# Ring Compute-and-Forward over Block-Fading Channels 

Shanxiang Lyu, Antonio Campello, and Cong Ling, Member, IEEE


#### Abstract

The Compute-and-Forward protocol in quasi-static channels normally employs lattice codes based on the rational integers $\mathbb{Z}$, Gaussian integers $\mathbb{Z}[i]$ or Eisenstein integers $\mathbb{Z}[\omega]$, while its extension to more general channels often assumes channel state information at transmitters (CSIT). In this paper, we propose a novel scheme for Compute-and-Forward in block-fading channels without CSIT, which is referred to as Ring Compute-andForward because the fading coefficients are quantized to the canonical embedding of a ring of algebraic integers. Thanks to the multiplicative closure of the algebraic lattices employed, a relay is able to decode an algebraic-integer linear combination of lattice codewords. We analyze its achievable computation rates and show it outperforms conventional Compute-and-Forward based on $\mathbb{Z}$ lattices. By investigating the effect of Diophantine approximation by algebraic conjugates, we prove that the degrees-of-freedom (DoF) of the optimized computation rate is $n / L$, where $n$ is the number of blocks and $L$ is the number of users.


Index Terms-Algebraic integers, block-fading channels, compute-and-forward, Diophantine approximation, lattice codes, number fields.

## I. Introduction

EFFICIENT information transmission over wireless relay networks has been extensively pursued in the past decades, in which the main issues to address include signal interference and fading. A number of relaying strategies have been proposed. The decode-and-forward protocol [1], [2] decodes at least some parts of the transmitted messages and removes the additive noise. Its main drawback is that the decoding performance deteriorates when the number of transmitters increases. The amplify-and-forward [3], [4] and compress-and-forward [5], [6] protocols maintain signal interference where the relay either transmits a scaled version of the received signal, or quantizes the received signal before passing it to the destination. The additive noise can however be amplified as signals traverse the network. The compute-and-forward (C\&F) [7] protocol harnesses signal interference introduced by the channel and removes the additive noise. It usually adopts lattice codes at source nodes so that the relay can decode a linear function of the messages. The C\&F paradigm has become a popular cooperative communication technique. In most cases, the underlying channel is assumed

[^0]to be quasi-static, which means that the (random) fading coefficients stay constant over the duration of each codeword.

There have been some works in the literature on C\&F dealing with more general channel models [8]-[10]. In this paper, we investigate C\&F for block-fading channels so as to achieve higher network throughput. Suppose that source nodes can transmit information with $n$ different resources (e.g., multiple carriers using orthogonal frequency-division multiplexing (OFDM)), and that channel coefficients also remain constant over the duration of each codeword. Our model of block-fading channels is essentially that of parallel independent fading channels defined in [11 Section 5.4.4], which assumes channel state information (CSI) at the receiver only. While the block length (or coherence time) $T$ in blockfading is dictated by properties of the physical world, and is a design parameter in parallel independent fading, the two models are equivalent if $T$ is large enough (see also [12]-[14] for using term "block-fading"). The crux here is that multiple resources offer diversity, which a coding scheme may utilize to improve performance.

Closely related to our work are [9], [10] where time-varying fading channels were investigated using lattice codes over the rational integers $\mathbb{Z}$. Yet, the channel model in [9], [10] is slightly different in that it consists of several blocks successive in time, which is better interpreted as time diversity. Also assuming multiple receive antennas at the relay, [9] derived the achievable rates of two integer-forcing decoders, namely, the arithmetic-mean (AM) decoder and geometric-mean (GM) decoder, for lattice codes over $\mathbb{Z}$. A practical $C \& F$ scheme based on root-LDA lattices was proposed in [10], where full diversity was observed for two-way relay channels and multiple-hop line networks. In a multi-input multi-output (MIMO) multipleaccess channel (MAC), [8] showed the multiplexing gain in MIMO C\&F is better than that provided by random coding if CSI is available at transmitters. Without CSI to perform precoding, however, the multiplexing gain in [8] is no better than that of a single antenna setting. For this reason, a coding technique with more algebraic structures is needed for C\&F over such channels. In this paper, we take a modest step by proposing algebraic lattice codes for $\mathrm{C} \& \mathrm{~F}$ over blockfading channels (which may be viewed as degenerated MIMO channels where channel matrices are diagonal), while leaving algebraic lattice codes for MIMO C\&F as future work.

In quasi-static fading channels, the structure of C\&F codes has been extended to rings and modules, initiated in [15]. This extension enlarges the space of code design, which brings several advantages to C\&F. For example, using more compact rings can result in higher computation rates, because
the rational integers $\mathbb{Z}$ or Gaussian integers $\mathbb{Z}[i]$ may not be the most suitable ring to quantize channel coefficients. It has been shown that using the Eisenstein integers $\mathbb{Z}[\omega]$ [16], [17] or rings from general quadratic number fields [18] can have better computation rates for complex channels. Since the lattice codes in these extensions are all $\mathcal{O}_{\mathbb{K}}$-modules $\left(\mathcal{O}_{\mathbb{K}}\right.$ refers to the ring of integers in number field $\mathbb{K}$ ), the message space can also be defined over $\mathcal{O}_{\mathbb{K}}$ due to the first isomorphism theorem of modules.

Our goal in this paper is to explore the fundamental limits of C\&F over block-fading channels by using algebraic lattices built from number fields of degree $n(n \geq 2)$. In quasi-static channels, the C\&F protocol essentially builds on capacityachieving lattice codes for the additive white Gaussian-noise (AWGN) channel [19]. To perform C\&F in block-fading channels, we employ universal lattice codes proposed in [13], [14] for compound block-fading channels. The celebrated Construction A has been extended to number fields in recent years [12], [13], [18], [20], [21]. In [12], the authors proposed algebraic lattice codes based on Construction A over $\mathcal{O}_{\mathbb{K}}$ so that the codes enjoy full diversity; subsequently it was proved in [13], [14] that such generalized Construction A can achieve the compound capacity of block-fading channels. It was also briefly suggested in [22] that number-field constructions as in [13], [14], [18] could be advantageous for C\&F in a blockfading scenario.

In this work, we propose a scheme termed Ring C\&F based on such algebraic lattices. As an extension of [23], we elaborate the construction of algebraic lattices for Ring C\&F, and provide a detailed analysis using the geometry of numbers and Diophantine approximation. The main contributions of this work are the following:

1) We propose Ring C\&F over block-fading channels based on lattice $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ from generalized Construction $A$, which satisfies relation $\mathcal{O}_{\mathbb{K}}^{T} / \Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C}) / \mathcal{I}_{\mathbb{K}}{ }^{T}$, where $T$ is the number of channel uses, $\mathcal{O}_{\mathbb{K}}^{T}, \Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ and $\mathcal{I}_{\mathbb{K}}{ }^{T}$ denote lattices built from ring $\mathcal{O}_{\mathbb{K}}$ itself, code $\mathcal{C}$ and ideal $\mathcal{I}_{\mathbb{K}}$, respectively. Such algebraic lattices are shown to be $\mathcal{O}_{\mathbb{K}}$-submodules so that they are multiplicatively closed. The relay aims to decode an algebraic-integer linear combination of lattice codewords, which means that the channel coefficient vectors are quantized to a lattice which is the canonical embedding of the ring of integers $\mathcal{O}_{\mathbb{K}}$. As a comparison, the lattice partition in a real quasi-static channel is $\mathbb{Z}^{T} / \Lambda^{\mathbb{Z}}(\mathcal{C}) /(p \mathbb{Z})^{T}$, in which $p$ is a prime number. Also note the difference from techniques in [16], [17] where channel coefficients are quantized to complex quadratic ring $\mathcal{O}_{\mathbb{K}}$ itself. Since the channel coefficients in different fading blocks are unequal with high probability, it is advantageous to employ the canonical embedding of $\mathcal{O}_{\mathbb{K}}$ so as to enjoy better quantization performance.
2) We analyze the computation rates in Ring C\&F based on the universal coding goodness and quantization goodness of algebraic lattices. The quantization goodness of algebraic lattices constructed from quadratic number fields [18] is extended to general number fields. The semi norm-ergodic metric in [24] is adopted to handle the effective noise. Regarding the equivalent block-fading channel, the universal lattice codes in [14] play an important role. In order to determine optimal algebraic-
integer coefficients, we resort to solving lattice problems over $\mathbb{Z}$-lattices and provide a means to assure linear independency of multiple equations over $\mathcal{O}_{\mathbb{K}}$.
3) We analyze the degrees-of-freedom (DoF) of our proposed coding scheme. The DoF of C\&F over quasi-static fading channels has been analyzed using the theory of Diophantine approximation in [25]-[27]. Our analysis of DoF for Ring C\&F requires a new result of Diophantine approximation by conjugates of an algebraic integer. The original contribution of our work is the proof of a Khintchin-type result for Diophantine approximation by conjugate algebraic integers (Lemma 4). It is well known that the standard Khintchine and Dirichlet theorems [28] only deal with the approximation of real numbers by rationals, which are algebraic numbers of degree one. Although some results on approximating a real number by an algebraic number are available in literature [29], [30], these results come with various restrictions which unfortunately do not lend themselves to our problem at hand. For instance, [29] only addresses simultaneous approximation of one number by algebraic conjugates or multiple numbers by non-conjugates of a bounded degree, while [30] requires the real numbers to be approximated lie in a field of transcendence degree one.

The rest of this paper is organized as follows. In Section II, we review some backgrounds on algebraic number theory and C\&F. In Sections III and IV, we present our Ring C\&F scheme and analyze its computation rates, respectively. In Section V, we analyze the achievable DoF without CSI at transmitters. Subsequently Section VI provides some simulation results. The last section concludes this paper.

Notation: The sets of all rationals, integers, real and complex numbers are denoted by $\mathbb{Q}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, respectively. log denotes logarithm with base 2 , and $\log ^{+}(x)=\max (\log (x), 0)$. Matrices and column vectors are denoted by uppercase and lowercase boldface letters, respectively. $\operatorname{dg}(\mathbf{x})$ represents a matrix filling vector x in the diagonal entries and zeros in the others. The operation of stacking the columns of matrix $\mathbf{X}$ one below the other is denoted by vec $(\mathbf{X}) .\|\mathbf{x}\|$ denotes the Euclidean norm of vector $\mathbf{x}$, while $\|\mathbf{X}\|$ denotes the Frobenius norm of matrix $\mathbf{X} . \otimes$ denotes the Kronecker tensor product, and $\oplus$ denotes the finite field summation. $\mathcal{Q}_{\Lambda}(\cdot)$ is the nearest neighbor quantizer to a lattice $\Lambda . \mathcal{V}(\Lambda) \triangleq$ $\left\{\mathbf{x} \in \mathbb{R}^{T} \mid \mathcal{Q}_{\Lambda}(\mathbf{x})=\mathbf{0}\right\}$ denotes the fundamental Voronoi region of lattice $\Lambda .[\mathbf{X}] \bmod \Lambda$ denotes $[\operatorname{vec}(\mathbf{X})] \bmod \Lambda$.

## II. Preliminaries

We first introduce necessary backgrounds on number fields and lattices (readers are referred to texts [31]-[33] for an introduction to these subjects), then review the protocol of C\&F over quasi-static channels.

## A. Number Fields and Lattices

Definition 1 (Number field). Let $\theta$ be a complex number with minimum polynomial $\mathfrak{m}_{\theta}$ of degree $n$. A number field is a field extension $\mathbb{K} \triangleq \mathbb{F}(\theta)$ that defines the minimum field containing the base field $\mathbb{F}$ and the primitive element $\theta$.

A number $c$ is called an algebraic integer if its minimal polynomial $\mathfrak{m}_{c}$ has integer coefficients. The maximal order of an algebraic number field is its ring of integers. Let $\mathbb{S}$ be the set of algebraic integers, then the ring of integers is $\mathcal{O}_{\mathbb{K}}=\mathbb{K} \cap \mathbb{S}$. The set $\left\{\theta_{1}, \theta_{2}, \ldots, \theta_{n}\right\} \in \mathcal{O}_{\mathbb{K}}^{n}$ is called an integral basis of $\mathcal{O}_{\mathbb{K}}$ if $\forall c \in \mathcal{O}_{\mathbb{K}}, c=c_{1} \theta_{1}+c_{2} \theta_{2}+\ldots+c_{n} \theta_{n}$ with $c_{i} \in \mathbb{Z}$. An element $u \in \mathcal{O}_{\mathbb{K}}$ is called a unit if it is invertible under multiplication. All the units of $\mathcal{O}_{\mathbb{K}}$ form a multiplicative group $\mathcal{U}$, referred to as the unit group.

An embedding of $\mathbb{K}$ into $\mathbb{C}$ is a homomorphism into $\mathbb{C}$ that fixes elements in $\mathbb{Q}$. For a number field of degree $n$, there are in total $n$ embeddings of $\mathbb{K}$ into $\mathbb{C}: \sigma_{i}: \mathbb{K} \rightarrow \mathbb{C}$, $i=1, \ldots, n$, referred to as canonical embedding. Canonical embedding establishes a correspondence between an element of an algebraic number field of degree $n$ and an $n$-dimensional vector in the Euclidean space. The embeddings of $\theta$, denoted by $\left\{\sigma_{i}(\theta)\right\}_{i=1}^{n}$, are determined by the roots of $\mathfrak{m}_{\theta}$. We denote by $r_{1}$ the number of embeddings with image in $\mathbb{R}$ and by $2 r_{2}$ the number of embeddings with image in $\mathbb{C}$. The pair $\left(r_{1}, r_{2}\right)$ is called the signature of $\mathbb{K}$. In a totally real number field, $\left(r_{1}, r_{2}\right)=(n, 0)$.

The following two quantities of an algebraic number are of particular interest:

1) The trace of $\theta: \operatorname{Tr}(\theta) \triangleq \sum_{i=1}^{n} \sigma_{i}(\theta) \in \mathbb{F}$;
2) The norm of $\theta: \operatorname{Nr}(\theta) \triangleq \prod_{i=1}^{n} \sigma_{i}(\theta) \in \mathbb{F}$.

In this work, we are only concerned with the scenario of real channels and hence totally real number fields, so we use $\mathbb{F}=$ $\mathbb{Q}$ as the base field. For an extension to complex channels, one can choose $\mathbb{F}=\mathbb{Q}(i)$ as the base field.

Definition 2 (Ideals and prime ideals). Let $R$ be a commutative ring with identity $1_{R} \neq 0$. An ideal $\mathfrak{I}$ of $R$ is a nonempty subset of $R$ that has the following two properties:

1) $c_{1}+c_{2} \in \mathfrak{I}$ if $c_{1}, c_{2} \in \mathfrak{I}$;
2) $c_{1} c_{2} \in \mathfrak{I}$ if $c_{1} \in \mathfrak{I}, c_{2} \in R$.

An ideal $\mathfrak{p}$ of $R$ is prime if it has the following two properties:

1) If $c_{1}$ and $c_{2}$ are two elements of $R$ such that their product $c_{1} c_{2}$ is an element of $\mathfrak{p}$, then either $c_{1} \in \mathfrak{p}$ or $c_{2} \in \mathfrak{p}$;
2) $\mathfrak{p}$ is not equal to $R$ itself.

Every ideal of $R$ can be decomposed into a product of prime ideals. In particular, if $p$ is a rational prime, we have $p R=$ $\prod_{i=1}^{g} \mathfrak{p}_{i}^{e_{i}}$ in which $e_{i}$ is the ramification index of prime ideal $\mathfrak{p}_{i}$. The inertial degree of $\mathfrak{p}_{i}$ is defined as $f_{i}=\left[R / \mathfrak{p}_{i}: \mathbb{Z} / p \mathbb{Z}\right]$, and it satisfies $\sum_{i=1}^{g} e_{i} f_{i}=n$. Each prime ideal $\mathfrak{p}_{i}$ is said to lie above $p$.

Definition 3 (Modules). A $R$-module is a set $M$ together with a binary operation under which $M$ forms an Abelian group, and an action of $R$ on $M$ which satisfies the same axioms as those for vector spaces.

Let $D$ be a subset of $R$-module $M . D$ forms an $R$-module basis of $M$ if every element in $M$ can be written as a finite linear combination of the elements of $D$. The order of the basis is called the rank of the module. A finite subset $\left\{d_{1}, \ldots, d_{m}\right\}$ of distinct elements of $M$ is said to be linearly independent over $R$ if whenever $\sum_{i=1}^{m} c_{i} d_{i}=\mathbf{0}$ for some $c_{1}, \ldots, c_{m} \in R$, then $c_{1}=\cdots=c_{m}=0$.

A real $\mathbb{Z}$-lattice is a discrete $\mathbb{Z}$-submodule of $\mathbb{R}^{m}$. Such a lattice $\Lambda^{\prime}$ generated by a basis $\mathbf{D}=\left[\mathbf{d}_{1}, \ldots, \mathbf{d}_{m}\right] \in \mathbb{R}^{m \times m}$ can be written as a direct sum:

$$
\Lambda^{\prime}(\mathbf{D})=\mathbb{Z} \mathbf{d}_{1}+\mathbb{Z} \mathbf{d}_{2}+\cdots+\mathbb{Z} \mathbf{d}_{m}
$$

With canonical embedding $\sigma$, an $\mathcal{O}_{\mathbb{K}}$-module $\Lambda$ of rank $m$ can be transformed into a $\mathbb{Z}$-lattice $\Lambda^{\prime}$, and we write $\Lambda^{\prime}=\sigma(\Lambda)$. If $\mathbb{K}$ is a totally real number field of degree $n$, then we have an embedded basis $\mathbf{D} \in \mathbb{R}^{m n \times m n}$, and we define its discriminant as $\operatorname{disc}_{\mathbb{K}}=|\operatorname{det}(\mathbf{D})|^{2}$. The successive minima $\lambda_{i}\left(\Lambda^{\prime}\right)$ of the $\mathbb{Z}$-lattice $\Lambda^{\prime}$ are defined in the usual manner. Analogously, we may define successive minima of $\Lambda$ over $\mathcal{O}_{\mathbb{K}}$.

Definition 4 (Successive minima of modules [34]). The $i$ th successive minimum of an $\mathcal{O}_{\mathbb{K}}$-module $\Lambda$ is the smallest real number $r$ such that the ball $\mathcal{B}(\mathbf{0}, r)$ contains the canonical embedding of $i$ linearly independent vectors of $\sigma(\Lambda)$ over $\mathbb{K}$ :
$\lambda_{i}(\Lambda)=\inf \left\{r \mid \operatorname{dim}\left(\operatorname{span}_{\mathbb{K}}\left(\sigma^{-1}(\sigma(\Lambda) \cap \mathcal{B}(\mathbf{0}, r))\right)\right) \geq i\right\}$.
Notice that $\lambda_{1}(\Lambda)=\lambda_{1}\left(\Lambda^{\prime}\right)$, and in general $\lambda_{i}(\Lambda) \geq \lambda_{i}\left(\Lambda^{\prime}\right)$ for $i>1$. Also, if $\mathbf{x}_{1}, \ldots, \mathbf{x}_{m}$ are linearly independent over $\mathbb{K}$ and achieve the successive minima of $\Lambda$, then the embeddings $\sigma\left(\mathbf{x}_{1}\right), \ldots, \sigma\left(\mathbf{x}_{m}\right)$ are linearly independent and primitive in the Euclidean lattice $\Lambda^{\prime}$.

For any real $\mathbb{Z}$-lattice $\Lambda^{\prime}(\mathbf{D})$ with $\mathbf{D} \in \mathbb{R}^{m \times m}$, Minkowski's first theorem states that [35]

$$
\begin{equation*}
\lambda_{1}^{2}\left(\Lambda^{\prime}\right) \leq \kappa_{m}|\operatorname{det}(\mathbf{D})|^{\frac{2}{m}} \tag{1}
\end{equation*}
$$

and Minkowski's second theorem states that

$$
\begin{equation*}
\prod_{i=1}^{m} \lambda_{i}^{2}\left(\Lambda^{\prime}\right) \leq \kappa_{m}^{m}|\operatorname{det}(\mathbf{D})|^{2} \tag{2}
\end{equation*}
$$

where $\kappa_{m} \triangleq \sup _{\Lambda^{\prime}(\mathbf{D})} \lambda_{1}\left(\Lambda^{\prime}\right)^{2} /|\operatorname{det}(\mathbf{D})|^{2 / m}$ is called Hermite's constant.

Analogous bounds exist for the successive minima of $\mathcal{O}_{\mathbb{K}^{-}}$ module $\Lambda$. Obviously,

$$
\begin{equation*}
\lambda_{1}^{2}(\Lambda) \leq \kappa_{m n}|\operatorname{det}(\mathbf{D})|^{\frac{2}{m n}} \tag{3}
\end{equation*}
$$

since the first minimum is identical. Applying Minkowski's second theorem to [36, Theorem 2] yields

$$
\begin{equation*}
\prod_{i=1}^{m} \lambda_{i}^{2 n}(\Lambda) \leq \kappa_{m n}^{m n}|\operatorname{det}(\mathbf{D})|^{2} \tag{4}
\end{equation*}
$$

## B. C\&F over Quasi-Static Fading Channels

Consider an AWGN network with $L$ source nodes which cannot collaborate with each other and are noiselessly connected to a final destination. We assume that all source nodes are operating with the same message space $W$ (over finite fields [7] or rings [15]), and the same message rate $R_{\text {mes }}=\frac{1}{T} \log (|W|)$. Let $\left(\Lambda_{c}^{\mathbb{Z}}, \Lambda_{f}^{\mathbb{Z}}\right)$ be a pair of nested lattices in the partition chain $\mathbb{Z}^{T} / \Lambda_{f}^{\mathbb{Z}} / \Lambda_{c}^{\mathbb{Z}} /(p \mathbb{Z})^{T}$, in which $p$ is a prime number growing with the lattice dimension. A message $\mathbf{w}_{l} \in W$ is mapped bijectively into a lattice code via $\mathbf{x}_{l}=$ $\mathcal{E}\left(\mathbf{w}_{l}\right) \in \gamma \Lambda_{f}^{\mathbb{Z}}$, satisfying a power constraint of $\left\|\mathbf{x}_{l}\right\|^{2} \leq T P$. $\gamma$ denotes a parameter to control the transmission power, and
$P$ denotes the signal power, hence the signal-to-noise ratio (SNR) if the noise variance is normalized.

The noisy observation at a relay is

$$
\begin{equation*}
\mathbf{y}=\sum_{l=1}^{L} h_{l} \mathbf{x}_{l}+\mathbf{z} \tag{5}
\end{equation*}
$$

where the channel coefficients $\mathbf{h}=\left[h_{1}, \ldots, h_{L}\right]^{\top} \in \mathbb{R}^{L}$, and the additive noise $\mathbf{z} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}_{T}\right)$. The relay aims to compute a finite field equation

$$
\begin{equation*}
\mathbf{u}=\bigoplus_{l=1}^{L} a_{l} \mathbf{w}_{l} \tag{6}
\end{equation*}
$$

with coefficient vector $\mathbf{a}=\left[a_{1}, \ldots, a_{L}\right]^{\top} \in \mathbb{Z}^{L}$ and forward $\mathbf{u}, \mathbf{a}$ to the destination. Each $\mathbf{u}$ corresponds to a lattice equation $\left[\sum_{l=1}^{L} a_{l} \mathbf{x}_{l}\right] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}$ as they are isomorphic. By first estimating the lattice equation and then map it to a finite field, the forwarded message from the relay is written as $\hat{\mathbf{u}}=\mathcal{D}(\mathbf{y} \mid \mathbf{h}, \mathbf{a})$. We say equation $\mathbf{u}=\bigoplus_{l=1}^{L} a_{l} \mathbf{w}_{l}$ is decoded with probability of error $\delta$ if $\operatorname{Pr}(\mathbf{u} \neq \hat{\mathbf{u}})<\delta$.

Definition 5 (Achievable Computation Rate for a Chosen a at a Relay). For a given channel coefficient vector $h$ and a chosen coefficient vector a, the computation rate $R_{\text {comp }}(\mathbf{h}, \mathbf{a})$ is achievable at a relay if for any $\delta>0$ and $T$ large enough, there exist encoders $\mathcal{E}_{1}, \ldots \mathcal{E}_{L}$ and decoders $\mathcal{D}$ such that the relay can recover its desired equation with error probability bound $\delta$ if the underlying message rate $R_{\text {mes }}$ satisfies:

$$
R_{\mathrm{mes}}<R_{\mathrm{comp}}(\mathbf{h}, \mathbf{a})
$$

Theorem 1 ([7]). There is a sequence of nested lattice codebooks $\left\{\Lambda_{f}^{\mathbb{Z}}, \Lambda_{c}^{\mathbb{Z}}\right\}$ of length $T$, such that by setting $T \rightarrow \infty$, the following computation rate is achievable:

$$
\begin{equation*}
R_{\text {comp }}(\mathbf{h}, \mathbf{a})=\frac{1}{2} \max _{\alpha \in \mathbb{R}} \log ^{+}\left(\frac{P}{|\alpha|^{2}+P\|\alpha \mathbf{h}-\mathbf{a}\|^{2}}\right) \tag{7}
\end{equation*}
$$

Upon receiving $L$ linearly independent equations in the form of (6), the destination estimates the messages by inverting the equations. The maximum information rate that the destination can receive through the AWGN network is dictated by the computation rates at the relays.

Definition 6 (Achievable Computation Rate of the AWGN Network). Given $\left\{\mathbf{h}_{l}\right\}_{l=1}^{L}$, and $\left\{\mathbf{a}_{l}\right\}_{l=1}^{L}$ from $L$ relays such that the morphism of $\left\{\mathbf{a}_{l}\right\}_{l=1}^{L}$ is invertible in the message space, the achievable computation rate of the AWGN network is $\min _{l} R_{\text {comp }}\left(\mathbf{h}_{l}, \mathbf{a}_{l}\right)$.

To characterize the the growth of computation rate w.r.t. SNR, define the DoF as

$$
\begin{equation*}
d_{\mathrm{comp}}=\lim _{P \rightarrow \infty} \frac{\max _{\mathbf{a}} R_{\mathrm{comp}}(\mathbf{h}, \mathbf{a})}{\frac{1}{2} \log (1+P)} \tag{8}
\end{equation*}
$$

Using the theory of Diophantine approximation, Niesen and Whiting [25] showed that

$$
d_{\mathrm{comp}} \leq \begin{cases}\frac{1}{2}, & L=2 \\ \frac{2}{L+1}, & L>2\end{cases}
$$



Fig. 1: Compute-and-Forward over block-fading channels with 2 users and 2 relays.

This has subsequently been improved by Ordenlitch, Erez and Nazer [26] to

$$
d_{\mathrm{comp}}=\frac{1}{L}
$$

## III. Ring C\&F

In this work, we consider a block-fading scenario where diversity is supplied in $n$ blocks and fading coefficients remain constant in each frame of coherence time $T$. That is, the fading process experienced by a codeword $\mathbf{x}_{l}$ of user $l$ consists of $n$ blocks $\{\underbrace{h_{1, l}, h_{1, l}, \ldots, h_{1, l}}_{T}\},\{\underbrace{h_{2, l}, h_{2, l}, \ldots, h_{2, l}}_{T}\}$,

a relay can be written in matrix form as

$$
\begin{equation*}
\mathbf{Y}=\sum_{l=1}^{L} \mathbf{H}_{l} \mathbf{X}_{l}+\mathbf{Z} \tag{9}
\end{equation*}
$$

where $\mathbf{Y} \in \mathbb{R}^{n \times T}, \mathbf{H}_{l}=\operatorname{dg}\left(h_{1, l}, \ldots, h_{n, l}\right)$ denotes the channel coefficients from user $l$ to the relay, $\mathbf{X}_{l} \in \mathbb{R}^{n \times T}$ denotes a transmitted codeword to be designed in the sequel, and $\mathbf{Z} \in \mathbb{R}^{n \times T}$ is the additive noise with entries drawn from $\mathcal{N}(0,1)$. The index of the relay is dropped in the equation for simplicity of notation. The C\&F diagram for this model with two users (source nodes) and two relays is shown in Fig. 1 In the figure, the encoded messages $\mathcal{E}\left(\mathbf{w}_{1}\right)$ and $\mathcal{E}\left(\mathbf{w}_{2}\right)$ are both transmitted by using two sub-channels in parallel, which are respectively denoted by black and blue arrows. Relays 1 and 2 forward two linearly independent equations to the destination which subsequently recovers message $\hat{\mathbf{w}}_{1}, \hat{\mathbf{w}}_{2}$ by inverting the equations.

Next, we present our Ring C\&F scheme, which contains message encoding based on algebraic lattices (such that the degree of the number field equals to the number of blocks in the block-fading model), and decoding algebraic-integer linear combinations of lattice codewords. The "goodness" properties of algebraic lattices are shown in the last subsection.

## A. Encoding

We follow [12], [13], [18] to build lattices from Construction A over number fields. Choose a prime ideal $\mathfrak{p}$ lying above rational prime $p$ with inertial degree $f$ so that we have an isomorphism $\mathcal{O}_{\mathbb{K}} / \mathfrak{p} \cong \mathbb{F}_{p^{f}}$. Let $\mathcal{C}$ be a $(T, k)$ linear code over $\mathbb{F}_{p^{f}}$ where $k<T$. Let $\rho: \mathcal{O}_{\mathbb{K}} \rightarrow \mathbb{F}_{p^{f}}$ be a component-wise
ring homomorphism defined by reduction modulo the ideal $\mathfrak{p}$. Generalized Construction A from code $\mathcal{C}$ is defined as

$$
\begin{equation*}
\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})=\rho^{-1}(\mathcal{C}) \tag{10}
\end{equation*}
$$

which is a free $\mathbb{Z}$-module ${ }^{1}$ of rank $n T$. The coding lattice $\Lambda^{\mathbb{Z}}(\mathcal{C})$ is the canonical embedding of $\mathcal{O}_{\mathbb{K}}$ module $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ into the Euclidean space.

We first build a pair of nested lattices $\left(\Lambda_{f}^{\mathbb{Z}}, \Lambda_{c}^{\mathbb{Z}}\right)$ based on a pair of nested linear codes $\left(\mathcal{C}_{f}, \mathcal{C}_{c}\right)$. Let $k_{c}<k_{f}<T$. Define $\mathcal{C}_{f}=\left\{\mathbf{G}_{f} \mathbf{w}_{f} \mid \mathbf{w}_{f} \in \mathbb{F}_{p^{f}}^{k_{f}}\right\}$ and $\mathcal{C}_{c}=\left\{\mathbf{G}_{c} \mathbf{w}_{c} \mid \mathbf{w}_{c} \in \mathbb{F}_{p^{f}}^{k_{c}}\right\}$, where $\mathbf{G}_{f}=\left[\mathbf{G}_{c}, \mathbf{G}^{\prime}\right] \in \mathbb{F}_{p^{f}}^{T \times k_{f}}$, and $\mathbf{G}_{c} \in \mathbb{F}_{p^{f}}^{T \times k_{c}}$. These codes are then lifted from $\mathbb{F}_{p^{f}}^{T}$ to $\mathcal{O}_{\mathbb{K}}^{T}$ :

$$
\Lambda_{f}^{\mathcal{O}_{\mathrm{K}}}=\rho^{-1}\left(\mathcal{C}_{f}\right), \quad \Lambda_{c}^{\mathcal{O}_{\mathrm{K}}}=\rho^{-1}\left(\mathcal{C}_{c}\right)
$$

which produce $\mathbb{Z}$-lattices $\Lambda_{f}^{\mathbb{Z}}$ and $\Lambda_{c}^{\mathbb{Z}}$ with canonical embeddings. The volumes of the Voronoi regions of $\Lambda_{f}^{\mathbb{Z}}$ and $\Lambda_{c}^{\mathbb{Z}}$ are $\operatorname{Vol}\left(\Lambda_{f}^{\mathbb{Z}}\right)=p^{\left(T-k_{f}\right) f} \operatorname{disc}_{\mathbb{K}}^{T / 2}$ and $\operatorname{Vol}\left(\Lambda_{c}^{\mathbb{Z}}\right)=$ $p^{\left(T-k_{c}\right) f} \operatorname{disc}_{\mathbb{K}}^{T / 2}$, respectively. Let $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ be an integral basis of $\mathcal{O}_{\mathbb{K}}$. Since every ideal of $\mathcal{O}_{\mathbb{K}}$ is a free $\mathbb{Z}$-module of rank $n$, a basis of ideal $\mathfrak{p}$ can be represented by $\left\{\mu_{1}, \ldots, \mu_{n}\right\}$ where $\mu_{i}=\sum_{j=1}^{n} \mu_{i j} \phi_{j}, \mu_{i j} \in \mathbb{Z}$. Thus the generator matrices of $\mathcal{O}_{\mathbb{K}}$ and $\mathfrak{p}$ are respectively given by

$$
\begin{aligned}
& \Phi=\left[\begin{array}{ccc}
\sigma_{1}\left(\phi_{1}\right) & \cdots & \sigma_{1}\left(\phi_{n}\right) \\
\sigma_{2}\left(\phi_{1}\right) & \cdots & \sigma_{2}\left(\phi_{n}\right) \\
\vdots & \vdots & \vdots \\
\sigma_{n}\left(\phi_{1}\right) & \cdots & \sigma_{n}\left(\phi_{n}\right)
\end{array}\right], \\
& \Phi_{\mathfrak{p}}=\left[\begin{array}{ccc}
\sum_{j=1}^{n} \mu_{1 j} \sigma_{1}\left(\phi_{j}\right) & \cdots & \sum_{j=1}^{n} \mu_{n j} \sigma_{1}\left(\phi_{j}\right) \\
\sum_{j=1}^{n} \mu_{1 j} \sigma_{2}\left(\phi_{j}\right) & \cdots & \sum_{j=1}^{n} \mu_{n j} \sigma_{2}\left(\phi_{j}\right) \\
\vdots & \vdots & \vdots \\
\sum_{j=1}^{n} \mu_{1 j} \sigma_{n}\left(\phi_{j}\right) & \cdots & \sum_{j=1}^{n} \mu_{n j} \sigma_{n}\left(\phi_{j}\right)
\end{array}\right] .
\end{aligned}
$$

Let the canonical representations of $\mathcal{C}_{f}, \mathcal{C}_{c}$ be $\mathbf{G}_{f}=$ $\left[\mathbf{I}_{k_{f}}, \mathbf{A}_{f}^{\top}\right]^{\top}, \mathbf{G}_{c}=\left[\mathbf{I}_{k_{c}}, \mathbf{A}_{c}^{\top}\right]^{\top}$, it was shown in [12] that the generator matrices of $\Lambda_{f}^{\mathbb{Z}}$ and $\Lambda_{c}^{\mathbb{Z}}$ are respectively given by

$$
\begin{aligned}
& \mathbf{M}_{f}=\left[\begin{array}{cc}
\mathbf{I}_{k_{f}} \otimes \Phi & \mathbf{0}_{n k_{f}, n\left(T-k_{f}\right)} \\
\mathbf{A}_{f} \otimes \Phi & \mathbf{I}_{T-k_{f}} \otimes \Phi_{\mathfrak{p}}
\end{array}\right] \\
& \mathbf{M}_{c}=\left[\begin{array}{cc}
\mathbf{I}_{k_{c}} \otimes \Phi & \mathbf{0}_{n k_{c}, n\left(T-k_{c}\right)} \\
\mathbf{A}_{c} \otimes \Phi & \mathbf{I}_{T-k_{c}} \otimes \Phi_{\mathfrak{p}}
\end{array}\right]
\end{aligned}
$$

For each user, a message $\mathbf{w} \in \mathbb{F}_{p_{f}^{k_{f}-k_{c}}}$ is encoded into $\tilde{\mathbf{x}} \in$ $\Lambda_{f}^{\mathcal{O}_{\mathrm{K}}}$ as

$$
\begin{equation*}
\tilde{\mathbf{x}}=\mathcal{E}(\mathbf{w}) \triangleq \gamma\left[\rho^{-1}\left(\mathbf{G}^{\prime} \mathbf{w}\right)\right] \quad \bmod \Lambda_{c}^{\mathcal{O}_{\mathrm{K}}} \tag{11}
\end{equation*}
$$

with a transmission rate $R_{\text {mes }}=\frac{\left(k_{f}-k_{c}\right) f}{T} \log (p)$. The actually transmitted codeword is obtained by apply component-wise canonical embedding to $\tilde{\mathbf{x}}$, which yields its matrix form

$$
\mathbf{X}=\gamma\left[\begin{array}{c}
\sigma_{1}\left(\tilde{\mathbf{x}}^{\top}\right)  \tag{12}\\
\sigma_{2}\left(\tilde{\mathbf{x}}^{\top}\right) \\
\vdots \\
\sigma_{n}\left(\tilde{\mathbf{x}}^{\top}\right)
\end{array}\right] \in \mathbb{R}^{n \times T}
$$

[^1]Again, $\gamma$ denotes a power scaling factor as before. In the construction, it is possible to map messages to lattice points and back while preserving linearity.

Proposition 1. The encoding function $\mathcal{E}(\mathbf{w})$ defines a bijection between messages $\mathbf{w} \in \mathbb{F}_{p^{f}}^{k_{f}-k_{c}}$ and lattice points inside $\Lambda_{f}^{\mathbb{Z}} \cap \mathcal{V}\left(\Lambda_{c}^{\mathbb{Z}}\right)$.

Proof: As $\rho^{-1}(\mathcal{C})$ defines a lattice, there is a unique correspondence between a codeword $\mathbf{G}^{\prime} \mathbf{w}_{i}$ and a lattice coset $\Lambda_{c}^{\mathbb{Z}}+\mathbf{x}_{i}^{*}$, where the set of representatives $\left\{\mathbf{x}_{i}^{*}\right\}$ satisfy $\left|\left\{\mathbf{x}_{i}^{*}\right\}\right|=$ $p^{\left(k_{f}-k_{c}\right) f}$, and $\mathbf{x}_{i}^{*} \notin \Lambda_{c}^{\mathbb{Z}}$ if $\mathbf{x}_{i}^{*} \neq \mathbf{0}$. We only need to show points in different cosets would not collide after modulo $\Lambda_{c}^{\mathbb{Z}}$, in which

$$
\left[\rho^{-1}\left(\mathbf{G}^{\prime} \mathbf{w}_{i}\right)\right] \quad \bmod \Lambda_{c}^{\mathbb{Z}}=\mathbf{x}_{i}^{*}+\arg \min _{\hat{\mathbf{x}}_{i} \in \Lambda_{c}^{\mathbb{Z}}}\left\|\mathbf{x}_{i}^{*}+\hat{\mathbf{x}}_{i}\right\|^{2}
$$

Since $\mathbf{x}_{i}^{*}-\mathbf{x}_{j}^{*} \notin \Lambda_{c}^{\mathbb{Z}}$ for $i \neq j$, there is no $\hat{\mathbf{x}}_{i} \in \Lambda_{c}^{\mathbb{Z}}$ such that $\mathbf{x}_{i}^{*}-\mathbf{x}_{j}^{*}+\hat{\mathbf{x}}_{i} \in \Lambda_{c}^{\mathbb{Z}}$, and the proposition is proved.

As usual, we apply dithering from the set $\left\{\operatorname{vec}\left(\mathbf{D}_{l}\right)\right\}_{l=1}^{L}$ where each vec $\left(\mathbf{D}_{l}\right)$ is uniformly distributed over $\mathcal{V}\left(\Lambda_{c}^{\mathbb{Z}}\right)$. To simplify the presentation, however, we defer their presence until Section IV.

## B. Decoding

The following lemma is the crux of our decoding algorithm, which says codewords $\mathbf{X}_{l}$ 's are not only closed in $\gamma \Lambda_{f}^{\mathbb{Z}}$ under $\mathbb{Z}$-linear combinations, but more generally under $\mathcal{O}_{\mathbb{K}}$-linear combinations.

Lemma 1. Let $a_{l} \in \mathcal{O}_{\mathbb{K}}$, and $\mathbf{A}_{l}=\operatorname{dg}\left(\sigma_{1}\left(a_{l}\right), \ldots, \sigma_{n}\left(a_{l}\right)\right)$ for $1 \leq l \leq L$. The physical layer codewords are closed under the action of ring elements, i.e., $\sum_{l=1}^{L}\left(\mathbf{A}_{l} \mathbf{X}_{l}\right) \in \gamma \Lambda_{f}^{\mathbb{Z}}$.

Proof: We let $\gamma=1$ for clarity. We first show that $\Lambda^{\mathcal{O}_{\mathbb{K}}}(\mathcal{C})$ constructed from (10) is an $\mathcal{O}_{\mathbb{K}}$-submodule. The definitions of rings and ideals show that $\mathcal{O}_{\mathbb{K}}, \mathfrak{p}$ are both $\mathcal{O}_{\mathbb{K}}$-modules of rank 1. It then follows from [37, p. 338] that the Cartesian product $\mathcal{O}_{\mathbb{K}}^{T}$ is a free $\mathcal{O}_{\mathbb{K}}$-module of rank $T$, based on componentwise addition and multiplication by elements of $\mathcal{O}_{\mathbb{K}}$. Since $\mathcal{O}_{\mathbb{K}} / \mathfrak{p} \cong \mathbb{F}_{p^{f}}$ and $\mathcal{C}$ is a subgroup of $\mathbb{F}_{p^{f}}^{T}, \rho^{-1}(\mathcal{C})$ becomes an $\mathcal{O}_{\mathbb{K}}$-submodule [37, p. 342] of $\mathcal{O}_{\mathbb{K}}^{T}$ which satisfies $a_{l} \rho^{-1}(\mathcal{C}) \subset$ $\rho^{-1}(\mathcal{C}), \forall a_{l} \in \mathcal{O}_{\mathbb{K}}$. It follows from a component-wise ring homomorphism $\sigma(\cdot): \mathbb{K} \rightarrow \mathbb{R}^{n}$ that $\mathbf{A}_{l} \mathbf{X}_{l} \in \Lambda_{f}^{\mathbb{Z}}$. Lastly, the additive closure of lattice points clearly holds.

Based on Lemma 10 the decoder aims to extract an algebraic combination of lattice codewords from the scaled signal

$$
\begin{equation*}
\mathbf{B Y}=\underbrace{\sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{X}_{l}}_{\text {lattice codeword }}+\underbrace{\mathbf{B} \sum_{l=1}^{L} \mathbf{H}_{l} \mathbf{X}_{l}-\sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{X}_{l}+\mathbf{B Z}}_{\text {effective noise }} \tag{13}
\end{equation*}
$$

where $\mathbf{B}=\operatorname{dg}\left(b_{1}, \ldots, b_{n}\right), b_{i} \in \mathbb{R}$ is an minimum mean square error (MMSE) matrix. We refer to

$$
\begin{equation*}
\left[\sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{X}_{l}\right] \quad \bmod \gamma \Lambda_{c}^{\mathbb{Z}} \tag{14}
\end{equation*}
$$

as an algebraic lattice equation. With some decoding procedures to be specified in the next section, we proceed by assuming (14) is available. Then each relay can extract a finite field equation

$$
\begin{align*}
\mathbf{u} & =\left[\left(\mathbf{G}^{\prime \top} \mathbf{G}^{\prime}\right)^{-1} \mathbf{G}^{\prime \top} \rho\left(\left[\gamma^{-1} \sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{X}_{l}\right] \bmod \Lambda_{c}^{\mathbb{Z}}\right)\right] \\
& =\sum_{l=1}^{L}\left[\left(\mathbf{G}^{\prime \top} \mathbf{G}^{\prime}\right)^{-1} \mathbf{G}^{\prime \top} \rho\left(a_{l}\right) \rho\left(\gamma^{-1} \mathbf{X}_{l}\right)\right] \bmod \mathbb{F}_{p^{f}} \\
& =\bigoplus_{l=1}^{L} \rho\left(a_{l}\right) \mathbf{w}_{l} \tag{15}
\end{align*}
$$

where the second equality is from the property of ring homomorphism $\rho(\cdot)$, and the third equality is due to Proposition 1 such that we have a bijection $\rho\left(\gamma^{-1} \mathbf{X}_{l}\right)=\mathbf{G}^{\prime} \mathbf{w}_{l}$.

In practice, all relays forward their decoded messages $\hat{\mathbf{u}}$ 's and coefficients $\left\{\mathbf{A}_{l}\right\}_{l=1}^{L}$ 's to the destination, where $\hat{\mathbf{u}}=$ $\mathcal{D}\left(\mathbf{Y} \mid\left\{\mathbf{H}_{l}\right\}_{l=1}^{L},\left\{\mathbf{A}_{l}\right\}_{l=1}^{L}\right)$ denotes an estimated message. Upon collecting $L$ linearly independent equations from those relays, the destination can estimate messages $\mathbf{w}_{1}, \ldots, \mathbf{w}_{L}$.

To explain the rationale, we give two examples below. Example 1demonstrates how multiplications are closed. Example 2 shows the information flow from users to a destination.

Example 1. Let $\operatorname{vec}\left(\mathbf{X}_{l}\right)=\mathbf{M}_{f} \mathbf{z}_{l}, \mathbf{z}_{l} \in \mathbb{Z}^{n T}$. The closure of $\mathbf{A}_{l} \times \Lambda^{\mathbb{Z}} \subset \Lambda^{\mathbb{Z}}$ implies that $\operatorname{vec}\left(\mathbf{A}_{l} \mathbf{X}_{l}\right)=\mathbf{M}_{f} \mathbf{z}_{l}^{\prime}, \mathbf{z}_{l}^{\prime} \in \mathbb{Z}^{n T}$, where $\mathbf{z}_{l}=\mathbf{z}_{l}^{\prime}$ if and only if $a_{l}=1$. For instance, in a quadratic field $\mathbb{K}=\mathbb{Q}(\sqrt{3})$, let the lattice basis be $\mathbf{M}_{f}=\left[\begin{array}{cc}1 & \sqrt{3} \\ 1 & -\sqrt{3}\end{array}\right]$ and the multiplication coefficient be $a_{l}=1+\sqrt{3}$. Then for any $\mathbf{z}_{l} \in \mathbb{Z}^{2}$, one has

$$
\left[\begin{array}{cc}
1+\sqrt{3} & 0 \\
0 & 1-\sqrt{3}
\end{array}\right] \mathbf{M}_{f} \mathbf{z}_{l}=\mathbf{M}_{f} \mathbf{z}_{l}^{\prime}
$$

with $\mathbf{z}_{l}^{\prime}=\left[\begin{array}{ll}1 & 3 \\ 1 & 1\end{array}\right] \mathbf{z}_{l} \in \mathbb{Z}^{2}$.
Example 2. Consider quadratic field $\mathbb{K}=\mathbb{Q}(\sqrt{5})$. Choose $p=5$, so the ideal factorization becomes $p \mathcal{O}_{\mathbb{K}}=\mathfrak{p}^{2}$, where $\mathfrak{p}=\frac{5-\sqrt{5}}{2} \mathbb{Z}+\frac{-5+3 \sqrt{5}}{2} \mathbb{Z}$. For the isomorphism $\mathbb{F}_{p} \cong \mathcal{O}_{\mathbb{K}} / \mathfrak{p}$, the five coset representatives in $\mathbb{R}^{2}$ corresponding to $\mathbb{F}_{5}$ are

$$
\begin{aligned}
& {[0,0]^{\top},[1,1]^{\top},\left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]^{\top},} \\
& {\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]^{\top},[-1,-1]^{\top} .}
\end{aligned}
$$

Let the two uncoded messages be $w_{1}=2$ for User 1 and $w_{2}=3$ for User 2 . For $\gamma=1$, the transmitted lattice points are

$$
\begin{aligned}
& \mathbf{X}_{1}=\mathcal{E}\left(w_{1}\right)=\left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]^{\top} \\
& \mathbf{X}_{2}=\mathcal{E}\left(w_{2}\right)=\left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]^{\top}
\end{aligned}
$$

For convenience, suppose the channel coefficients are exactly taken from $\mathcal{O}_{\mathbb{K}}$. In Relay 1, we receive $\mathbf{V}_{1}=\sum_{l=1}^{2} \mathbf{A}_{l}^{(1)} \mathbf{X}_{l}$ with

$$
\begin{aligned}
& \mathbf{A}_{1}^{(1)}=\operatorname{dg}(2+17 \sqrt{5}, 2-17 \sqrt{5}) \\
& \mathbf{A}_{2}^{(1)}=\operatorname{dg}(13+\sqrt{5}, 13-\sqrt{5})
\end{aligned}
$$

Its decoded message is $\hat{u}_{1}=\mathcal{D}\left(\mathbf{V}_{1}\right)=3$. Similarly in Relay 2, we receive $\mathbf{V}_{2}=\sum_{l=1}^{2} \mathbf{A}_{l}^{(2)} \mathbf{X}_{l}$ with

$$
\begin{aligned}
& \mathbf{A}_{1}^{(2)}=\operatorname{dg}\left(\frac{15+9 \sqrt{5}}{2}, \frac{15-9 \sqrt{5}}{2}\right) \\
& \mathbf{A}_{2}^{(2)}=\operatorname{dg}(2+17 \sqrt{5}, 2-17 \sqrt{5})
\end{aligned}
$$

Its decoded message is $\hat{u}_{2}=\mathcal{D}\left(\mathbf{V}_{2}\right)=1$. Then Relays 1 and 2 forward messages $\hat{u}_{1}, \hat{u}_{2}$ along with coefficients $\rho\left(a_{1}^{(1)}\right)$, $\rho\left(a_{2}^{(1)}\right), \rho\left(a_{1}^{(2)}\right)$ and $\rho\left(a_{2}^{(2)}\right)$. Namely, the destination also receives a finite field matrix

$$
\mathbf{A}_{p} \triangleq \rho(\mathbf{A})=\left[\begin{array}{ll}
\rho\left(a_{1}^{(1)}\right) & \rho\left(a_{2}^{(1)}\right) \\
\rho\left(a_{1}^{(2)}\right) & \rho\left(a_{2}^{(2)}\right)
\end{array}\right]=\left[\begin{array}{ll}
2 & 3 \\
0 & 2
\end{array}\right],
$$

and accordingly obtains a solution

$$
\left[\hat{w}_{1}, \hat{w}_{2}\right]^{\top}=\mathbf{A}_{p}^{-1}\left[\hat{u}_{1}, \hat{u}_{2}\right]^{\top}=[2,3]^{\top} .
$$

Remark 1. As in [7, Theorem. 11], we may choose large $p$ in Ring C\&F such that if $\mathbf{A}$ has full rank over $\mathcal{O}_{\mathbb{K}}$ (i.e., linear independence over a number field), then $\mathbf{A}_{p}$ also has full rank over $\mathbb{F}_{p}$ (i.e., linear independence over a finite field) with high probability. The sufficient and necessary condition for ensuring A has full rank over $\mathcal{O}_{\mathbb{K}}$ in Example 2 is

$$
\operatorname{det}(\mathbf{A})=\operatorname{det}\left[\begin{array}{cc}
a_{1}^{(1)} & a_{2}^{(1)}  \tag{16}\\
a_{1}^{(2)} & a_{2}^{(2)}
\end{array}\right] \neq 0
$$

Obviously, this condition can be extended to cases $L>2$.

## C. Goodness of Algebraic Lattices

Definition 7 (Moments). The second moment of a lattice $\Lambda^{\mathbb{Z}} \subseteq \mathbb{R}^{n T}$ is $\tilde{\sigma}^{2}\left(\Lambda^{\mathbb{Z}}\right) \triangleq \frac{\left.\int_{\mathcal{V}\left(\Lambda^{\mathbb{Z}}\right)}\right)\|\mathbf{x}\|^{2} \mathrm{dx}}{n T\left|\mathcal{V}\left(\Lambda^{\mathbb{Z}}\right)\right|}$, and the normalized second moment of $\Lambda^{\mathbb{Z}}$ is $G\left(\Lambda^{\mathbb{Z}}\right) \triangleq \frac{\tilde{\sigma}^{2}\left(\Lambda^{\mathbb{Z}}\right)}{\left|\mathcal{V}\left(\Lambda^{\mathbb{Z}}\right)\right|^{2 /(n T)}}$.
Definition 8 (Quantization goodness). A sequence of lattices $\Lambda^{\mathbb{Z}} \subseteq \mathbb{R}^{n T}$ is called good for MSE quantization if

$$
\lim _{T \rightarrow \infty} G\left(\Lambda^{\mathbb{Z}}\right)=\frac{1}{2 \pi e}
$$

The existence of such lattices has been shown in [38]. For lattices built from Construction A over quadratic fields, the quantization goodness has been proved in [18] following [24]. In the following theorem, we extend the quantization goodness to lattices constructed from general number fields, whose proof is given in Appendix A

Theorem 2. There exist a sequence of lattices in the ensemble (12) which are good for MSE quantization.

Definition 9 (Universal coding goodness). For a block-fading channel in the form of $\mathbf{y}=\mathbf{H x}+\mathbf{z}$, with channel $\mathbf{H} \in$ $\operatorname{dg}\left(\mathbb{R}^{n}\right) \otimes \mathbf{I}_{T}$, codeword $\mathbf{x} \in \Lambda^{\mathbb{Z}}$, and noise $\mathbf{z} \in \mathbb{R}^{n T}$ admitting $\mathcal{N}\left(\mathbf{0}, \sigma_{\mathbf{z}}^{2} \mathbf{I}_{n T}\right)$, define the generalized volume-to-noise ratio (VNR) as

$$
\mu\left(\mathbf{H} \Lambda^{\mathbb{Z}}\right) \triangleq \frac{\left(\operatorname{det}(\mathbf{H})\left|\mathcal{V}\left(\Lambda^{\mathbb{Z}}\right)\right|\right)^{\frac{2}{n T}}}{\sigma_{\mathbf{z}}^{2}}
$$

A sequence of lattices $\Lambda^{\mathbb{Z}} \subseteq \mathbb{R}^{n T}$ is called universally good for coding if for any $\mu\left(\mathbf{H} \Lambda^{\mathbb{Z}}\right)>2 \pi e$, the error probability of estimating x given $\mathbf{H}$ satisfies $P_{e}\left(\Lambda^{\mathbb{Z}}, \mathbf{H}\right) \rightarrow 0$ for all $\mathbf{H}$.

Theorem 3 ( [13], [14]). There exist a sequence of lattices in the ensemble (12) which are universally good for coding in block-fading channels.

Coding over algebraic lattices and coding over $\mathbb{Z}$-lattices have some differences, which we highlight in the following.

1) Relation to coding using a rank- $n T \mathbb{Z}$-lattice. The algebraic lattice $\Lambda^{\mathbb{Z}}$ is a special case of rank-nT $\mathbb{Z}$-lattices. Its extraordinary feature is that $\operatorname{dg}\left(\sigma\left(a_{l}\right)\right) \times \Lambda^{\mathbb{Z}} \subset \Lambda^{\mathbb{Z}}$. It also has a constant lower bound on $d_{\min }\left(\Lambda^{\mathbb{Z}}\right) \triangleq$ $\min _{\mathbf{x} \in \Lambda^{\mathbb{Z}} \backslash \mathbf{0}} \prod_{j=1}^{n}\left(\sum_{t=(j-1) T+1}^{j T} x_{t}^{2}\right)$, so the lattice enjoys full diversity in block fading channels [12]. On the contrary, for an arbitrary lattice constructed from a random Construction A over $\mathbb{Z}$, e.g., $\Lambda^{\prime}$, it may have $d_{\text {min }}\left(\Lambda^{\prime}\right)=0$.
2) Relation to coding using $n$ rank- $T \mathbb{Z}$-lattices. If we just transmit $n$ short lattice codewords of length $T$, then we will lose diversity and coding gain.

## IV. Achievable Computation Rate

The main results in this section are Theorems 4 and 55 whose proofs will be given in the subsections. We reemphasize here that our results only require channel knowledge at the receivers, not at the transmitters.

We begin by defining $\mathbf{a} \triangleq\left[a_{1}, \ldots, a_{L}\right]^{\top} \in \mathcal{O}_{\mathbb{K}}^{L}, \mathbf{h}_{j} \triangleq$ $\left[h_{j, 1}, \ldots, h_{j, L}\right]^{\top} \in \mathbb{R}^{L}$, and $\left\{\mathbf{H}_{l}\right\}$ as the shorthand notation of $\left\{\mathbf{H}_{l}\right\}_{l=1}^{L}$. The definitions of the achievable computation rates in one relay and the whole block-fading network are the same as those in Definitions 5] and 6, except that the channel coefficient here is $\left\{\mathbf{H}_{l}\right\}$, and the coefficient vector $\mathbf{a}$ is algebraic.
Theorem 4. With our coding scheme in block-fading channels, the following computation rate for a chosen a at a relay is achievable as $T \rightarrow \infty$ :

$$
\begin{align*}
& R_{\text {comp }}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a}\right)= \\
& \frac{n}{2} \max _{\mathbf{b}} \log ^{+}\left(\frac{n P}{\sum_{j=1}^{n}\left(\left|b_{j}\right|^{2}+P\left\|b_{j} \mathbf{h}_{j}-\sigma_{j}(\mathbf{a})\right\|^{2}\right)}\right) \tag{17}
\end{align*}
$$

and by optimizing $\mathbf{b}$ in (17), we have:

$$
R_{\mathrm{comp}}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a}\right)=\frac{n}{2} \log ^{+}\left(\frac{n}{\sum_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a})}\right)
$$

where $\mathbf{M}_{j}=\mathbf{I}-\frac{P}{P\left\|\mathbf{h}_{j}\right\|^{2}+1} \mathbf{h}_{j} \mathbf{h}_{j}^{\top}$.

Remark 2. If we confine $\mathbf{a} \in \mathbb{Z}^{L}$ in the above theorem, then obviously $R_{\text {comp }}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a} \mid \mathbf{a} \in \mathbb{Z}^{L}\right) \leq$ $R_{\text {comp }}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a} \mid \mathbf{a} \in \mathcal{O}_{\mathbb{K}}^{L}\right)$, namely, the rate achieved by $\mathbb{Z}$ lattice codes of length $n T$ can only be lower.

The above theorem leads to the computation rate of the block-fading network, which is simply the minimum computation rate among $L$ relays while making the set of combination coefficients invertible. In the following, we focus on understanding the computation rate at one relay, as well as its extension to the multiple access scenario.

Evaluating the $\mathcal{O}_{\mathbb{K}}$ coefficient vector $\mathbf{a}$ is crucial in understanding the performance limit of the computation rate. Our goal is to find one coefficient vector or multiple coefficient vectors minimizing the so-called additive Humbert form [39]

$$
\begin{equation*}
F(\mathbf{a})=\sum_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a}) \tag{19}
\end{equation*}
$$

With Cholesky decomposition of the $L \times L$ matrix $\mathbf{M}_{j}=$ $\overline{\mathbf{M}}_{j}^{\top} \overline{\mathbf{M}}_{j}$, we may write $F(\mathbf{a})=\sum_{j=1}^{n}\left\|\overline{\mathbf{M}}_{j} \sigma_{j}(\mathbf{a})\right\|^{2}$. This induces a squared distance over an $\mathcal{O}_{\mathbb{K}}$-module $\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$, whose generator matrix is given by the tuple $\left\{\overline{\mathbf{M}}_{j}\right\}$, and multiplication in the module is defined over the embedded space.

Let $\mathbf{a}_{1}, \ldots, \mathbf{a}_{L}$ be the coefficient vectors of the $L \mathcal{O}_{\mathbb{K}^{-}}$ successive minima of $\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$. Define the equation rate w.r.t. the $i$ th coefficient vector $\mathbf{a}_{i}$ as

$$
\begin{equation*}
R_{\mathrm{achv}, i}\left(\left\{\mathbf{H}_{l}\right\}\right)=\frac{n}{2} \log ^{+}\left(\frac{n}{F\left(\mathbf{a}_{i}\right)}\right) \tag{20}
\end{equation*}
$$

We refer to $R_{\text {achv, } 1}\left(\left\{\mathbf{H}_{l}\right\}\right)$ as the optimized (in the sense of optimizing the coefficient vectors) computation rate, and $\sum_{i=1}^{L} R_{\text {achv }, i}\left(\left\{\mathbf{H}_{l}\right\}\right)$ as the optimized computation sum-rate.
Theorem 5. The optimized computation rate satisfies

$$
\begin{align*}
& R_{\mathrm{achv}, 1}\left(\left\{\mathbf{H}_{l}\right\}\right) \geq \\
& \frac{1}{2 L} \sum_{j=1}^{n} \log ^{+}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)-\frac{n}{2} \log ^{+}\left(\frac{\kappa_{n L}}{n}\left(\operatorname{disc}_{\mathbb{K}}\right)^{1 / n}\right) \tag{21}
\end{align*}
$$

and the optimized computation sum-rate satisfies:

$$
\begin{align*}
& \sum_{i=1}^{L} R_{\text {achv }, i}\left(\left\{\mathbf{H}_{l}\right\}\right) \geq \\
& \frac{1}{2} \sum_{j=1}^{n} \log ^{+}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)-\frac{n L}{2} \log ^{+}\left(\frac{\kappa_{n L}}{n}\left(\operatorname{disc}_{\mathbb{K}}\right)^{1 / n}\right) \tag{22}
\end{align*}
$$

Remark 3. While Eq. (22) serves as a characterization of the performance of the $L$ best linearly independent combinations, our coding technique should be further generalized (for this equation) to allow for $L$ fine lattices (one per user) as well as a form of successive interference cancellation at the receiver in order to create effective channels that only involve the subset of lattices that can tolerate the increased varying noise faced when decoding each linear combination. For quasi-static
channels, such a scheme is developed by Ordentlich et al. in [26]. Our generalization follows in the same manner.
Remark 4. Theorem 5 resembles its quasi-static counterpart in [26, Theorem 3], [27, Theorem 6]. The sum-rate is understood in the context of block-fading MAC, whose sum capacity is

$$
\frac{1}{2} \sum_{j=1}^{n} \log ^{+}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)
$$

The theorem shows that, for any SNR, the computation rate and sum-rate are never much smaller than the symmetric capacity and sum-capacity of block-fading MAC. Since the gaps are determined by $n, L$ and $\operatorname{disc}_{\mathbb{K}}$, one should choose a number field with the smallest possible discriminant.

## A. Proof of Theorem 4

With dithering, the transmitted codeword is given by $\tilde{\mathbf{X}}_{l}=$ $\left[\mathbf{X}_{l}+\gamma \mathbf{D}_{l}\right] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}$. The signal $\operatorname{vec}\left(\tilde{\mathbf{X}}_{l}\right)$ is then uniformly distributed over $\gamma \mathcal{V}\left(\Lambda_{c}^{\mathbb{Z}}\right)$ and is statistically independent of $\operatorname{vec}\left(\mathbf{X}_{l}\right)$ according to the Crypto lemma [19, Lemma 1]. After MMSE scaling as well as removing the dithers, we have

$$
\begin{align*}
& \mathbf{B Y}-\gamma \sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{D}_{l} \\
& =\sum_{l=1}^{L} \mathbf{B H}_{l} \tilde{\mathbf{X}}_{l}+\mathbf{B Z}-\gamma \sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{D}_{l} \\
& =\sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{X}_{l}+\sum_{l=1}^{L} \mathbf{B} \mathbf{H}_{l} \tilde{\mathbf{X}}_{l}+\mathbf{B Z}-\sum_{l=1}^{L} \mathbf{A}_{l}\left(\mathbf{X}_{l}+\gamma \mathbf{D}_{l}\right) \tag{23}
\end{align*}
$$

To proceed, we need the following lemma.
Lemma 2. If $\mathbf{A}=\operatorname{dg}\left(\sigma_{1}(a), \ldots, \sigma_{n}(a)\right)$ with $a \in \mathcal{O}_{\mathbb{K}}$ and $\mathbf{S} \in \mathbb{R}^{n \times T}$, then
$[\mathbf{A S}] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}=\left[\mathbf{A}[\mathbf{S}] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}\right] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}$.

Proof: Write $\mathbf{S}=\mathbf{X}+\mathbf{S}^{\prime}$, where $\mathbf{X}$ is the closest lattice vector of $\mathbf{S}$ in $\gamma \Lambda_{c}^{\mathbb{Z}}$. Then clearly both sides of Eq. (24) equal $\left[\mathbf{A S}^{\prime}\right] \bmod \gamma \Lambda_{c}^{\mathbb{Z}}$, because $\Lambda_{c}^{\mathbb{Z}}$ is also multiplicatively closed, similarly to Lemma 1

Thus, the last term of Eq. (23) satisfies

$$
\sum_{l=1}^{L} \mathbf{A}_{l}\left(\mathbf{X}_{l}+\gamma \mathbf{D}_{l}\right) \quad \bmod \gamma \Lambda_{c}^{\mathbb{Z}}=\sum_{l=1}^{L} \mathbf{A}_{l} \tilde{\mathbf{X}}_{l} \quad \bmod \gamma \Lambda_{c}^{\mathbb{Z}}
$$

so we obtain

$$
\begin{align*}
& \mathbf{Y}_{\mathrm{eff}} \triangleq \mathbf{B Y}-\gamma \sum_{l=1}^{L} \mathbf{A}_{l} \mathbf{D}_{l} \quad \bmod \gamma \Lambda_{c}^{\mathbb{Z}} \\
& =\underbrace{\sum_{l=1}^{L=1} \mathbf{A}_{l} \mathbf{X}_{l}}_{\text {lattice codeword }}+ \\
& \mathbf{E}_{\mathbf{a}} \cdot \underbrace{\mathbf{E}_{\mathbf{a}}^{-1}\left(\sum_{l=1}^{L}\left(\mathbf{B H}_{l}-\mathbf{A}_{l}\right)\right.}_{\text {effective noise }} \tilde{\mathbf{X}}_{l}+\mathbf{B Z})  \tag{25}\\
& \bmod \gamma \Lambda_{c}^{\mathbb{Z}}
\end{align*}
$$

in which $\mathbf{E}_{\mathbf{a}}=\operatorname{dg}\left(\left[E_{1}, E_{2}, \ldots, E_{n}\right]\right)$ with

$$
E_{n}=\frac{\sqrt{\left|b_{n}\right|^{2}+P\left\|b_{n} \mathbf{h}_{n}-\sigma_{n}(\mathbf{a})\right\|^{2}}}{\prod_{j=1}^{n}\left(\sqrt{\left|b_{j}\right|^{2}+P\left\|b_{j} \mathbf{h}_{j}-\sigma_{j}(\mathbf{a})\right\|^{2}}\right)^{\frac{1}{n}}}
$$

and $\mathbf{Z}_{\text {eff }}=\mathbf{E}_{\mathbf{a}}^{-1}\left(\sum_{l=1}^{L}\left(\mathbf{B H}_{l}-\mathbf{A}_{l}\right) \tilde{\mathbf{X}}_{l}+\mathbf{B Z}\right)$ represents an effective noise. We then use the semi norm-ergodicity in [24] to characterize $\mathbf{Z}_{\text {eff }}$.
Definition 10 (Semi norm-ergodicity [24]). A random vector $\mathbf{x}$ of length $T$ is called semi norm-ergodic with effective variance $\frac{1}{T} \mathbb{E}\|\mathbf{x}\|^{2}$ if for any $\epsilon, \delta>0$, and $T$ large enough,

$$
\operatorname{Pr}\left(\mathbf{x} \notin \mathcal{B}\left(\mathbf{0}, \sqrt{(1+\delta) \mathbb{E}\left(\|\mathbf{x}\|^{2}\right)}\right)\right) \leq \epsilon
$$

In Appendix B we show that:
Lemma 3. The random vector vec $\left(\mathbf{Z}_{\mathrm{eff}}\right)$ is semi norm-ergodic with effective variance

$$
\begin{equation*}
\sigma_{\mathrm{eff}}^{2} \triangleq \prod_{j=1}^{n}\left(\left|b_{j}\right|^{2}+P\left\|b_{j} \mathbf{h}_{j}-\sigma_{j}(\mathbf{a})\right\|^{2}\right)^{\frac{1}{n}} \tag{26}
\end{equation*}
$$

The matrix $\mathbf{E}_{\mathbf{a}}^{-1}$ can be viewed as the channel matrix in Definition 9 . By inspection of the proof of Theorem 3] in [14], it is not difficult to see that Theorem 3 also holds for semi norm-ergodic noise, similarly to [24]. We omit the details. Therefore, there exist a sequence of lattices in the ensemble (12) such that the decoding error probability vanishes as $T \rightarrow$ $\infty$ as long as the VNR

$$
\begin{equation*}
\frac{\left(\operatorname{det}\left(\mathbf{E}_{\mathbf{a}}^{-1}\right) \operatorname{Vol}\left(\gamma \Lambda_{f}^{\mathbb{Z}}\right)\right)^{\frac{2}{n T}}}{\sigma_{\mathrm{eff}}^{2}}>2 \pi e \tag{27}
\end{equation*}
$$

On the other hand, the quantization goodness in Theorem 2 implies

$$
\begin{equation*}
\frac{P}{\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)^{\frac{2}{n T}}}<\frac{1+\delta}{2 \pi e} \tag{28}
\end{equation*}
$$

for any $\delta>0$ if $T$ is large enough. It follows from (27) and (28) that any computation rate up to

$$
\begin{equation*}
\frac{1}{T} \log \left(\frac{\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)}{\operatorname{Vol}\left(\gamma \Lambda_{f}^{\mathbb{Z}}\right)}\right)<\frac{n}{2} \log \left(\frac{P}{\sigma_{\text {eff }}^{2}}\right) \tag{29}
\end{equation*}
$$

is achievable.

The effective noise variance $\sigma_{\text {eff }}^{2}$ represents the geometric mean (GM) of the noise variances in all the blocks. The final rate expression based on this form is given by ${ }^{2}$ :

$$
\begin{align*}
& R_{\mathrm{comp}}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a}\right)=\frac{n}{2} \log ^{+}\left(\frac{1}{\prod_{j=1}^{n}\left(\sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a})\right)^{1 / n}}\right) \\
& =\frac{1}{2} \log ^{+}\left(\frac{1}{\prod_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a})}\right) . \tag{30}
\end{align*}
$$

Since the the optimization of the algebraic integer vector in a multiplicative form is complicated, we upper bound $\sigma_{\text {eff }}^{2}$ by the arithmetic mean (AM)

$$
\sigma_{\mathrm{AM}}^{2} \triangleq \frac{1}{n} \sum_{j=1}^{n}\left(\left|b_{j}\right|^{2}+P\left\|b_{j} \mathbf{h}_{j}-\sigma_{j}(\mathbf{a})\right\|^{2}\right)
$$

to reach (17), following (29). This enables the applications of a nice algorithmic framework based on successive minima in the next subsection. Lastly, the details of deriving (18) are given in Appendix C

## B. Searching the Optimal Coefficients

In this subsection, we show that $F(\mathbf{a})$ can be written as the squared distance of a $\mathbb{Z}$-lattice vector, and explain the relation between $\mathbb{Z}$-successive minima and $\mathcal{O}_{\mathbb{K}}$-successive minima. These results enable the application of conventional lattice algorithms over $\mathbb{Z}$ to find one or multiple coefficient vectors at a relay. We refer readers to [41]-[43] for these algorithms.

First, each $\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$ has a corresponding $\mathbb{Z}$-lattice $\Lambda^{\mathbb{Z}}\left(\Phi_{\overline{\mathbf{M}}}\right)$ that belongs to a submodule of $\mathbb{R}^{n L}$, whose generator matrix is

$$
\Phi_{\overline{\mathbf{M}}}=\overline{\mathbf{M}}\left(\Phi \otimes \mathbf{I}_{L}\right)
$$

where

$$
\overline{\mathbf{M}}=\left[\begin{array}{ccc}
\overline{\mathbf{M}}_{1} & \cdots & \mathbf{0} \\
\mathbf{0} & \cdots & \mathbf{0} \\
\vdots & \vdots & \vdots \\
\mathbf{0} & \cdots & \overline{\mathbf{M}}_{n}
\end{array}\right]
$$

and recall that $\Phi=\left[\sigma\left(\phi_{1}\right), \ldots, \sigma\left(\phi_{n}\right)\right]$ and $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ is an integral basis of $\mathcal{O}_{\mathbb{K}}$. To show this more explicitly, note that there exists a bijective mapping $\Psi: \mathbb{Z}^{n L} \rightarrow \mathcal{O}_{\mathbb{K}}^{L}$ defined by

$$
\begin{align*}
\mathbf{a} & =\Psi(\tilde{\mathbf{a}}) \\
& =\left[\sum_{k=1}^{n} \phi_{k} \tilde{a}_{(k-1) L+1}, \sum_{k=1}^{n} \phi_{k} \tilde{a}_{(k-1) L+2}, \ldots, \sum_{k=1}^{n} \phi_{k} \tilde{a}_{k L}\right]^{\top} ; \tag{31}
\end{align*}
$$

since $\sigma_{j}$ is a ring homomorphism, it follows that

$$
\sigma_{j}(\mathbf{a})=\left[\sum_{k=1}^{n} \sigma_{j}\left(\phi_{k}\right) \tilde{a}_{(k-1) L+1}, \ldots, \sum_{k=1}^{n} \sigma_{j}\left(\phi_{k}\right) \tilde{a}_{k L}\right]^{\top}
$$

Thus, $F(\mathbf{a})=\left\|\Phi_{\bar{M}} \tilde{\mathbf{a}}\right\|^{2}\left(\tilde{\mathbf{a}} \in \mathbb{Z}^{n L}\right)$ represents the squared distance of a point in $\Lambda^{\mathbb{Z}}\left(\Phi_{\overline{\mathrm{M}}}\right)$.

[^2]Second, if multiple message equations are required at one relay, a search algorithm over $\mathbb{Z}$-lattice $\Lambda^{\mathbb{Z}}\left(\Phi_{\overline{\mathbf{M}}}\right)$ has to ensure their coefficient vectors $\mathbf{a}_{1}, \ldots, \mathbf{a}_{L}$ are linearly independent over $\mathcal{O}_{\mathbb{K}}$. For the highest rates, it suffices to search for the $\mathcal{O}_{\mathbb{K}^{-}}$ successive minima. This constraint can be incorporated into an enumeration algorithm, which keeps increasing the search radius until linear independence is satisfied. The question that arises here is whether we can use the first few successive minima of a $\mathbb{Z}$-module to find those of an $\mathcal{O}_{\mathbb{K}}$-module.

Let $\tilde{\mathbf{a}}_{i}$ be the vector giving the $i$-th successive minima $\lambda_{i}\left(\Phi_{\overline{\mathbf{M}}}\right)$ of $\mathbb{Z}$-lattice $\Lambda^{\mathbb{Z}}\left(\Phi_{\overline{\mathbf{M}}}\right)$. It may happen that

$$
\operatorname{dim}\left(\operatorname{span}_{\mathcal{O}_{\mathbb{K}}}\left(\Psi\left(\left[\tilde{\mathbf{a}}_{1}, \ldots, \tilde{\mathbf{a}}_{L}\right]\right)\right)\right)<L
$$

For example, choose $\mathbb{K}=\mathbb{Q}(\sqrt{3})$. Let $\tilde{\mathbf{a}}_{1}=[1,2,1,1]^{\top}$, $\tilde{\mathbf{a}}_{2}=[6,9,4,5]^{\top}$; after mapping them back to $\mathcal{O}_{\mathbb{K}}^{2}$, we have $\mathbf{a}_{1}=[1+\sqrt{3}, 2+\sqrt{3}]^{\top}, \mathbf{a}_{2}=[6+4 \sqrt{3}, 9+5 \sqrt{3}]^{\top}$. Since $(3+\sqrt{3}) \mathbf{a}_{1}=\mathbf{a}_{2}$, one concludes that $\mathbf{a}_{1}$ and $\mathbf{a}_{2}$ are not independent over $\mathcal{O}_{\mathbb{K}}$.

Nevertheless, we have the following result:
Proposition 2. Let the mapping $\Psi$ be defined as in (31). Suppose $\mathbb{Z}$-coefficient vectors $\tilde{\mathbf{a}}_{1}, \ldots, \tilde{\mathbf{a}}_{n L}$ produce the $n L$ successive minima $\lambda_{1}\left(\Phi_{\overline{\mathbf{M}}}\right), \ldots, \lambda_{n L}\left(\Phi_{\overline{\mathbf{M}}}\right)$ of $\mathbb{Z}$-lattice $\Lambda^{\mathbb{Z}}\left(\Phi_{\overline{\mathbf{M}}}\right)$. Then $\left\{\Psi\left(\tilde{\mathbf{a}}_{1}\right), \ldots, \Psi\left(\tilde{\mathbf{a}}_{n L}\right)\right\}$ contains the $L \mathcal{O}_{\mathbb{K}}$-successive minima.

Proof: Write the $\mathbb{Z}$-coefficient matrix $\mathbf{T}=\left[\tilde{\mathbf{a}}_{1}, \ldots, \tilde{\mathbf{a}}_{n L}\right]$. From the definition of successive minima, $\mathbf{T} \in \mathbb{Z}^{n L \times n L}$ is a full-rank matrix such that $\Phi_{\overline{\mathbf{M}}} \mathbf{T}=\overline{\mathbf{M}}\left(\Phi \otimes \mathbf{I}_{L}\right) \mathbf{T}$ yields $\lambda_{1}\left(\Phi_{\bar{M}}\right), \ldots, \lambda_{n L}\left(\Phi_{\bar{M}}\right)$ of $\mathbb{Z}$-lattice $\Lambda^{\mathbb{Z}}\left(\Phi_{\bar{M}}\right)$. Notice that the $L \times n L$ algebraic-integer matrix $\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{n L}\right]=\Psi(\mathbf{T})$ simply consists of the the first $L$ rows of $\left(\Phi \otimes \mathbf{I}_{L}\right) \mathbf{T}$; in fact we have

$$
\begin{equation*}
\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{n L}\right]=\left[\phi_{1} \mathbf{I}_{L}, \ldots, \phi_{n} \mathbf{I}_{L}\right] \mathbf{T} \tag{32}
\end{equation*}
$$

Since $\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ is an integral basis of $\mathcal{O}_{\mathbb{K}}$, the matrix $\left[\phi_{1} \mathbf{I}_{L}, \ldots, \phi_{n} \mathbf{I}_{L}\right]$ obviously has rank $L$. Then it follows from the rank identity

$$
\operatorname{rank}\left(\mathbf{C}_{1} \mathbf{C}_{2}\right)=\operatorname{rank}\left(\mathbf{C}_{1}\right)
$$

for full-rank matrix $\mathbf{C}_{2}$ that the matrix $\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{n L}\right]$ is of rank $L$. Therefore, there exist exactly $L$ vectors in $\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{n L}\right\}$ which are linearly independent over $\mathcal{O}_{\mathbb{K}}$. Thus, the $L \mathcal{O}_{\mathbb{K}}$-successive minima must be contained in the set $\left\{\Psi\left(\tilde{\mathbf{a}}_{1}\right), \ldots, \Psi\left(\tilde{\mathbf{a}}_{n L}\right)\right\}$.

The proposition shows searching for $L \mathcal{O}_{\mathbb{K}}$-independent lattice points inside ball $\mathcal{B}\left(\mathbf{0}, \lambda_{n L}\left(\Phi_{\overline{\mathrm{M}}}\right)\right)$ is possible. We further explain Proposition 2 in Fig. 2] Suppose $L=3$ and $n=3$. There are 9 successive minima in the embedded real lattice $\Lambda\left(\Phi_{\bar{M}}\right)$, and their corresponding algebraic coefficient vectors are denoted by $\mathbf{a}_{(1), 1}, \ldots, \mathbf{a}_{(3), 3}$, where the vectors in the same row are linearly dependent over $\mathcal{O}_{\mathbb{K}}$. The $\mathbf{a}_{(1), 1}, \mathbf{a}_{(2), 1}, \mathbf{a}_{(3), 1}$ marked in red are coefficient vectors of the first three successive minima over $\mathcal{O}_{\mathbb{K}}$.

## C. Proof of Theorem 5

To derive the optimized computation rate and sum-rate, we only need to apply Minkowski's first and second theorems


Fig. 2: Illustration of $\mathcal{O}_{\mathbb{K}}$-successive minima for $L=3$ and $n=3$. Among the 9 successive minima of the embedded real lattice, those marked in red are coefficient vectors of the first three successive minima over $\mathcal{O}_{\mathbb{K}}$.
to $\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$. First, by applying Sylvester's determinant identity to each $\left|\operatorname{det}\left(\overline{\mathbf{M}}_{i}\right)\right|$, one has

$$
|\operatorname{det}(\overline{\mathbf{M}})|=\prod_{j=1}^{n}\left|\operatorname{det}\left(\overline{\mathbf{M}}_{i}\right)\right|=\prod_{j=1}^{n}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)^{-1 / 2}
$$

Consequently the volume of $\Lambda^{\mathcal{O}_{K}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$ becomes

$$
\begin{array}{r}
\left|\operatorname{det}\left(\Phi_{\overline{\mathbf{M}}}\right)\right|=|\operatorname{det}(\overline{\mathbf{M}})|\left|\operatorname{det}\left(\Phi \otimes \mathbf{I}_{L}\right)\right| \\
=\left(\operatorname{disc}_{\mathbb{K}}\right)^{L / 2} \prod_{j=1}^{n}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)^{-1 / 2}
\end{array}
$$

The shortest lattice vector of $\Lambda^{\mathbb{Z}}\left(\Phi_{\bar{M}}\right)$ is the embedding of the shortest lattice vector from $\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)$. Then it follows from Minkowski's first theorem over $\mathbb{Z}$-lattices that $\lambda_{1}^{2}\left(\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)\right) \leq \kappa_{n L}\left|\operatorname{det}\left(\Phi_{\overline{\mathbf{M}}}\right)\right|^{2 /(n L)}$, which yields
$\lambda_{1}^{2}\left(\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)\right) \leq \kappa_{n L}\left(\operatorname{disc}_{\mathbb{K}}\right)^{1 / n} \prod_{j=1}^{n}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)^{-1 /(n L)}$

By substituting (33) into the rate expression (18), we obtain

$$
\begin{align*}
& R_{\text {achv, } 1} \\
& =\frac{n}{2} \log ^{+}\left(\frac{n}{\lambda_{1}^{2}\left(\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)\right)}\right) \\
& \geq \frac{n}{2} \log ^{+}\left(\frac{n \prod_{j=1}^{n}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)^{1 /(n L)}}{\kappa_{n L}\left(\operatorname{disc}_{\mathbb{K}}\right)^{1 / n}}\right) \\
& =\underbrace{\frac{1}{2 L} \sum_{j=1}^{n} \log ^{+}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)}_{\frac{1}{L} \times \text { MAC capacity }}-\underbrace{\frac{n}{2} \log ^{+}\left(\frac{\kappa_{n L}}{n}\left(\operatorname{disc}_{\mathbb{K}}\right)^{1 / n}\right)}_{\text {constant }} \tag{34}
\end{align*}
$$

Meanwhile, from Minkowski's second theorem (4), we have

$$
\begin{equation*}
\prod_{j=1}^{L} \lambda_{j}^{2 n}\left(\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)\right) \leq \kappa_{n L}^{n L}\left|\operatorname{det}\left(\Phi_{\overline{\mathbf{M}}}\right)\right|^{2} \tag{35}
\end{equation*}
$$

Finally, after substituting (35) into (18), we have:

$$
\begin{align*}
& \sum_{i=1}^{L} R_{\mathrm{achv}, i}\left(\left\{\mathbf{H}_{l}\right\}\right) \\
& =\sum_{i=1}^{L} \frac{n}{2} \log ^{+}\left(\frac{n}{\lambda_{i}^{2}\left(\Lambda^{\mathcal{O}_{\mathbb{K}}}\left(\left\{\overline{\mathbf{M}}_{j}\right\}\right)\right)}\right) \\
& \geq \frac{n}{2} \log ^{+}\left(\frac{n^{L} \prod_{j=1}^{n}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)^{1 / n}}{\kappa_{n L}^{L}\left(\operatorname{disc}_{\mathbb{K}}\right)^{L / n}}\right) \\
& =\underbrace{\frac{1}{2} \sum_{j=1}^{n} \log ^{+}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)}_{\text {MAC capacity }}-\underbrace{\frac{1}{2} \log ^{+}\left(\frac{\kappa_{n L}^{n L}}{n^{n L}}\left(\operatorname{disc}_{\mathbb{K}}\right)^{L}\right)}_{\text {constant }} \tag{36}
\end{align*}
$$

## V. DoF Analysis

Define DoF associated with $R_{\text {achv }, i}$ as

$$
\begin{equation*}
d_{\mathrm{achv}, i}=\lim _{P \rightarrow \infty} \frac{R_{\mathrm{achv}, i}}{\frac{1}{2} \log (1+P)} \tag{37}
\end{equation*}
$$

The main result of this section is:
Theorem 6. For almost all $\left\{\mathbf{H}_{l}\right\}$ w.r.t. the Lebesgue measure, the DoF's of the optimized computation rate and sum-rate are respectively $d_{\mathrm{achv}, 1}=\frac{n}{L}$ and $\sum_{i=1}^{L} d_{\mathrm{achv}, i}=n$.

Proof of Theorem 6. As a direct consequence of Theorem [5] the lower bounds of DoF's are:

$$
d_{\mathrm{achv}, 1} \geq \frac{n}{L}, \quad \sum_{i=1}^{L} d_{\mathrm{achv}, i} \geq n
$$

${ }^{W}$ We will show in Theorem 7 that $d_{\text {achv, } 1} \leq \frac{n}{L}$, which is due to Lemma 4 on Diophantine approximation of a real vector by algebraic conjugates. The block-fading MAC capacity can upper bound the sum DoF's, which yields $\sum_{i=1}^{L} d_{\text {achv }, i} \leq n$. Consequently, along with $d_{\text {achv }, 1} \geq d_{\text {achv }, 2} \geq d_{\text {achv }, L}$, we have

$$
d_{\mathrm{achv}, 1}=\cdots=d_{\mathrm{achv}, L}=\frac{n}{L}
$$

Theorem 7. For almost all $\left\{\mathbf{H}_{l}\right\}$ w.r.t. the Lebesgue measure, the DoF associated to the first computation rate satisfies $d_{\mathrm{achv}, 1} \leq \frac{n}{L}$.
Lemma 4. Let $\psi: \mathbb{N} \rightarrow \mathbb{R}^{+}$be an approximation function. Then for almost all $\left\{\mathbf{H}_{l}\right\}$ w.r.t. the Lebesgue measure, and for all $q \in \mathcal{O}_{\mathbb{K}}$, there exists a constant $c_{\left\{\mathbf{H}_{l}\right\}}^{\prime}>0$ such that

$$
\begin{equation*}
\max _{l \in\{1, \ldots, L\}} \min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\operatorname{dg}(\sigma(a / q))\right\| \geq c_{\left\{\mathbf{H}_{l}\right\}}^{\prime} \psi(|\operatorname{Nr}(q)|) \tag{38}
\end{equation*}
$$

if $\sum_{k=1}^{\infty} \psi(k)^{n L} k^{L}<\infty$.
Lemma 4 generalizes the classical Khintchine-Groshev theorem from $\mathbb{Z}$ to $\mathcal{O}_{\mathbb{K}}$. The proof is given in Appendix $\square$ Note that the approximation function in (38) can decay as fast as $\psi(|\operatorname{Nr}(q)|)=|\operatorname{Nr}(q)|^{-\left(\frac{1+L}{n L}+\delta\right)}$ for any $\delta>0$. Lemma 4
also indicates that, all points in the set $q \mathcal{U}$ have the same approximation-error bound $c_{\left\{\mathbf{H}_{l}\right\}}^{\prime} \psi(|\mathrm{Nr}(q)|)$.

We proceed to prove Theorem 7 , where the technique is to generalize the approach in [26], [27] to vectors of algebraic conjugates.

Proof of Theorem 7. First rewrite the denominator $\sigma_{\text {AM }}^{2}$ in (17) explicitly as a trade-off between "range" and "accuracy":
$\frac{1}{n} \underbrace{\|\mathbf{B}\|^{2}}_{\text {range }}+\frac{P}{n} \underbrace{\left\|\mathbf{B}\left[\mathbf{H}_{1}, \ldots, \mathbf{H}_{L}\right]-\left[\operatorname{dg}\left(\sigma\left(a_{1}\right)\right), \ldots, \operatorname{dg}\left(\sigma\left(a_{L}\right)\right)\right]\right\|^{2}}_{\text {accuracy }}$
Let $\mathcal{V}_{0}$ stand for the Voronoi region of $\mathbf{0}$ in the embedded lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}\right)$. In the shortest vector problem (SVP), one aims to find a shortest nonzero vector, so the coefficients cannot be $\sigma\left(a_{1}\right)=\cdots=\sigma\left(a_{L}\right)=0$. By rearranging the order of $a_{1}, \ldots, a_{L}$ if necessary, we can assume that $\sigma\left(a_{1}\right) \neq \mathbf{0}$. Then the analysis falls into two cases depending on whether $\mathbf{B H}_{1} \in \mathcal{V}_{\mathbf{0}}$.
i) If $\mathbf{B H}_{1} \in \mathcal{V}_{\mathbf{0}}$, then $\left\|\mathbf{B H}_{1}-\operatorname{dg}\left(\sigma\left(a_{1}\right)\right)\right\|$ is lower bounded by the packing radius of lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}\right)$, which is $\frac{\lambda_{1}\left(\mathcal{O}_{\mathbb{K}}\right)}{2}$. Based on this, we have

$$
\begin{align*}
& \sigma_{\mathrm{AM}}^{2} \geq \frac{1}{n}\|\mathbf{B}\|^{2}+\frac{P}{n}\left\|\mathbf{B H}_{1}-\operatorname{dg}\left(\sigma\left(a_{1}\right)\right)\right\|^{2} \\
& \quad>P \frac{\lambda_{1}^{2}\left(\mathcal{O}_{\mathbb{K}}\right)}{4 n}>P^{\frac{L-1}{L}} \frac{\lambda_{1}^{2}\left(\mathcal{O}_{\mathbb{K}}\right)}{4 n} \tag{40}
\end{align*}
$$

where the first inequality is from

$$
\begin{aligned}
& \left\|\mathbf{B}\left[\mathbf{H}_{1}, \ldots, \mathbf{H}_{L}\right]-\left[\operatorname{dg}\left(\sigma\left(a_{1}\right)\right), \ldots, \operatorname{dg}\left(\sigma\left(a_{L}\right)\right)\right]\right\|^{2} \\
& \geq\left\|\mathbf{B H}_{1}-\operatorname{dg}\left(\sigma\left(a_{1}\right)\right)\right\|^{2} .
\end{aligned}
$$

ii) If $\mathbf{B H}_{1} \notin \mathcal{V}_{\mathbf{0}}$, we have $\mathbf{B H}_{1}=\operatorname{dg}(\sigma(q)+\varphi)$ for $\mathbf{0} \neq$ $\sigma(q) \in \sigma\left(\mathcal{O}_{\mathbb{K}}\right), \boldsymbol{\varphi} \in \mathcal{V}_{\mathbf{0}}$. The "accuracy" term for two vectors $\mathbf{H}_{1}$ and $\mathbf{H}_{l}$ satisfies

$$
\begin{align*}
& \left\|\mathbf{B}\left[\mathbf{H}_{1}, \mathbf{H}_{l}\right]-\left[\operatorname{dg}\left(\sigma\left(a_{1}\right)\right), \operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right]\right\|^{2} \\
& \geq\|\boldsymbol{\varphi}\|^{2}+\left\|\tilde{\mathbf{H}}_{l} \operatorname{dg}(\sigma(q)+\boldsymbol{\varphi})-\operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right\|^{2} \tag{41}
\end{align*}
$$

where $\tilde{\mathbf{H}}_{l}=\mathbf{H}_{1}^{-1} \mathbf{H}_{l}$. The r.h.s. of (41) is a quadratic function of $\varphi$. To attain its minimum, we solve the following equation

$$
\partial\left(\|\varphi\|^{2}+\left\|\tilde{\mathbf{H}}_{l} \operatorname{dg}(\sigma(q)+\boldsymbol{\varphi})-\operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right\|^{2}\right) / \partial \varphi=\mathbf{0}
$$

to get $\varphi=\left(\mathbf{I}+\tilde{\mathbf{H}}_{l}^{2}\right)^{-1}\left(\tilde{\mathbf{H}}_{l} \sigma\left(a_{l}\right)-\tilde{\mathbf{H}}_{l}^{2} \sigma(q)\right)$. Substitute this back into (41), we have

$$
\begin{aligned}
& \left\|\mathbf{B}\left[\mathbf{H}_{1}, \mathbf{H}_{l}\right]-\left[\operatorname{dg}\left(\sigma\left(a_{1}\right)\right), \operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right]\right\|^{2} \\
& \geq\left\|\left(\tilde{\mathbf{H}}_{l}^{2}+\mathbf{I}_{n}\right)^{-1}\left(\tilde{\mathbf{H}}_{l} \operatorname{dg}(\sigma(q))-\operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right)\right\|^{2} \\
& \geq h_{l}^{*}\left\|\tilde{\mathbf{H}}_{l} \operatorname{dg}(\sigma(q))-\operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right\|^{2},
\end{aligned}
$$

where $h_{l}^{*} \triangleq \min _{\tilde{h}_{l} \in \tilde{\mathbf{H}}_{l}} \frac{1}{\left(\tilde{h}_{l}^{2}+1\right)^{2}}$. Thus, for almost all channel realizations it holds that

$$
\begin{align*}
& \left\|\mathbf{B}\left[\mathbf{H}_{1}, \ldots, \mathbf{H}_{L}\right]-\left[\operatorname{dg}\left(\sigma\left(a_{1}\right)\right), \ldots, \operatorname{dg}\left(\sigma\left(a_{L}\right)\right)\right]\right\|^{2} \\
& \geq \max _{l \in\{2, \ldots, L\}}\left(h_{l}^{*}\left\|\tilde{\mathbf{H}}_{l} \operatorname{dg}(\sigma(q))-\operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right\|^{2}\right) \\
& \geq \max _{l \in\{2, \ldots, L\}}\left(h_{l}^{*} \min _{i}\left|\sigma_{i}(q)\right|^{2}\left\|\tilde{\mathbf{H}}_{l}-\operatorname{dg}(\sigma(q))^{-1} \operatorname{dg}\left(\sigma\left(a_{l}\right)\right)\right\|^{2}\right) \\
& \geq\left.\right|^{2} c_{\left\{\mathbf{H}_{l}\right\}}^{\prime \prime} \min _{i}\left|\sigma_{i}(q)\right|^{2}|\operatorname{Nr}(q)|^{-\frac{2}{n}-\frac{2}{n(L-1)}}  \tag{42}\\
& \text { where the last inequality is due to Lemma 4, and } c_{\left\{\mathbf{H}_{l}\right\}}^{\prime \prime} \\
& \text { depends on the realizations of }\left\{\mathbf{H}_{l}\right\} .
\end{align*}
$$

To analyze the "range" term of (39), we specify the gap among the embeddings of $q: \varrho \triangleq \frac{\min _{i}\left|\sigma_{i}(q)\right|^{2}}{\max _{i}\left|\sigma_{i}(q)\right|^{2}}$. Then the analysis follows that of [26]. Since $\left\|\mathbf{B H}_{1}\right\|^{2} \geq\|\sigma(q)\|^{2} / 4$ if $\mathbf{B H}_{1} \notin \mathcal{V}_{0}$, the first term of (39) satisfies

$$
\begin{equation*}
\frac{1}{n}\|\mathbf{B}\|^{2} \geq \frac{1}{n \max _{i}\left|h_{1, i}\right|^{2}} \sum_{i=1}^{n}\left|b_{i} h_{1, i}\right|^{2} \geq \frac{\varrho \max _{i}\left|\sigma_{i}(q)\right|^{2}}{4 \max _{i}\left|h_{1, i}\right|^{2}} \tag{43}
\end{equation*}
$$

Hereby we substitute (42) and (43) into (39):

$$
\begin{align*}
& \sigma_{\mathrm{AM}}^{2} \geq \frac{\varrho}{4 \max _{i}\left|h_{1, i}\right|^{2}} \max _{i}\left|\sigma_{i}(q)\right|^{2}+\frac{\varrho c_{\left\{\mathbf{H}_{l}\right\}}^{\prime \prime} P}{n} \max _{i}\left|\sigma_{i}(q)\right|^{-\frac{2}{L-1}} \\
& \geq \rho_{\min }^{*}\left(\max _{i}\left|\sigma_{i}(q)\right|^{2}+P \max _{i}\left|\sigma_{i}(q)\right|^{-\frac{2}{L-1}}\right) \\
& \geq \rho_{\min }^{*}\left(\left(\frac{1}{L-1}\right)^{\frac{L-1}{L}} P^{\frac{L-1}{L}}+\left(\frac{1}{L-1}\right)^{-\frac{1}{L}} P^{\frac{L-1}{L}}\right) \tag{44}
\end{align*}
$$

where $\rho_{\text {min }}^{*} \triangleq \min \left\{\frac{\varrho}{4 \max _{i}\left|h_{1, i}\right|^{2}}, \frac{\varrho c_{\left\{\mathbf{H}_{l}\right\}}^{\prime}}{n}\right\}$, and the last inequality follows from defining $x \triangleq \max _{i}\left|\sigma_{i}(q)\right|^{2}$ and noticing that the convex function $f(x) \triangleq x^{2}+P x^{-\frac{2}{L-1}}$ attains its minimum at root $x=\left(\frac{P}{L-1}\right)^{\frac{L-1}{2 L}}$.

Finally, the lower bounds (40) and (44) on noise variance $\sigma_{\mathrm{AM}}^{2}$ in both cases admit the inequality of $\sigma_{\mathrm{AM}}^{2} \geq c^{\prime \prime \prime} P^{\frac{L-1}{L}}$ for some constant $c^{\prime \prime \prime}$. Substitute this lower bound on noise into the rate expression (17) and the DoF expression (37), one can show that $d_{\text {achv }, 1} \leq \frac{n}{L}$.

## VI. Numerical Results

In this section, we present numerical results to evaluate the performance of Ring C\&F. Notice that there are many number fields [44], [45] available to construct lattice codes:
i) For relatively small $n$, we can enumerate all totally real number fields with small discriminants. Tables $\square$ to $\square \square$ in Appendix Eresent this enumeration from quadratic to quintic number fields. According to the principle of small discriminants shown in Theorem 5, the highest computation rates should come from quadratic to quintic number fields with minimal polynomials $\mathfrak{m}_{\theta}=\theta^{2}-\theta-1, \mathfrak{m}_{\theta}=\theta^{3}+\theta^{2}-2 \theta-1$, $\mathfrak{m}_{\theta}=\theta^{4}+\theta^{3}-3 \theta^{2}-\theta+1$ and $\mathfrak{m}_{\theta}=\theta^{5}+\theta^{4}-4 \theta^{3}-3 \theta^{2}+3 \theta+1$, respectively.
ii) For relatively large $n$, we can use the maximal real subfield of a cyclotomic number field. A cyclotomic field $\mathbb{Q}\left(\zeta_{k}\right)$


Fig. 3: The ergodic computation rates (dashed lines) and sum-rates (solid lines) based on number fields of different degrees.
is a number field obtained by adjoining $\zeta_{k}$ to $\mathbb{Q}$, where $\zeta_{k}$ represents a primitive $k$ th root of unity. Its degree is $n=$ $\varphi(k) / 2$, where $\varphi(\cdot)$ is Euler's totient function. Table $\square$ in Appendix E shows the properties of maximal real sub-fields $\mathbb{Q}\left(\zeta_{k}+\zeta_{k}^{-1}\right)$ with degrees $n=11,14$.

In Fig. 3, we compare the optimized computation rate and sum-rate of ring $\mathrm{C} \& \mathrm{~F}$ and classic $\mathrm{C} \& \mathrm{~F}$, in terms of ergodic rate metrics defined as $\mathbb{E}\left(R_{\text {achv, } 1}\left(\left\{\mathbf{H}_{l}\right\}\right)\right)$ and $\mathbb{E}\left(\sum_{i=1}^{L} R_{\mathrm{achv}, i}\left(\left\{\mathbf{H}_{l}\right\}\right)\right)$. The expectation is taken over $2 \times$ $10^{3}$ Monte Carlo runs, with channel coefficients admitting $\mathcal{N}(0,1)$ entries. The " $\mathbb{Z}$ " curve in Fig. 3 denotes the classic $C \& F$ using length- $n T \mathbb{Z}$-lattice codes. The " $\mathfrak{m}_{\theta}$ " curves, e.g., $\theta^{2}-\theta-1$, denote ring $C \& F$ based on field $\mathbb{Q}(\theta)$. For cyclotomic number fields, we mark them with $\mathbb{Q}\left(\zeta_{k}+\zeta_{k}^{-1}\right)$. The simulation starts by choosing $L=2, n=2$ in Fig. 3-(a), then repeats by choosing $n=3,4,5,11,14$ in Fig. 3(b) to Fig. 3-(f). Simulations can be made for the setting of larger $L$ in the same manner.

In Fig. 3-(a), significant performance gains can be observed for Ring C\&F. The quadratic field with minimal polynomial $\mathfrak{m}_{\theta}=\theta^{2}-\theta-1$ performs superiorly to all other quadratic fields, and its sum-rate is within 1 dB gap to the MAC capacity. The DoF's of Ring C\&F for computation rates and sum-rates are respectively 1 and 2 . The classic $C \& F$ using $\mathbb{Z}$ gives very poor rates. It falls behind Ring C\&F with $\mathfrak{m}_{\theta}=\theta^{2}-3$ by more than 25 dB and increasing SNR results in little performance gain. Similar observations can be made from Fig. 3-(b) to Fig. 3-(f). They confirm that fields with minimal polynomials $\mathfrak{m}_{\theta}=\theta^{3}+\theta^{2}-2 \theta-1, \mathfrak{m}_{\theta}=\theta^{4}+\theta^{3}-3 \theta^{2}-\theta+1$ and $\mathfrak{m}_{\theta}=\theta^{5}+$ $\theta^{4}-4 \theta^{3}-3 \theta^{2}+3 \theta+1$ are indeed the best for $n=3,4,5$. As predicted by the parameters in Theorem [5] the gaps between the computation sum-rates of $\mathbb{Q}\left(\zeta_{23}+\zeta_{23}^{-1}\right), \mathbb{Q}\left(\zeta_{29}+\zeta_{29}^{-1}\right)$ and MAC capacities are much larger than those of quadratic fields, but their optimality in DoF is preserved. We further explain why the classic C\&F has roughly 0 DoF . From the law of large numbers, we have approximation $\sum_{j=1}^{n} \mathbf{a}^{\top} \mathbf{M}_{j} \mathbf{a} \approx$ $\frac{n(n P+1-P)}{n P+1} \mathbf{a}^{\top} \mathbf{a}$ for relatively large $n$; thus increasing SNR $P$ does not improve the rate.

The Ring C\&F scheme can be extended to integer-forcing (IF) for time-varying channels [9]. Suppose the channel experiences $n$ successive blocks $\hat{\mathbf{H}}_{1}, \ldots, \hat{\mathbf{H}}_{n} \in \mathbb{R}^{L \times L}$ (i.e., $L$ single-antenna transmitters and one receiver with $L$ antennas) over the duration of a codeword. We can use our algebraic lattices to show that the following rate is achievable in IF:

$$
\begin{aligned}
& R_{\mathrm{IF}}\left(\left\{\hat{\mathbf{H}}_{l}\right\}\right)= \\
& \quad \max _{\substack{\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{L}\right] \in \mathcal{O}_{\mathrm{K}}^{L \times L} \\
\operatorname{rank}\left[\mathbf{a}_{1}, \ldots, \mathbf{a}_{L}\right]=L}} \min _{l \in\{1, \ldots L\}} \frac{1}{2} \log ^{+}\left(\frac{n P}{\sum_{j=1}^{n} \sigma_{j}\left(\mathbf{a}_{l}\right)^{\top} \mathbf{F}_{j} \sigma_{j}\left(\mathbf{a}_{l}\right)}\right)
\end{aligned}
$$

in which $\mathbf{F}_{j}=\left(P^{-1} \mathbf{I}+\hat{\mathbf{H}}_{j}^{\top} \hat{\mathbf{H}}_{j}\right)^{-1 / 2}$. The difference from [9, Theorem 1] is that $\mathbf{a}_{l} \in \mathcal{O}_{\mathbb{K}}^{L}$ rather than $\mathbb{Z}^{L}$. Again, we compare the ring-based IF and $\mathbb{Z}$-based IF in terms of ergodic rate $\mathbb{E}\left(R_{\mathrm{IF}}\left(\left\{\hat{\mathbf{H}}_{l}\right\}\right)\right)$. The channel capacity which equals the
rate of joint maximum likelihood (ML) decoding is

$$
\min _{\mathcal{S} \subset\{1, \ldots, L\}} \frac{1}{2 n|\mathcal{S}|} \sum_{j=1}^{n} \log \left(\operatorname{det}\left(\mathbf{I}+P \hat{\mathbf{H}}_{j, \mathcal{S}} \hat{\mathbf{H}}_{j, \mathcal{S}}^{\top}\right)\right)
$$

with $\hat{\mathbf{H}}_{j, \mathcal{S}}$ being a submatrix consists of the $\mathcal{S}$ columns of $\hat{\mathbf{H}}_{j}$. As shown in Fig. 4, unlike the C\&F setting, IF based on $\mathbb{Z}$ still has full DoF, thanks to the cooperation among all receive antennas. However, IF using $\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{5})$ in Fig. 4 (a) provides approximately $4-5 \mathrm{~dB}$ gain compared to that based on $\mathbb{Z}$. The gain rises to around 8 dB in Fig. 4 (b) for a large block size of $n=11$. Thus, similarly to C\&F, the ring structure offers significant gains in IF.

## VII. Conclusions

The class of algebraic lattices for C\&F proposed in this paper are built from Construction A over number fields. These lattices enjoy the advantage of closure under multiplication by algebraic integers. Since the embeddings of an algebraic integer are different, it helps to quantize block fading channels in a finer manner. Their achievable rates outperform those of $\mathbb{Z}$-lattices.

Although we relaxed the GM $\sigma_{\text {eff }}^{2}$ in (29) to the AM so that the problem was reduced to finding the successive minima of a lattice, an important open question is how to minimize the GM (a product form) efficiently, and how to analyze its Diophantine approximation.

Metric Diophantine approximation associated with $\mathcal{O}_{\mathbb{K}^{-}}$ modules studied in this paper is more involved than that associated with $\mathbb{Z}$-lattices. We only addressed the convergent part of the Khintchine-Groshev theorem, while the divergent part was not used. We leave the divergent part of Lemma 4 as another open problem.

## Appendix A

## Proof of Quantization goodness

Our proof follows the steps in [24] with some adjustments: i) The prime number $p$ is chosen to grow as $O\left(T^{3 n / 2}\right)$ rather than $O\left(T^{3 / 2}\right)$, to compensate for the factor $p^{T-k}$ in the volume of the coarse lattice, while it is $p^{n T-k}$ in [24]. ii) We count the number of lattice points inside a ball for a number field lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)$ rather than an integer lattice $\mathbb{Z}^{n T}$.

Let $V_{n T}$ be the volume of an $n T$-dimensional unit ball. Set the inertial degree $f=1$, and the scaling factor $\gamma=$ $p^{-1 / n} \operatorname{disc}_{\mathbb{K}^{-1 /(2 n)}}^{n T}$. Write (10) explicitly as $\rho^{-1}(\mathcal{C})=$ $\mathcal{M}(\mathcal{C})+\mathfrak{p}^{T}$, where $\mathcal{M}(\cdot)$ maps $\mathbb{F}_{p}$ onto the coset leaders of each $\mathcal{O}_{\mathbb{K}} / \mathfrak{p}$ based on component-wise isomorphism. The scaled lattice is

$$
\gamma \Lambda_{c}^{\mathcal{O}_{\mathbb{K}}}=\gamma \mathcal{M}(\mathbf{G w})+\gamma \mathfrak{p}^{T}
$$

where the volume of its embedded lattice satisfies $\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right) \geq \gamma^{n T} \operatorname{disc}_{\mathbb{K}}^{T / 2} p^{(T-k)}$ and the equality holds only if the generator matrix $\mathbf{G} \in \mathbb{F}_{p}^{T \times k}$ of $\mathcal{C}$ has full rank.

Since obviously $\lim _{T \rightarrow \infty} G\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right) \geq 1 /(2 \pi e)$, we are left with the task of showing that that for any $\delta>0, \epsilon>0$,

$$
\begin{equation*}
\operatorname{Pr}\left(\frac{\tilde{\sigma}^{2}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)}{\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)^{2 /(n T)}}>\frac{1}{2 \pi e}+\delta\right)<\epsilon \tag{45}
\end{equation*}
$$



Fig. 4: Ergodic rates of IF receivers with channel variation based on number fields of different degrees.
with large enough $T$. Letting $0<\alpha<\log (n T)$, and

$$
\begin{equation*}
k \triangleq \frac{n T}{2 \log (p)} \log \left(V_{n T}^{-2 /(n T)} 2^{\alpha}\right) \tag{46}
\end{equation*}
$$

we have

$$
\begin{equation*}
\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)^{2 /(n T)}=n T V_{n T}^{2 /(n T)} 2^{-\alpha} \tag{47}
\end{equation*}
$$

Denote by $r_{o}$ the covering radius of the embedded lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)$. Since $\operatorname{Vol}\left(\sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)\right)=\operatorname{disc}_{\mathbb{K}}^{T / 2}$, the number of points of $\operatorname{disc}_{\mathbb{K}}{ }^{-1 /(2 n)} \sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)$ inside a ball can be measured with volumes. Then we can adapt [24, Lemma 1] from $\mathbb{Z}^{n T}$ to $\sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)$ to get the following lemma.

Lemma 5. For any $\mathrm{x} \in \mathbb{R}^{n T}$ and $r>0$, the number of points of a scaled lattice $\operatorname{disc}_{\mathbb{K}}{ }^{-1 /(2 n)} \sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right)$ inside $\mathcal{B}(\mathbf{x}, r)$ can be bounded as

$$
\begin{aligned}
& \left|\operatorname{disc}_{\mathbb{K}}^{-1 /(2 n)} \sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right) \cap \mathcal{B}(\mathbf{x}, r)\right| \\
& \geq \operatorname{Vol}\left(\mathcal{B}\left(\mathbf{x}, r-\operatorname{disc}_{\mathbb{K}}^{-1 /(2 n)} r_{o}\right)\right)
\end{aligned}
$$

Assume the source $\mathbf{x}$ is uniformly distributed over a fundamental region of lattice $\sigma\left(\mathfrak{p}^{T}\right)$. For a target $\mathbf{x} \in \mathbb{R}^{n T}$, its distance to the closest lattice point equals to that modulo the coarse lattice:

$$
\begin{aligned}
d\left(\mathbf{x}, \gamma \Lambda_{c}^{\mathbb{Z}}\right) & =\min _{\mathbf{c} \in \mathcal{C}(\mathbf{G}), \lambda \in \sigma\left(\mathfrak{p}^{T}\right)} \frac{1}{n T}\|\mathbf{x}-\gamma \mathcal{M}(\mathbf{c})-\gamma \lambda\|^{2} \\
& =\min _{\mathbf{c} \in \mathcal{C}(\mathbf{G})} \frac{1}{n T}\left\|(\mathbf{x}-\gamma \mathcal{M}(\mathbf{c}))^{*}\right\|^{2}
\end{aligned}
$$

in which $(\cdot)^{*} \triangleq(\cdot) \bmod \sigma\left(\mathfrak{p}^{T}\right)$. Clearly, $d\left(\mathbf{x}, \gamma \Lambda_{c}^{\mathbb{Z}}\right) \leq \frac{\gamma^{2} r_{p}^{2}}{n T}$, where $r_{p}$ denotes the covering radius of ideal lattice $\sigma\left(\mathfrak{p}^{T}\right)$. Note that $\mathcal{M}(\mathbf{c})$ is uniformly distributed over the coset leaders $S^{T}$ of $\left(\mathcal{O}_{\mathbb{K}} / \mathfrak{p}\right)^{T}$ as the elements of $\mathbf{G}$ are uniform over $\mathbb{F}_{p}$, and
$\left|S^{T}\right|=p^{T}$. With $0<\rho<\alpha$, for any fixed $\mathbf{x}$, the probability of a small quantization distance is bounded as

$$
\begin{align*}
\varepsilon & \triangleq \operatorname{Pr}\left(d\left(\mathbf{x}, \gamma \Lambda_{c}^{\mathbb{Z}}\right) \leq 2^{-\rho}\right) \\
& =\operatorname{Pr}\left(\min _{\mathbf{c} \in \mathcal{C}(\mathbf{G})} \frac{1}{n T}\left\|(\mathbf{x}-\gamma \mathcal{M}(\mathbf{c}))^{*}\right\|^{2} \leq 2^{-\rho}\right) \\
& =p^{-T}\left|\gamma S^{T} \cap \mathcal{B}^{*}\left(\mathbf{x}, \sqrt{n T 2^{-\rho}}\right)\right| \\
& =p^{-T}\left|\gamma \sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right) \cap \mathcal{B}\left(\mathbf{x}, \sqrt{n T 2^{-\rho}}\right)\right| \\
& =p^{-T} \mid \operatorname{disc}_{\mathbb{K}}^{-1 /(2 n)} \sigma\left(\mathcal{O}_{\mathbb{K}}^{T}\right) \cap \mathcal{B}\left(\mathbf{x}, \gamma^{-1} \operatorname{disc}_{\mathbb{K}}\right. \\
& \stackrel{(a)}{\geq} V_{n T} p^{-T}\left(\gamma^{-1} \operatorname{disc}_{\mathbb{K}}-1 /(2 n) \sqrt{n T 2^{-\rho}}\right) \mid \\
& \stackrel{(b)}{\geq} V_{n T} p^{-k}\left(n T V_{n T}^{2 /(n T)} 2^{-\alpha}\right)^{-n T / 2}\left(n T 2^{-\rho}\right)^{n T / 2}\left(1-\gamma r_{o}\right)^{n T} \\
& \stackrel{(c)}{=} V_{n T} 2^{-\rho n T / 2} O(1), \tag{48}
\end{align*}
$$

where (a) is from Lemma 5, (b) is from using $\operatorname{disc}_{\mathbb{K}}-T / 2 \geq$ $p^{(T-k)} \gamma^{n T} \operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)^{-1}$ and Eq. (47), and (c) has used Eq. (46) and $\left(1-\gamma r_{o}\right)^{n T}=O(1)$. To see this, notice that $r_{o}=O(\sqrt{n T})$ as it is upper bounded by the length of a corner point of a Gram-Schmidt parallelepiped [43, Eq. (44)], and that $\operatorname{disc}_{\mathbb{K}}$ is independent of $T$. If we choose $p$ to grow with $T^{c n}, c>1$, e.g., $p=\xi T^{3 n / 2}$ and minimize $\xi \in[1,2)$ under the constraint that $p$ is a prime [24], then $\left(1-\gamma r_{o}\right)^{n T}=\left(1-\operatorname{disc}_{\mathbb{K}}^{-1 /(2 n)} p^{-1 / n} O(n T)\right)^{n T}=O(1)$ w.r.t. $T$.

For the $p^{k}-1$ non-zero random $\mathbf{w}_{i} \in \mathbb{F}_{p}^{k}$, define the indicator function

$$
\chi_{i}= \begin{cases}1, & \text { if }\left\|\left(\mathbf{x}-\gamma \mathcal{M}\left(\mathbf{G w}_{i}\right)\right)^{*}\right\|^{2} \leq 2^{-\rho} \\ 0, & \text { if }\left\|\left(\mathbf{x}-\gamma \mathcal{M}\left(\mathbf{G w}_{i}\right)\right)^{*}\right\|^{2}>2^{-\rho}\end{cases}
$$

which satisfies $\mathbb{E}\left(\chi_{i}\right)=\varepsilon$. From Chebyshev's inequality,

$$
\begin{aligned}
\operatorname{Pr}\left(d\left(\mathbf{x}, \gamma \Lambda_{c}^{\mathbb{Z}}\right)>2^{-\rho}\right) & \leq \operatorname{Pr}\left(\sum_{i=1}^{p^{k}-1} \chi_{i}=0\right) \\
& \leq \frac{\operatorname{Var}\left(\frac{1}{p^{k}-1} \sum_{i=1}^{p^{k}-1} \chi_{i}\right)}{\varepsilon^{2}} \\
& <\frac{p}{\left(p^{k}-1\right) \varepsilon}
\end{aligned}
$$

Together with Eqs. (46) and (48), one has

$$
\begin{equation*}
\operatorname{Pr}\left(d\left(\mathbf{x}, \gamma \Lambda_{c}^{\mathbb{Z}}\right)>2^{-\rho}\right)<2^{-\frac{n T}{2}(\alpha-\rho+O(1))} \tag{49}
\end{equation*}
$$

It follows from (49) that we can use the same arguments as in [24] to show the expected second moment is small. Finally, we complete the proof of (45) by using Markov's inequality.

Proof: i) First, we show that $\mathbf{z}_{\text {eff }}^{*}$ admits density $\mathcal{N}\left(\mathbf{0}, \sigma_{\text {eff }}^{2} \mathbf{I}_{n T}\right)$. As a linear combination of independent Gaussian random variables, $\sigma_{\text {eff }}^{-1} \mathbf{z}_{\text {eff }}^{*}$ has a density

$$
\begin{align*}
& f_{\tilde{\mathbf{z}}_{1}^{*}}\left(\sigma_{\text {eff }}^{-1} \mathbf{E}_{\mathbf{a}}^{-1}\left(\mathbf{B} \mathbf{H}_{1}-\mathbf{A}_{1}\right) \otimes \mathbf{I}_{T} \mathbf{z}\right) \circledast \cdots \circledast \\
& f_{\tilde{\mathbf{z}}_{L}^{*}}\left(\sigma_{\text {eff }}^{-1} \mathbf{E}_{\mathbf{a}}^{-1}\left(\mathbf{B} \mathbf{H}_{L}-\mathbf{A}_{L}\right) \otimes \mathbf{I}_{T} \mathbf{z}\right) \circledast f_{\mathbf{z}}\left(\sigma_{\text {eff }}^{-1} \mathbf{E}_{\mathbf{a}}^{-1} \mathbf{B} \otimes \mathbf{I}_{T} \mathbf{z}\right) \\
& =\mathcal{N}\left(\mathbf{0}, \mathbf{I}_{n T}\right), \tag{50}
\end{align*}
$$

where $\circledast$ refers to the convolution of density functions. Thus, we obtain

$$
\begin{equation*}
f_{\mathbf{z}_{\text {eff }}^{*}}(\mathbf{z})=\mathcal{N}\left(\mathbf{0}, \sigma_{\text {eff }}^{2} \mathbf{I}_{n T}\right) \tag{51}
\end{equation*}
$$

ii) Second, we upper bound the density of each dithered variable $\tilde{\mathbf{X}}_{l}$ by that of $\tilde{\mathbf{z}}_{l}^{*}$. The constrained Voronoi region for each $\tilde{\mathbf{X}}_{l}$ is $\mathcal{V}_{l} \triangleq \mathcal{V}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right) \cap \mathcal{B}(\mathbf{0}, \sqrt{(1+\delta) n T P})$, so the density function of $\tilde{\mathbf{X}}_{l}$ becomes

$$
f_{\tilde{\mathbf{x}}_{l}}(\mathbf{z})= \begin{cases}1 /\left|\mathcal{V}_{l}\right|, & \text { if } \mathbf{z} \in \mathcal{V}_{l} \\ 0, & \text { otherwise }\end{cases}
$$

Also, $\left|\mathcal{V}_{l}\right| \geq V_{n T} r_{\text {eff }}^{n T}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)\left(1-\delta^{\prime}\right)$ for any small $\delta^{\prime}>0$ as $\operatorname{vec}\left(\tilde{\mathbf{X}}_{l}\right)$ is semi-norm ergodic. Let $\tilde{\mathbf{b}}$ be a random vector uniformly distributed over a ball of volume $\operatorname{Vol}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)$; its $\operatorname{Pr}(\mathcal{T}=1) \operatorname{Pr}\left(\operatorname{vec}\left(\mathbf{Z}_{\text {eff }}\right) \notin \mathcal{B}\left(\mathbf{0}, \sqrt{(1+\delta) n T \sigma_{\text {eff }}^{2}}\right) \mid \mathcal{T}=1\right)$ density function $f_{\tilde{\mathbf{b}}}(\mathbf{z})$ upper bounds $f_{\tilde{\mathbf{x}}_{l}}(\mathbf{z})$ :

$$
+\operatorname{Pr}(\mathcal{T}=0) \operatorname{Pr}\left(\operatorname{vec}\left(\mathbf{Z}_{\mathrm{eff}}\right) \notin \mathcal{B}\left(\mathbf{0}, \sqrt{(1+\delta) n T \sigma_{\text {eff }}^{2}}\right) \mid \mathcal{T}=0\right), \quad \frac{f_{\tilde{\mathbf{x}}_{l}}(\mathbf{z})}{f_{\tilde{\mathbf{b}}}(\mathbf{z})}=\frac{V_{n T} r_{\mathrm{eff}}^{n T}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)}{\left|\mathcal{V}_{l}\right|} \leq 1-\delta^{\prime}
$$

where

$$
\mathcal{T}= \begin{cases}0, & \text { if } \exists \mathbf{x} \in\left\{\operatorname{vec}\left(\tilde{\mathbf{X}}_{l}\right)\right\}  \tag{52}\\ & \text { s.t., } \mathbf{x} \notin \mathcal{B}\left(\mathbf{0}, \sqrt{(1+\delta) \mathbb{E}\left(\|\mathbf{x}\|^{2}\right)}\right), \delta>0 \\ 1, & \text { otherwise. }\end{cases}
$$

For any $\epsilon>0$, we can make $\operatorname{Pr}(\mathcal{T}=0) \leq \epsilon$ by increasing $T$ because $\left\{\operatorname{vec}\left(\tilde{\mathbf{X}}_{l}\right)\right\}$ are all semi norm-ergodic. Then we can confine our discussion to the case of $\mathcal{T}=1$. This constraint enables us to show the density of the effective noise is tightly upper bounded by that of a Gaussian vector with the techniques in [19, Lemma 11], without proving the algebraic lattices are good for covering.

Proposition 3. Assume $\mathcal{T}=1$. Let
$\mathbf{z}_{\mathrm{eff}}=\left(\mathbf{E}_{\mathbf{a}}^{-1} \otimes \mathbf{I}_{T}\right)\left(\sum_{l=1}^{L}\left(\mathbf{B H}_{l}-\mathbf{A}_{l}\right) \otimes \mathbf{I}_{T} \operatorname{vec}\left(\tilde{\mathbf{X}}_{l}\right)+\mathbf{B} \otimes \mathbf{I}_{T} \mathbf{z}\right)$.
The Gaussian variable $\tilde{\mathbf{z}}_{l}^{*}$ with density $f_{\tilde{\mathbf{z}}_{l}^{*}}(\mathbf{z})$ has the same second moment as that of the fundamental Voronoi region $\mathcal{V}\left(\gamma \Lambda_{c}^{\mathbb{Z}}\right)$. Combining the above, we arrive at

$$
\frac{f_{\tilde{\mathbf{x}}_{l}}(\mathbf{z})}{f_{\tilde{\mathbf{z}}_{l}^{*}}(\mathbf{z})}=\frac{f_{\tilde{\mathbf{x}}_{l}}(\mathbf{z}) f_{\tilde{\mathbf{b}}^{( }}(\mathbf{z})}{f_{\tilde{\mathbf{b}}}(\mathbf{z}) f_{\tilde{\mathbf{z}}_{l}^{*}}(\mathbf{z})}<\left(1-\delta^{\prime}\right) e^{n T c(T)}
$$

where $f_{\tilde{\mathbf{b}}}(\mathbf{z}) / f_{\tilde{\mathbf{z}}_{l}^{*}}(\mathbf{z})<e^{n T c(T)}$ due to [19, Eq. (199)].
iii) Finally, notice that the density of $\mathbf{z}_{\mathrm{eff}}$ is:

$$
\begin{aligned}
& f_{\tilde{\mathbf{x}}_{1}}\left(\mathbf{E}_{\mathbf{a}}^{-1}\left(\mathbf{B} \mathbf{H}_{1}-\mathbf{A}_{1}\right) \otimes \mathbf{I}_{T} \mathbf{z}\right) \circledast \cdots \circledast \\
& f_{\tilde{\mathbf{x}}_{L}}\left(\mathbf{E}_{\mathbf{a}}^{-1}\left(\mathbf{B} \mathbf{H}_{L}-\mathbf{A}_{L}\right) \otimes \mathbf{I}_{T} \mathbf{z}\right) \circledast f_{\mathbf{z}}\left(\mathbf{E}_{\mathbf{a}}^{-1} \mathbf{B} \otimes \mathbf{I}_{T} \mathbf{z}\right),
\end{aligned}
$$

so combining this with the arguments in steps i) and ii) proves the proposition.

Since the Gaussian vector $\mathbf{z}_{\text {eff }}^{*}$ is semi norm-ergodic with effective variance $\sigma_{\text {eff }}^{2}$, we have

Then there exists an i.i.d. Gaussian vector

$$
\mathbf{z}_{\mathrm{eff}}^{*}=\left(\mathbf{E}_{\mathbf{a}}^{-1} \otimes \mathbf{I}_{T}\right)\left(\sum_{l=1}^{L}\left(\mathbf{B} \mathbf{H}_{l}-\mathbf{A}_{l}\right) \otimes \mathbf{I}_{T} \tilde{\mathbf{z}}_{l}^{*}+\mathbf{B} \otimes \mathbf{I}_{T} \mathbf{Z}\right)
$$

with density $f_{\mathbf{z}_{\text {eff }}^{*}}(\mathbf{z})=\mathcal{N}\left(\mathbf{0}, \sigma_{\text {eff }}^{2} \mathbf{I}_{n T}\right), \quad \sigma_{\text {eff }}^{2}=$ $\prod_{j=1}^{n}\left(\left|b_{j}\right|^{2}+P\left\|b_{j} \mathbf{h}_{j}-\sigma_{j}(\mathbf{a})\right\|^{2}\right)^{\frac{1}{n}}, \tilde{\mathbf{z}}_{l}^{*} \sim \mathcal{N}\left(\mathbf{0}, P \mathbf{I}_{n T}\right)$, such that the density of $\mathbf{z}_{\mathrm{eff}}$ is upper bounded as

$$
f_{\mathbf{z}_{\text {eff }}}(\mathbf{z}) \leq\left(1-\delta^{\prime}\right)^{L} e^{L c(T) n T} f_{\mathbf{z}_{\text {eff }}^{*}}(\mathbf{z})
$$

where $c(T) \triangleq \frac{1}{2} \log \left(2 \pi e G\left(\Lambda^{(n T)}\right)\right)+\frac{1}{n T}$, and $\delta^{\prime}, c(T) \rightarrow 0$ as $T \rightarrow \infty$.

Together with Proposition
3, we have $\operatorname{Pr}\left(\operatorname{vec}\left(\mathbf{Z}_{\text {eff }}\right) \notin \mathcal{B}\left(\mathbf{0}, \sqrt{(1+\delta) n T \sigma_{\text {eff }}^{2}}\right) \mid \mathcal{T}=1\right) \quad \rightarrow \quad 0$ and the proof is completed.

## Appendix C <br> Derivation of EQ. (18)

Note that $\sigma_{\mathrm{AM}}^{2}$ is a convex function of $\mathbf{b}$. By assuming a to be fixed, the minimum of $\sigma_{\mathrm{AM}}^{2}$ is reached by setting $\partial \sigma_{\mathrm{AM}}^{2} / \partial \mathbf{b}=\mathbf{0}$. From this we have, for $j=1,2, \ldots, n$,

$$
\begin{equation*}
b_{j}=\frac{P \sigma_{j}(\mathbf{a})^{\top} \mathbf{h}_{j}}{P\left\|\mathbf{h}_{j}\right\|^{2}+1} \tag{53}
\end{equation*}
$$

By plugging (53) into $\frac{1}{P} \sigma_{\text {AM }}^{2}$, we have

$$
\begin{aligned}
& \frac{1}{P} \sigma_{\mathrm{AM}}^{2} \\
= & \frac{1}{n P} \sum_{j=1}^{n}\left(b_{j}^{2}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)-2 P b_{j} \sigma_{j}(\mathbf{a})^{\top} \mathbf{h}_{j}+P\left\|\sigma_{j}(\mathbf{a})\right\|^{2}\right) \\
= & \frac{1}{n P} \sum_{j=1}^{n}\left(\left(\frac{P \sigma_{j}(\mathbf{a})^{\top} \mathbf{h}_{j}}{P\left\|\mathbf{h}_{j}\right\|^{2}+1}\right)^{2}\left(1+P\left\|\mathbf{h}_{j}\right\|^{2}\right)\right)+ \\
& \frac{1}{n P} \sum_{j=1}^{n}\left(-2 P\left(\frac{P \sigma_{j}(\mathbf{a})^{\top} \mathbf{h}_{j}}{P\left\|\mathbf{h}_{j}\right\|^{2}+1}\right) \sigma_{j}(\mathbf{a})^{\top} \mathbf{h}_{j}+P\left\|\sigma_{j}(\mathbf{a})\right\|^{2}\right) \\
= & \frac{1}{n} \sum_{j=1}^{n}\left(\left\|\sigma_{j}(\mathbf{a})\right\|^{2}-\sigma_{j}(\mathbf{a})^{\top}\left(\frac{P}{P\left\|\mathbf{h}_{j}\right\|^{2}+1} \mathbf{h}_{j} \mathbf{h}_{j}^{\top}\right) \sigma_{j}(\mathbf{a})\right) \\
= & \frac{1}{n} \sum_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top}(\underbrace{\mathbf{I}-\frac{P}{P\left\|\mathbf{h}_{j}\right\|^{2}+1} \mathbf{h}_{j} \mathbf{h}_{j}^{\top}}_{\triangleq \mathbf{M}_{j}}) \sigma_{j}(\mathbf{a}) .
\end{aligned}
$$

Then the computation rate in 17) can be written as

$$
R_{\mathrm{comp}}\left(\left\{\mathbf{H}_{l}\right\}, \mathbf{a}\right)=\frac{n}{2} \log ^{+}\left(\frac{1}{(1 / n) \sum_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a})}\right)
$$

in which the free parameter is $\mathbf{a} \in \mathcal{O}_{\mathbb{K}}^{L}$.

## Appendix D <br> DIOPHANTINE APPROXIMATION BY ALGEBRAIC CONJUGATES

Our proof may be viewed as an extension of Khintchine's theorem for complex numbers given in [46, Section 4], which dealt with Diophantine approximation by ratios of Gaussian integers. We first recall a result from [47, Theorem 5], [48, p. 132] to count the number of principal ideals in $\mathcal{O}_{\mathbb{K}}$.

Lemma 6. Let $J(k, \mathbb{K})$ be the number of principal ideals in $\mathcal{O}_{\mathbb{K}}$ with norm no larger than $k$. Then

$$
\left|J(k, \mathbb{K})-\rho_{\mathbb{K}} k\right| \leq \frac{\rho_{\mathbb{K}}}{w} 2^{n} k^{\frac{n-1}{n}} \max \left(1, \Phi_{0}^{n}\right),
$$

where $\Phi_{0}=2^{n-1} n^{2 n} \bar{\gamma}^{n} e^{r M(n-1)}, \rho_{\mathbb{K}}=\frac{2^{r_{1}}(2 \pi)^{r_{2}} R_{\mathbb{K}}}{w \sqrt{\mid \text { disc }_{\mathbb{K}} \mid}}, w$ denotes the number of roots of unity in $\mathbb{K}, R_{\mathbb{K}}$ denotes the regulator of the log-unit lattice, $\left(r_{1}, r_{2}\right)$ is the signature of $\mathbb{K}$, $r=r_{1}+r_{2}-1$, and $\bar{\gamma}, M$ are parameters of the log-unit lattice.

Proof of Lemma 4. Firstly assume that $\mathbf{H}_{l}$ belongs to $\mathcal{V}_{\mathcal{O}_{\mathbb{K}}}$, the fundamental Voronoi region of lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}\right)$, for $1 \leq l \leq L$. Let ties on the boundary of $\mathcal{V}_{\mathcal{O}_{\mathbb{K}}}$ be broken in an arbitrary manner. Define

$$
\begin{align*}
& \mathcal{A}_{q, \psi} \triangleq \\
& \left\{\left\{\mathbf{H}_{l}\right\} \mid \max _{l \in\{1, \ldots, L\}} \min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\operatorname{dg}(\sigma(a / q))\right\|<\psi(|\operatorname{Nr}(q)|)\right\} \tag{54}
\end{align*}
$$

for a fixed $q$. Note that $\mathcal{A}_{q, \psi}=\mathcal{A}_{u q, \psi}$ for any unit $u \in \mathcal{U}$, since

$$
\begin{equation*}
\min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\operatorname{dg}(\sigma(a /(u q)))\right\|=\min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\operatorname{dg}(\sigma(a / q))\right\| \tag{55}
\end{equation*}
$$

and since $|\operatorname{Nr}(q)|=|\operatorname{Nr}(u q)|$. This means that when investigating a sequence of $\{q\}$ with decreasing approximation error $\psi(|\operatorname{Nr}(q)|)$, we only have to pick $q$ modulo the unit group.

Denote by $\mathcal{B}(c, r)$ a ball of radius $r$ with centre at $c$. The Lebesgue measure of a ball of radius $\psi(|\operatorname{Nr}(q)|)$ centred at $\sigma(a / q)$ is given by

$$
\Upsilon(\mathcal{B}(\sigma(a / q), \psi(|\operatorname{Nr}(q)|)))=\frac{\pi^{n / 2}}{\Gamma\left(\frac{n}{2}+1\right)} \psi(|\operatorname{Nr}(q)|)^{n}
$$

A congruence consideration shows that the number of points $\sigma(a / q) \in \mathcal{V}_{\mathcal{O}_{\mathbb{K}}}$ in (54) is exactly $|\operatorname{Nr}(q)|$. We further elaborate counting lattice points inside the fundamental Voronoi region $\mathcal{V}_{\mathcal{O}_{\mathbb{K}}}$ in Fig. 5] Then the total measure of $\mathcal{A}_{q, \psi}$ is bounded by

$$
\begin{align*}
& \Upsilon\left(\mathcal{A}_{q, \psi}\right) \leq(\Upsilon(\mathcal{B}(\sigma(a / q), \psi(|\operatorname{Nr}(q)|)))|\operatorname{Nr}(q)|)^{L} \\
& =\left(\frac{\pi^{n / 2}}{\Gamma\left(\frac{n}{2}+1\right)} \psi(|\operatorname{Nr}(q)|)^{n}|\operatorname{Nr}(q)|\right)^{L} \tag{56}
\end{align*}
$$

Further define

$$
\begin{equation*}
\mathcal{W}_{\psi} \triangleq \limsup _{|\operatorname{Nr}(q)| \rightarrow \infty} \mathcal{A}_{q, \psi}=\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \bigcup_{q:|\operatorname{Nr}(q)|=k} \mathcal{A}_{q, \psi} \tag{57}
\end{equation*}
$$

as the subset of $\left\{\mathbf{H}_{l}\right\}$ for which 54) holds for infinitely many $q$ modulo the unit group.

Let $\mathfrak{q}=(q)$ denote the principal ideal generated by $q$. Since $(q)=(q u)$ for any unit $u \in \mathcal{U}$, the set of algebraic integers can be partitioned into different subsets indicated by principle ideals. Since $\operatorname{Nr}(\mathfrak{q})=|\operatorname{Nr}(q)|$, the number of subsets $q \mathcal{U}$ whose elements have absolute norm $k$ is equal to the number of principal ideals with norm $k$. Consequently, we have, for $N=1,2, \cdots, \infty$,

$$
\begin{aligned}
& \Upsilon\left(\bigcup_{k=N}^{\infty} \bigcup_{q:|\operatorname{Nr}(q)|=k} \mathcal{A}_{q, \psi}\right) \\
& \leq \sum_{k=N}^{\infty} \sum_{\mathfrak{q}: \operatorname{Nr}(\mathfrak{q})=k}\left(\frac{\pi^{n / 2}}{\Gamma\left(\frac{n}{2}+1\right)} \psi(\operatorname{Nr}(\mathfrak{q})) \operatorname{Nr}(\mathfrak{q})\right)^{L} \\
& =\left(\frac{\pi^{n / 2}}{\Gamma\left(\frac{n}{2}+1\right)}\right)^{L} \sum_{k=N}^{\infty} \psi(k)^{n L} k^{L} \sum_{\mathfrak{q}: \operatorname{Nr}(\mathfrak{q})=k} 1
\end{aligned}
$$

where $\mathfrak{q}: \operatorname{Nr}(\mathfrak{q})=k$ denotes a principal ideal with norm $k$. By Lemma6, we have $J(k, \mathbb{K})=\rho_{\mathbb{K}} k+O\left(k^{\frac{n-1}{n}}\right)$. As $O\left(k^{\frac{n-1}{n}}\right)$ grows no faster than $k$, we have $\sum_{\mathfrak{q}: \operatorname{Nr}(\mathfrak{q})=k} 1=O(1)$, which is bounded by a constant in the limit of $k$.

By the Borel-Cantelli lemma [28], the Lebesgue measure $\Upsilon\left(\mathcal{W}_{\psi}\right)=0$ if $\Upsilon\left(\bigcup_{k=N}^{\infty} \bigcup_{q:|\operatorname{Nr}(q)|=k} \mathcal{A}_{q, \psi}\right)<\infty$. Obviously, the convergence of the series $\sum_{k=1}^{\infty} \psi(k)^{n L} k^{L}$ implies that $\Upsilon\left(\mathcal{W}_{\psi}\right)=0$. Since a countably infinite number of the Voronoi regions cover the whole space, our result holds for all $\left\{\mathbf{H}_{l}\right\}$. In fact, it is readily verified that the set $\mathcal{W}_{\psi}$ is periodic with respect to lattice $\sigma\left(\mathcal{O}_{\mathbb{K}}\right)$. This establishes an algebraic version of Khintchin's theorem [28] in the convergent part.


Fig. 5: Approximating $\left[h_{1}, h_{2}\right]^{\top} \in \mathbb{R}^{2}$ with $\mathcal{O}_{\mathbb{K}}=\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$. The well approximable set $\left\{\left[h_{1}, h_{2}\right]^{\top} \mid \min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\left[h_{1}, h_{2}\right]^{\top}-\sigma(a / q)\right\|<\psi(|\operatorname{Nr}(q)|)\right\}$ is shaded in blue. Black dots denote lattice points in $\sigma\left(\mathcal{O}_{\mathbb{K}}\right)$. Orange dots $\cup_{a \in \mathcal{O}_{\mathbb{K}}} \sigma(a / q)$ denote the centers of balls in approximating $\left[h_{1}, h_{2}\right]^{\top}$.

Since for almost all $\left\{\mathbf{H}_{l}\right\}$, 54) holds for finitely many $q$ modulo the unit group, there exists a finite constant $c_{\left\{\mathbf{H}_{l}\right\}}$ such that

$$
\max _{l \in\{1, \ldots, L\}} \min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\sigma(a / q)\right\| \geq \psi(|\operatorname{Nr}(q)|)
$$

for all $|\operatorname{Nr}(q)| \geq c_{\left\{\mathbf{H}_{l}\right\}}$. So one can claim that

$$
\max _{l \in\{1, \ldots, L\}} \min _{a \in \mathcal{O}_{\mathbb{K}}}\left\|\mathbf{H}_{l}-\sigma(a / q)\right\| \geq c_{\left\{\mathbf{H}_{l}\right\}}^{\prime} \psi(|\operatorname{Nr}(q)|)
$$

for all algebraic integer $q$ with

$$
\begin{aligned}
& c_{\left\{\mathbf{H}_{l}\right\}}^{\prime}= \\
& \min \left\{1, \min _{q:|\operatorname{Nr}(q)|<c_{\left\{\mathbf{H}_{l}\right\}}} \frac{\max _{l \in\{1, \ldots, L\}} \min _{a \in \mathcal{O}_{\mathbb{K}}} \| \mathbf{H}_{l}-\sigma(a / q) \mid}{\psi(|\operatorname{Nr}(q)|)}\right\}
\end{aligned}
$$

## Appendix E

## REAL NUMBER FIELDS WITH SMALL DISCRIMINANTS

## AcKnowledgment

The authors acknowledge Dr. Yu-Chih (Jerry) Huang, Prof. Joseph J. Boutros and Prof. Jean-Claude Belfiore for fruitful discussions, as well as the anonymous reviewers whose comments improved the presentation of this work.

## REFERENCES

[1] T. M. Cover and A. E. Gamal, "Capacity theorems for the relay channel," IEEE Trans. Inf. Theory, vol. 25, no. 5, pp. 572-584, Sep. 1979.
[2] T. Wang, A. Cano, G. B. Giannakis, and J. N. Laneman, "Highperformance cooperative demodulation with decode-and-forward relays," IEEE Trans. Commun., vol. 55, no. 7, pp. 1427-1438, Jul. 2007.
[3] S. Borade, L. Zheng, and R. G. Gallager, "Amplify-and-forward in wireless relay networks: Rate, diversity, and network size," IEEE Trans. Inf. Theory, vol. 53, no. 10, pp. 3302-3318, Oct. 2007.
[4] C. S. Patel and G. L. Stüber, "Channel estimation for amplify and forward relay based cooperation diversity systems," IEEE Trans. Wirel. Commun., vol. 6, no. 6, pp. 2348-2356, Jun. 2007.
[5] S. H. Lim, Y. Kim, A. E. Gamal, and S. Chung, "Noisy network coding," IEEE Trans. Inf. Theory, vol. 57, no. 5, pp. 3132-3152, May 2011.
[6] Y. Song and N. Devroye, "Lattice codes for the Gaussian relay channel: Decode-and-forward and compress-and-forward," IEEE Trans. Inf. Theory, vol. 59, no. 8, pp. 4927-4948, Aug. 2013.
[7] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," IEEE Trans. Inf. Theory, vol. 57, no. 10, pp. 6463-6486, Oct. 2011.
[8] J. Zhan, U. Erez, M. Gastpar, and B. Nazer, "MIMO compute-andforward," in Proc. IEEE Int. Symp. Inf. Theory, ISIT 2009, Seoul, Korea. IEEE, 2009, pp. 2848-2852.
[9] I. E. Bakoury and B. Nazer, "The impact of channel variation on integerforcing receivers," in Proc. IEEE Int. Symp. Inf. Theory, ISIT 2015, Hong Kong, China. IEEE, 2015, pp. 576-580.
[10] P. Wang, Y. Huang, K. R. Narayanan, and J. J. Boutros, "Physicallayer network-coding over block fading channels with root-LDA lattice codes," in Proc. IEEE Int. Conf. Commun., ICC 2016, Kuala Lumpur, Malaysia. IEEE, 2016, pp. 1-6.
111] D. Tse and P. Viswanath, Fundamentals of Wireless Communication. Cambridge University Press, 2012.
[12] W. Kositwattanarerk, S. S. Ong, and F. E. Oggier, "Construction A of lattices over number fields and block fading (wiretap) coding," IEEE Trans. Inf. Theory, vol. 61, no. 5, pp. 2273-2282, May 2015.
[13] A. Campello, C. Ling, and J. Belfiore, "Algebraic lattice codes achieve the capacity of the compound block-fading channel," in Proc. IEEE Int. Symp. Inf. Theory, ISIT 2016, Barcelona, Spain. IEEE, 2016, pp. 910914.
[14] -, "Universal lattice codes for MIMO channels," IEEE Trans. Information Theory, vol. 64, no. 12, pp. 7847-7865, 2018.
[15] C. Feng, D. Silva, and F. R. Kschischang, "An algebraic approach to physical-layer network coding," IEEE Trans. Inf. Theory, vol. 59, no. 11, pp. 7576-7596, Nov 2013.
[16] Q. T. Sun, J. Yuan, T. Huang, and K. W. Shum, "Lattice network codes based on Eisenstein integers," IEEE Trans. Commun., vol. 61, no. 7, pp. 2713-2725, Jul. 2013.
[17] N. E. Tunali, Y. Huang, J. J. Boutros, and K. R. Narayanan, "Lattices over Eisenstein integers for compute-and-forward," IEEE Trans. Inf. Theory, vol. 61, no. 10, pp. 5306-5321, 102015.
[18] Y. Huang, K. R. Narayanan, and P. Wang, "Lattices over algebraic integers with an application to compute-and-forward," IEEE Trans. Information Theory, vol. 64, no. 10, pp. 6863-6877, 2018.
[19] U. Erez and R. Zamir, "Achieving $1 / 2 \log (1+\mathrm{SNR})$ on the AWGN channel with lattice encoding and decoding," IEEE Trans. Inf. Theory, vol. 50, no. 10, pp. 2293-2314, Oct. 2004.
[20] F. E. Oggier and J. Belfiore, "Enabling multiplication in lattice codes via construction A," in IEEE Information Theory Workshop, ITW 2013, Sevilla, Spain. IEEE, 2013, pp. 1-5.

TABLE I: Real quadratic fields with small discriminants.

| $\mathfrak{m}_{\theta}$ | $\theta^{2}-\theta-1$ | $\theta^{2}-2$ | $\theta^{2}-3$ | $\theta^{2}-\theta-3$ | $\theta^{2}-\theta-4$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{disc}_{\mathbb{K}}$ | 5 | 8 | 12 | 13 | 17 |
| basis $\phi$ | $\{1, \theta\}$ | $\{1, \theta\}$ | $\{1, \theta\}$ | $\{1, \theta\}$ | $\{1, \theta\}$ |

TABLE II: Real cubic fields with small discriminants.

| $\mathfrak{m}_{\theta}$ | $\theta^{3}+\theta^{2}-2 \theta-1$ | $\theta^{3}-3 \theta-1$ | $\theta^{3}+\theta^{2}-3 \theta-1$ | $\theta^{3}-\theta^{2}-4 \theta-1$ |
| :---: | :---: | :---: | :---: | :---: |
| $\operatorname{disc}_{\mathbb{K}}$ | 49 | 81 | 148 | 169 |
| basis $\phi$ | $\left\{1, \theta, \theta^{2}\right\}$ | $\left\{1, \theta, \theta^{2}\right\}$ | $\left\{1, \theta, \theta^{2}\right\}$ | $\left\{1, \theta, \theta^{2}\right\}$ |

TABLE III: Real quartic fields with small discriminants.

| $\mathfrak{m}_{\theta}$ | $\theta^{4}+\theta^{3}-3 \theta^{2}-\theta+1$ | $\theta^{4}+\theta^{3}-4 \theta^{2}-4 \theta+1$ |
| :---: | :---: | :---: |
| $\operatorname{disc}_{\mathbb{K}}$ | 725 | 1125 |
| basis $\phi$ | $\left\{1, \theta,-1+\theta+\theta^{2},-1-2 \theta+\theta^{2}+\theta^{3}\right\}$ | $\left\{1, \theta,-2+\theta^{2},-1-3 \theta+\theta^{3}\right\}$ |
| $\mathfrak{m}_{\theta}$ | $\theta^{4}-4 \theta^{3}+8 \theta-1$ | $\theta^{4}-4 \theta^{2}+\theta+1$ |
| $\operatorname{disc}_{\mathbb{K}}$ | 1600 | 1957 |
| basis $\phi$ | $\left\{1, \theta, \frac{1}{2}\left(-1-2 \theta+\theta^{2}\right), \frac{1}{2}\left(3-\theta-3 \theta^{2}+\theta^{3}\right)\right\}$ | $\left\{1, \theta,-2+\theta^{2}, 1-3 \theta+\theta^{3}\right\}$ |

TABLE IV: Real quintic fields with small discriminants.

| $\mathfrak{m}_{\theta}$ | $\theta^{5}+\theta^{4}-4 \theta^{3}-3 \theta^{2}+3 \theta+1$ | $\theta^{5}-5 \theta^{3}-\theta^{2}+3 \theta+1$ |
| :---: | :---: | :---: |
| $\operatorname{disc}_{\mathbb{K}}$ | 14641 | 24217 |
| $\operatorname{basis} \phi$ | $\left\{1, \theta,-2+\theta^{2},-3 \theta+\theta^{3}, 1-2 \theta-3 \theta^{2}+\theta^{3}+\theta^{4}\right\}$ | $\left\{1, \theta,-2+\theta^{2},-1-4 \theta+\theta^{3}, 2-5 \theta^{2}+\theta^{4}\right\}$ |
| $\mathfrak{m}_{\theta}$ | $\theta^{5}+\theta^{4}-5 \theta^{3}-3 \theta^{2}+2 \theta+1$ | $\theta^{5}+\theta^{4}-5 \theta^{3}-\theta^{2}+4 \theta-1$ |
| $\operatorname{disc}_{\mathbb{K}}$ | 36497 | 38569 |
| $\operatorname{basis} \phi$ | $\left\{1, \theta,-2+\theta^{2},-2-4 \theta+\theta^{2}+\theta^{3}, 1+2 \theta-5 \theta^{2}+\theta^{4}\right\}$ | $\left\{1, \theta,-2+\theta+\theta^{2},-3 \theta+\theta^{2}+\theta^{3}, 3-2 \theta-5 \theta^{2}+\theta^{3}+\theta^{4}\right\}$ |

TABLE V: Maximal real sub-fields of cyclotomic fields $\mathbb{Q}\left(\zeta_{k}\right)$.

| $\mathbb{Q}\left(\zeta_{k}+\zeta_{k}^{-1}\right)$ | $\mathbb{Q}\left(\zeta_{23}+\zeta_{23}^{-1}\right)$ | $\mathbb{Q}\left(\zeta_{29}+\zeta_{29}^{-1}\right)$ |
| :---: | :---: | :---: |
| $\phi(k) / 2$ | 11 | 14 |
| $\mathfrak{m}_{\zeta_{k}+\zeta_{k}^{-1}}$ | $\zeta^{11}+\zeta^{10}-10 \zeta^{9}-9 \zeta^{8}+36 \zeta^{7}+28 \zeta^{6}$ | $\zeta^{14}+\zeta^{13}-13 \zeta^{12}-12 \zeta^{11}+66 \zeta^{10}+55 \zeta^{9}-165 \zeta^{8}$ |
|  | $-56 \zeta^{5}-35 \zeta^{4}+35 \zeta^{3}+15 \zeta^{2}-6 \zeta-1$ | $-120 \zeta^{7}+210 \zeta^{6}+126 \zeta^{5}-126 \zeta^{4}-56 \zeta^{3}+28 \zeta^{2}+7 \zeta-1$ |
| $\operatorname{disc}_{\mathbb{Q}}\left(\zeta_{k}+\zeta_{k}^{-1}\right)$ | 41426511213649 | 10260628712958602189 |

[21] A. Campello, C. Ling, and J. Belfiore, "Algebraic lattices achieving the capacity of the ergodic fading channel," in IEEE Information Theory Workshop, ITW 2016, Cambridge, United Kingdom. IEEE, 2016, pp. 459-463.
[22] Y. Huang, (2016). "Construction $\pi_{a}$ lattices : A review and recent results." [Online]. Available: https://www.york.ac.uk/media/mathematics/documents/ Jerry_Huang_York2016.pdf
[23] S. Lyu, A. Campello, C. Ling, and J. Belfiore, "Compute-and-forward over block-fading channels using algebraic lattices," in Proc. IEEE Int. Symp. Inf. Theory, ISIT 2017, Aachen, Germany. IEEE, 2017, pp. 1848-1852.
[24] O. Ordentlich and U. Erez, "A simple proof for the existence of "good" pairs of nested lattices," IEEE Trans. Inf. Theory, vol. 62, no. 8, pp. 4439-4453, Aug. 2016.
[25] U. Niesen and P. Whiting, "The degrees of freedom of compute-andforward," IEEE Trans. Inf. Theory, vol. 58, no. 8, pp. 5214-5232, Aug. 2012.
[26] O. Ordentlich, U. Erez, and B. Nazer, "The approximate sum capacity of the symmetric Gaussian K-user interference channel," IEEE Trans. Inf. Theory, vol. 60, no. 6, pp. 3450-3482, Jun. 2014.
[27] B. Nazer and O. Ordentlich, "Diophantine approximation for network information theory: A survey of old and new results," in Proc. 54th Annu. Allert. Conf. Commun. Control. Comput., Allerton 2016, Monticello, IL, USA. IEEE, 2016, pp. 990-996.
[28] J. W. S. Cassels, An Introduction to Diophantine Approximation. Cambridge University Press, 1957.
[29] D. Roy and M. Waldschmidt, "Diophantine approximation by conjugate algebraic integers," Compositio Math., vol. 140, no. 3, pp. 593-612, May 2004.
[30] D. Roy, "Simultaneous approximation by conjugate algebraic numbers in fields of transcendence degree one," Int. J. Number Theory, vol. 1, no. 3, pp. 357-382, 2005.
[31] R. A. Mollin, Algebraic Number Theory, 2nd ed. Chapman and Hall/CRC, 2011.
[32] F. Oggier and E. Viterbo, "Algebraic number theory and code design for Rayleigh fading channels," Foundations and Trends on Communications and Information Theory, vol. 1, pp. 336-415, 2004.
[33] R. Zamir, Lattice Coding for Signals and Networks. Cambridge University Press, 2014.
[34] K. Rogers and H. P. F. Swinnerton-Dyer, "The geometry of numbers over algebraic number fields," Transactions of the American Mathematical Society, vol. 88, no. 1, pp. 227-242, 1958.
[35] C. G. Lekkerkerker and P. Gruber, Geometry of Numbers. Elsevier Science, 1987.
[36] C. Fieker and D. Stehlé, "Short bases of lattices over number fields," in Algorithmic Number Theory Symposium (ANTS), vol. 6197. Springer, 2010, pp. 157-173.
[37] D. S. Dummit and R. M. Foote, Abstract Algebra. John Wiley \& Sons, 2003.
[38] R. Zamir and M. Feder, "On lattice quantization noise," IEEE Trans. Inf. Theory, vol. 42, no. 4, pp. 1152-1159, 1996.
[39] A. Leibak, "On additive generalization of Voronoi's theory to algebraic number fields," Proceedings of the Estonian Academy of Science Physics/Mathematics, vol. 54, no. 4, pp. 195-212, 2005.
[40] R. Baeza and M. Icaza, "On Humbert-Minkowski’s constant for a number field," Proceedings of the American Mathematical Society, vol. 125, no. 11, pp. 3195-3202, 1997.
[41] D. Micciancio and S. Goldwasser, Complexity of Lattice Problems. Springer US, 2002.
[42] S. Sahraei and M. Gastpar, "Polynomially solvable instances of the shortest and closest vector problems with applications to compute-and-forward," IEEE Trans. Information Theory, vol. 63, no. 12, pp. 7780-7792, 2017.
[43] S. Lyu and C. Ling, "Boosted KZ and LLL algorithms," IEEE Trans. Signal Process., vol. 65, no. 18, pp. 4784-4796, Sep. 2017.
[44] M. Pohst and H. Zassenhaus, Algorithmic Algebraic Number Theory. Cambridge University Press, 1997.

Shanxiang Lyu received the B.Eng. and M.Eng. degrees in electronic and information engineering from South China University of Technology, Guangzhou, China, in 2011 and 2014, respectively, and the Ph.D. degree from the Electrical and Electronic Engineering Department, Imperial College London, in 2018. He is currently a lecturer with the College of Cyber Security, Jinan University. His research interests are in lattice theory, algebraic number theory, and their applications.

Antonio Campello received the Bachelor and PhD degrees in Applied Mathematics from the University of Campinas, Brazil, in 2009 and 2014, respectively. He was a visiting researcher at the Complutense University of Madrid in 2009, at the École Polytechnique fédérale de Lausanne (EPFL) 1025 in 2011, and at AT\&T Research Labs - Shannon Labs, New Jersey in 2013. He was as a postdoctoral researcher at Télécom ParisTech, France, in and at Imperial College London, UK. His research interests are in the interplay between discrete geometry, number theory, communications and machine learning.

Cong Ling (S'99-A'01-M'04) received the B.S. and M.S. degrees in electrical engineering from the Nanjing Institute of Communications Engineering, Nanjing, China, in 1995 and 1997, respectively, and the Ph.D. degree in electrical engineering from the Nanyang Technological University, Singapore, in 2005. He had been on the faculties of the Nanjing Institute of Communications Engineering and King's College. He is currently a Reader (Associate Professor) with the Electrical and Electronic Engineering Department, Imperial College London. His research interests are coding, information theory, and security, with a focus on lattices. Dr. Ling has served as an Associate Editor for the IEEE TRANSACTIONS ON COMMUNICATIONS and the IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY.
[48] S. Lang, Algebraic Number Theory. Springer-Verlag New York, 1994.


[^0]:    This work was presented in part at the International Symposium on Information Theory 2017, Aachen, Germany. The work of S. Lyu was supported by the China Scholarship Council.
    S. Lyu is with the College of Information Science and Technology, and the College of Cyber Security, Jinan University, Guangzhou 510632, China (e-mail: s.lyu14@imperial.ac.uk).
    A. Campello and C. Ling are with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, United Kingdom (e-mail: accampellojr@ gmail.com, cling@ieee.org).

[^1]:    ${ }^{1} \mathrm{~A}$ free module is a module that has a basis.

[^2]:    ${ }^{2}$ Here, $\prod_{j=1}^{n} \sigma_{j}(\mathbf{a})^{\top} \mathbf{M}_{j} \sigma_{j}(\mathbf{a})$ is called a multiplicative Humbert form (40).

