}}(v)\right)+|\Sigma| p \log _{2} p \\
& >H\left(\pi_{\{\mathbf{u}\}}(v)\right)-\frac{1}{3} \delta
\end{aligned}
$$

where the last inequality follows from (43). Hence, using (49),

$$
\frac{1}{n_{0}^{d}} \log _{2}|\varphi(\tilde{w})|=\frac{1}{n_{0}^{d}} \sum_{\mathbf{u} \in F_{n_{0}}^{d}} \log _{2}\left|\tilde{w}_{\mathbf{u}}\right|=\frac{1}{n_{0}^{d}} \sum_{\mathbf{u} \in F_{n_{0}}^{d}} H\left(\pi_{\{\mathbf{u}\}}(v)\right)-\frac{1}{3} \delta \geqslant h_{\text {ind }}\left(\Gamma_{\Phi}\right)-\frac{2}{3} \delta .
$$

Finally, using (42), this implies that,

$$
h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right) \geqslant \frac{1}{n_{0}^{d}} \max _{\tilde{w} \in \mathcal{B}_{n_{0}}\left(\tilde{X}_{\Phi}\right)}\left\{\log _{2}|\varphi(\tilde{w})|\right\}-\frac{1}{3} \delta \geqslant h_{\text {ind }}\left(\Gamma_{\Phi}\right)-\delta
$$

Since this holds for every $\delta>0$ we have $h_{\text {ind }}\left(\Gamma_{\Phi}\right) \leqslant h_{\text {ind }}^{\text {com }}\left(X_{\Phi}\right)$, as claimed.

## REFERENCES

[1] R. J. Baxter, "Hard hexagons: exact solution," J. Phys. A: Math. Gen., vol. 13, pp. L61-L70, 1980.
[2] R. Berger, "The undecidability of the domino problem," Mem. Amer. Math. Soc. No., vol. 66, p. 72, 1966.
[3] N. Calkin and H. Wilf, "The number of independent sets in the grid graph," SIAM J. Discrete Math., vol. 11, pp. 54-60, 1998.
[4] J.-R. Chazottes, J.-M. Gambaudo, M. Hochman, and E. Ugalde, "On the finite-dimensional marginals of shift-invariant measures," Ergodic Theory Dyn. Syst., vol. 32, no. 5, pp. 1485-1500, 2012.
[5] Y. M. Chee, J. Chrisnata, H. M. Kiah, S. Ling, T. T. Nguyen, and V. K. Vu, "Efficient encoding/decoding of capacity-achieving constant-composition ICI-free codes," in Proceedings of the 2016 IEEE International Symposium on Information Theory (ISIT2016), Barcelona, Spain, Jul. 2016, pp. 205-209.
[6] -_, "Rates of constant-composition codes that mitigate intercell interference," in Proceedings of the 2016 IEEE International Symposium on Information Theory (ISIT2016), Barcelona, Spain, Jul. 2016, pp. 200-204.
[7] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications. New York: Springer, 1998.
[8] P. Diaconis and D. Freedman, "Finite exchangeable sequences," The Annals of Probability, pp. 745-764, 1980.
[9] O. Elishco, T. Meyerovitch, and M. Schwartz, "Encoding semiconstrained systems," in Proceedings of the 2016 IEEE International Symposium on Information Theory (ISIT2016), Barcelona, Spain, Jul. 2016, pp. 395-399.
[10] -_, "Semiconstrained systems," IEEE Trans. Inform. Theory, vol. 62, no. 4, pp. 811-824, Apr. 2016.
[11] S. Halevy, J. Chen, R. M. Roth, P. H. Siegel, and J. K. Wolf, "Improved bit-stuffing bounds on two-dimensional constraints," IEEE Trans. Inform. Theory, vol. 50, no. 5, pp. 824-838, May 2004.
[12] H. Ito, A. Kato, Z. Nagy, and K. Zeger, "Zero capacity region of multidimensional run length constraints," Elec. J. of Comb., vol. 6, 1999.
[13] R. Karabed, D. Neuhoff, and A. Khayrallah, "The capacity of costly noiseless channels," IBM Research Report, Tech. Rep. RJ 6040 (59639), Jan. 1988.
[14] A. Kato and K. Zeger, "On the capacity of two-dimensional run-length constrained channels," IEEE Trans. Inform. Theory, vol. 45, pp. 1527-1540, Jul. 1999.
[15] S. Kayser and P. H. Siegel, "Constructions for constant-weight ICI-free codes," in Proceedings of the 2014 IEEE International Symposium on Information Theory (ISIT2014), Honolulu, HI, USA, Jul. 2014, pp. 1431-1435.
[16] A. S. Khayrallah and D. L. Neuhoff, "Coding for channels with cost constraints," IEEE Trans. Inform. Theory, vol. 42, no. 3, pp. 854-867, 1996.
[17] O. F. Kurmaev, "Constant-weight and constant-charge binary run-length limited codes," IEEE Trans. Inform. Theory, vol. 57, no. 7, pp. 4497-4515, Jul. 2011.
[18] E. H. Lieb, "Residual entropy of square ice," Physical Review, vol. 162, no. 1, pp. 162-172, 1967.
[19] E. Louidor, B. Marcus, and R. Pavlov, "Independence entropy of $\mathbb{Z}^{d}$-shift spaces," Acta Applicandae Mathematicae, vol. 126, no. 1, pp. 297-317, 2013.
[20] E. Louidor and B. H. Marcus, "Improved lower bounds on capacities of symmetric 2D constraints using Rayleigh quotients," IEEE Trans. Inform. Theory, vol. 56, no. 4, pp. 1624-1639, Apr. 2010.
[21] L. Lovász and B. Szegedy, "Szemerédi’s lemma for the analyst," GAFA Geometric And Functional Analysis, vol. 17, no. 1, pp. 252-270, Apr 2007. [Online]. Available: https://doi.org/10.1007/s00039-007-0599-6
[22] B. H. Marcus and R. M. Roth, "Improved Gilbert-Varshamov bound for constrained systems," IEEE Trans. Inform. Theory, vol. 38, no. 4, pp. 1213-1221, 1992.
[23] T. Meyerovitch and R. Pavlov, "On independence and entropy for high-dimensional isotropic subshifts," Proceedings of the London Mathematical Society, vol. 109, no. 4, pp. 921-945, 2014.
[24] Z. Nagy and K. Zeger, "Capacity bounds for the three-dimensional ( 0,1 ) run length limited channel," IEEE Trans. Inform. Theory, vol. 46, no. 3, pp. 1030-1033, May 2000.
[25] E. Ordentlich and R. M. Roth, "Independent sets in regular hypergraphs and multi-dimensional runlength-limited constraints," SIAM J. Discrete Math., vol. 17, no. 4, pp. 615-623, 2004.
[26] T. L. Poo, P. Chaichanavong, and B. Marcus, "Tradeoff functions for constrained systems with unconstrained positions," IEEE Trans. Inform. Theory, vol. 52, no. 4, pp. 1425-1449, Apr. 2006.
[27] M. Qin, E. Yaakobi, and P. H. Siegel, "Constrained codes that mitigate inter-cell interference in read/write cycles for flash memories," IEEE J. Select. Areas Comтип., vol. 32, no. 5, pp. 836-846, May 2014.
[28] M. Schwartz and J. Bruck, "Constrained codes as networks of relations," IEEE Trans. Inform. Theory, vol. 54, no. 5, pp. 2179-2195, May 2008.
[29] M. Schwartz and A. Vardy, "New bounds on the capacity of multidimensional run-length constraints," IEEE Trans. Inform. Theory, vol. 57, no. 7, pp. 4373-4382, Jul. 2011.
[30] A. Shafarenko, A. Skidin, and S. K. Turitsyn, "Weakly-constrained codes for suppression of patterning effects in digital communications," IEEE Trans. Communications, vol. 58, no. 10, pp. 2845-2854, Oct. 2010.
[31] A. Shafarenko, K. S. Turitsyn, and S. K. Turitsyn, "Information-theory analysis of skewed coding for suppression of pattern-dependent errors in digital communications," IEEE Trans. Communications, vol. 55, no. 2, pp. 237-241, Feb. 2007.
[32] A. Sharov and R. M. Roth, "Two-dimensional constrained coding based on tiling," IEEE Trans. Inform. Theory, vol. 56, no. 4, pp. 1800-1807, Apr. 2010.
[33] P. H. Siegel and J. K. Wolf, "Bit-stuffing bounds on the capacity of 2-dimensional constrained arrays," in Proceedings of the 1998 IEEE International Symposium on Information Theory (ISIT1998), Cambridge, MA, USA, Aug. 1998, p. 323.
[34] I. Tal and R. M. Roth, "Bounds on the rate of 2-D bit-stuffing encoders," IEEE Trans. Inform. Theory, vol. 56, no. 6, pp. 2561-2567, Jun. 2010.


[^0]:    The material in this paper was presented in part at the IEEE International Symposium on Information Theory (ISIT 2017), Aachen, Germany, June 2017. Ohad Elishco is with the Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer Sheva 8410501, Israel (e-mail: ohadeli@post.bgu.ac.il).

    Tom Meyerovitch is with the Department of Mathematics, Ben-Gurion University of the Negev, Beer Sheva 8410501, Israel (e-mail: mtom@math.bgu.ac.il).
    Moshe Schwartz is with the Department of Electrical and Computer Engineering, Ben-Gurion University of the Negev, Beer Sheva 8410501, Israel (e-mail: schwartz@ee.bgu.ac.il).

    This work was supported in part by the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement no. 333598 and by the Israel Science Foundation (grant no. 626/14).

