# Distortion Bounds for Source Broadcast Problems 

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#### Abstract

This paper investigates the joint source-channel coding problem of sending a memoryless source over a memoryless broadcast channel. An inner bound and several outer bounds on the admissible distortion region are derived, which respectively generalize and unify several existing bounds. As a consequence, we also obtain an inner bound and an outer bound for the degraded broadcast channel case. When specialized to the Gaussian or binary source broadcast, the inner bound and outer bound not only recover the best known inner bound and outer bound in the literature, but also generate some new results. Besides, we also extend the inner bound and outer bounds to the Wyner-Ziv source broadcast problem, i.e., source broadcast with side information available at decoders. Some new bounds are obtained when specialized to the Wyner-Ziv Gaussian and Wyner-Ziv binary cases.


Index Terms-Joint source-channel coding (JSCC), hybrid coding, Wyner-Ziv, side information, multivariate covering/packing, method of introducing remote channels, network information theory.

## I. Introduction

As stated in Shannon's source-channel separation theorem [2], cascading source coding and channel coding does not lose the optimality for point-to-point communication systems. This separation theorem does not only suggest a simple system architecture in which source coding and channel coding are separated by a universal digital interface, but also guarantees that such architecture does not incur any asymptotic performance loss. Consequently, it forms the basis of the architecture of today's communication systems. However, for many multi-user communication systems, the optimality of such a separation does not hold any more [3], [4]. Therefore, an increasing amount of literature focus on joint sourcechannel coding (JSCC) in multi-user setting.

One of the most classical problems in this area is JSCC of transmitting a Gaussian source over a $K$-user Gaussian broadcast channel with average transmitting power constrained. Goblick [3] observed that when the source bandwidth and the channel bandwidth are matched (i.e., one channel use per source sample) linear uncoded transmission (symbol-bysymbol mapping) is optimal. However, the optimality of such

[^0]a simple linear scheme cannot be extended to the bandwidth mismatch case. One way to approximately characterize the admissible distortion region is finding its inner bound and outer bound. For inner bound, analog coding or hybrid coding has been studied in a vast body of literature [4], [5], [6], [7]. For 2-user Gaussian broadcast communication, Prabhakaran et al. [7] gave the best known inner bound, which is achieved by a hybrid digital-analog (HDA) scheme. On the other hand, Reznic et al. [8] derived a non-trivial outer bound (tighter than the single-user bound) for 2-user Gaussian broadcast problem with bandwidth expansion (i.e., more than one channel uses per source sample) by introducing an auxiliary random variable (or a remote source). Tian et al. [9] extended this outer bound to the $K$-user case by introducing more than one auxiliary random variables. Similar to the results of Reznic et al., the outer bound given by Tian et al. is also nontrivial only for the bandwidth expansion case [10]. Beyond broadcast communication, Minero et al. [15] considered sending memoryless correlated sources over a memoryless multi-access channel, and derived an inner bound using a unified framework of hybrid coding. Lee et al. [20] derived a unified achievability result for memoryless network communication.

Besides, in [6], [17], [18] the Wyner-Ziv source communication problem was investigated, in which side information correlated with the source is available at decoder(s). Shamai et al. [6] studied the problem of sending a Wyner-Ziv source over a point-to-point channel, and proved that for such communication system, the separate coding (which cascades WynerZiv coding with channel coding) does not incur any loss of optimality. Nayak et al. [17] and Gao et al. [18] investigated the Wyner-Ziv source broadcast problem, and obtained the single-user outer bound by simply applying the cut-set bound (the minimum distortion achieved in point-to-point setting) for each receiver.

In this paper, we consider JSCC of transmitting a memoryless source over a $K$-user memoryless broadcast channel, and derive an inner bound and two outer bounds on the admissible distortion region. The inner bound is derived by using a unified framework of hybrid coding inspired by [15], and the outer bounds are derived by introducing remote sources at the sender side or introducing remote channels at receiver sides. The proof method of introducing remote sources at the sender side can be found in [8] and [9], and hence it is not new. However, the introducing remote channels method is different from existing methods. The existing method of introducing auxiliary random variables at receiver sides, named the genie-aided method, can be found in [9], [11], [16, Section 6.4.3]. Both the genie-aided method and the introducing remote channels method convert the original communication system into a new one. However, the genie-aided method constructs a "stronger" network such that the original network is a degraded version
of it, while the introducing remote channels method constructs a degraded ("weaker") version of the original network. Hence our introducing remote channels method is different from the existing genie-aided method. Furthermore, our distortion bounds are generalizations and unifications of several existing bounds in the literature. Besides, as a consequence, we also obtain an inner bound and an outer bound for the degraded broadcast channel case. When specialized to the Gaussian source broadcast and binary source broadcast, our inner bound could recover the best known performance achieved by hybrid coding, and our outer bound could recover the best known outer bounds given by Tian et al. [9] and Khezeli et al. [11]. Moreover, for these cases, our bounds can be also used to generate some new results. Besides, we also extend the inner bound and outer bounds to the Wyner-Ziv source broadcast problem, i.e., source broadcast with side information at decoders. When specialized to the Wyner-Ziv Gaussian and binary cases, our bounds reduce to some new bounds.

The rest of this paper is organized as follows. Section II summarizes basic notations, definitions and preliminaries, and formulates the problem. Section III gives the main results for the source broadcast problem, including the discrete, discrete and degraded, binary, and Gaussian cases. Section IV extends the results to the Wyner-Ziv source broadcast problem. Finally, Section V gives the concluding remarks.

## II. Problem Formulation and Preliminaries

## A. Notation

Throughout this paper, we follow the notation in [16]. For example, for a discrete random variable $X \sim p_{X}$ on alphabet $\mathcal{X}$ and $\epsilon \in(0,1)$, the set of $\epsilon$-typical $n$-sequences $x^{n}$ (or the typical set in short) is defined as

$$
\begin{aligned}
& \mathcal{T}_{\epsilon}^{(n)}(X) \\
& =\left\{x^{n}:\left|\frac{\left|\left\{i: x_{i}=x\right\}\right|}{n}-p_{X}(x)\right| \leq \epsilon p_{X}(x), \forall x \in \mathcal{X}\right\} .
\end{aligned}
$$

When it is clear from the context, we will use $\mathcal{T}_{\epsilon}^{(n)}$ to denote $\mathcal{T}_{\epsilon}^{(n)}(X)$.

In addition, we use $X_{\mathcal{A}}$ to denote the vector $\left(X_{j}: j \in \mathcal{A}\right)$, use $[i: j]$ to denote the set $\{\lfloor i\rfloor,\lfloor i\rfloor+1, \cdots,\lfloor j\rfloor\}$, and use 1 to denote an all-one vector (similarly, use 2 to denote an all-2 vector). We say vector $m_{[1: N]}$ is smaller than vector $m_{[1: N]}^{\prime}$ if $m_{j}=m_{j}^{\prime}$ for $k<j \leq K$ and $m_{k}<m_{k}^{\prime}$ for some $k$. For two vectors $m_{\mathcal{I}}$ and $m_{\mathcal{I}}^{\prime}$, we say $m_{\mathcal{I}}$ is component-wise unequal to $m_{\mathcal{I}}^{\prime}$, if $m_{i} \neq m_{i}^{\prime}$ for all $i \in \mathcal{I}$, and denote it as $m_{\mathcal{I}} \nLeftarrow m_{\mathcal{I}}^{\prime}$. Besides, we use $1\{\mathcal{A}\}$ to denote the indicator function of an event $\mathcal{A}$, i.e.,

$$
1\{\mathcal{A}\}= \begin{cases}1, & \text { if } \mathcal{A} \text { is true } \\ 0, & \text { if } \mathcal{A} \text { is false }\end{cases}
$$

We use exp and log to respectively denote the exponential and logarithmic functions with the base 2 .

## B. Problem Formulation

Consider the source broadcast system shown in Fig. 1. A discrete memoryless source (DMS) $S^{n}$ is first coded into $X^{n}$


Fig. 1. Source broadcast system.
using a source-channel code, then transmitted to $K$ receivers through a discrete memoryless broadcast channel (DM-BC) $p_{Y_{[1: K]} \mid X}$, and finally, the receiver $k$ produces a source reconstruction $\hat{S}_{k}^{n}$ from the received signal $Y_{k}^{n}$.

Definition 1 (Source). A discrete memoryless source (DMS) is specified by a probability mass function (pmf) $p_{S}$ on a finite alphabet $\mathcal{S}$. The DMS $p_{S}$ generates an i.i.d. random process $\left\{S_{i}\right\}$ with $S_{i} \sim p_{S}$.

Definition 2 (Broadcast Channel). A $K$-user discrete memoryless broadcast channel (DM-BC) is specified by a collection of conditional pmfs $p_{Y_{[1: K]} \mid X}$ on a finite output alphabet $\mathcal{Y}_{1} \times \cdots \times \mathcal{Y}_{K}$ for each $x$ in a finite input alphabet $\mathcal{X}$.

Definition 3 (Degraded Broadcast Channel). A DM-BC $p_{Y_{[1: K]} \mid X}$ is stochastically degraded (or simply degraded) if there exists a random vector $\tilde{Y}_{[1: K]}$ such that $p_{\tilde{Y}_{k} \mid X}=$ $p_{Y_{k} \mid X}, 1 \leq k \leq K$ (i.e., $\tilde{Y}_{[1: K]}$ has the same conditional marginal pmfs as $Y_{[1: K]}$ given $X$ ), and $X \rightarrow \tilde{Y}_{K} \rightarrow \tilde{Y}_{K-1} \rightarrow$ $\cdots \rightarrow \tilde{Y}_{1}{ }^{1}$ form a Markov chain. In addition, as a special case, if $X \rightarrow Y_{K} \rightarrow Y_{K-1} \rightarrow \cdots \rightarrow Y_{1}$, i.e., $\tilde{Y}_{k}=Y_{k}, 1 \leq k \leq K$, then $p_{Y_{[1: K]} \mid X}$ is physically degraded.
Definition 4. An $n$-length source-channel code is defined by a encoding function $x^{n}: \mathcal{S}^{n} \mapsto \mathcal{X}^{n}$ and $K$ decoding functions $\hat{s}_{k}: \mathcal{Y}_{k}^{n} \mapsto \hat{\mathcal{S}}_{k}^{n}, 1 \leq k \leq K$, where $\hat{\mathcal{S}_{k}}$ is the alphabet of source reconstruction at the receiver $k$.

For any $n$-length source-channel code, the induced distortion is defined as

$$
\begin{equation*}
\mathbb{E} d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right)=\frac{1}{n} \sum_{t=1}^{n} \mathbb{E} d_{k}\left(S_{t}, \hat{S}_{k, t}\right) \tag{1}
\end{equation*}
$$

for $1 \leq k \leq K$, where $d_{k}\left(s, \hat{s}_{k}\right): \mathcal{S} \times \hat{\mathcal{S}_{k}} \mapsto[0,+\infty]$ is a distortion measure function for the receiver $k$.

Definition 5. For transmitting a source $S$ over a channel $p_{Y_{[1: K]} \mid X}$, if there exists a sequence of source-channel codes such that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \mathbb{E} d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \leq D_{k} \tag{2}
\end{equation*}
$$

then we say that the distortion tuple $D_{[1: K]}$ is achievable.
Definition 6. For transmitting a source $S$ over a channel $p_{Y_{[1: K]} \mid X}$, the admissible distortion region is defined as

$$
\begin{equation*}
\mathcal{D} \triangleq\left\{D_{[1: K]}: D_{[1: K]} \text { is achievable }\right\} \tag{3}
\end{equation*}
$$

[^1]The admissible distortion region $\mathcal{R}$ only depends on the marginal distributions of $p_{Y_{[1: K]} \mid X}$, hence for the stochastically degraded channel case, it suffices to only consider the corresponding physically degraded channel case.

In addition, Shannon's source-channel separation theorem shows that the minimum distortion for transmitting a source over a point-to-point channel satisfies

$$
\begin{equation*}
R_{k}\left(D_{k}\right)=C_{k}, \tag{4}
\end{equation*}
$$

where $R_{k}(\cdot)$ is the rate-distortion function of the source with the distortion measure $d_{k}$, and $C_{k}$ is the capacity for the receiver $k$. Therefore, the optimal distortion (Shannon limit) is

$$
\begin{equation*}
D_{k}^{*}=R_{k}^{-1}\left(C_{k}\right) \tag{5}
\end{equation*}
$$

Obviously,

$$
\begin{equation*}
\mathcal{D} \subseteq \mathcal{D}^{*} \triangleq\left\{D_{[1: K]}: D_{k} \geq D_{k}^{*}, 1 \leq k \leq K\right\} \tag{6}
\end{equation*}
$$

where $\mathcal{D}^{*}$ is named single-user outer bound.
In the system above, the source bandwidth and channel bandwidth are matched. In this paper, we also consider the communication system with bandwidth mismatch, whereby $m$ samples of a DMS are transmitted through $n$ uses of a DMBC. For this case, the bandwidth mismatch factor is defined as $b=\frac{n}{m}$.

## C. Multivariate Covering/Packing Lemma

We first introduce the following multivariate covering and packing lemmas which are important in the achievability part in this work.

Let $\left(U, V_{[0: k]}\right) \sim p_{U, V_{[0: k]}}$. Let $\left(U^{n}, V_{0}^{n}\right) \sim p_{U^{n}, V_{0}^{n}}$ be a random vector sequence. For each $j \in[1: k]$, let $\mathcal{A}_{j} \subseteq$ $[1: j-1]$. Assume $\mathcal{A}_{j}$ satisfies if $i \in \mathcal{A}_{j}$, then $\mathcal{A}_{i} \subseteq \mathcal{A}_{j}$. For each $j \in[1: k]$ and each $m_{\mathcal{A}_{j}} \in \prod_{i \in \mathcal{A}_{j}}\left[1: 2^{n r_{i}}\right]$, let $V_{j}^{n}\left(m_{\mathcal{A}_{j}}, m_{j}\right), m_{j} \in\left[1: 2^{n r_{j}}\right]$, be pairwise conditionally independent random sequences, each distributed according to $\prod_{i=1}^{n} p_{V_{j} \mid V_{\mathcal{A}_{j}}, V_{0}}\left(v_{j, i} \mid v_{\mathcal{A}_{j}, i}\left(m_{\mathcal{A}_{j}}\right), v_{0, i}\right)$. Hence for each $j \in$ $[1: k], \mathcal{A}_{j} \cup\{0\}$ denotes the index set of the random variables on which the codeword $V_{j}^{n}$ is superposed. Based on the notations above, we have the following multivariate covering and packing lemmas.
Lemma 1 (Multivariate Covering Lemma). Let $\epsilon^{\prime}<\epsilon$. If $\lim _{n \rightarrow \infty} \mathbb{P}\left(\left(U^{n}, V_{0}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right)=1$, then there exists $\delta(\epsilon)$ that tends to zero as $\epsilon \rightarrow 0$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbb{P}\left(\left(U^{n}, V_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{[1: k]}\right) \\
& =1 \tag{7}
\end{align*}
$$

if $\sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid V_{0} U\right)+\delta(\epsilon)$ for all $\mathcal{J} \subseteq[1: k]$ such that $\mathcal{J} \neq \emptyset$ and if $j \in \mathcal{J}$, then $\mathcal{A}_{j} \subseteq \mathcal{J}$.
Lemma 2 (Multivariate Packing Lemma). There exists $\delta(\epsilon)$ that tends to zero as $\epsilon \rightarrow 0$ such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbb{P}\left(\left(U^{n}, V_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{[1: k]}\right) \\
& =0 \tag{8}
\end{align*}
$$

if $\sum_{j \in \mathcal{J}} r_{j}<\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid V_{0} U\right)-\delta(\epsilon)$ for some $\mathcal{J} \subseteq[1: k]$ such that $\mathcal{J} \neq \emptyset$ and if $j \in \mathcal{J}$, then $\mathcal{A}_{j} \subseteq \mathcal{J}$.

Note that all the existing covering and packing lemmas such as [16, Lem. 8.2] and [19, Lem. 4], involve only single-layer codebook. Our multivariate covering and packing lemmas generalize them to the case of multilayer codebooks.

## III. Source Broadcast

## A. Discrete Memoryless Broadcast

Now, we bound the distortion region for the source broadcast communication. To write the inner bound, for $1 \leq j \leq$ $N \triangleq 2^{K}-1$, we first introduce an auxiliary random variable $V_{j}$ for each of the $2^{K}-1$ nonempty subsets $\mathcal{G}_{j} \subseteq[1: K]$, and let $V_{j}$ denote a common message transmitted from the sender to all the receivers in $\mathcal{G}_{j}$. The $V_{j}$ corresponds to a subset $\mathcal{G}_{j}$ by the following one-to-one mapping. Sort all the nonempty subsets $\mathcal{G}_{j} \subseteq[1: K]$ in the decreasing order ${ }^{2}$. Map the $j$ th subset in the resulting sequence to $j$. Obviously this mapping is one-to-one corresponding. For example, if $K=3$, then $\mathcal{G}_{1}=\{1,2,3\}, \mathcal{G}_{2}=\{2,3\}, \mathcal{G}_{3}=\{1,3\}, \mathcal{G}_{4}=\{1,2\}, \mathcal{G}_{5}=$ $\{3\}, \mathcal{G}_{6}=\{2\}, \mathcal{G}_{7}=\{1\} ;$ see Fig. 2.

Besides, let

$$
\begin{align*}
& \mathcal{A}_{j} \triangleq\left\{i \in[1: N]: \mathcal{G}_{j} \varsubsetneqq \mathcal{G}_{i}\right\}, 1 \leq j \leq N  \tag{9}\\
& \mathcal{B}_{k} \triangleq\left\{i \in[1: N]: k \in \mathcal{G}_{i}\right\}, 1 \leq k \leq K \tag{10}
\end{align*}
$$

If $K=3$, then $\mathcal{A}_{1}=\emptyset, \mathcal{A}_{2}=\{1\}, \mathcal{A}_{3}=\{1\}, \mathcal{A}_{4}=$ $\{1\}, \mathcal{A}_{5}=\{1,2,3\}, \mathcal{A}_{6}=\{1,2,4\}, \mathcal{A}_{7}=\{1,3,4\}$ and $\mathcal{B}_{1}=\{1,3,4,7\}, \mathcal{B}_{2}=\{1,2,4,6\}, \mathcal{B}_{3}=\{1,2,3,5\}$. Later we will show that $\mathcal{A}_{j}$ and $\mathcal{B}_{k}$ respectively correspond to the index set of the random variables on which the codeword $V_{j}^{n}$ is superposed, and the index set of decodable codewords $V_{j}^{n}$,s for the receiver $k$ in the proposed hybrid coding scheme; see Appendix C-A. The decoder $k$ is able to recover correctly the $V_{\mathcal{B}_{k}}^{n}$ with probability approaching 1 as $n \rightarrow \infty$. In addition, it is easy to verify that if $j \in \mathcal{B}_{k}$, then $\mathcal{A}_{j} \subseteq \mathcal{B}_{k}$. It means that the proposed codebook satisfies that if the information $V_{j}^{n}$ can be recovered correctly by the receiver $k$ (i.e., $j \in \mathcal{B}_{k}$ ), then $V_{\mathcal{A}_{j}}^{n}$ can also be recovered correctly by it.

Based on the notations above, we define a distortion region (inner bound)

$$
\begin{align*}
\mathcal{D}^{(i)}= & \left\{D_{[1: K]}: \exists p_{V_{[1: N]} \mid S}, r_{[1: N]}, x\left(v_{[1: N]}, s\right), \hat{s}_{k}\left(v_{\mathcal{B}_{k}}, y_{k}\right),\right. \\
& k \in[1: K] \text { s.t. } \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K], \\
& \sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}} \mid S\right) \\
& \text { for all } \mathcal{J} \subseteq[1: N] \text { s.t. } \mathcal{J} \neq \emptyset \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}, \\
& \sum_{j \in \mathcal{J}^{c}} r_{j}<\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}^{c}} \mid Y_{k} V_{\mathcal{J}}\right), \\
& k \in[1: K] \text { for all } \mathcal{J} \subseteq \mathcal{B}_{k} \text { s.t. } \\
& \left.\mathcal{J}^{c} \triangleq \mathcal{B}_{k} \backslash \mathcal{J} \neq \emptyset \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}\right\} . \tag{11}
\end{align*}
$$

[^2]

Fig. 2. (a) The inclusion relation of the sets $\mathcal{G}_{j}, 1 \leq j \leq N$ with $K=3$; (b) The structure of the codebook for the unified hybrid coding with $K=3$.

Besides, for any positive integer $L$, define a distortion region (outer bound achieved by introducing remote sources at the sender)

$$
\begin{aligned}
\mathcal{D}_{1}^{(o)}= & \left\{D_{[1: K]}: \exists p_{\hat{S}_{[1: K]} \mid S}\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K],
\end{aligned}
$$

$$
\text { and for any } p_{U_{[1: L]} \mid S} \text {, one can find } p_{\tilde{Y}_{[1: K]} \tilde{U}_{[1: L]} X} \text { s.t. }
$$

$$
I\left(\hat{S}_{\mathcal{A}} ; U_{\mathcal{B}} \mid U_{\mathcal{C}}\right) \leq I\left(Y_{\mathcal{A}} ; \tilde{U}_{\mathcal{B}} \mid \tilde{U}_{\mathcal{C}} \tilde{Y}_{\mathcal{A}}\right)
$$

$$
\begin{equation*}
\text { for any } \mathcal{A} \subseteq[1: K], \mathcal{B}, \mathcal{C} \subseteq[1: L]\} \tag{12}
\end{equation*}
$$

and another distortion region (outer bound achieved by introducing remote channels at receivers)

$$
\begin{align*}
\mathcal{D}_{2}^{(o)}= & \left\{D_{[1: K]}: \exists p_{X}, \hat{s}_{k}\left(\tilde{y}_{k}\right), k \in[1: K]\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K], \\
& \text { and for any } p_{U_{[1: L]} \mid Y_{[1: K]}}, \\
& \text { one can find } p_{\tilde{Y}_{[1: K]} \mid S} p_{\tilde{U}_{[1: L]} \mid \tilde{Y}_{[1: K]}} \text { s.t. } \\
& I\left(S ; \tilde{Y}_{\mathcal{B}} \tilde{U}_{\mathcal{B}^{\prime}} \mid \tilde{Y}_{\mathcal{C}} \tilde{U}_{\mathcal{C}^{\prime}}\right) \leq I\left(X ; Y_{\mathcal{B}} U_{\mathcal{B}^{\prime}} \mid Y_{\mathcal{C}} U_{\mathcal{C}^{\prime}}\right) \\
& \text { for any } \left.\mathcal{B}, \mathcal{C} \subseteq[1: K], \mathcal{B}^{\prime}, \mathcal{C}^{\prime} \subseteq[1: L]\right\} . \tag{13}
\end{align*}
$$

Then we have the following theorem. The proof is given in Appendix C.

Theorem 1. For transmitting a DMS $S$ over a $D M-B C$ $p_{Y_{[1: K]} \mid X}$,

$$
\begin{equation*}
\mathcal{D}^{(i)} \subseteq \mathcal{D} \subseteq \mathcal{D}_{1}^{(o)} \cap \mathcal{D}_{2}^{(o)} \tag{14}
\end{equation*}
$$

Remark 1. The inner bound of Theorem 1 can be easily extended to Gaussian or any other well-behaved continuousalphabet source-channel pairs by the standard discretization method [16, Thm. 3.3]. Moreover for these cases the outer bounds still hold. Theorem 1 can be also extended to the bandwidth mismatch case, where $m$ samples of a DMS are transmitted through $n$ uses of a DM-BC. This can be accomplished by replacing the source and channel symbols in Theorem 1 by supersymbols of lengths $m$ and $n$, respectively. Besides, Theorem 1 could be also extended to the problems of broadcasting correlated sources (by modifying the distortion measures) and source broadcast with channel input cost constraints (by introducing channel input constraints).

The inner bound $\mathcal{D}^{(i)}$ in Theorem 1 is achieved by a unified hybrid coding scheme depicted in Fig. 3. In this scheme, the codebook has a layered (or superposition) structure (see Fig. 2), and consists of randomly and independently generated codewords $V_{[1: N]}^{n}\left(m_{[1: N]}\right), m_{[1: N]} \in \prod_{i=1}^{N}\left[1: 2^{n r_{i}}\right]$, where $r_{[1: N]}$ satisfies (11). At the encoder side, upon a source sequence $S^{n}$, the encoder produces digital messages $M_{[1: N]}$ with $M_{i}$ meant for all the receivers $k$ 's satisfying $i \in \mathcal{B}_{k}$. Then, the codeword $V_{[1: N]}^{n}\left(M_{[1: N]}\right)$ and the source sequence $S^{n}$ are used to generate a channel input $X^{n}$ by a symbol-bysymbol mapping $x\left(v_{[1: N]}, s\right)$. At the decoder sides, upon the received signal $Y_{k}^{n}$, the decoder $k$ reconstructs $M_{\mathcal{B}_{k}}$ (and also $V_{\mathcal{B}_{k}}^{n}\left(M_{\mathcal{B}_{k}}\right)$ ) losslessly, and then generates $\hat{S}_{k}^{n}$ by a symbol-bysymbol mapping $\hat{s}_{k}\left(v_{\mathcal{B}_{k}}, y_{k}\right)$. Such a scheme could achieve any $D_{[1: K]}$ in the inner bound $\mathcal{D}^{(i)}$.

To reveal essence of such hybrid coding, the digital transmission part of this hybrid coding can be roughly understood as the cascade of a $K$-user Gray-Wyner source-coding and a $K$-user Marton's broadcast channel-coding, which share a common codebook. According to [16, Thm. 13.3], the encoding operation of Gray-Wyner source-coding with rates $r_{[1: N]}$ is successful if $\sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-$ $H\left(V_{\mathcal{J}} \mid S\right)$ for all $\mathcal{J} \subseteq[1: N]$ such that $\mathcal{J} \neq \emptyset$ and if $j \in$ $\mathcal{J}$, then $\mathcal{A}_{j} \subseteq \mathcal{J}$, and according to [16, Thm 8.4] the decoding operation of Marton's broadcast channel-coding with rates $r_{[1: N]^{3}}{ }^{3}$ is successful if $\sum_{j \in \mathcal{J}^{c}} r_{j}<\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-$ $H\left(V_{\mathcal{J}^{c}} \mid Y_{k} V_{\mathcal{J}}\right)$ for all $1 \leq k \leq K$ and for all $\mathcal{J} \subseteq$ $\mathcal{B}_{k}$ such that $\mathcal{J}^{c} \neq \emptyset$ and if $j \in \mathcal{J}$, then $\mathcal{A}_{j} \subseteq \mathcal{J}$. Since the proposed hybrid coding satisfies the two sufficient conditions above, $V_{\mathcal{B}_{k}}^{n}\left(M_{\mathcal{B}_{k}}\right)$ can be correctly recovered by the receiver $k$. Note that such informal understanding is inaccurate owing to the use of symbol-by-symbol mappings, but it provides a rationale for our scheme. Besides, the design of such unified hybrid coding is inspired by the hybrid coding scheme for sending correlated sources over a multi-access channel in [15].

The outer bounds $\mathcal{D}_{1}^{(o)}$ and $\mathcal{D}_{2}^{(o)}$ in Theorem 1 are derived by introducing auxiliary random variables $U_{[1: L]}^{n}$ at the sender side or at receiver sides. The proof method of introducing auxiliary random variables (or remote sources) at sender side could be found in [9], [11, Thm. 2] and [12, Lem. 1]. In [9] it was used to derive the outer bound for Gaussian source broadcast, and in [11, Thm. 2] and [12, Lem. 1] it was used to derive the outer bounds for sending source over 2-user general broadcast channel. This proof method generalizes the one used to derive the single-user outer bound, but it does not always result in a strictly tighter outer bound than the single-user one [10]. On the other hand, introducing remote channels method is different from the existing genie-aided method [9], [11], [16, Section 6.4.3], since the genie-aided method constructs a "stronger" network such that the original network is a degraded version of it, but the introducing remote channels method con-

[^3]

Fig. 3. The unified hybrid coding used to prove the inner bound in Theorem 1.
structs a degraded ("weaker") version of the original network. A deeper understanding of these two proof methods has been given by Khezeli et al. in [12]. $p_{\hat{S}_{[1: K]} \mid S}$ can be considered as a virtual broadcast channel realized over physical broadcast channel $p_{Y_{[1: K]} \mid X}$, and hence certain mutual informations (e.g., channel capacity region) based on $p_{\hat{S}_{[1: K]} \mid S}$ are less than or equal to those based on $p_{Y_{[1: K]} \mid X}$. This leads to the desired necessary conditions. Besides, the necessary conditions can be also understood from the perspective of virtual sources. $X$ and $Y_{[1: K]}$ respectively can be considered as a virtual source and $K$ virtual reconstructions. Then the physical source $S$ and the physical reconstructions $\hat{S}_{[1: K]}$ are correlated through the virtual source and virtual reconstructions. Hence the physical source should be more "tractable" than the virtual one, and certain mutual informations (e.g., source-coding rate region) based on the physical source and reconstructions should be less than or equal to those based on the virtual source and reconstructions. The analysis above gives the reasons why $\mathcal{D}_{1}^{(o)}$ is expressed in form of comparison of the "capacity regions" of virtual broadcast channel and physical broadcast channel, while $\mathcal{D}_{2}^{(o)}$ is expressed in form of comparison of the "source-coding rate regions" of virtual source and physical source.

For the 2-user broadcast case, the inner bound in Theorem 1 reduces to

$$
\begin{align*}
\mathcal{D}^{(i)}= & \left\{\left(D_{1}, D_{2}\right): \exists p_{V_{0}, V_{1}, V_{2} \mid S}, x\left(v_{0}, v_{1}, v_{2}, s\right)\right. \\
& \hat{s}_{k}\left(v_{0}, v_{k}, y_{k}\right), k=1,2, \text { s.t. } \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, \\
& I\left(V_{0} V_{k} ; S\right)<I\left(V_{0} V_{k} ; Y_{k}\right), k=1,2 \\
& I\left(V_{0} V_{1} V_{2} ; S\right)+I\left(V_{1} ; V_{2} \mid V_{0}\right) \\
& <\min \left\{I\left(V_{0} ; Y_{1}\right), I\left(V_{0} ; Y_{2}\right)\right\} \\
& \quad+I\left(V_{1} ; Y_{1} \mid V_{0}\right)+I\left(V_{2} ; Y_{2} \mid V_{0}\right) \\
& I\left(V_{0} V_{1} ; S\right)+I\left(V_{0} V_{2} ; S\right)+I\left(V_{1} ; V_{2} \mid V_{0} S\right) \\
& \left.<I\left(V_{0} V_{1} ; Y_{1}\right)+I\left(V_{0} V_{2} ; Y_{2}\right)\right\} . \tag{15}
\end{align*}
$$

This inner bound was first given by Yassaee et. al [21]. On the other hand, letting $K=2, L=1, \mathcal{D}_{1}^{(o)}$ and $\mathcal{D}_{2}^{(o)}$ respectively reduce to

$$
\begin{aligned}
\mathcal{D}_{1}^{(o)}= & \left\{\left(D_{1}, D_{2}\right): \exists p_{\hat{S}_{1} \hat{S}_{2} \mid S}\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k=1,2
\end{aligned}
$$

and for any pmf $p_{U \mid S}$, one can find $p_{\tilde{U} X \tilde{Y}_{1} \tilde{Y}_{2}}$ s.t.

$$
I\left(\hat{S}_{1} ; U\right) \leq I\left(Y_{1} ; \tilde{U} \mid \tilde{Y}_{1}\right)
$$

$$
\begin{aligned}
& I\left(\hat{S}_{2} ; U\right) \leq I\left(Y_{2} ; \tilde{U} \mid \tilde{Y}_{2}\right) \\
& I\left(\hat{S}_{1} \hat{S}_{2} ; U\right) \leq I\left(Y_{1} Y_{2} ; \tilde{U} \mid \tilde{Y}_{1} \tilde{Y}_{2}\right) \\
& I\left(\hat{S}_{1} ; S \mid U\right) \leq I\left(Y_{1} ; X \mid \tilde{U} \tilde{Y}_{1}\right) \\
& I\left(\hat{S}_{2} ; S \mid U\right) \leq I\left(Y_{2} ; X \mid \tilde{U} \tilde{Y}_{2}\right) \\
& \left.I\left(\hat{S}_{1} \hat{S}_{2} ; S \mid U\right) \leq I\left(Y_{1} Y_{2} ; X \mid \tilde{U} \tilde{Y}_{1} \tilde{Y}_{2}\right)\right\}
\end{aligned}
$$

or a simpler but possibly looser one,

$$
\begin{aligned}
\mathcal{D}_{1}^{(o)}= & \left\{\left(D_{1}, D_{2}\right): \exists p_{\hat{S}_{1} \hat{S}_{2} \mid S}\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k=1,2
\end{aligned}
$$

and for any pmf $p_{U \mid S}$, one can find $p_{X \tilde{Y}_{1} \tilde{Y}_{2}}$ s.t.
$I\left(\hat{S}_{1} ; U\right) \leq I\left(Y_{1} ; \tilde{Y}_{1}\right)$,
$I\left(\hat{S}_{2} ; U\right) \leq I\left(Y_{2} ; \tilde{Y}_{2}\right)$,
$I\left(\hat{S}_{1} \hat{S}_{2} ; U\right) \leq I\left(Y_{1} Y_{2} ; \tilde{Y}_{1} \tilde{Y}_{2}\right)$,
$I\left(\hat{S}_{1} ; S \mid U\right) \leq I\left(Y_{1} ; X \mid \tilde{Y}_{1}\right)$,
$I\left(\hat{S}_{2} ; S \mid U\right) \leq I\left(Y_{2} ; X \mid \tilde{Y}_{2}\right)$,

$$
\left.I\left(\hat{S}_{1} \hat{S}_{2} ; S \mid U\right) \leq I\left(Y_{1} Y_{2} ; X \mid \tilde{Y}_{1} \tilde{Y}_{2}\right)\right\}
$$

and

$$
\begin{aligned}
\mathcal{D}_{2}^{(o)}= & \left\{\left(D_{1}, D_{2}\right): \exists p_{X}, \hat{s}_{k}\left(\tilde{y}_{k}\right), k=1,2\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k=1,2, \\
& \text { and for any pmf } p_{U \mid Y_{1} Y_{2}}, \\
& \text { one can find } p_{\tilde{Y}_{1} \tilde{Y}_{2} \mid S} p_{\tilde{U} \mid \tilde{Y}_{1} \tilde{Y}_{2}} \text { s.t. } \\
& I(S ; \tilde{U}) \leq I(X ; U) \\
& I\left(S ; \tilde{Y}_{1} \mid \tilde{U}\right) \leq I\left(X ; Y_{1} \mid U\right) \\
& I\left(S ; \tilde{Y}_{2} \mid \tilde{U}\right) \leq I\left(X ; Y_{2} \mid U\right) \\
& I\left(S ; \tilde{Y}_{1} \mid \tilde{Y}_{2} \tilde{U}\right) \leq I\left(X ; Y_{1} \mid Y_{2} U\right) \\
& I\left(S ; \tilde{Y}_{2} \mid \tilde{Y}_{1} \tilde{U}\right) \leq I\left(X ; Y_{2} \mid Y_{1} U\right) \\
& \left.I\left(S ; \tilde{Y}_{1} \tilde{Y}_{2} \mid \tilde{U}\right) \leq I\left(X ; Y_{1} Y_{2} \mid U\right)\right\} .
\end{aligned}
$$

Note that the outer bounds $\mathcal{D}_{1}^{(o)}$ and $\mathcal{D}_{2}^{(o)}$ are not trivial in general. The necessity of introducing nondegenerate variable(s) for $\mathcal{D}_{1}^{(o)}$ can be concluded from some special cases,
e.g., source broadcast over a degraded channel, quadratic Gaussian source broadcast, or Hamming binary source broadcast (see the details in the subsequent three subsections). To show the necessity of introducing a nondegenerate variable for $\mathcal{D}_{2}^{(o)}$, we consider the first three inequalities on mutual information in $\mathcal{D}_{2}^{(o)}$. Next we show that the necessary conditions

$$
\begin{align*}
& I(S ; \tilde{U}) \leq I(X ; U)  \tag{16}\\
& I\left(S ; \tilde{Y}_{1} \mid \tilde{U}\right) \leq I\left(X ; Y_{1} \mid U\right)  \tag{17}\\
& I\left(S ; \tilde{Y}_{2} \mid \tilde{U}\right) \leq I\left(X ; Y_{2} \mid U\right) \tag{18}
\end{align*}
$$

with a nondegenerate $U$ results in a tighter bound than that with a degenerate $U$ (for the latter case, the necessary conditions reduce to the single-user bound).

Suppose the broadcast channel $P_{Y_{1} Y_{2} \mid X}$ satisfies $Y_{1}=$ $\left(Y_{0}, Y_{1}^{\prime}\right), Y_{2}=\left(Y_{0}, Y_{2}^{\prime}\right)$ for some $Y_{0}, Y_{1}^{\prime}, Y_{2}^{\prime}$. Consider a lossless transmission case (Hamming distortion measure and $\left.D_{1}=D_{2}=0\right): S=\left(S_{1}, S_{2}\right), \hat{S}_{1}=S_{1}, \hat{S}_{2}=S_{2}$, and $H\left(S_{1}\right)=C_{1}$ and $H\left(S_{2}\right)=C_{2}$. Obviously, these conditions do not violate the single-user outer bound. Now we show that for some sources, these conditions violate the outer bound $\mathcal{D}_{2}^{(o)}$. Set $U=Y_{0}$ in $\mathcal{D}_{2}^{(o)}$. Then it is easy to obtain the following inequalities from (16)-(18).

$$
\begin{align*}
& I\left(S_{1} S_{2} ; V\right) \leq I\left(X ; Y_{0}\right)  \tag{19}\\
& H\left(S_{1} \mid V\right) \leq I\left(X ; Y_{1} \mid Y_{0}\right)  \tag{20}\\
& H\left(S_{2} \mid V\right) \leq I\left(X ; Y_{2} \mid Y_{0}\right) \tag{21}
\end{align*}
$$

for some $p_{V \mid S_{1} S_{2}}$. Therefore, we further have

$$
\begin{align*}
H\left(S_{1}\right) & \leq H\left(S_{1}\right)+I\left(S_{2} ; V \mid S_{1}\right)  \tag{22}\\
& =I\left(S_{1} S_{2} ; V\right)+H\left(S_{1} \mid V\right)  \tag{23}\\
& \leq I\left(X ; Y_{0}\right)+I\left(X ; Y_{1} \mid Y_{0}\right)  \tag{24}\\
& =I\left(X ; Y_{1}\right) \leq C_{1} \tag{25}
\end{align*}
$$

On the other hand, by the assumptions $H\left(S_{1}\right)=C_{1}$ and $H\left(S_{2}\right)=C_{2}$, the equalities hold in all the inequalities above, which implies $I\left(X ; Y_{1}\right)=C_{1}$ (i.e., $P_{X}$ is a capacity-achieving distribution), $I\left(S_{1} S_{2} ; V\right)=I\left(X ; Y_{0}\right)$, and $I\left(S_{2} ; V \mid S_{1}\right)=0$, i.e., $S_{2} \rightarrow S_{1} \rightarrow V$. Similarly, we have $S_{1} \rightarrow S_{2} \rightarrow V$. In addition, the Gács-Körner common information [22], [23] is defined as

$$
\begin{equation*}
C_{\mathrm{GK}}\left(S_{1} ; S_{2}\right)=\sup _{P_{V \mid S_{1} S_{2}}: S_{2} \rightarrow S_{1} \rightarrow V, S_{1} \rightarrow S_{2} \rightarrow V} I\left(S_{1} S_{2} ; V\right) \tag{26}
\end{equation*}
$$

Hence there exists $p_{V \mid S_{1} S_{2}}$ such that $S_{2} \rightarrow S_{1} \rightarrow V, S_{1} \rightarrow$ $S_{2} \rightarrow V$ and $I\left(S_{1} S_{2} ; V\right)=I\left(X ; Y_{0}\right)$, only if $C_{\mathrm{GK}}\left(S_{1} ; S_{2}\right) \geq$ $I\left(X ; Y_{0}\right)>0$ (suppose the channel $P_{Y_{0} \mid X}$ satisfies $I\left(X ; Y_{0}\right)>$ 0 for the capacity-achieving distribution $P_{X}$ ). However the Gács-Körner common information does not always exist for all source pairs $\left(S_{1}, S_{2}\right)$, e.g., $C_{\mathrm{GK}}\left(S_{1} ; S_{2}\right)=0$ for a doubly symmetric binary source. This implies the outer bound is tighter than the single-user one, which in turn implies $\mathcal{D}_{2}^{(o)}$ are not trivial in general.

## B. Discrete Memoryless Broadcast over Degraded Channel

If the channel is degraded, define

$$
\begin{align*}
\mathcal{D}_{\mathrm{DBC}}^{(i)}= & \left\{D_{[1: K]}: \exists p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}, x\left(v_{K}, s\right),\right. \\
& \hat{s}_{k}\left(v_{k}, y_{k}\right), k \in[1: K] \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k} \\
& I\left(S ; V_{k}\right) \leq \sum_{j=1}^{k} I\left(Y_{j} ; V_{j} \mid V_{j-1}\right), k \in[1: K] \\
& \text { where } \left.V_{0} \triangleq \emptyset\right\} \tag{27}
\end{align*}
$$

and

$$
\begin{align*}
& \mathcal{D}_{\mathrm{DBC}}^{(o)}= \\
& \left\{D_{[1: K]}: \exists p_{\hat{S}_{[1: K]} \mid S}, p_{X}\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K], \\
& \left(I\left(\hat{S}_{[1: k]} ; U_{k} \mid U_{k-1}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right) \\
& \text { for any } p_{U_{K-1} \mid S} p_{U_{K-2} \mid U_{K-1}} \cdots p_{U_{1} \mid U_{2}}, U_{0} \triangleq \emptyset, U_{K} \triangleq S, \\
& \text { and } \left.\left(I\left(S ; \hat{S}_{[1: k]}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)\right\}, \tag{28}
\end{align*}
$$

where

$$
\begin{align*}
& \mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)= \\
& \left\{R_{[1: K]}: R_{k} \geq 0, \exists p_{V_{K-1} \mid X} p_{V_{K-2} \mid V_{K-1}} \cdots p_{V_{1} \mid V_{2}}\right. \text { s.t. } \\
& \sum_{j=1}^{k} R_{j} \leq \sum_{j=1}^{k} I\left(Y_{j} ; V_{j} \mid V_{j-1}\right), k \in[1: K] \\
& \text { where } \left.V_{0} \triangleq \emptyset, V_{K} \triangleq X\right\} \tag{29}
\end{align*}
$$

denotes the capacity of the degraded broadcast channel $p_{Y_{[1: K]} \mid X}$ with the input $X$ following $p_{X}$, and

$$
\begin{align*}
& \mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)= \\
& \left\{R_{[1: K]}: R_{k} \geq 0, \sum_{j=1}^{k} R_{j} \geq I\left(X ; Y_{[1: k]}\right), k \in[1: K]\right\} \tag{30}
\end{align*}
$$

denotes the successive refinement coding rate region of source $X$ with reconstructions $Y_{[1: K]}$ following $p_{Y_{[1: K]} \mid X}$.

Then as a consequence of Theorem 1, the following theorem holds.

Theorem 2. For transmitting a DMS $S$ over a degraded $D M$ $B C p_{Y_{[1: K]} \mid X}$,

$$
\begin{equation*}
\mathcal{D}_{\mathrm{DBC}}^{(i)} \subseteq \mathcal{D} \subseteq \mathcal{D}_{\mathrm{DBC}}^{(o)} \tag{31}
\end{equation*}
$$

Remark 2. $\mathcal{D}_{\mathrm{DBC}}^{(o)}$ can be also expressed as

$$
\begin{aligned}
\mathcal{D}_{\mathrm{DBC}}^{(o)}= & \left\{D_{[1: K]}: \exists p_{\hat{S}_{[1: K]} \mid S}, p_{X}\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K] \\
& \mathcal{R}_{\mathrm{DBC}}\left(p_{S} p_{\hat{S}_{[1: K]}^{\prime} \mid S}\right) \subseteq \mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)
\end{aligned}
$$

$$
\begin{align*}
& \mathcal{R}_{\mathrm{SRC}}\left(p_{S} p_{\hat{S}_{[1: K]}^{\prime} \mid S}\right) \supseteq \mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right) \\
& \left.\hat{S}_{k}^{\prime} \triangleq \hat{S}_{[1: k]}, k \in[1: K]\right\} . \tag{32}
\end{align*}
$$

From the last constraint of $\mathcal{D}_{\text {DBC }}^{(o)}$, one can obtain an interesting conclusion: the single-user outer bound $D_{[1: K]}^{*}$ can be achieved for source broadcast over a degraded channel only if the source is successively refinable.
Remark 3. If $p_{S}$ is Gaussian and all $d_{k}\left(s, \hat{s}_{k}\right), k \in[1: K]$ are the quadratic distortion function (i.e., $d_{k}\left(s, \hat{s}_{k}\right)=(s-$ $\left.\hat{s}_{k}\right)^{2}, k \in[1: K]$ ), then the smallest capacity region (with input distribution restricted to $\left.p_{S}\right) \mathcal{R}_{\mathrm{DBC}}\left(p_{S} p_{\hat{S}_{[1: K]}^{\prime} \mid S}\right)$ over all $p_{\hat{S}_{[1: K]} \mid S}$ is obtained by setting $p_{\hat{S}_{[1: K]} \mid S}$ as the Gaussian broadcast channel such that for $k \in[1: K], S=\hat{S}_{k}+E_{k}$ and $\hat{S}_{k} \sim \mathcal{N}\left(0, N_{S}-D_{k}\right), E_{k} \sim \mathcal{N}\left(0, D_{k}\right)$ are independent. That is, setting $p_{\hat{S}_{[1: K]} \mid S}$ as the Gaussian broadcast channel is "optimal" for this case. This point is obtained by observing that: 1) the capacity region for this case is

$$
\begin{align*}
& \mathcal{C}=\bigcup_{+\infty=\tau_{0} \geq \tau_{1} \geq \cdots \geq \tau_{K}=0} \\
& \left\{R_{[1: K]}: R_{k} \leq \frac{1}{2} \log \frac{\left(D_{k}+\tau_{k-1}\right)\left(N_{S}+\tau_{k}\right)}{\left(D_{k}+\tau_{k}\right)\left(N_{S}+\tau_{k-1}\right)}, k \in[1: K]\right\} \tag{33}
\end{align*}
$$

2) for any other channels $p_{\hat{S}_{[1: K]} \mid S}$ satisfying $\mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq$ $D_{k}, k \in[1: K], \mathcal{C}$ is a subset (or an inner bound) of the capacity region (with input distribution restricted to $p_{S}$ ) of $p_{\hat{S}_{[1: K]} \mid S}$ [9]. Therefore, Gaussian broadcast channels have the smallest capacity regions given noise powers (here noise is not restricted to be independent of the channel input). This observation is consistent with the point-to-point case [14, Problem 10.8].

Note that the last constraint of $\mathcal{D}_{\text {DBC }}^{(i)}$ can be understood as the intersection between the successive refinement rate region of the source $S$ with reconstructions $V_{[1: K]}$ and the capacity of the degraded broadcast channel $p_{Y_{[1: K]} \mid X}$ with the input $X$ and auxiliary random variables $V_{[1: K]}$, is not empty. The second constraint of $\mathcal{D}_{\mathrm{DBC}}^{(o)}$ can be understood as the capacity of the virtual degraded broadcast channel $p_{\hat{S}_{[1: K]}^{\prime} \mid S}$ with the input $S$ is included in the capacity of the physical degraded broadcast channel $p_{Y_{[1: K]} \mid X}$ with the input $X$. Similarly, the last constraint of $\mathcal{D}_{\mathrm{DBC}}^{(o)}$ can be understood as the successive refinement rate region of the physical source $S$ with reconstructions $\hat{S}_{[1: K]}^{\prime}$ includes the successive refinement rate region of the virtual source $X$ with reconstructions $Y_{[1: K]}$.

## C. Hamming Binary Broadcast

Consider sending a binary source $S \sim \operatorname{Bern}\left(\frac{1}{2}\right)$ with the Hamming distortion measure $d_{k}(s, \hat{s})=d(s, \hat{s}) \triangleq 0$, if $s=$ $\hat{s} ; 1$, otherwise, over a binary broadcast channel $Y_{k}=X \oplus$ $W_{k}, 1 \leq k \leq K$ with $W_{k} \sim \operatorname{Bern}\left(p_{k}\right), \frac{1}{2} \geq p_{1} \geq p_{2} \geq \cdots \geq$ $p_{K} \geq 0$. Assume the bandwidth mismatch factor is $b$.

We first consider the inner bound part. For bandwidth expansion $(b>1)$ case, as a special case of hybrid coding, systematic source-channel coding (or Uncoded Systematic

Coding) was first investigated in [6]. For any point-to-point lossless communication, such systematic coding does not lose the optimality; however, for some lossy transmission cases such as Hamming binary source communication, it is not optimal any more [6]. To retain the optimality, we can first quantize the source $S$, and then transmit the quantized signal using Uncoded Systematic Coding. The performance of such code can be obtained directly from Theorem 2.

Specifically, let $U_{2}=S \oplus E_{2}, U_{1}=U_{2} \oplus E_{1}$ with $E_{2} \sim \operatorname{Bern}\left(D_{2}\right), E_{1} \sim \operatorname{Bern}\left(d_{1}\right)$. Let $V_{2}=\left(U_{2}, X^{b-1}\right), V_{1}=$ $\left(U_{1}, X_{1}^{b-1}\right), X_{1}^{b-1}=X^{b-1} \oplus B^{b-1}$, where $X_{1}^{b-1}$ and $X^{b-1}$ are independent of $U_{2}$ and $U_{1}$, and $X^{b-1}$ and $B^{b-1}$ follow $b-1$ dimensional $\operatorname{Bern}\left(\frac{1}{2}\right)$ and $\operatorname{Bern}(\theta)$, respectively. Let $x^{b}\left(v_{2}, s\right)=\left(u_{2}, x^{b-1}\right), \hat{s}_{2}\left(v_{2}, y_{2}^{b}\right)=u_{2}$, and $\hat{s}_{1}\left(v_{1}, y_{1}^{b}\right)=$ $u_{1}$, if $d_{1}<p_{1} ; y_{1}$, otherwise, where $y_{1}$ is the first letter of $y_{1}^{b}$. Substituting these variables and functions into the inner bound $\mathcal{D}_{\text {DBC }}^{(i)}$ in Theorem 2, we get the following corollary.
Corollary 1 (Coded Systematic Coding). For transmitting a binary source $S$ with the Hamming distortion measure over a 2-user binary broadcast channel with the bandwidth mismatch factor $b$,
$\mathcal{D} \supseteq \mathcal{D}_{\mathrm{CSC}}^{(i)} \triangleq$ convexhull $\left\{\left(D_{1}, D_{2}\right):\right.$
$0 \leq \theta, d_{1} \leq \frac{1}{2}$,
$D_{1} \geq \min \left\{d_{1} \star D_{2}, p_{1} \star D_{2}\right\}$,
$r_{1} \triangleq 1-H_{2}\left(d_{1} \star p_{1}\right)+(b-1)\left[1-H_{2}\left(\theta \star p_{1}\right)\right]$,
$r_{2} \triangleq H_{2}\left(d_{1} \star p_{2}\right)-H_{2}\left(p_{2}\right)+(b-1)\left[H_{2}\left(\theta \star p_{2}\right)-H_{2}\left(p_{2}\right)\right]$,
$1-H_{2}\left(d_{1} \star D_{2}\right) \leq r_{1}$,
$\left.1-H_{2}\left(D_{2}\right) \leq r_{1}+r_{2}\right\}$,
where $\star$ denotes an binary operation such that

$$
\begin{equation*}
x \star y=(1-x) y+x(1-y) \tag{35}
\end{equation*}
$$

and $\mathrm{H}_{2}$ denotes the binary entropy function, i.e.,

$$
\begin{equation*}
H_{2}(p)=-p \log p-(1-p) \log (1-p) \tag{36}
\end{equation*}
$$

Remark 4. Coded Systematic Coding without timesharing does not always lead to a convex distortion region, hence a timesharing mechanism is needed to improve the performance. This is equivalent to adding a timesharing variable $Q$ into $V_{2}$ and $V_{1}$, before substituting them into the inner bound $\mathcal{D}_{\mathrm{DBC}}^{(i)}$. Besides, note that unlike Uncoded Systematic Coding, the Coded Systematic Coding could always achieve the optimal distortion for at least one of the receivers. Moreover, unlike separate coding the Coded Systematic Coding weakens the cliff effect, and results in a slope-cliff effect.

The outer bound of Theorem 2 reduces to the following outer bound for the Hamming binary source broadcast problem. This outer bound was first given in [11, Eqn (41)] for the 2-user case.
Theorem 3. [11, Eqn (41)] For transmitting a binary source $S$ with the Hamming distortion measure over a $K$-user binary broadcast channel with the bandwidth mismatch factor $b$,
$\mathcal{D} \subseteq \mathcal{D}_{\mathrm{DBC}}^{(o)} \triangleq$


Fig. 4. Distortion bounds for sending a binary source over a binary broadcast channel with $b=2, p_{1}=0.18, p_{2}=0.12$. Outer Bounds 1 and 2 respectively correspond to the outer bound of Theorem 3 and the outer bound of Theorem 6. Separate Coding, Uncoded Systematic Coding, and Coded Systematic Coding respectively correspond to the separate scheme (combining successive-refinement code [16, Example 13.3] with superposition code [16, Example 5.3]), the inner bound in Corollary 2, and the inner bound in Corollary 1. Single-user Outer Bounds 1 and 2 correspond to the single-user outer bound (6) and the single-user Wyner-Ziv outer bound (41), respectively. Besides, Single-user Outer Bound 2, Outer Bound 2 and Uncoded Systematic Coding can be considered as outer bounds and inner bound for the Wyner-Ziv source broadcast problem with $b=1, \beta_{1}=p_{1}, \beta_{2}=p_{2}$.

$$
\begin{align*}
& \left\{D_{[1: K]}: \text { For any values } \frac{1}{2}=\tau_{0} \geq \tau_{1} \geq \cdots \geq \tau_{K}=0\right. \\
& \left.\frac{1}{b}\left(H_{2}\left(\tau_{k-1} \star D_{k}\right)-H_{2}\left(\tau_{k} \star D_{k}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{BBC}}\right\} \tag{37}
\end{align*}
$$

where $\mathcal{R}_{\mathrm{BBC}}$ denotes the capacity of the binary broadcast channel given by

$$
\begin{align*}
& \mathcal{R}_{\mathrm{BBC}}= \\
& \left\{R_{[1: K]}: \exists \text { some values } \frac{1}{2}=\theta_{0} \geq \theta_{1} \geq \cdots \geq \theta_{K}=0\right. \text { s.t. } \\
& \left.0 \leq R_{k} \leq H_{2}\left(\theta_{k-1} \star p_{k}\right)-H_{2}\left(\theta_{k} \star p_{k}\right), k \in[1: K]\right\} \tag{38}
\end{align*}
$$

The bounds in Corollary 1 and Theorem 3 are illustrated in Fig. 4.

## D. Quadratic Gaussian Broadcast

Consider sending a Gaussian source $S \sim \mathcal{N}\left(0, N_{S}\right)$ with the quadratic distortion measure $d_{k}(s, \hat{s})=d(s, \hat{s}) \triangleq(s-\hat{s})^{2}$ over a power-constrained Gaussian broadcast channel $Y_{k}=$ $X+W_{k}, 1 \leq k \leq K$ with $\mathbb{E}\left(X^{2}\right) \leq P$ and $W_{k} \sim$ $\mathcal{N}\left(0, N_{k}\right), N_{1} \geq N_{2} \geq \cdots \geq N_{K}$. Assume the bandwidth mismatch factor is $b$. As stated in Remark 1, Theorem 1 (or 2) holds for the Gaussian case as well. The inner bound $\mathcal{D}_{\text {DBC }}^{(i)}$ in Theorem 2 could recover the best known inner bound [7, Thm. 5] by setting the random variables and symbol-bysymbol mappings to some suitable ones. On the other hand, setting $U_{[1: K-1]}$ to be jointly Gaussian with $S$, the outer bound


Fig. 5. Distortion bounds for sending a Gaussian source over a Gaussian broadcast channel with $b=2, N_{S}=1, P=50, N_{1}=10, N_{2}=1$. Outer Bounds 1 and 2 and Inner Bounds 1 and 2 respectively correspond to the outer bound in [9, Thm. 2], the outer bound in Theorem 7, the inner bound in [7, Thm. 5], and the inner bound achieved by Wyner-Ziv separate coding (uncoded systematic code) [17, Lem. 3]. Single-user Outer Bound corresponds to the single-user outer bound (6). Besides, Outer Bound 2 and Inner Bound 2 can be considered as an outer bound and an inner bound for the Wyner-Ziv source broadcast problem with $b=1, \beta_{1}=\frac{N_{S} N_{1}}{P+N_{1}}, \beta_{2}=\frac{N_{S} N_{2}}{P+N_{2}}$. For this case, Single-user Outer Bound corresponds to the single-user Wyner-Ziv outer bound (41).


Fig. 6. Wyner-Ziv source broadcast system: a broadcast communication system with side information at decoders.
$\mathcal{D}_{\text {DBC }}^{(o)}$ in Theorem 2 could recover the best known outer bound [9, Thm. 2]. The bounds in [7, Thm. 5] and [9, Thm. 2] are illustrated in Fig. 5.

## IV. Wyner-Ziv Source Broadcast: Source Broadcast with Side Information

We now extend the source broadcast problem by allowing decoders to access side information correlated with the source. As depicted in Fig. 6, receiver $k$ observes memoryless side information $Z_{k}^{n}$, and it produces a source reconstruction $\hat{S}_{k}^{n}$ from the received signal $Y_{k}^{n}$ and side information $Z_{k}^{n}$.
Definition 7. An $n$-length Wyner-Ziv source-channel code is defined by the encoding function $x^{n}: \mathcal{S}^{n} \mapsto \mathcal{X}^{n}$ and $K$ decoding functions $\hat{s}_{k}^{n}: \mathcal{Y}_{k}^{n} \times \mathcal{Z}_{k}^{n} \mapsto \hat{\mathcal{S}}_{k}^{n}, 1 \leq k \leq K$.

Definition 8. If there exists a sequence of Wyner-Ziv sourcechannel codes satisfying

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \mathbb{E} d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \leq D_{k} \tag{39}
\end{equation*}
$$

then we say that the distortion tuple $D_{[1: K]}$ is achievable.
Definition 9. The admissible distortion region for the WynerZiv broadcast problem is defined as

$$
\begin{equation*}
\mathcal{D}_{\mathrm{SI}} \triangleq\left\{D_{[1: K]}: D_{[1: K]} \text { is achievable }\right\} \tag{40}
\end{equation*}
$$

Shamai et al. [6, Thm. 2.1] showed that for transmitting a source over a point-to-point channel $p_{Y_{k} \mid X}$ with side information $Z_{k}$ available at the decoder, the minimum achievable distortion for the receiver $k$ satisfies $R_{S \mid Z_{k}}\left(D_{k}\right)=C_{k}$, where $R_{S \mid Z_{k}}(\cdot)$ is the Wyner-Ziv rate-distortion function of the source $S$ given that the decoder observes $Z_{k}$ [16]. Therefore, the optimal achievable distortion is $D_{\mathrm{SI}, k}^{*}=R_{S \mid Z_{k}}^{-1}\left(C_{k}\right)$. Obviously,

$$
\begin{equation*}
\mathcal{D}_{\mathrm{SI}} \subseteq \mathcal{D}_{\mathrm{SI}}^{*} \triangleq\left\{D_{[1: K]}: D_{k} \geq D_{\mathrm{SI}, k}^{*}, 1 \leq k \leq K\right\} \tag{41}
\end{equation*}
$$

where $\mathcal{D}_{\mathrm{SI}}^{*}$ is named single-user Wyner-Ziv outer bound.
Besides, we also consider the bandwidth mismatch case, whereby $m$ samples of a DMS are transmitted through $n$ uses of a DM-BC with $l$ samples of side information available at each decoder. For simplicity, we assume $m=l$. For this case, the bandwidth mismatch factor is defined as $b=\frac{n}{m}$.

## A. Discrete Memoryless Wyner-Ziv Broadcast

If consider $S$ and $Z_{[1: K]}$ as the input and outputs of a virtual broadcast channel $p_{Z_{[1: K]} \mid S}$, then the Wyner-Ziv source broadcast problem with the encoding function $x^{n}\left(s^{n}\right)$ is equivalent to the source broadcast problem of sending $S$ over $p_{Z_{[1: K]} \mid S} p_{Y_{[1: K]} \mid X}$ with the encoding function $\left(s^{n}, x^{n}\left(s^{n}\right)\right)$. Correspondingly, the Wyner-Ziv source broadcast system can be considered as a (uncoded) systematic source-channel coding system. Hence by setting the symbol-by-symbol function as $\left(s, x\left(v_{[1: N]}, s\right)\right)$, from Theorem 1, we obtain the following inner bound for the Wyner-Ziv source broadcast problem.

$$
\begin{align*}
\mathcal{D}_{\mathrm{Sl}}^{(i)}= & \left\{D_{[1: K]}: \exists p_{V_{[1: N]} \mid S}, r_{[1: N]}, x\left(v_{[1: N]}, s\right),\right. \\
& \hat{s}_{k}\left(v_{\mathcal{B}_{k}}, y_{k}, z_{k}\right), k \in[1: K] \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K] \\
& \sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}} \mid S\right) \\
& \text { for all } \mathcal{J} \subseteq[1: N] \text { s.t. } \mathcal{J} \neq \emptyset \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}, \\
& \sum_{j \in \mathcal{J}^{c}} r_{j}<\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J} c} \mid Y_{k} Z_{k} V_{\mathcal{J}}\right), \\
& k \in[1: K] \text { for all } \mathcal{J} \subseteq \mathcal{B}_{k} \text { s.t. } \\
& \left.\mathcal{J}^{c} \triangleq \mathcal{B}_{k} \backslash \mathcal{J} \neq \emptyset \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}\right\} . \tag{42}
\end{align*}
$$

In addition, regard $\left(U_{[1: L]}, Z_{[1: K]}\right)$ as auxiliary random variables following $p_{U_{[1: L]} \mid S} p_{Z_{[1: K]} \mid S}$ given $S$, then following steps similar to the proof of the outer bound $\mathcal{D}_{1}^{(o)}$ of Theorem 1 , we can obtain the following outer bound on $\mathcal{D}_{\mathrm{SI}}$.

$$
\mathcal{D}_{\mathrm{SI}, 1}^{(o)}=\left\{D_{[1: K]}: \exists p_{\hat{S}_{[1: K]} \mid S, Z_{[1: K]}}\right. \text { s.t. }
$$

$$
\begin{align*}
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K], \\
& \text { and for any } p_{U_{[1: L]} \mid S, Z_{[1: K]}}, \\
& \text { one can find } p_{\tilde{U}_{[1: L]}, \tilde{Z}_{[1: K]}, X} \text { s.t. } \\
& I\left(\hat{S}_{\mathcal{A}} ; U_{\mathcal{B}} \mid U_{\mathcal{C}} Z_{\mathcal{A}}\right) \leq I\left(Y_{\mathcal{A}} ; \tilde{U}_{\mathcal{B}} \mid \tilde{U}_{\mathcal{C}} \tilde{Z}_{\mathcal{A}}\right) \\
& \text { for any } \mathcal{A} \subseteq[1: K], \mathcal{B}, \mathcal{C} \subseteq[1: L]\}, \tag{43}
\end{align*}
$$

Similarly, following steps similar to the proof of the outer bound $\mathcal{D}_{2}^{(o)}$ of Theorem 1, we can prove another outer bound on $\mathcal{D}_{\mathrm{SI}}$.

$$
\begin{align*}
\mathcal{D}_{\mathrm{SI}, 2}^{(o)}= & \left\{D_{[1: K]}: \exists p_{X}, \hat{s}_{k}^{n}\left(\tilde{y}_{k}, z_{k}^{n}\right), k \in[1: K]\right. \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, k \in[1: K], \\
& \text { and for any pmf } p_{U_{[1: L]} \mid Y_{[1: K]}}, \\
& \text { one can find } p_{\tilde{Y}_{[1: K]} \mid S} p_{\tilde{U}_{[1: L]} \mid \tilde{Y}_{[1: K]}} \text { s.t. } \\
& I\left(S^{n} ; \tilde{Y}_{\mathcal{B}} \tilde{U}_{\mathcal{B}^{\prime}} \mid \tilde{Y}_{\mathcal{C}} \tilde{U}_{\mathcal{C}^{\prime}}\right) \leq I\left(X ; Y_{\mathcal{B}} U_{\mathcal{B}^{\prime}} \mid Y_{\mathcal{C}} U_{\mathcal{C}^{\prime}}\right) \\
& \text { for any } \left.\mathcal{B}, \mathcal{C} \subseteq[1: K], \mathcal{B}^{\prime}, \mathcal{C}^{\prime} \subseteq[1: L]\right\} . \tag{44}
\end{align*}
$$

Therefore, the following theorem holds. The proof is omitted.

Theorem 4. For transmitting a DMS $S$ over a $D M-B C$ $p_{Y_{[1: K]} \mid X}$ with side information $Z_{k}$ at the decoder $k(k \in[1:$ $K]$ ),

$$
\begin{equation*}
\mathcal{D}_{\mathrm{SI}}^{(i)} \subseteq \mathcal{D}_{\mathrm{SI}} \subseteq \mathcal{D}_{\mathrm{SI}, 1}^{(o)} \cap \mathcal{D}_{\mathrm{SI}, 2}^{(o)} \tag{45}
\end{equation*}
$$

Remark 5. Similar to Theorem 1, Theorem 4 could be extended to a Gaussian or any other well-behaved continuousalphabet source-channel pair. It also can be extended to the problems of broadcasting Wyner-Ziv correlated sources, and Wyner-Ziv source broadcast with channel input cost.

## B. Discrete Memoryless Wyner-Ziv Broadcast over Degraded Channel with Degraded Side Information

Theorem 4 can be used to derive an inner bound and an outer bound for the degraded channel and degraded side information case. Define

$$
\begin{align*}
\mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(i)}= & \left\{D_{[1: K]}: \exists p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}, x\left(v_{K}, s\right),\right. \\
& \hat{s}_{k}\left(v_{k}, y_{k}, z_{k}\right), k \in[1: K] \text { s.t. } \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k} \\
& I\left(S ; V_{k}\right) \leq \sum_{j=1}^{k} I\left(Y_{j} Z_{j} ; V_{j} \mid V_{j-1}\right), k \in[1: K] \tag{46}
\end{align*}
$$

where $\left.V_{0} \triangleq \emptyset\right\}$,
and

$$
\begin{aligned}
& \mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)}=\left\{D_{[1: K]}: \exists p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}, p_{X},\right. \\
& \hat{s}_{k}\left(v_{k}, z_{k}\right), k \in[1: K] \mathrm{s.t.} \\
& \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k}, \\
& \left(I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)
\end{aligned}
$$

for any pmf $p_{U_{K-1} \mid S} p_{U_{K-2} \mid U_{K-1}} \cdots p_{U_{1} \mid U_{2}}$,
$U_{0} \triangleq \emptyset, U_{K} \triangleq S$,
and $\left.\left(I\left(V_{k} ; S \mid Z_{k}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)\right\}$,
where $\mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)$ and $\mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)$ are given in (29) and (30), respectively. Then we have the following theorem. The proof is analogous to that of Theorem 2, and therefore omitted.

Theorem 5. For transmitting a DMS $S$ over a degraded DM$B C p_{Y_{[1: K]} \mid X}\left(X \rightarrow Y_{K} \rightarrow Y_{K-1} \rightarrow \cdots \rightarrow Y_{1}\right)$ with degraded side information $Z_{k}\left(S \rightarrow Z_{K} \rightarrow Z_{K-1} \rightarrow \cdots \rightarrow Z_{1}\right)$ at the decoder $k(k \in[1: K])$,

$$
\begin{equation*}
\mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(i)} \subseteq \mathcal{D}_{\mathrm{SI}} \subseteq \mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)} \tag{48}
\end{equation*}
$$

Remark 6. $\mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)}$ can be also expressed as

$$
\begin{align*}
\mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)}= & \left\{D_{[1: K]}: \exists p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}, \hat{s}_{k}\left(v_{k}, z_{k}\right)\right. \\
& k \in[1: K] \text { s.t. } \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq D_{k} \\
& \mathcal{R}_{\mathrm{DBC}-\mathrm{SI}}\left(p_{S} p_{Z_{[1: K]} \mid S} p_{V_{[1: K]} \mid S, Z_{[1: K]}}\right) \\
& \subseteq \mathcal{R}_{\mathrm{DBC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right) \\
& \mathcal{R}_{\mathrm{SRC}-\mathrm{SI}}\left(p_{S} p_{Z_{[1: K]} \mid S} p_{V_{[1: K]} \mid S, Z_{[1: K]}}\right) \\
& \left.\supseteq \mathcal{R}_{\mathrm{SRC}}\left(p_{X} p_{Y_{[1: K]} \mid X}\right)\right\} \tag{49}
\end{align*}
$$

where

$$
\begin{align*}
& \mathcal{R}_{\mathrm{DBC}-\mathrm{SI}}\left(p_{X} p_{Z_{[1: K]} \mid X} p_{Y_{[1: K]} \mid X, Z_{[1: K]}}\right) \triangleq \\
& \left\{R_{[1: K]}: R_{k} \geq 0, \exists p_{V_{K-1} \mid X} p_{V_{K-2} \mid V_{K-1}} \cdots p_{V_{1} \mid V_{2}}\right. \text { s.t. } \\
& \sum_{j=1}^{k} R_{j} \leq \sum_{j=1}^{k} I\left(Y_{j} ; V_{j} \mid V_{j-1} Z_{j}\right) \\
& \left.k \in[1: K], \text { where } V_{0} \triangleq \emptyset, V_{K} \triangleq X\right\} \tag{50}
\end{align*}
$$

and

$$
\begin{align*}
& \mathcal{R}_{\mathrm{SRC}-\mathrm{SI}}\left(p_{X} p_{Z_{[1: K]} \mid X} p_{Y_{[1: K]} \mid X, Z_{[1: K]}}\right) \triangleq \\
& \left\{R_{[1: K]}: R_{k} \geq 0, \sum_{j=1}^{k} R_{j} \geq I\left(X ; Y_{k} \mid Z_{k}\right), k \in[1: K]\right\} \tag{51}
\end{align*}
$$

## C. Wyner-Ziv Binary Broadcast

Consider sending a binary source $S \sim \operatorname{Bern}\left(\frac{1}{2}\right)$ with the Hamming distortion measure $d_{k}(s, \hat{s})=d(s, \hat{s}) \triangleq 0$, if $s=$ $\hat{s} ; 1$, otherwise, over a binary broadcast channel $Y_{k}=X \oplus$ $W_{k}, 1 \leq k \leq K$ with $W_{k} \sim \operatorname{Bern}\left(p_{k}\right), \frac{1}{2} \geq p_{1} \geq p_{2} \geq \cdots \geq$ $p_{K} \geq 0$. Assume the side information $Z_{k}$ observed by the receiver $k$ satisfies $S=Z_{k} \oplus B_{k}$ with independent variables $Z_{k} \sim \operatorname{Bern}\left(\frac{1}{2}\right)$ and $B_{k} \sim \operatorname{Bern}\left(\beta_{k}\right)$. Assume the bandwidth mismatch factor is $b$.

Let $V_{1}=\left(U_{1}, X_{1}^{b}\right), V_{2}=\left(U_{2}, X^{b}\right), V_{3}=\emptyset$ with $U_{1}$ and $U_{2}$ are independent of $X_{1}^{b}$ and $X^{b} . S, U_{2}$ and $U_{1}$ satisfy the distribution $p_{S} p_{U_{2} \mid S} p_{U_{1} \mid U_{2}}$, where

$$
\begin{gather*}
p_{U_{2} \mid S}=\begin{array}{c}
0 \\
1
\end{array}\left(\begin{array}{ccc}
q_{2} \bar{\alpha}_{2} & q_{2} \alpha_{2} & 2 \\
\bar{q}_{2} \\
q_{2} \alpha_{2} & q_{2} \bar{\alpha}_{2} & \bar{q}_{2}
\end{array}\right)  \tag{52}\\
\begin{array}{ccc}
0 & 1 & 2 \\
p_{U_{1} \mid U_{2}}= & 1 \\
2
\end{array}\left(\begin{array}{ccc}
q_{1}^{\prime} \bar{\alpha}_{1}^{\prime} & q_{1}^{\prime} \alpha_{1}^{\prime} & \bar{q}_{1}^{\prime} \\
q_{1}^{\prime} \alpha_{1}^{\prime} & q_{1}^{\prime} \bar{\alpha}_{1}^{\prime} & \bar{q}_{1}^{\prime} \\
0 & 0 & 1
\end{array}\right), \tag{53}
\end{gather*}
$$

with $0 \leq q_{2}, q_{1}^{\prime} \leq 1,0 \leq \alpha_{2}, \alpha_{1}^{\prime} \leq \frac{1}{2}$. $X^{b}$ and $X_{1}^{b}$ satisfy $X_{1}^{b}=X^{b} \oplus B^{b}, X^{b} \sim b$-dimensional $\operatorname{Bern}\left(\frac{1}{2}\right)$, and $B^{b} \sim$ $b$-dimensional $\operatorname{Bern}(\theta)$ with $0 \leq \theta \leq \frac{1}{2}$. Denote $\alpha_{1}=\alpha_{2} \star$ $\alpha_{1}^{\prime}, q_{1}=q_{2} q_{1}^{\prime}$, and set $x^{b}\left(v_{2}, s\right)=x^{b}$ and for $k=1,2$,

$$
\hat{s}_{k}\left(v_{k}, y_{k}^{b}, z_{k}\right)= \begin{cases}z_{k}, & \text { if } \alpha_{k} \geq \beta_{k} \text { or } \alpha_{k}<\beta_{k}, u_{k}=2  \tag{54}\\ u_{k}, & \text { if } \alpha_{k}<\beta_{k}, u_{k}=0,1\end{cases}
$$

Substituting these random variables and functions into $\mathcal{D}_{\mathrm{SI}}^{(i)}$ in Theorem 4 (for this case, the hybrid coding reduces to a layered digital coding), we get the following performance, which is tighter than that of the Layered Description Scheme (LDS) [17, Lem. 4].
Corollary 2 (Layered Digital Coding). For transmitting a binary source $S$ with the Hamming distortion measure over a 2-user binary broadcast channel with side information $Z_{k}$ at the decoder $k(k \in[1: K])$,

$$
\begin{align*}
& \mathcal{D}_{\mathrm{SI}} \supseteq \mathcal{D}_{\mathrm{LDC}}^{(i)} \triangleq \\
& \left\{\left(D_{1}, D_{2}\right): 0 \leq q_{1} \leq q_{2} \leq 1\right. \\
& 0 \leq \alpha_{2} \leq \alpha_{1} \leq \frac{1}{2}, 0 \leq \theta \leq \frac{1}{2} \\
& q_{1} r\left(\alpha_{1}, \beta_{1}\right) \leq b\left(1-H_{2}\left(\theta \star p_{1}\right)\right) \\
& q_{1} r\left(\alpha_{1}, \beta_{2}\right) \leq b\left(1-H_{2}\left(\theta \star p_{2}\right)\right) \\
& q_{2} r\left(\alpha_{2}, \beta_{2}\right) \leq b\left(1-H_{2}\left(p_{2}\right)\right) \\
& q_{1} r\left(\alpha_{1}, \beta_{1}\right)+\left(q_{2} r\left(\alpha_{2}, \beta_{2}\right)-q_{1} r\left(\alpha_{1}, \beta_{2}\right)\right) \\
& \quad \leq b\left(1-H_{2}\left(\theta \star p_{1}\right)\right)+b\left(H_{2}\left(\theta \star p_{2}\right)-H_{2}\left(p_{2}\right)\right) \\
& \left.D_{i} \leq q_{i} \min \left\{\alpha_{i}, \beta_{i}\right\}+\left(1-q_{i}\right) \beta_{i}, i=1,2\right\} \tag{55}
\end{align*}
$$

where

$$
\begin{equation*}
r(\alpha, \beta)=H_{2}(\alpha \star \beta)-H_{2}(\alpha) \tag{56}
\end{equation*}
$$

$\star$ denotes the binary operation given in (35), and $\mathrm{H}_{2}$ denotes the binary entropy function given in (36).

In addition, the outer bound of Theorem 5 reduces to the following one for the Wyner-Ziv binary case. The proof is given in Appendix D.

Theorem 6. For transmitting a binary source $S$ with the Hamming distortion measure over a $K$-user binary broadcast channel with degraded side information $Z_{k}\left(\frac{1}{2} \geq \beta_{1} \geq \beta_{2} \geq\right.$ $\left.\cdots \geq \beta_{K} \geq 0\right)$ at the decoder $k(k \in[1: K])$,

$$
\begin{aligned}
& \mathcal{D}_{\mathrm{SI}} \subseteq \mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)} \triangleq \\
& \left\{D_{[1: K]}: \exists 0 \leq \alpha_{1}, \alpha_{2}, \cdots, \alpha_{K} \leq \frac{1}{2}\right. \text { s.t. }
\end{aligned}
$$

$$
\begin{align*}
& \alpha_{k} \leq D_{k}^{\prime} \triangleq \min \left\{D_{k}, \beta_{k}\right\}, k \in[1: K] \\
& \text { and for any values } \frac{1}{2}=\tau_{0} \geq \tau_{1} \geq \cdots \geq \tau_{K}=0 \\
& \frac{1}{b}\left(\eta _ { k } \left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right.\right. \\
& \left.\quad-\left(H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)-H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k-1}\right)\right)\right): \\
& \left.\quad k \in[1: K]) \in \mathcal{R}_{\mathrm{BBC}}\right\} \tag{57}
\end{align*}
$$

where $\mathcal{R}_{\mathrm{BBC}}$ denotes the capacity region of the binary broadcast channel given in (38),

$$
\begin{align*}
& \eta_{k} \triangleq \begin{cases}\frac{\beta_{k}-D_{k}^{\prime}}{\beta_{k}-\alpha_{k}}, & \text { if } \alpha_{k}<\beta_{k} \\
0, & \text { otherwise },\end{cases}  \tag{58}\\
& H_{4}(x, y, z) \triangleq-(x y z+\overline{x y z}) \log (x y z+\overline{x y z}) \\
&-(x \bar{y} z+\bar{x} y \bar{z}) \log (x \bar{y} z+\bar{x} y \bar{z}) \\
&-(x y \bar{z}+\overline{x y} z) \log (x y \bar{z}+\overline{x y} z) \\
&-(x \overline{y z}+\bar{x} y z) \log (x \overline{y z}+\bar{x} y z), \tag{59}
\end{align*}
$$

and $\bar{x} \triangleq 1-x$.
The bounds in Corollary 2 and Theorem 6 are shown in Fig. 4.

## D. Wyner-Ziv Gaussian Broadcast

Consider sending a Gaussian source $S \sim \mathcal{N}\left(0, N_{S}\right)$ with the quadratic distortion measure $d_{k}(s, \hat{s})=d(s, \hat{s}) \triangleq(s-\hat{s})^{2}$ over a power-constrained Gaussian broadcast channel $Y_{k}=$ $X+W_{k}, 1 \leq k \leq K$ with $\mathbb{E}\left(X^{2}\right) \leq P$ and $W_{k} \sim$ $\mathcal{N}\left(0, N_{k}\right), N_{1} \geq N_{2} \geq \cdots \geq N_{K}$. Assume the side information $Z_{k}$ observed by the receiver $k$ satisfies $S=Z_{k}+B_{k}$ with independent Gaussian variables $Z_{k} \sim \mathcal{N}\left(0, N_{S}-\beta_{k}\right)$ and $B_{k} \sim \mathcal{N}\left(0, \beta_{k}\right)$. Assume the bandwidth mismatch factor is $b$. As stated in Remark 5, Theorem 4 (or 5) holds for the Gaussian case as well. The inner bound of Theorem 5 recovers the existing results in [17], [18], and the outer bound of Theorem 5 generates the following outer bound for the Wyner-Ziv Gaussian source broadcast problem. The proof is given in Appendix E.
Theorem 7. For transmitting a Gaussian source $S$ over a Gaussian broadcast channel with degraded side information $Z_{k}\left(\beta_{1} \geq \beta_{2} \geq \cdots \geq \beta_{K}\right)$ at the decoder $k(k \in[1: K])$, $\mathcal{D}_{\mathrm{SI}} \subseteq \mathcal{D}_{\mathrm{SI}-\mathrm{D}}^{(o)} \triangleq$
$\left\{D_{[1: K]}:\right.$ For any values $+\infty=\tau_{0} \geq \tau_{1} \geq \cdots \geq \tau_{K}=0$,
$\left.\frac{1}{b}\left(\frac{1}{2} \log \frac{\left(D_{k}+\tau_{k-1}\right)\left(\beta_{k}+\tau_{k}\right)}{\left(D_{k}+\tau_{k}\right)\left(\beta_{k}+\tau_{k-1}\right)}: k \in[1: K]\right) \in \mathcal{R}_{\mathrm{GBC}}\right\}$,
where $\mathcal{R}_{\mathrm{GBC}}$ denotes the capacity of Gaussian broadcast channel given by

$$
\begin{align*}
\mathcal{R}_{\mathrm{GBC}}= & \left\{R_{[1: K]}: R_{k} \geq 0, k \in[1: K], N_{K+1}=0,\right. \\
& \left.\sum_{k=1}^{K}\left(N_{k}-N_{k+1}\right) \exp \left(2 \sum_{j=1}^{k} R_{j}\right) \leq P+N_{1}\right\} . \tag{61}
\end{align*}
$$

The bound of Theorem 7 is shown in Fig. 5.

## V. Concluding Remarks

In this paper, we focused on the joint source-channel coding problem of sending a memoryless source over a memoryless broadcast channel, and developed an inner bound and two outer bounds for this problem. The inner bound is achieved by using a unified hybrid coding scheme, and it can recover the best known performance of hybrid coding. Similarly, our outer bounds can also recover the best known outer bound in the literature. Besides, we extend the results to the Wyner-Ziv source broadcast problem.

The inner bounds achieved by the proposed hybrid coding is established by using generalized multivariate covering and packing lemmas, and the outer bounds are derived by introducing auxiliary random variables (at the sender side or receiver sides). These lemmas and tools are expected to be exploited to derive more and stronger achievability and converse results for network information theory.

## Appendix A

## Proof of Lemma 1

We follow similar steps to the proof of mutual covering lemma [15]. Let
$\mathcal{B}=\left\{m_{[1: k]} \in \prod_{i=1}^{k}\left[1: 2^{n r_{i}}\right]:\left(U^{n}, V_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)}\right\}$.
Then we only need to show $\lim _{n \rightarrow \infty} \mathbb{P}(|\mathcal{B}|=0)=0$. On the other hand,

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \mathbb{P}(|\mathcal{B}|=0) \\
& =\lim _{n \rightarrow \infty} \sum_{u^{n}, v_{0}^{n}} p_{U^{n}, V_{0}^{n}}\left(u^{n}, v_{0}^{n}\right) \mathbb{P}\left(|\mathcal{B}|=0 \mid u^{n}, v_{0}^{n}\right)  \tag{63}\\
& \leq \lim _{n \rightarrow \infty} \sum_{\left(u^{n}, v_{0}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}} p_{U^{n}, V_{0}^{n}}\left(u^{n}, v_{0}^{n}\right) \mathbb{P}\left(|\mathcal{B}|=0 \mid u^{n}, v_{0}^{n}\right) \\
& \quad+\lim _{n \rightarrow \infty} \mathbb{P}\left(\left(u^{n}, v_{0}^{n}\right) \notin \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right)  \tag{64}\\
& =\lim _{n \rightarrow \infty} \sum_{\left(u^{n}, v_{0}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}} p_{U^{n}, V_{0}^{n}}\left(u^{n}, v_{0}^{n}\right) \mathbb{P}\left(|\mathcal{B}|=0 \mid u^{n}, v_{0}^{n}\right) \tag{65}
\end{align*}
$$

To prove $\lim _{n \rightarrow \infty} \mathbb{P}(|\mathcal{B}|=0)=0$, it is sufficient to show $\lim _{n \rightarrow \infty} \mathbb{P}\left(|\mathcal{B}|=0 \mid u^{n}, v_{0}^{n}\right)=0$ for any $\left(u^{n}, v_{0}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}$. Utilizing the Chebyshev lemma [16, App. B], we can bound the probability as

$$
\begin{align*}
& \mathbb{P}\left(|\mathcal{B}|=0 \mid u^{n}, v_{0}^{n}\right) \\
& \leq \mathbb{P}\left((|\mathcal{B}|-\mathbb{E}|\mathcal{B}|)^{2} \geq(E|\mathcal{B}|)^{2} \mid u^{n}, v_{0}^{n}\right)  \tag{66}\\
& \leq \frac{\operatorname{Var}\left(|\mathcal{B}| \mid u^{n}, v_{0}^{n}\right)}{\left(\mathbb{E}\left(|\mathcal{B}| \mid u^{n}, v_{0}^{n}\right)\right)^{2}} \tag{67}
\end{align*}
$$

Next we prove the upper bound $\frac{\operatorname{Var}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right)}{\left(\mathbb{E}\left(|\mathcal{B}| \mid u^{n}, v_{0}^{n}\right)\right)^{2}}$ tends to zero as $n \rightarrow \infty$. Define

$$
E\left(m_{[1: k]}\right) \triangleq \begin{cases}1, & \text { if }\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)}  \tag{68}\\ 0, & \text { otherwise }\end{cases}
$$

for each $m_{[1: k]} \in \prod_{i=1}^{k}\left[1: 2^{n r_{i}}\right]$, then $|\mathcal{B}|$ can be expressed as

$$
\begin{equation*}
|\mathcal{B}|=\sum_{m_{[1: k]} \in \prod_{i=1}^{k}\left[1: 2^{n r_{i}}\right]} E\left(m_{[1: k]}\right) . \tag{69}
\end{equation*}
$$

Denote

$$
\begin{align*}
p_{0}= & \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right)  \tag{70}\\
p_{\mathcal{I}}= & \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)}\right. \\
& \left.\quad\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right), \tag{71}
\end{align*}
$$

for $m_{[1: k]}=\mathbf{1}$, and $m_{[1: k]}^{\prime}=\mathbf{2}$. Obviously, $p_{[1: k]}=p_{0}$. Then

$$
\begin{align*}
& \mathbb{E}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right) \\
& =\sum_{m_{[1: k]}} \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right)  \tag{72}\\
& =2^{n \sum_{j=1}^{k} r_{j}} p_{0} \tag{73}
\end{align*}
$$

and

$$
\begin{align*}
& \mathbb{E}\left(|\mathcal{B}|^{2} \mid u^{n}, v_{0}^{n}\right) \\
& =\sum_{\mathcal{I} \subseteq[1: k]} \sum_{m_{[1: k]}} \sum_{m_{\mathcal{I}^{c}}^{\prime}: m_{\mathcal{I}^{c} \nless}^{\prime} m_{\mathcal{I}^{c}}} \\
& \quad \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)},\right. \\
& \left.\quad\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right)  \tag{74}\\
& = \\
& 2^{n \sum_{j=1}^{k} r_{j}} p_{0}+\sum_{\mathcal{I}_{\mathcal{Z}}[1: k]} \sum_{m_{[1: k]}} \sum_{m_{\mathcal{I}^{c}}^{\prime}: m_{\mathcal{I}^{c} \nLeftarrow}^{\prime} m_{\mathcal{I}^{c}}} \\
& \quad \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)},\right.  \tag{75}\\
& \left.\quad\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right) .
\end{align*}
$$

Define

$$
\begin{equation*}
\mathbb{J} \triangleq\left\{\mathcal{J} \varsubsetneqq[1: k]: \text { if } j \in \mathcal{J}, \text { then } \mathcal{A}_{j} \subseteq \mathcal{J}\right\} \tag{76}
\end{equation*}
$$

Then any set $\mathcal{I} \varsubsetneqq[1: k]$ can transform into a $\mathcal{J}(\mathcal{I}) \in \mathbb{J}$ by removing all the elements $j$ 's such that $\mathcal{A}_{j} \nsubseteq \mathcal{I}$. According to generation of random codebook, we can observe that $p_{\mathcal{I}}=$ $p_{\mathcal{J}(\mathcal{I})}$. Therefore,

$$
\begin{align*}
& \sum_{\mathcal{I} \varsubsetneqq[1: k]} \sum_{m_{[1: k]}} \sum_{m_{\mathcal{I}}^{\prime}: m_{\mathcal{I}^{c}}^{\prime} \nless m_{\mathcal{I}^{c}}} \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)},\right. \\
& \left.\left(u^{n}, v_{0}^{n}, V_{[1: k]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right) \\
& =\sum_{\mathcal{I} \varsubsetneqq[1: k]} \sum_{m_{[1: k]}} \sum_{m_{\mathcal{I}^{c}}^{\prime}: m_{\mathcal{I}^{c}}^{\prime} \not m_{\mathcal{I}^{c}}} p_{\mathcal{J}(\mathcal{I})}  \tag{77}\\
& \leq \sum_{\mathcal{I} \subsetneq[1: k]} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{\left.j \in \mathcal{I}^{c} r_{j}\right)} p_{\mathcal{J}(\mathcal{I})}, ~\right.}  \tag{78}\\
& \leq \sum_{\mathcal{I} \varsubsetneqq[1: k]} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{\left.j \in(\mathcal{J}(\mathcal{I}))^{c} r_{j}\right)} p_{\mathcal{J}(\mathcal{I})}, ~\right.}  \tag{79}\\
& \leq \sum_{\mathcal{J} \in \mathbb{J}} 2^{k-|\mathcal{J}|} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{j \in \mathcal{J}^{c}} r_{j}\right)} p_{\mathcal{J}}  \tag{80}\\
& \leq \sum_{\mathcal{J} \in \mathbb{J}} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{j \in \mathcal{J}}{ }^{c} r_{j}+o(1)\right)} p_{\mathcal{J}}, \tag{81}
\end{align*}
$$

where (79) follows from $\mathcal{J}(\mathcal{I}) \subseteq \mathcal{I}$, (80) follows from that for each $\mathcal{J} \subseteq \mathbb{J}$, there are at most $2^{k-|\mathcal{J}|}$ of $\mathcal{I}$ 's that could transform into $\mathcal{J}$, and $o(1)$ denotes a term that vanishes as $n \rightarrow \infty$. Hence

$$
\begin{align*}
& \operatorname{Var}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right) \\
& \leq \mathbb{E}\left(|\mathcal{B}|^{2} \mid u^{n}, v_{0}^{n}\right)  \tag{82}\\
& \leq 2^{n \sum_{j=1}^{k} r_{j}} p_{0}+\sum_{\mathcal{J} \in \mathbb{J}} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{\left.j \in \mathcal{J}^{c} r_{j}+o(1)\right)} p_{\mathcal{J}} .\right.} \tag{83}
\end{align*}
$$

Furthermore we have

$$
\begin{align*}
& \frac{\operatorname{Var}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right)}{\left(\mathbb{E}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right)\right)^{2}} \\
& \leq \frac{2^{n \sum_{j=1}^{k} r_{j}} p_{0}+\sum_{\mathcal{J} \in \mathbb{J}} 2^{n\left(\sum_{j=1}^{k} r_{j}+\sum_{j \in \mathcal{J}^{c}} r_{j}+o(1)\right)} p_{\mathcal{J}}}{\left(2^{n \sum_{j=1}^{k} r_{j}} p_{0}\right)^{2}}  \tag{84}\\
& =2^{-n \sum_{j=1}^{k} r_{j}} \frac{1}{p_{0}}+\sum_{\mathcal{J} \in \mathbb{J}} 2^{n\left(-\sum_{j \in \mathcal{J}} r_{j}+o(1)\right)} \frac{p_{\mathcal{J}}}{p_{0}^{2}} . \tag{85}
\end{align*}
$$

According to the generation process of random codebook, we can observe that

$$
\begin{align*}
p_{0} & =\sum_{v_{[1: k]}^{n}:\left(u^{n}, v_{0}^{n}, v_{[1: k]}^{n}\right) \in \mathcal{T}_{\epsilon}^{(n)}} \mathbb{P}\left(V_{[1: k]}^{n}\left(m_{[1: k]}\right)=v_{[1: k]}^{n} \mid u^{n}, v_{0}^{n}\right)  \tag{86}\\
& \geq 2^{-n\left(\sum_{j=1}^{k} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{[1: k]} \mid U V_{0}\right)+2 \delta(\epsilon)\right)} \tag{87}
\end{align*}
$$

where (87) follows from that for any $\left(u^{n}, v_{0}^{n}, v_{[1: k]}^{n}\right) \in \mathcal{T}_{\epsilon}^{(n)}$,

$$
\begin{align*}
& \mathbb{P}\left(V_{[1: k]}^{n}\left(m_{[1: k]}\right)=v_{[1: k]}^{n} \mid u^{n}, v_{0}^{n}\right) \\
& \geq 2^{-n\left(\sum_{j=1}^{k} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)+\delta(\epsilon)\right)}, \tag{88}
\end{align*}
$$

and for any $\left(u^{n}, v_{0}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}$,

$$
\begin{align*}
& \left|\left\{v_{[1: k]}^{n}:\left(u^{n}, v_{0}^{n}, v_{[1: k]}^{n}\right) \in \mathcal{T}_{\epsilon}^{(n)}\right\}\right| \\
& \geq 2^{n\left(H\left(V_{[1: k]} \mid U V_{0}\right)+\delta(\epsilon)\right)} \tag{89}
\end{align*}
$$

Similarly, we also can get

$$
\begin{align*}
p_{\mathcal{J}} \leq & \exp
\end{align*} \quad-n\left(\sum_{j=1}^{k} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)+\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right) .\right.
$$

Substituting (87) and (90) into (85), we have

$$
\begin{aligned}
& \frac{\operatorname{Var}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right)}{\left(\mathbb{E}\left(\mid \mathcal{B} \| u^{n}, v_{0}^{n}\right)\right)^{2}} \\
& \leq \exp \left\{-n\left(\sum_{j=1}^{k} r_{j}-\left(\sum_{j=1}^{k} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)\right.\right.\right. \\
& \left.\left.\left.\quad-H\left(V_{[1: k]} \mid U V_{0}\right)+2 \delta(\epsilon)\right)\right)\right\} \\
& \quad+\sum_{\mathcal{J} \in \mathbb{J}} \exp \left\{-n\left(\sum_{j=1}^{k} r_{j}-\left(\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)\right.\right.\right.
\end{aligned}
$$

$$
\begin{equation*}
\left.\left.\left.-H\left(V_{\mathcal{J}} \mid U V_{0}\right)+6 \delta(\epsilon)+o(1)\right)\right)\right\} \tag{91}
\end{equation*}
$$

(91) tends to zero if

$$
\begin{align*}
& \sum_{j=1}^{k} r_{j}>\sum_{j=1}^{k} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{[1: k]} \mid U V_{0}\right)+2 \delta(\epsilon)  \tag{92}\\
& \sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid U V_{0}\right)+6 \delta(\epsilon)+o(1), \tag{93}
\end{align*}
$$

i.e., $\sum_{j \in \mathcal{J}} r_{j}>\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid U V_{0}\right)+\delta^{\prime}(\epsilon)$ for some $\delta^{\prime}(\epsilon)$ that tends to zero as $\epsilon \rightarrow 0$. This completes the proof.

## Appendix B

## Proof of Lemma 2

For any $\mathcal{J}$ such that $\mathcal{J} \neq \emptyset$ and if $j \in \mathcal{J}$ then $\mathcal{A}_{j} \subseteq \mathcal{J}$,

$$
\begin{align*}
& \mathbb{P}\left(\left(U^{n}, V_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{[1: k]}\right) \\
& \leq \mathbb{P}\left(\left(U^{n}, V_{0}^{n}, V_{\mathcal{J}}^{n}\left(m_{\mathcal{J}}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{\mathcal{J}}\right)  \tag{94}\\
& =\sum_{u^{n}, v_{0}^{n}} p_{U^{n}, V_{0}^{n}}\left(u^{n}, v_{0}^{n}\right) \\
& \quad \times \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{\mathcal{J}}^{n}\left(m_{\mathcal{J}}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{\mathcal{J}} \mid u^{n}, v_{0}^{n}\right) \tag{95}
\end{align*}
$$

$$
\begin{align*}
& \leq \sum_{u^{n}, v_{0}^{n}} p_{U^{n}, V_{0}^{n}}\left(u^{n}, v_{0}^{n}\right)  \tag{E}\\
&  \tag{96}\\
& \quad \times \sum_{m_{\mathcal{J}}} \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{\mathcal{J}}^{n}\left(m_{\mathcal{J}}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right) .
\end{align*}
$$

Similar to (87), we can obtain that

$$
\begin{align*}
& \mathbb{P}\left(\left(u^{n}, v_{0}^{n}, V_{\mathcal{J}}^{n}\left(m_{\mathcal{J}}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \mid u^{n}, v_{0}^{n}\right) \\
& \leq 2^{-n\left(\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid U V_{0}\right)-2 \delta(\epsilon)\right)} . \tag{97}
\end{align*}
$$

Substituting it into (96), we have

$$
\begin{align*}
& \mathbb{P}\left(\left(U^{n}, V_{0}^{n}, V_{[1: k]}^{n}\left(m_{[1: k]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)} \text { for some } m_{[1: k]}\right) \\
& \leq 2^{n\left(\sum_{j \in \mathcal{J}} r_{j}-\left(\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid U V_{0}\right)-2 \delta(\epsilon)\right)\right)} . \tag{98}
\end{align*}
$$

(98) tends to zero if

$$
\begin{equation*}
\sum_{j \in \mathcal{J}} r_{j}<\sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}} V_{0}\right)-H\left(V_{\mathcal{J}} \mid U V_{0}\right)-2 \delta(\epsilon) \tag{99}
\end{equation*}
$$

This completes the proof.

## Appendix C

## Proof of Theorem 1

## A. Inner Bound

Actually the inner bound can be seen as a corollary to [20, Thm. 1] by choosing a proper set of network topology, transit probability and symbol-by-symbol functions. For completeness and clarity, next we provide a direct description of the proposed hybrid coding scheme and a direct proof for it.

Codebook Generation: Fix the conditional pmf $p_{V_{[1: N]} \mid S}$, vector $r_{[1: N]}$, encoding function $x\left(v_{[1: N]}, s\right)$ and decoding functions $\hat{s}_{k}\left(v_{\mathcal{B}_{k}}, y_{k}\right)$ that satisfy

$$
\begin{align*}
\mathbb{E} d_{k}\left(S, \hat{S}_{k}\right) \leq & D_{k}, 1 \leq k \leq K  \tag{100}\\
\sum_{j \in \mathcal{J}} r_{j}> & \sum_{j \in \mathcal{J}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}} \mid S\right) \\
& \text { for all } \mathcal{J} \subseteq[1: N] \text { s.t. } \mathcal{J} \neq \emptyset \\
& \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}  \tag{101}\\
\sum_{j \in \mathcal{J}^{c}} r_{j}< & \sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}^{c}} \mid Y_{k} V_{\mathcal{J}}\right) \\
& k \in[1: K] \text { for all } \mathcal{J} \subseteq \mathcal{B}_{k} \text { s.t. } \\
& \mathcal{J}^{c} \triangleq \mathcal{B}_{k} \backslash \mathcal{J} \neq \emptyset \text { and } \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J} \tag{102}
\end{align*}
$$

For each $j \in[1: N]$ and each $m_{\mathcal{A}_{j}} \in \prod_{i \in \mathcal{A}_{j}}\left[1: 2^{n r_{i}}\right]$, randomly and independently generate a set of sequences $v_{j}^{n}\left(m_{\mathcal{A}_{j}}, m_{j}\right), m_{j} \in\left[1: 2^{n r_{j}}\right]$, with each distributed according to $\prod_{i=1}^{n} p_{V_{j} \mid V_{\mathcal{A}_{j}}}\left(v_{j, i} \mid v_{\mathcal{A}_{j}, i}\left(m_{\mathcal{A}_{j}}\right)\right)$. The codebook

$$
\begin{equation*}
\mathcal{C}=\left\{v_{[1: N]}^{n}\left(m_{[1: N]}\right): m_{[1: N]} \in \prod_{i=1}^{N}\left[1: 2^{n r_{i}}\right]\right\} \tag{103}
\end{equation*}
$$

is revealed to the encoder and all the decoders.
Encoding: We use joint typicality encoding. Given $s^{n}$, encoder finds the smallest index vector $m_{[1: N]}$ such that $\left(s^{n}, v_{[1: N]}^{n}\left(m_{[1: N]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)}$. If there is no such index vector, let $m_{[1: N]}=\mathbf{1}$. Then the encoder transmits the signal

$$
\begin{equation*}
x_{i}=x\left(v_{[1: N], i}\left(m_{[1: N]}\right), s_{i}\right), 1 \leq i \leq n . \tag{104}
\end{equation*}
$$

Decoding: We use joint typicality decoding. Let $\epsilon^{\prime}>\epsilon$. Upon receiving the signal $y_{k}^{n}$, the decoder of the receiver $k$ finds the smallest index vector $\hat{m}_{\mathcal{B}_{k}}^{(k)}$ such that

$$
\begin{equation*}
\left(v_{\mathcal{B}_{k}}^{n}\left(\hat{m}_{\mathcal{B}_{k}}^{(k)}\right), y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)} \tag{105}
\end{equation*}
$$

If there is no such index vector, let $\hat{m}_{\mathcal{B}_{k}}^{(k)}=\mathbf{1}$. The decoder reconstructs the source as

$$
\begin{equation*}
\hat{s}_{k, i}=\hat{s}_{k}\left(v_{\mathcal{B}_{k}, i}\left(\hat{m}_{\mathcal{B}_{k}}^{(k)}\right), y_{k, i}\right), 1 \leq i \leq n \tag{106}
\end{equation*}
$$

Analysis of Expected Distortion: We bound the distortions averaged over $S^{n}$ and the random codebook $\mathcal{C}$. Define the "error" event

$$
\begin{equation*}
\mathcal{E}=\mathcal{E}_{1} \cup\left(\bigcup_{k} \mathcal{E}_{2, k}\right) \cup\left(\bigcup_{k} \mathcal{E}_{3, k}\right) \tag{107}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{E}_{1}=\{ & \left.\left(S^{n}, V_{[1: N]}^{n}\left(m_{[1: N]}\right)\right) \notin \mathcal{T}_{\epsilon}^{(n)} \text { for all } m_{[1: N]}\right\}  \tag{108}\\
\mathcal{E}_{2, k}=\{ & \left.\left(S^{n}, V_{[1: N]}^{n}\left(M_{[1: N]}\right), Y_{k}^{n}\right) \notin \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\},  \tag{109}\\
\mathcal{E}_{3, k}=\{ & \left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{B}_{k}}^{\prime}\right), Y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)} \\
& \text { for some } \left.m_{\mathcal{B}_{k}}^{\prime}, m_{\mathcal{B}_{k}}^{\prime} \neq M_{\mathcal{B}_{k}}\right\}, \tag{110}
\end{align*}
$$

for $1 \leq k \leq K$. Using the union bound, we have

$$
\begin{equation*}
\mathbb{P}(\mathcal{E}) \leq \mathbb{P}\left(\mathcal{E}_{1}\right)+\sum_{k=1}^{K} \mathbb{P}\left(\mathcal{E}_{1}^{c} \bigcap \mathcal{E}_{2, k}\right)+\sum_{k=1}^{K} \mathbb{P}\left(\mathcal{E}_{3, k}\right) . \tag{111}
\end{equation*}
$$

Now we claim that if (101) and (102) hold, then $\mathbb{P}(\mathcal{E})$ tends to zero as $n \rightarrow \infty$. Before proving it, we show that this claim implies the inner bound of Theorem 1.

Define

$$
\begin{equation*}
\mathcal{E}_{4, k}=\left\{\left(S^{n}, V_{\mathcal{B}_{k}}^{n}\left(\hat{M}_{\mathcal{B}_{k}}^{(k)}\right), Y_{k}^{n}\right) \notin \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\} \tag{112}
\end{equation*}
$$

then we have $\mathcal{E}^{c} \subseteq \mathcal{E}_{4, k}^{c}$, i.e., $\mathcal{E}_{4, k} \subseteq \mathcal{E}$. This implies that $\mathbb{P}\left(\mathcal{E}_{4, k}\right) \leq \mathbb{P}(\mathcal{E}) \rightarrow 0$ as $n \rightarrow \infty$. Then utilizing the typical average lemma [16], we have

$$
\begin{align*}
& \limsup _{n \rightarrow \infty} \mathbb{E} d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \\
& =\limsup _{n \rightarrow \infty}\left\{\mathbb{P}\left(\mathcal{E}_{4, k}\right) \mathbb{E}\left[d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \mid \mathcal{E}_{4, k}\right]\right. \\
& \left.\quad+\mathbb{P}\left(\mathcal{E}_{4, k}^{c}\right) \mathbb{E}\left[d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \mid \mathcal{E}_{4, k}^{c}\right]\right\}  \tag{113}\\
& =\limsup _{n \rightarrow \infty} \mathbb{E}\left[d_{k}\left(S^{n}, \hat{S}_{k}^{n}\right) \mid \mathcal{E}_{4, k}^{c}\right]  \tag{114}\\
& \leq\left(1+\epsilon^{\prime}\right) \mathbb{E} d_{k}\left(S, \hat{S}_{k}\right)  \tag{115}\\
& \leq\left(1+\epsilon^{\prime}\right) D_{k} \tag{116}
\end{align*}
$$

Therefore, the desired distortions are achieved for sufficiently small $\epsilon^{\prime}$.

Next we turn back to prove the claim above. Following from the multivariate covering lemma (Lemma 1), the first term of (111), $\mathbb{P}\left(\mathcal{E}_{1}\right)$, vanishes as $n \rightarrow \infty$, and according to conditional typicality lemma [16, Sec. 3.7], the second item tends to zero as $n \rightarrow \infty$.

Now we focus on the third term of (111). $\mathcal{E}_{3, k}$ can be writen as

$$
\begin{equation*}
\mathcal{E}_{3, k}=\bigcup_{\mathcal{I} \subseteq \mathcal{B}_{k}} \mathcal{E}_{3, k}^{\mathcal{I}} \tag{117}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{E}_{3, k}^{\mathcal{I}}=\{ & \left(V_{\mathcal{B}_{k}}^{n}\left(M_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), Y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)} \\
& \text { for some } \left.m_{\mathcal{I}^{c}}^{\prime}, m_{\mathcal{I}^{c}}^{\prime} \nLeftarrow M_{\mathcal{I}^{c}}\right\}, \tag{118}
\end{align*}
$$

with $\mathcal{I}^{c} \triangleq \mathcal{B}_{k} \backslash \mathcal{I}$. Using the union bound we have

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{E}_{3, k}\right) \leq \sum_{\mathcal{I} \subseteq \mathcal{B}_{k}} \mathbb{P}\left(\mathcal{E}_{3, k}^{\mathcal{I}}\right) \tag{119}
\end{equation*}
$$

Each $\mathcal{B}_{k}$ has a finite number of subsets, hence we only need to show for each $\mathcal{I} \subseteq \mathcal{B}_{k}, \mathbb{P}\left(\mathcal{E}_{3, k}^{\mathcal{I}}\right)$ vanishes as $n \rightarrow \infty$. To show this, it is needed to analyze the correlation between coding index $M_{[1: N]}$ and nonchosen codewords. Specifically, $M_{[1: N]}$ depends on the source sequence and the entire codebook, and hence the standard packing lemma cannot be applied directly. This problem has been resolved by the technique developed in [15], [20].

$$
\begin{aligned}
& \mathbb{P}\left(\left(V_{\mathcal{B}_{k}}^{n}\left(M_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), Y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right. \\
& \left.\quad \text { for some } m_{\mathcal{I}^{c}}^{\prime}, m_{\mathcal{I}^{c}}^{\prime} \nLeftarrow M_{\mathcal{I}^{c}}\right)
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{m_{[1: N]}} \sum_{y_{k}^{n}} \mathbb{P}\left(M_{[1: N]}=m_{[1: N]}, Y_{k}^{n}=y_{k}^{n}\right. \\
& \quad\left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)} \\
& \left.\quad \text { for some } m_{\mathcal{I}^{c}}^{\prime}, m_{\mathcal{I}^{c}}^{\prime} \nLeftarrow m_{\mathcal{I}^{c}}\right)  \tag{120}\\
& \leq \sum_{m_{[1: N]}} \sum_{y_{k}^{n}} \sum_{m_{\mathcal{I}^{c}}^{\prime}: m_{\mathcal{I}^{c} \nLeftarrow}^{\prime} m_{\mathcal{I}^{c}}} \mathbb{P}\left(M_{[1: N]}=m_{[1: N]}\right. \\
& \left.\quad Y_{k}^{n}=y_{k}^{n},\left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right) \tag{121}
\end{align*}
$$

where (121) follows from the union bound.
Define a sub-codebook as

$$
\begin{align*}
\mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}= & \left\{V_{[1: N]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime \prime}, m_{\mathcal{B}_{k}^{c}}^{\prime \prime}\right):\right. \\
& \left.\forall\left(m_{\mathcal{I}^{c}}^{\prime \prime}, m_{\mathcal{B}_{k}^{c}}^{\prime \prime}\right), m_{\mathcal{I}^{c}}^{\prime \prime} \nRightarrow m_{\mathcal{I}^{c}}^{\prime}\right\} . \tag{122}
\end{align*}
$$

Define another coding index as $\tilde{M}_{[1: N]}$ which is generated by performing the same coding process as $M_{[1: N]}$ but on the codebook $\mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I} c}^{\prime}\right)}$, i.e., given the source sequence $s^{n}$, the encoder finds the smallest index vector $\tilde{m}_{[1: N]}$ such that $\left(s^{n}, v_{[1: N]}^{n}\left(\tilde{m}_{[1: N]}\right)\right) \in \mathcal{T}_{\epsilon}^{(n)}$; if there is no such index vector, let $\tilde{m}_{[1: N]}=1$. Then according to the generation process of $M_{[1: N]}$ and $\tilde{M}_{[1: N]}$, we have if $M_{[1: N]}=m_{[1: N]}$, then $\tilde{M}_{[1: N]}=m_{[1: N]}$. Now continuing with (121), we have

$$
\begin{align*}
& \mathbb{P}\left(M_{[1: N]}=m_{[1: N]}, Y_{k}^{n}=y_{k}^{n},\left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right) \\
& =\sum_{v_{\mathcal{B}_{k}}^{\prime n}, c, s^{n}} \mathbb{P}\left(M_{[1: N]}=m_{[1: N]}, \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c,\right. \\
& \left.S^{n}=s^{n}, V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{D}_{k}}^{\prime \prime n}\right) \\
& \times \prod_{i=1}^{n} p_{Y_{k} \mid X}\left(y_{k, i} \mid x\left(v_{[1: N], i}\left(m_{[1: N]}\right), s_{i}\right)\right) \\
& \times 1\left\{\left(v_{\mathcal{B}_{k}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}  \tag{123}\\
& =\sum_{v_{\mathcal{B}_{k}^{\prime}, c, s^{n}} \mathbb{P}\left(M_{[1: N]}=m_{[1: N]}, \tilde{M}_{[1: N]}=m_{[1: N]}, ~\right.}^{\text {, }} \\
& \left.\mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c, S^{n}=s^{n}, V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{D}_{k}}^{\prime \prime}\right) \\
& \times \prod_{i=1}^{n} p_{Y_{k} \mid X}\left(y_{k, i} \mid x\left(v_{[1: N], i}\left(m_{[1: N]}\right), s_{i}\right)\right) \\
& \times 1\left\{\left(v_{\mathcal{B}_{k}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}  \tag{124}\\
& \leq \sum_{v_{\mathcal{B}_{k}^{\prime}, c, s^{n}}} \mathbb{P}\left(\tilde{M}_{[1: N]}=m_{[1: N]}, \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c,\right. \\
& \left.S^{n}=s^{n}, V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{D}_{k}}^{\prime n}\right) \\
& \times \prod_{i=1}^{n} p_{Y_{k} \mid X}\left(y_{k, i} \mid x\left(v_{[1: N], i}\left(m_{[1: N]}\right), s_{i}\right)\right) \\
& \times 1\left\{\left(v_{\mathcal{B}_{k}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}  \tag{125}\\
& =\sum_{v_{\mathcal{B}_{k}}^{\prime \prime}, c, s^{n}} \mathbb{P}\left(\tilde{M}_{[1: N]}=m_{[1: N]}, \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c, S^{n}=s^{n}\right) \\
& \times \prod_{i=1}^{n} p_{Y_{k} \mid X}\left(y_{k, i} \mid x\left(v_{[1: N], i}\left(m_{[1: N]}\right), s_{i}\right)\right)
\end{align*}
$$

$$
\begin{align*}
& \times \mathbb{P}\left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{B}_{k}}^{\prime n} \mid \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c\right) \\
& \times 1\left\{\left(v_{\mathcal{B}_{k}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\} \tag{126}
\end{align*}
$$

where $c=\left\{v_{[1: N]}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime \prime}, m_{\mathcal{B}_{k}^{c}}^{\prime \prime}\right)\right.$
$\left.\forall\left(m_{\mathcal{I}^{c}}^{\prime \prime}, m_{\mathcal{B}_{k}^{c}}^{\prime \prime}\right), m_{\mathcal{I}^{c}}^{\prime \prime} \nLeftarrow m_{\mathcal{I}^{c}}^{\prime}\right\}$, and (126) follows from the fact that $V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right) \rightarrow \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)} \rightarrow\left(S^{n}, \tilde{M}_{[1: N]}\right)$ forms a Markov chain.

Define

$$
\begin{equation*}
\mathbb{J} \triangleq\left\{\mathcal{J} \subseteq \mathcal{B}_{k}: \mathcal{A}_{j} \subseteq \mathcal{J}, \forall j \in \mathcal{J}\right\} \tag{127}
\end{equation*}
$$

Then any set $\mathcal{I} \subseteq \mathcal{B}_{k}$ can be transformed into a $\mathcal{J}(\mathcal{I}) \in \mathbb{J}$ by removing all the elements $j$ 's such that $\mathcal{A}_{j} \nsubseteq \mathcal{I}$. Denote $\mathcal{J}^{c} \triangleq \mathcal{B}_{k} \backslash \mathcal{J}$. Then according to the generation process of the codebook, continuing with (126), we have

$$
\begin{align*}
& \sum_{v_{\mathcal{B}_{k}}^{\prime n}} \mathbb{P}\left(V_{\mathcal{B}_{k}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{B}_{k}}^{\prime n} \mid \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c\right) \\
& \quad \times 1\left\{\left(v_{\mathcal{B}_{k}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\} \\
& =\sum_{v_{\mathcal{J}^{c}}^{\prime n}} \mathbb{P}\left(V_{\mathcal{J}^{c}}^{n}\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)=v_{\mathcal{J}^{c}}^{\prime n} \mid \mathcal{C}_{\left(m_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right)}=c\right) \\
& \quad \times 1\left\{\left(v_{\mathcal{J}^{\prime}}^{n}\left(m_{\mathcal{J}}\right), v_{\mathcal{J}^{c}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}  \tag{128}\\
& =\sum_{v_{\mathcal{J}^{c}}^{\prime n}} \prod_{j \in \mathcal{J}^{c}} \prod_{i=1}^{n} p_{V_{j} \mid V_{\mathcal{A}_{j}}}\left(v_{j_{j, i}}^{\prime} \mid v_{\mathcal{A}_{j} \cap \mathcal{J}_{, i}}\left(m_{\mathcal{J}}\right), v_{\mathcal{A}_{j} \cap \mathcal{J}^{c}, i}^{\prime}\right) \\
& \quad \times 1\left\{\left(v_{\mathcal{J}^{\prime}}^{n}\left(m_{\mathcal{J}}\right), v_{\mathcal{J}^{c}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}  \tag{129}\\
& \leq 2^{n\left(H \left(V_{\left.\left.\mathcal{J}^{c} \mid Y_{k} V_{\mathcal{J}}\right)-\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)+\left(\left|\mathcal{J}^{c}\right|+1\right) \delta\left(\epsilon^{\prime}\right)\right)}\right.\right.} \tag{130}
\end{align*}
$$

where $\delta\left(\epsilon^{\prime}\right)$ is a term that tends to zero as $\epsilon^{\prime} \rightarrow 0$, and (130) follows from the fact that $\prod_{i=1}^{n} p_{V_{j} \mid V_{\mathcal{A}_{j}}}\left(v_{j, i} \mid v_{\mathcal{A}_{j}, i}\right) \leq$ $2^{-n\left(H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-\delta\left(\epsilon^{\prime}\right)\right)}$ for any $\left(v_{j}^{n}, v_{\mathcal{A}_{j}}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}$ and $\left|\left\{v_{\mathcal{J}^{c}}^{\prime n}:\left(v_{\mathcal{J}}^{n}, v_{\mathcal{J}^{c}}^{\prime n}, y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right\}\right| \leq 2^{n\left(H\left(V_{\mathcal{J}} \mid Y_{k} V_{\mathcal{J}}\right)+\delta\left(\epsilon^{\prime}\right)\right)}$ for any $\left(y_{k}^{n}, v_{\mathcal{J}}^{n}\right)$.

Combining (121), (126) and (130) gives

$$
\begin{align*}
& \mathbb{P}\left(\left(V_{\mathcal{B}_{k}}^{n}\left(M_{\mathcal{I}}, m_{\mathcal{I}^{c}}^{\prime}\right), Y_{k}^{n}\right) \in \mathcal{T}_{\epsilon^{\prime}}^{(n)}\right. \\
& \left.\quad \text { for some } m_{\mathcal{I}^{c}}^{\prime}, m_{\mathcal{I}^{c}}^{\prime} \leftrightarrow M_{\mathcal{I}^{c}}\right) \\
& \leq \exp \left\{n \left(\sum_{j \in \mathcal{J}^{c}} r_{j}-\left(\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)\right.\right.\right. \\
& \left.\left.\left.\quad-H\left(V_{\mathcal{J}^{c}} \mid Y_{k} V_{\mathcal{J}}\right)-\left(\left|\mathcal{J}^{c}\right|+1\right) \delta\left(\epsilon^{\prime}\right)\right)\right)\right\} . \tag{131}
\end{align*}
$$

Hence if $\sum_{j \in \mathcal{J}^{c}} r_{j}<\sum_{j \in \mathcal{J}^{c}} H\left(V_{j} \mid V_{\mathcal{A}_{j}}\right)-H\left(V_{\mathcal{J}^{c}} \mid Y_{k} V_{\mathcal{J}}\right)-$ $\left(\left|\mathcal{J}^{c}\right|+1\right) \delta\left(\epsilon^{\prime}\right)$ for all $\mathcal{J} \in \mathbb{J}$, then the third term of (111) tends to zero as $n \rightarrow \infty$. Letting $\epsilon^{\prime}$ small enough, this completes the proof of the inner bound.

It is worth noting that although the multivariate packing lemma (Lemma 2) has not been employed directly in the proof, the derivation after (129) is essentially the same as that of the multivariate packing lemma.

## B. Outer Bound $\mathcal{D}_{1}^{(o)}$

For fixed $p_{U_{[1: L]} \mid S}$, we first introduce a set of auxiliary random variables $U_{[1: L]}^{n}$ that follow the distribution $\prod_{i=1}^{n} p_{U_{[1: L]} \mid S}\left(u_{[1: L], i} \mid s_{i}\right)$. Hence the Markov chains
$\underset{[1: L]}{U_{\text {l }}^{n}} \rightarrow S^{n} \rightarrow X^{n} \rightarrow Y_{k}^{n} \rightarrow \hat{S}_{k}^{n}, 1 \leq k \leq K$ hold. Consider that

$$
\begin{align*}
& I\left(Y_{\mathcal{A}}^{n} ; U_{\mathcal{B}}^{n} \mid U_{\mathcal{C}}^{n}\right) \\
& =\sum_{t=1}^{n} I\left(Y_{\mathcal{A}}^{n} ; U_{\mathcal{B}, t} \mid U_{\mathcal{C}}^{n} U_{\mathcal{B}}^{t-1}\right)  \tag{132}\\
& =\sum_{t=1}^{n} H\left(U_{\mathcal{B}, t} \mid U_{\mathcal{C}}^{n} U_{\mathcal{B}}^{t-1}\right)-H\left(U_{\mathcal{B}, t} \mid U_{\mathcal{C}}^{n} U_{\mathcal{B}}^{t-1} Y_{\mathcal{A}}^{n}\right)  \tag{133}\\
& =\sum_{t=1}^{n} H\left(U_{\mathcal{B}, t} \mid U_{\mathcal{C}, t}\right)-H\left(U_{\mathcal{B}, t} \mid U_{\mathcal{C}}^{n} U_{\mathcal{B}}^{t-1} Y_{\mathcal{A}}^{n}\right)  \tag{134}\\
& =\sum_{t=1}^{n} I\left(U_{\mathcal{B}, t} ; U_{\mathcal{C}}^{n} U_{\mathcal{B}}^{t-1} Y_{\mathcal{A}}^{n} \mid U_{\mathcal{C}, t}\right)  \tag{135}\\
& \geq \sum_{t=1}^{n} I\left(U_{\mathcal{B}, t} ; \hat{S}_{\mathcal{A}, t} \mid U_{\mathcal{C}, t}\right)  \tag{136}\\
& =n I\left(U_{\mathcal{B}, Q} ; \hat{S}_{\mathcal{A}, Q} \mid U_{\mathcal{C}, Q} Q\right)  \tag{137}\\
& =n I\left(U_{\mathcal{B}, Q} ; \hat{S}_{\mathcal{A}, Q} Q \mid U_{\mathcal{C}, Q}\right)  \tag{138}\\
& \geq n I\left(U_{\mathcal{B}, Q} ; \hat{S}_{\mathcal{A}, Q} \mid U_{\mathcal{C}, Q}\right)  \tag{139}\\
& =n I\left(U_{\mathcal{B}} ; \hat{S}_{\mathcal{A}} \mid U_{\mathcal{C}}\right) \tag{140}
\end{align*}
$$

where the time-sharing random variable $Q$ is defined to be uniformly distributed $[1: n]$ and independent of all other random variables, and in (140), $U_{l} \triangleq U_{l, Q}, \hat{S}_{k} \triangleq \hat{S}_{k, Q}, 1 \leq$ $l \leq L, 1 \leq k \leq K$.

On the other hand,

$$
\begin{align*}
& I\left(Y_{\mathcal{A}}^{n} ; U_{\mathcal{B}}^{n} \mid U_{\mathcal{C}}^{n}\right) \\
& =\sum_{t=1}^{n} I\left(Y_{\mathcal{A}, t} ; U_{\mathcal{B}}^{n} \mid U_{\mathcal{C}}^{n} Y_{\mathcal{A}}^{t-1}\right)  \tag{141}\\
& =n I\left(Y_{\mathcal{A}, Q} ; U_{\mathcal{B}}^{n} \mid U_{\mathcal{C}}^{n} Y_{\mathcal{A}}^{Q-1} Q\right)  \tag{142}\\
& =n I\left(Y_{\mathcal{A}} ; \tilde{U}_{\mathcal{B}} \mid \tilde{U}_{\mathcal{C}} \tilde{Y}_{\mathcal{A}}\right) \tag{143}
\end{align*}
$$

Set $Y_{k} \triangleq Y_{k, Q}, \tilde{U}_{l} \triangleq U_{l}^{n}, \tilde{Y}_{k} \triangleq Y_{k}^{Q-1} Q, 1 \leq l \leq L, 1 \leq k \leq$ $K$, then combining (140) and (143) gives us the outer bound $\mathcal{D}_{1}^{(o)}$ 。

## C. Outer Bound $\mathcal{D}_{2}^{(o)}$

For fixed $p_{U_{[1: L]} \mid Y_{[1: K]}}$, we first introduce a set of auxiliary random variables $U_{[1: L]}^{n}$ that follow distribution $\prod_{i=1}^{n} p_{U_{[1: L]}^{n} \mid Y_{[1: K]}}\left(u_{[1: L], i} \mid y_{[1: K], i}\right)$. Hence the Markov chains $S^{n} \rightarrow X^{n} \rightarrow Y_{[1: K]}^{n} \rightarrow U_{[1: L]}^{n}$ hold. Note that different from the proof of $\mathcal{R}_{1}^{(o)}$, the auxiliary random variables $U_{[1: L]}^{n}$ here is introduced at receiver sides, and $p_{Y_{[1: K]} U_{[1: L]} \mid X}=$ $p_{U_{[1: L]} \mid Y_{[1: K]}} p_{Y_{[1: K]} \mid X}$ forms a new memoryless broadcast channel. Consider that

$$
\begin{align*}
& I\left(S^{n} ; Y_{\mathcal{B}}^{n} U_{\mathcal{B}^{\prime}}^{n} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n}\right) \\
& \leq I\left(X^{n} ; Y_{\mathcal{B}}^{n} U_{\mathcal{B}^{\prime}}^{n} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n}\right)  \tag{144}\\
& =\sum_{t=1}^{n} I\left(X^{n} ; Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n} Y_{\mathcal{B}}^{t-1} U_{\mathcal{B}^{\prime}}^{t-1}\right) \tag{145}
\end{align*}
$$

$$
\begin{align*}
= & \sum_{t=1}^{n} H\left(Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n} Y_{\mathcal{B}}^{t-1} U_{\mathcal{B}^{\prime}}^{t-1}\right) \\
& \quad-H\left(Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n} Y_{\mathcal{B}}^{t-1} U_{\mathcal{B}^{\prime}}^{t-1} X^{n}\right)  \tag{146}\\
\leq & \sum_{t=1}^{n} H\left(Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} \mid Y_{\mathcal{C}, t} U_{\mathcal{C}^{\prime}, t}\right) \\
& \quad-H\left(Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} \mid Y_{\mathcal{C}, t} U_{\mathcal{C}^{\prime}, t} X_{t}\right)  \tag{147}\\
= & \sum_{t=1}^{n} I\left(Y_{\mathcal{B}, t} U_{\mathcal{B}^{\prime}, t} ; X_{t} \mid Y_{\mathcal{C}, t} U_{\mathcal{C}^{\prime}, t}\right)  \tag{148}\\
= & n I\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} ; X_{Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q} Q\right)  \tag{149}\\
= & n H\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q} Q\right) \\
& \quad-n H\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q} X_{Q} Q\right)  \tag{150}\\
\leq & n H\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q}\right) \\
& \quad-n H\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q} X_{Q}\right)  \tag{151}\\
= & n I\left(Y_{\mathcal{B}, Q} U_{\mathcal{B}^{\prime}, Q} ; X_{Q} \mid Y_{\mathcal{C}, Q} U_{\mathcal{C}^{\prime}, Q}\right)  \tag{152}\\
= & n I\left(X ; Y_{\mathcal{B}} U_{\mathcal{B}^{\prime}} \mid Y_{\mathcal{C}} U_{\mathcal{C}^{\prime}}\right) \tag{153}
\end{align*}
$$

where (147) follows from the Markov chain $U_{[1: L]}^{t-1} U_{[1: L], t+1}^{n} Y_{[1: K]}^{t-1} Y_{[1: K], t+1}^{n} X^{t-1} X_{t+1}^{n} \quad \rightarrow \quad X_{t} \quad \rightarrow$ $Y_{[1: K], t} U_{[1: L], t}$ and the fact conditioning reduces entropy, (151) follows from the Markov chain $Q \rightarrow X_{Q} \rightarrow Y_{[1: K], Q} U_{[1: L], Q}$ and the fact conditioning reduces entropy, the time-sharing random variable $Q$ is defined to be uniformly distributed $[1: n]$ and independent of all other random variables, and in (153), $U_{l} \triangleq U_{l, Q}, Y_{k} \triangleq Y_{k, Q}, X \triangleq X_{Q}, 1 \leq l \leq L, 1 \leq k \leq K$.

On the other hand,

$$
\begin{align*}
& I\left(S^{n} ; Y_{\mathcal{B}}^{n} U_{\mathcal{B}^{\prime}}^{n} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n}\right) \\
& =\sum_{t=1}^{n} I\left(S_{t} ; Y_{\mathcal{B}}^{n} U_{\mathcal{B}^{\prime}}^{n} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n} S^{t-1}\right)  \tag{154}\\
& =n I\left(S_{Q} ; Y_{\mathcal{B}}^{n} U_{\mathcal{B}^{\prime}}^{n} \mid Y_{\mathcal{C}}^{n} U_{\mathcal{C}^{\prime}}^{n} S^{Q-1} Q\right)  \tag{155}\\
& =n I\left(S ; \tilde{Y}_{\mathcal{B}} \tilde{U}_{\mathcal{B}^{\prime}} \mid \tilde{Y}_{\mathcal{C}} \tilde{U}_{\mathcal{C}^{\prime}}\right) \tag{156}
\end{align*}
$$

Set $S \triangleq S_{Q}, \tilde{U}_{l} \triangleq U_{l}^{n} S^{Q-1} Q, \tilde{Y}_{k} \triangleq Y_{k}^{n} S^{Q-1} Q, 1 \leq l \leq$ $L, 1 \leq k \leq K$, then combining (153) and (156) gives us the outer bound $\mathcal{D}_{2}^{(o)}$.

## Appendix D <br> Proof of Theorem 6

Observe that if there is no information transmitted over the channel, receiver $k$ could produce a reconstruction within the distortion $\beta_{k}$. Hence we only need consider the case of $D_{[1: K]}$ with

$$
\begin{equation*}
D_{k} \leq \beta_{k}, 1 \leq k \leq K \tag{157}
\end{equation*}
$$

For the Wyner-Ziv binary broadcast with bandwidth mismatch case (the bandwidth mismatch factor $b$ ), Theorem 5 states that if $D_{[1: K]}$ is achievable, then there exists some pmf $p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}$ and functions $\hat{s}_{k}\left(v_{k}, z_{k}\right), 1 \leq k \leq$ $K$ such that

$$
\begin{equation*}
\mathbb{E} d\left(S, \hat{S}_{k}\right)=\mathbb{P}\left(\hat{S}_{k} \oplus S=1\right) \leq D_{k} \tag{158}
\end{equation*}
$$

and for any pmf $p_{U_{K-1} \mid S} p_{U_{K-2} \mid U_{K-1}} \cdots p_{U_{1} \mid U_{2}}$,

$$
\begin{equation*}
\frac{1}{b}\left(I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{BBC}} \tag{159}
\end{equation*}
$$

holds, where the capacity of binary broadcast channel $\mathcal{R}_{\mathrm{BBC}}$ is given in (38) [13].

Define the sets

$$
\begin{equation*}
\mathcal{A}_{k}=\left\{v_{k}: \hat{s}_{k}\left(v_{k}, 0\right)=\hat{s}_{k}\left(v_{k}, 1\right)\right\}, 1 \leq k \leq K \tag{160}
\end{equation*}
$$

so that their complements

$$
\begin{equation*}
\mathcal{A}_{k}^{c}=\left\{v_{k}: \hat{s}_{k}\left(v_{k}, 0\right) \neq \hat{s}_{k}\left(v_{k}, 1\right)\right\}, 1 \leq k \leq K \tag{161}
\end{equation*}
$$

By hypothesis,

$$
\begin{align*}
& \mathbb{E} d\left(S, \hat{S}_{k}\right) \\
& =\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}\right) \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k} \in \mathcal{A}_{k}\right] \\
& \quad+\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}^{c}\right) \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k} \in \mathcal{A}_{k}^{c}\right]  \tag{162}\\
& \leq D_{k} \tag{163}
\end{align*}
$$

We first show that

$$
\begin{equation*}
\mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k} \in \mathcal{A}_{k}^{c}\right] \geq \beta_{k} \tag{164}
\end{equation*}
$$

To do this, we write

$$
\begin{align*}
& \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k} \in \mathcal{A}_{k}^{c}\right] \\
& =\sum_{v_{k} \in \mathcal{A}_{k}^{c}} \frac{\mathbb{P}\left(V_{k}=v_{k}\right)}{\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}^{c}\right)} \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k}=v_{k}\right] \tag{165}
\end{align*}
$$

If $v_{k} \in \mathcal{A}_{k}^{c}$ and $\hat{s}_{k}\left(v_{k}, 0\right)=0$ then $\hat{s}_{k}\left(v_{k}, 1\right)=1$. Therefore, for such $v_{k}$,

$$
\begin{align*}
& \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k}=v_{k}\right] \\
& =\mathbb{P}\left(Z_{k}=0, S=1 \mid V_{k}=v_{k}\right) \\
& \quad \quad+\mathbb{P}\left(Z_{k}=1, S=0 \mid V_{k}=v_{k}\right)  \tag{166}\\
& =\mathbb{P}\left(S=1 \mid V_{k}=v_{k}\right) \mathbb{P}\left(Z_{k}=0 \mid S=1\right) \\
& \quad \quad+\mathbb{P}\left(S=0 \mid V_{k}=v_{k}\right) \mathbb{P}\left(Z_{k}=1 \mid S=0\right)  \tag{167}\\
& =\beta_{k} \tag{168}
\end{align*}
$$

where (167) follows that $Z_{k} \rightarrow S \rightarrow V_{k}$ forms a Markov chain. If $v_{k} \in \mathcal{A}_{k}^{c}$ but $\hat{s}_{k}\left(v_{k}, 0\right)=1$, then for such $v_{k}$,

$$
\begin{equation*}
\mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k}=v_{k}\right]=1-\beta_{k} \geq \beta_{k} \tag{169}
\end{equation*}
$$

since $\beta_{k} \leq \frac{1}{2}$. Therefore, (164) follows from (168) and (169).
Now we write

$$
\begin{align*}
& \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k} \in \mathcal{A}_{k}\right] \\
& =\sum_{v_{k} \in \mathcal{A}_{k}} \frac{\mathbb{P}\left(V_{k}=v_{k}\right)}{\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}\right)} \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k}=v_{k}\right] \tag{170}
\end{align*}
$$

and define $g_{k}\left(v_{k}\right) \triangleq \hat{s}_{k}\left(v_{k}, 0\right), \lambda_{v_{k}} \triangleq \frac{\mathbb{P}\left(V_{k}=v_{k}\right)}{\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}\right)}, \mu_{k} \triangleq$ $\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}\right)$,

$$
\begin{align*}
d_{v_{k}} & \triangleq \mathbb{E}\left[d\left(S, \hat{S}_{k}\right) \mid V_{k}=v_{k}\right]  \tag{171}\\
& =\mathbb{P}\left(S \neq g_{k}\left(v_{k}\right) \mid V_{k}=v_{k}\right) \tag{172}
\end{align*}
$$

then utilizing (163) and (164), we have

$$
\begin{equation*}
d_{k}^{\prime} \triangleq \mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} d_{v_{k}}+\left(1-\mu_{k}\right) \beta_{k} \leq D_{k} \tag{173}
\end{equation*}
$$

Next we will show

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid Z_{k}\right) \\
& \geq \frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)\right. \\
& \left.\quad-\left(H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)-H_{2}\left(\alpha_{k} \star \beta_{k}\right)\right)\right) \tag{174}
\end{align*}
$$

Choose $U_{K-1}=S \oplus E_{K-1}^{\prime}$ and $U_{k}=U_{k+1} \oplus E_{k}^{\prime}, 1 \leq k \leq$ $K-2$, where $E_{k}^{\prime} \sim \operatorname{Bern}\left(\tau_{k}^{\prime}\right)$ is independent of all the other random variables. Define $E_{k}=E_{K-1}^{\prime} \oplus E_{K-2}^{\prime} \oplus \cdots \oplus E_{k}^{\prime} \sim$ $\operatorname{Bern}\left(\tau_{k}\right)$ with $\tau_{k}=\tau_{K-1}^{\prime} \star \tau_{K-2}^{\prime} \star \cdots \star \tau_{k}^{\prime}$. Then

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid Z_{k}\right) \\
& =H\left(U_{k} \mid Z_{k}\right)-H\left(U_{k} \mid V_{k}, Z_{k}\right)  \tag{175}\\
& =H_{2}\left(\beta_{k} \star \tau_{k}\right)-\mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) \\
& \quad-\left(1-\mu_{k}\right) \sum_{v_{k} \in \mathcal{A}_{k}^{c}} \frac{\mathbb{P}\left(V_{k}=v_{k}\right)}{\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}^{c}\right)} H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) . \tag{176}
\end{align*}
$$

For fixed $v_{k}$, define a set of random variables $\left(V_{k}^{\prime}, S^{\prime}, U_{k}^{\prime}, Z_{k}^{\prime}\right) \sim 1\left\{v_{k}^{\prime}=v_{k}\right\} p_{S U_{k} Z_{k} \mid V_{k}}\left(s^{\prime}, u_{k}^{\prime}, z_{k}^{\prime} \mid v_{k}^{\prime}\right)$, then $H\left(U_{k}^{\prime} Z_{k}^{\prime} \mid V_{k}^{\prime}\right) \quad=\quad H\left(U_{k} Z_{k} \mid V_{k}=v_{k}\right) \quad$ and $H\left(Z_{k}^{\prime} \mid V_{k}^{\prime}\right)=H\left(Z_{k} \mid V_{k}=v_{k}\right)$. Since $p_{S U_{k} Z_{k} \mid V_{k}}$ satisfies

$$
\begin{align*}
& p_{S U_{k} Z_{k} \mid V_{k}}\left(s^{\prime}, u_{k}^{\prime}, z_{k}^{\prime} \mid v_{k}^{\prime}\right) \\
& =p_{S \mid V_{k}}\left(s^{\prime} \mid v_{k}^{\prime}\right) p_{Z_{k} \mid S}\left(z_{k}^{\prime} \mid s^{\prime}\right) p_{U_{k} \mid S}\left(u_{k}^{\prime} \mid s^{\prime}\right) \tag{177}
\end{align*}
$$

it holds that $Z_{k}^{\prime}=S^{\prime} \oplus B_{k}, U_{k}^{\prime}=S^{\prime} \oplus E_{k}$. Hence $Z_{k}^{\prime} \oplus U_{k}^{\prime}=$ $B_{k} \oplus E_{k}$.

For fixed $v_{k}$, consider

$$
\begin{align*}
& H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) \\
& =H\left(U_{k} Z_{k} \mid V_{k}=v_{k}\right)-H\left(Z_{k} \mid V_{k}=v_{k}\right)  \tag{178}\\
& =H\left(U_{k}^{\prime} Z_{k}^{\prime} \mid V_{k}^{\prime}\right)-H\left(Z_{k}^{\prime} \mid V_{k}^{\prime}\right)  \tag{179}\\
& =H\left(U_{k}^{\prime} \mid Z_{k}^{\prime} V_{k}^{\prime}\right)  \tag{180}\\
& =H\left(U_{k}^{\prime} \oplus Z_{k}^{\prime} \mid Z_{k}^{\prime} V_{k}^{\prime}\right)  \tag{181}\\
& =H\left(B_{k} \oplus E_{k} \mid Z_{k}^{\prime} V_{k}^{\prime}\right)  \tag{182}\\
& \leq H\left(B_{k} \oplus E_{k}\right)  \tag{183}\\
& =H_{2}\left(\beta_{k} \star \tau_{k}\right) \tag{184}
\end{align*}
$$

Combine (176) and (184), then we have

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid Z_{k}\right) \\
& \geq H_{2}\left(\beta_{k} \star \tau_{k}\right)-\mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) \\
& \quad-\left(1-\mu_{k}\right) H_{2}\left(\beta_{k} \star \tau_{k}\right)  \tag{185}\\
& =\mu_{k} H_{2}\left(\beta_{k} \star \tau_{k}\right)-\mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) . \tag{186}
\end{align*}
$$

Now we consider the second term of (186).

$$
\begin{align*}
& \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right) \\
& =\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}}\left(H\left(U_{k} Z_{k} \mid V_{k}=v_{k}\right)-H\left(Z_{k} \mid V_{k}=v_{k}\right)\right) \tag{187}
\end{align*}
$$

$$
\begin{align*}
& =\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}}\left(H_{4}\left(d_{v_{k}}, \beta_{k}, \tau_{k}\right)-H_{2}\left(d_{v_{k}} \star \beta_{k}\right)\right)  \tag{188}\\
& =\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} G_{1}\left(d_{v_{k}}, \beta_{k}, \tau_{k}\right) \tag{189}
\end{align*}
$$

where the function $H_{4}(x, y, z)$ is defined in (59) and

$$
\begin{equation*}
G_{1}(x, y, z) \triangleq H_{4}(x, y, z)-H_{2}(x \star y) \tag{190}
\end{equation*}
$$

Equality (188) follows from calculating the entropies according to the definition.

Now we show that $G_{1}(x, y, z)$ is concave in $x$. To do this, we consider

$$
\begin{align*}
& \frac{\partial^{2}}{\partial x^{2}} G_{1}(x, y, z) \\
& =-\frac{(y z-\overline{y z})^{2}}{x y z+\overline{x y z}}-\frac{(\bar{y} z-y \bar{z})^{2}}{x \bar{y} z+\overline{x y} y}-\frac{(y \bar{z}-\bar{y} z)^{2}}{x y \bar{z}+\overline{x y} z} \\
& \quad-\frac{(\overline{y z}-y z)^{2}}{x \overline{y z}+\bar{x} y z}+\frac{(y-\bar{y})^{2}}{x \bar{y}+\bar{x} y}+\frac{(y-\bar{y})^{2}}{x y+\overline{x y}}  \tag{191}\\
& = \\
& -\left(\frac{(y z-\overline{y z})^{2}}{x y z+\overline{x y z}}+\frac{(y \bar{z}-\bar{y} z)^{2}}{x y \bar{z}+\overline{x y} z}-\frac{(y-\bar{y})^{2}}{x y+\overline{x y}}\right)  \tag{192}\\
&  \tag{193}\\
& \quad-\left(\frac{(\bar{y} z-y \bar{z})^{2}}{x \bar{y} z+\bar{x} y \bar{z}}+\frac{(\overline{y z}-y z)^{2}}{x \overline{y z}+\bar{x} y z}-\frac{(y-\bar{y})^{2}}{x \bar{y}+\bar{x} y}\right) \\
& \leq 0
\end{align*}
$$

where (193) follows from the following inequality

$$
\begin{align*}
\frac{a_{1}^{2}}{b_{1}}+\frac{a_{2}^{2}}{b_{2}} & =\frac{1}{b_{1}+b_{2}}\left(b_{1}+b_{2}\right)\left(\frac{a_{1}^{2}}{b_{1}}+\frac{a_{2}^{2}}{b_{2}}\right)  \tag{194}\\
& =\frac{1}{b_{1}+b_{2}}\left(a_{1}^{2}+a_{2}^{2}+\frac{b_{2} a_{1}^{2}}{b_{1}}+\frac{b_{1} a_{2}^{2}}{b_{2}}\right)  \tag{195}\\
& \geq \frac{1}{b_{1}+b_{2}}\left(a_{1}^{2}+a_{2}^{2}+2 a_{1} a_{2}\right)  \tag{196}\\
& =\frac{\left(a_{1}+a_{2}\right)^{2}}{b_{1}+b_{2}} \tag{197}
\end{align*}
$$

for $b_{1}, b_{2}>0$ and arbitrary real numbers $a_{1}, a_{2}$. (193) implies $G_{1}(x, y, z)$ is concave in $x$.

Then combining the concavity of $G_{1}(x, y, z)$ with (186) and (189), we have

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid Z_{k}\right) \\
& \geq \mu_{k}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-G_{1}\left(\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} d_{v_{k}}, \beta_{k}, \tau_{k}\right)\right)  \tag{198}\\
& =\mu_{k}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-G_{1}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)\right) \tag{199}
\end{align*}
$$

where

$$
\begin{equation*}
\alpha_{k} \triangleq \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} d_{v_{k}} \tag{200}
\end{equation*}
$$

From (173), $\alpha_{k}$ satisfies

$$
\begin{equation*}
\mu_{k} \alpha_{k}+\left(1-\mu_{k}\right) \beta_{k} \leq D_{k} \tag{201}
\end{equation*}
$$

Combine (201) with $D_{k} \leq \beta_{k}$ (i.e., (157)), then we have

$$
\begin{equation*}
0 \leq \alpha_{k} \leq D_{k} \leq \beta_{k} \tag{202}
\end{equation*}
$$

Therefore,

$$
I\left(V_{k} ; U_{k} \mid Z_{k}\right)
$$

$$
\begin{align*}
\geq & \frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-G_{1}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)\right)  \tag{203}\\
= & \frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)\right. \\
& \left.\quad-\left(H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)-H_{2}\left(\alpha_{k} \star \beta_{k}\right)\right)\right), \tag{204}
\end{align*}
$$

i.e., (174) holds.

Next we will show

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& \geq \frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right. \\
& \left.\quad \quad-\left(H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)-H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k-1}\right)\right)\right) . \tag{205}
\end{align*}
$$

## Consider

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& =H\left(U_{k} \mid U_{k-1} Z_{k}\right)-H\left(U_{k} \mid U_{k-1} Z_{k} V_{k}\right)  \tag{206}\\
& =H\left(U_{k-1} \mid U_{k}\right)+H\left(U_{k} \mid Z_{k}\right) \\
& \quad-H\left(U_{k-1} \mid Z_{k}\right)-H\left(U_{k} \mid U_{k-1} Z_{k} V_{k}\right)  \tag{207}\\
& =H_{2}\left(\tau_{k-1}^{\prime}\right)+H_{2}\left(\beta_{k} \star \tau_{k}\right) \\
& \quad \quad-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)-H\left(U_{k} \mid U_{k-1} Z_{k} V_{k}\right) . \tag{208}
\end{align*}
$$

Write the last term as

$$
\begin{align*}
& H\left(U_{k} \mid U_{k-1} Z_{k} V_{k}\right) \\
& =-\left(1-\mu_{k}\right) \sum_{v_{k} \in \mathcal{A}_{k}^{c}} \frac{\mathbb{P}\left(V_{k}=v_{k}\right)}{\mathbb{P}\left(V_{k} \in \mathcal{A}_{k}^{c}\right)} H\left(U_{k} \mid U_{k-1}, Z_{k}, V_{k}=v_{k}\right) \\
& \quad-\mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid U_{k-1}, Z_{k}, V_{k}=v_{k}\right) . \tag{209}
\end{align*}
$$

For fixed $v_{k}$, define $\left(V_{k}^{\prime}, S^{\prime}, U_{k}^{\prime}, U_{k-1}^{\prime}, Z_{k}^{\prime}\right)$ $1\left\{v_{k}^{\prime}=v_{k}\right\} p_{S U_{k} U_{k-1} Z_{k} \mid V_{k}}\left(s^{\prime}, u_{k}^{\prime}, u_{k-1}^{\prime}, z_{k}^{\prime} \mid v_{k}^{\prime}\right)$. Since

$$
\begin{align*}
& p_{S U_{k} U_{k-1} Z_{k} \mid V_{k}}\left(s^{\prime}, u_{k}^{\prime}, u_{k-1}^{\prime}, z_{k}^{\prime} \mid v_{k}^{\prime}\right) \\
& =p_{S \mid V_{k}}\left(s^{\prime} \mid v_{k}^{\prime}\right) p_{Z_{k} \mid S}\left(z_{k}^{\prime} \mid s^{\prime}\right) p_{U_{k} \mid S}\left(u_{k}^{\prime} \mid s^{\prime}\right) \\
& \quad \times p_{U_{k-1} \mid U_{k}}\left(u_{k-1}^{\prime} \mid u_{k}^{\prime}\right) \tag{210}
\end{align*}
$$

we have $Z_{k}^{\prime}=S^{\prime} \oplus B_{k}, U_{k}^{\prime}=S^{\prime} \oplus E_{k}, U_{k-1}^{\prime}=U_{k}^{\prime} \oplus E^{\prime}{ }_{k-1}$. Hence $Z_{k}^{\prime} \oplus U_{k}^{\prime}=B_{k} \oplus E_{k}, Z_{k}^{\prime} \oplus U_{k-1}^{\prime}=B_{k} \oplus E_{k-1}$. Similar to the derivation for $H\left(U_{k} \mid U_{k-1}, V_{k}=v_{k}\right)$, we can write

$$
\begin{align*}
H & \left(U_{k} \mid U_{k-1}, Z_{k}, V_{k}=v_{k}\right) \\
= & H\left(U_{k}^{\prime} \mid U_{k-1}^{\prime} Z_{k}^{\prime} V_{k}^{\prime}\right)  \tag{211}\\
= & H\left(U_{k}^{\prime} \oplus Z_{k}^{\prime} \mid U_{k-1}^{\prime} \oplus Z_{k}^{\prime}, Z_{k}^{\prime}, V_{k}^{\prime}\right)  \tag{212}\\
\leq & H\left(U_{k}^{\prime} \oplus Z_{k}^{\prime} \mid U_{k-1}^{\prime} \oplus Z_{k}^{\prime}\right)  \tag{213}\\
= & H\left(B_{k} \oplus E_{k} \mid B_{k} \oplus E_{k-1}\right)  \tag{214}\\
= & H\left(B_{k} \oplus E_{k}\right)+H\left(B_{k} \oplus E_{k-1} \mid B_{k} \oplus E_{k}\right) \\
& \quad-H\left(B_{k} \oplus E_{k-1}\right)  \tag{215}\\
= & H_{2}\left(\tau_{k-1}^{\prime}\right)+H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right) . \tag{216}
\end{align*}
$$

Combine (208), (209) and (216), then we have

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& \geq \mu_{k}\left(H_{2}\left(\tau_{k-1}^{\prime}\right)+H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right) \\
& \quad-\mu_{k} \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid U_{k-1}, Z_{k}, V_{k}=v_{k}\right) \tag{217}
\end{align*}
$$

Consider the last term of (217),

$$
\begin{align*}
& \sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} H\left(U_{k} \mid U_{k-1}, Z_{k}, V_{k}=v_{k}\right) \\
& =\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}}\left(H\left(U_{k} \mid Z_{k}, V_{k}=v_{k}\right)\right. \\
& \quad+H\left(U_{k-1} \mid U_{k}, Z_{k}, V_{k}=v_{k}\right) \\
& \left.\quad-H\left(U_{k-1} \mid Z_{k}, V_{k}=v_{k}\right)\right)  \tag{218}\\
& =\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}}\left(H\left(U_{k}, Z_{k} \mid V_{k}=v_{k}\right)+H_{2}\left(\tau_{k-1}^{\prime}\right)\right. \\
& \left.\quad \quad-H\left(U_{k-1}, Z_{k} \mid V_{k}=v_{k}\right)\right)  \tag{219}\\
& = \\
& \quad H_{2}\left(\tau_{k-1}^{\prime}\right)  \tag{220}\\
& \quad+\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}}\left(H_{4}\left(d_{v_{k}}, \beta_{k}, \tau_{k}\right)-H_{4}\left(d_{v_{k}}, \beta_{k}, \tau_{k-1}\right)\right)  \tag{221}\\
& = \\
& H_{2}\left(\tau_{k-1}^{\prime}\right)+\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} G_{2}\left(d_{v_{k}}, \beta_{k}, \tau_{k}, \tau_{k-1}\right)
\end{align*}
$$

where (220) is by directly calculating the entropies according to the definition, and

$$
\begin{equation*}
G_{2}(x, y, z, t) \triangleq H_{4}(x, y, z)-H_{4}(x, y, t) \tag{222}
\end{equation*}
$$

Note that function $G_{1}(x, y, z)$ is a special case of function $G_{2}(x, y, z, t)$ given $t=\frac{1}{2}$, i.e.,

$$
\begin{equation*}
G_{1}(x, y, z)=G_{2}\left(x, y, z, \frac{1}{2}\right) \tag{223}
\end{equation*}
$$

Now we show that $G_{2}(x, y, z, t)$ is concave in $x$ when $0 \leq$ $z \leq t \leq \frac{1}{2}$, which generalizes the concavity of $G_{1}(x, y, z)$. To do this, we consider

$$
\begin{align*}
& \frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, t) \\
& =-\frac{(y z-\overline{y z})^{2}}{x y z+\overline{x y z}}-\frac{(\bar{y} z-y \bar{z})^{2}}{x \bar{y} z+\bar{x} y \bar{z}}-\frac{(y \bar{z}-\bar{y} z)^{2}}{x y \bar{z}+\overline{x y} z}-\frac{(\overline{y z}-y z)^{2}}{x \overline{y z}+\bar{x} y z} \\
& \quad+\frac{(y t-\bar{y} \bar{t})^{2}}{x y t+\overline{x y} \bar{t}}+\frac{(\bar{y} t-y \bar{t})^{2}}{x \bar{y} t+\bar{x} y \bar{t}}+\frac{(y \bar{t}-\bar{y} t)^{2}}{x y \bar{t}+\overline{x y} t}+\frac{(\bar{y} \bar{t}-y t)^{2}}{x \bar{y} \bar{t}+\bar{x} y t}, \tag{224}
\end{align*}
$$

and

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(\frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, t)\right) \\
& =\frac{\partial}{\partial t}\left(\frac{(y t-\bar{y} \bar{t})^{2}}{x y t+\overline{x y} \bar{t}}+\frac{(y \bar{t}-\bar{y} t)^{2}}{x y \bar{t}+\overline{x y} t}\right) \\
& \quad+\frac{\partial}{\partial t}\left(\frac{(\bar{y} t-y \bar{t})^{2}}{x \bar{y} t+\bar{x} y \bar{t}}+\frac{(\bar{y} \bar{t}-y t)^{2}}{x \bar{y} \bar{t}+\bar{x} y t}\right)  \tag{225}\\
& =\frac{-y^{2} \cdot \bar{y}^{2} \cdot(x y+\overline{x y}) \cdot(1-2 t)}{(x y t+\overline{x y t})^{2}(x y \bar{t}+\overline{x y} t)^{2}} \\
& \quad+\frac{-y^{2} \cdot \bar{y}^{2} \cdot(x \bar{y}+\bar{x} y) \cdot(1-2 t)}{(x \bar{y} t+\bar{x} y \bar{t})^{2}(x \bar{y} \bar{t}+\bar{x} y t)^{2}} \tag{226}
\end{align*}
$$

Hence for $0 \leq t \leq \frac{1}{2}$,

$$
\frac{\partial}{\partial t}\left(\frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, t)\right) \leq 0
$$

(227)
i.e., $\frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, t)$ is decreasing in $t$. Then we have for $0 \leq z \leq t \leq \frac{1}{2}$,

$$
\begin{equation*}
\frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, t) \leq \frac{\partial^{2}}{\partial x^{2}} G_{2}(x, y, z, z)=0 \tag{228}
\end{equation*}
$$

It implies $G_{2}(x, y, z, t)$ is concave in $x$ when $0 \leq z \leq t \leq \frac{1}{2}$.
Combining (217) and (221), and utilizing the concavity of $G_{2}(x, y, z, t)$, we have

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& \geq \mu_{k}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right. \\
& \left.\quad-G_{2}\left(\sum_{v_{k} \in \mathcal{A}_{k}} \lambda_{v_{k}} d_{v_{k}}, \beta_{k}, \tau_{k}, \tau_{k-1}\right)\right)  \tag{229}\\
& =\mu_{k}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right. \\
& \left.\quad-G_{2}\left(\alpha_{k}, \beta_{k}, \tau_{k}, \tau_{k-1}\right)\right) \tag{230}
\end{align*}
$$

where $\alpha_{k}$ is given by (200) and satisfies (201) and (202). Therefore,

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& \geq \frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right. \\
& \left.\quad-G_{2}\left(\alpha_{k}, \beta_{k}, \tau_{k}, \tau_{k-1}\right)\right)  \tag{231}\\
& =\frac{\beta_{k}-D_{k}}{\beta_{k}-\alpha_{k}}\left(H_{2}\left(\beta_{k} \star \tau_{k}\right)-H_{2}\left(\beta_{k} \star \tau_{k-1}\right)\right. \\
& \left.\quad \quad-\left(H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k}\right)-H_{4}\left(\alpha_{k}, \beta_{k}, \tau_{k-1}\right)\right)\right) \tag{232}
\end{align*}
$$

i.e., (205) holds.

Combining (159), (174) and (205) gives Theorem 6.

## Appendix E <br> Proof of Theorem 7

For the Wyner-Ziv Gaussian broadcast with bandwidth mismatch case (the bandwidth mismatch factor $b$ ), Theorem 5 states that if $D_{[1: K]}$ is achievable, then there exist some pmf $p_{V_{K} \mid S} p_{V_{K-1} \mid V_{K}} \cdots p_{V_{1} \mid V_{2}}$ and functions $\hat{s}_{k}\left(v_{k}, z_{k}\right), 1 \leq k \leq$ $K$ such that

$$
\begin{equation*}
\mathbb{E} d\left(S, \hat{S}_{k}\right) \leq D_{k} \tag{233}
\end{equation*}
$$

and for any pmf $p_{U_{K-1} \mid S} p_{U_{K-2} \mid U_{K-1}} \cdots p_{U_{1} \mid U_{2}}$,

$$
\begin{equation*}
\frac{1}{b}\left(I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right): k \in[1: K]\right) \in \mathcal{R}_{\mathrm{GBC}} \tag{234}
\end{equation*}
$$

holds, where the capacity of Gaussian broadcast channel $\mathcal{R}_{\mathrm{GBC}}$ is given in (61).

Choose $U_{K-1}=S+E_{K-1}^{\prime}$ and $U_{k}=U_{k+1}+E_{k}^{\prime}, 1 \leq k \leq$ $K-2$, where $E_{k}^{\prime} \sim \mathcal{N}\left(0, \tau_{k}^{\prime}\right)$ is independent of all the other random variables. Define $E_{k}=\sum_{j=k}^{K-1} E_{j}^{\prime} \sim \mathcal{N}\left(0, \tau_{k}\right)$ with $\tau_{k}=\sum_{j=k}^{K-1} \tau_{j}^{\prime}$. Then

$$
\begin{align*}
& I\left(V_{1} ; U_{1} \mid Z_{1}\right) \\
& \geq I\left(\hat{S}_{1} ; U_{1} \mid Z_{1}\right)  \tag{235}\\
& =h\left(U_{1} \mid Z_{1}\right)-h\left(U_{1} \mid \hat{S}_{1} Z_{1}\right)  \tag{236}\\
& =h\left(U_{1} \mid Z_{1}\right)-h\left(U_{1}-\hat{S}_{1} \mid \hat{S}_{1} Z_{1}\right) \tag{237}
\end{align*}
$$

$$
\begin{align*}
& \geq h\left(U_{1} \mid Z_{1}\right)-h\left(U_{1}-\hat{S}_{1}\right)  \tag{238}\\
& \geq \frac{1}{2} \log \left(2 \pi e\left(\beta_{1}+\tau_{1}\right)\right)-\frac{1}{2} \log \left(2 \pi e\left(D_{1}+\tau_{1}\right)\right)  \tag{239}\\
& =\frac{1}{2} \log \frac{\beta_{1}+\tau_{1}}{D_{1}+\tau_{1}} \tag{240}
\end{align*}
$$

where (239) follows from the fact that a Gaussian distribution maximizes the differential entropy for a given second moment.

On the other hand,

$$
\begin{align*}
& I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \\
& \geq I\left(\hat{S}_{k} ; U_{k} \mid U_{k-1} Z_{k}\right)  \tag{241}\\
& =I\left(\hat{S}_{k} ; U_{k} \mid Z_{k}\right)-I\left(\hat{S}_{k} ; U_{k-1} \mid Z_{k}\right)  \tag{242}\\
& =h\left(U_{k} \mid Z_{k}\right)-h\left(U_{k-1} \mid Z_{k}\right) \\
& \quad+h\left(U_{k-1} \mid Z_{k} \hat{S}_{k}\right)-h\left(U_{k} \mid Z_{k} \hat{S}_{k}\right) \tag{243}
\end{align*}
$$

The first two terms of (243)

$$
\begin{equation*}
h\left(U_{k} \mid Z_{k}\right)-h\left(U_{k-1} \mid Z_{k}\right)=\frac{1}{2} \log \frac{\beta_{k}+\tau_{k}}{\beta_{k}+\tau_{k-1}} . \tag{244}
\end{equation*}
$$

The last two terms of (243)

$$
\begin{align*}
& h\left(U_{k-1} \mid Z_{k} \hat{S}_{k}\right)-h\left(U_{k} \mid Z_{k} \hat{S}_{k}\right) \\
& =h\left(U_{k-1} \mid Z_{k} \hat{S}_{k}\right)-h\left(U_{k} \mid Z_{k} \hat{S}_{k} E_{k-1}^{\prime}\right)  \tag{245}\\
& =h\left(U_{k-1} \mid Z_{k} \hat{S}_{k}\right)-h\left(U_{k-1} \mid Z_{k} \hat{S}_{k} E_{k-1}^{\prime}\right)  \tag{246}\\
& =I\left(U_{k-1} ; E_{k-1}^{\prime} \mid Z_{k} \hat{S}_{k}\right)  \tag{247}\\
& =h\left(E_{k-1}^{\prime}\right)-h\left(E_{k-1}^{\prime} \mid Z_{k} \hat{S}_{k} U_{k-1}\right)  \tag{248}\\
& =h\left(E_{k-1}^{\prime}\right)-h\left(E_{k-1}^{\prime} \mid Z_{k}, \hat{S}_{k}, U_{k-1}-\hat{S}_{k}\right)  \tag{249}\\
& \geq h\left(E_{k-1}^{\prime}\right)-h\left(E_{k-1}^{\prime} \mid U_{k-1}-\hat{S}_{k}\right)  \tag{250}\\
& =I\left(E_{k-1}^{\prime} ; S-\hat{S}_{k}+E_{k}+E_{k-1}^{\prime}\right)  \tag{251}\\
& \geq \frac{1}{2} \log \frac{D_{k}+\tau_{k-1}}{D_{k}+\tau_{k}} \tag{252}
\end{align*}
$$

where (252) is by applying the mutual information game result that Gaussian noise is the worst additive noise under a variance constraint [14, p. 298, Problem 9.21] and taking $E_{k-1}^{\prime}$ as the channel input.

Combining (243), (244) and (252), we have
$I\left(V_{k} ; U_{k} \mid U_{k-1} Z_{k}\right) \geq \frac{1}{2} \log \frac{\left(\beta_{k}+\tau_{k}\right)\left(D_{k}+\tau_{k-1}\right)}{\left(\beta_{k}+\tau_{k-1}\right)\left(D_{k}+\tau_{k}\right)}$.
(234), (240) and (253) imply Theorem 7.

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[^1]:    ${ }^{1}$ For brevity, the Markov chain is assumed in this direction.This differs from that in the conference version [1].

[^2]:    ${ }^{2}$ We say a set $\mathcal{G}$ is larger than another $\mathcal{H}$ if $|\mathcal{G}|>|\mathcal{H}|$, or $|\mathcal{G}|=|\mathcal{H}|$ and there exists some $1 \leq i \leq|\mathcal{G}|$ such that $\mathcal{G}[i]>\mathcal{H}[i]$ and $\mathcal{G}[l]=\mathcal{H}[l]$ for all $1 \leq l \leq i-1$, where $\mathcal{G}[i]$ (or $\mathcal{H}[i]$ ) denotes the $i$ th largest element in $\mathcal{G}$ (or $\mathcal{H}$ ).

[^3]:    ${ }^{3}$ Note that for Marton's broadcast channel-coding [16, p. 212-213], $r_{[1: N]}$ here does not correspond to the regular broadcast-rates $R_{[1: N]}$, but corresponds to the total rates $\widetilde{R}_{[1: N]}$ of each subcodebooks. In addition, here we only require the decoding operation is successful, hence the condition that the chosen codewords and the source sequence are jointly typical, which was required in the encoding operation of Gray-Wyner source-coding, is not repeatedly required here.

