Semi-Robust Communications over a Broadcast Channel

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Abstract

We establish the deterministic-code capacity region of a network with one transmitter and two receivers: an "ordinary receiver" and a "robust receiver." The channel to the ordinary receiver is a given (known) discrete memoryless channel, whereas the channel to the robust receiver is an arbitrarily varying channel. Both receivers are required to decode the "common message," whereas only the ordinary receiver is required to decode the "private message."

Index Terms

Arbitrarily varying channel, binary symmetric, broadcast channel, degraded message set, robust communications.

I. Introduction

As in Figure 1, two independent data streams—a rate- $R_{\rm c}$ common data stream and a rate- $R_{\rm p}$ private data stream—are to be transmitted over a broadcast channel with two receivers: an "ordinary receiver" and a "robust receiver." The channel to the ordinary receiver, the receiver that is required to recover both streams reliably, is a given (known) discrete memoryless channel (DMC) W(y|x). The channel to the robust receiver, the receiver that is only required to recover the common stream, is an arbitrarily varying channel (AVC) [1]. The set of rate pairs $(R_{\rm c}, R_{\rm p})$ that can be communicated reliably under these requirements is the *capacity region*, which we derive here.

The scenario where one receiver must recover both streams and the other only one, falls under the heading of *degraded message sets*. The capacity region of the broadcast channel with degraded message sets was established by Körner and Marton in [2]. But their model differs from ours because their broadcast channel is fixed and given: there is nothing "varying" about it.

Our network can be viewed as an arbitrarily varying broadcast channel (AVBC) of a special kind: one where the channel to one of the receivers is degenerate in the sense of being given and not depending on the state. General AVBCs where studied by Jahn [3] who derived an inner bound on their capacity regions, and our achievability result essentially follows from his. Our converse shows that in our setting the inner bound is tight.

More recent results on the AVBC for settings with causal and noncausal side information were obtained by Pereg and Steinberg [4]–[6].

II. THE MAIN RESULT

A discrete memoryless state-dependent broadcast channel $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{S}, \mathsf{W}_{Y,Z|X,S})$ consists of a finite input alphabet \mathcal{X} , finite output alphabets \mathcal{Y} and \mathcal{Z} , a (not necessarily finite) state set \mathcal{S} , and a collection of transition probability matrices $\mathsf{W}_{Y,Z|X,S}$. A semi-AVBC (SAVBC) is a state-dependent broadcast channel where the conditional law of the output Y given the input x and the state s does not depend on the state. For such a channel, we denote the marginal conditional distributions of the outputs Y and Z given the input x and the state s by $\mathsf{W}(y|x)$ and $\mathsf{V}_s(z|x)$ respectively:

$$W(y|x) = W_{Y|X,S}(y|x,s), \tag{1a}$$

$$V_s(z|x) = W_{Z|X,S}(z|x,s). \tag{1b}$$

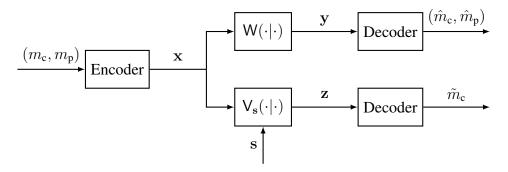


Fig. 1. The semi-arbitrarily-varying broadcast channel (semi-AVBC) with common message m_c , private message m_p , and state sequence $s \in \mathcal{S}^n$.

Given a blocklength n, an input sequence $\mathbf{x} \in \mathcal{X}^n$, and a state sequence $\mathbf{s} \in \mathcal{S}^n$,

$$W_{Y^n,Z^n|X^n,S^n}(\mathbf{y},\mathbf{z}|\mathbf{x},\mathbf{s}) = \prod_{i=1}^n W_{Y,Z|X,S}(y_i,z_i|x_i,s_i), \quad (\mathbf{y},\mathbf{z}) \in \mathcal{Y}^n \times \mathcal{Z}^n.$$
 (2)

We consider the transmission from degraded message sets: the encoder sends a common message m_c to both receivers and a private message m_p to the receiver observing Y. The receiver observing Z is thus only required to decode the common message.

Given a blocklength n, a deterministic code C for the SAVBC consists of a common message set \mathcal{M}_c with 2^{nR_c} messages, a private message set \mathcal{M}_p with 2^{nR_p} messages, an encoder mapping

$$f: \mathcal{M}_{c} \times \mathcal{M}_{p} \to \mathcal{X}^{n},$$
 (3)

and decoding mappings

$$\phi_y \colon \mathcal{Y}^n \to \mathcal{M}_c \times \mathcal{M}_p$$
 (4a)

$$\phi_z \colon \mathcal{Z}^n \to \mathcal{M}_c.$$
 (4b)

The message-averaged probability of error of a code $\mathcal C$ given a state sequence $\mathbf s \in \mathcal S^n$ is

$$P_{e|\mathbf{s}}^{(n)}(\mathcal{C}) = \frac{1}{|\mathcal{M}_{c}||\mathcal{M}_{p}|} \sum_{(m_{c}, m_{p}) \in \mathcal{M}_{c} \times \mathcal{M}_{p}} \sum_{(\mathbf{y}, \mathbf{z}) \notin \mathcal{D}(m_{c}, m_{p})} W_{Y^{n}, Z^{n}|X^{n}, S^{n}}(\mathbf{y}, \mathbf{z}|\mathbf{x}, \mathbf{s}),$$
(5)

where

$$\mathcal{D}(m_{c}, m_{p}) = \{ (\mathbf{y}, \mathbf{z}) \in \mathcal{Y}^{n} \times \mathcal{Z}^{n} : \phi_{y}(\mathbf{y}) = (m_{c}, m_{p}), \phi_{z}(\mathbf{z}) = m_{c} \}.$$

$$(6)$$

We say that the rate pair (R_c, R_p) is achievable with deterministic codes, if there exists a sequence of codes $\{C_n\}$ with rates (R_c, R_p) such that

$$\lim_{n \to \infty} \sup_{\mathbf{s} \in \mathcal{S}^n} P_{\mathbf{e}|\mathbf{s}}^{(n)}(\mathcal{C}_n) = 0.$$
 (7)

The deterministic code capacity \mathcal{C}_{det} (under the average-probability-of-error criterion) of the SAVBC is the closure of the set of rate pairs that are achievable with deterministic codes.

As in [7, Corollary 12.3], it can be shown that the capacity region depends on the states only via the convex-closure of the channels they induce. We shall thus make the following assumption without any loss of generality:

Assumption: We shall assume throughout that $\{V_s(z|x)\}_{s\in\mathcal{S}}$ is compact¹ and convex in the sense that for every $0 < \lambda < 1$ and $s_1, s_2 \in \mathcal{S}$, there exists a state $\bar{s} \in \mathcal{S}$ such that

$$V_{\bar{s}}(z|x) = \lambda V_{s_1}(z|x) + (1-\lambda)V_{s_2}(z|x), \quad (x,z) \in \mathcal{X} \times \mathcal{Z}.$$
(8)

¹If the set $\{V_s(z|x)\}_{s\in\mathcal{S}}$ is not compact our result still holds, but with infima replacing the minima in the characterizations of the capacity region.

Following [3, Remark IIB2] or using a time-sharing argument we note:

Remark 1. The interior of \mathcal{C}_{det} is nonempty if, and only if, the capacity of the channel W(y|x) to Y and the capacity (under the average-probability-of-error criterion) of the AVC to Z are both positive. The latter is positive if, and only if, the AVC is nonsymmetrizable [8], [9].

We next define the region $\mathscr{C}_{\text{det}}^{(I)}$ that will turn out to equal the capacity region when the latter is not empty. It is defined as the closure of the union over all PMFs $p_{U,X}$ of the set of rate pairs (R_c, R_p) that satisfy

$$R_{c} \le \min_{s \in \mathcal{S}} I(U; Z)$$
 (9a)

$$R_{\rm p} \le I(X;Y|U) \tag{9b}$$

$$R_{\rm c} + R_{\rm p} \le I(X;Y),\tag{9c}$$

where the mutual informations are computed w.r.t. the joint distribution

$$p_{U,X}(u,x) \mathsf{W}(y|x) \mathsf{V}_s(z|x), \tag{10}$$

and where U is an auxiliary chance variable taking values in a finite set \mathcal{U} . Our main result is the following theorem.

Theorem 2. Under the above assumption, if the deterministic-code capacity \mathcal{C}_{det} of a SAVBC is not empty, then it equals $\mathcal{C}_{det}^{(I)}$:

$$\left(\mathscr{C}_{\det} \neq \emptyset\right) \implies \left(\mathscr{C}_{\det} = \mathscr{C}_{\det}^{(I)}\right).$$
 (11)

III. PROOF OF THE MAIN RESULT

The achievability result—that $\mathscr{C}_{\text{det}} \neq \emptyset$ implies that every rate pair $(R_{\text{c}}, R_{\text{p}})$ satisfying (9) for some $p_{U,X}$ is achievable—follows directly from Jahn [3, Theorem 2]. We therefore focus on the converse, i.e., on showing that the achievability of a rate pair $(R_{\text{c}}, R_{\text{p}})$ implies that it lies in $\mathscr{C}_{\text{det}}^{(I)}$. But before proving this, we study $\mathscr{C}_{\text{det}}^{(I)}$.

Proposition 3. The region $\mathscr{C}^{(I)}_{det}$ can also be expressed as the closure of the union over all PMFs $p_{U,X,Q}$ of the set of rate pairs (R_c, R_p) that satisfy

$$R_{c} \le \min_{s \in \mathcal{S}} I(U; Z|Q) \tag{12a}$$

$$R_{\mathbf{p}} \le I(X; Y|U, Q) \tag{12b}$$

$$R_{c} + R_{p} \le I(X; Y|Q), \tag{12c}$$

where the mutual informations are computed w.r.t. the joint distribution

$$p_{U,X,Q}(u,x,q) \mathsf{W}(y|x) \mathsf{V}_s(z|x), \tag{13}$$

and where U and Q are auxiliary chance variables taking values in the finite sets U and Q.

Proof: One inclusion is obvious and simply follows by setting Q to be deterministic. We therefore focus on the other, namely, on showing that if there exists some joint PMF $p_{U,X,Q}$ under which the pair (R_c, R_p) satisfies (12), then there exists some auxiliary chance variable \tilde{U} and a PMF $p_{\tilde{U},X}$ under which the pair satisfies (9) when we substitute \tilde{U} for U. To this end we choose $\tilde{U} = (U,Q)$ and show that the

results of substituting \tilde{U} for U on the RHS of each of the inequalities in (9) is at least as high as the RHS of the corresponding inequality in (12):

$$\begin{split} \min_{s \in \mathcal{S}} I(\tilde{U}; Z) &= \min_{s \in \mathcal{S}} I(U, Q; Z) \\ &= \min_{s \in \mathcal{S}} \left\{ I(U; Z|Q) + I(Q; Z) \right\} \\ &\geq \min_{s \in \mathcal{S}} I(U; Z|Q); \end{split}$$

$$I(X; Y|\tilde{U}) = I(X; Y|U, Q);$$

and

$$I(X;Y) = I(X,Q;Y)$$

$$= I(Q;Y) + I(X;Y|Q)$$

$$\geq I(X;Y|Q),$$
(14)

where (14) follows from the Markovity $Q \rightarrow X \rightarrow Y$.

From Proposition 3 we obtain:

Proposition 4. The region $\mathscr{C}_{\det}^{(I)}$ is a compact convex set containing the rate pairs

$$\left(\min\left\{C_{Sh}(\mathsf{W}), \min_{s \in \mathcal{S}} C_{Sh}(\mathsf{V}_s)\right\}, 0\right)$$
(15a)

and

$$(0, C_{\operatorname{Sh}}(\mathsf{W})), \tag{15b}$$

where $C_{Sh}(W)$ denotes the Shannon capacity of the channel W. Moreover, $\mathscr{C}_{det}^{(I)}$ is included in the triangle with vertices

$$(0,0), (C_{Sh}(W),0), (0,C_{Sh}(W)).$$
 (16)

Proof: The convexity is due to the auxiliary chance variable Q. To see that the pair (15a) is in $\mathscr{C}^{(I)}_{\det}$, consider choosing Q to be deterministic and U to equal X. To see that the pair (15b) is in $\mathscr{C}^{(I)}_{\det}$, consider choosing both U and Q to be deterministic. The inclusion in the triangle (16) follows from $I(X;Y|Q) \leq C_{\operatorname{Sh}}(W)$.

As a final step before proving the converse, we next provide one last characterization of $\mathscr{C}_{\text{det}}^{(I)}$. To that end, let $\mathscr{C}_{\text{det}}^{(O)}$ denote the set of rate pairs $(R_{\text{c}}, R_{\text{p}})$ that satisfy

$$R_{c} \le \min_{s \in \mathcal{S}} I(U; Z|Q) \tag{17a}$$

$$R_{p} + R_{c} \le I(X; Y|U, Q) + \min_{s \in \mathcal{S}} I(U; Z|Q)$$

$$\tag{17b}$$

$$R_{\rm p} + R_{\rm c} \le I(X; Y|Q),\tag{17c}$$

for some PMF $p_{U,X,Q}$, where the mutual informations are computed w.r.t. the joint PMF of (13). The inequalities in the definition of $\mathscr{C}_{\det}^{(O)}$ thus differ from those in (12) in that we have replaced (12b) with (17b). As we shall next show, this replacement does not change the region, and $\mathscr{C}_{\det}^{(O)} = \mathscr{C}_{\det}^{(I)}$. Once we show this, we will prove the converse for $\mathscr{C}_{\det}^{(O)}$.

Proposition 5. The region $\mathscr{C}_{det}^{(O)}$, which is defined in (17), is equal to $\mathscr{C}_{det}^{(I)}$:

$$\mathscr{C}_{\text{det}}^{(O)} = \mathscr{C}_{\text{det}}^{(I)}. \tag{18}$$

Proof: We shall prove this result using the characterization of $\mathscr{C}^{(I)}_{det}$ of Proposition 3. To see why the two regions are equivalent, fix some PMF $p_{U,X,Q}$ and consider the bounds in (12) and (17). If a rate pair satisfies (12), then it also satisfies (17), because (17b) is the result of adding (12a) and (12b). Thus, $\mathscr{C}^{(I)}_{det} \subseteq \mathscr{C}^{(O)}_{det}$.

To establish the reverse inclusion we consider two cases separately.

Case I: $\min_{s \in \mathcal{S}} I(U; Z|Q) + I(X; Y|U, Q) \ge I(X; Y|Q)$.

In this case Inequality (17b) is implied by (17c) and is thus redundant. Consequently, we need to show that the trapezoid defined by (17a) and (17c) of vertices

$$(0,0), \quad \left(\min_{s \in \mathcal{S}} I(U;Z|Q), 0\right), \quad \left(\min_{s \in \mathcal{S}} I(U;Z|Q), \ I(X;Y|Q) - \min_{s \in \mathcal{S}} I(U;Z|Q)\right), \quad \left(0, I(X;Y|Q)\right)$$

is contained in $\mathscr{C}_{\text{det}}^{(I)}$. This can be shown by noting that $\mathscr{C}_{\text{det}}^{(I)}$ contains the pentagon defined by (12) of vertices

$$(0,0), \quad \left(\min_{s \in \mathcal{S}} I(U;Z|Q), 0 \right), \quad \left(\min_{s \in \mathcal{S}} I(U;Z|Q), \ I(X;Y|Q) - \min_{s \in \mathcal{S}} I(U;Z|Q) \right),$$

$$\left(I(X;Y|Q) - I(X;Y|U,Q), \ I(X;Y|U,Q) \right), \quad \left(0, I(X;Y|U,Q) \right),$$

that $\mathscr{C}_{\det}^{(I)}$ also contains the point $\{0, I(X;Y|Q)\}$ (Proposition 4); and that it thus also contains the convex hull of the union of the pentagon and the point (Proposition 4).

Case II: $\min_{s \in \mathcal{S}} I(U; Z|Q) + I(X; Y|U, Q) < I(X; Y|Q)$.

In this case (17c) is implied by (17b), and we need to show that the trapezoid defined by (17a) and (17b) of vertices

$$(0,0), \quad \left(\min_{s\in\mathcal{S}}I(U;Z|Q),0\right), \quad \left(\min_{s\in\mathcal{S}}I(U;Z|Q),I(X;Y|U,Q)\right), \quad \left(0,\ I(X;Y|U,Q)+\min_{s\in\mathcal{S}}I(U;Z|Q)\right),$$

is contained in $\mathscr{C}_{\text{det}}^{(I)}$. This can be shown by noting that $\mathscr{C}_{\text{det}}^{(I)}$ contains the rectangle defined by (12) of vertices

$$(0,0), \quad \left(\min_{s\in\mathcal{S}}I(U;Z|Q),0\right), \quad \left(\min_{s\in\mathcal{S}}I(U;Z|Q),I(X;Y|U,Q)\right), \quad \left(0,\ I(X;Y|U,Q)\right);$$

it contains the point $\{(0, I(X;Y|U,Q) + \min_{s \in \mathcal{S}} I(U;Z|Q))\}$ (because in the case under consideration $I(X;Y|U,Q) + \min_{s \in \mathcal{S}} I(U;Z|Q)$ is smaller than I(X;Y|Q) so the achievability of this point follows from Proposition 4); and because $\mathscr{C}_{\text{det}}^{(I)}$ is convex.

We shall now prove the converse by proving that no rate pair outside $\mathscr{C}_{\text{def}}^{(O)}$ is achievable.

Proof of the converse part of Theorem 2: We first note that, as in [10, Theorem 15.6.1], the capacity region depends only on the marginals (namely the given channel W(y|x) and the AVC). There is thus no loss in generality in assuming, as we shall, that Y and Z are conditionally independent given the channel input and state.

Fix finite sets \mathcal{M}_c , \mathcal{M}_p , a blocklength-n $(n, \mathcal{M}_c, \mathcal{M}_p)$ encoder

$$f: \mathcal{M}_{c} \times \mathcal{M}_{p} \to \mathcal{X}^{n}$$
 (19)

and decoders as in (4) with $\sup_{\mathbf{s} \in \mathcal{S}^n} P_{\mathbf{e}|\mathbf{s}}^{(n)}(\mathcal{C})$ tending to zero. Fix also a state sequence $\mathbf{s} = (s_1, \dots, s_n)$. Draw the message pair $(M_{\mathbf{c}}, M_{\mathbf{p}})$ uniformly over $\mathcal{M}_{\mathbf{c}} \times \mathcal{M}_{\mathbf{p}}$, and denote its distribution $p_{M_{\mathbf{c}}, M_{\mathbf{p}}}$. In view of our conditional independence assumption, for every $(m_{\mathbf{c}}, m_{\mathbf{p}}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathcal{M}_{\mathbf{c}} \times \mathcal{M}_{\mathbf{p}} \times \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$

$$P[(M_{c}, M_{p}, X^{n}, Y^{n}, Z^{n}) = (m_{c}, m_{p}, \mathbf{x}, \mathbf{y}, \mathbf{z})]$$

$$= p_{M_{c}, M_{p}}(m_{c}, m_{p}) p_{X^{n}|M_{c}, M_{p}}(\mathbf{x}|m_{c}, m_{p}) W^{n}(\mathbf{y}|\mathbf{x}) V_{s}(\mathbf{z}|\mathbf{x}),$$
(20)

where

$$p_{X^n|M_c,M_p}(\mathbf{x}|m_c,m_p) = \begin{cases} 1 & \text{if } \mathbf{x} = f(m_c,m_p) \\ 0 & \text{otherwise} \end{cases}$$
 (21)

is determined by the encoder mapping;

$$W^{n}(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} W(y_{i}|x_{i}); \tag{22}$$

and

$$V_{\mathbf{s}}(\mathbf{z}|\mathbf{x}) = \prod_{i=1}^{n} V_{s_i}(z_i|x_i). \tag{23}$$

Fano's inequality yields that for any state sequence $\mathbf{s} \in \mathcal{S}^n$

$$\log|\mathcal{M}_{c}| \le I(M_{c}; Z^{n}) + n\epsilon_{n} \tag{24a}$$

$$\log(|\mathcal{M}_{c}||\mathcal{M}_{p}|) \le I(M_{c}; Z^{n}) + I(M_{p}; Y^{n}) + n\epsilon_{n}$$
(24b)

$$\log(|\mathcal{M}_{c}||\mathcal{M}_{p}|) \le I(M_{c}, M_{p}; Y^{n}) + n\epsilon_{n}, \tag{24c}$$

where ϵ_n approaches zero uniformly in s as n tends to infinity.

For each $i \in [1:n]$ define

$$U_i = (M_c, Y_{i+1}^n, Z_1^{i-1}). (25)$$

Let Q be independent of $(M_c, M_p, X^n, Y^n, Z^n)$ and uniformly distributed over the integers [1:n]. Denote the joint PMF of $(Q, M_c, M_p, X^n, Y^n, Z^n)$ also by P, so

$$P[(Q, M_{c}, M_{p}, X^{n}, Y^{n}, Z^{n}) = (q, m_{c}, m_{p}, \mathbf{x}, \mathbf{y}, \mathbf{z})]$$

$$= \frac{1}{n} p_{M_{c}, M_{p}}(m_{c}, m_{p}) p_{X^{n}|M_{c}, M_{p}}(\mathbf{x}|m_{c}, m_{p}) W^{n}(\mathbf{y}|\mathbf{x}) V_{s}(\mathbf{z}|\mathbf{x}),$$
(26)

Define the chance variables

$$U = (U_Q, Q), \quad X = X_Q, \quad Y = Y_Q, \quad Z = Z_Q,$$
 (27)

and note that their joint distribution factorizes as in (13).

Continuing from (24), we upper-bound $R_c = \frac{1}{n} \log |\mathcal{M}_c|$ and $R_c + R_p = \frac{1}{n} \log \left(|\mathcal{M}_c| |\mathcal{M}_p| \right)$ by the following calculations under the joint PMF P of (26). By (24a)

$$R_{c} - \epsilon_{n} \leq \frac{1}{n} I(M_{c}; Z_{1}^{n})$$

$$= \frac{1}{n} \sum_{i=1}^{n} I(M_{c}; Z_{i} | Z_{1}^{i-1})$$
(28a)

$$\leq \frac{1}{n} \sum_{i=1}^{n} I(M_{c}, Y_{i+1}^{n}, Z_{1}^{i-1}; Z_{i})$$
(28b)

$$= I(U; Z|Q), (28c)$$

where (28a) and (28b) follow from the chain rule and the nonnegativity of mutual information, and (28c) follows from the definitions in (27).

By (24b),

$$R_{\rm c} + R_{\rm p} - \epsilon_n \le \frac{1}{n} \left(I(M_{\rm c}; Z_1^n) + I(M_{\rm p}; Y_1^n | M_{\rm c}) \right)$$
 (29a)

$$= \frac{1}{n} \sum_{i=1}^{n} \left(I(M_{c}; Z_{i} | Z_{1}^{i-1}) + I(M_{p}; Y_{i} | M_{c}, Y_{i+1}^{n}) \right)$$
 (29b)

$$\leq \frac{1}{n} \sum_{i=1}^{n} \left(I(M_{c}; Z_{i} | Z_{1}^{i-1}) + I(X_{i}; Y_{i} | M_{c}, Y_{i+1}^{n}) \right)$$
(29c)

$$\leq \frac{1}{n} \sum_{i=1}^{n} \left(I(M_{c}, Z_{1}^{i-1}; Z_{i}) + I(X_{i}, Z_{1}^{i-1}; Y_{i} | M_{c}, Y_{i+1}^{n}) \right) \tag{29d}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left(I(M_{c}, Y_{i+1}^{n}, Z_{1}^{i-1}; Z_{i}) - I(Y_{i+1}^{n}; Z_{i} | M_{c}, Z_{1}^{i-1}) \right)$$

$$+I(Z_1^{i-1}; Y_i|M_c, Y_{i+1}^n) + I(X_i; Y_i|M_c, Y_{i+1}^n, Z_1^{i-1}))$$
 (29e)

$$= \frac{1}{n} \sum_{i=1}^{n} \left(I(M_{c}, Y_{i+1}^{n}, Z_{1}^{i-1}; Z_{i}) + I(X_{i}; Y_{i} | M_{c}, Y_{i+1}^{n}, Z_{1}^{i-1}) \right)$$
(29f)

$$= I(U; Z|Q) + I(X; Y|U, Q), (29g)$$

where (29a) follows from (24b) (because M_c is independent of M_p); Equality (29b) follows from the chain rule; Inequality (29c) follows because—conditional on (M_c, Y_{i+1}^n) —the chance variables $M_p \multimap -X_i \multimap -Y_i$ form a Markov chain, i.e.,²

$$M_{p} \multimap (X_{i}, M_{c}, Y_{i+1}^{n}) \multimap Y_{i};$$

(29d) and (29e) follow from the chain rule and the nonnegativity of mutual information; (29f) follows from Csiszár's sum-identity; and (29g) follows from the definitions in (27).

And finally, by (24c)

$$R_{\rm c} + R_{\rm p} - \epsilon_n \le \frac{1}{n} I(M_{\rm c}, M_{\rm p}; Y_1^n)$$
 (30a)

$$= \frac{1}{n}I(X_1^n; Y_1^n)$$
 (30b)

$$\leq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Y_i) \tag{30c}$$

$$=I(X;Y|Q), (30d)$$

where (30b) holds because X^n is a deterministic function of (M_c, M_p) ; and (30c) holds because the channel to Y is memoryless and without feedback.

Inequalities (28), (29), and (30) hold for any state sequence $s \in S^n$. We next construct a specific state sequence and from it a joint PMF $\tilde{p}_{U,X,Q}$. We will then show that the rates $(R_c - \epsilon_n, R_p - \epsilon_n)$ must satisfy (17) when the latter is evaluated w.r.t. $\tilde{p}_{U,X,Q}$. To construct the state sequence, we begin by expressing I(U; Z|Q) as a sum

$$I(U;Z|Q) = \frac{1}{n} \sum_{i=1}^{n} I(U;Z|Q=i) = \frac{1}{n} \sum_{i=1}^{n} I(U_i;Z_i),$$
(31)

and consider each term separately in increasing order, starting with i = 1.

²In fact, we can replace the inequality with equality, because X_i is computable from M_c and M_p .

The joint distribution p_{U_1,X_1} is determined by the message distribution p_{M_c,M_p} , the encoder mapping (21), and the channel law W; it is uninfluenced by the state sequence. We choose $s_1 \in \mathcal{S}$ so that

$$I(U_1, Z_1) = \min_{s \in S} I(U_1; Z),$$
 (32a)

where the mutual informations are computed w.r.t. the joint distribution

$$p_{U_1,X_1}(u_1,x_1) \mathsf{V}_{s_1}(z_1|x_1) \mathsf{V}_{s}(z|x_1).$$
 (32b)

Suppose now that we have chosen the first i-1 states s_1, \ldots, s_{i-1} for some $i \in [2:n]$. These states together with the message distribution p_{M_c, M_p} , the encoder mapping (21), and the channel law W fully specify the joint distribution p_{U_i, X_i} of (U_i, X_i) . We can then choose $s_i \in \mathcal{S}$ so that

$$I(U_i, Z_i) = \min_{s \in S} I(U_i; Z), \tag{33a}$$

where the mutual informations are computed w.r.t. the joint distribution

$$p_{U_i,X_i}(u_i, x_i) \, \mathsf{V}_{s_i}(z_i | x_i) \, \mathsf{V}_s(z | x_i).$$
 (33b)

Once the entire state sequence s_1, \ldots, s_n has been chosen, the joint distributions $\{p_{U_i, X_i}\}_{i=1}^n$ are fully determined, and we can define the joint distribution $\tilde{p}_{U,X,Q}$ as one where Q is uniform over [1:n] and

$$\tilde{p}_{U,X|Q=i} = p_{U_i,X_i}, \qquad i \in [1:n].$$
 (34)

We will next show that, under this joint distribution $\tilde{p}_{U,X,Q}$, the rates $(R_c - \epsilon_n, R_p - \epsilon_n)$ must satisfy (17). Indeed, under this joint distribution, I(U; Z|Q) can be upper-bounded as

$$I(U;Z|Q) = \frac{1}{n} \sum_{i=1}^{n} I(U;Z|Q=i)$$
 (35a)

$$= \frac{1}{n} \sum_{i=1}^{n} I(U_i; Z_i)$$
 (35b)

$$= \frac{1}{n} \sum_{i=1}^{n} \min_{s \in \mathcal{S}} I(U_i; Z)$$
(35c)

$$\leq \min_{s \in \mathcal{S}} \frac{1}{n} \sum_{i=1}^{n} I(U_i; Z) \tag{35d}$$

$$= \min_{s \in \mathcal{S}} I(U; Z|Q), \tag{35e}$$

where (35c) follows from our choice of s_i in (32) and (33), and Inequality (35d) holds because for the state $s^* \in \mathcal{S}$ that minimizes the whole sum, the mutual information $I(U_i; Z^*)$ w.r.t. the joint $p_{U_i, X_i}(u_i, x_i) \bigvee_{s^*} (z^*|x_i)$ is greater or equal to $\min_{s \in \mathcal{S}} I(U_i; Z)$ w.r.t. the joint $p_{U_i, X_i}(u_i, x_i) \bigvee_{s} (z|x_i)$ for all $i \in [1:n]$.

The upper bounds (28c), (29g), (30d), and (35e) together yield that, under $\tilde{p}_{U,X,Q}$ of (34), the rates of the coding scheme are upper-bounded by

$$R_{c} \le \min_{s \in S} I(U; Z|Q) + \epsilon_{n} \tag{36a}$$

$$R_{p} + R_{c} \le I(X; Y|U, Q) + \min_{s \in \mathcal{S}} I(U; Z|Q) + \epsilon_{n}$$
(36b)

$$R_{\rm p} + R_{\rm c} \le I(X; Y|Q) + \epsilon_n, \tag{36c}$$

where the mutual informations are computed w.r.t. joint PMF

$$\tilde{p}_{U,X,Q}(u,x,q) \,\mathsf{W}(y|x) \,\mathsf{V}_s(z|x). \tag{37}$$

Having established that $(R_{\rm c}-\epsilon_n,R_{\rm p}-\epsilon_n)$ must satisfy (17), it now follows from the fact that ϵ_n tend to zero and that $\mathscr{C}_{\rm det}^{(O)}$ is closed that $(R_{\rm c},R_{\rm p})$ must be in $\mathscr{C}_{\rm det}^{(O)}$. And since the latter is equal to $\mathscr{C}_{\rm det}^{(I)}$ (Proposition 5), the pair must also be in $\mathscr{C}_{\rm det}^{(I)}$.

IV. EXAMPLE

Consider the binary symmetric semi-arbitrarily-varying broadcast channel (BS-SAVBC), where the channel to Y is a BSC(p), i.e., a binary symmetric channel (BSC) with crossover probability p, and the channel to Z is a BSC with a state-dependent crossover probability between p_{\min} and p_{\max} . The state alphabet S is the closed interval $[p_{\min}, p_{\max}]$, and we identify a state $s \in S$ with its corresponding crossover probability p_s . Thus, when the state is s, the channel from S to S is a BSC (p_s) . We focus on the case³

$$0$$

$$0 \le p_{\min} \le p_{\max} < 1/2. \tag{39}$$

In this case the capacity of the DMC to Y and of the AVC to Z are both positive (c.f. [8], [9]), and therefore (by Remark 1 and Theorem 2) the capacity region of the BS-SAVBC is $\mathscr{C}^{(I)}_{\text{det}}$. Evaluating (9) for the joint PMF $p_{U,X}$ under which

$$U \sim \text{Bernoulli}(1/2)$$
 (40a)

$$V \sim \text{Bernoulli}(\alpha)$$
 (40b)

$$X = U + V \mod 2, \tag{40c}$$

proves that $\mathscr{C}_{\text{det}}^{(I)}$ contains all the rate pairs $(R_{\text{c}}, R_{\text{p}})$ satisfying

$$R_{c} \le \min_{s \in \mathcal{S}} \left(1 - H_{b}(\alpha * p_{s}) \right) \tag{41a}$$

$$R_{\mathsf{p}} \le H_{\mathsf{b}}(\alpha * p) - H_{\mathsf{b}}(p) \tag{41b}$$

$$R_{\rm c} + R_{\rm p} \le 1 - H_{\rm b}(p) \tag{41c}$$

for some $\alpha \in [0,1/2]$. Here $H_{\rm b}(\cdot)$ denotes the binary entropy function, and $\alpha * \delta = \alpha(1-\delta) + (1-\alpha)\delta$. For a fixed $\alpha \in [0,1/2]$ the mapping $\delta \mapsto \alpha * \delta$ is nondecreasing on $(0<\delta<1/2)$, and so is $H_{\rm b}(\cdot)$. Consequently, the minimum on the RHS of (41a) is achieved when p_s equals $p_{\rm max}$, and (41) simplifies to

$$R_{\rm c} \le 1 - H_{\rm b}(\alpha * p_{\rm max}) \tag{42a}$$

$$R_{\mathbf{p}} \le H_{\mathbf{b}}(\alpha * p) - H_{\mathbf{b}}(p) \tag{42b}$$

$$R_{\rm c} + R_{\rm p} \le 1 - H_{\rm b}(p).$$
 (42c)

We next show that $\mathscr{C}_{\text{det}}^{(I)}$ contains no other rate pairs, and that it thus equals the union over all $\alpha \in [0, 1/2]$ of the polytopes defined by (42). This region is depicted in Figure 2. We do so by fixing the state to be p_{max} throughout the block and by then showing that every achievable rate pair $(R_{\text{c}}, R_{\text{p}})$ must satisfy (42) for some $\alpha \in [0, 1/2]$. To this end, we distinguish between two cases, depending on whether or not p exceeds p_{max} .

But first we note that, by the above monotonicity argument, the relation between p and p_{max} translates to the relation between $H_{\text{b}}(\alpha * p)$ and $H_{\text{b}}(\alpha * p_{\text{max}})$ as follows:

$$(p \le p_{\text{max}}) \iff (H_{\text{b}}(\alpha * p) \le H_{\text{b}}(\alpha * p_{\text{max}}))$$
 (43a)

$$(p > p_{\text{max}}) \iff (H_{\text{b}}(\alpha * p) > H_{\text{b}}(\alpha * p_{\text{max}})).$$
 (43b)

Case I: $p \leq p_{\text{max}}$.

In this case fixing the state at p_{max} results in a stochastically degraded binary-symmetric broadcast channel (BS-BC), where Z is a stochastically degraded version of Y. Since Receiver Y recovers $(M_{\text{c}}, M_{\text{p}})$, and

³When p equals 1/2 the capacity from X to Y is zero, and if we exclude this case, then—by possibily inverting Y—we can guarantee that p be in [0,1/2). Likewise, if the interval $[p_{\min}, p_{\max}]$ includes 1/2, then the capacity of the AVC from X to Z is zero. And if this is excluded, then—again by possibly inverting Z—we can restrict ourselves to the case where this interval is a subset of [0,1/2).

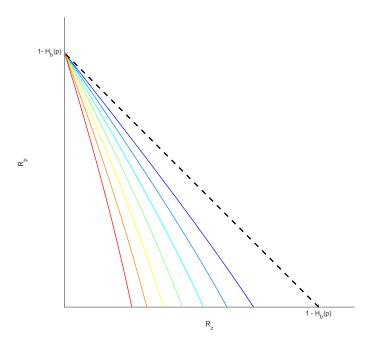


Fig. 2. The boundary of the capacity region of the binary symmetric semi-arbitrarily varying broadcast channel for various values of $p_{\text{max}} > p$. The capacity region shrinks (and eventually has an empty interior) as p_{max} increases to 1/2. If $p_{\text{max}} \le p$, the capacity region is the triangle defined by the sum-rate constraint $R_c + R_p \le 1 - H_b(p)$.

Receiver Z recovers M_c , any achievable rate pair (R_c, R_p) must be in the private-message capacity region of the above BS-BC. The latter is given by the set of rate pairs (R_c, R_p) that satisfy

$$R_{\mathbf{p}} \le I(X;Y|U) \tag{44a}$$

$$R_{\rm c} \le I(U; Z) \tag{44b}$$

for some PMF $p_{U,X}$ [11, Theorem 5.2]. For the stochastically degraded BS-BC with the stronger receiver Y observing the BSC(p) and the degraded receiver Z observing the BSC(p_s), the capacity region (44) simplifies to the set of rate pairs (R_c , R_p) that satisfy

$$R_{\mathbf{p}} \le H_{\mathbf{b}}(\alpha * p) - H_{\mathbf{b}}(p) \tag{45a}$$

$$R_{\rm c} \le 1 - H_{\rm b}(\alpha * p_{\rm max}) \tag{45b}$$

for some $\alpha \in [0,1/2]$ [11, Section 5.4.2]. Since these inequalities coincide with (42a) and (42b), it follows that to every rate pair $(R_c,R_p)\in\mathscr{C}^{(I)}_{\text{det}}$ there corresponds some $\alpha\in[0,1/2]$ for which (42a) and (42b) are satisfied. The sum-rate constraint (42c) is satisfied automatically because, in the case at hand, (42a) and (42b) imply (42c). Indeed, adding (42a) and (42b) yields

$$R_{c} + R_{p} \le 1 - H_{b}(\alpha * p_{max}) + H_{b}(\alpha * p) - H_{b}(p)$$
 (46)

$$\leq 1 - H_{\mathsf{b}}(p),\tag{47}$$

where the second inequality follows from (43a).

Case II: $p > p_{\text{max}}$.

In this case fixing the state at $p_{\rm max}$ again results in a stochastically degraded BS-BC, but in reverse order: now Y is a degraded version of Z. To show that any achievable rate pair $(R_{\rm c},R_{\rm p})$ must satisfy (42), we first note that—since it is now the weaker receiver, namely Receiver Y, that must recover both $M_{\rm c}$ and $M_{\rm p}$ —the sum-rate $R_{\rm c}+R_{\rm p}$ must not exceed the Shannon capacity of the BSC(p) from X to Y

$$R_{\rm c} + R_{\rm p} \le 1 - H_{\rm b}(p).$$
 (48)

Every rate pair in $\mathscr{C}_{\det}^{(I)}$ must thus satisfy (48).

We next show that, to every rate pair (R_c, R_p) satisfying (48), there corresponds some $\alpha \in [0, 1/2]$ for which (42) hold. To see why note that, for the case at hand, for every $\alpha \in [0, 1/2]$ the pair

$$R_{\rm c} = 1 - H_{\rm b}(\alpha * p) \tag{49a}$$

$$R_{p} = H_{b}(\alpha * p) - H_{b}(p) \tag{49b}$$

satisfies (42) (because, by (43b), $1 - H_b(\alpha * p)$ cannot exceed $1 - H_b(\alpha * p_{max})$ and (42a) must therefore hold). As we vary α from 0 to 1/2, the rate pair (49) traces the line $R_c + R_p = 1 - H_b(p)$.

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