On the covering radius of small codes versus dual distance

Louay Bazzi *†

June 16, 2021

Abstract

Tietäväinen's upper and lower bounds assert that for block-length-n linear codes with dual distance d, the covering radius R is at most $\frac{n}{2} - (\frac{1}{2} - o(1))\sqrt{dn}$ and typically at least $\frac{n}{2} - \Theta(\sqrt{dn\log\frac{n}{d}})$. The gap between those bounds on $R - \frac{n}{2}$ is an $\Theta(\sqrt{\log\frac{n}{d}})$ factor related to the gap between the worst covering radius given d and the sphere-covering bound. Our focus in this paper is on the case when d = o(n), i.e., when the code size is subexponential and the gap is w(1). We show that up to a constant, the gap can be eliminated by relaxing the covering requirement to allow for missing o(1) fraction of points. Namely, if the dual distance d = o(n), then for sufficiently large d, almost all points can be covered with radius $R \leq \frac{n}{2} - \Theta(\sqrt{dn\log\frac{n}{d}})$. Compared to random linear codes, our bound on $R - \frac{n}{2}$ is asymptotically tight up to a factor less than 3. We give applications to dual BCH codes. The proof builds on the author's previous work on the weight distribution of cosets of linear codes, which we simplify in this paper and extend from codes to probability distributions on $\{0,1\}^n$, thus enabling the extension of the above result to (d-1)-wise independent distributions.

1 Introduction

The covering radius of a subset C of the Hamming cube $\{0,1\}^n$ is the minimum r such that any vector in $\{0,1\}^n$ is within Hamming distance at most r from C. Throughout the paper, $n \ge 1$ is an integer and the Hamming weight of a vector $x \in \{0,1\}^n$, which we denote by |x|, is the number of nonzero coordinates of x. If $r \ge 0$ is a real number and $x \in \{0,1\}^n$, let $\mathcal{H}_n(x;r)$ be the Hamming ball of radius r centered at x, i.e., $\mathcal{H}_n(x;r) = \{x \in \{0,1\}^n : |x+y| \le r\}$, where + is addition modulo 2. If C is a subset of $\{0,1\}^n$, let $\mathcal{H}_n(C;r)$ be the the r-neighborhood of C, i.e., $\mathcal{H}_n(C;r) = \bigcup_{x \in C} \mathcal{H}_n(x;r)$. Thus, the covering radius of C is the minimum r such that $\mathcal{H}_n(C;r) = \{0,1\}^n$. See [1] for a comprehensive reference on covering codes.

In 1990, Tietäväinen derived an upper bound on the covering radius R of a block-length-n linear code C in terms of only its minimum dual distance d. The minimum distance of a non-zero \mathbb{F}_2 -linear¹ code is the minimum Hamming weight of a nonzero codeword.

Theorem 1.1 (Tietäväinen [2, 3]) (Upper bound on covering radius of codes in terms of dual distance) Let $C \subset \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual has minimum distance $d \geq 2$. Then the covering radius R of C is at most

$$\begin{cases} \frac{n}{2} - \sqrt{s(n-s)} + s^{1/6}\sqrt{n-s} & \text{if } d = 2s \text{ is even} \\ \frac{n}{2} - \sqrt{s(n-1-s)} + s^{1/6}\sqrt{n-1-s} - \frac{1}{2} & \text{if } d = 2s+1 \text{ is odd.} \end{cases}$$

^{*}Department of Electrical and Computer Engineering, American University of Beirut, Beirut, Lebanon. E-mail: louay.bazzi@aub.edu.lb.

[†]Research supported by MSFEA URB grant, American University of Beirut.

 $^{{}^{1}\}mathbb{F}_{2}$ is the finite field structure on $\{0, 1\}$.

Tietäväinen's argument is based on studying the dual linear program in the context of Delsarte's linear programming framework [4]. In particular, Tietäväinen proved Theorem 1.1 by establishing the existence of certain univariate low degree polynomials constructed from Krawtchouk polynomials.

Prior to Tietäväinen's work, the relation between the covering radius and dual distance was investigated in [4]- [6]. In the $d = \Theta(n)$ regime, Tietäväinen's bound was later improved in a sequence of works [7] - [17] (see also [18]). For small values of d including the d = o(n) regime, it is still the best known upper bound. In [19], we showed that for $d \leq \frac{n^{1/3}}{\log^2 n}$, Tietäväinen's bound on $R - \frac{n}{2}$ is asymptotically tight up to a factor of 2 for (d - 1)-wise independent probability distributions on $\{0, 1\}^n$, of which linear codes with dual distance d are special cases.

By combining the sphere-covering bound and Gilbert-Varshamov bound, Tietäväinen [3] established also a simple lower bound on the covering radius as function of the dual distance. For comparison purposes, we need the following version of the lower bound tailored to the small d regime.

Lemma 1.2 (Small codes version of Tietäväinen's lower bound on the covering radius in terms of dual distance) Let $n \ge 1$ be an integer and $n \le K \le 2^{n-1}$ be an integer power of 2. Then, almost all \mathbb{F}_2 -linear codes $C \subset \mathbb{F}_2^n$ of size K have covering radius $R \ge \frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d}})$, where d is the minimum distance of C^{\perp} . More specifically, all but at most $\frac{1}{n}$ fraction of \mathbb{F}_2 -linear codes $C \subset \mathbb{F}_2^n$ of size K have covering radius

$$R \ge \frac{n}{2} - \sqrt{\frac{dn}{2}\log\frac{en}{d} + n\log(n+1)}.$$

For a proof of Lemma 1.2, see Lemma 1.8 with $\varepsilon = 0$. Note that throughout the paper log means \log_{e} .

The difference between Tietäväinen's upper and lower bounds on $R - \frac{n}{2}$ is a $\Theta(\sqrt{\log \frac{n}{d}})$ factor. The focus of this brief paper is on linear codes with dual distance d = o(n), which corresponds to the case when the code size is subexponential and the factor $\Theta(\sqrt{\log \frac{n}{d}})$ grows with n. Our study is motivated by this gap which is related to the gap between the typical and the worst possible covering radius given d. In what follows, we highlight the gap by comparing the covering radius of dual BCH codes with the typical covering radius of linear codes of the same size.

It follows from the work of Cohen and Blinovskii that the typical covering radius of linear codes achieves the sphere-covering bound. Cohen showed that there are linear codes up to the sphere-covering bound:

Theorem 1.3 (Cohen [20]; see also [1, Chapter 12]) (Linear codes up to the spherecovering bound) For any $n \ge 1$ and $0 < R \le n$, there exists an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$ of covering radius R and dimension

$$k \le \left\lceil \log_2 \frac{n(\log 2)}{v_n(R)} \right\rceil,$$

where $v_n(R)$ is the probability with respect to the uniform distribution of the radius-R Hamming ball $\mathcal{H}_n(0; R)$.

Later, Blinovskii [21, 22] showed that almost all linear codes achieve the sphere-covering bound. See also [1, Chapter 12] and the references therein.

To illustrate the gap in the case of dual BCH code, we need the following immediate corollary to Cohen's theorem customized to small codes. We include a proof in Appendix A for completeness. **Corollary 1.4 (Explicit version for small codes)** If $n \ge 1$, s > 1, and $s = o(\frac{n}{\log n})$, then for n large enough, there exists an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$ of dimension at most $\lceil s \log_2 n \rceil$ and covering radius $R \le \frac{n}{2} - \sqrt{\frac{(s-1)n \log n}{2+o(1)}}$.

More specifically, for each $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $n \ge 1$ be an integer and s > 1 be such that $s \log_2 n \le \delta n$. Then for n large enough, there exists an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$ of dimension at most $\lceil s \log_2 n \rceil$ and covering radius

$$R \leq \frac{n}{2} - \sqrt{\frac{(s-1)n\log n}{2+\epsilon}} + \sqrt{2n} + 2.$$

Before moving to the next section, we note that a related work is an explicit construction due to Alon – attributed to Alon by Rabani and Shpilka [23] – of polynomial size codes of covering radius $\frac{n}{2} - c\sqrt{n \log n}$, for any constant c.

1.1 The gap in the case of dual BCH codes

Consider the block-length-*n* dual BCH code $C = BCH(s,m)^{\perp}$, where $m \geq 2$ and $s \geq 1$ are integers such that $2s - 2 < 2^{m/2}$, i.e., $s < \frac{1}{2}\sqrt{n+1} + 1$, and $n = 2^m - 1$. The dimension of *C* is $k = sm = s \log_2(n+1) > s \log_2 n$, the minimum distance of C^{\perp} is at least d = 2s + 1, and the covering radius *R* of *C* satisfies:

$$R \leq \frac{n}{2} - (1 - o(1))\sqrt{sn} \tag{1}$$

$$R \geq \frac{n}{2} - (s-1)\sqrt{n+1} - \frac{1}{2}.$$
 (2)

The upper bound (1) is Tietäväinen's bound (Theorem 1.1) and the lower bound (2) is Weil-Carlitz-Uchiyama bound (see Section 2.5). Applying Corollary 1.4 to linear codes of dimension comparable to the dimension k of the dual BCH code C, we get that there exists an \mathbb{F}_2 -linear code $C' \subset \mathbb{F}_2^n$ of dimension $k' \leq \lceil s \log_2 n \rceil \leq k$ and covering radius

$$R' \le \frac{n}{2} - \sqrt{\frac{(s-1)n\log n}{2+o(1)}}.$$
(3)

Comparing (1) and (3), we see that the upper bound on $R - \frac{n}{2}$ in (1) is worse than that in (3) by a factor of $\Theta(\sqrt{\log n})$. The same factor appears if we compare the lower bound (2) with the upper bound (3) when $s = \Theta(1)$. That is, in the $s = \Theta(1)$ regime, while the actual covering radius of $BCH(s,m)^{\perp}$ is $R = \frac{n}{2} - \Theta(\sqrt{n})$, linear codes of smaller dimension have covering radius $R = \frac{n}{2} - \Theta(\sqrt{n \log n})$.

1.2 Summary of results

For dual BCH codes $BCH(s,m)^{\perp}$, where $s \geq 3$ and $2s - 2 < 2^{m/2}$, we show that the $\Theta(\sqrt{\log n})$ gap can be eliminated by relaxing the covering requirement: instead of covering all the vectors in $\{0,1\}^n$, we can guarantee covering all but o(1) fraction of them with radius $\frac{n}{2} - \Theta(\sqrt{sn \log n})$. More generally, we show that if a linear code has dual minimum distance at least d, where d = o(n), then for sufficiently large d, almost all points can be covered with radius $R \leq \frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d}})$. This bound on $R - \frac{n}{2}$ asymptotically matches Tietäväinen lower bound up to factor less than 3. It also asymptotically matches up to the same factor an adaptation of Tietäväinen lower bound to almost-all-coverings, i.e., compared to random linear codes, it is tight up to a constant factor less than 3. The proof builds on the author's previous work on the weight distribution of cosets of linear codes with given bilateral minimum distance [24].

The bilateral minimum distance of a non-zero \mathbb{F}_2 -linear code D is the maximum b such that all nonzero codewords have weights between b and n - b, i.e., $b \leq |z| \leq n - b$, for each nonzero $z \in D$.

We also simplify in this paper a part of the proof in [24] which makes it possible to extend the results in [24] as well as the above results from codes to probability distributions. In particular, we extend the above results on the almost-all covering radius from codes with dual distance d to (d-1)-wise independent distributions, of which linear codes with dual distance d are special cases. A probability distribution μ on $\{0,1\}^n$ is called k-wise independent if for $(x_1,\ldots,x_n) \sim \mu$, each x_i is equally likely to be 0 or 1 and any k of the x_i 's are statistically independent [25, 26]. Linear codes with dual distance at least k + 1 are special cases of k-wise independent distributions. Namely, if μ is uniformly distributed on an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$, then μ being k-wise independent is equivalent to C having dual minimum distance at least k+1. Note that the covering radius of a probability distribution on $\{0,1\}^n$ is interpreted as the covering radius of its support.

We elaborate below on the results in the case of linear codes. Their extensions to distributions are presented in Section 5.

Definition 1.5 (Almost-all covering) Let $0 \le \varepsilon \le 1$. The ε -covering radius of a subset C of the Hamming cube $\{0,1\}^n$ is the minimum r such that the fraction of points of the Hamming cube not contained in the r-Hamming-neighborhood $\mathcal{H}_n(C;r)$ of C is at most ε .

Thus the covering radius corresponds to $\varepsilon = 0$. The notion of almost-all-covering goes back to the argument of Blinovskii [21, 22] in his proof that almost all linear codes achieve the sphere-covering bound.

First we establish the following nonasymptotic bound.

Theorem 1.6 (Upper bound on the almost-all-covering radius of small codes in terms of dual distance) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d, where $d \ge 7$ be an odd integer. If R > 0, then the fraction of points in Hamming cube not covered by $\mathcal{H}_n(C; R)$ is at most

$$\varepsilon = \frac{d}{v_{n+d}(R)} \left(e \log \frac{n+d}{d-1} \right)^{\frac{d-1}{2}} \left(\frac{d-1}{n+d} \right)^{\frac{d-5}{4}},$$

where $v_{n+d}(R)$ is the probability with respect to the uniform distribution of the radius-R Hamming ball $\mathcal{H}_{n+d}(0; R)$ in $\{0, 1\}^{n+d}$. That is, if $\varepsilon \leq 1$, then the ε -covering radius of C is at most R.

The proof of Theorem 1.6 builds on [24], where it is shown that for an \mathbb{F}_2 -linear code Q with dual bilateral minimum distance at least b, the average L_1 -distance between the weight distribution of a random cosets of Q and the binomial distribution decays quickly in b, and namely, it is bounded by $b\left(e\log\frac{n}{b-1}\right)^{\frac{b-1}{2}}\left(\frac{b-1}{n}\right)^{\frac{b-5}{4}}$, if $b \ge 7$ is odd. The proof of Theorem 1.6 boils down to using Markov Inequality and applying the above result to the block-length n + d code Q constructed from C by appending d independent bits to C. This simple construction turns the lower bound d on the minimum distance of C^{\perp} into a lower bound on the bilateral minimum distance of Q^{\perp} .

Then, based on the entropy estimate of the binomial coefficients, we conclude the following bound in the d = o(n) regime.

Corollary 1.7 (Explicit asymptotic version) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d.

If d = o(n), then for sufficiently large d, the o(1)-covering radius of C is at most $\frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d}})$.

More specifically, if $d \ge 7$ is an odd integer such that d = o(n), then, for sufficiently large n, the $\left(\frac{d-1}{n}\right)^{\frac{d-5}{13}}$ -covering radius of C is at most $R = \frac{n}{2} - \Delta$, where

$$\Delta = \sqrt{\frac{1}{13}(d-5)n\log\frac{n}{d-1}}.$$

Comparing the bounds on $R - \frac{n}{2}$ in Corollary 1.7 and Lemma 1.2, we see that the guarantee given by Corollary 1.7 on $R - \frac{n}{2}$ is asymptotically not far from Tietäväinen's lower bound on the covering radius of random linear codes by more than a factor of $\sqrt{\frac{13}{2}} \approx 2.55 < 3$. Actually, for almost-all-coverings, the upper bound of Corollary 1.7 is asymptotically tight up to the same factor in comparison to random linear codes. This follows from the following simple variation of Lemma 1.2:

Lemma 1.8 (Variation of Tietäväinen's lower bound: lower bound on the almostall-covering radius of small codes in terms of dual distance) Consider any $0 \le \varepsilon < 1$ and let $n \ge 1$ be an integer and $n \le K \le 2^{n-1}$ be an integer power of 2. Then, almost all \mathbb{F}_2 -linear codes $C \subset \mathbb{F}_2^n$ of size K have ε -covering radius $R \ge \frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d} + n \log \frac{n}{1-\varepsilon}})$, where d is the minimum distance of C^{\perp} . More specifically, all but at most $\frac{1}{n}$ fraction of \mathbb{F}_2 -linear codes $C \subset \mathbb{F}_2^n$ of size K have ε -covering radius

$$R \ge \frac{n}{2} - \sqrt{\frac{dn}{2}\log\frac{en}{d} + n\log\frac{n+1}{1-\varepsilon}}.$$

See Appendix B for a proof of Lemma 1.8.

Applying Corollary 1.7 to the dual BCH codes $BCH(s,m)^{\perp}$ with d = 2s + 1, where $s \ge 3$ so that $d \ge 7$, we get the following:

Corollary 1.9 (Application to dual BCH codes) Let $m \ge 2$ be an integer and $n = 2^m - 1$. Let $s \ge 3$ be an integer such that $2s - 2 < 2^{m/2}$, i.e., $s < \frac{1}{2}\sqrt{n+1} + 1$ and consider the dual BCH code $C = BCH(s,m)^{\perp}$. Then, the o(1)-covering radius of C is at most $\frac{n}{2} - \Theta(\sqrt{sn \log \frac{n}{s}})$. More specifically, for sufficiently large n, the $\left(\frac{2s}{n}\right)^{\frac{2s-5}{13}}$ -covering radius of C is at most

$$R = \frac{n}{2} - \sqrt{\frac{1}{13}(2s-4)n\log\frac{n}{2s}}.$$

For instance, for s = 3, we have $R = \frac{n}{2} - \sqrt{\frac{2}{13}n\log\frac{n}{6}}$. Thus, for $BCH(3,m)^{\perp}$, even though we need an $\frac{n}{2} - \Theta(\sqrt{n})$ radius to cover all points in $\{0,1\}^n$, we can cover almost all of them using an $\frac{n}{2} - \sqrt{\frac{2}{13}n\log\frac{n}{6}}$ radius.

Using Cohen's iterative argument for showing the existence of linear coverings up the spherecovering bound [20], we conclude from Corollary 1.7 that there exists a small $\lceil \log_2 n \rceil$ -dimensional linear code which can be added to C to turn the almost cover into a total cover.

Corollary 1.10 (Adding a small code) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d, where $d \ge 7$ is an odd integer such that d = o(n). Then there exists an \mathbb{F}_2 -linear code D of dimension at most $\lceil \log_2 n \rceil$ such that, for sufficiently large n, the covering radius of C + D is at most $\frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d}})$.

See Section 6 for a related open problem on dual BCH codes.

Before outlining the rest of the paper in the next section, we highlight additional links with the existing literature.

Turning an almost-all linear covering into a total covering goes back to the work of Blinovskii [21, 22].

The notion of bilateral minimum distance b of a linear code is equivalent to its width σ which is given by $\sigma = n - 2b$. For small values of b, it is more convenient to work with b rather than σ . In the high rate regime, the relation between the covering radius and the dual width was studied by Sole and Stokes [7].

Finally, we compare with the related work of Navon and Samorodnitsy [27], who recovered the first linear programming bound using a covering argument and Fourier analysis techniques. The related result in [27] is a bound that relates the dual distance to the minimum radius which guarantees covering a significant fraction of the Hamming cube. Namely, in terms of ε -covering, Corollary 1.5 in [27] asserts that if C is a block-length- $n \mathbb{F}_2$ -linear code with dual distance d, then the $(1-\frac{1}{n})$ -covering radius R of C is at most $\frac{n}{2} - \sqrt{d(n-d)} + o(n)$. Thus, in the context of $(1-\frac{1}{n})$ -coverings, Navon-Samorodnitsky's upper bound on $R - \frac{n}{2}$ is stronger than Tietäväinen's upper bound (Theorem 1.1) by factor of $\sqrt{2}$. It is however weaker than our bound in Corollary 1.7 by factor of $\Theta(\sqrt{\log \frac{n}{d}})$ in the d = o(n) regime. Also, Corollary 1.7 allows for smaller values of ε .

1.3 Paper outline

After introducing some preliminaries in Section 2, we prove Theorem 1.6 in Section 3. In Section 4, we prove Corollaries 1.7 and 1.10. In Section 5, we extend the results from codes to distributions.

2 Preliminaries

In this section, we compile some notations and definitions used throughout the paper.

2.1 Notations

We will use the following notations as in [19]. The set $\{0, \ldots, n\}$ is denoted by [0 : n]. The binomial distribution on [0 : n] is denoted by B_n , i.e., $B_n(w) = \frac{1}{2^n} \binom{n}{w}$. The uniform distribution on $\{0, 1\}^n$ is denoted by U_n , i.e., $U_n(x) = \frac{1}{2^n}$, for all $x \in \{0, 1\}^n$.

Thus, in terms of the above notations, the ε -covering radius of a subset $C \subset \{0,1\}^n$ is the minimum r such that $U_n(\mathcal{H}_n(C;r)) \geq 1 - \varepsilon$.

If μ is a probability distribution, \mathbb{E}_{μ} denotes the expectation with respect to μ and " $x \sim \mu$ " denotes the process of sampling a random vector x according to μ .

Weight distributions We will also use the following notations as in [24]. If μ is a probability distribution on $\{0, 1\}^n$, $\overline{\mu}$ denotes the corresponding weight distribution on [0:n], i.e., for all $w \in [0:n]$, $\overline{\mu}(w) \stackrel{\text{def}}{=} \mu(x \in \{0, 1\}^n : |x| = w)$.

If $A \subset \{0,1\}^n$, μ_A denotes the probability distribution on $\{0,1\}^n$ uniformly distributed on A. Thus $\overline{\mu_A}(w)$ is the fraction of points in A of weight w.

2.2 Hamming Ball Volume

Let $v_n(R)$ denote the probability with respect to the uniform distribution of the radius-R Hamming ball, i.e.,

$$v_n(R) = U_n(\mathcal{H}_n(0; R)) = \sum_{w \le R} B_n(w).$$

The proofs of Corollaries 1.4 and 1.7 use the lower bound on $v_n(R)$ in Lemma 2.1 below. The lower bound is based on the following estimate of the binomial coefficients: if $n \ge 1$ is an integer

and $0 < \lambda < 1$ is such that λn is an integer, then

$$\binom{n}{\lambda n} \ge \frac{2^{nH(\lambda)}}{\sqrt{8n\lambda(1-\lambda)}},$$

where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function (see, e.g., [1, Lemma 2.4.2]).

Lemma 2.1 For each $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $R = \frac{n}{2} - \Delta$, where $\Delta > 0$ is such that $\Delta \leq \delta n$. Then, for n large enough,

$$\upsilon_n(R) \ge e^{-(2+\epsilon)\frac{(\Delta+\sqrt{2n}+2)^2}{n}}.$$

Proof: With $\lambda = \frac{1}{n} \lfloor \frac{n}{2} - \Delta - \sqrt{2n} - 1 \rfloor$, we have

$$\upsilon_n(R) = \sum_{w \le \frac{n}{2} - \Delta} B_n(w) \ge \sqrt{2n} B_n(\lambda n) \ge \sqrt{\frac{2n}{8n\lambda(1-\lambda)}} 2^{-n(1-H(\lambda))} \ge 2^{-n(1-H(\lambda))}.$$

Let $x = \frac{\Delta + \sqrt{2n+2}}{n}$, hence $\lambda \ge \frac{1}{2} - x$. Since $H(\frac{1}{2} - x) = 1 - \frac{2x^2}{\log 2} - O(x^4)$, let $\delta > 0$ so that $H(\frac{1}{2} - x) \ge 1 - \frac{(2+\epsilon)x^2}{\log 2}$ for each $0 \le x \le 2\delta$. Thus

$$v_n(R) \ge 2^{-n(1-H(\lambda))} \ge 2^{-n(1-H(1-x))} \ge e^{-(2+\epsilon)nx^2} = e^{-(2+\epsilon)\frac{(\Delta+\sqrt{2n}+2)^2}{n}}.$$

The claim then holds for n sufficiently large so that $\frac{\sqrt{2n+2}}{n} \leq \delta$.

2.3 Fourier transform preliminaries

We compile in this section harmonic analysis preliminaries as in [24, 19]. See [24, Section IV] for a more detailed treatment. The notions in this section are used in Sections 2.4 and 5.

Consider the abelian group structure $\mathbb{Z}_2^n = (\mathbb{Z}/2\mathbb{Z})^n$ on the hypercube $\{0,1\}^n$ and the \mathbb{C} -vector space $\mathcal{L}(\mathbb{Z}_2^n) = \{f : \mathbb{Z}_2^n \to \mathbb{C}\}$ endowed with the inner product:

$$\langle f,g \rangle = \mathbb{E}_{U_n} f \overline{g} = \frac{1}{2^n} \sum_x f(x) \overline{g(x)}.$$

The characters of \mathbb{Z}_2^n are $\{\chi_z\}_{z\in\mathbb{Z}_2^n}$, where $\chi_z:\mathbb{Z}_2^n \to \{-1,1\}$ is given by $\chi_z(x) = (-1)^{\langle x,z \rangle}$ and $\langle x,z \rangle = \sum_{i=1}^n x_i z_i$. They form an orthonormal basis of $\mathcal{L}(\mathbb{Z}_2^n)$, i.e., $\langle \chi_z, \chi_{z'} \rangle = \delta_{z,z'}$, for each $z, z' \in \{0,1\}^n$, where δ is the Kronecker delta function.

The Fourier transform of a function $f \in \mathcal{L}(\mathbb{Z}_2^n)$ is the function $\hat{f} \in \mathcal{L}(\mathbb{Z}_2^n)$ given by the coefficients of the unique expansion of f in terms of the characters:

$$f(x) = \sum_{z} \widehat{f}(z)\chi_{z}(x)$$
 and $\widehat{f}(z) = \langle f, \chi_{z} \rangle = \mathbb{E}_{U_{n}}f\chi_{z}.$

2.4 Limited independence, Fourier transform, and bilateral limited independence

In this section, we highlight the limited independence property in the Fourier domain and we define the notion of bilateral limited independence. The notions in this section are used in Section 5.

Let μ be probability distribution on $\{0,1\}^n$. In terms of the characters $\{\chi_z\}_z$ of \mathbb{Z}_2^n , μ being k-wise independent is equivalent to $\mathbb{E}_{\mu}\chi_z = 0$ for each nonzero $z \in \{0,1\}^n$ such that $|z| \leq k$.

We define the notion of bilateral k-wise independence to match the dual bilateral minimum distance in the case of linear codes. Recall that, if μ is uniformly distributed on an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$, i.e., $\mu = \mu_C$, then μ being k-wise independent is equivalent to C having dual minimum distance at least k + 1. We call a probability distribution μ on $\{0,1\}^n$ bilaterally k-wise independent if $\mathbb{E}_{\mu}\chi_z = 0$ for each nonzero $z \in \{0,1\}^n$ such that $|z| \leq k$ or $|z| \geq n - k$. Thus, if μ is uniformly distributed on an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$, then μ being bilaterally k-wise independent is equivalent to C having bilateral dual minimum distance at least k + 1.

2.5 Dual BCH codes

For a general reference on dual BCH codes, see [28]. We compile in this section some of their basic properties used in the introduction. Let $m \ge 2$ be an integer and $n = 2^m - 1$. Consider the finite field \mathbb{F}_{2^m} on 2^m elements and let $\mathbb{F}_{2^m}^{\times}$ be \mathbb{F}_{2^m} excluding zero. Let $s \ge 1$ be an integer such that $2s - 2 < 2^{m/2}$, i.e., $s < \frac{1}{2}\sqrt{n+1} + 1$. Consider the BCH code $BCH(s,m) \subset \mathbb{F}_2^n$:

$$BCH(s,m) = \{(f(a))_{a \in \mathbb{F}_{2^m}^{\times}} : f \in \mathbb{F}_{2^m}[x] \text{ such that } deg(f) < 2^m - 2s - 1\} \cap \mathbb{F}_2^{\mathbb{F}_{2^m}^{\times}}.$$

We have:

- a) $dim(BCH(s,m)^{\perp}) = ms$
- b) The minimum distance of BCH(s,m) is at least 2s + 1
- c) (Weil-Carlitz-Uchiyama Bound) For each non-zero codeword $x \in BCH(s,m)^{\perp}$, we have $||x| 2^{m-1}| \leq (s-1)2^{m/2}$, hence $||x| \frac{n+1}{2}| \leq (s-1)\sqrt{n+1}$.

Let R be the covering radius of dual BCH code $BCH(s,m)^{\perp}$. It follows from (c) that

$$R \ge \frac{n}{2} - (s-1)\sqrt{n+1} - \frac{1}{2}.$$

This holds because with $\vec{1}$ denoting the all-ones vector, we have for each $x \in BCH(s,m)^{\perp}$, $|\vec{1}+x| = n - |x| \ge n - (\frac{n+1}{2} + (s-1)\sqrt{n+1}) = \frac{n}{2} - (s-1)\sqrt{n+1} - \frac{1}{2}$.

3 Proof of Theorem 1.6

The statement of Theorem 1.6 is restated below for convenience.

Theorem 1.6 (Upper bound on the almost-all-covering radius of small codes in terms of dual distance) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d, where $d \ge 7$ be an odd integer. If R > 0, then the fraction of points in Hamming cube not covered by $\mathcal{H}_n(C; R)$ is at most

$$\frac{d}{v_{n+d}(R)} \left(e \log \frac{n+d}{d-1} \right)^{\frac{d-1}{2}} \left(\frac{d-1}{n+d} \right)^{\frac{d-5}{4}}.$$

The proof builds on [24]:

Theorem 3.1 [24, Corollary 3] (Dual bilateral minimum distance versus weight distribution of cosets of small codes; L_1 -bound) Let $Q \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual Q^{\perp} has bilateral minimum distance at least b, where $b \ge 7$ is an odd integer. Then

$$\mathbb{E}_{u \sim U_n} \| \overline{\mu_{Q+u}} - B_n \|_1 \le b \left(e \log \frac{n}{b-1} \right)^{\frac{b-1}{2}} \left(\frac{b-1}{n} \right)^{\frac{b-5}{4}}.$$

See Section 2.1 for weight distribution notations. At a high level, the proof of Theorem 3.1 uses Fourier analysis techniques to establish a mean-square-error identity. Then the argument proceeds by estimating the dual linear program in the context of Delsarte's linear programming framework [4]. The dual estimate is based on Taylor approximation of the exponential function.

Using Markov Inequality², we obtain the following corollary to Theorem 3.1.

Corollary 3.2 (Upper bound on the almost-all-covering radius of small codes in terms of dual bilateral distance) Let $Q \subseteq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual Q^{\perp} has bilateral minimum distance at least b, where $b \geq 7$ is an odd integer. If R > 0, then the fraction p of points in the Hamming cube not covered by $\mathcal{H}_n(Q; R)$ is at most

$$\frac{b}{\upsilon_n(R)} \left(e \log \frac{n}{b-1} \right)^{\frac{b-1}{2}} \left(\frac{b-1}{n} \right)^{\frac{b-5}{4}}.$$

Proof: Choose a uniformly random $u \in \{0,1\}^n$, thus p is the probability that $Q \cap \mathcal{H}_n(u; R) = \emptyset$. Let $f : \{0,1\}^n \to \{0,1\}$ be the indicator function of $\mathcal{H}_n(0; R)$, i.e., f(x) = 1 if $|x| \leq R$ and f(x) = 0 otherwise. If $Q \cap \mathcal{H}_n(u; R) = \emptyset$, i.e., $\mathbb{E}_{Q+u}f = 0$, then $|\mathbb{E}_{Q+u}f - \mathbb{E}_{U_n}f| \geq \mathbb{E}_{U_n}f$. Therefore, by Markov Inequality,

$$p \le \frac{1}{\mathbb{E}_{U_n} f} \mathbb{E}_{u \sim U_n} |\mathbb{E}_{Q+u} f - \mathbb{E}_{U_n} f|.$$

Note that f is a symmetric function in the sense that its value on x depends only on the weight |x| of x. Thus, for any $u \in \{0,1\}^n$, $\mathbb{E}_{Q+u}f = \mathbb{E}_{\overline{\mu}Q+u}\overline{f}$ and $\mathbb{E}_{U_n}f = \mathbb{E}_{B_n}\overline{f}$, where $\overline{f}: [0:n] \to \{0,1\}$ is 1 iff $w \leq R$ and zero otherwise. Therefore,

$$|\mathbb{E}_{Q+u}f - \mathbb{E}_{U_n}f| = |\mathbb{E}_{\overline{\mu}Q+u}\overline{f} - \mathbb{E}_{B_n}\overline{f}| \le ||\overline{\mu}Q+u} - B_n||_1.$$

Noting that $\mathbb{E}_{U_n} f = v_n(R)$, we get

$$p \le \frac{1}{\upsilon_n(R)} \mathbb{E}_{u \sim U_n} \| \overline{\mu_{Q+u}} - B_n \|_1$$

The lemma then follows from Theorem 3.1.

Proof of Theorem 1.6 Let m = n+d and extend C to the \mathbb{F}_2 -linear code $Q = C \times \{0,1\}^d \subset \{0,1\}^m$. Thus $Q^{\perp} = C^{\perp} \times \vec{0}_J$, where $J = \{n+1,\ldots,n+d\}$ and $\vec{0}_J \in \{0,1\}^J$ is the all-zeros vector. By construction, the bilateral minimum distance of Q^{\perp} is at least d since $d \leq |z| \leq n = m - d$ for each nonzero $z \in Q^{\perp}$. Applying Corollary 3.2 to Q, we get

$$U_m(\mathcal{H}_m(Q;R)) \ge 1 - \frac{d}{\upsilon_{n+d}(R)} \left(e \log \frac{n+d}{d-1} \right)^{\frac{d-1}{2}} \left(\frac{d-1}{n+d} \right)^{\frac{d-5}{4}}.$$
 (4)

On the other hand,

$$\mathcal{H}_m(Q;R)|_I \subset \mathcal{H}_n(C;R),\tag{5}$$

where $I = \{1, \ldots, n\}$ and $\mathcal{H}_m(Q; R)|_I$ is the restriction of $\mathcal{H}_m(Q; R)$ to I. To see why (5) holds, note that for any $x \in \mathcal{H}_m(Q; R)$, we have $|x + (y', y'')| \leq R$ for some $y' \in C$ and $y'' \in \{0, 1\}^d$. Thus $|x|_I + y'| \leq R$.

The claim then follows from (5) and (4) via the bounds:

$$U_n(\mathcal{H}_n(C;R)) \ge U_n(\mathcal{H}_m(Q;R)|_I) = U_m(\mathcal{H}_m(Q;R)|_I \times \{0,1\}^J) \ge U_m(\mathcal{H}_m(Q;R))$$

²If X is a random variable taking nonnegative values and a > 0, then the probability that $X \ge a$ is at most $\frac{\mathbb{E}[X]}{a}$.

4 Proofs of Corollary 1.7 and 1.10

The statement of Corollary 1.7 is restated below for convenience. Corollary 4.1 below is a nonasymptotic version of Corollary 1.10.

Corollary 1.7 (Explicit asymptotic version) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d, where $d \ge 7$ is an odd integer such that d = o(n). Let $R = \frac{n}{2} - \Delta$, where

$$\Delta = \sqrt{\frac{1}{13}(d-5)n\log\frac{n}{d-1}}$$

Then, for sufficiently large n, the fraction of points in Hamming cube not covered by $\mathcal{H}_n(C; R)$ is at most $\left(\frac{d-1}{n}\right)^{\frac{d-5}{13}}$.

Proof: Write d = 2t + 5, where $t \ge 1$ is an integer. Let m = n + d, $R' = \frac{m}{2} - \Delta'$, where

$$\Delta' = \sqrt{\frac{(d-5)m}{12}\log\frac{m}{d-1}} - \sqrt{2m} - 2,$$

and

$$p' = \frac{d}{v_m(R')} \left(e \log \frac{m}{d-1} \right)^{t+2} \left(\frac{d-1}{m} \right)^{\frac{t}{2}}.$$

By Theorem 1.6, it is enough to show that that for n large enough,

$$R' \le R \tag{6}$$

and

$$p' \le \left(\frac{d-1}{n}\right)^{t/6.5}.$$
(7)

Note that since d = o(n), we have m = n(1 + o(1)) and d = o(m).

Proof of (7): We have $\Delta' = o(m)$ since d = o(m). To see why this holds, note that $\Delta' \leq \sqrt{\frac{(d-1)m}{12} \log \frac{m}{d-1}} = \frac{m}{\sqrt{12}} \sqrt{\frac{d-1}{m} \log \frac{m}{d-1}}$ and that the function $x \log \frac{1}{x}$ is zero at x = 0. Hence, by Corollary 2.1, for any $\epsilon > 0$ and for m large enough

$$v_m(R') \ge e^{-(2+\epsilon)\frac{(\Delta'+\sqrt{2m}+2)^2}{m}} = \left(\frac{d-1}{m}\right)^{\frac{(2+\epsilon)t}{6}}$$

Therefore

$$p' \le d\left(e\log\frac{m}{d-1}\right)^{t+2} \left(\frac{d-1}{m}\right)^{\frac{(1-\epsilon)t}{6}}$$

Let $a = \frac{m}{d-1}$ and note that a = w(1) since d = o(m). We have $d = 2t + 5 = 2(t+2) + 1 \le 2^{t+2}$, hence $d\left(e \log \frac{m}{d-1}\right)^{t+2} \le (2e \log a)^{t+2} \le (2e \log a)^{3t}$. It follows that

$$p' \le \left(\frac{(2e\log a)^{\frac{18}{1-\epsilon}}}{a}\right)^{\frac{(1-\epsilon)t}{6}} \le \left(\frac{1}{a}\right)^{\frac{t}{6.5}} = \left(\frac{d-1}{n+d}\right)^{\frac{t}{6.5}} \le \left(\frac{d-1}{n}\right)^{\frac{t}{6.5}}$$

where the second inequality holds for ϵ sufficiently small and for a sufficiently large, i.e., for n sufficiently large.

Proof of (6): We have

$$R' = \frac{n}{2} - \sqrt{\frac{(d-5)(n+d)}{12}\log\frac{n+d}{d-1}} + \sqrt{2(n+d)} + \frac{d}{2} + 2.$$

Since d = o(n), we have $\frac{d}{2} + 2 = o\left(\sqrt{(d-5)(n+d)}\right)$ and $\sqrt{2(n+d)} = o\left(\sqrt{(n+d)\log\frac{n+d}{d-5}}\right)$, hence, for *n* large enough,

$$R' \le \frac{n}{2} - \sqrt{\frac{(d-5)(n+d)}{13}\log\frac{n+d}{d-1}} \le \frac{n}{2} - \sqrt{\frac{(d-5)n}{13}\log\frac{n}{d-1}}.$$

Corollary 4.1 (Adding a small code) Let $C \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual C^{\perp} has minimum distance at least d, where $d \ge 7$ is an odd integer such that d = o(n). Then there exists an \mathbb{F}_2 -linear code D of dimension at most $\lceil \log_2 n \rceil$ such that, for sufficiently large n, the covering radius of C + D is at most $R = \frac{n}{2} - \sqrt{\frac{1}{13}(d-5)n\log\frac{n}{d-1}}$.

Proof: Cohen's argument is based on iteratively augmenting the code by adding points in \mathbb{F}_2^n to minimize the number of uncovered points ([20]; see also [1, Section 12.3]). Consider the set of points not *R*-covered by *C*, i.e., $\mathcal{H}_n(C;R)^c$. Choose $x^{(1)} \in \mathbb{F}_2^n$ to minimize the number of points not *R*-covered by $C^{(1)} = C \cup (C + x^{(1)})$. Thus $U_n(\mathcal{H}_n(C^{(1)};R)^c) = U_n(\mathcal{H}_n(C;R)^c \cap (\mathcal{H}_n(C;R) + x^{(1)})^c)$. By [1, Lemma 12.3.1], for each $A \subset \mathbb{F}_2^n$, there exists $x \in \mathbb{F}_2^n$ such that $U_n(A \cap (A + x)) \leq U_n(A)^2$. Thus $U_n(\mathcal{H}_n(C^{(1)};R)^c) \leq U_n(\mathcal{H}_n(C;R)^c)^2$.

By repeating this process *i* steps, we get that there exists an \mathbb{F}_2 -linear code *D* of dimension *i* such that $U_n(\mathcal{H}_n(C+D;R)^c) \leq U_n(\mathcal{H}_n(C;R)^c)^{2^i} < 2^{-2^i}$ assuming that *n* is large enough so that $\left(\frac{d-1}{n}\right)^{\frac{d-5}{13}} < \frac{1}{2}$. Thus, for $i = \lceil \log_2 n \rceil$, $U_n(\mathcal{H}_n(C+D;R)^c)) < 2^{-n}$, i.e., $\mathcal{H}_n(C+D;R) = \{0,1\}^n$.

5 Extension from codes to distributions

In this section, we simplify a part of the proof in [24], which makes it possible to extend the results in [24] and accordingly the results reported in this paper from codes to distributions. Namely, we extend the results in [24] on the weight distribution of cosets of codes with bilateral dual distance at least b to translations of bilaterally k-wise independent probability distributions, where k = b - 1. In particular, we show that if μ is a bilaterally k-wise independent probability distribution of a random translation of $\{0, 1\}^n$, then the average L_1 -distance between the weight distribution of a random translation of μ and the binomial distribution decays quickly in b. The decay is exactly the same as in [24] with b-1 replaced with k. This immediately extends the results reported in this paper on the ε -covering radius from codes with dual distance d to k-wise independent distributions on $\{0, 1\}^n$, where k = d - 1 and the ε -covering radius of a distribution is interpreted as that of its support.

In this section, we use the weight distributions, Fourier transform, and limited independence notations and definitions given in Sections 2.1, 2.3, 2.4, respectively. We also need the following notations for translation and convolution. If μ is a probability distribution on $\{0,1\}^n$ and $u \in \{0,1\}^n$, define $\sigma_u \mu$ to be the *translation* over \mathbb{F}_2 of μ by u, i.e., $(\sigma_u \mu)(x) = \mu(x+u)$. If $f, g: \{0,1\}^n \to \mathbb{C}$, define their convolution $f \star g: \{0,1\}^n \to \mathbb{C}$ with respect to addition in \mathbb{Z}_2^n by $(f \star g)(x) = \sum_y f(y)g(x+y)$. If μ_1, μ_2 are probability distributions on $\{0,1\}^n$, their convolution $\mu_1 \star \mu_2$, sample $a \sim \mu_1, b \sim \mu_2$, and return a + b.

In the proofs of the main results in [24], the only part which relies on the linearity of the code is the following lemma.

Lemma 5.1 [24, Lemma 14] If $0 \le \theta < 2\pi$, define $e_{\theta} : \{0,1\}^n \to \mathbb{C}$ by $e_{\theta}(x) = e^{i\theta|x|}$. Let $Q \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code and $0 \le \theta < 2\pi$. Then

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\mu_{Q+u}} e_{\theta} - \mathbb{E}_{U_n} e_{\theta}|^2 = \mathbb{E}_{y \sim \mu_Q} (\cos \theta)^{|y|} - \left(\frac{\cos \theta + 1}{2}\right)^n.$$

Lemma 5.1 is used in the proof of [24, Theorem 5]:

Theorem 5.2 [24, Theorem 5] (Mean-square-error bound) Let $Q \subsetneq \mathbb{F}_2^n$ be an \mathbb{F}_2 -linear code whose dual Q^{\perp} has bilateral minimum distance at least b = 2t+1, where $t \geq 1$ is an integer. Then, for each $0 \leq \theta < 2\pi$, we have the bounds:

a) (Small dual distance bound)

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\mu_{Q+u}} e_{\theta} - \mathbb{E}_{U_n} e_{\theta}|^2 \le \left(e \log \frac{n}{2t}\right)^{2t} \left(\frac{2t}{n}\right)^{t}$$

b) (Large dual distance bound)

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\mu_{Q+u}} e_\theta - \mathbb{E}_{U_n} e_\theta|^2 \le 2e^{-\frac{t}{5}}.$$

Lemma 5.3 below extends Lemma 5.1 from codes to probability distributions and it has a simpler proof.

Lemma 5.3 Let μ be a probability distribution on $\{0,1\}^n$ and $0 \leq \theta < 2\pi$. Then

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_\theta - \mathbb{E}_{U_n} e_\theta|^2 = \mathbb{E}_{y \sim \mu \star \mu} (\cos \theta)^{|y|} - \mathbb{E}_{y \sim U_n} (\cos \theta)^{|y|}$$

Note that $\mathbb{E}_{y \sim U_n}(\cos \theta)^{|y|} = \left(\frac{\cos \theta + 1}{2}\right)^n$. Moreover, if $\mu = \mu_Q$, where $Q \subset \mathbb{F}_2^n$ is an \mathbb{F}_2 -linear code, then $\mu_Q \star \mu_Q = \mu_Q$.

Proof We have

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_\theta - \mathbb{E}_{U_n} e_\theta|^2 = \mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_\theta|^2 - |\mathbb{E}_{U_n} e_\theta|^2$$

and

$$\begin{split} \mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_{\theta}|^2 &= \mathbb{E}_{u \sim U_n} |\sum_x \mu(x) e_{\theta}(x+u)|^2 \\ &= \mathbb{E}_{u \in \{0,1\}^n} \sum_{x,y} \mu(x) \mu(y) e_{\theta}(u+x) \overline{e_{\theta}(u+y)} \\ &= \sum_{x,y} \mu(x) \mu(y) \mathbb{E}_{u \in \{0,1\}^n} e_{\theta}(u) \overline{e_{\theta}(u+x+y)} \\ &= \mathbb{E}_{\mu \star \mu} e_{\theta} \circledast \overline{e_{\theta}}, \end{split}$$

where \circledast is the weighted convolution operator: if $f, g: \{0, 1\}^n \to \mathbb{C}$, their weighted convolution $f \circledast g = \frac{1}{2^n} f \star g$, i.e., $(f \circledast g)(x) = \mathbb{E}_y f(y) g(x+y)$, hence $\widehat{f \circledast g} = \widehat{f}\widehat{g}$. Then the Lemma follows from the fact that

$$(e_{\theta} \circledast \overline{e_{\theta}})(x) = (\cos \theta)^{|x|}. \tag{8}$$

Note that $\mathbb{E}_{U_n} e_{\theta} \circledast \overline{e_{\theta}} = |\mathbb{E}_{U_n} e_{\theta}|^2$. Thus, by (8), $|\mathbb{E}_{U_n} e_{\theta}|^2 = \mathbb{E}_{y \sim U_n} (\cos \theta)^{|y|}$. To verify (8), we go to the Fourier domain. In the Fourier domain, (8) is equivalent to

 $\widehat{e_{\theta} \circledast \overline{e_{\theta}}} = \widehat{g_{\cos\theta}}$, where $g_r(x) = r^{|x|}$. Since $\widehat{f \circledast g} = \widehat{fg}$, we have to show that $|\widehat{e_{\theta}}|^2 = \widehat{g_{\cos\theta}}$. We need the following basic lemma about the Fourier transform of exponential function, e.g., [24, Lemma 11]:

Lemma 5.4 Let r be complex number and $g_r: \{0,1\}^n \to \mathbb{C}$ be given by $g_r(x) = r^{|x|}$. Then $\widehat{g_r}(z) = \left(\frac{1+r}{2}\right)^n \left(\frac{1-r}{1+r}\right)^{|z|}. \text{ Moreover, if } r = e^{i\theta}, \text{ then } \widehat{g_r}(z) = e^{in\theta/2} \left(\cos\frac{\theta}{2}\right)^n \left(-i\tan\frac{\theta}{2}\right)^{|z|}.$

Therefore

$$|\widehat{e_{\theta}}(z)|^{2} = \left(\cos\frac{\theta}{2}\right)^{2n} \left(\tan\frac{\theta}{2}\right)^{2|z|} = \left(\frac{1+\cos\theta}{2}\right)^{n} \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{|z|} = g_{\cos\theta}(z),$$

where the second equality follows from the trigonometric identities $\left(\cos\frac{\theta}{2}\right)^2 = \frac{1+\cos\theta}{2}$ and $(\tan \frac{\theta}{2})^2 = \frac{1-\cos \theta}{1+\cos \theta}$. As explicitly noted in [24, Section V p. 6499], the proof of Theorem 5.2 bounds the term

 $\mathbb{E}_{y \in \mu_Q}(\cos \theta)^{|y|} - \left(\frac{\cos \theta + 1}{2}\right)^h$ by ignoring the linearity of Q and using only the bilateral k-wise independence property of μ_Q , where k = b - 1. This property holds also for $\mu \star \mu$; if μ is bilaterally k-wise independent, then so is $\mu \star \mu$. This follows from the definition of bilateral k-wise independence since $\mathbb{E}_{\mu \star \mu} \chi_z = (\mathbb{E}_{\mu} \chi_z)^2$, for each $z \in \{0, 1\}^n$.

Accordingly, using Lemma 5.3, Theorem 5.2 as well as [24, Theorem 2] (L_{∞} -bound) and [24, Corollary 3] (L₁-bound, i.e., Theorem 3.1 in this paper) extend as follows from codes with bilateral minimum distance at least b to bilaterally k-wise independent probability distributions, where k = b - 1.

Theorem 5.5 (Bilateral limited independence versus weight distribution of translates; mean-square-error bound) Let μ be a bilaterally k-wise independent probability distribution on $\{0,1\}^n$, where $k \ge 2$ is an even integer. Then, for each $0 \le \theta \le 2\pi$, we have the bounds:

a) (Small k bound)

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_\theta - \mathbb{E}_{U_n} e_\theta|^2 \le \left(e \log \frac{n}{k}\right)^k \left(\frac{k}{n}\right)^{\frac{n}{2}}$$

b) (Large k bound)

$$\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e_\theta - \mathbb{E}_{U_n} e_\theta|^2 \le 2e^{-\frac{k}{10}}.$$

Theorem 5.6 (Bilateral limited independence versus weight distribution of trans**lates;** L_{∞} -bound) Let μ be a bilaterally k-wise independent probability distribution on $\{0,1\}^n$, where $k \geq 2$ is an even integer. Then, we have the bounds:

a) (Small k bound)

$$\mathbb{E}_{u \sim U_n} \|\overline{\sigma_u \mu} - B_n\|_{\infty} \le \left(e \log \frac{n}{k}\right)^{\frac{k}{2}} \left(\frac{k}{n}\right)^{\frac{k}{4}}$$

b) (Large k bound)

$$\mathbb{E}_{u \sim U_n} \|\overline{\sigma_u \mu} - B_n\|_{\infty} \le \sqrt{2}e^{-\frac{k}{20}}$$

Theorem 5.7 (Bilateral limited independence versus weight distribution of trans**lates;** L_1 -bound) Let μ be a bilaterally k-wise independent probability distribution on $\{0,1\}^n$, where $k \ge 6$ is an even integer. Then, we have the bounds:

a) (Small k bound)

$$\mathbb{E}_{u \sim U_n} \|\overline{\sigma_u \mu} - B_n\|_1 \le (k+1) \left(e \log \frac{n}{k}\right)^{\frac{k}{2}} \left(\frac{k}{n}\right)^{\frac{k}{4}-1}$$

b) (Large k bound)

$$\mathbb{E}_{u \sim U_n} \|\overline{\sigma_u \mu} - B_n\|_1 \le \sqrt{2}(n+1)e^{-\frac{k}{20}}$$

As in [24], Theorem 5.7 follows from Theorem 5.6, which in turns follows from Theorem 5.5.

Accordingly, Theorem 1.6 (Nonasymptotic bound) and Corollaries 1.7 (Explicit asymptotic version) and 4.1 (Adding a small code) extend as follows from codes with minimum distance at least d to k-wise independent probability distributions, where k = d - 1.

Definition 5.8 (ε -covering radius of a probability distribution) Let μ be a probability distribution on $\{0,1\}^n$. The covering radius of μ is the covering radius of its support. Equivalently, the covering radius of μ is the minimum r such that $\mu(\mathcal{H}_n(x;r)) \neq 0$ for each $x \in \{0,1\}^n$.

More generally, if $0 \le \varepsilon \le 1$, the ε -covering radius of μ is the ε -covering radius of its support. Equivalently, the ε -covering radius of μ is the minimum r such that the fraction of points $x \in \{0,1\}^n$ such that $\mu(\mathcal{H}_n(x;r)) = 0$ is at most ε .

Theorem 5.9 (Limited independence versus almost-all-covering radius) Let μ be a kwise independent probability distribution on $\{0,1\}^n$, where $k \ge 6$ is an even integer. If R > 0, let

$$\varepsilon = \frac{d}{v_{n+k+1}(R)} \left(e \log \frac{n+k+1}{k} \right)^{\frac{k}{2}} \left(\frac{k}{n+k+1} \right)^{\frac{k}{4}-1}$$

and assume that $\varepsilon \leq 1$. Then the ε -covering radius of μ is most R.

To adapt the proof of Theorem 1.6 into a the setup of Theorem 5.9, given a k-wise independent probability distribution μ on $\{0, 1\}^n$, consider the probability distribution $\gamma = \mu \times U_d$ on $\{0, 1\}^m$, where d = k + 1 and m = n + d. Then γ is bilaterally k-wise independent. The reason is that if $z \in \{0, 1\}^m$ is such that |z| > m - d = n, then with $I = \{1, \ldots, n\}$ and $J = \{n + 1, \ldots, n + d\}$, we have $z|_J \neq 0$, hence $\mathbb{E}_{U_d} \chi_{z|_J} = 0$, and accordingly $\mathbb{E}_{\gamma} \chi_z = 0$ since $\mathbb{E}_{\gamma} \chi_z = \mathbb{E}_{\mu} \chi_{z|_J} \mathbb{E}_{U_d} \chi_{z|_J}$.

Corollary 5.10 (Explicit asymptotic version) Let μ be a k-wise independent probability distribution on $\{0,1\}^n$, where $k \ge 6$ is an even integer such that k = o(n). Then, for sufficiently large n, the $\left(\frac{k}{n}\right)^{\frac{k-4}{13}}$ -covering radius of μ is at most $R = \frac{n}{2} - \Delta$, where

$$\Delta = \sqrt{\frac{1}{13}(k-4)n\log\frac{n}{k}}.$$

Corollary 5.11 (Convolution with a small code) Let μ be a k-wise independent probability distribution on $\{0,1\}^n$, where $k \ge 6$ is an even integer such that k = o(n). Then there exists an \mathbb{F}_2 -linear code D of dimension at most $\lceil \log_2 n \rceil$ such that, for sufficiently large n, the covering radius of $\mu \star \mu_D$ is at most $R = \frac{n}{2} - \sqrt{\frac{1}{13}(k-4)n\log\frac{n}{k}}$.

6 Open problems

We conclude with the following open questions:

- As noted in the introduction, after Corollary 1.7, the upper bound of Corollary 1.7 on R n/2, where R is the almost-all covering radius, is asymptotically tight up to a factor of √13/2 in comparison to random linear codes (see Lemma 1.8). The proofs of Theorem 1.6 and Corollary 1.7 can be easily tuned to bring the √13/2 factor down to 2 + ε, for any ε > 0. The gain is at the cost of increasing the fraction of uncovered points while keeping it o(1). Is it possible to go below 2?
- Corollary 1.7 assumes that the dual distance d is at least 7. Is it possible to extend it to smaller values of d?

• Consider the block-length-*n* dual BCH code $C = BCH(s, m)^{\perp}$, where $m \ge 2$ is an integer, $n = 2^m - 1$, and *s* is an integer such that $2s - 2 < 2^{m/2}$. If $s \ge 3$, we know from Corollary 4.1 that there exists a small \mathbb{F}_2 -linear code *D* of dimension at most $\lceil \log_2 n \rceil = m$ such that, for sufficiently large *n*, the covering radius of C + D is at most $\frac{n}{2} - \sqrt{\frac{1}{13}(2s - 4)n\log\frac{n}{2s}} = \frac{n}{2} - \Theta(\sqrt{sn\log n})$. It would be interesting to explicitly construct such a code *D* using algebraic tools.

Appendix

A Proof of Corollary 1.4

The corollary is restated below for convenience.

Corollary 1.4 For each $\epsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $n \ge 1$ be an integer and s > 1 be such that $s \log_2 n \le \delta n$. Then for n large enough, there exists an \mathbb{F}_2 -linear code $C \subset \mathbb{F}_2^n$ of dimension at most $\lceil s \log_2 n \rceil$ and covering radius

$$R \le \frac{n}{2} - \sqrt{\frac{(s-1)n\log n}{2+\epsilon}} + \sqrt{2n} + 2.$$

Let $\Delta = \sqrt{\frac{(s-1)n\log n}{2+\epsilon}} - \sqrt{2n} - 2$. By Theorem 1.3, it is enough to show that $\log_2 \frac{n(\log 2)}{\upsilon_n(\frac{n}{2}-\Delta)} \leq s\log_2 n$, i.e., $\upsilon_n(\frac{n}{2}-\Delta) \geq \frac{\log 2}{n^{s-1}}$. Since $s\log_2 n \leq \delta n$, we have $\Delta \leq \sqrt{\frac{\delta\log 2}{2+\epsilon}}n$. Applying Lemma 2.1, we get that for sufficiently small δ and sufficiently large n,

$$v_n(n/2 - \Delta) \ge e^{-(2+\epsilon)\frac{(\Delta + \sqrt{2n} + 2)^2}{n}} = \frac{1}{n^{s-1}} > \frac{\log 2}{n^{s-1}}$$

B Proof of Lemma 1.8

The lemma is restated below for convenience.

Lemma 1.8 Consider any $0 \le \varepsilon < 1$ and let $n \ge 1$ be an integer and $n \le K \le 2^{n-1}$ be an integer power of 2. Then, all but at most $\frac{1}{n}$ fraction of \mathbb{F}_2 -linear codes $C \subset \mathbb{F}_2^n$ of size K have ε -covering radius

$$R \ge \frac{n}{2} - \sqrt{\frac{dn}{2}\log\frac{en}{d} + n\log\frac{n+1}{1-\varepsilon}},$$

where d is the minimum distance of C^{\perp} .

We need the following simple variation of the sphere-covering bound:

Lemma B.1 (Sphere-covering bound adaptation to almost-all covers) Let $0 \le \varepsilon < 1$ and $n \ge 1$. Then for any code $C \subset \{0,1\}^n$ of size K, where $K \ge 1$, the ε -covering radius of C is at least

$$R \ge \frac{n}{2} - \sqrt{\frac{1}{2}n\log\frac{K}{1-\varepsilon}}.$$

The proof of Lemma B.1 follows from exactly the same counting argument used to establish the sphere-covering bound [1, Theorem 12.5.1].

The upper bound on K in terms of d comes from Gilbert-Varshamov bound. Choose the generator matrix $G_{k^{\perp} \times n}$ of the dual code C^{\perp} uniformly at random, where $k^{\perp} = n - \log_2 K$. Let

d be the minimum distance of C^{\perp} and let $1 \leq d_0 \leq \frac{n}{2} + 1$ be an integer. The probability that $d < d_0$ is at most

$$(|C^{\perp}| - 1)2^{-n} |\mathcal{H}_n(0; d_0 - 1)| \le \frac{d_0}{K} \binom{n}{d_0 - 1} \le \frac{d_0}{K} \left(\frac{en}{d_0 - 1}\right)^{d_0 - 1} \le \frac{1}{n}$$

if $K \ge f(d_0)$, where $f(x) = nx \left(\frac{en}{x-1}\right)^{x-1}$. Since f(x) is increasing in x for all $1 \le x \le n+1$, we conclude that, with probability at least $1 - \frac{1}{n}$, $d \ge \lfloor f^{-1}(K) \rfloor$ if $1 \le \lfloor f^{-1}(K) \rfloor \le \frac{n}{2} + 1$, hence $K \le f(d+1)$ if $f(1) \le K \le f(\frac{n}{2}+1)$. We have $f(1) = n \le K$ and $f(\frac{n}{2}+1) > 2^n$ for all $n \ge 1$. Therefore,

$$R \ge \frac{n}{2} - \sqrt{\frac{1}{2}n\log\left(\frac{n(d+1)}{1-\varepsilon}\left(\frac{en}{d}\right)^d\right)} \ge \frac{n}{2} - \sqrt{\frac{dn}{2}\log\frac{en}{d} + n\log\frac{n+1}{1-\varepsilon}}.$$

Acknowledgment

The author would like to thank the anonymous reviewers for their useful and constructive comments.

References

- [1] G. Cohen, I. Honkala, S. Litsyn, A. Lobstein, Covering Codes, Elsevier, 1997
- [2] A. Tietäväinen: An upper bound on the covering radius as a function of the dual distance, IEEE Trans. Inform. Th., vol. 36, pp. 1472-1474, 1990.
- [3] A. Tietäväinen: Covering radius and dual distance, Designs, Codes and Cryptography, vol. 1, pp. 31-46, 1991
- [4] P. Delsarte: Four fundamental parameters of a code and their combinatorial significance, Information and Control, vol. 23, pp. 407-438, 1973.
- [5] T. Helleseth, T. Kløve, and J. Mykkeltveit: On the covering radius of binary codes, IEEE Trans. Inform. Th., vol. 24, pp. 627-628, 1978
- [6] P. Sole: Asymptotic bounds on the covering radius of binary codes, IEEE Trans. Inform. Th., vol. 36, pp. 1470-1472, 1990.
- [7] P. Sole and P. Stokes: Covering radius, codimension, and dual-distance width, IEEE Trans. Inform. Th., vol. 39, pp. 1195-1203, 1993.
- [8] I. S. Honkala, S. Litsyn, and A. Tietäväinen: On algebraic methods in covering radius problems, Lecture Notes in Computer Science, No. 948, pp. 21-32, Springer-Verlag, 1995
- [9] S. Litsyn and A. Tietäväinen: Upper bounds on the covering radius of a code with a given dual distance, European J. Combinatorics, vol. 17, pp. 265-270, 1996
- [10] S. Litsyn, P. Sole and R. Struik: On the covering radius of an unrestricted code as a function of the rate and dual distance, Discrete Applied Mathematics, vol. 82, pp. 177-192, 1998.
- [11] I. S. Honkala, T. Laihonen and S. Litsyn: On covering radius and discrete Chebyshev polynomials, Applicable Algebra in Engineering, Communication and Computing, vol. 8, pp. 395-401, 1997.
- [12] T. Laihonen: Estimates on the covering radius when the dual distance is known, Ph. D. Thesis, University of Turku, Finland, 1998.

- [13] T. Laihonen, Covering radius of self-complementary codes and BCH codes, Proc. IEEE Int'l Sympos. Inform. Theory (ISIT 1998), Cambridge, MA.
- [14] T. Laihonen and S. Litsyn: New bounds on covering radius as a function of dual distance, SIAM J. Discrete Mathematics, vol. 12, pp. 243-251, 1999.
- [15] A. A. Ashikhmin, I. S. Honkala, T. Laihonen and S. Litsyn: On relations between covering radius and dual distance, IEEE Trans. Inform. Th., vol. 45, pp. 1808-1816, 1999.
- [16] T. Laihonen: On an algebraic method for bounding the covering radius, Proc. DIMACS Workshop on Codes and Association Schemes, vol. 56, pp. 213-221, 2001.
- [17] A. Ashikhmin and A. Barg: Bounds on the covering radius of linear codes, Designs, Codes and Cryptography, vol. 27, pp. 261-269, 2002.
- [18] G. Fazekas and V. I. Levenshtein: On upper bounds for code distance and covering radius of designs in polynomial metric spaces, J. Combinatorial Th., Ser. A, vol. 70, pp. 267-288, 1995.
- [19] L. Bazzi: On the tightness of Tietäväinen's bound for distributions with limited independence, arXiv:1707.00552.
- [20] G. D. Cohen: A nonconstructive upper bound on covering radius, IEEE Trans. Inform. Th., vol. 29, pp. 352-353, 1983.
- [21] V. M. Blinovskii: Lower asymptotic bound on the number of linear code words in a sphere of given radius in F_q^n , Problemy Peredachi Informatsii, vol. 23, No. 2, pp. 50-53, 1987. Translated in: Problems of Inform. Transm., vol. 23, No. 2, pp. 130-132.
- [22] V. M. Blinovskii: Asymptotically exact uniform bounds for spectra of cosets of linear codes, Problemy Peredachi Informatsii, vol. 26, No. 1, pp. 99-103, 1990. Translated in: Problems of Inform. Transm., vol. 26, No. 1, pp. 83-86.
- [23] Y. Rabani and A. Shpilka: Explicit Construction of a Small ϵ -Net for Linear Threshold Functions, SIAM Journal on Computing, Volume 39 Issue 8, pp. 3501-3520, 2010.
- [24] L. Bazzi: Weight distribution of cosets of small codes with good dual properties, IEEE Transactions on Information Theory. vol. 61, pp. 6493 - 6504, 2015. Corrections submitted, 2017, available at arXiv:1408.5681.
- [25] M. Luby: A simple parallel algorithm for the maximal independent set problem, In Proc. 17th Annual ACM Symposium on the Theory of Computing, pp. 1-10, 1985.
- [26] U. Vazirani: Randomness, adversaries, and computation. Ph.D. dissertation, University of California, Berkeley, 1986.
- [27] M. Navon, A. Samorodnitsky: Linear programming bounds for codes via a covering argument, Discrete and Computational Geometry, vol. 41, no. 2, 2009.
- [28] F. MacWilliams and N. Sloane: The Theory of Error-Correcting Codes, New York: North-Holland, 1977.