Message Transmission over Classical Quantum Channels with a Jammer with Side Information: Message Transmission Capacity and Resources

Holger Boche Lehrstuhl für Theoretische Informationstechnik, Technische Universität München, Munich, Germany, boche@tum.de Minglai Cai Lehrstuhl für Theoretische Informationstechnik, Technische Universität München, Munich, Germany, minglai.cai@tum.de Ning Cai School of Information Science and Technology, ShanghaiTech University, Shanghai, China, ningcai@shanghaitech.edu.cn

Abstract

In this paper we propose a new model for arbitrarily varying classical-quantum channels. In this model a jammer has side information. We consider two scenarios. In the first scenario the jammer knows the channel input, while in the second scenario the jammer knows both the channel input and the message. The transmitter and receiver share a secret random key with a vanishing key rate. We determine the capacity for both average and maximum error criteria for both scenarios. We also establish the strong converse. We show that all these corresponding capacities are equal, which means that additionally revealing the message to the jammer does not change the capacity.

I. INTRODUCTION

Quantum information theory has developed into a very active field of reseach in the last years and its study provide an enormous amount of potential advantages. Quantum channels differs significantly from communication over classical channels. Quantum communication allow us to exploit possibilities for new applications for communications. To name a few: message transmission, secret message transmission, entanglement transmission, entanglement generation. secure communications over quantum channels is one of the first practical applications of quantum communications. In such systems one usually consider active jamming and passive eavesdropping attacks.

Communication models including a jammer who tries to disturb the legal parties' communication have received a lot of attention in recent years. These publications concentrated on the model of message transmission over an arbitrarily varying channel where a third channel user, the jammer, may change his input in every channel use. This model captures completely all possible jamming attacks and is not restricted to use a repetitive probabilistic strategy. The arbitrarily varying channel was introduced in [9]. In the model of message transmission over arbitrarily varying channels it is understood that the sender and the receiver have to select their coding scheme first. In the conventional model it is assumed that this coding scheme is known by the jammer, and he may choose the most advantaged jamming attacking strategy depending on his knowledge, but the jammer has neither knowledge about the transmitted codeword nor knowledge about the message. Ahlswede showed in [2] the surprising result, that either the deterministic capacity of an arbitrarily varying channel is zero or it is equal to its random correlated capacity (Ahlswede dichotomy). For this dichotomy it is essential that the average error criterion was used. After that discovery, it remained an open question exactly when the deterministic capacity is nonzero. In [17] Ericson gave a sufficient condition for that, and in [16] Csiszár and Narayan proved that this is condition is also necessary. Ahlswede dichotomy demonstrates the importance of resources (shared randomness) in a very clear form. It is required that both sender and receiver have access to a perfect copy of the outcome of a random experiment, and thus we should assume an additional perfect channel. The legal channel users' knowledge about the shared randomness is very helpful for message transmission through an arbitrarily varying channel (random correlated capacity), where we assume that the resource is only known by the legal channel users, since otherwise it will be completely useless (cf. [12]).

In this work we consider classical quantum channels, i.e., the sender's inputs are classical symbols and the receiver's outputs are quantum systems. The capacity of classical-quantum channels under average error criterion has been determined in [19], [23], and [24]. The capacity of arbitrarily varying classical-quantum channels has been delivered in [5]. An alternative proof of [5]'s result and a proof of the strong converse have been given in [7]. In [4] Ahlswede dichotomy for the arbitrarily varying classical-quantum channels was established, and a sufficient and necessary condition for the zero deterministic capacity has been given. In [13] a simplification of this condition was delivered. See also [20] and [21] for a classical quantum channel model with a benevolent third channel user instead of with a jammer. These results are basis tools for secure communication over arbitrarily varying wiretap channel is a channel with both a jammer and an eavesdropper. Classical arbitrarily varying wiretap channels have been studied extensively in the context of classical information theory.

The secrecy capacity of arbitrarily varying wiretap classical quantum channels has been determined in [12].

As already mentioned the message transmission capacity of an arbitrarily varying channel depends on the demanded error criterion. The deterministic capacities of classical arbitrarily varying channel under maximal error criterion and under the average error criterion are in general, not equal. The deterministic capacity formula of classical arbitrarily varying channels under average error criterion is already well studied in the context of classical information theory. The deterministic capacity formula of classical arbitrarily varying channels under maximal error criterion is still an open problem. It has been shown by Ahlswede in [1] that the capacity under maximal error criterion of certain arbitrarily varying channels can be equal to the zero-error capacity of related discrete memoryless channels. Furthermore the random correlated capacities of arbitrarily varying quantum to quantum channels under maximal error criterion and under the average error criterion are equal. Interestingly, [13] shows that the deterministic capacities of arbitrarily varying quantum to quantum channels, i.e., quantum encoding is very powerful. By the above facts there is no Ahlswede dichotomy for arbitrarily varying channels under maximal error criterion: It may occur that the deterministic capacity of a classical arbitrarily varying channel under maximal error criterion is not zero, but on the other hand, unequal to its random correlated capacity. We will provide a example in Section III.

In all the above mentioned works it is assumed that the jammer knows the coding scheme, but has neither side information about the codeword nor side information about the message of the legal transmitters. In many applications, especially for secure communications, it is too optimistic to assume this. Thus in this paper we want to consider two scenarios, where the jammer has side information: In the first one the jammer knows both coding scheme and input codeword. In the second one the jammer knows additionally the message (cf. Figure 1 and 2). The jammer can make use of this knowledge in each scenario to advance his attacking strategy. We require that information transmission can be guaranteed even in the worst case, when the jammer chooses the most advantageous attacking strategy according to his knowledge. For classical arbitrarily varying channels this was first considered by [22]. In this paper we extend this result to arbitrarily varying classical-quantum channels, where we use techniques different to these used in [22] (cf. Section IV). In this work we consider for both scenarios the random correlated capacities under average and maximal error criteria. Detailed descriptions for both scenarios are given in Section II. In Section III the message transmission capacities for both scenarios and both error criteria are completely characterized. In Section IV, Section V, and Section VI we deliver proofs for the capacities results for both scenarios and both error criteria. A vanishing rate of the key is sufficient for our codes since the resource we use here is only of polynomial size of the code length (cf. Remark 2, and also [13] and [11] for a discussion about the difference between various forms of shared randomness).

II. PROBLEM FORMULATION

A: Basic notations

Throughout the paper random variables will be denoted by capital letters e. g., S, X, Y, and their realizations (or values) and domains (or alphabets) will be denoted by corresponding lower case letters e. g., s, x, y, and script

letters e.g., S, X, Y, respectively. Random sequences will be denoted a by capital bold-face letters, whose lengths are understood by the context, e. g., $\mathbf{S} = (S_1, S_2, \dots, S_n)$ and $\mathbf{X} = (X_1, X_2, \dots, X_n)$, and deterministic sequences are written as lower case bold-face letters e. g., $\mathbf{s} = (s_1, s_2, \dots, s_n), \mathbf{x} = (x_1, x_2, \dots, x_n)$.

 P_X is distribution of random variable X. Joint distributions and conditional distributions of random variables X and S will be written as P_{SX} , etc and $P_{S|X}$ etc, respectively and P_{XS}^n and $P_{S|X}^n$ are their product distributions i. e., $P_{XS}^n(\mathbf{x}, \mathbf{s}) := \prod_{t=1}^n P_{XS}(x_t, s_t)$, and $P_{S|X}^n(\mathbf{s}|\mathbf{x}) := \prod_{t=1}^n P_{S|X}(s_t|x_t)$. Moreover $\mathcal{T}_X^n, \mathcal{T}_{XS}^n$ and $\mathcal{T}_{S|X}^n(\mathbf{x})$ are sets of (strongly) typical sequences of the type P_X , joint type P_{XS} and conditional type $P_{S|X}$, respectively. The cardinality of a set \mathcal{X} will be denoted by $|\mathcal{X}|$. For a positive integer L, $[L] := \{1, 2, \dots, L\}$. "Q is a classical channel, or a conditional probability distribution, from set \mathcal{X} to set \mathcal{Y} " is abbreviated to " $Q : \mathcal{X} \to \mathcal{Y}$ ". "Random variables X, Y and Z form a Markov chain" is abbreviated to " $X \leftrightarrow Y \leftrightarrow Z$ ". \mathbb{E} will standard for the operator of mathematical expectation.

Throughout the paper dimensions of all Hilbert spaces are finite, and the identity operator in a Hilbert space \mathcal{H} is denoted by $\mathbb{I}_{\mathcal{H}}$.

Throughout the paper the base(s) of logarithm is 2. For a discrete random variable X on a finite set X and a discrete random variable Y on a finite set Y, we denote the Shannon entropy of X by $H(X) = -\sum_{x \in \mathcal{X}} p_x(x) \log p_x(x)$ and the mutual information between X and Y by $I(X;Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{x,y}(x,y) \log \left(\frac{p_{x,y}(x,y)}{p_x(x)p_y(y)}\right)$. Here $p_{x,y}$ is the joint probability distribution function of X and Y, and p_x and p_y are the marginal probability distribution functions of X and Y respectively.

Let \mathfrak{P} and \mathfrak{Q} be quantum systems. We denote the Hilbert space of \mathfrak{P} and \mathfrak{Q} by $G^{\mathfrak{P}}$ and $G^{\mathfrak{Q}}$, respectively. Let $\phi^{\mathfrak{PQ}}$ be a bipartite quantum state in $\mathcal{S}(G^{\mathfrak{PQ}})$. We denote the partial trace over $G^{\mathfrak{P}}$ by

$$\mathrm{tr}_{\mathfrak{P}}(\phi^{\mathfrak{PQ}}) := \sum_{l} \langle l|_{\mathfrak{P}} \phi^{\mathfrak{PQ}} |l \rangle_{\mathfrak{P}} \; ,$$

where $\{|l\rangle_{\mathfrak{P}}: l\}$ is an orthonormal basis of $G^{\mathfrak{P}}$. We denote the conditional entropy by

$$S(\mathfrak{P} \mid \mathfrak{Q})_{\rho} := S(\phi^{\mathfrak{PQ}}) - S(\phi^{\mathfrak{Q}})$$
.

Here $\phi^{\mathfrak{Q}} = \operatorname{tr}_{\mathfrak{P}}(\phi^{\mathfrak{PQ}}).$

For a finite-dimensional complex Hilbert space \mathcal{H} , we denote the (convex) set of density operators on \mathcal{H} by

$$\mathcal{S}(\mathcal{H}) := \{ \rho \in \mathcal{L}(\mathcal{H}) : \rho \text{ is Hermitian, } \rho \ge 0_{\mathcal{H}} \text{ , } \operatorname{tr}(\rho) = 1 \} \text{ ,}$$

where $\mathcal{L}(\mathcal{H})$ is the set of linear operators on \mathcal{H} , and $0_{\mathcal{H}}$ is the null matrix on \mathcal{H} . Note that any operator in $\mathcal{S}(\mathcal{H})$ is bounded.

For finite-dimensional complex Hilbert spaces \mathcal{H} and \mathcal{H}' a **quantum channel** $N: S(\mathcal{H}) \to S(\mathcal{H}'), S(\mathcal{H}) \ni \rho \to N(\rho) \in S(\mathcal{H}')$ is represented by a completely positive trace-preserving map which accepts input quantum states in $S(\mathcal{H})$ and produces output quantum states in $S(\mathcal{H}')$.

B: Code definitions

If the sender wants to transmit a classical message of a finite set X to the receiver using a quantum channel N, his encoding procedure will include a classical-to-quantum encoder to prepare a quantum message state $\rho \in S(\mathcal{H})$ suitable as an input for the channel. If the sender's encoding is restricted to transmit an indexed finite set of quantum states $\{\rho_x : x \in \mathcal{X}\} \subset \mathcal{S}(\mathcal{H})$, then we can consider the choice of the signal quantum states ρ_x as a component of the channel. Thus, we obtain a channel $\sigma_x := N(\rho_x)$ with classical inputs $x \in \mathcal{X}$ and quantum outputs, which we call a classical-quantum channel. This is a map $\mathbf{N}: \mathcal{X} \to \mathcal{S}(\mathcal{H}'), \ \mathcal{X} \ni x \to \mathcal{N}(x) \in \mathcal{S}(\mathcal{H}')$ which is represented by the set of $|\mathcal{X}|$ possible output quantum states $\{\sigma_x = \mathbf{N}(x) := N(\rho_x) : x \in \mathcal{X}\} \subset \mathcal{S}(\mathcal{H}')$, meaning that each classical input of $x \in \mathcal{X}$ leads to a distinct quantum output $\sigma_x \in \mathcal{S}(\mathcal{H}')$. In view of this, we have the following definition.

Definition 1: Let \mathcal{H} be a finite-dimensional complex Hilbert space. A **classical-quantum channel** is a mapping $W : \mathcal{X} \to \mathcal{S}(\mathcal{H})$, specified by a set of quantum states $\{\rho(x), x \in \mathcal{X}\} \subset \mathcal{S}(\mathcal{H})$, indexed by "input letters" x in a finite set \mathcal{X} . \mathcal{X} and \mathcal{H} are called input alphabet and output space respectively. We define the n-th extension of classical-quantum channel W as follows. The channel outputs a quantum state $\rho^{\otimes n}(\mathbf{x}) := \rho(x_1) \otimes \rho(x_2) \otimes \ldots, \otimes \rho(x_n)$, in the n-th tensor power $\mathcal{H}^{\otimes n}$ of the output space \mathcal{H} , when an input codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ of length n is input into the channel.

Let $V: \mathcal{X} \to \mathcal{S}(\mathcal{H})$ be a classical-quantum channel. For $P \in P(\mathcal{X})$, the conditional entropy of the channel for V with input distribution P is denoted by

$$S(\mathbb{V}|P) := \sum_{x \in \mathcal{X}} P(x) S(\mathbb{V}(x)) \; .$$

Let $\Phi := \{\rho_x : x \in \mathcal{X}\}$ be a be a classical-quantum channel, i.e., a set of quantum states labeled by elements of \mathcal{X} . For a probability distribution Q on \mathcal{X} , the Holevo χ quantity is defined as

$$\chi(Q;\Phi) := S\left(\sum_{x \in \mathbf{A}} Q(x)\rho_x\right) - \sum_{x \in \mathbf{A}} Q(x)S\left(\rho_x\right) \ .$$

For a probability distribution P on a finite set X and a positive constant δ , we denote the set of typical sequences by

$$\mathcal{T}_{P,\delta}^{n} := \left\{ x^{n} \in \mathcal{X}^{n} : \left| \frac{1}{n} N(x' \mid x^{n}) - P(x') \right| \le \frac{\delta}{|\mathcal{X}|} \forall x' \in \mathcal{X} \right\} ,$$

where $N(x' \mid x^n)$ is the number of occurrences of the symbol x' in the sequence x^n .

Let \mathcal{H} be a finite-dimensional complex Hilbert space. Let $n \in \mathbb{N}$ and $\alpha > 0$. We suppose $\rho \in S(\mathcal{H})$ has the spectral decomposition $\rho = \sum_{x} P(x)|x\rangle\langle x|$, its α -typical subspace is the subspace spanned by $\{|x^n\rangle, x^n \in \mathcal{T}_{P,\alpha}^n\}$, where $|x^n\rangle := \bigotimes_{i=1}^n |x_i\rangle$. The orthogonal subspace projector which projected onto the typical subspace is

$$\Pi_{\rho,\alpha} = \sum_{x^n \in \mathcal{T}_{P,\alpha}^n} |x^n\rangle \langle x^n| \; .$$

Similarly, let \mathcal{X} be a finite set, and G be a finite-dimensional complex Hilbert space. Let $\mathbb{V}: \mathcal{X} \to \mathcal{S}(\mathcal{H})$ be a classical-quantum channel. For $x \in \mathcal{X}$, suppose $\mathbb{V}(x)$ has the spectral decomposition $\mathbb{V}(x) = \sum_{j} V(j|x)|j\rangle\langle j|$ for a stochastic matrix $V(\cdot|\cdot)$. The α -conditional typical subspace of \mathbb{V} for a typical sequence x^n is the subspace spanned by $\left\{\bigotimes_{x\in\mathcal{X}} |j^{\mathbb{I}_x}\rangle, j^{\mathbb{I}_x} \in \mathcal{T}_{V(\cdot|x),\delta}^{\mathbb{I}_x}\right\}$. Here $\mathbb{I}_x := \{i \in \{1, \cdots, n\} : x_i = x\}$ is an indicator set that selects the indices i in the sequence $x^n = (x_1, \cdots, x_n)$ for which the i-th symbol x_i is equal to $x \in \mathcal{X}$. The subspace is often referred to as the α -conditional typical subspace of the state $\mathbb{V}^{\otimes n}(x^n)$. The orthogonal subspace projector which projected onto it is defined as

$$\Pi_{\mathbf{V},\alpha}(x^n) = \bigotimes_{x \in \mathcal{X}} \sum_{j^{\mathbf{I}_x} \in \mathcal{T}_{\mathbf{V}(\cdot \mid x^n),\alpha}^{\mathbf{I}_x}} |j^{\mathbf{I}_x}\rangle \langle j^{\mathbf{I}_x}| \ .$$

The typical subspace has following properties:

For $\sigma \in S(\mathcal{H}^{\otimes n})$ and $\alpha > 0$ there are positive constants $\beta(\alpha)$, $\gamma(\alpha)$, and $\delta(\alpha)$, depending on α and tending to zero when $\alpha \to 0$ such that

$$\operatorname{tr}\left(\sigma\Pi_{\sigma,\alpha}\right) > 1 - 2^{-n\beta(\alpha)} , \qquad (1)$$

$$2^{n(S(\sigma)-\delta(\alpha))} \le \operatorname{tr}\left(\Pi_{\sigma,\alpha}\right) \le 2^{n(S(\sigma)+\delta(\alpha))} , \qquad (2)$$

$$2^{-n(S(\sigma)+\gamma(\alpha))}\Pi_{\sigma,\alpha} \le \Pi_{\sigma,\alpha}\sigma\Pi_{\sigma,\alpha} \le 2^{-n(S(\sigma)-\gamma(\alpha))}\Pi_{\sigma,\alpha} .$$
(3)

For $a^n \in \mathcal{T}_{P,\alpha}^n$ there are positive constants $\beta(\alpha)'$, $\gamma(\alpha)'$, and $\delta(\alpha)'$, depending on α and tending to zero when $\alpha \to 0$ such that

$$\operatorname{tr}\left(\mathbb{V}^{\otimes n}(x^{n})\Pi_{\mathbb{V},\alpha}(x^{n})\right) > 1 - 2^{-n\beta(\alpha)'},\tag{4}$$

$$2^{-n(S(\mathbf{V}|P)+\gamma(\alpha)')}\Pi_{\mathbf{V},\alpha}(x^{n}) \leq \Pi_{\mathbf{V},\alpha}(x^{n})\mathbf{V}^{\otimes n}(x^{n})\Pi_{\mathbf{V},\alpha}(x^{n})$$
$$\leq 2^{-n(S(\mathbf{V}|P)-\gamma(\alpha)')}\Pi_{\mathbf{V},\alpha}(x^{n}) , \qquad (5)$$

$$2^{n(S(\mathfrak{V}|P)-\delta(\alpha)')} \le \operatorname{tr}\left(\Pi_{\mathfrak{V},\alpha}(x^n)\right) \le 2^{n(S(\mathfrak{V}|P)+\delta(\alpha)')} .$$
(6)

For the classical-quantum channel $\mathbb{V} : \mathcal{X} \to \mathcal{S}(\mathcal{H})$ and a probability distribution P on \mathcal{X} we define a quantum state $P\mathbb{V} := \sum_{x} P(x)\mathbb{V}(x)$ on $\mathcal{S}(\mathcal{H})$. For $\alpha > 0$ we define an orthogonal subspace projector $\Pi_{P\mathbb{V},\alpha}$ fulfilling (1), (2), and (3). Let $x^n \in \mathcal{T}_{P,\alpha}^n$. For $\Pi_{P\mathbb{V},\alpha}$ there is a positive constant $\beta(\alpha)''$ such that following inequality holds:

$$\operatorname{tr}\left(\rho^{\otimes n}(x^{n})\cdot\Pi_{P\mathbf{V},\alpha}\right) \ge 1 - 2^{-n\beta(\alpha)^{\prime\prime}} . \tag{7}$$

We give here a sketch of the proof. For a detailed proof please see [26].

proof

(1) holds because tr $(\sigma \Pi_{\sigma,\alpha}) = \text{tr} (\Pi_{\sigma,\alpha} \sigma \Pi_{\sigma,\alpha}) = P^n(\mathcal{T}_{P,\alpha}^n)$. (2) holds because tr $(\Pi_{\sigma,\alpha}) = |\mathcal{T}_{P,\alpha}^n|$. (3) holds because $2^{-n(S(\sigma)+\gamma(\alpha))} \leq P^n(x^n) \leq 2^{-n(S(\sigma)-\gamma(\alpha))}$ for $x \in \mathcal{T}_{P,\alpha}^n$ and a positive $\gamma(\alpha)$. (4), (5), and (6) can be obtained in a similar way. (7) follows from the permutation-invariance of $\Pi_{PV,\alpha}$.

Definition 2:

A arbitrarily varying classical-quantum channel (AVCQC) W is specified by a set $\{\{\rho(x,s), x \in \mathcal{X}\}, s \in \mathcal{S}\}$ of classical quantum channels with a common input alphabet \mathcal{X} and output space \mathcal{H} , which are indexed by elements s in a finite set \mathcal{S} . Elements $s \in \mathcal{S}$ usually are called the states of the channel. W outputs a quantum state

$$\rho^{\otimes n}(\mathbf{x}, \mathbf{s}) := \rho(x_1, s_1) \otimes \rho(x_2, s_2) \otimes \dots, \otimes \rho(x_n, s_n),$$
(8)

if an input codeword $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is input into the channel, and the channel is governed by a state sequence $\mathbf{s} = (s_1, s_2, \dots, s_n)$, while the state varies from symbol to symbol in an arbitrary manner.

We assume that the channel state s is in control of the jammer. Without loss of generality we also assume that the jammer always chooses the most advantageous attacking strategy according to his knowledge.

Definition 3: A code $\gamma := (\mathcal{U}, \{\mathcal{D}(i), i \in \mathcal{I}\})$ of length n for a classical quantum channel consists of its code book \mathcal{U} and decoding measurement $\{\mathcal{D}(i), i \in \mathcal{I}\}$, where the code book $\mathcal{U} := \{\mathbf{u}(i), i \in \mathcal{I}\}$ is a subset of input alphabet \mathcal{X}^n indexed by messages i in the message set \mathcal{I} , and the decoding measurement $\{\mathcal{D}(i), i \in \mathcal{I}\}$ is a quantum measurement in the output space $\mathcal{H}^{\otimes n}$ that is, $\mathcal{D}(i) \geq 0$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}}, \mathcal{D}(i) = \mathbb{I}_{\mathcal{H}}$.

Definition 4:

A random correlated code Γ for a AVCQC W is a uniformly distributed random variable taking values in a set of codes $\{(\mathcal{U}(k), \{\mathcal{D}(j,k), j \in \mathcal{J}\}), k \in \mathcal{K}\}$ with a common message set \mathcal{J} , where $\mathcal{U}(k) = \{\mathbf{u}(j,k), j \in \mathcal{J}\}$ and $\{\mathcal{D}(j,k), j \in \mathcal{J}\}$ are the code book and decoding measurement of the kth code in the set respectively. $|\mathcal{K}|$ is called the key size.

Remark 1: Usually a random correlated code is defined as any random variable taking values in a set of codes. Here we restrict ourselves to uniformly distributed random variables, since it is sufficiently for our purpose (cf. [25]).

C: Capacity definitions and basic relations

One of the fundamental task of quantum Shannon theory is to characterize performance measurements maximizing the efficiency of quantum communication. Hence we introduce here capacity for message transmission and simple relations between different quantities.

As already mentioned this work concentrates on message transmission over classical quantum channels with a jammer with additonal side information. It is clear that this side information are encoded by the same coding scheme, which is known by the jammer by assumption, as the legal transmitters use for their communication. We assume that the jammer chooses the most advantageous attacking strategy according to his side information. We now distinguish two scenarios depending on the jammer's knowledge (cf. Figure 1 and 2). We consider for each scenario both average and maximum error criteria.



Fig. 1. The jammer knows both the coding scheme and the input codeword (scenario 1)

Scenario 1

In this scenario jammer knows coding scheme and input codeword but not the message to be sent.

Definition 5: By assuming that the random message J is uniformly distributed, we define the average probability of error by

$$p_{a}(\Gamma)$$

$$= \max_{\mathbf{s}} \mathbb{E}tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s}(\mathbf{u}(J,K)))(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(J,K))]$$

$$= \max_{\mathbf{s}} \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{K}} Pr\{K = k\}$$

$$tr[\rho^{\otimes n}(\mathbf{u}(i,k),\mathbf{s}(\mathbf{u}(j,k)))(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j,k))].$$
(9)

This can be also rewritten as

$$p_{a}(\Gamma)$$

$$= \sum_{\mathbf{x}} Pr\{\mathbf{u}(J,K) = \mathbf{x}\} \max_{\mathbf{s}\in\mathcal{S}^{n}} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s}) | (\mathbb{I}_{\mathcal{H}} - \mathcal{D}(J,K))] | \mathbf{u}(J,K) = \mathbf{x}\}.$$
(10)

The maximum probability of error is defined as

$$p_m(\Gamma)$$

= $\max_{j \in \mathcal{J}} \max_{\mathbf{s}} \mathbb{E}tr[\rho^{\otimes n}(\mathbf{u}(j,K),\mathbf{s}(\mathbf{u}(j,K)))(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j,K))].$ (11)

Definition 6: A non-negative number R is an achievable rate for the arbitrarily varying classical-quantum channel W under random correlated coding in scenario 1 under the average error criterion and under the maximal

error criterion if for every $\delta > 0$ and $\epsilon > 0$, if n is sufficiently large, there is an random correlated code Γ of length n such that $\frac{\log |\mathcal{J}|}{n} > R - \delta$, and $p_a(\Gamma) < \epsilon$ and $p_m(\Gamma) < \epsilon$, respectively.

The supremum on achievable rate under random correlated coding of W under the average error criterion and under the maximal error criterion in scenario 1 is called the random correlated capacity of W under the average error criterion and under the maximal error criterion in scenario 1, denoted by $C^*(W)$ and $C^*_m(W)$, respectively.

Definition 7: Let $\epsilon \in [0, 1)$. A non-negative number R is an ϵ - achievable rate for the arbitrarily varying classical-quantum channel W under random correlated coding in scenario 1 under the average error criterion and under the maximal error criterion if for every $\delta > 0$ if n is sufficiently large, there is an random correlated code Γ of length n such that $\frac{\log |\mathcal{J}|}{n} > R - \delta$, and $p_a(\Gamma) < \epsilon$ and $p_m(\Gamma) < \epsilon$, respectively.

The supremum on achievable rate under random correlated coding of W under the average error criterion and under the maximal error criterion in scenario 1 is called the random correlated ϵ - capacity of W under the average error criterion and under the maximal error criterion in scenario 1, denoted by $C^*(W, \epsilon)$ and $C^*_m(W, \epsilon)$, respectively.

By (10) it is clear, that to employ a "mixed strategy" for the jammer may not do better than only to use deterministic strategy. That is, the jammer may not enlarge the average probability of error, if he randomly chooses a state sequence with any conditional distribution $Q : \mathcal{X}^n \to \mathcal{S}^n$, according to the input codeword, instead chooses a fixed state sequence with the best deterministic strategy, because

$$\sum_{\mathbf{s}\in\mathcal{S}^n} Q(\mathbf{s}|\mathbf{x}) \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))]|\mathbf{u}(J,K)=\mathbf{x}\}$$

$$\leq \max_{\mathbf{s}\in\mathcal{S}^n} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))]|\mathbf{u}(J,K)=\mathbf{x}\}$$

for all Q and all \mathbf{x} (with $Pr{\mathbf{u}(J, K) = \mathbf{x}} > 0$).



Fig. 2. The jammer knows coding scheme, input codeword, and message (scenario 2)

Scenario 2

Now the jammer has more benefit and he can choose the state sequence according to both input codeword and message which sender wants to transmit, or a function $\psi : \bigcup_{k \in \mathcal{K}} \mathcal{U}(k) \times \mathcal{J} \to \mathcal{S}^n$.

Definition 8: We define the average probability of error in scenario 2 by

$$p_a^{**}(\Gamma) = \max_{\psi} \sum_{j \in \mathcal{J}} \frac{1}{|\mathcal{J}|} \mathbb{E}tr[\rho^{\otimes n} \\ (\mathbf{u}(j, K), \psi(\mathbf{u}(j, K), j))(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j, K))].$$
(12)

The maximum probability of error in scenario 2 is defined as

$$p_m^{**}(\Gamma) = \max_{j \in \mathcal{J}} \max_{\psi} \mathbb{E}tr[\rho^{\otimes n} \\ (\mathbf{u}(j,K), \psi(\mathbf{u}(j,K),j))(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j,K))].$$
(13)

Definition 9: A non-negative number R is an achievable rate for the arbitrarily varying classical-quantum channel \mathcal{W} under random correlated coding in scenario 2 under the average error criterion and under the maximal error criterion if for every $\delta > 0$ and $\epsilon > 0$, if n is sufficiently large, there is an random correlated code Γ of length n such that $\frac{\log |\mathcal{J}|}{n} > R - \delta$, and $p_a^{**}(\Gamma) < \epsilon$ and $p_m^{**}(\Gamma) < \epsilon$, respectively.

The supremum on achievable rate under random correlated coding of W under the average error criterion and under the maximal error criterion in scenario 2 is called the random correlated capacity of W under the average error criterion and under the maximal error criterion in scenario 2, denoted by $C^{**}(W)$ and $C^{**}_m(W)$, respectively.

Definition 10: Let $\epsilon \in [0, 1)$. A non-negative number R is an ϵ - achievable rate for the arbitrarily varying classical-quantum channel W under random correlated coding in scenario 2 under the average error criterion and under the maximal error criterion if for every $\delta > 0$, if n is sufficiently large, there is an random correlated code Γ of length n such that $\frac{\log |\mathcal{J}|}{n} > R - \delta$, and $p_a^{**}(\Gamma) < \epsilon$ and $p_m^{**}(\Gamma) < \epsilon$, respectively.

The supremum on ϵ - achievable rate under random correlated coding of W under the average error criterion and under the maximal error criterion in scenario 2 is called the random correlated ϵ - capacity of W under the average error criterion and under the maximal error criterion in scenario 2, denoted by $C^{**}(W, \epsilon)$ and $C^{**}_m(W, \epsilon)$, respectively.

Obviously

$$C^{**}(\mathcal{W}) \le C^*(\mathcal{W}).$$

It is easy to show that

$$C_m^*(\mathcal{W}) = C_m^{**}(\mathcal{W}),$$

because both (11) and (13) are equal to

$$\max_{j} \sum_{\mathbf{x}} \Pr\{\mathbf{u}(j,K) = \mathbf{x}\} \max_{\mathbf{s} \in \mathcal{S}^{n}} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(j,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j,K))] | \mathbf{u}(j,K) = \mathbf{x}\}.$$

Moreover, the average probability of error (12) can rewritten as

$$\sum_{j\in\mathcal{J}}\frac{1}{|\mathcal{J}|}\sum_{\mathbf{x}}\Pr\{\mathbf{u}(j,K)=\mathbf{x}\}\max_{\mathbf{s}\in\mathcal{S}^n}\mathbb{E}tr[\rho^{\otimes n}(\mathbf{u}(j,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))|\mathbf{u}(j,K)=\mathbf{x}].$$

11

Thus, in the standard way, by Markov inequality one may conclude that the message set \mathcal{J} of any code with average probability of error λ in scenario 2 contains a subset \mathcal{J}' such that $|\mathcal{J}'| \geq \frac{|\mathcal{J}|}{2}$ and

$$\max_{j} \sum_{\mathbf{x}} \Pr\{\mathbf{u}(j,K) = \mathbf{x}\} \max_{\mathbf{s} \in S^{n}} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(j,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(j,K))] | \mathbf{u}(j,K) = \mathbf{x}\} \le 2\lambda,$$

for all $j \in \mathcal{J}'$. That is,

$$C^{**}(\mathcal{W}) = C_m^{**}(\mathcal{W}),$$

thus

$$C^*(\mathcal{W}) \ge C^*_m(\mathcal{W}) = C^{**}(\mathcal{W}) = C^{**}_m(\mathcal{W}).$$
(14)

III. MAIN RESULTS

For a given AVCQC $\mathcal{W} = \{\{\rho(x,s), x \in \mathcal{X}\}, s \in \mathcal{S}\}$ with set of state \mathcal{S} , let

$$\bar{\bar{\mathcal{W}}} := \{\{\bar{\bar{\rho}}_Q(x) := \sum_s Q(s|x)\rho(x,s), x \in \mathcal{X}\}: \text{ for all } Q: \mathcal{X} \to \mathcal{S}\}.$$
(15)

Theorem 1: (Direct Coding Theorem for Scenario 1) Given a AVCQC $\mathcal{W} = \{\{\rho(x,s), x \in \mathcal{X}\}, s \in \mathcal{S}\}$ and a type P_X , for all $\epsilon > 0$, and $\lambda > 0$, there is a b > 0, such that for all sufficiently large n, there exists a code Γ of length n with a rate larger than $\min_{\bar{\rho}(\cdot)\in \bar{W}} \chi(P_X, \bar{\rho}(\cdot)) - \epsilon$, average probability of error in scenario 1 smaller than λ , and key size of the random correlated code smaller then bn^2 . Moreover codewords of code books in support set of the random correlated code Γ , all are in \mathcal{T}_X^n .

Remark 2: In particular, there is a constant a > 0 (depending only on the AVCQC) such that for any sequence of positive real numbers $\{\lambda_n\}$, lower bounded by $\lambda_n \ge 2^{-n\alpha}$ for an $\alpha > 0$ (depending on ϵ), with $\lim_{n\to\infty} \lambda_n = 0$, there exists a sequence of random correlated codes with a rate larger than $\min_{\bar{\rho}(\cdot)\in \bar{W}} \chi(P_X, \bar{\bar{\rho}}(\cdot)) - \epsilon$, average probability of error smaller than λ_n and the amount of common randomness upper bounded by $\frac{an^2}{\lambda_n^2}$.

Theorem 2: (Strong Converse Coding Theorem for Scenario 1)

For every $\epsilon \in [0,1)$ we have

$$C^*(\mathcal{W},\epsilon) \le \max_{P} \min_{\bar{\rho}(\cdot)\in\bar{\mathcal{W}}} \chi(P,\bar{\rho}(\cdot)).$$
(16)

Let

$$\bar{\mathcal{W}} := \{\{\bar{\rho}_P(x) := \sum_s P(s)\rho(x,s), x \in \mathcal{X}\}: \text{ for all probability distributions } P \text{ on } \mathcal{S}\}.$$
(17)

Then obviously

$$\max_{P} \min_{\bar{\rho}(\cdot)\in\bar{\mathcal{W}}} \chi(P,\bar{\rho}(\cdot)) \le \max_{P} \min_{\bar{\rho}(\cdot)\in\bar{\mathcal{W}}} \chi(P,\bar{\rho}(\cdot)).$$
(18)

The following Example 1 shows that the inequality is strict already in classical arbitrarily varying channels, as a special case of AVCQC. It was shown the random correlated capacities of a AVCQC under maximum error probability and average error probability when the jammer does not know the channel input are the same and both equal to $\max_{P} \min_{\bar{\rho}(\cdot) \in \bar{W}} \chi(P, \bar{\rho}(\cdot))$. Recalling that to employ the criterion of average probability of error corresponds to scenario 1 and the criterion of maximum probability of error corresponds to scenario 2, we conclude that knowing the message to be sent may not help a jammer who only know the coding scheme, for reduction the capacity, if random correlated codes are allowed to be used by the communicators side.

Example 1: Let $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and $\mathcal{S} = \{s_0, s_1\}$. We define a classical arbitrarily varying channel \mathcal{W} represented by the transmission matrices

$$\left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right) , \left(\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{array}\right)$$

The jammer may choose Q by setting $Q(s_0|0) = Q(s_1|0) = \frac{1}{2}$, $Q(s_0|1) = 1$ and $Q(s_1|1) = 0$. Since

$$\frac{1}{2} \cdot \left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \end{array}\right) + \frac{1}{2} \cdot \left(\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \end{array}\right) = 1 \cdot \left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right) + 0 \cdot (0, 1),$$

we have

 $C^*(\mathcal{W}) = 0.$

But when the jammer has no knowledge about the channel input, we can always achieve positive capacity, since zero capacity means there is a $a \in (0, 1)$ such that

$$a \cdot \left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{array}\right) + (1-a) \cdot \left(\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \\ 0 & 1 \end{array}\right)$$

has rank 1, which can only be true when

$$a \cdot \left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \end{array}\right) + (1-a) \cdot \left(\begin{array}{cc} \frac{1}{4} & \frac{3}{4} \end{array}\right) = a \cdot \left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right) + (1-a) \cdot (0,1).$$

But there is clearly no such $a \in (0, 1)$ since else we would have

$$\begin{aligned} \frac{3}{4}a + \frac{1}{4}(1-a) &= \frac{1}{2}a\\ \Rightarrow \frac{1}{4} &= \frac{1}{2}a + \frac{1}{4}a - \frac{3}{4}a\\ \Rightarrow \frac{1}{4}. \end{aligned}$$

Thus when the jammer has no knowledge about the channel input, this channel has a positive deterministic capacity.

Example 1 shows that the jammer really benefits from his knowledge about the channel input.

The following example was first presented at the IEEE International Symposium on Information Theory 2010 in a talk by N. Cai, T. Chen, and A, Grant.

Example 2: Let $\mathcal{X} = \mathcal{Y} = \{a, 0, 1, 2\}$ and $\mathcal{S} = \{s_0, s_1\}$. We define a classical arbitrarily varying channel \mathcal{W} such that $W(a|a, s_0) = W(a|a, s_1) = 1$, $W(y|x; s_i) = 1$ if $y = x + i \pmod{3}$ for $x, y \in \{0, 1, 2\}$. That is the transmission matrices in \mathcal{W} are

(1	0	0	0)		1	0	0	0)	
	0	1	0	0	,	0	0	1	0	
	0	0	1	0		0	0	0	1	
	0	0	0	1)		0	1	0	0)	

At first we have that the deterministic capacity of W under maximum error probability is larger or equal to 2 because for all $n, \{a, 0\}^n$ there is a zero-error code of length n and therefore a code with criterion of maximum probability of error. Secondly let g be a mapping from $\mathcal{X}^n \to \{a, 0\}^n$ for arbitrary n sending x^n to y^n such that $y_i = a$ if $x_i = a$ and otherwise $x_i = 0$, for $i = 1, 2, \dots, n$. Then no pair of codewords in a code with criterion of maximum probability of error have the same image under the mapping g because in probability one the decoder may not separate the two codewords with the same image if the jammer properly chooses the state sequence according to the input codeword. Thus the deterministic capacity of W under maximum error probability is equal to 2.

On other hand let P be a input distribution such that $P(a) = \frac{2}{5}$ and $P(i) = \frac{1}{5}$ for i = 0, 1, 2. Let X and Y be the input and output random variables for P and \overline{W} , the channel in \overline{W} , minimizing $I(P; \overline{W})$. Then $H(X) = \frac{2}{5} \log \frac{5}{2} + \frac{3}{5} \log 5$. Next by considering the support sets of conditional distributions, we have H(X|Y = a) = 0 and $H(X|Y = i) \le 1$ for i = 0, 1, 2. Thus $H(X|Y) \le \frac{3}{5}$ and therefore $I(X; Y) = H(X) - H(X|Y) = \log \frac{5}{2}$. Moreover by simple calculation, $I(P; W) = \log \frac{5}{2}$ for $W(\cdot|\cdot) := \frac{1}{2}W(\cdot|\cdot, s_0) + \frac{1}{2}W(\cdot|\cdot, s_1)$. Thus $\min_{\overline{W} \in \overline{W}} I(P; \overline{W}) = \log \frac{5}{2}$. and $\max_{P \in P(X)} \min_{\overline{W} \in \overline{W}} I(P; \overline{W}) \ge \log \frac{5}{2}$.

Example 2 show that the legal transmitters really benefits from the resource even when the deterministic capacity under the maximal error criterion is positive.

Now one may concern the same question in scenario 2. This is answered by the following Theorem, which can be proven by modifying the proof of Theorem 1:

Theorem 3: The same conclusion for scenario 2, as that for scenario 1 in Theorem1, holds.

The above three Theorems and the facts that

$$C^{*}(\mathcal{W}) \leq C^{*}(\mathcal{W}, \epsilon), \quad C^{*}_{m}(\mathcal{W}) \leq C^{*}_{m}(\mathcal{W}, \epsilon) \leq C^{*}(\mathcal{W}, \epsilon),$$
$$C^{**}(\mathcal{W}) \leq C^{**}(\mathcal{W}, \epsilon) \leq C^{*}(\mathcal{W}, \epsilon), \quad C^{**}_{m}(\mathcal{W}) \leq C^{**}_{m}(\mathcal{W}, \epsilon) \leq C^{*}(\mathcal{W}, \epsilon)$$

yield the coding theorem:

Corollary 1: For all $\epsilon \in [0, 1)$ we have

$$C^{*}(\mathcal{W}) = C^{**}(\mathcal{W}) = C^{*}_{m}(\mathcal{W}) = C^{**}_{m}(\mathcal{W})$$
$$= C^{*}(\mathcal{W}, \epsilon) = C^{**}(\mathcal{W}, \epsilon) = C^{**}_{m}(\mathcal{W}, \epsilon) = C^{**}_{m}(\mathcal{W}, \epsilon)$$
$$= \max_{P} \min_{\bar{\rho}(\cdot) \in \bar{\mathcal{W}}} \chi(P, \bar{\bar{\rho}}(\cdot)).$$
(19)

Moreover the both capacity $C^{**}(W)$ and $C^{*}(W)$ can be achieved by codes with vanishing key rates.

Thus we conclude that:

- Further knowing message to be sent, may help a jammer to reduce the capacity neither in the scenario that the jammer knows coding scheme nor in the scenario that the jammer knows both coding scheme and input codeword.
- knowing input codeword is more effectual than knowing the message for a jammer, who knows coding scheme, for attack the communication.

IV. PROOF THEOREM 1

Although coding for classical arbitrarily varying channels is already a challenging topic with a lot of open problems, coding for AVCQC is even much harder. Due to the non-commutativity of quantum operators, many

techniques, concepts and methods of classical information theory, for instance, non-standard decoder and list decoding, may not be extended to quantum information theory. Sarwate used in [22] list decoding to prove the coding theorem for classical arbitrarily varying channels when the jammer knows input codeword. However since how to apply list decoding for quantum channels is still an open problem, the technique for classical channels in [22] can not be extended to AVCQC. We need a different approach for our scenario 1.

If the jammer would have some information about the outcome k of the random key through the input codeword, to which he has access in scenario 1, he could apply a strategy against the kth deterministic coding for AVCQC by choosing the worst state sequence to attack the communication, which we do not want. To this end a codeword must be used by "many" outcomes $\gamma(k)$ of a random correlated code Γ , if it is used by at least one of $\gamma(k)$. This is the main idea of our proof. We divide the proof into 5 steps. At the first step we derive a useful auxiliary result from known results. Next with the auxiliary result and Chernoff bound, we shall generate a ground set of code books from a typical set \mathcal{T}_X^n . Then our code Γ is constructed through the ground set and analyzed at the 3th and 4th steps, respectively. To simplify the statement, we shall not fix the values of parameters at the 2-4th steps exactly, but only set up necessary constraints to them. So finally we have to assign values to the parameters appearing in the proof at the last step.

A. An Auxiliary Result

We first derive a useful auxiliary result from known projections in previous work.

To construct decoding measurements of codes for classical quantum compound channel the authors in [6] and [18] introduced two kinds of projections for a set of classical quantum channels and input codewords $\mathbf{x} \in \mathcal{T}_X^n$ respectively. Although the two projections are quite different, they share the same properties. We summary their properties, which will be used in the paper, as the following lemma.

Lemma 1: For a set of classical quantum channels $\tilde{\mathcal{W}}$ with a common input alphabet \mathcal{X} and a common output Hilbert space \mathcal{H} and any an input codeword $\mathbf{x} \in \mathcal{T}_X^n$, there exits a projection $\mathcal{P}(\mathbf{x})$ in \mathcal{H} such that,

(i) For all $\tilde{\rho}(\cdot) \in \tilde{\mathcal{W}}$,

$$tr(\tilde{\rho}^{\otimes n}(\mathbf{x})\mathcal{P}(\mathbf{x})) > 1 - 2^{-n\eta}$$
⁽²⁰⁾

for an $\eta > 0$;

(ii)

$$tr(\tilde{\rho}_X^{\otimes n}\mathcal{P}(\mathbf{x})) < 2^{-n[\min_{\tilde{\rho}(\cdot)\in\tilde{\mathcal{W}}}\chi(P_X,\tilde{\rho}(\cdot))-\nu]},\tag{21}$$

for all $\nu > 0$, $\tilde{\rho}(\cdot) \in \tilde{\mathcal{W}}$ and sufficiently large n, where

$$\tilde{\rho}_X := \sum_{x \in \mathcal{X}} P_X(x) \tilde{\rho}(x).$$

(iii) Moreover, for all permutation π on $[n] = \{1, 2, ..., n\}$ with $\mathbf{x} = (x_1, x_2, ..., x_n) = (x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$, $\mathcal{P}(\mathbf{x})$ keeps invariant when permutation π acts on coordinates of *n*th tensor power \mathcal{H}^n of Hilbert space \mathcal{H} .

Let $\mathcal{W} = \{\rho(\cdot, s) = \{\rho(x, s), x \in \mathcal{X}\}, s \in \mathcal{S}\}$ be a finite set of classical quantum channels, indexed by elements of \mathcal{S} and let $\overline{\mathcal{W}}$ is defined by (15). Then

Corollary 2: Let $\mathcal{P}(\mathbf{x}')$ be the projection in Lemma 1 for $\tilde{\mathcal{W}} = \overline{\tilde{\mathcal{W}}}$, $\mathbf{x} \in \mathcal{T}_X^n$, $\mathbf{s} \in \mathcal{S}^n$ and \mathbf{X}' be randomly and uniformly distributed on \mathcal{T}_X^n , then

$$\mathbb{E}tr(\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{X}')) < 2^{-n[\min_{\bar{\rho}(\cdot)\in\bar{\mathcal{W}}}\chi(P_X, \bar{\bar{\rho}}(\cdot)) - \nu - \xi]},\tag{22}$$

for all $\xi > 0$ and sufficiently large n.

Proof: Let P_{XS} be joint type of (\mathbf{x}, \mathbf{s}) . Let (\mathbf{X}, \mathbf{S}) be randomly and uniformly distributed on \mathcal{T}_{XS}^n and \mathbf{X}' be random variable with uniform distribution on \mathcal{T}_X^n , and independent of (\mathbf{X}, \mathbf{S}) . Then by Lemma 1 (ii), we have that

$$\mathbb{E}tr(\rho^{\otimes n}(\mathbf{X}, \mathbf{S})\mathcal{P}(\mathbf{X}'))$$

$$= \sum_{\mathbf{x}'\in\mathcal{T}_{X}^{n}} Pr(\mathbf{X}'=\mathbf{x}') \sum_{(\mathbf{x},\mathbf{s})\in\mathcal{T}_{XS}^{n}} Pr[(\mathbf{X}, \mathbf{S}) = (\mathbf{x}, \mathbf{s})]tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}')]$$

$$< \sum_{\mathbf{x}'\in\mathcal{T}_{X}^{n}} Pr(\mathbf{X}'=\mathbf{x}')2^{n\xi} \sum_{\mathbf{x}\in\mathcal{X}^{n}\mathbf{s}\in\mathcal{S}^{n}} P_{XS}^{n}(\mathbf{x}, \mathbf{s})]tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}')]$$

$$= 2^{n\xi} \sum_{\mathbf{x}'\in\mathcal{T}_{X}^{n}} Pr(\mathbf{X}'=\mathbf{x}')tr\{[\sum_{\mathbf{x}\in\mathcal{X}^{n}\mathbf{s}\in\mathcal{S}^{n}} \prod_{t=1}^{n} P_{XS}(x_{t}, s_{t}) \bigotimes_{t=1}^{n} \rho(x_{t}, s_{t})]\mathcal{P}(\mathbf{x}')\}$$

$$= 2^{n\xi} \sum_{\mathbf{x}'\in\mathcal{T}_{X}^{n}} Pr(\mathbf{X}'=\mathbf{x}')tr\{[\sum_{x\in\mathcal{X}} P_{X}(x)(\sum_{s\in\mathcal{S}} P_{S|X}(s|x)\rho(x, s))]^{\otimes n}\mathcal{P}(\mathbf{x}')\}$$

$$< 2^{n\xi} \sum_{\mathbf{x}'\in\mathcal{T}_{X}^{n}} Pr(\mathbf{X}'=\mathbf{x}')2^{-n[\min_{\bar{\rho}(\cdot)\in\bar{W}}\chi(P_{X},\bar{\rho}(\cdot))-\nu]}$$

$$= 2^{-n[\min_{\bar{\rho}(\cdot)\in\bar{W}}\chi(P_{X},\bar{\rho}(\cdot))-\nu-\xi]},$$
(23)

for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{s} = (s_1, s_2, \dots, s_n)$, where the first inequality holds because

$$Pr[(\mathbf{X}, \mathbf{S}) = (\mathbf{x}, \mathbf{s})] = \frac{1}{|\mathcal{T}_{XS}^n|} < 2^{-n(H(X, S) - \frac{\xi}{2})} < 2^{n\xi} P_{XS}^n(\mathbf{x}, \mathbf{s})$$

for all $\xi > 0$ and sufficiently large n, if $(\mathbf{x}, \mathbf{s}) \in \mathcal{T}_{XS}^n$, and equal to zero otherwise; and by (21) the last inequality holds, because by (15), $\{\sum_{s \in S} P_{S|X}(s|x)\rho(x,s), x \in \mathcal{X}\} \in \overline{\mathcal{W}}$.

Now by Lemma 1 (iii), we note that for all $(\mathbf{x}, \mathbf{s}) \in \mathcal{T}_{XS}^n, \mathbf{x}' \in \mathcal{T}_X^n$, the value of $tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}')]$ depends only on the joint type of $(\mathbf{x}, \mathbf{x}', \mathbf{s})$, and therefore for all $(\mathbf{x}, \mathbf{s}) \in \mathcal{T}_{XS}^n$, the value of

$$\sum_{\mathbf{x}' \in \mathcal{T}_X^n} Pr(\mathbf{X}' = \mathbf{x}') tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s}) \mathcal{P}(\mathbf{x}')]$$

is a constant (only depending on the joint type of (x,s). Thus (22) follows from (23) and the fact that

$$\mathbb{E}tr(\rho^{\otimes n}(\mathbf{X}, \mathbf{S})\mathcal{P}(\mathbf{X}')) = \sum_{(\mathbf{x}, \mathbf{s}) \in \mathcal{T}_{XS}^n} Pr[(\mathbf{X}, \mathbf{S}) = (\mathbf{x}, \mathbf{s})] \{\sum_{\mathbf{x}' \in \mathcal{T}_X^n} Pr(\mathbf{X}' = \mathbf{x}')tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}')]\}.$$

Thus, the proof is completed.

B. Generation Ground Set for Code books

Let

$$A_n \ge 2^{-n[\min_{\bar{\rho}(\cdot)\in\bar{\mathcal{W}}}\chi(P_X,\bar{\rho}(\cdot))-\nu-\xi]}$$
(24)

and \mathcal{I}_n be a finite index set with the cardinality

$$|\mathcal{I}_n| > \frac{n\log_e |\mathcal{X}||\mathcal{S}|}{(3-e)A_n},\tag{25}$$

which will be specified in Subsection IV-E. Let $\mathbf{X}(i), i \in \mathcal{I}_n$ be randomly, independently and uniformly distributed on \mathcal{T}_X^n . Then by Corollary 2 and Chernoff bound, we have that for all $\mathbf{x} \in \mathcal{T}_X^n$, $\mathbf{s} \in \mathcal{S}^n$

$$Pr\{\sum_{i\in\mathcal{I}_{n}}tr[\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{X}(i))] > 3A_{n}I_{n}\}\$$

$$= Pr\{\exp_{e}[-3A_{n}I_{n} + \sum_{i\in\mathcal{I}_{n}}tr[\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{X}(i))]] > 1\}\$$

$$\leq e^{-3A_{n}I_{n}}\prod_{i\in\mathcal{I}_{n}}\mathbb{E}e^{tr[\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{X}(i))]}\$$

$$\leq e^{-3A_{n}I_{n}}\prod_{i\in\mathcal{I}_{n}}[1 + e\mathbb{E}\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{X}(i))]\$$

$$\leq e^{-3A_{n}I_{n}}[1 + eA_{n}]^{|\mathcal{I}_{n}|}\$$

$$\leq \exp_{e}\{-3A_{n}|\mathcal{I}_{n}| + eA_{n}|\mathcal{I}_{n}|\} = e^{-(3-e)A_{n}|\mathcal{I}_{n}|},$$
(26)

where the first inequality is Chernoff bound, the second inequality holds because e^z is a monotone increasing and convex function and so $e^z \le 1 + ez$ for $z \in (0, 1)$; the third inequality holds by (22) and (24); and the last inequality follows from inequality $1 + z \le e^z$. Thus by union bound and (25), we obtain that

$$Pr\{\bigcup_{\mathbf{x}\in\mathcal{T}_X^n,\mathbf{s}\in\mathcal{S}^n}[\sum_{i\in\mathcal{I}_n}tr[\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{X}(i))]>3A_n|\mathcal{I}_n|]\}<|\mathcal{X}|^n|\mathcal{S}|^ne^{-(3-e)A_n|\mathcal{I}_n|}<1.$$

Consequently we have that there exists a subset $\mathcal{B} = \{\mathbf{x}(i), i \in \mathcal{I}_n\} \subset \mathcal{T}_X^n$, with

$$\sum_{\mathbf{x}(i)\in\mathcal{B}} tr[\rho^{\otimes n}(\mathbf{x},\mathbf{s})\mathcal{P}(\mathbf{x}(i))] \le 3A_n|\mathcal{I}_n|,\tag{27}$$

for all $\mathbf{x} \in \mathcal{T}_X^n$, $\mathbf{s} \in \mathcal{S}^n$.

C. Construction of Code

1) Generation of Code books: Let \mathcal{J}_n and \mathcal{K}_n be two finite set and their cardinalities (depending on n) will be specified in Subsection IV-E, but at this moment, we only assume that

$$\mathcal{J}_n| \le A_n^{-1}.\tag{28}$$

Let $(\mathbf{U}(j,k), j \in \mathcal{J}_n), k \in \mathcal{K}_n$ be randomly uniformly and independently generated from

$$\{(\mathbf{x}(i_1), \mathbf{x}(i_2), \dots, \mathbf{x}(i_{|\mathcal{J}_n|})) : i_j \in \mathcal{I}_n, \text{ for } j = 1, 2, \dots, |\mathcal{J}_n|, \text{ with } i_j \neq i_{j'} \text{ for } j \neq j'\}$$

Then by (27) we have that for all $i \in \mathcal{I}_n$, $\mathbf{s} \in \mathcal{S}^n$, $j, j' \in \mathcal{J}_n$, with $j \neq j'$ and $k \in \mathcal{K}_n$

$$\mathbb{E}tr[\rho^{\otimes n}(\mathbf{U}(j,k)), \mathbf{s})\mathcal{P}(\mathbf{U}(j',k))|\mathbf{U}(j,k)) = \mathbf{x}(i)]$$

$$= \sum_{i'\in\mathcal{I}_n\setminus\{i\}} Pr[\mathbf{U}(j',k) = \mathbf{x}(i')|\mathbf{U}(j,k)) = \mathbf{x}(i)]tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{P}(\mathbf{x}(i'))]$$

$$= \frac{1}{|\mathcal{I}_n| - 1} \sum_{i'\in\mathcal{I}_n\setminus\{i\}} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{P}(\mathbf{x}(i'))]$$

$$\leq \frac{1}{|\mathcal{I}_n| - 1} \sum_{i'\in\mathcal{I}_n} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{P}(\mathbf{x}(i'))] \leq \frac{3A_n|\mathcal{I}_n|}{|\mathcal{I}_n| - 1}.$$
(29)

Consequently by Markov inequality we have that

$$Pr\{\sum_{j'\in\mathcal{J}_{n}\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{U}(j,k),\mathbf{s})\mathcal{P}(\mathbf{U}(j',k))] > \mu_{n}|\mathbf{U}(j,k) = \mathbf{x}(i)\}]$$

$$\leq \frac{\mathbb{E}\{\sum_{j'\in\mathcal{J}_{n}\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{U}(j,k),\mathbf{s})\mathcal{P}(\mathbf{U}(j',k))]|\mathbf{U}(j,k) = \mathbf{x}(i)\}}{\mu_{n}}$$

$$= \frac{\sum_{j'\in\mathcal{J}_{n}\setminus\{j\}} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{U}(j,k),\mathbf{s})\mathcal{P}(\mathbf{U}(j',k))]|\mathbf{U}(j,k) = \mathbf{x}(i)\}}{\mu_{n}}$$

$$\leq \frac{3A_{n}(|\mathcal{J}_{n}|-1)||\mathcal{I}_{n}|}{(|\mathcal{I}_{n}|-1)\mu_{n}} < \frac{3A_{n}|\mathcal{J}_{n}|}{\mu_{n}}$$
(30)

for all $i \in \mathcal{I}_n$, $\mathbf{s} \in \mathcal{S}^n$, $j \in \mathcal{J}_n$, $k \in \mathcal{K}_n$ and $\mu_n \in (0, 1)$, where the last inequality holds because by (25) and (28), $|\mathcal{J}_n| < |\mathcal{I}_n|$ and therefore $\frac{|\mathcal{J}_n|-1}{|\mathcal{I}_n|-1} < \frac{|\mathcal{J}_n|}{|\mathcal{I}_n|}$. Therefore

$$Pr\{\mathcal{E}(i, \mathbf{s}, k; \mu_n)\} = \sum_{j \in \mathcal{J}_n} Pr(\mathbf{U}(j, k) = \mathbf{x}(i)) Pr\{\sum_{j' \in \mathcal{J}_n \setminus \{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i), \mathbf{s}) \mathcal{P}(\mathbf{U}(j', k))] > \mu_n | \mathbf{U}(j, k) = \mathbf{x}(i)\}$$

$$< \frac{3A_n |\mathcal{J}_n|^2}{|\mathcal{I}_n|\mu_n},$$
(31)

for all $i \in \mathcal{I}_n$, $\mathbf{s} \in \mathcal{S}$, $k \in \mathcal{K}_n$ and $\mu_n \in (0, 1)$, if we define $\mathcal{E}(i, \mathbf{s}, k; \mu_n)$ as the random event that there exists a $j \in \mathcal{J}_n$ such that $\mathbf{U}(j, k) = \mathbf{x}(i)$ and

$$\sum_{j' \in \mathcal{J}_n \setminus \{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i), \mathbf{s}) \mathcal{P}(\mathbf{U}(j', k))] > \mu_n.$$

In the sequel, we shall use the following version of well known Chernoff Bound.

Lemma 2: (Chernoff Bound) Let B_1, B_2, \ldots, B_L be i.i.d. random binary sequence taking values in $\{0, 1\}$, with $Pr(B_l = 1) = p$. Then for all $\alpha \in (0, 1), p_0 \le p \le p_1$

$$Pr\{\sum_{l=1}^{L} B_l > Lp_1(1+\alpha)\} < e^{-\frac{\alpha^2}{8}Lp_1},$$
(32)

and

$$Pr\{\sum_{l=1}^{L} B_l < Lp_0(1-\alpha)\} < e^{-\frac{3\alpha^2}{8}Lp_0}.$$
(33)

For self-contained we prove it in Appendix A, although (32) was shown in [15] and (33) can be shown in a similar way.

Next for a fixed $i \in \mathcal{I}_n$, we define random sets

$$\mathfrak{K}(i) := \{ (k: \text{ there exists a } j \in \mathcal{J}_n \text{ with } \mathbf{U}(j,k) = \mathbf{x}(i) \}$$
(34)

18

and for all $\mathbf{s} \in \mathcal{S}^n$,

$$\mathfrak{K}_{0}(i,\mathbf{s})) := \{k : \text{ there exists a } j \text{ with } \mathbf{U}(j,k) = \mathbf{x}(i) \text{ and } \sum_{j' \in \mathcal{J}_{n} \setminus \{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{P}(\mathbf{U}(j',k))] > \mu_{n}\}.$$
(35)

Let $\iota(\mathcal{E}(i, \mathbf{s}, k; \mu_n))$ be the indicator of the random event of $\mathcal{E}(i, \mathbf{s}, k; \mu_n)$ (i.e., $\iota(\mathcal{E}(i, \mathbf{s}, k; \mu_n)) = 1$ if $\mathcal{E}(i, \mathbf{s}, k; \mu_n)$ occurs and otherwise $\iota(\mathcal{E}(i, \mathbf{s}, k; \mu_n)) = 0$) and random variables

$$Z_i(k) = \begin{cases} 1 & \text{if exists a } j \text{ with } \mathbf{U}(j,k) = \mathbf{x}(i) \\ 0 & \text{else.} \end{cases}$$

Then by (31) we have that $Pr(\iota(\mathcal{E}(i,k;\mu_n)) = 1) < \frac{3A_n|\mathcal{J}_n|^2}{|\mathcal{I}_n|\mu_n}$. By the definition of $Z_i(k)$ we have that

$$Pr(Z_i(k) = 1) = \sum_{j \in \mathcal{J}_n} Pr[\mathbf{U}(j, k) = \mathbf{x}(i)] = \frac{|\mathcal{J}_n|}{|\mathcal{I}_n|},$$

as by the definition of U(j,k)'s, the random events $\{U(j,k) = x(i)\}, j \in \mathcal{J}_n$ are pairwise disjoint.

For each fixed $i \in \mathcal{I}_n$, we apply (33) to $[L] = \mathcal{K}_n, B_k = Z_i(k), k \in \mathcal{K}_n$ and $p_0 = \frac{|\mathcal{I}_n|}{|\mathcal{I}_n|}$ and obtain that

$$Pr\{|\mathfrak{K}(i)| < \frac{|\mathcal{K}_{n}||\mathcal{J}_{n}|}{|\mathcal{I}_{n}|}(1-\alpha)\}$$

$$= Pr\{\sum_{k\in\mathcal{K}_{n}} Z_{i}(k) < |\mathcal{K}_{n}|\frac{|\mathcal{J}_{n}|}{|\mathcal{I}_{n}|}(1-\alpha)\}$$

$$< \exp_{e}\{-\frac{3\alpha^{2}}{8}\frac{|\mathcal{K}_{n}||\mathcal{J}_{n}|}{|\mathcal{I}_{n}|}\}.$$
(36)

Similarly, by apply (32) to $[L] = \mathcal{K}_n, B_k = \iota(\mathcal{E}(i, \mathbf{s}, k; \mu)), k \in \mathcal{K}_n \text{ and } p_1 = \frac{3A_n |\mathcal{J}_n|^2}{|\mathcal{I}_n|\mu_n}$, we have that

$$Pr\{|\mathfrak{K}_{0}(i,\mathbf{s}))| > \frac{3A_{n}|\mathcal{J}_{n}|^{2}|\mathcal{K}_{n}|}{|\mathcal{I}_{n}|\mu_{n}}(1+\alpha)\}$$

$$= Pr\{\sum_{k\in\mathcal{K}_{n}}\iota(\mathcal{E}(i,\mathbf{s},k;\mu_{n})) > |\mathcal{K}_{n}|\frac{3A_{n}|\mathcal{J}_{n}|^{2}}{|\mathcal{I}_{n}|\mu_{n}}(1+\alpha)\}$$

$$< \exp_{e}\{-\frac{\alpha^{2}}{8}\frac{3A_{n}|\mathcal{J}_{n}|^{2}|\mathcal{K}_{n}|}{|\mathcal{I}_{n}|\mu_{n}}\},$$
(37)

for all $i \in \mathcal{I}_n$, $\mathbf{s} \in \mathcal{S}$ and $\mu_n \in (0, 1)$. Now choose $\alpha = \frac{1}{2}$, $|\mathcal{J}_n|$ and μ_n properly such that (28) holds and

$$\lambda_n' := \frac{A_n |\mathcal{J}_n|}{\mu_n} < 1 \tag{38}$$

sufficiently small, $|\mathcal{K}_n|$ sufficiently large such that

$$\frac{3}{32}\lambda_n'\frac{|\mathcal{K}_n||\mathcal{J}_n|}{|\mathcal{I}_n|} > 2n\log_e|\mathcal{S}||\mathcal{X}|,\tag{39}$$

and

$$|\mathcal{I}_n| < |\mathcal{X}|^n,\tag{40}$$

(all to be specified in Subsection IV-E)). Thus, by the union bound and (36), (37), (38), (39) and (40), we have that

$$Pr\{\bigcup_{i\in\mathcal{I}_n}[|\mathfrak{K}(i)|<\frac{|\mathcal{K}_n||\mathcal{J}_n|}{2|\mathcal{I}_n|}]\}<\frac{1}{2},$$

and

$$Pr\{\cup_{\mathbf{s}\in\mathcal{S}}\cup_{i\in\mathcal{I}_n}[|\mathfrak{K}_0(i,\mathbf{s}))| > \frac{9|\mathcal{J}_n||\mathcal{K}_n|\lambda_n'}{2|\mathcal{I}_n|}\} < \frac{1}{2}$$

respectively. Consequently

$$Pr\{\left[\cap_{i\in\mathcal{I}_{n}}(|\mathfrak{K}(i)|\geq\frac{|\mathcal{K}_{n}||\mathcal{J}_{n}|}{2|\mathcal{I}_{n}|})\right]\cap\left[\cap_{\mathbf{s}\in\mathcal{S}}\cap_{i\in\mathcal{I}_{n}}(|\mathfrak{K}_{0}(i,\mathbf{s})|\leq\frac{9|\mathcal{J}_{n}||\mathcal{K}_{n}|\lambda_{n}'}{2|\mathcal{I}_{n}|})\right]\}>0$$
(41)

Thus $\{\mathbf{U}(j,k) \in \mathcal{J}_n\}, k \in \mathcal{K}_n$ has a realization

$$\mathcal{U}(k) := \{\mathbf{u}(j,k), j \in \mathcal{J}_n\}, k \in \mathcal{K}_n$$

such that

for all
$$k \in \mathcal{K}_n$$
 and $j \neq j', [\mathbf{u}(j,k) = \mathbf{x}(i), \mathbf{u}(j',k) = \mathbf{x}(i')] \Rightarrow i \neq i'$ (42)

$$|\mathcal{K}(i)| \ge \frac{|\mathcal{K}_n||\mathcal{J}_n|}{2|\mathcal{I}_n|} \text{ and } |\mathcal{K}_0(i,\mathbf{s})| \le \frac{9|\mathcal{J}_n||\mathcal{K}_n|\lambda'_n}{2|\mathcal{I}_n|}$$
(43)

for all $i \in \mathcal{I}_n$ and $\mathbf{s} \in \mathcal{S}^n$, where

$$\mathcal{K}(i) := \{k : \text{ there exists a } j \in \mathcal{J}_n \text{ with } \mathbf{u}(j,k) = \mathbf{x}(i)\}$$
(44)

and

 $\mathcal{K}_{0}(i,\mathbf{s}) := \{k : \text{ there exists a } j \text{ with } \mathbf{u}(j,k) = \mathbf{x}(i) \text{ and } \sum_{j' \in \mathcal{J}_{n} \setminus \{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{P}(\mathbf{u}(j',k))] > \mu_{n}\}.$ (45)

Now we choose $\mathcal{U}(k)$ as the code book of our kth code $\gamma(k)$.

2) Define Decoding Measurements: We define its decoding measurement $\{\mathcal{D}(j,k), j \in \mathcal{J}_n\}$ for the kth code $\gamma(k)$, such that

$$\mathcal{D}(j,k) := \left[\sum_{j' \in \mathcal{J}_n} \mathcal{P}(\mathbf{u}(j',k))\right]^{-\frac{1}{2}} \mathcal{P}(\mathbf{u}(j,k) \left[\sum_{j' \in \mathcal{J}_n} \mathcal{P}(\mathbf{u}(j',k))\right]^{-\frac{1}{2}}$$
(46)

for its *j*th codeword $\mathbf{u}(j, k)$.

3) Define the Random Correlated Code: Let our random code Γ be randomly uniformly generated from the set of codes $\{\gamma(k), k \in \mathcal{K}_n\}$.

D. Error Analysis

At first we have to estimate $tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s})\mathcal{P}(\mathbf{u}(j,k))]$ for all $j \in \mathcal{J}_n, k \in \mathcal{K}_n$ and $\mathbf{s} \in \mathcal{S}^n$. To the end let us first fix $j \in \mathcal{J}_n, k \in \mathcal{K}_n$ and $\mathbf{s} \in \mathcal{S}^n$. Let P_{XS} be joint type of $(\mathbf{u}(j,k),\mathbf{s})$ and

$$\bar{\rho}_{S|X}(x) := \sum_{s' \in \mathcal{S}} P_{S|X}(s'|x)\rho(x,s'), \tag{47}$$

for all $x \in \mathcal{X}$. Then by (15) we have that $\{\bar{\rho}_{S|X}(x), x \in \mathcal{X}\} \in \bar{\mathcal{W}}$. Therefore by (20) we obtain that for $\mathbf{u}(j,k) := (u_1(j,k), u_2(j,k), \dots, u_n(j,k))$

$$\sum_{\mathbf{s}' \in \mathcal{S}^n} P_{S|X}^n(\mathbf{s}'|\mathbf{u}(j,k))tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s}')\mathcal{P}(\mathbf{u}(j,k))]$$

$$= tr\{\{\sum_{\mathbf{s}' \in \mathcal{S}^n} [\prod_{t=1}^n P_{S|X}(s_t'|u_t(j,k))] [\bigotimes_{t=1}^n \rho(u_t(j,k),s_t')]\} \mathcal{P}(\mathbf{u}(j,k))\}$$

$$= tr\{[\bigotimes_{t=1}^n (\sum_{s_t' \in \mathcal{S}} P_{S|X}(s_t'|u_t(j,k))\rho(u_t(j,k),s_t'))]\mathcal{P}(\mathbf{u}(j,k))\}$$

$$= tr[\bar{\rho}_{S|X}^{\otimes n}(\mathbf{u}(j,k))\mathcal{P}(\mathbf{u}(j,k))] > 1 - 2^{-n\eta}, \qquad (48)$$

where

$$\bar{\bar{\rho}}_{S|X}^{\otimes n}(\mathbf{u}(j,k)) = \bar{\bar{\rho}}_{S|X}(u_1(j,k)) \otimes \bar{\bar{\rho}}_{S|X}(u_2(j,k)) \otimes \dots \otimes \bar{\bar{\rho}}_{S|X}(u_n(j,k))$$

However by Lemma 1 (iii), the value of $tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s}')\mathcal{P}(\mathbf{u}(j,k))]$ depends only on the joint type of $(\mathbf{u}(j,k),\mathbf{s}')$ and so does $tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s}')(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j,k)))]$. Therefore (48) yields that

$$2^{-n\eta} > \sum_{\mathbf{s}' \in \mathcal{S}^n} P_{S|X}^n(\mathbf{s}'|\mathbf{u}(j,k)) tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s}')(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j,k)))]$$

$$\geq \sum_{\mathbf{s}' \in \mathcal{T}_{S|X}^n(\mathbf{u}(j,k))} P_{S|X}^n(\mathbf{s}'|\mathbf{u}(j,k)) tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s}')(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j,k)))]$$

$$= P_{S|X}^n[\mathcal{T}_{S|X}^n(\mathbf{u}(j,k))|\mathbf{u}(j,k)] tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j,k)))], \qquad (49)$$

for the particular $\mathbf{u}(j,k)$ and s, since P_{XS} is the joint type of $\mathbf{u}(j,k)$ and s. That is,

$$tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j,k)))] < 2^{-\frac{n\eta}{2}}$$
(50)

or

$$tr[\rho^{\otimes n}(\mathbf{u}(j,k),\mathbf{s})\mathcal{P}(\mathbf{u}(j,k))] \ge 1 - 2^{-\frac{n\eta}{2}},$$

 $\text{for all } \mathbf{u}(j,k) \text{ and } \mathbf{s}, \text{ as } P^n_{S|X}[\mathcal{T}^n_{S|X}(\mathbf{u}(j,k)) | \mathbf{u}(j,k)] > 2^{-\frac{n\eta}{2}} \text{ for any } \eta > 0 \text{ and sufficiently large } n.$

Let J and K be two independent random variables taking values in \mathcal{J}_n and \mathcal{K}_n according uniform distributions, respectively. Since (42) and (44) yield that for very $k \in \mathcal{K}(i)$ there is exactly one j := j(i,k) (say) in \mathcal{J}_n , such that $\mathbf{u}(k, j(i, k)) = \mathbf{x}(i)$, by (43) we have that for all $\mathbf{x}(i) \in \mathcal{B}$, $\mathbf{s} \in \mathcal{S}^n Pr[\mathbf{u}(J, K) = \mathbf{x}(i)] = \frac{|\mathcal{K}(i)|}{|\mathcal{J}_n||\mathcal{K}_n|} > 0$ for all $i \in \mathcal{I}_n$ and

$$\mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})\mathcal{D}(J,K)]|\mathbf{u}(J,K) = \mathbf{x}(i)\} = \frac{1}{|\mathcal{K}(i)|} \sum_{k \in \mathcal{K}(i)} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})\mathcal{D}((j(i,k)),k)].$$
(51)

Next we shall apply Hayashi-Nagaoka inequality

$$\mathbb{I}_{\mathcal{H}} - (S+T)]^{-\frac{1}{2}} S(S+T)^{-\frac{1}{2}} \le 2(\mathbb{I}_{\mathcal{H}} - S) + 4T$$
(52)

for any positive operators S and T with $0 \le S \le \mathbb{I}_{\mathcal{H}}$ and $T \ge 0$, to estimate

$$\max_{\mathbf{s}\in\mathcal{S}^n} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))]|\mathbf{u}(J,K)=\mathbf{x}(i)\}$$

To this end let $\mathcal{K}_1(i, \mathbf{s}) := \mathcal{K}(i) \setminus \mathcal{K}_0(i, \mathbf{s})$ for all *i* and **s**. Then it follows from (43) that

$$\frac{|\mathcal{K}_0(i,\mathbf{s})|}{|\mathcal{K}(i)|} \le 9\lambda'_n \text{ and } \frac{|\mathcal{K}_1(i,\mathbf{s})|}{|\mathcal{K}(i)|} \ge 1 - 9\lambda'_n.$$
(53)

Consequently we have

$$\frac{1}{|\mathcal{K}(i)|} \sum_{k \in \mathcal{K}_{0}(i,\mathbf{s})} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{D}((j(i,k)),k))] \\
\leq \frac{|\mathcal{K}_{0}(i,\mathbf{s})|}{|\mathcal{K}(i)|} \leq 9\lambda'_{n},$$
(54)

for all $\mathbf{x} \in \mathcal{B}$ and $\mathbf{s} \in \mathcal{S}^n$. On the other hand, by (45), (46), (50), (52) and the definitions of $\mathcal{K}_1(i, \mathbf{s})$ and j(i, k), we obtain that

$$\frac{1}{|\mathcal{K}(i)|} \sum_{k \in \mathcal{K}_{1}(i,\mathbf{s})} tr[\rho^{\otimes n}(\mathbf{x}(i), \mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{D}((j(i,k)), k))] \\
\leq \frac{1}{|\mathcal{K}(i)|} \sum_{k \in \mathcal{K}_{1}(i,\mathbf{s})} \left\{ 2tr[\rho^{\otimes n}(\mathbf{u}(j(i,k)), \mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{P}(\mathbf{u}(j(i,k))))] + 4tr[\rho^{\otimes n}(\mathbf{x}(i), \mathbf{s}) \sum_{j' \in \mathcal{J}_{n} \setminus \{j(i,k)\}} \mathcal{P}(\mathbf{u}(j', k))] \right\} \\
< \frac{1}{|\mathcal{K}(i)|} \sum_{k \in \mathcal{K}_{1}(i,\mathbf{s})} \left\{ 2^{-\frac{n\eta}{2}+1} + 4\mu_{n} \right\} \leq 2^{-\frac{n\eta}{2}+1} + 4\mu_{n},$$
(55)

where to have the first inequality, we first apply (46) and (52) to break $(\mathbb{I}_{\mathcal{H}} - \mathcal{D}((j(i,k)),k))$ to two terms and then by the definition of j(i,k) substitute $\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})$ by $\rho^{\otimes n}(\mathbf{u}(j(i,k)),\mathbf{s})$ in the first term; the second inequality holds by (50), (45) and the facts that $\mathcal{K}_1(i,\mathbf{s}) := \mathcal{K}(i) \setminus \mathcal{K}_0(i,\mathbf{s})$ and $\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s}) = \rho^{\otimes n}(\mathbf{u}(j(i,k)),\mathbf{s})$; and finally the last inequality follows from that $|\mathcal{K}_1(i,\mathbf{s})| \leq |\mathcal{K}(i)|$. Now (51), (54) and (55) together yield that

$$\mathbb{E}\left\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))]|\mathbf{u}(J,K)=\mathbf{x}(i)\right\} \\
= \frac{1}{|\mathcal{K}(i)|} \sum_{k\in\mathcal{K}_{0}(i,\mathbf{s})} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}((j(i,k)),k))] \\
+ \frac{1}{|\mathcal{K}(i)|} \sum_{k\in\mathcal{K}_{1}(i,\mathbf{s})} tr[\rho^{\otimes n}(\mathbf{x}(i),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}((j(i,k)),k))] \\
< 9\lambda'_{n} + 2^{-\frac{n\eta}{2}+1} + 4\mu_{n},$$
(56)

for all $\mathbf{x}(i) \in \mathcal{B}$ and all $\mathbf{s} \in \mathcal{S}^n$. That is,

$$\max_{\mathbf{s}\in\mathcal{S}^n} \mathbb{E}\{tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}}-\mathcal{D}(J,K))]|\mathbf{u}(J,K)=\mathbf{x}(i)\} < 9\lambda'_n + 2^{-\frac{n\eta}{2}+1} + 4\mu_n,\tag{57}$$

for all $\mathbf{x}(i) \in \mathcal{B}$, or

$$\sum_{\mathbf{x}(i)} \Pr\{\mathbf{u}(J,K) = \mathbf{x}(i)\} \max_{\mathbf{s}\in\mathcal{S}^n} \mathbb{E}[tr[\rho^{\otimes n}(\mathbf{u}(J,K),\mathbf{s})(\mathbb{I}_{\mathcal{H}} - \mathcal{D}(J,K))] | \mathbf{u}(J,K) = \mathbf{x}(i)] < 9\lambda'_n + 2^{-\frac{n\eta}{2}+1} + 4\mu_n$$
(58)

Consequently, by (10), we conclude that

$$p_a(\Gamma) < 9\lambda'_n + 2^{-\frac{n\eta}{2}+1} + 4\mu_n.$$
(59)

Finally we notice that like in the standard way to apply random choice for showing direct coding theorem in classical and quantum Shannon Theory, we have not excluded the case that for $i \neq i'$ in \mathcal{I}_n , $\mathbf{x}(i)$ and $\mathbf{x}(i')$ take the same input codeword as their values, formally distinguish them by their indices, and consider them as different members of \mathcal{B} even in the case that it occurs. (It is the reason why we do not write " $\mathbf{u}(j,k) \neq \mathbf{u}(j',k)$ for $j \neq j'$ " in (42).) This slightly makes a difference in (57) and (58). That is, if $\mathbf{x}(i) = \mathbf{x}(i') = \mathbf{x}$ and \mathbf{x} is sent, by our assumption jammer only knows the input codeword \mathbf{x} , but does not know which index in \mathcal{B} leads to the input codeword. On the other hand the expressions at left hand sides of (57) and (58) mean that jammer may choose state sequence according to the index, which implies the jammer has more information than our assumption. Thus, in this case left hand side of (58) in fact is an upper bound of conditional expectation at right hand side of (10). Clearly this does not impede us to have (59).

E. Set up the Parameters

Now we have to fix the parameters A_n , $|\mathcal{I}_n|$, $|\mathcal{S}_n|$, $|\mathcal{K}_n|$, μ_n and λ'_n and they must satisfy our previous assumptions (24), (25), (28), (38), (39) and (40). Given $\epsilon > 0$ (independent of n) and λ_n with $\lambda_n \ge \max(2^{-\frac{n\eta}{3}}, 2^{-\frac{n\epsilon}{5}})$ (for η in (20)), (which may or may not depend on n,) we hope to have a code with rate $\frac{1}{n} \log |\mathcal{J}_n| > \min_{\bar{\rho}(\cdot) \in \bar{\mathcal{W}}} \chi(P_X, \bar{\rho}(\cdot)) - \epsilon$ and probability of error smaller than λ_n to minimize *the order*, of the size of random code $|\mathcal{K}_n|$.

At first we note that ξ and ν in (22) can be arbitrary positive numbers, then we choose them such that $0 < \xi + \nu < \frac{\epsilon}{2}$. Let $A_n = 2^{-n[\min_{\bar{\rho}(\cdot)\in\bar{W}}\chi(P_X,\bar{\rho}(\cdot))-\frac{\epsilon}{2}]}$ and then (24) holds. Next we choose a_1 as positive real larger than $\frac{1}{3-e}$ such that $a_1 \frac{n \log_e |\mathcal{X}|S|}{A_n}$ is a integer and let $|\mathcal{I}_n| = a_1 \frac{n \log_e |\mathcal{X}||S|}{A_n}$. Thus (25) and (40) hold. Let $a_2 = \frac{1}{27}, \mu_n = \lambda'_n = a_2\lambda_n$ so that the upper bound to the average probability of error at the right hand side of (59) is smaller than λ_n when n is sufficiently large. Let $|\mathcal{J}_n| = \frac{(a_2\lambda_n)^2}{A_n} = \frac{\lambda'_n\mu_n}{A_n}$ (or its integer part) and then (28) and (38) hold, and $\frac{1}{n} \log |\mathcal{J}_n| > \min_{\bar{\rho}(\cdot)\in\bar{W}}\chi(P_X,\bar{\rho}(\cdot)) - \epsilon$ (since by our assumption $(a_2\lambda_n)^2 > 2^{-\frac{n\epsilon}{2}}$). Finally to satisfy (39), we choose

$$|\mathcal{K}_n| = \frac{32n|\mathcal{I}_n|\log_e|\mathcal{X}||\mathcal{S}|}{\lambda'_n|\mathcal{J}_n|} = \frac{32a_1(n\log_e|\mathcal{X}||\mathcal{S}|)^2}{(a_2\lambda_n)^3} = \frac{an^2}{\lambda_n^3},\tag{60}$$

(or its integer part) for a constant $a := \frac{32a_1(\log_e |\mathcal{X}||\mathcal{S}|)^2}{a_2^3}$ depending only on $|\mathcal{X}||\mathcal{S}|$, where the second equality is obtained by substitute $|\mathcal{I}_n| = \frac{a_1 n \log_e |\mathcal{X}||\mathcal{S}|}{A_n}$, $\lambda'_n = a_2 \lambda_n$ and $|\mathcal{J}_n| = \frac{(a_2 \lambda_n)^2}{A_n}$. Thus the proof is completed.

V. PROOF OF THEOREM 2

In this section we prove Theorem 2. At first, we show Theorem 2 for codes with vanishing key rate as those in Theorem 1, i.e., when there is a positive constant B such that $|\mathcal{K}| \leq bn^2$.

Suppose that we are given a random correlated code Γ taking value on $\{(\{\mathbf{u}(j,k), j \in \mathcal{J}\}, \{\mathcal{D}(j,k), j \in \mathcal{J}\}), k \in \mathcal{K}\}$ such that the random message J is randomly uniformly distributed on \mathcal{J} and the random key K is randomly distributed on \mathcal{K} with any distribution. Denote the rate and the average probability of error of the code Γ by R and λ respectively.

As a randomizing or so-called mixed strategy may not enlarge the probability of error, without loss of generality we assume the jammer randomly chooses state sequences, according to the input codeword. More specifically let $\mathbf{X}' = \mathbf{u}(J, K)$ be the random input of the AVCQC and $P_{X'}$ be its distribution. Then the jammer knows both the input distribution $P_{\mathbf{X}'}$ and the outcome \mathbf{x} of $\mathbf{X}' = \mathbf{u}(J, K)$, since we assume he knows that both coding scheme and input codeword. Let $P_{X'_{i}}$ be the *t*th marginal distribution of $P_{X'}$.

Let

$$\bar{\bar{\rho}}(x) := \sum_{s} Q(s|x)\rho(x,s) \in \bar{\bar{\mathcal{W}}}$$
(61)

be an arbitrary classical quantum channel in $\bar{\mathcal{W}}$ component wise independently. That is,

$$Pr\{\mathbf{S} = \mathbf{s} | \mathbf{X}' = \mathbf{x}\} = \prod_{t=1}^{n} Q_t(s_t | x_t),$$

where Q_t is the *t*th marginal distribution of Q. Let R be a ϵ -achievable rate for $\{\bar{\rho}_t(x) : t, x\}$ with a $\epsilon \in [0, 1)$, where $\bar{\rho}_t(x) := \sum_{s_t} Q(s_t | x_t) \rho(x_t, s_t)$. By Winters strong converse for the single memoryless classical quantum channel in [27] for any δ when n is sufficiently large it holds

$$nR \leq \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \chi\left(P_{X'}; \bar{\rho}^{\otimes n}(x(\cdot, k))\right) + n\delta$$

Let X be the random variable taking value on $u(\mathcal{J})$ such that $P_X^n(x(j)) = \sum k \in \mathcal{K}P_{X'}(x(j,k))$. Let G_{uni} be the uniformly distributed random variable with value in \mathcal{K} . When n is sufficiently large we have

$$\frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \chi\left(P_{X'}; \bar{\rho}^{\otimes n}(x(\cdot, k))\right) - \chi\left(P_X; \bar{\rho}^{\otimes n}(x(\cdot))\right) \\
= \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} S\left(\frac{1}{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} x(j, k)\right) - \frac{1}{|\mathcal{K}|} \frac{1}{|\mathcal{J}|} \sum_{k \in \mathcal{K}} \sum_{j=1}^{|\mathcal{J}|} S\left(x(j, k)\right) \\
- S\left(\frac{1}{|\mathcal{K}|} \frac{1}{|\mathcal{J}|} \sum_{k \in \mathcal{K}} \sum_{j=1}^{|\mathcal{J}|} x(j, k)\right) + \frac{1}{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} S\left(\frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} x(j, k)\right) \\
= \frac{1}{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} \chi\left(G_{uni}, \bar{\rho}^{\otimes n}(x(j, k))\right) - \chi\left(G_{uni}, \sum_{j=1}^{|\mathcal{J}|} P_X(j)\bar{\rho}^{\otimes n}(x(j, k))\right) \\
\leq \frac{1}{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} \chi\left(G_{uni}, \bar{\rho}^{\otimes n}(x(j, k))\right) \leq \frac{1}{|\mathcal{J}|} \sum_{j=1}^{|\mathcal{J}|} H\left(G_{uni}\right) \\
= 2\log n + \log b \leq n\delta.$$
(62)

Now we assume that the jammer chooses the *t*th component s_t of random state sequence **S** according to the *t*th outcome of the random input **X'** and the conditional distribution Q_t for t = 1, 2, ..., n. By applying first Holevo bound to the ensemble $\{(P_{\mathbf{X}}(\mathbf{x}), \bar{\rho}^{\otimes n*}(\mathbf{x})), \mathbf{x} \in \mathcal{X}^n\}$, for the classical quantum channel

$$\bar{\bar{\rho}}^{\otimes n*}(\mathbf{x}) = \bigotimes_{t=1}^{n} \left[\sum_{s_t} Q_t(s_t|x_t)\rho(x_t,s_t)\right] = \bigotimes_{t=1}^{n} \bar{\bar{\rho}}_t(x_t)$$
(63)

for $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{s} = (s_1, s_2, \dots, x_n)$, and then subadditivity of von Neumann entropy we obtain that

$$nR \leq \chi(P_{\mathbf{X}}, \bar{\rho}^{\otimes n*}(\cdot)) + n\delta(\lambda) = S(\sum_{\mathbf{x}} P_{\mathbf{X}}(\mathbf{x})\bar{\rho}^{\otimes n*}(\mathbf{x})) - \sum_{\mathbf{x}} P_{\mathbf{X}}(\mathbf{x})S(\bar{\rho}^{\otimes n*}(\mathbf{x})) + 2n\delta$$

$$\leq \sum_{t=1}^{n} S(\sum_{x_{t}} P_{X_{t}}(x_{t}) \sum_{s_{t}} Q_{t}(s_{t}|x_{t})\rho(x_{t},s_{t})) - \sum_{\mathbf{x}} P_{\mathbf{X}}(\mathbf{x})S(\bar{\rho}^{\otimes n*}(\mathbf{x})) + 2n\delta$$

$$= \sum_{t=1}^{n} S(\sum_{x_{t}} P_{X_{t}}(x_{t}) \sum_{s_{t}} Q_{t}(s_{t}|x_{t})\rho(x_{t},s_{t})) - \sum_{t=1}^{n} [\sum_{x_{t}} P_{X_{t}}(x_{t})S(\sum_{s} Q_{t}(s_{t}|x_{t})\rho(x_{t},s_{t}))] + 2n\delta$$

$$= \sum_{t=1}^{n} [S(\sum_{x_{t}} P_{X_{t}}(x_{t}) \sum_{s_{t}} Q_{t}(s_{t}|x_{t})\rho(x_{t},s_{t})) - \sum_{x_{t}} P_{X_{t}}(x_{t})S(\sum_{s} Q_{t}(s_{t}|x_{t})\rho(x_{t},s_{t}))] + 2n\delta$$

$$= \sum_{t=1}^{n} [S(\sum_{x_{t}} P_{X_{t}}(x_{t}) \bar{\rho}_{t}(x_{t})) - \sum_{x_{t}} P_{X_{t}}(x_{t})S(\bar{\rho}_{t}(x_{t}))] + 2n\delta = \sum_{t=1}^{n} \chi(P_{X_{t}}, \bar{\rho}_{t}(\cdot)) + 2n\delta \qquad (64)$$

where the first and the last equalities follow from the definition of Holevo quantity; the first inequality holds by (63) and the subadditivity of von Neumann entropy; and the second equality follows from (63); the second last equality follows from (61).

 \bar{W} is a compact set, and $\chi(\cdot, \cdot)$ is a concave-convex function, therefore by the Minimax Theorem we have

$$\max_{P} \min_{\bar{\rho}(\cdot)} \chi(P, \bar{\bar{\rho}}(\cdot)) = \min_{\bar{\rho}(\cdot)} \max_{P} \chi(P, \bar{\bar{\rho}}(\cdot)).$$

From (64) and (62) we have that

$$R \leq \min_{\bar{\rho}(\cdot)} \frac{1}{n} \sum_{t=1}^{n} \chi(P_{X_t}, \bar{\rho}(\cdot)) + 2n\delta$$

$$\leq \min_{\bar{\rho}(\cdot)} \max_{P_X} \chi(P_{P_X}, \bar{\rho}(\cdot)) + 2n\delta$$

$$= \max_{P_X} \min_{\bar{\rho}(\cdot)} \chi(P_{P_X}, \bar{\rho}(\cdot)) + 2n\delta.$$
(65)

(65) proves Theorem 2 for codes with a vanishing key rate.

Now we want to prove Theorem 2 for codes with an arbitrary key rate For the proof of (62) we assume that the key rate is vanishing. In fact (62) also holds with arbitrary key size $|\mathcal{K}|$ when we limit the amount of common randomness. similar to the results for classical arbitrarily varying wiretap channel in [25].

Lemma 3 (cf. [11]): Let c > 0. For every $q \in P(S)$ and $s^n \in S^n$, let a function $I_{q,s^n} : \Gamma \to (0,c)$ be given. Assume these functions satisfy the following: for every $\gamma \in \Gamma$, $s^n \in \theta^n$, and $q, q' \in P(\theta)$ satisfy $||q - q'||_1 \le \delta$

$$|I_{q,s^n}(\gamma) - I_{q',s^n}(\gamma)| \le f(\delta)$$
,

for some $f(\delta)$ which tends to 0 as δ tends to 0. We write $\mu(I_{q,s^n}) := \sum_{\gamma \in \Gamma} \mu(\gamma) I_{q,s^n}(\gamma)$, where $\mu(\gamma)$ is the probability of γ . Then for every $\varepsilon > 0$ and sufficiently large n, there are $L = n^2$ realizations $\gamma_1, \dots, \gamma_L$ such that

$$\frac{1}{L}\sum_{l=1}^{L} I_{q,s^n}(\gamma_l) \ge (1-\varepsilon)\mu(I_{q,s^n}) - \varepsilon$$

for every $q \in P(\theta)$ and $s^n \in \theta^n$.

For a conditional distribution Q on ${\mathcal S}$ and $\bar{\bar{\rho}}(x)=\sum_s Q(\cdot|x)\rho(x,s)$ we define

$$I_{Q,s^n}(k) := \frac{1}{n} \chi(P_X; \chi(P_X; \bar{\rho}^{\otimes n}(x(j,k))))$$

In [14] the continuity of $Q(\cdot|x) \to \sum_{s} Q(\cdot|x)\rho(x,s)$ has been shown; thus when for any conditional distribution Q' on S fulfilling $\|Q(\cdot|x) - Q'(\cdot|x)\|_1 = \delta \to 0$ for all x there is a $f(\delta)$ such that $|I_{Q,s^n}(k) - I_{Q',s^n}(k)| = \frac{1}{n} \frac{1}{|\mathcal{K}|} \sum_{k=1}^{|\mathcal{K}|} \chi(P_X; \bar{\rho}^{\otimes n}(x(j,k)) - \frac{1}{n} \frac{1}{|\mathcal{K}|} \sum_{k=1}^{|\mathcal{K}|} (\chi(P_X; \bar{\rho}^{\otimes n'}(x(j,k))) \leq f(\delta)$ for a $f(\delta)$ that fulfills $f(\delta) \to 0$, where $\bar{\rho}'(x) := \sum_s Q'(s|x)\rho(x,s)$. By Lemma 3 there is a set $\mathcal{K}' \subset \mathcal{K}$ such that $|\mathcal{K}'| = n^2$ and

$$\frac{1}{|\mathcal{K}'|} \frac{1}{n} \sum_{k' \in \mathcal{K}'} \chi(P_X \bar{\rho}^{\otimes n}(x(j,k')))$$

$$\geq (1 - \varepsilon) \frac{1}{n} \frac{1}{|\mathcal{K}|} \sum_{k \in \mathcal{K}} \chi(P_X; \bar{\rho}^{\otimes n}(x(j,k))) .$$

Thus

$$\frac{1}{n} \log |\mathcal{J}| \leq \frac{1}{1 - \varepsilon} \frac{1}{n} \frac{1}{|\mathcal{K}'|} \sum_{k \in \mathcal{K}'} \left(\chi(P_X; \bar{\rho}^{\otimes n}(x(j,k))) + \delta \right) \\ \leq \frac{1}{1 - \varepsilon} \frac{1}{n} \max_{P_X} \min_{\bar{\rho}(\cdot)} \chi(P_{P_X}, \bar{\rho}(\cdot)) + 2\delta .$$
(66)

(66) shows that (62) is even then true if we do not have a vanishing key rate, i.e., when we do not have $|\mathcal{K}| \leq bn^2$.

VI. PROOF OF THEOREM 3

The proof will be done by modification of the step 3 of proof of Theorem 1 in Subsection IV-C of Section IV to have a code achieving the full capacity not only in scenario 1, but also in scenario 2, as follows.

Let a ground set of codeword $\mathcal{B} = \{\mathbf{x}(i), i \in \mathcal{I}\}$ be generated in Subsection IV-B and $A_n, |\mathcal{I}_n|, |\mathcal{J}_n|, |\mathcal{K}_n|, \mu_n$ and λ'_n be given in Subsection IV-E. Additionally, without loss of generality, we require $|\mathcal{I}_n|$ is divided by $|\mathcal{J}_n|$, i. e., $B_n := \frac{|\mathcal{I}_n|}{|\mathcal{J}_n|} = \frac{na_1 \log_e |\mathcal{X}||\mathcal{S}|}{(a_2\lambda_n)^2}$ is an integer. Thus we may partition $|\mathcal{I}_n|$ into $|\mathcal{J}_n|$ subsets, $\mathcal{I}_n(j), j \in \mathcal{J}_n$ with equal size $B_n = \frac{|\mathcal{I}_n|}{|\mathcal{J}_n|}$ in an arbitrary way. Let $\mathcal{B}(j) = \{\mathbf{x}(i) : i \in \mathcal{I}_n(j)\}$ for $j \in \mathcal{J}_n$. Let $\mathbf{U}'(j,k)$ be independently and uniformly generated from $\mathcal{B}(j)$ for $j \in \mathcal{J}_n$ respectively and all $k \in \mathcal{K}_n$. Then for all $\mathbf{x} \in \mathcal{T}_X^n, \mathbf{s} \in \mathcal{S}^n$ and $k \in \mathcal{K}_n$, we have that

$$\mathbb{E}tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s}) \sum_{j \in \mathcal{J}_n} \mathcal{P}(\mathbf{U}'(j, k))] = \sum_{j \in \mathcal{J}_n} \mathbb{E}tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{U}'(j, k))]$$
$$= \sum_{j \in \mathcal{J}_n} [\sum_{i(j) \in \mathcal{I}_n(j)} \frac{1}{B_n} tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}(i(j)))]$$
$$= \frac{1}{B_n} \sum_{i \in \mathcal{I}_n} tr[\rho^{\otimes n}(\mathbf{x}, \mathbf{s})\mathcal{P}(\mathbf{x}(i))] \leq 3A_n |\mathcal{J}_n|,$$
(67)

where the last equality holds because $\{\mathcal{I}_n(j), j \in \mathcal{J}_n\}$ is a partition of \mathcal{I}_n ; and the last inequality follows from (27) and $B_n = \frac{|\mathcal{I}_n|}{|\mathcal{J}_n|}$. Because of the independence of $\mathbf{U}'(j,k), j \in \mathcal{J}_n$, by Markov inequality we have that for all $j \in \mathcal{J}_n, i(j) \in \mathcal{I}_n(j)$ and $\mathbf{s} \in S^n$,

$$Pr\{\sum_{j'\in\mathcal{J}_n\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{U}'(j,k),\mathbf{s})\mathcal{P}(\mathbf{U}'(j',k))] > \mu_n | \mathbf{U}'(j,k) = \mathbf{x}(i(j))\}]$$

$$\leq \frac{\mathbb{E}\{\sum_{j'\in\mathcal{J}_n\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i(j)),\mathbf{s})\mathcal{P}(\mathbf{U}'(j',k))] | \mathbf{U}'(j,k) = \mathbf{x}(i(j))\}}{\mu_n}$$

$$\leq \frac{\mathbb{E}\{\sum_{j'\in\mathcal{J}_n} tr[\rho^{\otimes n}(\mathbf{x}(i(j))),\mathbf{s})\mathcal{P}(\mathbf{U}'(j',k))]\}}{\mu_n}$$

$$\leq \frac{3A_n |\mathcal{J}_n|}{\mu_n}, \tag{68}$$

which is analogue to (30), where the fist inequality is Markov inequality; the second inequality holds because $\mathbf{U}'(j,k), j \in \mathcal{J}_n$ are independent and each with probability one not small than 0; the last inequality follows from (67).

Next for all $j \in \mathcal{J}_n, i(j) \in \mathcal{I}_n(j), k \in \mathcal{K}_n$ and $\mathbf{s} \in \mathcal{S}^n$, let $\mathcal{E}'(i(j), \mathbf{s}, k; \mu_n)$ be the random event that $\mathbf{U}'(j, k) = \mathbf{x}(i(j))$ and

$$\sum_{j'\in\mathcal{J}_n\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i(j)),\mathbf{s})\mathcal{P}(\mathbf{U}'(j',k))] > \mu_n,$$

and

$$Z'_{i(j)}(k) = \begin{cases} 1 & \text{if } \mathbf{U}'(j,k) = \mathbf{x}(i(j)) \\ 0 & \text{else.} \end{cases}$$

Then we have that for all $j \in \mathcal{J}_n, i(j) \in \mathcal{I}_n(j)$ and $k \in \mathcal{K}_n$

$$Pr\{Z'_{i(j)}(k) = 1\} = \frac{1}{B_n} = \frac{|\mathcal{I}_n|}{|\mathcal{J}_n|},\tag{69}$$

and analogously to (31)

$$Pr\{\mathcal{E}'(i(i), \mathbf{s}, k; \mu_n)\}$$

$$= Pr(\mathbf{U}'(j, k) = \mathbf{x}(i(j)))Pr\{\sum_{j'\in\mathcal{J}_n\setminus\{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i(j)), \mathbf{s})\mathcal{P}(\mathbf{U}'(j', k))] > \mu_n | \mathbf{U}'(j, k) = \mathbf{x}(i(j))\}$$

$$< \frac{3A_n |\mathcal{J}_n|}{B_n \mu_n} = \frac{3A_n |\mathcal{J}_n|^2}{|\mathcal{I}_n|\mu_n}.$$
(70)

Thus as we did in Subsection IV-C, by Lemma 2, $\mathbf{U}'(j,k), j \in \mathcal{J}_n, k \in \mathcal{K}_n$ has a realization $\mathbf{u}'(j,k), j \in \mathcal{J}_n, k \in \mathcal{K}_n$ with

$$\mathbf{u}'(j,k) \in \mathcal{B}(j) \tag{71}$$

for all $j \in \mathcal{J}_n$ and $k \in \mathcal{K}_n$ (which implies that $i \neq i'$ if $\mathbf{u}'(j,k) = \mathbf{x}(i)$ and $\mathbf{u}'(j',k) = \mathbf{x}(i')$ for $j \neq j'$),

$$|\mathcal{K}'(i(j))| \geq \frac{|\mathcal{K}_n||\mathcal{J}_n|}{2|\mathcal{I}_n|} \text{ and } |\mathcal{K}'_0(i(i),\mathbf{s})| \leq \frac{9|\mathcal{J}_n||\mathcal{K}_n|\lambda'_n}{2|\mathcal{I}_n|}$$

for

$$\mathcal{K}'(i(j)) := \{k : \mathbf{u}(j,k) = \mathbf{x}(i(j))\}$$

and

$$\mathcal{K}_0'(i(j),\mathbf{s}) := \{k : \mathbf{u}(j,k) = \mathbf{x}(i(j)) \text{ and } \sum_{j' \in \mathcal{J}_n \setminus \{j\}} tr[\rho^{\otimes n}(\mathbf{x}(i(j)),\mathbf{s})\mathcal{P}(\mathbf{u}(j',k))] > \mu_n\}$$

Then it follows the rest part of proof of Theorem 1 in Section IV, we obtain a RCWJKI code with rate $\min_{\bar{\rho}(\cdot)\in\bar{W}}\chi(P_X,\bar{\rho}(\cdot))-\epsilon$, average probability of error λ_n and size $\frac{an^2}{\lambda^3}$. Now the scenario 1 here, for which we have now constructed a code, is actually scenario 2, too, because by (71), that the jammer knows the input codeword $\mathbf{u}'(j,k)$ implies that he knows the message j as well. Thus our proof is completed.

APPENDIX A

PROOF OF LEMMA 2

Now let us show Lemma 2

$$\begin{split} ⪻\{\sum_{l=1}^{L}B_{j} > Lp_{1}(1+\alpha)\}\\ &= Pr\{exp_{e}[-\frac{\alpha}{2}Lp_{1}(1+\alpha) + \frac{\alpha}{2}\sum_{j=1}^{L}B_{l}] > 1\}\\ &\leq exp_{e}[-\frac{\alpha}{2}Lp_{1}(1+\alpha)]\prod_{l=1}^{L}\mathbb{E}e^{\frac{\alpha}{2}B_{l}}\\ &= exp_{e}[-\frac{\alpha}{2}Lp_{1}(1+\alpha)]\prod_{l=1}^{L}[(1-p) + e^{\frac{\alpha}{2}}p]\\ &\leq exp_{e}[-\frac{\alpha}{2}Lp_{1}(1+\alpha)][1 + (e^{\frac{\alpha}{2}} - 1)p_{1}]^{L}\\ &< exp_{e}[-\frac{\alpha}{2}Lp_{1}(1+\alpha)][1 + (\frac{\alpha}{2} + \frac{e\alpha^{2}}{8})p_{1}]^{L}\\ &< exp_{e}\{[-\frac{\alpha}{2}Lp_{1}(1+\alpha)] + (\frac{\alpha}{2} + \frac{e\alpha^{2}}{8})Lp_{1}\}\\ &= exp_{E}\{-\frac{\alpha}{2}Lp_{1}[(1+\alpha) - (1 + \frac{e\alpha}{4})]\}\\ &< e^{-\frac{\alpha^{2}}{8}Lp_{1}}, \end{split}$$

where the first inequality follows from Markov inequality and the assumption B_1, B_2, \ldots, B_L are independent; the third and fourth inequalities follows from the inequalities $e^x < 1 + x + \frac{e}{2}x^2$ for $x \in (0, 1)$ and $1 + x < e^x$ for x > 0 respectively. That is (32). Similarly instead of the inequalities $e^x < 1 + x + \frac{e}{2}x^2$ for $x \in (0, 1)$ and $1 + x < e^x$ for x > 0 we use $e^{-x} < 1 - x + \frac{1}{2}x^2$ for $x \in (0, 1)$ and $1 - x < e^{-x}$ for x > 0 and have

$$\begin{aligned} ⪻\{\sum_{l=1}^{L}B_{l} < Lp_{0}(1-\alpha)\} \\ &= Pr\{exp_{e}[\frac{\alpha}{2}Lp_{0}(1-\alpha) - \frac{\alpha}{2}\sum_{l=1}^{L}B_{l}] > 1\} \\ &\leq exp_{e}[\frac{\alpha}{2}Lp_{0}(1-\alpha)] \prod_{l=1}^{l} \mathbb{E}e^{-\frac{\alpha}{2}B_{l}} \\ &= exp_{e}[\frac{\alpha}{2}Lp_{0}(1-\alpha)] \prod_{l=1}^{L}[(1-p) + e^{-\frac{\alpha}{2}}p] \\ &\leq exp_{e}[\frac{\alpha}{2}Lp_{0}(1+\alpha)][1 - (1 - e^{\frac{-\alpha}{2}})p_{0}]^{L} \\ &< exp_{e}[\frac{\alpha}{2}Lp_{0}(1-\alpha)][1 - (\frac{\alpha}{2} - \frac{\alpha^{2}}{8})p_{0}]^{L} \\ &< exp_{e}\{[\frac{\alpha}{2}Lp_{0}(1-\alpha)] - (\frac{\alpha}{2} - \frac{\alpha^{2}}{8})Lp_{0}\} \\ &= exp_{E}\{\frac{\alpha}{2}Lp_{0}[(1-\alpha) - (1 - \frac{\alpha}{4})]\} \\ &< e^{-\frac{3\alpha^{2}}{8}Lp_{0}}. \end{aligned}$$

that is (33).

ACKNOWLEDGMENT

Support by the Bundesministerium für Bildung und Forschung (BMBF) via Grant 16KIS0118K is gratefully acknowledged.

REFERENCES

- R. Ahlswede, A note on the existence of the weak capacity for channels with arbitrarily varying channel probability functions and its relation to Shannon's zero error capacity, The Annals of Mathematical Statistics, Vol. 41, No. 3, 1970.
- [2] R. Ahlswede, Elimination of correlation in random codes for arbitrarily varying channels, Z. Wahrscheinlichkeitstheorie verw. Gebiete, Vol. 44, pp. 159-175, 1978.
- [3] R. Ahlswede, The maximal error capacity of arbitrarily varying channels for constant list sizes, IEEE Trans. Inform Theory, Vol. IT-39, pp. 1416-1417, 1993.

- [4] R. Ahlswede, I. Bjelaković, H. Boche, and J. Nötzel, Quantum capacity under adversarial quantum noise: arbitrarily varying quantum channels, Comm. Math. Phys. A, Vol. 317, No. 1, pp. 103-156, 2013.
- [5] R. Ahlswede and V. Blinovsky, Classical capacity of classical-quantum arbitrarily varying channels, IEEE Trans. Inform. Theory, Vol. 53, No. 2, pp. 526-533, 2007.
- [6] I. Bjelaković and H. Boche, Classical capacities of averaged and compound quantum channels. IEEE Trans. Inform. Theory, Vol. 57, No. 7, pp. 3360-3374, 2009.
- [7] I. Bjelaković, H. Boche, G. Janßen, and J. Nötzel, Arbitrarily varying and compound classical-quantum channels and a note on quantum zero-error capacities, Information Theory, Combinatorics, and Search Theory, in Memory of Rudolf Ahlswede, H. Aydinian, F. Cicalese, and C. Deppe eds., LNCS Vol.7777, pp. 247-283, arXiv:1209.6325, 2012.
- [8] I. Bjelaković, H. Boche, and J. Sommerfeld, Capacity results for arbitrarily varying wiretap channels, Information Theory, Combinatorics, and Search Theory, in Memory of Rudolf Ahlswede, H. Aydinian, F. Cicalese, and C. Deppe eds., LNCS Vol.7777, pp. 114-129, arXiv:1209.5213, 2012.
- [9] D. Blackwell, L. Breiman, and A. J. Thomasian, The capacities of a certain channel classes under random coding, Ann. Math. Statist. Vol. 31, No. 3, pp. 558-567, 1960.
- [10] H. Boche, M. Cai, N. Cai, Message transmission over classical quantum channels with a jammer with side information, message transmission capacity and results, on arXiv, 2018.
- [11] H. Boche, M. Cai, C. Deppe, and J. Nötzel, Classical-quantum arbitrarily varying wiretap channel: cCommon randomness assisted code and continuity, Quantum Information Processing, Vol. 16, No. 1, 1-48, 2016.
- [12] H. Boche, M. Cai, C. Deppe, and J. Nötzel, Classical-quantum arbitrarily varying wiretap channel: secret message transmission under jamming attacks, Journal of Mathematical Physics, Vol. 58, pp. 102203, 2017.
- [13] H. Boche and J. Nötzel, Arbitrarily small amounts of correlation for arbitrarily varying quantum channel, J. Math. Phys., Vol. 54, No. 11, pp. 112202, arXiv 1301.6063, 2013.
- [14] H. Boche and J. Nötzel, Positivity, discontinuity, finite resources, and nonzero error for arbitrarily varying quantum channels, J. Math. Phys., Vol. 55, 122201, 2014.
- [15] N. Cai, Localized error correction in projective space, IEEE Trans. Inform Theory, Vol.59, pp. 3282-3294, 2013.
- [16] I. Csiszár and P. Narayan, The capacity of the arbitrarily varying channel revisited: positivity, constraints, IEEE Trans. Inform. Theory, Vol. 34, No. 2, 181-193, 1988.
- [17] T. Ericson, Exponential error bounds for random codes in the arbitrarily varying channel, IEEE Trans. Inform. Theory, Vol. 31, No. 1, 42-48, 1985.
- [18] M. Hayashi, Universal coding for classical-quantum channel, Comm. Math. Phys., Vol. 289, No. 3, pp. 1087-1098, 2009.
- [19] A. S. Holevo, The capacity of quantum channel with general signal states, IEEE Trans. Inform. Theory, Vol. 44, pp. 269-273, 1998.
- [20] S. Karumanchi, S. Mancini, A. Winter, and D. Yang, Quantum channel capacities with passive environment sssistance, IEEE Trans. Inf. Theory, Vol. 62, No.4, pp. 1733-1747, arXiv: 1407.8160v2, 2016
- [21] S. Karumanchi, S. Mancini, A. Winter, and D. Yang, classical capacities of quantum channels with environment assistance, Problems Inf. Transm., Vol. 52, No. 3, pp. 214-238, arXiv: 1602.02036v2, 2016.
- [22] Anand D. Sarwate, Robust and adaptive communication under uncertain interference, Technical Report No. UCB/EECS-2008-86, University of California at Berkeley, 2008.
- [23] B. Schumacher and M. A. Nielsen, Quantum data processing and error correction, Phys. Rev. A, Vol. 54, pp. 2629, 1996.
- [24] B. Schumacher and M. D. Westmoreland, Sending classical information via noisy quantum channels, Phys. Rev., Vol. 56, pp. 131-138, 1997.
- [25] M. Wiese, J. Nötzel, and H. Boche, The arbitrarily varying wiretap channel-deterministic and correlated random coding capacities under the strong secrecy criterion, IEEE Trans. Inform. Theory, Vol. 62, No. 7, pp. 3844-3862, arXiv:1410.8078, 2016.
- [26] M. Wilde, Quantum Information Theory, Cambridge University Press, 2013.
- [27] A. Winter, Coding theorem and strong converse for quantum channels, IEEE Trans. Inform. Theory, Vol. 45, No. 7, pp. 2481-2485, 1999.