An Innovations Approach to Viterbi Decoding of Convolutional Codes

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Abstract-We introduce the notion of innovations for Viterbi decoding of convolutional codes. First we define a kind of innovation corresponding to the received data, i.e., the input to a Viterbi decoder. Then the structure of a Scarce-State-Transition (SST) Viterbi decoder is derived in a natural manner. It is shown that the newly defined innovation is just the input to the main decoder in an SST Viterbi decoder and generates the same syndrome as the original received data does. A similar result holds for Quick-Look-In (QLI) codes as well. In this case, however, the precise innovation is not defined. We see that this innovationlike quantity is related to the linear smoothed estimate of the information. The essence of innovations approach to a linear filtering problem is first to whiten the observed data, and then to treat the resulting simpler white-noise observations problem. In our case, this corresponds to the reduction of decoding complexity in the main decoder in an SST Viterbi decoder. We show the distributions related to the main decoder (i.e., the input distribution and the state distribution in the code trellis for the main decoder) are much biased under moderately noisy conditions. We see that these biased distributions actually lead to the complexity reduction in the main decoder. Furthermore, it is shown that the proposed innovations approach can be extended to maximum-likelihood (ML) decoding of block codes as well.

Index Terms—Convolutional codes, Viterbi decoding, innovations, linear filtering, linear smoothing, Scarce-State-Transition (SST) Viterbi decoder.



I. INTRODUCTION

Fig. 1. The structure of an SST Viterbi decoder (pre-decoder: G^{-1}).

I N 1985, Kubota, Kohri, and Kato [17] proposed a Viterbi decoding scheme named Scarce-State-Transition (SST) for the purpose of decoding of Quick-Look-In (QLI) codes [23]. They also extended the scheme to general codes. The corresponding Viterbi decoder consists of a pre-decoder and a main decoder (i.e., a conventional Viterbi decoder). The structure

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of an SST Viterbi decoder is shown in Fig.1 [38], where the inverse encoder is used as a pre-decoder. At the first stage, the transmitted information is estimated using a rather simple decoder (i.e., a pre-decoder) such as the inverse encoder, and then at the second stage, the estimation error at the first stage is decoded using a main decoder. Finally, two decoder outputs are combined to produce the final decoder output. The SST scheme was devised mainly for the purpose of hardware and power-consumption reduction in Viterbi decoder VLSI implementation. More precisely [18], [19],

- A likelihood concentration to the all-zero state¹ occurs in the main decoder.
- 2) In the main decoder, a maximum-likelihood decision circuit, which is used to determine the most likely survivor from among all survivors at each depth, is omitted within a very small performance degradation.
- On-off switching rarely occurs in the path-memory circuit in the main decoder when a decoder LSI is implemented using the CMOS technology.

Since the estimation "error" is decoded in the main decoder, it is natural to think that the SST scheme is closely related to syndrome decoding [2], [3], [4], [28], [29], [30] based on an error trellis. Later [37], [38], we showed that SST Viterbi decoding based on a code trellis and syndrome decoding based on the corresponding error trellis are equivalent under a general condition.

On the other hand, in connection with stochastic processes, the problem of extracting the innovations [1], [10], [11], [16], [20], [41] from a given (complex) process has been discussed for a long time (see [10], [11]). Let X(t) be a stochastic process. Suppose that during an infinitesimal interval [t, t+dt), X(t) obtains new information which is independent of the information obtained by X(t) prior to time t. The newly obtained information is called the "innovation" associated with X(t). Kailath [14] applied the notion of innovations to a linear filtering problem [5], [12], [14], [20], [27], [40]. Also, Kailath and Frost [15] extended the idea to a linear smoothing problem [12], [15], [27]. In the linear filtering theory, the innovation associated with an observation is defined by the difference between the observation and the estimate of a signal, or equivalently, the sum of the estimation error and a noise [14], [15]. Hence, we thought the notion of innovations has some connection with SST Viterbi decoding in the coding theory.

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¹The state in the code trellis for the main decoder consists of errors and is regarded as a discrete random variable. We call its distribution simply a *state distribution*. Then a *likelihood concentration* means that the state distribution is not uniform but *biased*.

In this paper, by comparing with the results in the linear filtering theory, we define a kind of innovation corresponding to the received data for a Viterbi decoder. Then the structure of an SST Viterbi decoder is derived in a natural manner. We see that the newly defined innovation is just the input to the main decoder in an SST Viterbi decoder. A similar result is obtained in connection with QLI codes as well. In the latter case, however, the precise innovation is not defined. It is shown that the obtained innovation-like quantity is related to linear smoothing of the information. Moreover, for a QLI code, we examine the relationship between the two estimates of the information, i.e., the linear filtered estimate and the linear smoothed estimate. Then it is shown that the latter has higher accuracy as compared with the former. These are discussed in Section II.

Now the main purpose of introducing the innovations in a filtering problem is to whiten the observed data [14]. As a result, the given problem is transformed to a simpler whitenoise observations problem. We thought this corresponds to the reduction of decoding complexity in the main decoder in an SST Viterbi decoder. The reduction of hardware and power-consumption of an LSI is also considered as a related simplification. Then we thought all of these reductions are caused by biased distributions related to the main decoder. Hence, in Section III, we focus our arguments mainly on these distributions. We see that the distribution of the input to the main decoder is biased under moderately noisy conditions. The state distribution in the code trellis for the main decoder is also biased under the same channel conditions. Moreover, we observe that the state distribution in the error trellis is equally biased.

Subsequently, in Section IV, we show those biased distributions actually lead to the reduction of decoding complexity in the main decoder. Since there have been several related works [2], [4], [25], [33], [35], [36], the discussion is mainly based on these known works. We remark that syndrome decoding based on an error trellis has less complexity as compared with Viterbi decoding based on a code trellis [2], [4]. Since the SST scheme is equivalent to syndrome decoding based on the error trellis, this is quite reasonable. In connection with the subject, we derive an approximate criterion for complexity reduction in the main decoder.

The fundamental feature of the SST scheme lies in its structure where an *estimation error* is decoded in the main decoder. Then we see that a similar scheme (i.e., two-stage decoding) can be applied to block codes as well. In Section V, it is shown that a kind of innovation can also be extracted in connection with maximum-likelihood (ML) decoding of block codes [22].

Let us close this section by introducing the basic notions needed for this paper. We always assume that the underlying field is GF(2). Let G(D) be a generator matrix for an (n_0, k_0) convolutional code, where G(D) is assumed to be *canonical* [13], [24] (i.e., *minimal* [6]). A corresponding check matrix H(D) is also assumed to be canonical. Hence, they have the same constraint length, denoted ν . Denote by $i = \{i_k\}$ and $y = \{y_k\}$ an information sequence and the corresponding code sequence, respectively, where $i_k = (i_k^{(1)}, \dots, i_k^{(k_0)})$ is the information block at t = k and $y_k = (y_k^{(1)}, \dots, y_k^{(n_0)})$ is the encoded block at t = k. In this paper, it is assumed that a code sequence y is transmitted symbol by symbol over a memoryless AWGN channel using BPSK modulation [9]. Let $z = \{z_k\}$ be a received sequence, where $z_k = (z_k^{(1)}, \dots, z_k^{(n_0)})$ is the received block at t = k. Each component z_j of z is modeled as

$$z_j = x_j \sqrt{2E_s/N_0} + w_j.$$
 (1)

Here, x_j takes ± 1 depending on whether the code symbol y_j is 0 or 1. E_s and N_0 denote the energy per channel symbol and the single-sided noise spectral density, respectively. (Let E_b be the energy per information bit. Then the relationship between E_b and E_s is defined by $E_s = RE_b$, where R is the code rate.) Also, w_j is a zero-mean unit variance Gaussian random variable with probability density function

$$q(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$
 (2)

Each w_j is independent of all others. Let $p(z_j|y_j)$ be the conditional probability density function of z_j given y_j . The hard-decision (denoted "*h*") data of z_j is defined by

$$z_{j}^{h} \stackrel{\triangle}{=} \begin{cases} 0, & L(z_{j}|y_{j}) \ge 0\\ 1, & L(z_{j}|y_{j}) < 0, \end{cases}$$
(3)

where

$$L(z_j|y_j) \stackrel{\triangle}{=} \log \frac{p(z_j|y_j=0)}{p(z_j|y_j=1)} \tag{4}$$

is the log-likelihood ratio conditioned on y_j ("log" denotes the natural logarithm). In our case, this is equivalent to

$$z_j^h \stackrel{\triangle}{=} \begin{cases} 0, & z_j \ge 0\\ 1, & z_j < 0. \end{cases}$$
(5)

Note that in Fig.1, the main decoder input $r_k^{(l)}$ $(1 \le l \le n_0)$ is given by

$$r_k^{(l)} = \begin{cases} |z_k^{(l)}|, & r_k^{(l)h} = 0\\ -|z_k^{(l)}|, & r_k^{(l)h} = 1. \end{cases}$$
(6)

Let $v_k = (v_k^1, \dots, v_k^n)$ be an *n*-tuple of variables. Also, let $p(D) = (p_1(D), \dots, p_n(D))$ be an *n*-tuple of polynomials in *D*. Since each $p_i(D)$ is a delay operator with respect to $k, \sum_{i=1}^n p_i(D)v_k^i$ is well defined, where $D^m v_k^i = v_{k-m}^i$. In this paper, noting that v_k is a row vector, we express the above variable as $v_k p^T(D)$ ("T" means transpose). Using this notation, we have

$$\boldsymbol{y}_k = \boldsymbol{i}_k G(D). \tag{7}$$

Also, the syndrome at t = k is defined by

$$\boldsymbol{\zeta}_k = \boldsymbol{z}_k^h H^T(D). \tag{8}$$

Note that $\zeta_k = e_k H^T(D)$ holds, where $e_k = (e_k^{(1)}, \cdots, e_k^{(n_0)})$ is the error at t = k.

II. AN INNOVATIONS APPROACH TO VITERBI DECODING OF CONVOLUTIONAL CODES

As stated in the preceding section, it seems that the notion of innovations introduced for linear filtering/smoothing problems has some connection with SST Viterbi decoding of convolutional codes. In the following, based on this conjecture, we investigate Viterbi decoding of convolutional code from an innovation viewpoint.

A. Innovations Associated with the Received Data for a Viterbi Decoder

First consider a linear filtering problem [5], [12], [14], [20], [27], [40]. Let

$$y(t) = C(t)x(t) + w(t)$$
 (9)

be the observation corresponding to a signal x(t), where C(t)is a coefficient matrix and w(t) is a white Gaussian noise. In this case, the innovation $\nu(t)$ [14] associated with y(t) is defined as

$$\nu(t) = y(t) - C(t)\hat{x}(t|t),$$
(10)

where $\hat{x}(t|t)$ is a linear function of all the data $\{y(s), s < t\}$ t} that minimizes the mean-square error E[(x(t) - t)] $\hat{x}(t|t))^T(x(t) - \hat{x}(t|t))]$ (" $E[\cdot]$ " is the expectation) [14].

Next, consider convolutional encoding based on G(D). Let

$$\boldsymbol{z}_k^h = \boldsymbol{i}_k \boldsymbol{G}(\boldsymbol{D}) + \boldsymbol{e}_k \tag{11}$$

be the received data, where i_k and e_k are an information block and an error, respectively. By comparison with the linear filtering theory, it is reasonable to think that

$$\boldsymbol{r}_{k}^{h} = \boldsymbol{z}_{k}^{h} - \hat{\boldsymbol{i}}(k|k)G(D)$$

$$= \boldsymbol{z}_{k}^{h} + \hat{\boldsymbol{i}}(k|k)G(D)$$
(12)

corresponds to $\nu(t)$, where $\hat{i}(k|k)$ denotes an estimate of i_k based on $\{z_s^h, s \leq k\}$. Suppose that $\hat{i}(k|k)$ is a linear combination of the received data $\{z_s^h, s \leq k\}$ and has the form

$$\hat{\boldsymbol{i}}(k|k) = \boldsymbol{z}_k^h P(D), \tag{13}$$

where P(D) is a polynomial matrix. Then we have

$$\begin{aligned} \boldsymbol{r}_k^h &= \boldsymbol{z}_k^h + \boldsymbol{z}_k^h P(D) G(D) \\ &= (\boldsymbol{i}_k G(D) + \boldsymbol{e}_k) + (\boldsymbol{i}_k G(D) + \boldsymbol{e}_k) P(D) G(D) \\ &= \boldsymbol{i}_k (I_{k_0} + G(D) P(D)) G(D) + \boldsymbol{e}_k P(D) G(D) + \boldsymbol{e}_k \end{aligned}$$

where I_{k_0} is the identity matrix of size $k_0 \times k_0$. Note that if

$$(I_{k_0} + G(D)P(D))G(D) = G(D) + G(D)P(D)G(D) = 0$$

or
$$G(D)P(D)G(D) = G(D)$$
(14)

holds, then
$$r_k^h$$
 is independent of i_k . Here $G(D)P(D)G(D) = G(D)$ implies that $P(D)$ is a generalized inverse [26] of $G(D)$. Then a right inverse $G^{-1}(D)$ of $G(D)$ can be taken as $P(D)$. In this case, r_k^h is independent of i_k and we have

$$\boldsymbol{r}_k^h = (\boldsymbol{e}_k G^{-1}) G + \boldsymbol{e}_k \tag{15}$$

$$= \boldsymbol{u}_k \boldsymbol{G} + \boldsymbol{e}_k \tag{16}$$

$$= e_k (G^{-1}G + I_{n_0}), (17)$$

where $u_k \stackrel{\triangle}{=} e_k G^{-1}$. We think this quantity corresponds to an innovation in the linear filtering theory. We remark that the right-hand side is just the input to the main decoder in an SST Viterbi decoder, where the inverse encoder G^{-1} is used as a pre-decoder (see Fig.1). Also, note that

$$\boldsymbol{r}_{k}^{h}\boldsymbol{H}^{T}(D) = \boldsymbol{z}_{k}^{h}\boldsymbol{H}^{T}(D) + \boldsymbol{z}_{k}^{h}\boldsymbol{P}(D)\boldsymbol{G}(D)\boldsymbol{H}^{T}(D)$$

$$= \boldsymbol{z}_{k}^{h}\boldsymbol{H}^{T}(D) = \boldsymbol{\zeta}_{k}$$
(18)

holds irrespective of P(D). Hence, r_k^h and z_k^h generate the same syndrome ζ_k .

On the other hand, r_k^h has another expression. Let

$$G = A \times \Gamma \times B \tag{19}$$

be an invariant-factor decomposition [6] of G(D). Since G(D)is canonical (accordingly, basic), we can assume [6] that the first k_0 rows of B coincide with G(D) and the last $(n_0 - k_0)$ columns of B^{-1} coincide with the syndrome former $H^T(D)$. As a result, we have

Then

is obtained. Thus we have again

$$\boldsymbol{r}_k^h \boldsymbol{H}^T = \boldsymbol{\zeta}_k (\boldsymbol{H}^{-1})^T \boldsymbol{H}^T = \boldsymbol{\zeta}_k.$$

Therefore, r_k^h has the following properties:

- 1) $r_k^h = e_k(G^{-1}G + I_{n_0})$ holds. Hence, r_k^h consists of errors $\{e_s, s \leq k\}$. There is a correspondence between e_k and r_k^h in the sense that they generate the same syndrome ζ_k . 2) $\{r_s^h, s \leq k\}$ and $\{z_s^h, s \leq k\}$ generate the same
- syndrome sequence $\{\boldsymbol{\zeta}_s, s \leq k\}$.

Property 1) corresponds to the fact that an innovation process is a white-noise process in the linear filtering theory. Property 2) is the most important one and corresponds to the fact that the original received data and the associated innovations have the same information. In the case of error correction, if two quantities generate the same syndrome sequence, then we can conclude that they have the equal information. Here we remark that $\{r_k^h\}$ does not have the same properties as those of innovations in the linear filtering theory. Hence, we may call $\{r_k^h\}$ the innovations associated with $\{z_k^h\}$ in a weak sense [20]. All of this leads to the following notation.

Definition 2.1: Let $\{z_k^h\}$ be the received data. Here assume the following: For \boldsymbol{z}_k^h , there exists \boldsymbol{r}_k^h which consists of errors $\{e_s, s \leq k\}$ such that for each $k, \{r_s^h, s \leq k\}$ and $\{z_s^h, s \leq k\}$ k} generate the same syndrome sequence $\{\boldsymbol{\zeta}_s, s \leq k\}$. In this case, we call $\{r_k^h\}$ the innovations associated with $\{z_k^h\}$.

The above argument implies that we may call

$$m{r}_{k}^{h} = m{z}_{k}^{h} + (m{z}_{k}^{h}G^{-1})G$$

= $m{z}_{k}^{h}(I_{n_{0}} + G^{-1}G)$ (22)

the *innovation* corresponding to z_k^h .

Here note the mapping: $z_k^h \mapsto r_k^h$. In the innovations approach to linear filtering problems, the observed data is whitened by a causal [6] and invertible operation. With respect to the above mapping, we have the following.

Proposition 2.2: The mapping: $\boldsymbol{z}_k^h \mapsto \boldsymbol{r}_k^h = \boldsymbol{z}_k^h(I_{n_0} + G^{-1}G)$ is not invertible.

Proof: We will show that $\det(I_{n_0} + G^{-1}G) = \det(H^T(H^{-1})^T) = 0$ ("det(\cdot)" is the determinant). Since *H* is assumed to be canonical (accordingly, basic), we have a following invariant-factor decomposition:

$$H = \hat{A} \times \hat{\Gamma} \times \hat{B},$$

where

$$\hat{\Gamma} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}$$
$$\stackrel{\triangle}{=} \left(I_{n_0-k_0} \quad O_{n_0-k_0,k_0} \right).$$

Here, $O_{n_0-k_0,k_0}$ denotes the zero matrix of size $(n_0-k_0) \times k_0$. Then [13] we have

$$H^{-1} = \hat{B}^{-1} \times \hat{\Gamma}^{-1} \times \hat{A}^{-1},$$

where

$$\hat{\Gamma}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \hline 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$
$$\stackrel{\triangle}{=} \begin{pmatrix} I_{n_0 - k_0} \\ O_{k_0, n_0 - k_0} \end{pmatrix}.$$

Hence, it follows that

$$H^{-1}H = \hat{B}^{-1}\hat{\Gamma}^{-1}\hat{A}^{-1}\hat{A}\hat{\Gamma}\hat{B}$$

$$= \hat{B}^{-1}\hat{\Gamma}^{-1}\hat{\Gamma}\hat{B}$$

$$= \hat{B}^{-1}\begin{pmatrix} I_{n_0-k_0} \\ O_{k_0,n_0-k_0} \end{pmatrix}$$

$$\times \begin{pmatrix} I_{n_0-k_0} & O_{n_0-k_0,k_0} \\ O_{k_0,n_0-k_0} & O_{k_0,k_0} \end{pmatrix} \hat{B}$$

$$= \hat{B}^{-1}\begin{pmatrix} I_{n_0-k_0} & O_{n_0-k_0,k_0} \\ O_{k_0,n_0-k_0} & O_{k_0,k_0} \end{pmatrix} \hat{B}.$$

Accordingly,

$$H^{T}(H^{-1})^{T} = \hat{B}^{T} \begin{pmatrix} I_{n_{0}-k_{0}} & O_{n_{0}-k_{0},k_{0}} \\ O_{k_{0},n_{0}-k_{0}} & O_{k_{0},k_{0}} \end{pmatrix} (\hat{B}^{-1})^{T}$$

Hence, we have

$$det(H^{T}(H^{-1})^{T}) = det(\hat{B}^{T})det\begin{pmatrix} I_{n_{0}-k_{0}} & O_{n_{0}-k_{0},k_{0}} \\ O_{k_{0},n_{0}-k_{0}} & O_{k_{0},k_{0}} \end{pmatrix} \times det((\hat{B}^{-1})^{T}) = det(\hat{B})det\begin{pmatrix} I_{n_{0}-k_{0}} & O_{n_{0}-k_{0},k_{0}} \\ O_{k_{0},n_{0}-k_{0}} & O_{k_{0},k_{0}} \end{pmatrix} \times det(\hat{B}^{-1}) = det\begin{pmatrix} I_{n_{0}-k_{0}} & O_{n_{0}-k_{0},k_{0}} \\ O_{k_{0},n_{0}-k_{0}} & O_{k_{0},k_{0}} \end{pmatrix}.$$

Finally, note that

$$\det \left(\begin{array}{cc} I_{n_0-k_0} & O_{n_0-k_0,k_0} \\ O_{k_0,n_0-k_0} & O_{k_0,k_0} \end{array}\right) = 0.$$

The following shows that the innovation r_k^h corresponding to z_k^h cannot be further reduced.

Proposition 2.3: In the relation $r_k^h = z_k^h(I_{n_0} + G^{-1}G)$, replace z_k^h on the right-hand side by r_k^h . Then we have r_k^h again.

Proof:

$$\boldsymbol{r}_{k}^{h}(I_{n_{0}} + G^{-1}G)$$

$$= \boldsymbol{r}_{k}^{h}H^{T}(H^{-1})^{T}$$

$$= \boldsymbol{\zeta}_{k}(H^{-1})^{T} = \boldsymbol{r}_{k}^{h}.$$
(23)

B. Relationship Between General Codes and QLI Codes



Fig. 2. The structure of an SST Viterbi decoder for a QLI code (pre-decoder: $F = (1, 1)^T$).

We remark that the first paper [17] on SST Viterbi decoding dealt with QLI codes. Let

$$G(D) = (g_1(D), g_2(D))$$
(24)
$$(g_1 + g_2 = D^L, \ 1 \le L \le \nu - 1)$$

be a generator matrix for a QLI code, where ν is the constraint length of G(D). The corresponding SST Viterbi decoder is shown in Fig.2 [38].

Here consider the following quantity:

$$\begin{aligned} \boldsymbol{\eta}_{k-L}^h &= \boldsymbol{z}_{k-L}^h - \hat{i}(k-L|k)G(D) \\ &= \boldsymbol{z}_{k-L}^h + \hat{i}(k-L|k)G(D), \end{aligned} \tag{25}$$

where $\hat{i}(k - L|k)$ denotes an estimate of i_{k-L} based on $\{\boldsymbol{z}_s^h, s \leq k\}$. In the linear filtering/smoothing theory, this corresponds to

$$y(t) - C(t)\hat{x}(t|b) \ (t < b).$$
 (26)

Hence, η_{k-L}^{h} is slightly different from the innovation associated with the observation \boldsymbol{z}_{k-L}^{h} . We can call $\hat{i}(k-L|k)$ a linear *smoothed* estimate of i_{k-L} . Note that $\hat{x}(t|b)$ is the estimate of x(t) (t < b) based on the observations y(s) (s < b) [15]. That is, more observations are used for the estimation of x(t) as compared with $\hat{x}(t|t)$. Accordingly, the accuracy of $\hat{x}(t|b)$ may increase as compared with $\hat{x}(t|t)$. Then it is reasonable to think a similar result holds with respect to $\hat{i}(k-L|k)$ (see Proposition 2.8).

Now suppose that $\hat{i}(k - L|k)$ has the form

$$\hat{i}(k-L|k) = \boldsymbol{z}_k^h Q(D), \qquad (27)$$

where Q(D) is a polynomial matrix. Then we have

$$\begin{split} \eta_{k-L}^{h} &= \mathbf{z}_{k-L}^{h} + \mathbf{z}_{k}^{h} Q(D) G(D) \\ &= (i_{k-L} G(D) + \mathbf{e}_{k-L}) \\ &+ (i_{k} G(D) + \mathbf{e}_{k}) Q(D) G(D) \\ &= i_{k} (D^{L} + G(D) Q(D)) G(D) \\ &+ \mathbf{e}_{k} Q(D) G(D) + \mathbf{e}_{k-L}. \end{split}$$

Note that if

$$(D^{L}+G(D)Q(D))G(D) = D^{L}G(D) + G(D)Q(D)G(D) = 0$$

$$G(D)D^{-L}Q(D)G(D) = G(D)$$
 (28)

holds, then η_{k-L}^h is independent of i_k . Here $G(D)D^{-L}Q(D)G(D) = G(D)$ implies that $D^{-L}Q(D)$ is a generalized inverse [26] of G(D). Then we can take $F \stackrel{\triangle}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as Q(D). In this case, η_{k-L}^h is independent of i_k and we have

$$\boldsymbol{\eta}_{k-L}^h = (\boldsymbol{e}_k F) G + \boldsymbol{e}_{k-L}$$
(29)

$$= u_k G + \boldsymbol{e}_{k-L} \tag{30}$$

$$= \boldsymbol{e}_k(FG + D^L I_2), \qquad (31)$$

where $u_k \stackrel{\triangle}{=} e_k F$. We remark that the right-hand side is just the input to the main decoder in an SST Viterbi decoder, where F is used as a pre-decoder (see Fig.2). Also, note that

$$\boldsymbol{\eta}_{k-L}^{h} \boldsymbol{H}^{T}(D) = \boldsymbol{z}_{k-L}^{h} \boldsymbol{H}^{T}(D) + \boldsymbol{z}_{k}^{h} \boldsymbol{Q}(D) \boldsymbol{G}(D) \boldsymbol{H}^{T}(D)$$
$$= \boldsymbol{z}_{k-L}^{h} \boldsymbol{H}^{T}(D) = \zeta_{k-L}$$
(32)

holds irrespective of Q(D). Hence, η_{k-L}^h and z_{k-L}^h generate the same syndrome ζ_{k-L} .

On the other hand, η_{k-L}^h has another expression. We have

$$FG + D^{L}I_{2} = \begin{pmatrix} g_{1} + D^{L} & g_{2} \\ g_{1} & g_{2} + D^{L} \end{pmatrix}$$
$$= \begin{pmatrix} g_{2} & g_{2} \\ g_{1} & g_{1} \end{pmatrix}$$
$$= (H^{T}, H^{T}), \qquad (33)$$

where $H^T = \begin{pmatrix} g_2 \\ g_1 \end{pmatrix}$ is the syndrome former corresponding to $G = (g_1, g_2)$. Then

$$\boldsymbol{\eta}_{k-L}^{h} = \boldsymbol{e}_{k}(H^{T}, H^{T}) = (\zeta_{k}, \zeta_{k})$$
(34)

is obtained. Thus we have again

$$\boldsymbol{\eta}_{k-L}^{h} \boldsymbol{H}^{T} = (\zeta_{k}, \zeta_{k}) \begin{pmatrix} g_{2} \\ g_{1} \end{pmatrix}$$

$$= \zeta_{k} (g_{1} + g_{2})$$

$$= \zeta_{k} \boldsymbol{D}^{L} = \zeta_{k-L}.$$

Therefore, η_{k-L}^h has the following properties:

- 1) $\eta_{k-L}^{h} = e_k(FG + D^L I_2)$ holds. Hence, η_{k-L}^{h} depends not only on errors $\{e_s, s \leq k - L\}$ but also on errors $\{e_s, k - L < s \leq k\}$ in general. There is a correspondence between e_k and η_{k-L}^{h} in the sense that the former generates the syndrome ζ_k and the latter generates the syndrome ζ_{k-L} .
- 2) $\{\eta_s^h, s \le k L\}$ and $\{z_s^h, s \le k L\}$ generate the same syndrome sequence $\{\zeta_s, s \le k L\}$.

The above argument implies that

$$\boldsymbol{\eta}_{k-L}^{h} = \boldsymbol{z}_{k-L}^{h} + (\boldsymbol{z}_{k}^{h}F)G$$

$$= \boldsymbol{z}_{k}^{h}(D^{L}I_{2} + FG)$$
(35)

is not the innovation corresponding to z_{k-L}^h in the meaning of Definition 2.1.

Now with respect to the mapping: $\boldsymbol{z}_k^h \mapsto \boldsymbol{\eta}_{k-L}^h$, we have the following.

Proposition 2.4: The mapping: $\mathbf{z}_k^h \mapsto \boldsymbol{\eta}_{k-L}^h = \mathbf{z}_k^h(D^L I_2 + FG)$ is not invertible.

Proof: It follows from

$$D^L I_2 + FG = \left(\begin{array}{cc} g_2 & g_2\\ g_1 & g_1 \end{array}\right)$$

that $\det(D^L I_2 + FG) = 0.$

The following shows that η_{k-L}^h cannot be further reduced as in the case of r_k^h .

Proposition 2.5: In the relation $\eta_{k-L}^h = z_k^h(D^L I_2 + FG)$, replace z_k^h on the right-hand side by η_k^h . Then we have η_{k-L}^h again.

Proof:

$$\eta_k^h(D^L I_2 + FG) = \eta_k^h(H^T, H^T) = (\zeta_k, \zeta_k) = \eta_{k-L}^h.$$
(36)

Consider a QLI code defined by G(D). It can be regarded as a general code as well. Hence, we can apply the argument in the preceding section to it. Let $\hat{i}(k - L|k)$ be the estimate of i_{k-L} derived as a QLI code, whereas let $\hat{i}(k - L|k - L)$ be the estimate of i_{k-L} derived as a general code. Then we have the following.

Proposition 2.6: Let $G = (g_1, g_2) (g_1 + g_2 = D^L)$ be a generator matrix for a QLI code. Define as follows:

$$\hat{i}(k-L|k) \stackrel{\triangle}{=} \boldsymbol{z}_k^h F$$
 (37)

$$\hat{i}(k-L|k-L) \stackrel{\triangle}{=} \boldsymbol{z}_{k-L}^{h} G^{-1}.$$
(38)

Then we have

$$\hat{i}(k-L|k) = \hat{i}(k-L|k-L) + \zeta_k,$$
 (39)

where $\zeta_k = e_k H^T = e_k \begin{pmatrix} g_2 \\ g_1 \end{pmatrix}$ is the syndrome. *Proof:* From

$$\hat{i}(k-L|k) = \mathbf{z}_{k}^{h}F = i_{k-L} + \mathbf{e}_{k}F$$
$$\hat{i}(k-L|k-L) = \mathbf{z}_{k-L}^{h}G^{-1} = i_{k-L} + \mathbf{e}_{k-L}G^{-1},$$

the difference between $\hat{i}(k-L|k)$ and $\hat{i}(k-L|k-L)$ is given by

$$e_k F + e_{k-L} G^{-1} = e_k (F + D^L G^{-1}).$$
 (40)

Let

 $G^{-1} = \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right).$

Then we have

$$F + D^{L}G^{-1} = \begin{pmatrix} 1 + D^{L}b_{1} \\ 1 + D^{L}b_{2} \end{pmatrix}.$$
 (41)

We show that the above is equal to H^T . In fact, we have

$$(g_1, g_2) \begin{pmatrix} 1+D^L b_1 \\ 1+D^L b_2 \end{pmatrix}$$

= $(g_1 + g_2) + D^L (g_1 b_1 + g_2 b_2)$
= $D^L + D^L = 0.$

Corollary 2.7: Under the same conditions as in Proposition 2.6,

$$\boldsymbol{\eta}_{k-L}^h = \boldsymbol{r}_{k-L}^h + \zeta_k G \tag{42}$$

holds.

Proof: From

$$\boldsymbol{z}_k^h F = \boldsymbol{z}_{k-L}^h G^{-1} + \zeta_k,$$

it follows that

$$\boldsymbol{z}_{k-L}^h + (\boldsymbol{z}_k^h F)G = \boldsymbol{z}_{k-L}^h + (\boldsymbol{z}_{k-L}^h G^{-1})G + \zeta_k G$$

Here, it suffices to note the following equalities:

$$\begin{aligned} \boldsymbol{\eta}_{k-L}^h &= \boldsymbol{z}_{k-L}^h + (\boldsymbol{z}_k^h F) G \\ \boldsymbol{r}_{k-L}^h &= \boldsymbol{z}_{k-L}^h + (\boldsymbol{z}_{k-L}^h G^{-1}) G. \end{aligned}$$

On the analogy of the linear filtering/smoothing theory, it is expected that the linear smoothed estimate $\hat{i}(k - L|k)$ has higher accuracy as compared with the linear filtered estimate $\hat{i}(k - L|k - L)$. In the following, $P(\cdot)$ denotes the probability and

$$\epsilon = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E_s/N_0}}^{\infty} e^{-\frac{y^2}{2}} dy \stackrel{\triangle}{=} Q(\sqrt{2E_s/N_0}) \tag{43}$$

is the channel error probability. We have the following. *Proposition 2.8:* Let

$$p_f \stackrel{\triangle}{=} P(\hat{i}(k-L|k-L) \neq i_{k-L})$$
$$= P(\boldsymbol{e}_{k-L}G^{-1} = 1)$$
(44)

$$p_s \stackrel{\simeq}{=} P(i(k-L|k) \neq i_{k-L})$$

= $P(e_k F = 1).$ (45)

Then $p_s \leq p_f$ for $0 \leq \epsilon \leq 1/2$.

Proof: Let
$$G^{-1} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
. Then $e_{k-L}G^{-1} = 1$ is essed as

expressed as

$$e_{k-L}^{(1)}b_1(D) + e_{k-L}^{(2)}b_2(D) = 1.$$

We can rewrite the above as $e_1+e_2+\cdots+e_m=1$, where errors e_j $(1 \le j \le m, 3 \le m)$ are statistically independent of each other. Also, note that under this condition,

$$P(e_k^{(1)} + e_k^{(2)} = 1) = P(e_1 + e_2 = 1)$$

holds. Hence, the comparison between p_f and p_s is reduced to that between $P(e_1+e_2+\cdots+e_m=1)$ and $P(e_1+e_2=1)$. Now we have

$$P(e_1 + e_2 + \dots + e_m = 1)$$

= $P(e_1 + e_2 + e_b = 1)$
= $P(e_1 + e_2 = 1, e_b = 0) + P(e_1 + e_2 = 0, e_b = 1)$
= $P(e_1 + e_2 = 1)P(e_b = 0) + P(e_1 + e_2 = 0)P(e_b = 1),$

where $e_b \stackrel{\triangle}{=} e_3 + \cdots + e_m$. Hence, we have

$$P(e_{1}+e_{2}+e_{b}=1) - P(e_{1}+e_{2}=1)$$

$$= -P(e_{1}+e_{2}=1)(1 - P(e_{b}=0))$$

$$+P(e_{1}+e_{2}=0)P(e_{b}=1)$$

$$= P(e_{b}=1)(P(e_{1}+e_{2}=0) - P(e_{1}+e_{2}=1))$$

$$= P(e_{b}=1)(1 - 2\epsilon)^{2} \ge 0 \ (0 \le \epsilon \le 1/2).$$
(46)

Example 1: Consider the QLI code C_1 defined by $G(D) = (1 + D + D^2, 1 + D^2)$ (L = 1). From an invariant-factor decomposition of G(D),

$$G^{-1}(D) = \begin{pmatrix} D\\ 1+D \end{pmatrix}$$
(47)

is obtained. Hence, we have

 \hat{i}

$$F + D^{L}G^{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D \begin{pmatrix} D \\ 1+D \end{pmatrix}$$
$$= \begin{pmatrix} 1+D^{2} \\ 1+D+D^{2} \end{pmatrix}$$
$$= H^{T}.$$
 (48)

First compare the two estimates of i_{k-1} . Note the following:

$$\hat{i}(k-1|k) = z_k^{(1)h} + z_k^{(2)h} = i_{k-1} + \boldsymbol{e}_k F$$

(k-1|k-1) = $z_{k-2}^{(1)h} + z_{k-2}^{(2)h} + z_{k-1}^{(2)h} = i_{k-1} + \boldsymbol{e}_{k-1} G^{-1}.$

From the first equation, the error probability of $\hat{i}(k-1|k)$ is given by

$$p_s \stackrel{\triangle}{=} P(e_k^{(1)} + e_k^{(2)} = 1)$$
$$= 2\epsilon - 2\epsilon^2.$$
(49)

On the other hand, from the second equation, the error probability of $\hat{i}(k-1|k-1)$ is given by

$$p_f \stackrel{\triangle}{=} P(e_{k-2}^{(1)} + e_{k-2}^{(2)} + e_{k-1}^{(2)} = 1) = 3\epsilon - 6\epsilon^2 + 4\epsilon^3.$$
(50)

TABLE I AN EXAMPLE OF ENCODING BASED ON $G(D) \!=\! (1\!+\!D\!+\!D^2, 1\!+\!D^2)$

k	1	2	3	4	5	6	7	8
i_k	1	0	0	1	0	1	0	0
$oldsymbol{y}_k$	11	10	11	11	10	00	10	11
$oldsymbol{e}_k$	00	10	00	01	00	10	00	00
$oldsymbol{z}_k^h$	11	00	11	10	10	10	10	11
ζ_k^n	0	1	0	0	1	0	0	1
$\hat{i}(k-1 k)$	0	0^*	0	1*	1	1*	1	0
$\hat{i}(k-1 k-1)$	0	1	0	1^*	0^*	1^{*}	1	1^{*}
i_{k-1}	0	1	0	0	1	0	1	0

Hence, we have

$$p_f - p_s = (3\epsilon - 6\epsilon^2 + 4\epsilon^3) - (2\epsilon - 2\epsilon^2)$$
$$= \epsilon(1 - 2\epsilon)^2 \ge 0.$$
(51)

This inequality implies that $\hat{i}(k-1|k)$ has higher accuracy as compared with $\hat{i}(k-1|k-1)$.

Next, we show an example of encoding (see Table I). In this example, the encoder is terminated in state (00) at k = 8. In Table I, "*" denotes that the information i_{k-1} and its estimate are different. We observe that the relation

$$i(k-1|k) = i(k-1|k-1) + \zeta_k$$

actually holds.

III. DISTRIBUTIONS RELATED TO THE MAIN DECODER IN AN SST VITERBI DECODER

It is stated [14] that the innovations approach to linear filtering problems is first to convert the observed process to a white-noise process, and then to treat the resulting simpler white-noise observations problem. In our case, we think this corresponds to the reduction of decoding complexity in the main decoder in an SST Viterbi decoder. We also think the reduction is caused by biased distributions related to the main decoder. First we show that the distribution of the input to the main decoder is biased under low to moderate channel noise level. Next, we show that the state distribution in the code trellis for the main decoder is also biased under the same channel conditions. In either case, a QLI code is used in the discussion. This is because a QLI code is regarded as a general code as well and then we can compare two distributions, i.e., the one obtained as a general code and the other obtained as a OLI code. Furthermore, we show that the state distribution in the error trellis is equally biased.

A. Information Obtained through Observations [5]

Consider the channel model in Section I:

$$z_j = x_j \sqrt{2E_s/N_0} + w_j = cx_j + w_j,$$

where $c \stackrel{\triangle}{=} \sqrt{2E_s/N_0}$. The conditional entropy H[z|x] of the observation z_j given x_j is equal to the entropy H[w] of w_j , where H[w] is given by

$$H[w] = \frac{1}{2}\log(2\pi e).$$
 (52)

Suppose that y_j has values 0 and 1 with equal probability. Then the probability density function of z_j , denoted p(y), is given by

$$p(y) = \frac{1}{2}q(y-c) + \frac{1}{2}q(y+c),$$
(53)

where

$$q(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$$

Remark 1: When there is no danger of confusion, we call the *probability density function* of a random variable X simply the *distribution* of X.

Let us calculate the entropy H[z] of z_j [39]. Since

$$\int_{-\infty}^{\infty} yq(y)dy = \frac{c}{2} + \frac{(-c)}{2} = 0$$

and

$$\int_{-\infty}^{\infty} y^2 q(y) dy = \frac{1+c^2}{2} + \frac{1+c^2}{2} = 1+c^2$$

the entropy H[z] associated with p(y) [39] is computed as

$$H[z] = -\int_{-\infty}^{\infty} p(y) \log p(y) dy \le \frac{1}{2} \log \left(2\pi e(1+c^2)\right),$$
(54)

with equality when p(y) is Gaussian.

Hence, we have

$$H[x; z] = H[z] - H[w]$$

$$\leq \frac{1}{2} \log(2\pi e(1+c^2)) - \frac{1}{2} \log(2\pi e)$$

$$= \frac{1}{2} \log(1+c^2), \qquad (55)$$

where H[x; z] represents the information obtained through the observation [5].

Remark 2: H[x;z] is the channel capacity of the binary-input AWGN channel [39]DSuppose that $c \rightarrow 0$ ($\sqrt{2E_s/N_0} \rightarrow 0$). Then the inequality almost becomes an equality. Also, note that

$$\log(1+c^2) \approx c^2 \ (c \to 0).$$

Then we have

$$H[x;z] \approx \frac{1}{2} 2E_s/N_0 = E_s/N_0 \ (c \to 0).$$
 (56)

B. Entropy Associated with the Distribution of the Input to the Main Decoder

1) General codes: Suppose that the inverse encoder $G^{-1}(D)$ is used as a pre-decoder. Let $\mathbf{r}_k = (r_k^{(1)}, \cdots, r_k^{(n_0)})$ be the input to the main decoder in an SST Viterbi decoder. We have the following.

Proposition 3.1: The distribution of $r_k^{(l)}$ $(1 \le l \le n_0)$ is given by

$$p_r(y) = (1 - \alpha)q(y - c) + \alpha q(y + c),$$
 (57)

where

$$\alpha \stackrel{\triangle}{=} P(e_k^{(l)} = 0, r_k^{(l)h} = 1) + P(e_k^{(l)} = 1, r_k^{(l)h} = 0).$$
(58)

Proof: We can assume that the all-zero code sequence is transmitted. In this case, the distribution of $z_k^{(l)}$ is given by q(y-c) and we have

$$z_k^{(l)} = \left\{ \begin{array}{ll} |z_k^{(l)}|, & e_k^{(l)} = 0 \\ -|z_k^{(l)}|, & e_k^{(l)} = 1. \end{array} \right.$$

On the other hand, from the structure of the SST Viterbi decoder (cf. Fig.1), it follows that

$$r_k^{(l)} = \begin{cases} |z_k^{(l)}|, & r_k^{(l)h} = 0\\ -|z_k^{(l)}|, & r_k^{(l)h} = 1. \end{cases}$$

Hence, there are four cases:

Hence, there are four cases: 1) $e_k^{(l)} = 0, r_k^{(l)h} = 0 \rightarrow z_k^{(l)} = |z_k^{(l)}|, r_k^{(l)} = |z_k^{(l)}|$ 2) $e_k^{(l)} = 0, r_k^{(l)h} = 1 \rightarrow z_k^{(l)} = |z_k^{(l)}|, r_k^{(l)} = -|z_k^{(l)}|$ 3) $e_k^{(l)} = 1, r_k^{(l)h} = 0 \rightarrow z_k^{(l)} = -|z_k^{(l)}|, r_k^{(l)} = |z_k^{(l)}|$ 4) $e_k^{(l)} = 1, r_k^{(l)h} = 1 \rightarrow z_k^{(l)} = -|z_k^{(l)}|, r_k^{(l)} = -|z_k^{(l)}|.$ In cases 2) and 3), $r_k^{(l)} = -z_k^{(l)}$ holds and the distribution of $r_k^{(l)} = -z_k^{(l)}$ becomes q(y + c). Hence, the distribution of $r_k^{(l)}$

is given by

$$p_r(y) = (1 - \alpha)q(y - c) + \alpha q(y + c),$$

where

$$\alpha = P(e_k^{(l)} = 0, r_k^{(l)h} = 1) + P(e_k^{(l)} = 1, r_k^{(l)h} = 0).$$

Next, let us calculate the entropy of $r_k^{(l)}$, denoted H[r]. For the purpose, we calculate the variance σ_r^2 of $p_r(y)$. Note the following:

$$m_r = \int_{-\infty}^{\infty} y p_r(y) dy$$

= $(1-\alpha) \int_{-\infty}^{\infty} y q(y-c) dy + \alpha \int_{-\infty}^{\infty} y q(y+c) dy$
= $(1-\alpha)c + \alpha(-c)$
= $c(1-2\alpha)$

$$\int_{-\infty}^{\infty} y^2 p_r(y) dy$$

= $(1 - \alpha) \int_{-\infty}^{\infty} y^2 q(y - c) dy + \alpha \int_{-\infty}^{\infty} y^2 q(y + c) dy$
= $(1 - \alpha)(1 + c^2) + \alpha(1 + c^2)$
= $1 + c^2$.

Then

$$\begin{aligned} \sigma_r^2 &= \int_{-\infty}^{\infty} y^2 p_r(y) dy - m_r^2 \\ &= (1+c^2) - c^2 (1-2\alpha)^2 \\ &= 1 + 4c^2 \alpha (1-\alpha) \end{aligned}$$

is obtained. Hence, we have

$$H[r] = -\int_{-\infty}^{\infty} p_r(y) \log p_r(y) dy$$

$$\leq \frac{1}{2} \log \left(2\pi e (1 + 4c^2 \alpha (1 - \alpha)) \right), \quad (59)$$

with equality when $p_r(y)$ is Gaussian. We remark that the right-hand side contains a parameter α which depends on $e_k^{(l)}$ and $r_k^{(l)h}$. Hence, α inevitably depends on G(D) (cf. $r_k^h = e_k(G^{-1}G + I_{n_0})$).

We have already calculated H[z] and H[r]. However, all of the obtained expressions are inequalities. First consider the difference H[z] - H[r]. Let $\epsilon = Q(\sqrt{2E_s/N_0})$ be the channel error probability. We need the following.

Lemma 3.2: For $0 \le \epsilon \le 1/2$, we have $0 \le \alpha \le 1/2$. Proof: See Appendix A.

Note that $p_r(y)$ is biased and that the smaller ϵ becomes (i.e., $\alpha \to 0$), the more $p_r(y)$ is biased. Hence, it is expected that $H[z] - H[r] \ge 0$ and H[z] - H[r] increases as ϵ decreases. On the other hand, let us evaluate the difference between

the right-hand sides of H[z] and H[r], i.e., 1

$$\frac{1}{2}\log(2\pi e(1+c^2)) -\frac{1}{2}\log(2\pi e(1+4c^2\alpha(1-\alpha)))) = \frac{1}{2}\log\left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)}\right).$$
 (60)

Since $0 \le \alpha \le 1/2$, we have

$$0 \le 4\alpha(1-\alpha) \le 1.$$

Hence, from

$$1 + 4c^2\alpha(1 - \alpha) \le 1 + c^2$$
,

it follows that

$$\frac{1}{2}\log\frac{1+c^2}{1+4c^2\alpha(1-\alpha)} = \frac{1}{2}\log(1+\theta) \ (\theta \ge 0).$$

Moreover, consider the special cases, 1) $\epsilon \rightarrow 0$ and 2) $\epsilon \rightarrow$ 1/2.

1) $\epsilon \to 0$: We see that $p_r(y) \to q(y-c)$, where q(y-c) is Gaussian. Hence, we have

$$H[r] \approx \frac{1}{2} \log(2\pi e(1 + 4c^2\alpha(1 - \alpha)))).$$

Then we approximately have

$$H[z] - H[r] \leq \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)}\right)$$
$$\approx \frac{1}{2} \log(1+c^2) \ (c \to \infty).$$

2) $\epsilon \to 1/2$: We see that $p(y) \to q(y)$, where q(y) is Gaussian. Hence, we have

$$H[z] \approx \frac{1}{2} \log(2\pi e(1+c^2)).$$

Then we approximately have

$$H[z] - H[r] \geq \frac{1}{2} \log \left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)} \right)$$
$$\approx \frac{1}{2} \log \left(\frac{1+c^2}{1+c^2} \right) = 0 \ (c \to 0)$$

Furthermore, observe that as $\epsilon \ (0 \le \epsilon \le 1/2)$ decreases, $\frac{1}{2}\log\left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)}\right)$ increases (cf. Table II). We see that this is consistent with the expected behavior of H[z] - H[r].

We have not derived the exact value of H[z] - H[r]. However, the above argument implies that the two quantities H[z]-H[r] and $\frac{1}{2}\log\left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)}\right)$ have a close relation and the latter can be regarded as an approximation of H[z] - H[r]. Hence, in the following, we will compute the latter in order to evaluate H[z] - H[r]. Also, the relationship between the two quantities is denoted as

$$H[z] - H[r] \approx \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\alpha(1-\alpha)}\right),$$
 (61)

where the notation " \approx " is used in the above meaning.

We remark that the above calculation applies to a single component of the branch code. However, in order to know the bias of the composite distribution, we should calculate the entropy corresponding to the whole branch. Note that in our channel model, the branch code is transmitted symbol by symbol. Then the distributions corresponding to each code symbol are statistically independent of each other. Hence, the entropy associated with the composite distribution, denoted $H[r_1, r_2, \cdots, r_{n_0}]$, is the sum of the entropies associated with the distributions corresponding to each code symbol. That is, we have

$$H[r_1, r_2, \cdots, r_{n_0}] = H[r_1] + H[r_2] + \dots + H[r_{n_0}].$$
(62)

2) QLI codes: Let

$$G(D) = (g_1(D), g_2(D)) \ (g_1 + g_2 = D^L)$$

be a generator matrix for a QLI code. Suppose that $F = (1, 1)^T$ is used as a pre-decoder. Let $\eta_{k-L} = (\eta_{k-L}^{(1)}, \eta_{k-L}^{(2)})$ be the input to the main-decoder in an SST Viterbi decoder (see Fig.2). We have the following.

Proposition 3.3: The distribution of $\eta_{k-L}^{(l)}$ (l=1,2) is given by

$$p_{\eta}(y) = (1 - \beta)q(y - c) + \beta q(y + c),$$
 (63)

where

$$\beta \stackrel{\Delta}{=} P(e_{k-L}^{(l)} = 0, \zeta_k = 1) + P(e_{k-L}^{(l)} = 1, \zeta_k = 0).$$
(64)

Proof: Suppose that the all-zero code sequence is transmitted as before. In this case, the distribution of $z_{k-L}^{(l)}$ is given by q(y-c) and we have

$$z_{k-L}^{(l)} = \begin{cases} |z_{k-L}^{(l)}|, & e_{k-L}^{(l)} = 0\\ -|z_{k-L}^{(l)}|, & e_{k-L}^{(l)} = 1. \end{cases}$$

On the other hand, we already have $\eta_{k-L}^h = (\zeta_k, \zeta_k)$. Then it follows that

$$\eta_{k-L}^{(l)} = \begin{cases} |z_{k-L}^{(l)}|, & \zeta_k = 0\\ -|z_{k-L}^{(l)}|, & \zeta_k = 1. \end{cases}$$

Hence, there are four cases:

 $\begin{array}{l} 1) \quad e_{k-L}^{(l)} = 0, \zeta_k = 0 \to z_{k-L}^{(l)} = |z_{k-L}^{(l)}|, \eta_{k-L}^{(l)} = |z_{k-L}^{(l)}| \\ 2) \quad e_{k-L}^{(l)} = 0, \zeta_k = 1 \to z_{k-L}^{(l)} = |z_{k-L}^{(l)}|, \eta_{k-L}^{(l)} = -|z_{k-L}^{(l)}| \\ 3) \quad e_{k-L}^{(l)} = 1, \zeta_k = 0 \to z_{k-L}^{(l)} = -|z_{k-L}^{(l)}|, \eta_{k-L}^{(l)} = |z_{k-L}^{(l)}| \\ 4) \quad e_{k-L}^{(l)} = 1, \zeta_k = 1 \to z_{k-L}^{(l)} = -|z_{k-L}^{(l)}|, \eta_{k-L}^{(l)} = -|z_{k-L}^{(l)}|. \end{array}$

In cases 2) and 3), $\eta_{k-L}^{(l)} = -z_{k-L}^{(l)}$ holds and the distribution of $\eta_{k-L}^{(l)} = -z_{k-L}^{(l)}$ becomes q(y+c). Hence, the distribution of $\eta_{k-L}^{(l)}$ is given by

$$p_{\eta}(y) = (1 - \beta)q(y - c) + \beta q(y + c),$$

where

$$\beta = P(e_{k-L}^{(l)} = 0, \zeta_k = 1) + P(e_{k-L}^{(l)} = 1, \zeta_k = 0).$$

The rest of the argument follows as in the preceding section. Let $H[\eta]$ be the entropy of $\eta_{k-L}^{(l)}$. Then we have

$$H[\eta] = -\int_{-\infty}^{\infty} p_{\eta}(y) \log p_{\eta}(y) dy \\ \leq \frac{1}{2} \log(2\pi e(1 + 4c^{2}\beta(1 - \beta))), \quad (65)$$

with equality when $p_{\eta}(y)$ is Gaussian. Also, we have

$$H[z] - H[\eta] \approx \frac{1}{2} \log(2\pi e(1+c^2)) \\ -\frac{1}{2} \log(2\pi e(1+4c^2\beta(1-\beta))) \\ = \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\beta(1-\beta)}\right), \quad (66)$$

where the notation " \approx " is employed in the same meaning as in the case of general codes. Furthermore, we have used the following (cf. Lemma 3.2).

Lemma 3.4: For $0 \le \epsilon \le 1/2$, we have $0 \le \beta \le 1/2$.

Proof: See Appendix B.

3) An example: Consider the QLI code C_1 defined in Example 1. First we regard C_1 as a general code (G^{-1} is used as a pre-decoder). Let us evaluate the parameter α defined in the previous section. For the first component of a branch, we have

$$\alpha_1 = 5\epsilon - 20\epsilon^2 + 40\epsilon^3 - 40\epsilon^4 + 16\epsilon^5,$$

where $\epsilon = Q(\sqrt{2E_s/N_0}) = Q(\sqrt{E_b/N_0})$ is the channel error probability. Similarly, for the second component of the branch, we have

$$\alpha_2 = 6\epsilon - 30\epsilon^2 + 80\epsilon^3 - 120\epsilon^4 + 96\epsilon^5 - 32\epsilon^6.$$

Hence,

$$H_r^{(1)} \stackrel{\triangle}{=} H[z_1] - H[r_1] \approx \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\alpha_1(1-\alpha_1)}\right)$$
(67)

$$H_r^{(2)} \stackrel{\triangle}{=} H[z_2] - H[r_2] \approx \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\alpha_2(1-\alpha_2)}\right)$$
(68)

are obtained, where $c = \sqrt{2E_s/N_0} = \sqrt{E_b/N_0}$.

Next, we regard C_1 as a QLI code (F is used as a predecoder) and evaluate the parameter β . In this case, we have

$$\beta_1 = 6\epsilon - 30\epsilon^2 + 80\epsilon^3 - 120\epsilon^4 + 96\epsilon^5 - 32\epsilon^6 \quad (=\alpha_2)$$

for the first component of a branch. Similarly, for the second component of the branch, we have

$$\beta_2 = 4\epsilon - 12\epsilon^2 + 16\epsilon^3 - 8\epsilon^4.$$

 TABLE II

 ENTROPIES ASSOCIATED WITH INPUT DISTRIBUTIONS (AS A GENERAL CODE)

E_b/N_0 (dB)	c	ϵ	α_1	α_2	$H_{r}^{(1)}$	$H_{r}^{(2)}$	$H_r^{(1)} + H_r^{(2)}$
0	1.000	0.1587	0.4259	0.4494	0.0055	0.0026	0.0081
1	1.122	0.1309	0.3904	0.4191	0.0136	0.0075	0.0211
2	1.259	0.1040	0.3442	0.3766	0.0307	0.0190	0.0497
3	1.413	0.0788	0.2879	0.3213	0.0639	0.0445	0.1084
4	1.585	0.0565	0.2255	0.2565	0.1214	0.0929	0.2143
5	1.778	0.0377	0.1621	0.1876	0.2131	0.1759	0.3890
6	1.995	0.0230	0.1049	0.1231	0.3456	0.3027	0.6483
7	2.239	0.0126	0.0599	0.0710	0.5191	0.4756	0.9947
8	2.512	0.00600	0.0293	0.0349	0.7241	0.6870	1.4111
9	2.818	0.00242	0.0120	0.0143	0.9355	0.9103	1.8458
10	3.162	0.00078	0.0039	0.0047	1.1266	1.1131	2.2397

 TABLE III

 ENTROPIES ASSOCIATED WITH INPUT DISTRIBUTIONS (AS A QLI CODE)

$E_b/N_0 (\mathrm{dB})$	c	ϵ	β_1	β_2	$H_{\eta}^{(1)}$	$H_{\eta}^{(2)}$	$H_{\eta}^{(1)} + H_{\eta}^{(2)}$
0	1.000	0.1587	0.4494	0.3914	0.0026	0.0119	0.0145
1	1.122	0.1309	0.4191	0.3515	0.0075	0.0252	0.0327
2	1.259	0.1040	0.3766	0.3033	0.0190	0.0498	0.0688
3	1.413	0.0788	0.3213	0.2482	0.0445	0.0926	0.1371
4	1.585	0.0565	0.2565	0.1905	0.0929	0.1602	0.2531
5	1.778	0.0377	0.1876	0.1346	0.1759	0.2602	0.4361
6	1.995	0.0230	0.1231	0.0858	0.3027	0.3975	0.7002
7	2.239	0.0126	0.0710	0.0485	0.4756	0.5694	1.0450
8	2.512	0.00600	0.0349	0.0236	0.6870	0.7654	1.4524
9	2.818	0.00242	0.0143	0.0096	0.9103	0.9634	1.8737
10	3.162	0.00078	0.0047	0.0031	1.1131	1.1406	2.2536

Hence,

$$H_{\eta}^{(1)} \stackrel{\triangle}{=} H[z_1] - H[\eta_1] \approx \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\beta_1(1-\beta_1)}\right)$$
(69)

$$H_{\eta}^{(2)} \stackrel{\triangle}{=} H[z_2] - H[\eta_2] \approx \frac{1}{2} \log\left(\frac{1+c^2}{1+4c^2\beta_2(1-\beta_2)}\right)$$
(70)

are obtained.

Tables II and III show entropy versus E_b/N_0 . From these tables, we observe that

$$H_r^{(1)} + H_r^{(2)} < H_\eta^{(1)} + H_\eta^{(2)}.$$
(71)

That is, when C_1 is regarded as a QLI code, the distribution of the input to the main decoder is more biased.

C. State Distribution in the Code Trellis for the Main Decoder

In the preceding section, it was shown that the distribution of the input to the main decoder is biased under moderately noisy conditions. In this section, we show that the state distribution in the code trellis for the main decoder is also biased under the same channel conditions. For the purpose, we will take a QLI code. Since a QLI code can be regarded as a general code as well, we have two state expressions for the main decoder. Hence, we can evaluate a likelihood concentration in the main decoder more precisely by comparing the two state distributions.

Remark 1: Note that the code trellis module can be constructed as an error trellis module based on the syndrome former. We remark that for a high-rate code, the resulting code trellis module has less complexity than that of the conventional one [31], [42]. Lee et al. [21] used this method when they applied the SST scheme to $(n_0, n_0 - 1)$ convolutional codes.

Consider a QLI code defined by $G(D) = (g_1(D), g_2(D))$. A likelihood concentration in the main decoder depends heavily on the choice of a pre-decoder. Roughly speaking, if the information u_k for the main decoder consists of smaller number of error terms, then a higher likelihood concentration occurs. First apply F as a pre-decoder. Then we have

$$u_k = e_k^{(1)} + e_k^{(2)} \tag{72}$$

and u_k consists of two error terms. Next, apply the inverse encoder G^{-1} as a pre-decoder. Suppose that

$$G^{-1} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{73}$$

where b_1 and b_2 are polynomials in D. If these polynomials consist of small number of terms, then $u_k = e_k G^{-1}$ also consists of small number of error terms, which results in a high likelihood concentration in the main decoder. Let n_e be the number of error terms in u_k . Since $n_e > 2$ in general, QLI codes are preferable from a likelihood concentration viewpoint. On the other hand, for any fixed ν , the free distance, denoted d_{free} , of the best QLI codes is a little less than that of the best overall codes. (Here the optimality criterion first maximizes d_{free} and then minimizes $N_{d_{free}}$, where $N_{d_{free}}$ is the number of codewords with weight d_{free} [22].) In order to cope with this problem in application of the SST scheme, Ping et al. [25] searched for a good non-systematic encoder whose inverse consists of polynomials with small number of terms. For $\nu = 6$, they found the generator matrix

$$G(D) = (1 + D + D^4 + D^5 + D^6, 1 + D^2 + D^3 + D^4 + D^6)$$
(74)

with

$$G^{-1} = \left(\begin{array}{c} D\\ 1+D \end{array}\right).$$

Note that the above G(D) is an *optimum distance profile* (ODP) encoding matrix [13, Table 8.1] and the corresponding code has $d_{free} = 10$. It is shown that

$$G(D) = (1 + D + D^4, 1 + D^2 + D^3 + D^4)$$
(75)

has the same inverse encoder. Note that the above is also an ODP encoding matrix.

Example 2: Consider the QLI code C_1 defined in Example 1. First we regard C_1 as a general code $(G^{-1}$ is used as a pre-decoder). In this case, the information u_k for the main decoder is given by

$$u_{k} = e_{k} \begin{pmatrix} D \\ 1+D \end{pmatrix}$$

= $e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)}.$ (76)

Accordingly, the trellis state becomes

$$\boldsymbol{s}_k \!=\! (u_{k-1}, u_k) \!=\! (e_{k-2}^{(1)} \!+\! e_{k-2}^{(2)} \!+\! e_{k-1}^{(2)}, e_{k-1}^{(1)} \!+\! e_{k-1}^{(2)} \!+\! e_k^{(2)}).$$

Hence, we have

$$P_{00} \stackrel{\triangle}{=} P(\mathbf{s}_{k} = (00)) = 1 - 5\epsilon + 12\epsilon^{2} - 12\epsilon^{3} + 4\epsilon^{4}$$

$$P_{01} \stackrel{\triangle}{=} P(\mathbf{s}_{k} = (01)) = 2\epsilon - 6\epsilon^{2} + 8\epsilon^{3} - 4\epsilon^{4}$$

$$P_{10} \stackrel{\triangle}{=} P(\mathbf{s}_{k} = (10)) = 2\epsilon - 6\epsilon^{2} + 8\epsilon^{3} - 4\epsilon^{4}$$

$$P_{11} \stackrel{\triangle}{=} P(\mathbf{s}_{k} = (11)) = \epsilon - 4\epsilon^{3} + 4\epsilon^{4},$$

where $\epsilon = Q(\sqrt{E_b/N_0})$ is the channel error probability.

Next, we regard C_1 as a QLI code (F is used as a predecoder). Then the information u_k for the main decoder is given by

$$u_k = e_k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= e_k^{(1)} + e_k^{(2)}.$$

Accordingly, the trellis state becomes

$$\mathbf{s}_k = (u_{k-1}, u_k) = (e_{k-1}^{(1)} + e_{k-1}^{(2)}, e_k^{(1)} + e_k^{(2)}).$$

Hence, we have

$$P_{00} = 1 - 4\epsilon + 8\epsilon^2 - 8\epsilon^3 + 4\epsilon^4$$

$$P_{01} = 2\epsilon - 6\epsilon^2 + 8\epsilon^3 - 4\epsilon^4$$

$$P_{10} = 2\epsilon - 6\epsilon^2 + 8\epsilon^3 - 4\epsilon^4$$

$$P_{11} = 4\epsilon^2 - 8\epsilon^3 + 4\epsilon^4.$$

In either case, the entropy H associated with the state distribution is given by

$$H = -P_{00} \log_2 P_{00} - P_{01} \log_2 P_{01} -P_{10} \log_2 P_{10} - P_{11} \log_2 P_{11}.$$
(77)

The results are shown in Tables IV and V. We observe that a higher likelihood concentration to state (00) occurs when the code is regarded as a QLI code. Denote by s_k^p and s_k^q the states for the main decoder obtained as a general code and as a QLI code, respectively. Note that u_k consists of three error terms in s_k^p , whereas u_k consists of two error terms in s_k^q . As was stated above, a likelihood concentration in the main decoder depends on the number of error terms (n_e) forming u_k in general. Hence, the results are reasonable.

Remark 2: Note that the components of the state are not statistically independent of each other in general. For example, take $s_k^p = (u_{k-1}, u_k) = (e_{k-2}^{(1)} + e_{k-2}^{(2)} + e_{k-1}^{(2)}, e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_k^{(2)})$. We see that $e_{k-1}^{(2)}$ is contained in both components. Hence, n_e alone does not affect the state distribution. Nevertheless, n_e provides useful information about a likelihood concentration in the main decoder.

D. State Distribution in the Error Trellis

It has been shown [37], [38] that SST Viterbi decoding based on a code trellis and syndrome decoding based on the corresponding error trellis are equivalent. In the following, k_0 is assumed to be (n_0-1) for simplicity. Then the size of H(D)is $1 \times n_0$. Let ν be the constraint length of H(D). Denote by s_k and σ_k the state at t = k in the code trellis for the main decoder and the state at t = k in the error trellis, respectively. Based on an adjoint-obvious realization (observer canonical

TABLE IV STATE DISTRIBUTIONS FOR THE MAIN DECODER (AS A GENERAL CODE)

E_b/N_0 (dB)	ϵ	P_{00}	P_{01}	P_{10}	P_{11}	H
0	0.1587	0.4633	0.1957	0.1957	0.1452	1.8398
1	0.1309	0.5253	0.1758	0.1758	0.1231	1.7418
2	0.1040	0.5968	0.1516	0.1516	0.1000	1.6019
3	0.0788	0.6746	0.1243	0.1243	0.0768	1.4153
4	0.0565	0.7536	0.0953	0.0953	0.0558	1.1864
5	0.0377	0.8279	0.0673	0.0673	0.0375	0.9273
6	0.0230	0.8912	0.0429	0.0429	0.0230	0.6631
7	0.0126	0.9389	0.0243	0.0243	0.0126	0.4255
8	0.00600	0.9704	0.0118	0.0118	0.0060	0.2376
9	0.00242	0.9880	0.0048	0.0048	0.0024	0.1121
10	0.00078	0.9961	0.0016	0.0016	0.0008	0.0436

 TABLE V

 State distributions for the main decoder (as a QLI code)

E_b/N_0 (dB)	ϵ	P_{00}	P_{01}	P_{10}	P_{11}	H
0	0.1587	0.5372	0.1957	0.1957	0.0713	1.6745
1	0.1309	0.5967	0.1758	0.1758	0.0518	1.5476
2	0.1040	0.6620	0.1516	0.1516	0.0347	1.3875
3	0.0788	0.7306	0.1243	0.1243	0.0209	1.1953
4	0.0565	0.7981	0.0953	0.0953	0.0113	0.9790
5	0.0377	0.8601	0.0673	0.0673	0.0053	0.7511
6	0.0230	0.9121	0.0429	0.0429	0.0020	0.5288
7	0.0126	0.9509	0.0243	0.0243	0.00062	0.3363
8	0.00600	0.9763	0.0118	0.0118	0.00014	0.1868
9	0.00242	0.9904	0.0048	0.0048	0.000023	0.0882
10	0.00078	0.9969	0.0016	0.0016	0.000003	0.0344

form [7]) of the syndrome former H^T , σ_k can be expressed as

$$\boldsymbol{\sigma}_k = \boldsymbol{e}_k U(D), \tag{78}$$

where U(D) is an $n_0 \times \nu$ matrix whose entries are polynomials in D. Then we have

$$\sigma_k = (\boldsymbol{u}_k G + \boldsymbol{r}_k^h) U$$

= $\boldsymbol{u}_k G U + \zeta_k (H^{-1})^T U.$ (79)

Note that the first term $u_k GU$ corresponds to the syndrome former state obtained by inputting the encoder output $u_k G$ directly to the syndrome former H^T . That is, $u_k GU$ is the *dual (physical) state* [7] corresponding to the encoder state s_k . Since the space of encoder states and that of the corresponding dual states are isomorphic, the correspondence between s_k and $u_k GU$ is one-to-one. Here note that the term $\zeta_k (H^{-1})^T U$ is common to every state s_k . Hence, the correspondence between s_k and σ_k is also one-to-one. This fact implies that the state distribution in a code trellis for the main decoder is closely related to that in the corresponding error trellis.

Example 2 (Continued): Consider the QLI code C_1 again. Based on an adjoint-obvious realization of the syndrome former $H^T = \begin{pmatrix} 1+D^2 \\ 1+D+D^2 \end{pmatrix}$, the state in the error trellis becomes

$$\sigma_{k} = (\sigma_{k1}, \sigma_{k2}) = (e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)}, e_{k}^{(1)} + e_{k}^{(2)})$$

$$= (e_{k}^{(1)}, e_{k}^{(2)}) \begin{pmatrix} D & 1 \\ 1+D & 1 \end{pmatrix}$$

$$\stackrel{\triangle}{=} e_{k}U(D).$$
(80)

TABLE VI STATE DISTRIBUTIONS IN THE ERROR TRELLIS

E_b/N_0 (dB)	ε	\tilde{P}_{00}	\tilde{P}_{01}	\tilde{P}_{10}	\tilde{P}_{11}	\tilde{H}
0	0.1587	0.5255	0.1335	0.2075	0.1335	1.7344
1	0.1309	0.5874	0.1138	0.1851	0.1138	1.6150
2	0.1040	0.6552	0.0932	0.1584	0.0932	1.4590
3	0.0788	0.7263	0.0726	0.1285	0.0726	1.2649
4	0.0565	0.7956	0.0533	0.0978	0.0533	1.0416
5	0.0377	0.8589	0.0363	0.0685	0.0363	0.8009
6	0.0230	0.9117	0.0225	0.0434	0.0225	0.5645
7	0.0126	0.9507	0.0124	0.0244	0.0124	0.3570
8	0.00600	0.9763	0.0060	0.0118	0.0060	0.1980
9	0.00242	0.9904	0.0024	0.0048	0.0024	0.0926
10	0.00078	0.9969	0.0008	0.0016	0.0008	0.0358

Hence, we have

$$\tilde{P}_{00} \stackrel{\Delta}{=} P(\boldsymbol{\sigma}_k = (00)) = 1 - 4\epsilon + 7\epsilon^2 - 4\epsilon^3$$
$$\tilde{P}_{01} \stackrel{\Delta}{=} P(\boldsymbol{\sigma}_k = (01)) = \epsilon - \epsilon^2$$
$$\tilde{P}_{10} \stackrel{\Delta}{=} P(\boldsymbol{\sigma}_k = (10)) = 2\epsilon - 5\epsilon^2 + 4\epsilon^3$$
$$\tilde{P}_{11} \stackrel{\Delta}{=} P(\boldsymbol{\sigma}_k = (11)) = \epsilon - \epsilon^2,$$

where $\epsilon = Q(\sqrt{E_b/N_0})$. The entropy \tilde{H} associated with the above distribution is given by

$$\tilde{H} = -\tilde{P}_{00} \log_2 \tilde{P}_{00} - \tilde{P}_{01} \log_2 \tilde{P}_{01} -\tilde{P}_{10} \log_2 \tilde{P}_{10} - \tilde{P}_{11} \log_2 \tilde{P}_{11}.$$
(81)

The result is shown in Table VI. From Tables IV, V, and VI, we see that \tilde{H} lies between the value of entropy obtained by regarding C_1 as a general code and that obtained by regarding C_1 as a QLI code. This observation comes from the state expressions for s_k^p , s_k^q , and σ_k :

$$\begin{aligned} \boldsymbol{s}_{k}^{p} &= (u_{k-1}, u_{k}) \\ &= (e_{k-2}^{(1)} + e_{k-2}^{(2)} + e_{k-1}^{(2)}, e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)}) \\ \boldsymbol{s}_{k}^{q} &= (u_{k-1}, u_{k}) = (e_{k-1}^{(1)} + e_{k-1}^{(2)}, e_{k}^{(1)} + e_{k}^{(2)}) \\ \boldsymbol{\sigma}_{k} &= (\sigma_{k1}, \sigma_{k2}) = (e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)}, e_{k}^{(1)} + e_{k}^{(2)}). \end{aligned}$$

(Also, see Remark 2 at the end of Section III-C.)

Finally, examine the correspondence between the state in the code trellis for the main decoder and that in the error trellis. First consider the correspondence between $s_k^p = (u_{k-1}, u_k)$ and σ_k . Note the relation

$$\sigma_k = (u_k G + r_k^h) U$$

= $u_k G U + \zeta_k (H^{-1})^T U.$

Since

$$GU = (1+D+D^{2}, 1+D^{2}) \begin{pmatrix} D & 1 \\ 1+D & 1 \end{pmatrix}$$
$$= (1,D)$$
$$(H^{-1})^{T}U = (1+D,D) \begin{pmatrix} D & 1 \\ 1+D & 1 \end{pmatrix}$$

= (0,1),

it follows that

$$\sigma_k = u_k G U + \zeta_k (H^{-1})^T U = u_k (1, D) + \zeta_k (0, 1) = (u_k, u_{k-1} + \zeta_k).$$

Hence, we have

$$\boldsymbol{s}_{k}^{p} = (u_{k-1}, u_{k}) \leftrightarrow \boldsymbol{\sigma}_{k} = (u_{k}, u_{k-1} + \zeta_{k}), \qquad (82)$$

where

$$u_{k-1} + \zeta_k = (e_{k-2}^{(1)} + e_{k-2}^{(2)} + e_{k-1}^{(2)}) + (e_{k-2}^{(1)} + e_k^{(1)} + e_{k-2}^{(2)} + e_{k-1}^{(2)} + e_k^{(2)}) = e_k^{(1)} + e_k^{(2)}.$$

Next, consider the correspondence between $s_k^q = (u_{k-1}, u_k)$ and σ_k . This time (cf. Section II-B), note the relation

$$\sigma_k = (u_{k+L}G + \eta_k^h)U$$

= $u_{k+L}GU + (\zeta_{k+L}, \zeta_{k+L})U$

Letting L=1, it follows that

$$\sigma_{k} = u_{k+1}GU + (\zeta_{k+1}, \zeta_{k+1})U$$

= $u_{k+1}(1, D) + (\zeta_{k+1}, \zeta_{k+1}) \begin{pmatrix} D & 1 \\ 1+D & 1 \end{pmatrix}$
= $(u_{k+1} + \zeta_{k+1}, u_{k}).$

Hence, we have

$$\boldsymbol{s}_{k}^{q} = (u_{k-1}, u_{k}) \leftrightarrow \boldsymbol{\sigma}_{k} = (u_{k+1} + \zeta_{k+1}, u_{k}), \quad (83)$$

where

$$u_{k+1} + \zeta_{k+1} = (e_{k+1}^{(1)} + e_{k+1}^{(2)}) + (e_{k-1}^{(1)} + e_{k+1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)} + e_{k+1}^{(2)}) = e_{k-1}^{(1)} + e_{k-1}^{(2)} + e_{k}^{(2)}.$$

These results are consistent with the concrete state expressions for s_k^p , s_k^q , and σ_k .

IV. COMPLEXITY REDUCTION IN THE MAIN DECODER IN AN SST VITERBI DECODER

We have shown that the state distribution in the code trellis for the main decoder in an SST Viterbi decoder is biased under moderately noisy conditions. In this section, we show that those biased distributions actually lead to complexity reduction in the main decoder. Two reduction methods will be discussed. In the first one, biased state distributions are directly used for complexity reduction, whereas in the second one, those distributions are indirectly used. There have been several related works [2], [4], [25], [33], [35], [36] since the SST scheme was proposed. Hence, the discussion in the former part is mainly based on these known works. The known material is also dealt with in the latter part, but some original results are contained. In particular, we give an approximate criterion for complexity reduction in the main decoder in relation to the second reduction method.

A. Complexity Reduction Using State Distributions

So far biased state distributions have been directly used in order to reduce the decoder complexity [25], [33]. In the following, $k_0 = 1$ is assumed for simplicity. First we briefly review the *generalized Viterbi algorithm* (GVA) [8]. Let

$$\boldsymbol{u}^k \stackrel{\triangle}{=} u_1 u_2 \cdots u_k \tag{84}$$

be the transmitted information sequence, where k is the current depth. In the usual Viterbi algorithm, a trellis diagram is drawn by regarding the latest ν symbols $(u_{k-\nu+1}\cdots u_k)$ as a state (i.e., encoder state). On the other hand, in the GVA, the latest $\tilde{\nu}$ symbols $(u_{k-\tilde{\nu}+1}\cdots u_k)$ is considered as an algorithm's state (i.e., decoder state), where $\tilde{\nu} (> 0)$ can be chosen independent of ν . $\tilde{\nu}$ is called a constraint length of the algorithm. By choosing $\tilde{\nu}$ smaller that ν , the number of decoder states can be reduced. In this case, however, it is not guaranteed that the overall ML path can be chosen if a single survivor is preserved for each decoder states. Note that a decoder state consists of multiple encoder states. Hence, when a survivor for the decoder state is determined, the most likely path for each component encoder state has to be selected beforehand. This procedure is called *pre-selection* [8].

In [33], the GVA was applied to the main decoder by taking account of a biased state distribution. The method is based on the conjecture that, if a likelihood concentration to some particular states is occurring in the main decoder, then a great deal of decoding complexity reduction can be realized by applying the GVA to the main decoder with $\tilde{\nu}$ smaller than ν and by slightly increasing the number of total survivors as compared with that of decoder states. The method is formulated as follows:

- 1) The SST scheme is used to produce a likelihood concentration in the main decoder.
- 2) The GVA is applied to the main decoder with $\tilde{\nu}$ smaller than ν .
- In order to avoid a performance degradation due to choosing ν̃ smaller than ν, more than one survivors are preserved for those decoder states with high probabilities.

The above method was applied to the QLI code C_2 defined by

$$G(D) = (1 + D + D^3 + D^4 + D^6, 1 + D + D^2 + D^3 + D^4 + D^6).$$
(85)

Note that this code has $d_{free} = 9$. We observe that there occurs a likelihood concentration to the all-zero state and the states containing only one "1" (e.g., (000001)). Then $\tilde{\nu}$ is set to 5 and two survivors are preserved for each of the decoder states with high probabilities and only one survivor for each of the other decoder states. Hence, the number of decoder states is 32 and 38 survivors are preserved. Simulation results show that the method can reduce the decoding complexity to almost 1/2 of that of the conventional one within a very small performance degradation, where 8-level receiver quantization is assumed. It is also shown that a small increase of the number of survivors (i.e., additional 6 survivors) significantly improves the performance. This fact comes from a much biased state distribution in the code trellis for the main decoder. Ping et al. [25] also used the SST scheme to reduce the decoder complexity. Note that C_2 is a QLI code and has not the best d_{free} with $\nu = 6$. On the other hand, the number of error terms in $u_k = e_k G^{-1}$ must be small in order to produce a high likelihood concentration in the main decoder. As a result (see Section III-C), they chose the generator matrix

$$G(D) = (1+D+D^4+D^5+D^6, 1+D^2+D^3+D^4+D^6)$$

with

$$G^{-1} = \left(\begin{array}{c} D\\ 1+D \end{array}\right).$$

Note that the corresponding code C_3 has $d_{free} = 10$. Next, they applied a simplifying scheme to the main decoder. Since the state distribution in the code trellis for the main decoder is biased, they eliminated those states whose occurring probabilities are nearly zero. (Hence, the scheme is called PSS (probability selecting states).) More precisely, from among $2^6 = 64$ states, 22 states with lowest probabilities are eliminated for the above code. Then the number of states used for decoding is 42 and 42 survivors are preserved. Computer simulations show that the performance of a PSS-type decoder is as good as that of the conventional Viterbi decoder, whereas the hardware complexity of the former decoder is almost 1/2 of that of the latter one.

B. Trellis Degeneration Using Zero-Strings

There exists a method where biased state distributions are indirectly used for complexity reduction in the main decoder. First consider an error trellis. Given a received data $z = \{z_k\}$, let T_e be the corresponding error trellis. Note that unlike the code trellis T_c , the paths through T_e have different a priori probabilities in general. Consequently, when T_e is constructed based on the syndrome $\zeta = \{\zeta_k\}$ (which is computed using $z^h = \{z_k^h\}$), T_e usually has many redundant paths that can be deleted in advance. Using this fundamental property of error trellises, Ariel and Snyders [2], [4] proposed several methods to simplify T_e . Among them trellis degeneration using zerostrings [2], [4] is most effective.

In the following, k_0 is assumed to be (n_0-1) for simplicity. Let $\zeta = \{\zeta_k\}$ be the syndrome. An interval [t, t'] is called a *zero-string* if $\zeta_k = 0$, $t + 1 \leq k \leq t'$. Note that within a zero-string, any two consecutive zero states (denoted **0**) are connected by a zero-weight branch. Hence, if state **0** has the least weight at $s \in [t, t']$, then state **0** continues to have the least weight in [s + 1, t']. We remark that this principle also holds in the reverse direction. Here suppose that we can identify a sub-interval $[\tau, \tau']$ of [t, t'] such that the all-zero path connecting state **0** at depth τ and state **0** at depth τ' is a portion of the overall ML path. In this case, all but the all-zero path connecting those states can be deleted. That is, T_e is simplified in the interval $[\tau, \tau']$. This procedure is called *trellis degeneration* [2], [4].

On the other hand, we already know that SST Viterbi decoding based on a code trellis and syndrome decoding based on the corresponding error trellis are equivalent. Hence, it is reasonable to think that trellis degeneration is equally possible in the code trellis for the main decoder in an SST Viterbi decoder [35], [36].

Remark 1: The following argument is almost the same as that in [36]. Also, the material is taken from it. To the best of our knowledge, however, when the work of [36] was published (1997), the equivalence between SST Viterbi decoding based on a code trellis and syndrome decoding based on the corresponding error trellis had not been obtained. On the other hand, since the equivalence between the two schemes has been shown by now, the results about an error trellis can be transformed to the associated code trellis for the main decoder. That is, the application of the results in [2], [4] to the code trellis for the main decoder is justified.

First (see Section II) note that the hard-decision input to the main decoder is given by

$$\boldsymbol{r}_k^h = \zeta_k (H^{-1})^T$$

Also, in the case of QLI codes, the hard-decision input to the main decoder becomes

$$\boldsymbol{\eta}_{k-L}^h = (\zeta_k, \zeta_k).$$

Hence, an interval with $\zeta_k = 0$ is transformed to an interval with $\mathbf{r}_k^h = \mathbf{0}$ (or $\boldsymbol{\eta}_{k-L}^h = \mathbf{0}$). In this paper, we call the latter (i.e., an interval where the hard-decision input to the main decoder is consecutively zero) a *zero-string* as well. We describe the trellis degeneration in the code trellis for the main decoder in more detail.

Code trellis degeneration using zero-strings [36]:

- Given a zero-string [t, t'], decode forward the code trellis from state x (≠ 0) at depth t. Let τ(x) be the first depth at which the metric of state 0 is largest.
- 2) Similarly, decode backward the code trellis from state $\mathbf{x}' \ (\neq \mathbf{0})$ at depth t'. Let $\tau'(\mathbf{x}')$ be the first depth at which the metric of state $\mathbf{0}$ is largest.
- 3) Let $\tau \triangleq \max_{\boldsymbol{x}} \tau(\boldsymbol{x})$. Also, let $\tau' \triangleq \min_{\boldsymbol{x}'} \tau'(\boldsymbol{x}')$. If $\tau, \tau' \in [t, t']$ and $\tau < \tau'$, then delete all the sub-paths in $[\tau, \tau']$ except for the all-zero sub-path. (That is, the code trellis is simplified in the interval $[\tau, \tau']$. In this case, we call trellis degeneration "successful".)

Remark 2: The starting depths of the forward and the backward decoding can be chosen as $\tilde{t}(\leq t)$ and $\tilde{t}'(\geq t')$, respectively.

Remark 3: Suppose that the length of a zero-string [t, t'](denoted by ℓ) has an appropriate value. Then for harddecision data, the length $\ell_H \stackrel{\triangle}{=} (\tau - t) + (t' - \tau')$ can be determined in advance. Hence, for hard-decision data, if $\ell > \ell_H$ holds, then trellis degeneration is successful. For example, consider the code defined by $G = (1+D+D^2, 1+D^2)$. We have $\ell_H = (\tau - t) + (t' - \tau') = 5 + 5 = 10$.

Next, evaluate the complexity of Viterbi decoding where the trellis degeneration procedure is employed. Since trellis degeneration is rather complicated in a general case, we apply the procedure to those zero-strings whose lengths are larger than or equal to ℓ_0 , where ℓ_0 is a predetermined value. Let $[t_j, t'_j]$ be any such zero-string (j is used to distinguish zerostrings). It is assumed that trellis degeneration is successful for each $[t_j, t'_j]$. Let N_s be the number of states in the trellis. Also, let M be the section length of the trellis. We regard the computational complexity needed to decode one trellis section as one unit. (Then the Viterbi decoding complexity required to decode the whole trellis is given by M.) Under these conditions, let us evaluate the complexity of Viterbi decoding. Since trellis degeneration is successful for each zero-string $[t_j, t'_j]$, the decoding complexity is reduced by

$$\Delta \stackrel{\triangle}{=} \sum_{j} (\tau'_j - \tau_j) \tag{86}$$

as compared with the conventional decoding. On the other hand, in order to identify the sub-interval $[\tau_j, \tau'_j]$ of $[t_j, t'_j]$, the forward and the backward decoding are performed while changing the starting state. Let Δ' be the required computational complexity. Then the decoding complexity increases by

$$\Delta' \approx \sum_{j} ((N_s - 1) \times (\tau_j - t_j) + (N_s - 1) \times (t'_j - \tau'_j))$$

=
$$\sum_{j} (N_s - 1) ((\tau_j - t_j) + (t'_j - \tau'_j)).$$
(87)

Therefore, the overall decoding complexity is estimated as

$$Q_c \approx M + \Delta' - \Delta. \tag{88}$$

Hence, if $\Delta' < \Delta$, then complexity reduction is realized. In particular, if

$$(N_s - 1) \left((\tau_j - t_j) + (t'_j - \tau'_j) \right) < \tau'_j - \tau_j$$
(89)

holds for each j, then we have $\Delta' < \Delta$. Here note that for hard-decision data, we have

$$\tau_j - t_j) + (t'_j - \tau'_j) = \ell_H.$$

Hence, if

i.e.,

$$(N_s - 1)\ell_H < \tau'_j - \tau_j$$

holds approximately, then we can expect to have $\Delta' < \Delta$. In this case, the length of the corresponding zero-string (i.e., ℓ) becomes

$$\ell \approx (\tau_j' - \tau_j) + \ell_H$$

That is, if the condition

$$(N_s - 1)\ell_H + \ell_H < (\tau'_j - \tau_j) + \ell_H,$$
$$N \times \ell_H < \ell$$

 $N_s \times \ell_H < \ell \tag{90}$

holds, then complexity reduction is expected to occur. We can use the above inequality as a criterion for the length of a zerostring required for complexity reduction.

Example 3 [36]: In connection with the above subject, computer simulations have been done using the QLI code C_1 defined in Example 1, where $M = 10^5$ and 8-level receiver quantization is assumed. Under these conditions, the behavior of the main decoder was investigated. Table VII shows the number of zero-strings whose lengths are larger than or equal to ℓ_0 . Table VIII shows the average length of zero-strings counted in Table VII. We observe that as the SNR increases, the zero-strings become less numerous and longer.

TABLE VII Number of zero-strings

$E_b/N_0 (dB)$	$\ell_0 = 10$	$\ell_0 = 15$	$\ell_0 = 20$	$\ell_0 = 25$	$\ell_0 = 30$
4	2761	1527	879	490	287
5	2948	2003	1373	936	651
6	2602	2056	1634	1290	1040
7	1808	1590	1398	1236	1080
8	1006	953	907	851	792
9	427	425	415	407	395
10	148	148	145	145	144

TABLE VIII Average length of zero-strings

$E_b/N_0~(\mathrm{dB})$	$\ell_0 = 10$	$\ell_0 = 15$	$\ell_0 = 20$	$\ell_0 = 25$	$\ell_0 = 30$
4	18.1	23.3	28.1	33.1	37.7
5	22.6	27.7	32.7	37.8	42.6
6	31.2	36.4	41.4	46.6	51.3
7	50.3	55.5	60.8	65.9	71.5
8	95.4	100.1	104.3	109.7	115.8
9	230.7	231.7	236.9	241.1	247.6
10	672.4	672.4	685.9	685.9	685.9

The normalized decoding complexity Q_c/M obtained from simulations is given in Table IX. Since trellis degeneration is successful for almost all zero-strings of length $\ell \ge 15$, $\ell_0 \ge 20$ is assumed. In this example, the starting depths of the forward and the backward decoding for a zero-string [t, t'] are chosen as t-1 and t'+1, respectively.

Now evaluate the length of a zero-string required for complexity reduction. Taking into account the starting depths of the forward and the backward decoding, we have

$$(N_s - 1)((\tau_j - t_j) + (t'_j - \tau'_j) + 2) < \tau'_j - \tau_j.$$

(i.e., $(N_s - 1)(\ell_H + 2) + \ell_H < (\tau'_j - \tau_j) + \ell_H.$)

Note that $N_s = 4$ and $\ell_H = 10$. Hence, if

$$46 < \ell \tag{91}$$

holds, then we can expect that complexity reduction is realized. Accordingly, using Table VIII, let us search for the SNR at which the average length of zero-strings is nearly equal to 46. We see that this value is attained at an SNR of $E_b/N_0 = 6 \sim 7 \text{dB}$ for $\ell_0 = 20$. Similarly, we see that $E_b/N_0 \approx 6 \text{dB}$ for $\ell_0 = 25$ and $E_b/N_0 = 5 \sim 6 \text{dB}$ for $\ell_0 = 30$. From Table IX, it is confirmed that these values are almost equal to the SNRs at which the decoding complexity is less than 1 for the first time. Hence, the derived criterion for complexity reduction (i.e., $N_s \times \ell_H < \ell$) seems to be reasonable.

We remark that the derived criterion can be loosened. Note that for a trellis with large N_s , the condition seems to be strict. On the other hand, we already know that the state distribution in the main decoder is much biased under moderately noisy conditions. For example, consider the code trellis associated with the QLI code C_2 (cf. (85)). Here note the all-zero state and the states containing only one "1" (e.g., (000001)). We examined the total probability of these 7 states. As a result (cf. [33]), we have 87% at $E_b/N_0 = 4$ dB, 94% at $E_b/N_0 =$

TABLE IX Normalized decoding complexity

E_b/N_0 (dB)	$\ell_0 = 20$	$\ell_0 = 25$	$\ell_0 = 30$
4	1.22	1.10	1.04
5	1.25	1.12	1.05
6	1.11	1.02	0.97
7	0.79	0.76	0.73
8	0.45	0.44	0.43
9	0.18	0.18	0.18
10	0.06	0.06	0.04

5dB, and 97% at $E_b/N_0 = 6$ dB. Hence, in order to identify the sub-interval $[\tau_j, \tau'_j]$ of a zero-string $[t_j, t'_j]$, we need not use all states ($\neq 0$) in the trellis as the starting state. That is, we can restrict the starting state to those 6 states (the all-zero state is not used) under low to moderate noise level within a very small degradation. In this way, N_s can be replaced by some smaller number. In this case, the values of τ_j and τ'_j may be slightly changed. A modified inequality can ease the criterion for complexity reduction.

V. AN INNOVATIONS APPROACH TO ML DECODING OF BLOCK CODES

In Section II, we have introduced the notion of innovations for Viterbi decoding of convolutional codes. The derived innovation is closely related to an SST Viterbi decoder which consists of a pre-decoder and a main decoder. The fundamental feature of the SST scheme lies in its structure where an *estimation error* is decoded in the main decoder. Here we see that a similar scheme (i.e., two-stage decoding) can be applied to block codes as well. Then it is reasonable to think that a kind of innovation can also be extracted in connection with ML decoding of block codes [22]. In the following, we will show that this is actually possible.

A. Two-Stage ML Decoding

Let G be a generator matrix for an (n, k) block code, where its rank is assumed to be k. Let H be a corresponding check matrix, where its rank is assumed to be (n - k). Denote by $\mathbf{i} = \{i_j\}_{j=1}^k$ and $\mathbf{i}G = \mathbf{y} = \{y_j\}_{j=1}^n$ a message and the corresponding codeword, respectively. Here consider a twostage ML decoding algorithm.

i) First stage: Let $z = \{z_j\}_{j=1}^n$ be a received data. The harddecision received data is expressed as

$$\boldsymbol{z}^{h} = \boldsymbol{y} + \boldsymbol{e} = \boldsymbol{i}\boldsymbol{G} + \boldsymbol{e}, \tag{92}$$

where $e = \{e_j\}_{j=1}^n$ is an error. The transmitted message is estimated by using the inverse encoder G^{-1} . We have

$$z^h G^{-1} = i + e G^{-1}.$$
 (93)

ii) Second stage: The estimated message is re-encoded by G and then the re-encoded data is added to the original received data z. Let $\boldsymbol{\xi} = \{\xi_j\}_{j=1}^n$ be the result. We have

$$\boldsymbol{\xi}^{h} = \boldsymbol{z}^{h} + (\boldsymbol{z}^{h} \boldsymbol{G}^{-1})\boldsymbol{G}$$
(94)

$$\xi_j = \begin{cases} |z_j|, & \xi_j^h = 0\\ -|z_j|, & \xi_j^h = 1. \end{cases}$$
(95)

At the second stage, ML decoding is performed by regarding where P is an $n \times k$ matrix. Then we have $\boldsymbol{\xi}$ as a received data. Note that $\boldsymbol{\xi}^h$ is expressed as

$$\boldsymbol{\xi}^{h} = (\boldsymbol{i}G + \boldsymbol{e}) + (\boldsymbol{i} + \boldsymbol{e}G^{-1})G$$
$$= (\boldsymbol{e}G^{-1})G + \boldsymbol{e}$$
(96)

$$= uG + e, \qquad (97)$$

where $\boldsymbol{u} \stackrel{\triangle}{=} \boldsymbol{e} G^{-1}$ is a message for the second-stage decoder and uG is the corresponding codeword. Hence, $u = eG^{-1}$ is decoded by the second-stage ML decoder. Finally, two decoder outputs are combined to produce the final decoder output, i.e.,

$$(i + u) + u = i$$

On the other hand, $\boldsymbol{\xi}^h$ has another expression. Since the rank of G is k, G can be decomposed as

$$G = A \times \Gamma \times B, \tag{98}$$

where $A = I_k$, $\Gamma = (I_k \quad O_{k,n-k})$, and B is an $n \times n$ nonsingular matrix. Here the first k rows of B are equal to G and the last (n-k) columns of B^{-1} are equal to H^{T} . As a result, we have

$$I_{n} = B^{-1}B$$

= $(G^{-1} H^{T}) \begin{pmatrix} G \\ (H^{-1})^{T} \end{pmatrix}$
= $G^{-1}G + H^{T}(H^{-1})^{T}.$ (99)

Then

$$\boldsymbol{\xi}^{h} = \boldsymbol{e}(G^{-1}G + I_{n}) = \boldsymbol{e}H^{T}(H^{-1})^{T} = \boldsymbol{\zeta}(H^{-1})^{T}$$
 (100)

is obtained, where $\zeta = z^h H^T = e H^T$ is the syndrome.

In particular, let $G = (I_k \ S)$, where S is a $k \times (n-k)$ matrix. In this case, since

$$(H^{-1})^T = (O_{n-k,k} \quad I_{n-k}),$$

we have

$$\boldsymbol{\xi}^{h} = \boldsymbol{\zeta} (H^{-1})^{T}$$

= $\boldsymbol{\zeta} (O_{n-k,k} \quad I_{n-k})$
= $(O_{1,k}, \boldsymbol{\zeta}).$ (101)

B. Innovations Associated with the Received Data for an ML Decoder

The proposed two-stage ML decoding of block codes can also be discussed from an innovation viewpoint. In fact, the following argument is almost the same as that in Section II-A.

Let

$$\boldsymbol{z}^h = \boldsymbol{i}G + \boldsymbol{e}$$

be the hard-decision received data. By comparison with the linear filtering theory, consider the quantity

$$\begin{aligned} \boldsymbol{r}^{h} &= \boldsymbol{z}^{h} - \boldsymbol{\hat{i}}G \\ &= \boldsymbol{z}^{h} + \boldsymbol{\hat{i}}G, \end{aligned}$$
 (102)

where \hat{i} denotes an estimate of i based on z^h . Suppose that \hat{i} has the form

$$\hat{\boldsymbol{i}} = \boldsymbol{z}^h \boldsymbol{P},\tag{103}$$

 $r^h = z^h + z^h PG$

$$= (iG + e) + (iG + e)PG$$

= $i(G + GPG) + ePG + e.$

Note that if

or

$$GPG = G$$

G + GPG = 0

holds, then r^h is independent of *i*. Here GPG = G implies that P is a generalized inverse [26] of G. Then a right inverse G^{-1} can be taken as P. In this case, r^h is independent of *i* and we have

$$\boldsymbol{r}^h = (\boldsymbol{e}G^{-1})G + \boldsymbol{e} \tag{104}$$

$$= uG + e \tag{105}$$

$$= e(G^{-1}G + I_n), (106)$$

where $\boldsymbol{u} \stackrel{\triangle}{=} \boldsymbol{e} G^{-1}$. We think this quantity corresponds to an innovation in the linear filtering theory. We remark that the right-hand side is just the input to the second-stage decoder in a two-stage ML decoder. Also, note that

$$r^{h}H^{T} = z^{h}H^{T} + z^{h}PGH^{T}$$
$$= z^{h}H^{T} = \zeta$$
(107)

holds irrespective of P, where ζ is the syndrome. Hence, r^h and z^h generate the same syndrome ζ .

On the other hand, r^h has another expression, i.e.,

$$r^{h} = e(G^{-1}G + I_{n})$$

= $eH^{T}(H^{-1})^{T} = \zeta(H^{-1})^{T}.$ (108)

Therefore, with respect to r^h , we have the following:

- 1) $\mathbf{r}^h = \mathbf{e}(G^{-1}G + I_n)$ holds and there is a correspondence between e and r^h in the sense that they generate the same syndrome ζ .
- 2) r^h and z^h generate the same syndrome ζ .

These properties imply that we can regard r^h as the innovation corresponding to z^h . We remark that the variable which represents time (or depth) is not assumed explicitly in block codes. That is, a codeword may not be regarded as a time series. Hence, we may call the extracted quantity the innovation in a weak sense [20].

Moreover, consider the mapping: $z^h \mapsto r^h = z^h (G^{-1}G^+)$ I_n). It is shown that it is not invertible and the innovation \boldsymbol{r}^h corresponding to \boldsymbol{z}^h cannot be further reduced. Proofs are almost the same as those given in Section II-A.

VI. CONCLUSION

In this paper, by comparing the results in the linear filtering theory, we have introduced the notion of innovations for Viterbi decoding of convolutional codes. It has been shown that the newly defined innovations are closely related to the structure of an SST Viterbi decoder. We have also shown that a similar result holds with respect to QLI codes. In this case, we have seen that the innovation-like quantity has a connection

with linear smoothing of the information. Moreover, for a QLI code, we have clarified the relationship between the filtered estimate and the smoothed estimate of the information. We think the obtained results are due to having introduced innovations associated with the received data. With respect to innovations, it is written in [10], [11] as follows:

Consider a complex system. Suppose that we have generated some simpler system composed of mutually independent elements. Also, suppose that for a given time t, the new system has the same information as the original one has by time t. Then the newly generated simpler system is called the innovations. It is not easy to obtain such an ideal system. For typical problems, however, the corresponding innovations have been derived. Obtaining innovations for a given complex system provides a method for the *reduction* of time series or stochastic processes.

In those books, the innovations method is regarded as an essentially important tool for reduction \rightarrow synthesis \rightarrow analysis of a given complex system. In our case, the known SST scheme has been more clarified using innovations. Furthermore, we have shown the proposed innovations approach can be extended to block codes as well. In fact, a kind of innovation has been extracted in connection with ML decoding of block codes.

APPENDIX A PROOF OF LEMMA 3.2

Without loss of generality, for

$$\alpha_1 \stackrel{\triangle}{=} P(e_k^{(1)} \!=\! 0, r_k^{(1)h} \!=\! 1) + P(e_k^{(1)} \!=\! 1, r_k^{(1)h} \!=\! 0),$$

we will show that $0 \le \alpha_1 \le 1/2$. In the following, we omit the delay operator D for simplicity. Let

$$G = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n_0} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n_0} \\ \cdots & \cdots & \cdots & \cdots \\ g_{k_0,1} & g_{k_0,2} & \cdots & g_{k_0,n_0} \end{pmatrix}$$
(A.1)

be the generator matrix. Also, let

$$G^{-1} = \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,k_0} \\ b_{2,1} & b_{2,2} & \dots & b_{2,k_0} \\ \dots & \dots & \dots & \dots \\ b_{n_0,1} & b_{n_0,2} & \dots & b_{n_0,k_0} \end{pmatrix}$$
(A.2)

be a right inverse of G. Then the first column of

$$G^{-1}G + I_{n_0} = \begin{pmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,k_0} \\ b_{2,1} & b_{2,2} & \dots & b_{2,k_0} \\ \dots & \dots & \dots & \dots \\ b_{n_0,1} & b_{n_0,2} & \dots & b_{n_0,k_0} \end{pmatrix} \\ \times \begin{pmatrix} g_{1,1} & g_{1,2} & \dots & g_{1,n_0} \\ g_{2,1} & g_{2,2} & \dots & g_{2,n_0} \\ \dots & \dots & \dots & \dots \\ g_{k_0,1} & g_{k_0,2} & \dots & g_{k_0,n_0} \end{pmatrix} \\ + \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is given by

$$\begin{pmatrix} b_{1,1}g_{1,1}+b_{1,2}g_{2,1}+\dots+b_{1,k_0}g_{k_0,1}+1\\ b_{2,1}g_{1,1}+b_{2,2}g_{2,1}+\dots+b_{2,k_0}g_{k_0,1}\\ \dots\\ b_{n_0,1}g_{1,1}+b_{n_0,2}g_{2,1}+\dots+b_{n_0,k_0}g_{k_0,1} \end{pmatrix}.$$

Hence, it follows that

$$r_{k}^{(1)h} = e_{k}^{(1)}(b_{1,1}g_{1,1} + b_{1,2}g_{2,1} + \dots + b_{1,k_{0}}g_{k_{0},1} + 1) + e_{k}^{(2)}(b_{2,1}g_{1,1} + b_{2,2}g_{2,1} + \dots + b_{2,k_{0}}g_{k_{0},1}) \dots + e_{k}^{(n_{0})}(b_{n_{0},1}g_{1,1} + b_{n_{0},2}g_{2,1} + \dots + b_{n_{0},k_{0}}g_{k_{0},1}) = \tilde{r}_{k}^{(1)h} + e_{k}^{(1)},$$
(A.3)

where

$$\tilde{r}_{k}^{(1)h} \stackrel{\triangle}{=} e_{k}^{(1)}(b_{1,1}g_{1,1}+b_{1,2}g_{2,1}+\dots+b_{1,k_{0}}g_{k_{0},1}) \\ +e_{k}^{(2)}(b_{2,1}g_{1,1}+b_{2,2}g_{2,1}+\dots+b_{2,k_{0}}g_{k_{0},1}) \\ \dots \\ +e_{k}^{(n_{0})}(b_{n_{0},1}g_{1,1}+b_{n_{0},2}g_{2,1}+\dots \\ +b_{n_{0},k_{0}}g_{k_{0},1}).$$
(A.4)

 $\begin{array}{l} \text{Here note the definition of } \alpha_1. \\ 1) \ \ e_k^{(1)} \!=\! 0, r_k^{(1)h} \!=\! 1 \text{: This is equivalent to } e_k^{(1)} \!=\! 0, \tilde{r}_k^{(1)h} \!=\! 1. \\ 2) \ \ e_k^{(1)} \!=\! 1, r_k^{(1)h} \!=\! 0 \text{: This is equivalent to } e_k^{(1)} \!=\! 1, \tilde{r}_k^{(1)h} \!=\! 1. \end{array}$ Hence, we have

$$\begin{aligned} \alpha_1 &= P(e_k^{(1)} = 0, r_k^{(1)h} = 1) + P(e_k^{(1)} = 1, r_k^{(1)h} = 0) \\ &= P(e_k^{(1)} = 0, \tilde{r}_k^{(1)h} = 1) + P(e_k^{(1)} = 1, \tilde{r}_k^{(1)h} = 1) \\ &= P(\tilde{r}_k^{(1)h} = 1). \end{aligned}$$
(A.5)

Since $\tilde{r}_k^{(1)h}$ is the sum of error terms, we can assume that α_1 has the form

$$\alpha_1 = P(e_1 + e_2 + \dots + e_n = 1), \tag{A.6}$$

where errors e_i are mutually independent. In the following, nis assumed to be even without loss of generality.

In order to evaluate the right-hand side, consider the binominal expansion:

$$\begin{pmatrix} (1-\epsilon) + \epsilon \end{pmatrix}^n \\ = {}_n C_0 (1-\epsilon)^n + {}_n C_1 \epsilon (1-\epsilon)^{n-1} + \cdots \\ + {}_n C_{n-1} \epsilon^{n-1} (1-\epsilon) + {}_n C_n \epsilon^n \\ = \begin{pmatrix} {}_n C_0 (1-\epsilon)^n + {}_n C_2 \epsilon^2 (1-\epsilon)^{n-2} + \cdots \\ + {}_n C_{n-2} \epsilon^{n-2} (1-\epsilon)^2 + {}_n C_n \epsilon^n \end{pmatrix} \\ + \begin{pmatrix} {}_n C_1 \epsilon (1-\epsilon)^{n-1} + {}_n C_3 \epsilon^3 (1-\epsilon)^{n-3} + \cdots \\ + {}_n C_{n-1} \epsilon^{n-1} (1-\epsilon) \end{pmatrix} \\ = h(\epsilon) + f(\epsilon),$$
 (A.7)

where

$$h(\epsilon) \stackrel{\Delta}{=} {}_{n}C_{0}(1-\epsilon)^{n} + {}_{n}C_{2}\epsilon^{2}(1-\epsilon)^{n-2} + \cdots + {}_{n}C_{n-2}\epsilon^{n-2}(1-\epsilon)^{2} + {}_{n}C_{n}\epsilon^{n} \qquad (A.8)$$

$$f(\epsilon) \stackrel{\Delta}{=} {}_{n}C_{1}\epsilon(1-\epsilon)^{n-1} + {}_{n}C_{3}\epsilon^{3}(1-\epsilon)^{n-3} + \cdots + {}_{n}C_{n-1}\epsilon^{n-1}(1-\epsilon). \qquad (A.9)$$

Note that $\alpha_1 = f(\epsilon)$. We will show the following:

- 1) f(0) = 0
- 2) f(1/2) = 1/2
- 3) $f(\epsilon)$ is monotone increasing for $0 \le \epsilon \le 1/2$.
- 1) is obvious. Let us show 2). Note that

$$f(1/2) = {}_{n}C_{1}\left(\frac{1}{2}\right)^{n} + {}_{n}C_{3}\left(\frac{1}{2}\right)^{n} + \dots + {}_{n}C_{n-1}\left(\frac{1}{2}\right)^{n}$$
$$= ({}_{n}C_{1} + {}_{n}C_{3} + \dots + {}_{n}C_{n-1})\left(\frac{1}{2}\right)^{n}$$
$$= 2^{n-1} \times \left(\frac{1}{2}\right)^{n} = 1/2, \qquad (A.10)$$

where the equality ${}_{n}C_{1} + {}_{n}C_{3} + \dots + {}_{n}C_{n-1} = 2^{n-1}$ [32] is used.

Finally, we will show 3). Since $h(\epsilon) + f(\epsilon) = 1$,

$$h'(\epsilon) + f'(\epsilon) = 0$$

holds ("'" means differentiation with respect to ϵ). Hence, $f'(\epsilon) \geq 0$ is equivalent to $h'(\epsilon) \leq 0$. Then we will show the latter for $0 \le \epsilon \le 1/2$. From the definition of $h(\epsilon)$, we have

$$\begin{aligned} h'(\epsilon) &= -n(1-\epsilon)^{n-1} + n(n-1)\epsilon(1-\epsilon)^{n-2} \\ &- \frac{n(n-1)(n-2)}{2\times 1} \epsilon^2 (1-\epsilon)^{n-3} \\ &+ \dots + \frac{n(n-1)(n-2)}{2\times 1} \epsilon^{n-3} (1-\epsilon)^2 \\ &- n(n-1)\epsilon^{n-2} (1-\epsilon) + n\epsilon^{n-1} \\ &= (-n) \times \left((1-\epsilon)^{n-1} - (n-1)\epsilon(1-\epsilon)^{n-2} \\ &+ \frac{(n-1)(n-2)}{2\times 1} \epsilon^2 (1-\epsilon)^{n-3} \\ &- \dots - \frac{(n-1)(n-2)}{2\times 1} \epsilon^{n-3} (1-\epsilon)^2 \\ &+ (n-1)\epsilon^{n-2} (1-\epsilon) - \epsilon^{n-1} \right) \\ &= -n((1-\epsilon)-\epsilon)^{n-1} \\ &= -n(1-2\epsilon)^{n-1} \le 0 \ (0 \le \epsilon \le 1/2). \end{aligned}$$
(A.11)

Thus 3) is proved. This completes the proof of the lemma.

APPENDIX B **PROOF OF LEMMA 3.4**

Without loss of generality, for

$$\beta_1 \stackrel{\triangle}{=} P(e_{k-L}^{(1)} = 0, \zeta_k = 1) + P(e_{k-L}^{(1)} = 1, \zeta_k = 0),$$

we will show that $0 \le \beta_1 \le 1/2$. Let

$$G = (g_1, g_2), \ g_1 + g_2 = D^L$$
 (B.12)

be a generator matrix of a QLI code. Since the check matrix is given by $H = (g_2, g_1)$, we have

$$\begin{aligned} \zeta_k &= e_k H^T = (e_k^{(1)}, e_k^{(2)}) \begin{pmatrix} g_2 \\ g_1 \end{pmatrix} \\ &= e_k^{(1)} g_2 + e_k^{(2)} g_1. \end{aligned}$$

First consider the case 1) $e_{k-L}^{(1)} = 0$, $\zeta_k = 1$. Since ζ_k is rewritten as

$$\begin{aligned} \mathbf{x}_{k} &= e_{k}^{(1)}(g_{1} + g_{2}) + e_{k}^{(1)}g_{1} + e_{k}^{(2)}g_{1} \\ &= e_{k-L}^{(1)} + e_{k}^{(1)}g_{1} + e_{k}^{(2)}g_{1}, \end{aligned}$$

ζi

1) is equivalent to $e_{k-L}^{(1)} = 0$, $e_k^{(1)}g_1 + e_k^{(2)}g_1 = 1$. Next, consider the case 2) $e_{k-L}^{(1)} = 1$, $\zeta_k = 0$. We see that this is equivalent to $e_{k-L}^{(1)} = 1$, $e_k^{(1)}g_1 + e_k^{(2)}g_1 = 1$. Hence, we have

$$\beta_{1} = P(e_{k-L}^{(1)} = 0, \zeta_{k} = 1) + P(e_{k-L}^{(1)} = 1, \zeta_{k} = 0)$$

$$= P(e_{k-L}^{(1)} = 0, e_{k}^{(1)}g_{1} + e_{k}^{(2)}g_{1} = 1)$$

$$+ P(e_{k-L}^{(1)} = 1, e_{k}^{(1)}g_{1} + e_{k}^{(2)}g_{1} = 1)$$

$$= P(e_{k}^{(1)}g_{1} + e_{k}^{(2)}g_{1} = 1). \quad (B.13)$$

As in the case of Lemma 3.2, the right-hand side is less than or equal to 1/2 for $0 \le \epsilon \le 1/2$. This proves the lemma.

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