# A Correlation Measure Based on Vector-Valued $L_{p}$-Norms 

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#### Abstract

In this paper, we introduce a new measure of correlation for bipartite quantum states. This measure depends on a parameter $\alpha$, and is defined in terms of vector-valued $L_{p}$-norms. The measure is within a constant of the exponential of $\alpha$-Rényi mutual information, and reduces to the trace norm (total variation distance) for $\alpha=1$. We will prove some decoupling type theorems in terms of this measure of correlation, and present some applications in privacy amplification as well as in bounding the random coding exponents. In particular, we establish a bound on the secrecy exponent of the wiretap channel (under the total variation metric) in terms of the $\alpha$-Rényi mutual information according to Csiszár's proposal.


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## 1 Introduction

In this paper, for any $\alpha \geq 1$ we introduce a new measure of correlation $V_{\alpha}(A ; B)$ by

$$
\begin{equation*}
V_{\alpha}(A ; B)=\left\|\left(I_{B} \otimes \rho_{A}^{-(\alpha-1) / 2 \alpha}\right) \rho_{B A}\left(I_{B} \otimes \rho_{A}^{-(\alpha-1) /(2 \alpha)}\right)-\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)}, \tag{1}
\end{equation*}
$$

where $\|\cdot\|_{(1, \alpha)}$ denotes a certain norm which for $\alpha=1$ reduces to the 1 -norm. Since Rényi mutual information (according to Sibson's proposal) can also be expressed in terms of the ( $1, \alpha$ )-norm our measure of correlation is also related Rényi mutual information.

The main motivation for introducing these measures of correlation, particularly for $1 \leq \alpha \leq 2$, is their applications in decoupling theorems. The point is that the average of $V_{\alpha}\left(A_{0} ; B\right)$, when $\rho_{A_{0} B}$ is the outcome of a certain random CPTP map $\Phi_{A \rightarrow A_{0}}$ applied on the bipartite quantum state $\rho_{A B}$, can be bounded by $c V_{\alpha}(A ; B)$ where $c<1$ is a constant. Thus our measures of correlation can be used to prove decoupling type theorems in information theory.

Decoupling theorems have already found several applications in information theory. Most achievability results in quantum information theory are based on the phenomenon of decoupling (see [1] and references therein). Also, in classical information theory the OSRB method of [2] provides a similar decoupling-type tool for proving achievability results. The advantage of our decoupling theorem based on the measure $V_{\alpha}$, comparing to previous ones, is that it works for all values of $\alpha \in(1,2]$. Given the relation between $V_{\alpha}$ and Rényi mutual information mentioned above, the parameters appearing in our decoupling theorem would be related to $\alpha$-Rényi mutual information, which for $\alpha=1$ reduces to Shannon's mutual information. Therefore, we can use our decoupling theorems not only for proving achievability results but also for proving interesting bounds on the random coding exponents. We demonstrate this application via the examples of entanglement generation via a noisy quantum communication channel, and secure communication over a (classical) wiretap channel. In particular, we show a bound on the secrecy exponent of random coding over a wiretap channel in terms of Rényi mutual information according to Csiszár's proposal.

Another application of our new measures of correlation is in secrecy. To measure the security of a communication system, one has to quantify the amount of information leaked to an eavesdropper. While the common security metric for measuring the leakage is mutual information (see e.g., see [3]) or the total variation distance [2,4], there have been few recent works that motivate and define other measures of correlation to quantify leakage [5-12]. Herein, we suggest the use of our metric instead of mutual information because it is a stronger metric and has a better rate-security tradeoff curve. To explain the rate-security tradeoff, consider a secure transmission protocol over a communication channel, achieving a communication rate of $R$ with certified leakage of at most $L$ according to the mutual information metric. Now, if the transmitter obtains a classified message for which leakage $L$ is no longer acceptable, it can sacrifice communication rate for improved transmission security. We show that the rate-security tradeoff with the mutual information metric is far worse than that of our metric. We will discuss this fact in more details via the problem of privacy amplification.

The definition of our measure of correlation $V_{\alpha}(A ; B)$ is based on the theory of vector-valued $L_{p}$ spaces. These spaces are generalizations of the $L_{p}$ spaces and are defined via the theory of complex interpolation. Then the proofs of our main theorems are heavily based on the interpolation theory. In particular, we use the Riesz-Thorin interpolation theorem several times, in order to establish an inequality for all $\alpha \in[1,2]$ by interpolating between $\alpha=1$ and $\alpha=2$.

In the following section, we review some notations and introduce vector-valued $L_{p}$ norms. Section 3 introduces our new measure of correlation and presents some of its properties. Section 4
contains the main technical results of this paper. Section 5 and Section 6 contain some applications of our results in privacy amplification as well as in bounding the random coding exponents.

## 2 Vector-valued $L_{p}$ norms

For a finite set $\mathcal{A}$ let $\ell(A)$ to be the vector space of functions $f: \mathcal{A} \rightarrow \mathbb{C}$. For any $p>0$ and $f \in \ell(A)$ we define

$$
\|f\|_{p}:=\left(\sum_{a \in \mathcal{A}}|f(a)|^{p}\right)^{\frac{1}{p}} .
$$

This quantity for $p \geq 1$ satisfies the triangle inequality and turns $\ell(A)$ into a normed space. The dual of $p$-norm is the $p^{\prime}$-norm where $p^{\prime}$ is the Hölder conjugate of $p$ given by

$$
\begin{equation*}
\frac{1}{p}+\frac{1}{p^{\prime}}=1 \tag{2}
\end{equation*}
$$

More generally, for any $p, q, r>0$ with $1 / p=1 / q+1 / r$ and any $f, g \in \ell(A)$ we have

$$
\|f g\|_{p} \leq\|f\|_{q} \cdot\|g\|_{r},
$$

where $(f g)(a)=f(a) g(a)$.
Suppose that $\mathcal{B}$ is another set and we equip the vector space $\ell(B)$ with the $q$-norm. The question is how we can naturally define a $(p, q)$-norm on the space $\ell(A B):=\ell(A \times B)=\ell(A) \otimes \ell(B)$ that is compatible with the norm of the individual spaces $\ell(A), \ell(B)$. By compatible we mean that if $h=f \otimes g$ with $f \in \ell(A)$ and $g \in \ell(B)$ (i.e., $h(a, b)=f(a) g(b))$ then

$$
\begin{equation*}
\|f \otimes g\|_{(p, q)}=\|f\|_{p} \cdot\|g\|_{q} . \tag{3}
\end{equation*}
$$

To this end, any vector $h \in \ell(A B)$ can be taught of as a collection of $|\mathcal{A}|$ vectors $h_{a} \in \ell(B)$ for any $a \in \mathcal{A}$, where $h_{a}(b)=h(a, b)$. Let us denote $t(a)=\left\|h_{a}\right\|_{q}$. Then we may define

$$
\|h\|_{(p, q)}:=\|t\|_{p}=\left(\sum_{a}\left\|h_{a}\right\|_{q}^{p}\right)^{1 / p}
$$

This definition of the $(p, q)$-norm satisfies (3). Moreover, when $p=q$, this $(p, p)$-norm coincides with the usual $p$-norm. Finally, it is not hard to verify that the $(p, q)$-norm, for $p, q \geq 1$, is indeed a norm and satisfies the triangle inequality.

The $p$-norm can also be defined in the non-commutative case. Suppose that $\mathcal{H}_{A}$ is a Hilbert space of finite dimension $d_{A}=\operatorname{dim} \mathcal{H}_{A}$. Let $\mathbf{L}(A)=\mathbf{L}\left(\mathcal{H}_{A}\right)$ to be the space of linear operators $M: \mathcal{H}_{A} \rightarrow \mathcal{H}_{A}$ acting on $\mathcal{H}_{A}$. Again we can define

$$
\|M\|_{p}=\left(\operatorname{tr}\left(|M|^{p}\right)\right)^{\frac{1}{p}}
$$

where $|M|=\sqrt{M^{\dagger} M}$, and $M^{\dagger}$ is the adjoint of $M$. For $p \geq 1$ this equips $\mathbf{L}(A)$ with a norm, called the Schatten norm, that satisfies the triangle inequality. Hölder's inequality is also satisfied for Schatten norms [13: if $p, q, r>0$ with $1 / p=1 / q+1 / r$, then for $M, N \in \mathbf{L}(A)$ we have

$$
\begin{equation*}
\|M N\|_{p} \leq\|M\|_{q} \cdot\|N\|_{r} . \tag{4}
\end{equation*}
$$

Our notation in the non-commutative case can be made compatible with the commutative case. By abuse of notation, an element $f_{A} \in \ell(A)$ can be taught of as a diagonal matrix of the form

$$
f_{A}=\sum_{a} f(a)|a\rangle\langle a|,
$$

acting on the Hilbert space $\mathcal{H}_{A}$ with the orthonormal basis $\{|a\rangle: a \in \mathcal{A}\}$. Therefore, $\ell(A)$ can be taught of as a subspace of $\mathbf{L}(A)$. We also have

$$
\left\|f_{A}\right\|_{p}=\left(\operatorname{tr}\left(\left|f_{A}\right|^{p}\right)\right)^{\frac{1}{p}}=\left(\sum_{a}|f(a)|^{p}\right)^{\frac{1}{p}} .
$$

Now the question is how we can define the $(p, q)$-norm in the non-commutative case. Let us start with the easy case of $M_{A B} \in \ell(A) \otimes \mathbf{L}(B)$. Then, following the above notation, $M_{A B}$ can be written as

$$
M_{A B}=\sum_{a}|a\rangle\langle a| \otimes M_{a},
$$

with $M_{a} \in \mathbf{L}(B)$. Similar to the fully commutative case we can define

$$
\begin{equation*}
\left\|M_{A B}\right\|_{(p, q)}=\left(\sum_{a}\left\|M_{a}\right\|_{q}^{p}\right)^{1 / p} . \tag{5}
\end{equation*}
$$

Now let us turn to the fully non-commutative case. In this case, the definition of the ( $p, q$ )-norm is not easy and is derived from interpolation theory [14. Here, we present an equivalent definition provided in [15] (see also [16]). We also focus on the case of $p \leq q$ that we need in this paper. In this case, since $p \leq q$ there exists $r \in(0,+\infty]$ such that $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. Then for any $M_{A B} \in \mathbf{L}(A B)$ we define

$$
\begin{equation*}
\left\|M_{A B}\right\|_{(p, q)}=\inf _{\sigma_{A}, \tau_{A}}\left\|\left(\sigma_{A}^{-\frac{1}{2 r}} \otimes I_{B}\right) M_{A B}\left(\tau_{A}^{-\frac{1}{2 r}} \otimes I_{B}\right)\right\|_{q}, \tag{6}
\end{equation*}
$$

where the infimum is taken over all density matrice $\sqrt{1}^{1} \sigma_{A}, \tau_{A} \in \mathbf{L}(A)$ and $I_{B} \in \mathbf{L}(B)$ is the identity operator. In the following, for simplicity we sometimes suppress the identity operators in expressions of the form $\left(\sigma_{A}^{-1 / 2 r} \otimes I_{B}\right) M_{A B}$ and write $\sigma_{A}^{-1 / 2 r} M_{A B}$. Therefore,

$$
\left\|M_{A B}\right\|_{(p, q)}=\inf _{\sigma_{A}, \tau_{A}}\left\|\sigma_{A}^{-\frac{1}{2 r}} M_{A B} \tau_{A}^{-\frac{1}{2 r}}\right\|_{q}
$$

When $q \geq p \geq 1$, the ( $p, q$ )-norm satisfies the triangle inequality and is a norm. Some remarks are in line.

Remark 1. As in the commutative case, the order of subsystems in the above definition is important, i.e., $\left\|M_{A B}\right\|_{(p, q)}$ and $\left\|M_{B A}\right\|_{(p, q)}$ are different.

Remark 2. From Hölder's inequality (4), one can derive that if $M_{A B}=M_{A} \otimes M_{B}$, then

$$
\left\|M_{A} \otimes M_{B}\right\|_{(p, q)}=\left\|M_{A}\right\|_{p}\left\|M_{B}\right\|_{q}
$$

[^1]Remark 3. When $M_{A B} \in \ell(A) \otimes \mathbf{L}(B)$, the above definition of $(p, q)$-norm coincides with that of (5). This can be shown by trying to optimize the choices of $\tau_{A}, \sigma_{A}$ in (6), which can be taken to be diagonal.

Remark 4. When $p=q$ the $(p, p)$-norm coincides with the usual $p$-norm [14, 15]:

$$
\left\|M_{A B}\right\|_{(p, p)}=\left\|M_{A B}\right\|_{p}
$$

Remark 5. When $M_{A B} \geq 0$ is positive semidefinite, in (6) we may assume that $\sigma_{A}=\tau_{A}$, see [16]. That is, when $M_{A B}$ is positive semidefinite we have

$$
\left\|M_{A B}\right\|_{p, q}=\inf _{\sigma_{A}}\left\|\sigma_{A}^{-\frac{1}{2 r}} M_{A B} \sigma_{A}^{-\frac{1}{2 r}}\right\|_{q}=\inf _{\sigma_{A}}\left\|\Gamma_{\sigma_{A}}^{-\frac{1}{r}}\left(M_{A B}\right)\right\|_{q}
$$

where

$$
\begin{equation*}
\Gamma_{\sigma}(X)=\sigma^{\frac{1}{2}} X \sigma^{\frac{1}{2}} \tag{7}
\end{equation*}
$$

We will compare our measure of correlation with Rényi mutual information which interestingly can also be written in terms of $(1, p)$-norms. For $\alpha \geq 1$ the sandwiched $\alpha$-Rényi relative entropy is defined by ${ }^{2}$

$$
D_{\alpha}(\rho \| \sigma)=\alpha^{\prime} \log \left\|\Gamma_{\sigma}^{-1 / \alpha^{\prime}}(\rho)\right\|_{\alpha}
$$

where $\alpha^{\prime}=\alpha /(\alpha-1)$ is the Hölder conjugate of $\alpha$ given by (2). The $\alpha$-Rényi mutual information (Sibson's proposal) for $\alpha>1$ is given by ${ }^{3}$

$$
I_{\alpha}(A ; B)=\inf _{\sigma_{B}} D_{\alpha}\left(\rho_{A B} \| \rho_{A} \otimes \sigma_{B}\right)
$$

Using the definition of $D_{\alpha}\left(\rho_{A B} \| \rho_{A} \otimes \sigma_{B}\right)$ and Remark 5 we find that

$$
I_{\alpha}(A ; B)=\alpha^{\prime} \log \left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)\right\|_{(1, \alpha)}
$$

In particular, for classical random variables $A$ and $B$ with joint distribution $p_{A B}$ we have

$$
\begin{equation*}
I_{\alpha}(A ; B)=\alpha^{\prime} \log \left(\sum_{b}\left[\sum_{a} p(a) p(b \mid a)^{\alpha}\right]^{1 / \alpha}\right) \tag{8}
\end{equation*}
$$

Finally the $\alpha$-Rényi conditional entropy is defined by

$$
\begin{equation*}
H_{\alpha}(A \mid B)=-\inf _{\sigma_{B}} D_{\alpha}\left(\rho_{A B} \| I_{A} \otimes \sigma_{B}\right)=-\alpha^{\prime} \log \left\|\rho_{B A}\right\|_{(1, \alpha)} \tag{9}
\end{equation*}
$$

We finish this section by stating a lemma about the monotonicity of the $(1, \alpha)$-norm.
Lemma 6. For any $M_{A B}$ and any density matrix $\xi_{A}$ the function $\alpha \mapsto\left\|\Gamma_{\xi_{A}}^{-1 / \alpha^{\prime}}\left(M_{B A}\right)\right\|_{(1, \alpha)}$ is non-decreasing on $[1,+\infty)$.

[^2]Proof. Let $\beta>\alpha \geq 1$, and let $\gamma>0$ be such that $1 / \alpha=1 / \beta+1 / \gamma$. Using Hölder's inequality for arbitrary density matrices $\sigma_{B}, \tau_{B}$ we have

$$
\begin{aligned}
\| \sigma_{B}^{-1 /\left(2 \alpha^{\prime}\right)} \Gamma_{\xi_{A}}^{-1 / \alpha^{\prime}} & \left(M_{B A}\right) \tau_{B}^{-1 /\left(2 \alpha^{\prime}\right)} \|_{\alpha} \\
& =\left\|\left(\sigma_{B} \otimes \xi_{A}\right)^{1 /(2 \gamma)} \sigma_{B}^{-1 /\left(2 \beta^{\prime}\right)} \Gamma_{\xi_{A}}^{-1 / \beta^{\prime}}\left(M_{B A}\right) \tau_{B}^{-1 /\left(2 \beta^{\prime}\right)}\left(\tau_{B} \otimes \xi_{A}\right)^{1 /(2 \gamma)}\right\|_{\alpha} \\
& \leq\left\|\left(\sigma_{B} \otimes \xi_{A}\right)^{1 /(2 \gamma)}\right\|_{2 \gamma} \cdot\left\|\sigma_{B}^{-1 /\left(2 \beta^{\prime}\right)} \Gamma_{\xi_{A}}^{-1 / \beta^{\prime}}\left(M_{B A}\right) \tau_{B}^{-1 /\left(2 \beta^{\prime}\right)}\right\|_{\beta} \cdot\left\|\left(\tau_{B} \otimes \xi_{A}\right)^{1 /(2 \gamma)}\right\|_{2 \gamma} \\
& =\left\|\sigma_{B}^{-1 /\left(2 \beta^{\prime}\right)} \Gamma_{\xi_{A}}^{-1 / \beta^{\prime}}\left(M_{B A}\right) \tau_{B}^{-1 /\left(2 \beta^{\prime}\right)}\right\|_{\beta} .
\end{aligned}
$$

Taking infimum over $\sigma_{B}, \tau_{B}$ we obtain the desired result.

### 2.1 Completely bounded norm

The completely bounded norm of a super-operator $\Phi: \mathbf{L}(A) \rightarrow \mathbf{L}(B)$ is defined by

$$
\|\Phi\|_{\mathrm{cb}, p \rightarrow q}:=\sup _{d_{C}}\left\|\mathcal{I}_{C} \otimes \Phi\right\|_{(\infty, p) \rightarrow(\infty, q)}=\sup _{X_{C A}} \frac{\left\|\mathcal{I}_{C} \otimes \Phi\left(X_{C A}\right)\right\|_{(\infty, q)}}{\left\|X_{C A}\right\|_{(\infty, p)}},
$$

where the supremum is taken over all auxiliary Hilbert spaces $\mathcal{H}_{C}$ with arbitrary dimension $d_{C}$ and $\mathcal{I}_{C}: \mathbf{L}(C) \rightarrow \mathbf{L}(C)$ is the identity super-operator. In the above definition, we may replace $\infty$ with any $1 \leq t \leq \infty$, see [14]. That is, for any $t \geq 1$ we have

$$
\begin{equation*}
\|\Phi\|_{\mathrm{cb}, p \rightarrow q}:=\sup _{d_{C}}\left\|\mathcal{I}_{C} \otimes \Phi\right\|_{(t, p) \rightarrow(t, q)} \tag{10}
\end{equation*}
$$

We say that a super-operator between spaces with certain norms is a complete contraction if its completely bounded norm is at most 1 .
Lemma 7. For any $M_{B C A} \in \mathbf{L}(B C A)$ and $1 \leq \alpha \leq \infty$ we have

$$
\left\|M_{B C A}\right\|_{(1,1, \alpha)} \geq\left\|M_{B C A}\right\|_{(1, \alpha, \alpha)} .
$$

Proof. First of all the swap super-operator is a complete contraction [14], i.e.,

$$
\left\|M_{B C A}\right\|_{(1,1, \alpha)} \geq\left\|M_{B A C}\right\|_{(1, \alpha, 1)}
$$

Therefore, it suffices to show that

$$
\left\|M_{B A C}\right\|_{(1, \alpha, 1)} \geq\left\|M_{B C A}\right\|_{(1, \alpha, \alpha)}=\left\|M_{B A C}\right\|_{(1, \alpha, \alpha)} .
$$

Equivalently we need to show that

$$
\left\|\mathcal{I}_{A C}\right\|_{\mathrm{cb},(\alpha, \alpha) \rightarrow(\alpha, 1)} \leq 1
$$

Using (10) we have

$$
\left\|\mathcal{I}_{A C}\right\|_{\mathrm{cb},(\alpha, \alpha) \rightarrow(\alpha, 1)}=\sup _{d_{E}}\left\|\mathcal{I}_{E A C}\right\|_{(\alpha, \alpha, \alpha) \rightarrow(\alpha, \alpha, 1)}=\sup _{d_{D}}\left\|\mathcal{I}_{D C}\right\|_{(\alpha, \alpha) \rightarrow(\alpha, 1)}=\left\|\mathcal{I}_{C}\right\|_{\mathrm{cb}, \alpha \rightarrow 1} .
$$

Next since $\mathcal{I}_{C}$ is completely positive and $\alpha \geq 1$ we have [16]

$$
\left\|\mathcal{I}_{C}\right\|_{\mathrm{cb}, \alpha \rightarrow 1}=\left\|\mathcal{I}_{C}\right\|_{\alpha \rightarrow 1}=1 .
$$

We are done.

## 3 A new measure of correlation

In this section, we define our measure of correlation and study some of its properties.
Definition 1. Let $\rho_{A B}$ be an arbitrary bipartite density matrix. For any $\alpha \geq 1$ we defind

$$
\begin{align*}
V_{\alpha}(A ; B) & :=\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)-\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)},  \tag{11}\\
W_{\alpha}(A \mid B) & :=\left\|\rho_{B A}-\rho_{B} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1, \alpha)}, \tag{12}
\end{align*}
$$

where $1 / \alpha+1 / \alpha^{\prime}=1$, and $\rho_{A}=\operatorname{tr}_{B}\left(\rho_{A B}\right), \rho_{B}=\operatorname{tr}_{A}\left(\rho_{A B}\right)$ are the marginal states on $A$ and $B$ subsystems, respectively.

As will be seen below, $V_{\alpha}(A ; B)$ is a measure of correlation while $W_{\alpha}(A \mid B)$ is a related quantity that may be thought of as a conditional entropy.

By Remark 4 when $\alpha=1, V_{\alpha}$ and $W_{\alpha}$ can be expressed in terms of the 1-norm:

$$
\begin{align*}
V_{1}(A ; B) & =\left\|\rho_{A B}-\rho_{A} \otimes \rho_{B}\right\|_{1}  \tag{13}\\
W_{1}(A \mid B) & =\left\|\rho_{B A}-\rho_{B} \otimes \frac{I_{A}}{d_{A}}\right\|_{1} . \tag{14}
\end{align*}
$$

In the classical case when $p_{A B}$ is a joint probability distribution we have

$$
V_{\alpha}(A ; B)=\sum_{b}\left(\sum_{a} p(a)|p(b \mid a)-p(b)|^{\alpha}\right)^{1 / \alpha}
$$

and

$$
W_{\alpha}(A \mid B)=\sum_{b} p(b)\left(\sum_{a}\left|p(a \mid b)-\frac{1}{|\mathcal{A}|}\right|^{\alpha}\right)^{1 / \alpha}
$$

As an immediate property of the above definitions, both $V_{\alpha}(A ; B)$ and $W_{\alpha}(A \mid B)$ are nonnegative. Moreover, since they are defined in terms of a norm, we have $V_{\alpha}(A ; B)=0$ if and only if $\rho_{A B}=\rho_{A} \otimes \rho_{B}$, and $W_{\alpha}(A \mid B)=0$ if and only if $\rho_{A B}=\frac{I_{A}}{d_{A}} \otimes \rho_{B}$.

Proposition 8. For any $\rho_{A B}$ the functions

$$
\alpha \mapsto V_{\alpha}(A ; B),
$$

and

$$
\alpha \mapsto d_{A}^{\frac{1}{\alpha^{j}}} W_{\alpha}(A ; B),
$$

are non-decreasing. In particular, for any $\alpha \geq 1$ we have

$$
V_{\alpha}(A ; B) \geq\left\|\rho_{A B}-\rho_{A} \otimes \rho_{B}\right\|_{1}, \quad \text { and } \quad W_{\alpha}(A ; B) \geq d_{A}^{-\frac{1}{\alpha^{\prime}}}\left\|\rho_{A B}-\frac{I_{A}}{d_{A}} \otimes \rho_{B}\right\|_{1} .
$$

Proof. For the monotonicity of $\alpha \mapsto V_{\alpha}(A ; B)$, in Lemma 6 put $M_{A B}=\rho_{A B}-\rho_{A} \otimes \rho_{B}$ and $\xi_{A}=\rho_{A}$. For the other monotonicity let $M_{A B}=\rho_{A B}-I_{A} / d_{A} \otimes \rho_{B}$ and $\xi_{A}=I_{A} / d_{A}$.

[^3]We now prove the main property of $V_{\alpha}(A ; B)$ and $W_{\alpha}(A \mid B)$, namely their monotonicity under local operations.

Theorem 9 (Monotonicity under local operations). (i) For any $\rho_{A B}$ and all CPTP maps $\Phi_{A \rightarrow X}$ and $\Psi_{B \rightarrow Y}$ we have

$$
V_{\alpha}(X ; Y) \leq V_{\alpha}(A ; B),
$$

where $\rho_{X Y}=\Phi \otimes \Psi\left(\rho_{A B}\right)$.
(ii) For any $\rho_{A B}$ and any CPTP map $\Psi_{B \rightarrow Y}$

$$
W_{\alpha}(A \mid Y) \leq W_{\alpha}(A \mid B),
$$

where $\rho_{A Y}=\mathcal{I}_{A} \otimes \Psi\left(\rho_{A B}\right)$ and $\mathcal{I}_{A}$ is the identity super-operator.
Proof. For (i) we compute

$$
\begin{aligned}
V_{\alpha}(X ; Y) & =\left\|\Gamma_{\rho_{X}}^{-1 / \alpha^{\prime}}\left(\rho_{Y X}\right)-\rho_{Y} \otimes \rho_{X}^{1 / \alpha}\right\|_{(1, \alpha)} \\
& =\left\|\left(\Psi \otimes \Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right)\left(\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)-\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right)\right\|_{(1, \alpha)} \\
& \leq\left\|\Psi \otimes \Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{(1, \alpha) \rightarrow(1, \alpha)} \cdot\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)-\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)} \\
& =\left\|\Psi \otimes \Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{(1, \alpha) \rightarrow(1, \alpha)} \cdot V_{\alpha}(A ; B) \\
& =\left\|\Psi \otimes \mathcal{I}_{A}\right\|_{(1, \alpha) \rightarrow(1, \alpha)} \cdot\left\|\mathcal{I}_{B} \otimes \Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{(1, \alpha) \rightarrow(1, \alpha)} \cdot V_{\alpha}(A ; B),
\end{aligned}
$$

where $(1, \alpha) \rightarrow(1, \alpha)$ denotes the super-operator norm:

$$
\|\mathcal{T}\|_{(1, \alpha) \rightarrow(1, \alpha)}:=\sup _{M \neq 0} \frac{\|\mathcal{T}(M)\|_{(1, \alpha)}}{\|M\|_{(1, \alpha)}}
$$

Now using equation (3.5) and Theorem 13 of [16] we have

$$
\left\|\mathcal{I}_{B} \otimes \Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{(1, \alpha) \rightarrow(1, \alpha)} \leq\left\|\Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{\alpha \rightarrow \alpha} .
$$

On the other hand, using Lemma 9 of [18] (see also [19]) we have

$$
\left\|\Gamma_{\Phi\left(\rho_{A}\right)}^{-1 / \alpha^{\prime}} \circ \Phi \circ \Gamma_{\rho_{A}}^{1 / \alpha^{\prime}}\right\|_{\alpha \rightarrow \alpha} \leq 1 .
$$

Moreover, by Lemma 5 of [16] we have

$$
\left\|\Psi \otimes \mathcal{I}_{A}\right\|_{(1, \alpha) \rightarrow(1, \alpha)}=\|\Psi\|_{1 \rightarrow 1}=1,
$$

since $\Psi$ is CPTP. We conclude that, $V_{\alpha}(X ; Y) \leq V_{\alpha}(A ; B)$.
The proof of (ii) is similar, so we skip it.
We now state the relation between $V_{\alpha}, W_{\alpha}$ and Rényi information measures.

Proposition 10. For any bipartite density matrix $\rho_{A B}$ we have

$$
2^{\frac{1}{\alpha^{\prime}} I_{\alpha}(A ; B)}-1 \leq V_{\alpha}(A ; B) \leq 2^{\frac{1}{\alpha^{\prime}} I_{\alpha}(A ; B)}+1
$$

where $\alpha^{\prime}$ is the Hölder conjugate of $\alpha$. For $W_{\alpha}(A \mid B)$ we have

$$
2^{-\frac{1}{\alpha^{\prime}} H_{\alpha}(A \mid B)}-d_{A}^{-\frac{1}{\alpha^{\prime}}} \leq W_{\alpha}(A \mid B) \leq 2^{-\frac{1}{\alpha^{\prime}} H_{\alpha}(A \mid B)}+d_{A}^{-\frac{1}{\alpha^{\prime}}}
$$

where $d_{A}=\operatorname{dim} \mathcal{H}_{A}$.
Proof. By the triangle inequality we have

$$
\begin{aligned}
\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)\right\|_{(1, \alpha)}-\left\|\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)} \leq \| & \Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)-\rho_{B} \otimes \rho_{A}^{1 / \alpha} \|_{(1, \alpha)} \\
& \leq\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}\right)\right\|_{(1, \alpha)}+\left\|\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)}
\end{aligned}
$$

Moreover, by Remark 2] we have

$$
\left\|\rho_{B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)}=\left\|\rho_{B}\right\|_{1} \cdot\left\|\rho_{A}^{1 / \alpha}\right\|_{\alpha}=1
$$

These give the first inequality. The proof of the second inequality is similar.
Theorem 11. Let $\rho_{A B C}$ be a tripartite density matrix. Then the followings hold:
(i) For any $1 \leq \alpha \leq 2$ we have

$$
W_{\alpha}(A \mid B C) \leq 2^{\frac{2}{\alpha}-1} d_{C}^{\frac{1}{\alpha^{\prime}}} W_{\alpha}(A C \mid B)
$$

(ii) Assume that $\rho_{A C}=\frac{1}{d_{A} d_{C}} I_{A} \otimes I_{C}$. Then for any $1 \leq \alpha \leq 2$ we have

$$
V_{\alpha}(A ; B C) \leq 2^{\frac{2}{\alpha}-1} V_{\alpha}(A C ; B)
$$

Moreover, if $C$ is classical (and $\rho_{A C}=\frac{1}{d_{A} d_{C}} I_{A} \otimes I_{C}$ ) then

$$
V_{\alpha}(A ; B \mid C) \leq 2^{\frac{2}{\alpha}-1} V_{\alpha}(A C ; B)
$$

where we define

$$
V_{\alpha}(A ; B \mid C)=\sum_{c} p(c) V_{\alpha}(A ; B \mid C=c)
$$

Proof. The proof of (ii) is immediate once we have (i) since if $\rho_{A}=I_{A} / d_{A}$ then

$$
V_{\alpha}(A ; B)=d_{A}^{1 / \alpha^{\prime}} W_{\alpha}(A \mid B)
$$

Moreover, when $C$ is classical and

$$
\rho_{A B C}=\sum_{c} p(c) \rho_{A B \mid c} \otimes|c\rangle\langle c|
$$

with $\rho_{A \mid c}=\operatorname{tr}_{B}\left(\rho_{A B \mid c}\right)=\rho_{A}$, we have

$$
\begin{aligned}
V_{\alpha}(A ; B \mid C) & =\sum_{c} p(c)\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{A B \mid c}\right)-\rho_{B \mid c} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1, \alpha)} \\
& =\| \sum_{c} p(c)|c\rangle\langle c| \otimes \Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{A B \mid c}\right)-\sum_{c} p(c)|c\rangle\langle c| \otimes \rho_{B \mid c} \otimes \rho_{A}^{1 / \alpha} \|_{(1,1, \alpha)} \\
& =\left\|\Gamma_{\rho_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{C B A}\right)-\rho_{C B} \otimes \rho_{A}^{1 / \alpha}\right\|_{(1,1, \alpha)} \\
& =V_{\alpha}(A ; B C)
\end{aligned}
$$

So we only need to prove (i).
Define $\Xi: \mathbf{L}(B C A) \rightarrow \mathbf{L}(B C A)$ by

$$
\Xi\left(M_{B C A}\right)=M_{B C A}-\operatorname{tr}_{A}\left(M_{B C A}\right) \otimes I_{A} / d_{A} .
$$

We claim that

$$
\begin{equation*}
\|\Xi\|_{(1, \alpha, \alpha) \rightarrow(1,1, \alpha)} \leq 2^{\frac{2}{\alpha}-1} d_{C}^{\frac{1}{\alpha}} . \tag{15}
\end{equation*}
$$

Since vector valued $L_{p}$-spaces form an interpolation family [14], by the Riesz-Thorin theorem (see Appendix (A) it suffices to prove this for $\alpha=1$ and $\alpha=2$. For $\alpha=1$ by the triangle inequalality we have

$$
\begin{aligned}
\left\|M_{B C A}-\operatorname{tr}_{A}\left(M_{B C A}\right) \otimes I_{A} / d_{A}\right\|_{1} & \leq\left\|M_{B C A}\right\|_{1}+\left\|\operatorname{tr}_{A}\left(M_{B C A}\right) \otimes I_{A} / d_{A}\right\|_{1} \\
& =\left\|M_{B C A}\right\|_{1}+\left\|\operatorname{tr}_{A}\left(M_{B C A}\right)\right\|_{1} \cdot\left\|I_{A} / d_{A}\right\|_{1} \\
& =\left\|M_{B C A}\right\|_{1}+\left\|\operatorname{tr}_{A}\left(M_{B C A}\right)\right\|_{1} \\
& \leq 2\left\|M_{B C A}\right\|_{1},
\end{aligned}
$$

where the second inequality comes from the fact that $\left\|\operatorname{tr}_{A}\right\|_{1 \rightarrow 1} \leq 1$ that is easy to verify. We now prove the inequality for $\alpha=2$. We compute

$$
\begin{aligned}
\| M_{B C A}- & \operatorname{tr}_{A}\left(M_{B C A}\right) \otimes I_{A} / d_{A} \|_{(1,1,2)}^{2} \\
& =\inf _{\tau_{B C}, \sigma_{B C}}\left\|\tau_{B C}^{-1 / 4} M_{B C A} \sigma_{B C}^{-1 / 4}-\tau_{B C}^{-1 / 4} \operatorname{tr}_{A}\left(M_{B C A} \sigma_{B C}\right) \sigma_{B C}^{-1 / 4} \otimes I_{A} / d_{A}\right\|_{2}^{2} \\
& =\inf _{\tau_{B C}, \sigma_{B C}}\left\|\tau_{B C}^{-1 / 4} M_{B C A} \sigma_{B C}^{-1 / 4}\right\|_{2}^{2}-\frac{1}{d_{A}}\left\|\tau_{B C}^{-1 / 4} \operatorname{tr}_{A}\left(M_{B C A} \sigma_{B C}\right) \sigma_{B C}^{-1 / 4}\right\|_{2}^{2} \\
& \leq \inf _{\tau_{B C}, \sigma_{B C}}\left\|\tau_{B C}^{-1 / 4} M_{B C A} \sigma_{B C}^{-1 / 4}\right\|_{2}^{2} \\
& \leq \inf _{\tau_{B}, \sigma_{B}}\left\|\left(\tau_{B} \otimes I_{C} / d_{C}\right)^{-1 / 4} M_{B C A}\left(\sigma_{B} \otimes I_{C} / d_{C}\right)^{-1 / 4}\right\|_{2}^{2} \\
& =\inf _{\tau_{B}, \sigma_{B}} d_{C}\left\|\tau_{B}^{-1 / 4} M_{B C A} \sigma_{B}^{-1 / 4}\right\|_{2}^{2} \\
& =d_{C}\left\|M_{B C A}\right\|_{(1,2,2)}^{2} .
\end{aligned}
$$

Then (15)) holds for all $1 \leq \alpha \leq 2$ and for any $M_{B C A}$ we have

$$
\left\|M_{B C A}-\operatorname{tr}_{A}\left(M_{B C A}\right) \otimes I_{A} / d_{A}\right\|_{(1,1, \alpha)} \leq 2^{\frac{2}{\alpha}-1} d_{C}^{\frac{1}{\alpha^{\alpha}}}\left\|M_{B C A}\right\|_{(1, \alpha, \alpha)} .
$$

Letting

$$
M_{B C A}=\rho_{B C A}-\rho_{B} \otimes \frac{I_{C}}{d_{C}} \otimes \frac{I_{A}}{d_{A}}
$$

in the above inequality we obtain the desired result.

The next theorem gives a "weak converse" of the above inequalities.
Theorem 12. For every tripartite density matrix $\rho_{A B C}$ and $\alpha \geq 1$ the followings hold:
(i) $W_{\alpha}(A C \mid B) \leq W_{\alpha}(A \mid B C)+d_{A}^{-1 / \alpha^{\prime}} W_{\alpha}(C \mid B)$.
(ii) If $\rho_{A C}=\frac{1}{d_{A} d_{C}} I_{A} \otimes I_{C}$ then

$$
V_{\alpha}(A C ; B) \leq d_{C}^{-1 / \alpha^{\prime}} V_{\alpha}(A ; B C)+V_{\alpha}(C ; B)
$$

Moreover if $C$ is classical (and $\rho_{A C}=\frac{1}{d_{A} d_{C}} I_{A} \otimes I_{C}$ ) then

$$
V_{\alpha}(A C ; B) \leq d_{C}^{-1 / \alpha^{\prime}} V_{\alpha}(A ; B \mid C)+V_{\alpha}(C ; B) .
$$

Proof. Again we only need to prove (i). To this end we use the triangle inequality as follows:

$$
\begin{aligned}
W_{\alpha}(A C \mid B) & =\left\|\rho_{B C A}-\rho_{B} \otimes \frac{I_{C}}{d_{C}} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1, \alpha, \alpha)} \\
& \leq\left\|\rho_{B C A}-\rho_{B C} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1, \alpha, \alpha)}+\left\|\rho_{B C} \otimes \frac{I_{A}}{d_{A}}-\rho_{B} \otimes \frac{I_{C}}{d_{C}} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1, \alpha, \alpha)} \\
& =\left\|\rho_{B C A}-\rho_{B C} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1, \alpha, \alpha)}+\left\|\frac{I_{A}}{d_{A}}\right\|_{\alpha} \cdot\left\|\rho_{B C}-\rho_{B} \otimes \frac{I_{C}}{d_{C}}\right\|_{(1, \alpha)} \\
& \leq\left\|\rho_{B C A}-\rho_{B C} \otimes \frac{I_{A}}{d_{A}}\right\|_{(1,1, \alpha)}+d_{A}^{-1 / \alpha^{\prime}} \cdot\left\|\rho_{B C}-\rho_{B} \otimes \frac{I_{C}}{d_{C}}\right\|_{(1, \alpha)} \\
& =W_{\alpha}(A \mid B C)+d_{A}^{-1 / \alpha^{\prime}} W_{\alpha}(C \mid B),
\end{aligned}
$$

where the last inequality follows from Lemma 7 .

### 3.1 Special case of $\alpha=2$

The case of $\alpha=2$ is of particular interest for us since computing $V_{\alpha}(A ; B)$ and $W_{\alpha}(A \mid B)$ are easier in this case. So we focus on this special case here, and find equivalent expressions for $V_{2}, W_{2}$.

Lemma 13. We have

$$
V_{2}(A ; B)=\inf _{\tau_{B}, \sigma_{B}}\left(\operatorname{tr}\left[\left(\rho_{A}^{-1 / 2} \otimes \tau_{B}^{-1 / 2}\right) \rho_{A B}\left(\rho_{A}^{-1 / 2} \otimes \sigma_{B}^{-1 / 2}\right) \rho_{A B}\right]-\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right)^{1 / 2}
$$

and

$$
\begin{equation*}
W_{2}(A \mid B)=\inf _{\tau_{B}, \sigma_{B}}\left(\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{A B} \sigma_{B}^{-1 / 2} \rho_{A B}\right]-\frac{1}{d_{A}} \operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right)^{1 / 2} \tag{16}
\end{equation*}
$$

Proof. We compute

$$
\begin{aligned}
V_{2}^{2}(A ; B)= & \left\|\Gamma_{\rho_{A}}^{-1 / 2}\left(\rho_{B A}\right)-\rho_{B} \otimes \rho_{A}^{1 / 2}\right\|_{(1,2)}^{2} \\
= & \inf _{\tau_{B}, \sigma_{B}}\left\|\Gamma_{\rho_{A}}^{-1 / 2}\left(\tau_{B}^{-1 / 4} \rho_{B A} \sigma_{B}^{-1 / 4}\right)-\tau_{B}^{-1 / 4} \rho_{B} \sigma_{B}^{-1 / 4} \otimes \rho_{A}^{1 / 2}\right\|_{2}^{2} \\
= & \inf _{\tau_{B}, \sigma_{B}}\left(\operatorname{tr}\left[\left(\rho_{A}^{-1 / 2} \otimes \tau_{B}^{-1 / 2}\right) \rho_{A B}\left(\rho_{A}^{-1 / 2} \otimes \sigma_{B}^{-1 / 2}\right) \rho_{A B}\right]+\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right. \\
& \left.\quad-\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{A B} \sigma_{B}^{-1 / 2}\left(I_{A} \otimes \rho_{B}\right)\right]-\operatorname{tr}\left[\tau_{B}^{-1 / 2}\left(I_{A} \otimes \rho_{B}\right) \sigma_{B}^{-1 / 2} \rho_{A B}\right]\right) \\
= & \inf _{\tau_{B}, \sigma_{B}}\left(\operatorname{tr}\left[\left(\rho_{A}^{-1 / 2} \otimes \tau_{B}^{-1 / 2}\right) \rho_{A B}\left(\rho_{A}^{-1 / 2} \otimes \sigma_{B}^{-1 / 2}\right) \rho_{A B}\right]+\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right. \\
& \left.\quad-2 \operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right) \\
= & \inf _{\tau_{B}, \sigma_{B}}\left(\operatorname{tr}\left[\left(\rho_{A}^{-1 / 2} \otimes \tau_{B}^{-1 / 2}\right) \rho_{A B}\left(\rho_{A}^{-1 / 2} \otimes \sigma_{B}^{-1 / 2}\right) \rho_{A B}\right]-\operatorname{tr}\left[\tau_{B}^{-1 / 2} \rho_{B} \sigma_{B}^{-1 / 2} \rho_{B}\right]\right) .
\end{aligned}
$$

The proof of the second expression is similar.

It is also instructive to write down $V_{2}(A ; B)$ for classical distributions $p_{A B}$ :

$$
V_{2}(A ; B)=\sum_{b}\left(\sum_{a} p(a)(p(b \mid a)-p(b))^{2}\right)^{1 / 2}
$$

Given any realization $b \in \mathcal{B}$, we can view $p_{b \mid A}$ as a random variable (a function of the random variable $A$ with $\left.p_{b \mid A}(a)=p(b \mid a)\right)$. We have $\mathbb{E}_{A}\left[p_{b \mid A}\right]=\sum_{a} p(a) p(b \mid a)=p(b)$. Thus,

$$
V_{2}(A ; B)=\sum_{b} \sqrt{\operatorname{Var}_{A}\left[p_{b \mid A}\right]}
$$

Another characterization of $V_{2}(A ; B)$ can be found using the Bayes' rule:

$$
\begin{aligned}
V_{2}(A ; B) & =\sum_{b} \sqrt{\sum_{a} p(a)(p(b \mid a)-p(b))^{2}} \\
& =\sum_{b} p(b) \sqrt{\sum_{a} p(a)\left(\frac{p(b \mid a)}{p(b)}-1\right)^{2}} \\
& =\sum_{b} p(b) \sqrt{\sum_{a} p(a)\left(\frac{p(a \mid b)}{p(a)}-1\right)^{2}} \\
& =\sum_{b} p(b) \sqrt{\sum_{a}\left(\frac{p^{2}(a \mid b)}{p(a)}-2 p(a \mid b)+p(a)\right)} \\
& =\sum_{b} p(b) \sqrt{\sum_{a} \frac{p^{2}(a \mid b)}{p(a)}-1 .}
\end{aligned}
$$

Thus,

$$
\begin{equation*}
V_{2}(A ; B)=\sum_{b} p(b) \sqrt{\sum_{a} \frac{p^{2}(a \mid b)}{p(a)}-1}=\mathbb{E}_{B}\left[\sqrt{\chi^{2}\left(p_{A \mid B} \| p_{A}\right)}\right], \tag{17}
\end{equation*}
$$

where $\chi^{2}(\cdot \| \cdot)$ is the $\chi$-square distance. The above formula has some interesting consequences:
(i) Note that

$$
\begin{align*}
2^{\frac{1}{2} I_{2}(A ; B)} & =\sum_{b} \sqrt{\sum_{a} p(a) p^{2}(b \mid a)} \\
& =\sum_{b} \sqrt{\sum_{a} \frac{p^{2}(a, b)}{p(a)}} \\
& =\sum_{b} p(b) \sqrt{\sum_{a} \frac{p^{2}(a \mid b)}{p(a)}} . \tag{18}
\end{align*}
$$

Comparing (17) and (18), and utilizing the inequality $\sqrt{x} \geq \sqrt{x-1} \geq \sqrt{x}-1$ for $x \geq 1$, we obtain that

$$
\begin{equation*}
2^{\frac{1}{2} I_{2}(A ; B)} \geq V_{2}(A ; B) \geq 2^{\frac{1}{2} I_{2}(A ; B)}-1 . \tag{19}
\end{equation*}
$$

The above inequality is stronger than the one given in Proposition 10 for $\alpha=2$ in the classical case.
(ii) Using the above expressions, proving the property of the monotonicity under local operations (Theorem (9) would be easier. For example, since the $\chi$-square distance retains monotonicity under local operations (the data processing inequality) [20], we conclude that $V_{2}(X ; B) \leq$ $V_{2}(A ; B)$.
(iii) When the marginal distribution $p_{A}$ is uniform over $\mathcal{A}$, we have

$$
\begin{aligned}
V_{2}(A ; B) & =\mathbb{E}_{B}\left[\sqrt{\chi^{2}\left(p_{A \mid B} \| p_{A}\right)}\right] \\
& \leq \mathbb{E}_{B}\left[\frac{\left\|p_{A \mid B}-p_{A}\right\|_{1}}{\sqrt{\frac{2}{|\mathcal{A}|}}}\right] \\
& =\left\|p_{A B}-p_{A} p_{B}\right\|_{1} \cdot \sqrt{|\mathcal{A}| / 2}
\end{aligned}
$$

where for the inequality we use equation (25) of [21]. This can be taught as a converse of Proposition 8 .

Finally, another characterization of $V_{2}(A ; B)$ for classical systems is given in Appendix B where it is shown in Theorem 34 that $V_{2}^{2}(A ; B)$ equals a Tsallis mutual information of order two.

## 4 A decoupling theorem

Our main motivation for defining $V_{\alpha}$ is in its applications in decoupling type theorems. To explain this let us for example, think of the average of the so called purity of $\rho_{A_{0}}=\operatorname{tr}_{C}\left(U \rho_{A} U^{\dagger}\right)$, i.e., $\mathbb{E}_{U}\left[\operatorname{tr}\left(\rho_{A_{0}}^{2}\right)\right]$, where the quantum system $A$ is composed of two subsystems $A_{0}, C$ and $U_{A} \in \mathbf{L}(A)$ is a random unitary distributed according to the Haar measure. Computing this average (using techniques that will be explained below) the result would be a multiple of $\operatorname{tr}\left(\rho_{A}^{2}\right)$ plus a constant. Thus $\mathbb{E}_{U}\left[\operatorname{tr}\left(\rho_{A_{0}}^{2}\right)\right]$ cannot be naturally bounded by $c \operatorname{tr}\left(\rho_{A}^{2}\right)$ for some constant $c<1$. We conclude that for this problem it is more natural to replace purity with purity plus an appropriate constant. This simple modification is exactly what we do in using $V_{\alpha}(A ; B)$ and $W_{\alpha}(A \mid B)$ instead of $I_{\alpha}(A ; B)$ and $H_{\alpha}(A \mid B)$. The statement and the proof of the following decoupling theorem will clear up our point here.

In the following, we use (say) $A^{\prime}$ to denote a copy of the system $A$. That is, $\mathcal{H}_{A^{\prime}}$ is a Hilbert space isomorphic to $\mathcal{H}_{A}$, and $\mathcal{A}^{\prime}=\mathcal{A}$ as sets. Let

$$
F_{A A^{\prime}}: \mathcal{H}_{A} \otimes \mathcal{H}_{A^{\prime}} \rightarrow \mathcal{H}_{A} \otimes \mathcal{H}_{A^{\prime}}
$$

to be the swap operator given by

$$
\begin{equation*}
F_{A A^{\prime}}|\psi\rangle_{A} \otimes|\varphi\rangle_{A^{\prime}}=|\varphi\rangle_{A} \otimes|\psi\rangle_{A^{\prime}} \tag{20}
\end{equation*}
$$

Observe that $F_{A A^{\prime}}^{2}=I_{A A^{\prime}}$ and $\operatorname{tr}\left(F_{A A^{\prime}}\right)=d_{A}$.
Theorem 14. Let $\rho_{A B}$ be an arbitrary quantum state and $\Phi: \mathbf{L}(A) \rightarrow \mathbf{L}\left(A_{0}\right)$ be an arbitrary completely positive map (not necessarily trace preserving) satisfying

$$
\Phi\left(\frac{I_{A}}{d_{A}}\right)=\frac{I_{A_{0}}}{d_{A_{0}}} .
$$

For a given unitary $U_{A} \in \mathbf{L}\left(\mathcal{H}_{A}\right)$ define

$$
\rho_{A_{0} B}=\Phi_{A} \otimes \mathcal{I}_{B}\left(U_{A} \rho_{A B} U_{A}^{\dagger}\right),
$$

that is not necessarily normalized. Then for every $1 \leq \alpha \leq 2$ the followings hold:
(i) We have

$$
\mathbb{E}_{U}\left[W_{\alpha}\left(A_{0} \mid B\right)\right] \leq 2^{\frac{2}{\alpha}-1}\left(\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1}\right)^{\frac{1}{\alpha^{\prime}}} W_{\alpha}(A \mid B)
$$

where the expectation is taken with respect to the Haar measure and

$$
\gamma=\operatorname{tr}\left(F_{A_{0} A_{0}^{\prime}} \Phi^{\otimes 2}\left(F_{A A^{\prime}}\right)\right) .
$$

(ii) Suppose that $\rho_{A}=I_{A} / d_{A}$ is maximally mixed. Then we have

$$
\mathbb{E}_{U}\left[V_{\alpha}\left(A_{0} ; B\right)\right] \leq 2^{\frac{2}{\alpha}-1}\left(\frac{d_{A_{0}}}{d_{A}}\right)^{\frac{1}{\alpha^{\prime}}}\left(\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1}\right)^{\frac{1}{\alpha^{\prime}}} V_{\alpha}(A ; B) .
$$

This theorem in the special case of $\alpha=2$ (together with Proposition (8) resembles the oneshot decoupling theorem of [1] with similar proof ideas. See also [22] for a similar decoupling type theorem.

The following corollary presents two important especial cases of this theorem.
Corollary 15. For an arbitrary quantum state $\rho_{A B}$ and $1 \leq \alpha \leq 2$ the followings hold:
(a) If $A$ is composed of two subsystems $A_{0}, C$ and for a unitary $U_{A}$ we define $\rho_{A_{0} B}=\operatorname{tr}_{C}\left(\left(U_{A} \otimes\right.\right.$ $\left.\left.I_{B}\right) \rho_{A B}\left(U_{A}^{\dagger} \otimes I_{B}\right)\right)$ then

$$
\mathbb{E}_{U}\left[W_{\alpha}\left(A_{0} \mid B\right)\right] \leq 2^{\frac{2}{\alpha}-1} d_{C}^{-\frac{1}{\alpha^{\prime}}} W_{\alpha}(A \mid B),
$$

where the expectation is taken with respect to the Haar measure. Moreover, if $\rho_{A}=I_{A} / d_{A}$ then

$$
\mathbb{E}_{U}\left[V_{\alpha}\left(A_{0} ; B\right)\right] \leq 2^{\frac{2}{\alpha}-1} d_{C}^{-\frac{2}{\alpha^{\prime}}} V_{\alpha}(A ; B) .
$$

(b) Suppose that $\mathcal{H}_{A_{0}} \subseteq \mathcal{H}_{A}$ is a subspace and $P: \mathcal{H}_{A} \rightarrow \mathcal{H}_{A_{0}}$ is the orthogonal projection onto this subspace. Then for a unitary $U_{A}$ defining

$$
\rho_{A_{0} B}=\frac{d_{A}}{d_{A_{0}}}\left(P \otimes I_{B}\right) \rho_{A B}\left(P_{A} \otimes I_{B}\right),
$$

we have

$$
\mathbb{E}_{U}\left[W_{\alpha}\left(A_{0} \mid B\right)\right] \leq 2^{\frac{2}{\alpha}-1} W_{\alpha}(A \mid B) .
$$

Moreover, if $\rho_{A}=I_{A} / d_{A}$ then

$$
\mathbb{E}_{U}\left[V_{\alpha}\left(A_{0} ; B\right)\right] \leq 2^{\frac{2}{\alpha}-1}\left(\frac{d_{A_{0}}}{d_{A}}\right)^{\frac{1}{\alpha^{\prime}}} V_{\alpha}(A ; B)
$$

Part (b) of this corollary gives the following generalization of the decoupling result of [23]. To prove this corollary use part (b) of the above corollary together with Proposition 8,

Corollary 16. Let $\rho_{A B}$ be bipartite quantum state and let $P: \mathcal{H}_{A} \rightarrow \mathcal{H}_{A_{0}}$ be an orthonormal projection. Then we have

$$
\mathbb{E}_{U}\left[\left\|\frac{d_{A}}{d_{A_{0}}}\left(P \otimes I_{B}\right) U_{A} \rho_{A B} U_{A}^{\dagger}\left(P \otimes I_{B}\right)-\frac{I_{A_{0}}}{d_{A_{0}}} \otimes \rho_{B}\right\|_{1}\right] \leq 2^{\frac{2}{\alpha}-1} d_{A_{0}}^{\frac{1}{\alpha^{\top}}} W_{\alpha}(A \mid B)
$$

Before getting into the proof of Theorem 14 let us explain the classical counterpart of this theorem in which $A$ denotes a classical system. Due to its applications, we present only the classical counterpart of part (a) of Corollary [15.

Theorem 17. Let $\mathcal{A}=\mathcal{A}_{0} \times \mathcal{C}$ be arbitrary sets, and let

$$
\rho_{A B}=\sum_{a} p(a)|a\rangle\langle a| \otimes \rho_{a},
$$

be an arbitrary classical-quantum state. For a function $f: \mathcal{A} \rightarrow \mathcal{A}_{0}$ define

$$
\rho_{A_{0} B}=\sum_{a} p(a)|f(a)\rangle\langle f(a)| \otimes \rho_{a} .
$$

Then for every $1 \leq \alpha \leq 2$ the followings hold:
(i) We have

$$
\mathbb{E}_{f}\left[W_{\alpha}\left(A_{0} \mid B\right)\right] \leq 2^{\frac{2}{\alpha}-1} W_{\alpha}(A \mid B),
$$

where the expectation is taken with respect to the uniform distribution over all $|\mathcal{C}|$-to- 1 function. $5 f: \mathcal{A} \rightarrow \mathcal{A}_{0}$.
(ii) Suppose that $p(a)=1 /|\mathcal{A}|$ is the uniform distribution. Then we have

$$
\mathbb{E}_{f}\left[V_{\alpha}\left(A_{0} ; B\right)\right] \leq 2^{\frac{2}{\alpha}-1}|\mathcal{C}|^{-\frac{1}{\alpha^{\prime}}} V_{\alpha}(A ; B)
$$

where the expectation is taken with respect to the uniform distribution over all $|\mathcal{C}|$-to- 1 functions $f: \mathcal{A} \rightarrow \mathcal{A}_{0}$.

To prove the above theorems we first use the Riesz-Thorin theorem to reduce the statement for a general $1 \leq \alpha \leq 2$ to the special cases of $\alpha=1$ and $\alpha=2$. The proof for $\alpha=1$ follows from a simple application of the triangle inequality. To prove the theorem for $\alpha=2$ we need to compute certain averages over a Haar random unitary (random permutation). In the following, we first explain some tools for computing these averages and then present the proof of the above theorems.

Lemma 18. [1] For any $M_{A} \otimes N_{A^{\prime}} \in \mathbf{L}\left(\mathcal{H}_{A} \otimes \mathcal{H}_{A^{\prime}}\right)$ we have

$$
\operatorname{tr}\left[F_{A A^{\prime}}(M \otimes N)\right]=\operatorname{tr}[M N],
$$

where $F_{A A^{\prime}}$ is the swap operator defined by (20).
Lemma 19. [1] Let $M_{A A^{\prime}} \in \mathbf{L}\left(\mathcal{H}_{A} \otimes \mathcal{H}_{A^{\prime}}\right)$. Then we have

$$
\mathbb{E}_{U}\left[(U \otimes U) M_{A A^{\prime}}\left(U^{\dagger} \otimes U^{\dagger}\right)\right]=\alpha I_{A A^{\prime}}+\beta F_{A A^{\prime}}
$$

where the expectation is taken with respect to the Haar measure and $\alpha, \beta$ are determined by

$$
\begin{aligned}
\operatorname{tr}[M] & =\alpha d_{A}^{2}+\beta d_{A}, \\
\operatorname{tr}[M F] & =\alpha d_{A}+\beta d_{A}^{2} .
\end{aligned}
$$

Corollary 20. Let $M_{A A^{\prime} B B^{\prime}} \in \mathbf{L}\left(\mathcal{H}_{A B} \otimes \mathcal{H}_{A^{\prime} B^{\prime}}\right)$. Then we have

$$
\begin{aligned}
\mathbb{E}_{U_{A}}\left[\left(U_{A} \otimes U_{A^{\prime}}\right) M_{A A^{\prime} B B^{\prime}}\left(U_{A}^{\dagger} \otimes U_{A^{\prime}}^{\dagger}\right)\right]=\frac{1}{d_{A}^{2}-1} & {\left[I_{A A^{\prime}} \otimes \operatorname{tr}_{A A^{\prime}}(M)-\frac{1}{d_{A}} I_{A A^{\prime}} \otimes \operatorname{tr}_{A A^{\prime}}\left(F_{A A^{\prime}} M\right)\right.} \\
& \left.+F_{A A^{\prime}} \otimes \operatorname{tr}\left(F_{A A^{\prime}} M\right)-\frac{1}{d_{A}} F_{A A^{\prime}} \otimes \operatorname{tr}_{A A^{\prime}}(M)\right]
\end{aligned}
$$

Proof. To simplify the expressions let us denote $d=d_{A}$. Decompose $M$ as

$$
M_{A A^{\prime} B B^{\prime}}=\sum_{j}\left(X_{j}\right)_{A A^{\prime}} \otimes\left(Y_{j}\right)_{B B^{\prime}}
$$

[^4]Define

$$
\begin{aligned}
& \alpha_{j}=\frac{1}{d^{2}-1} \operatorname{tr}\left(X_{j}\right)-\frac{1}{d\left(d^{2}-1\right)} \operatorname{tr}\left(F_{A A^{\prime}} X_{j}\right) \\
& \beta_{j}=\frac{1}{d^{2}-1} \operatorname{tr}\left(F_{A A^{\prime}} X_{j}\right)-\frac{1}{d\left(d^{2}-1\right)} \operatorname{tr}\left(X_{j}\right) .
\end{aligned}
$$

Note that $\alpha_{j}, \beta_{j}$ satisfy

$$
\begin{aligned}
\operatorname{tr}\left[X_{j}\right] & =\alpha_{j} d^{2}+\beta_{j} d, \\
\operatorname{tr}\left[F_{A A^{\prime}} X_{j}\right] & =\alpha_{j} d+\beta_{j} d^{2} .
\end{aligned}
$$

Thus by Lemma 19 we have

$$
\begin{aligned}
\mathbb{E}_{U}\left[U^{\otimes 2} M\left(U^{\dagger}\right)^{\otimes 2}\right] & =\sum_{j} \mathbb{E}_{U}\left[U^{\otimes 2} X_{j}\left(U^{\dagger}\right)^{\otimes 2}\right] \otimes Y_{j} \\
& =\sum_{j}\left(\alpha_{j} I_{A A^{\prime}}+\beta_{j} F_{A A^{\prime}}\right) \otimes Y_{j} .
\end{aligned}
$$

Then the desired result follows once we note that

$$
\begin{aligned}
& \sum_{j} \alpha_{j} Y_{j}=\frac{1}{d^{2}-1} \operatorname{tr}_{A A^{\prime}}(M)-\frac{1}{d\left(d^{2}-1\right)} \operatorname{tr}_{A A^{\prime}}\left(F_{A A^{\prime}} M\right), \\
& \sum_{j} \beta_{j} Y_{j}=\frac{1}{d^{2}-1} \operatorname{tr}_{A A^{\prime}}\left(F_{A A^{\prime}} M\right)-\frac{1}{d\left(d^{2}-1\right)} \operatorname{tr}_{A A^{\prime}}(M) .
\end{aligned}
$$

Proof of Theorem [14. The proof of part (ii) is immediate once we have (i). The point is that when $\rho_{A}=I / d_{A}$, then $\rho_{A_{0}}=I / d_{A_{0}}$. In this case we have

$$
V_{\alpha}(A ; B)=\left\|\Gamma_{I / d_{A}}^{-1 / \alpha^{\prime}}\left(\rho_{B A}-\rho_{B} \otimes I / d_{A}\right)\right\|_{(1, \alpha)}=d_{A}^{\frac{1}{\alpha^{\prime}}}\left\|\rho_{B A}-\rho_{B} \otimes I / d_{A}\right\|_{(1, \alpha)}=d_{A}^{\frac{1}{\alpha^{\prime}}} W_{\alpha}(A \mid B)
$$

and similarly $V_{\alpha}\left(A_{0} ; B\right)=d_{A_{0}}^{\frac{1}{\alpha^{\prime}}} W_{\alpha}\left(A_{0} \mid B\right)$. Using these in (i), part (ii) will be implied. So we focus on the proof of (i).

Let $\mathcal{U}_{A} \subset \mathbf{L}(A)$ be the space of unitary operators (equipped with the Haar measure). Define $\Xi: \mathbf{L}(B A) \rightarrow \ell\left(\mathcal{U}_{A}\right) \otimes \mathbf{L}\left(B A_{0}\right)$ by

$$
\Xi\left(M_{A B}\right)\left(U_{A}\right):=\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)-\operatorname{tr}_{A}\left(M_{A B}\right) \otimes \frac{I_{A_{0}}}{d_{A_{0}}}
$$

Suppose that for every $1 \leq \alpha \leq 2$ we have

$$
\begin{equation*}
\|\Xi\|_{(1, \alpha) \rightarrow(1,1, \alpha)} \leq 2^{\frac{2}{\alpha}-1}\left(\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1}\right)^{-\frac{1}{\alpha^{\prime}}} \tag{21}
\end{equation*}
$$

That is, for every $M_{A B}$ we have

$$
\begin{equation*}
\mathbb{E}_{U_{A}}\left[\left\|\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)-\operatorname{tr}_{A}\left(M_{B A}\right) \otimes \frac{I_{A_{0}}}{d_{A_{0}}}\right\|_{(1, \alpha)}\right] \leq 2^{\frac{2}{\alpha}-1}\left(\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1}\right)^{-\frac{1}{\alpha^{\prime}}}\left\|M_{B A}\right\|_{(1, \alpha)} \tag{22}
\end{equation*}
$$

Then part (i) follows once in the above inequality we put $M_{A B}=\rho_{A B}-\rho_{B} \otimes I / d_{A}$. So we just need to prove (21). Now the point is that the ( $1, \alpha$ )-norms as well as $(1,1, \alpha)$-norms for $1 \leq \alpha \leq 2$ form an interpolation family [14]. Thus by the Riesz-Thorin theorem (see Appendix (A) proving (21) for values of $\alpha=1$ and $\alpha=2$ implies it for all $1 \leq \alpha \leq 2$. So in the following, we focus on the proof of (22) for special cases of $\alpha=1$ and $\alpha=2$.

First let $\alpha=1$. Let us writte $\Xi=\Xi_{0}-\Xi_{1}$ where $\Xi_{0}\left(M_{B A}\right)\left(U_{A}\right)=\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)$ and $\Xi_{0}\left(M_{B A}\right)\left(U_{A}\right)=\operatorname{tr}_{A}\left(M_{B A}\right) \otimes I_{A_{0}} / d_{A_{0}}$. Then by the triangle inequality we have

$$
\|\Xi\|_{1 \rightarrow 1} \leq\left\|\Xi_{0}\right\|_{1 \rightarrow 1}+\left\|\Xi_{1}\right\|_{1 \rightarrow 1}
$$

So it suffices to show that each term on right hand side is at most 1. That is, we need to show that for every $M_{A B}$ we have

$$
\left\|\Xi_{j}\left(M_{B A}\right)\right\|_{1} \leq\left\|M_{B A}\right\|_{1}, \quad j=0,1
$$

Since $\Xi_{j}$ for $j=0,1$ are completely positive, by [16, Corollary 6$]$, it suffices to prove the above inequality for $M_{B A} \geq 0$ positive semidefinite. For $j=0$ we have

$$
\begin{aligned}
\left\|\Xi_{0}\left(M_{B A}\right)\right\|_{1} & =\mathbb{E}_{U}\left[\left\|\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)\right\|_{1}\right] \\
& =\mathbb{E}_{U}\left[\operatorname{tr}\left(\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)\right)\right] \\
& =\operatorname{tr}\left(\Phi_{A \rightarrow A_{0}} \mathbb{E}_{U}\left(U_{A} M_{B A} U_{A}^{\dagger}\right)\right) \\
& =\operatorname{tr}\left(M_{A B}\right) \operatorname{tr}\left(\Phi\left(\frac{I_{A B}}{d_{A} d_{B}}\right)\right) \\
& =\operatorname{tr}\left(M_{A B}\right) \operatorname{tr}\left(\frac{I_{A_{0} B}}{d_{A_{0}} d_{B}}\right) \\
& =\left\|M_{A B}\right\|_{1}
\end{aligned}
$$

For $j=1$ we have

$$
\left\|\Xi_{1}\left(M_{B A}\right)\right\|_{1}=\left\|\operatorname{tr}_{A}\left(M_{B A}\right) \otimes I_{A_{0}} / d_{A_{0}}\right\|_{1}=\left\|\operatorname{tr}_{A}\left(M_{B A}\right)\right\|_{1}=\operatorname{tr}\left(M_{B A}\right)=\left\|M_{B A}\right\|_{1}
$$

We are done with the case $\alpha=1$.
Proof of (22) for $\alpha=2$ needs more work. For given density matrices $\tau_{B}, \sigma_{B}$ define

$$
\begin{equation*}
\hat{M}_{A B}=\tau_{B}^{-1 / 4} M_{A B} \sigma_{B}^{-1 / 4}, \quad \hat{M}_{B}=\operatorname{tr}_{A}\left(\hat{M}_{A B}\right)=\tau_{B}^{-1 / 4} \operatorname{tr}_{A}\left(M_{A B}\right) \sigma_{B}^{-1 / 4} \tag{23}
\end{equation*}
$$

and for a unitary $U_{A}$ define

$$
M_{A_{0} B}=\Phi_{A \rightarrow A_{0}}\left(U_{A} M_{A B} U_{A}^{\dagger}\right), \quad \hat{M}_{A_{0} B}=\tau_{B}^{-1 / 4} M_{A_{0} B} \sigma_{B}
$$

Following similar computations as in the proof of Lemma 13 we have

$$
\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{d_{A_{0}}}\right\|_{(1,2)}^{2}=\inf _{\tau_{B}, \sigma_{B}} \operatorname{tr}\left[\hat{M}_{B A_{0}} \hat{M}_{B A_{0}}^{\dagger}\right]-\frac{1}{d_{A_{0}}} \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)
$$

For fix $\tau_{B}, \sigma_{B}$, by Lemma 18 we have

$$
\begin{align*}
\mathbb{E}_{U_{A}} \operatorname{tr}\left[\tau_{B}^{-1 / 2} M_{A_{0} B} \sigma_{B}^{-1 / 2} M_{A_{0} B}^{\dagger}\right] & =\mathbb{E}_{U_{A}} \operatorname{tr}\left[\Phi\left(U_{A} \hat{M}_{A B} U_{A}^{\dagger}\right) \cdot \Phi\left(U_{A} \hat{M}_{A B}^{\dagger} U_{A}^{\dagger}\right)\right] \\
& =\mathbb{E}_{U_{A}} \operatorname{tr}\left[F_{A_{0} B A_{0} B^{\prime}} \Phi\left(U \hat{M}_{A B} U^{\dagger}\right) \otimes \Phi\left(U \hat{M}_{A^{\prime} B^{\prime}}^{\dagger} U^{\dagger}\right)\right] \\
& =\mathbb{E}_{U_{A}} \operatorname{tr}\left[\Phi^{*} \otimes \Phi^{*}\left(F_{A_{0} B A_{0} B^{\prime}}\right) U \hat{M}_{A B} U^{\dagger} \otimes U \hat{M}_{A^{\prime} B^{\prime}}^{\dagger} U^{\dagger}\right] \tag{24}
\end{align*}
$$

where $\Phi_{A_{0} \rightarrow A}^{*}$ is the adjoint of $\Phi$ with respect to the Hilbert-Schmidt inner product. Now using Corollary 20 we compute

$$
\begin{aligned}
\mathbb{E}_{U}\left[U^{\otimes 2} \hat{M}_{A B} \otimes \hat{M}_{A^{\prime} B^{\prime}}^{\dagger}\left(U^{\dagger}\right)^{\otimes 2}\right]=\frac{1}{d_{A}^{2}-1} & {\left[I_{A A^{\prime}} \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}-\frac{1}{d_{A}} I_{A A^{\prime}} \otimes \hat{\mu}_{B B^{\prime}}\right.} \\
& \left.+F_{A A^{\prime}} \otimes \hat{\mu}_{B B^{\prime}}-\frac{1}{d_{A}} F_{A A^{\prime}} \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}\right],
\end{aligned}
$$

where

$$
\begin{equation*}
\hat{\mu}_{B B^{\prime}}=\operatorname{tr}_{A A^{\prime}}\left[F_{A A^{\prime}} \hat{M}_{A B} \otimes \hat{M}_{A^{\prime} B^{\prime}}^{\dagger}\right] . \tag{25}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
& \mathbb{E}_{U_{A}} \operatorname{tr}\left[\tau_{B}^{-1 / 2} M_{A_{0} B} \sigma_{B}^{-1 / 2} M_{A_{0} B}\right]=\frac{1}{d_{A}^{2}-1} \operatorname{tr}\left[\Phi ^ { * } \otimes \Phi ^ { * } ( F _ { A _ { 0 } B A _ { 0 } ^ { \prime } B ^ { \prime } } ) \left(I_{A A^{\prime}} \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}-\frac{1}{d_{A}} I_{A A^{\prime}} \otimes \hat{\mu}_{B B^{\prime}}\right.\right. \\
& \left.\left.+F_{A A^{\prime}} \otimes \hat{\mu}_{B B^{\prime}}-\frac{1}{d_{A}} F_{A A^{\prime}} \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}\right)\right] \\
& =\frac{1}{d_{A}^{2}-1} \operatorname{tr}\left[F _ { A _ { 0 } B A _ { 0 } ^ { \prime } B ^ { \prime } } \left(\Phi^{\otimes 2}\left(I_{A A^{\prime}}\right) \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}-\frac{1}{d_{A}} \Phi^{\otimes 2}\left(I_{A A^{\prime}}\right) \otimes \hat{\mu}_{B B^{\prime}}\right.\right. \\
& \left.\left.+\Phi^{\otimes 2}\left(F_{A A^{\prime}}\right) \otimes \hat{\mu}_{B B^{\prime}}-\frac{1}{d_{A}} \Phi^{\otimes 2}\left(F_{A A^{\prime}}\right) \otimes \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}\right)\right] \\
& =\frac{1}{d_{A}^{2}-1}\left[\frac{d_{A}^{2}}{d_{A_{0}}} \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)-\frac{d_{A}}{d_{A_{0}}} \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right)\right. \\
& \left.+\gamma \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right)-\frac{\gamma}{d_{A}} \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)\right] \\
& =\frac{1}{d_{A}^{2}-1}\left[\left(\gamma-\frac{d_{A}}{d_{A_{0}}}\right) \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right)+\left(\frac{d_{A}^{2}}{d_{A_{0}}}-\frac{\gamma}{d_{A}}\right) \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)\right] \text {. }
\end{aligned}
$$

Therefore, using the convexity of the square function we have

$$
\begin{aligned}
\left(\mathbb { E } _ { U } \left[\| M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right)\right.\right. & \left.\left.\otimes \frac{I_{A_{0}}}{d_{A_{0}}} \|_{(1,2)}\right]\right)^{2} \leq \mathbb{E}_{U}\left[\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{d_{A_{0}}}\right\|_{(1,2)}^{2}\right] \\
& =\frac{1}{d_{A}^{2}-1}\left[\left(\gamma-\frac{d_{A}}{d_{A_{0}}}\right) \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right)+\left(\frac{d_{A}^{2}}{d_{A_{0}}}-\frac{\gamma}{d_{A}}\right) \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)-\frac{1}{d_{A_{0}}} \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)\right. \\
& \left.=\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1} \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right)-\frac{1}{d_{A}} \operatorname{tr}\left(\hat{M}_{B} \hat{M}_{B}^{\dagger}\right)\right] \\
& \leq \frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1} \operatorname{tr}\left(\hat{M}_{A B} \hat{M}_{A B}^{\dagger}\right) .
\end{aligned}
$$

Taking infimum over the choice of $\tau_{B}, \sigma_{B}$ we find that

$$
\mathbb{E}_{U}\left[\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{d_{A_{0}}}\right\|_{(1,2)}\right] \leq\left(\frac{\gamma-d_{A} / d_{A_{0}}}{d_{A}^{2}-1}\right)^{1 / 2}\left\|M_{B A}\right\|_{(1,2)}
$$

Proof of Theorem 17. The proof is similar to that of Theorem 14. Again part (ii) is an immediate consequence of part (i). Also, for part (i) it suffices to prove that for every

$$
M_{A B}=\sum_{a}|a\rangle\langle a| \otimes N_{a}
$$

the inequality

$$
\mathbb{E}_{f}\left[\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{\left|\mathcal{A}_{0}\right|}\right\|_{(1, \alpha)}\right] \leq 2^{\frac{2}{\alpha}-1}\left\|M_{B A}\right\|_{(1, \alpha)}
$$

holds for all $1 \leq \alpha \leq 2$, where

$$
M_{A_{0} B}=\sum_{a}|f(a)\rangle\langle f(a)| \otimes N_{a}
$$

and the average is with respect to the uniform distribution over all $|\mathcal{C}|$-to- 1 functions $f$. Moreover, by the Riesz-Thorin theorem it suffices to prove the above inequality for $\alpha=1$ and $\alpha=2$.

Again the proof for $\alpha=1$ is a simple consequence of the triangle inequality which we do not repeat. For $\alpha=2$ we first use

$$
\left(\mathbb{E}_{f}\left[\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{\left|\mathcal{A}_{0}\right|}\right\|_{(1,2)}\right]\right)^{2} \leq \mathbb{E}_{f}\left[\left\|M_{B A_{0}}-\operatorname{tr}_{A_{0}}\left(M_{B A_{0}}\right) \otimes \frac{I_{A_{0}}}{\left|\mathcal{A}_{0}\right|}\right\|_{(1,2)}^{2}\right]
$$

and then to estimate the left hand side we follow similar steps as in the proof of Theorem 14. We only need to replace the average with respect to the Haar measure with another average.

A uniformly random $|\mathcal{C}|$-to- 1 function $f: \mathcal{A} \rightarrow \mathcal{A}_{0}$ can be chosen as follows: let $\pi$ be a uniformly random permutation on $\mathcal{A}=\mathcal{A}_{0} \times \mathcal{C}$. Then $f(a)=\pi_{0}(a)$ is a uniformly random $|\mathcal{C}|$-to- 1 function where by $\pi_{0}(a)$ we mean the first coordinate of $\pi(a) \in \mathcal{A}=\mathcal{A}_{0} \times \mathcal{C}$. With this choice of $f$, the operator $M_{A_{0} B}$ can be written as

$$
M_{A_{0} B}=\operatorname{tr}_{C}\left(\left(U_{\pi} \otimes I_{B}\right) M_{A B}\left(U_{\pi}^{\dagger} \otimes I_{B}\right)\right)
$$

where $U_{\pi}$ is the permutation matrix associated with $\pi$. Now we can follow the proof of Theorem 14 , Fixing $\sigma_{B}, \tau_{B}$, using notations in (23) and replacing $U$ with $U_{\pi}$, equation (24) is still valid for the choice of $\Phi$ being the partial trace with respect to $C$. Nevertheless, instead of Corollary 20 we should use

$$
\begin{align*}
\mathbb{E}_{\pi}\left[U_{\pi}^{\otimes 2} \hat{M}_{A B} \otimes \hat{M}_{A^{\prime} B^{\prime}}^{\dagger}\left(U_{\pi}^{\dagger}\right)^{\otimes 2}\right]= & \frac{1}{|\mathcal{A}|^{2}-|\mathcal{A}|} I_{A A^{\prime}} \otimes\left(\hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}-\hat{\mu}_{B B^{\prime}}\right) \\
& +J_{A A^{\prime}} \otimes\left(\left(\frac{1}{|\mathcal{A}|^{2}-|\mathcal{A}|}+\frac{1}{|\mathcal{A}|}\right) \hat{\mu}_{B B^{\prime}}-\frac{1}{|\mathcal{A}|^{2}-|\mathcal{A}|} \hat{M}_{B} \otimes \hat{M}_{B^{\prime}}^{\dagger}\right) \tag{26}
\end{align*}
$$

where $\hat{\mu}_{B B^{\prime}}$ is given by (25) and

$$
J_{A A^{\prime}}=\sum_{a}|a\rangle\langle a| \otimes|a\rangle\langle a|
$$

This equation can be proven using

$$
\mathbb{E}_{\pi}\left[U_{\pi}^{\otimes 2}|a\rangle\langle a| \otimes\left|a^{\prime}\right\rangle\left\langle a^{\prime}\right| U_{\pi}^{\otimes 2}\right]= \begin{cases}\frac{1}{|\mathcal{A}|^{2}-|\mathcal{A}|}\left(I_{A A^{\prime}}-J_{A A^{\prime}}\right) & a \neq a^{\prime}, \\ \frac{1}{|\mathcal{A}|^{\prime}} J_{A A^{\prime}} & a=a^{\prime} .\end{cases}
$$

Then the proof follows by putting (26) in (24), using $F_{A A^{\prime}} J_{A A^{\prime}}=J_{A A^{\prime}}$, $\operatorname{tr} J_{A A^{\prime}}=|\mathcal{A}|$, and a straightforward computation.

The ratio $|\mathcal{C}|^{-\frac{1}{\alpha^{\prime}}}$ in Theorem 17 is asymptotically tight up to a constant as the following example shows:

Example 21. Let $p_{A}$ be uniform on $\mathcal{A}$, and $p_{B \mid A}$ be a classical erasure channel, i.e., the alphabet set of $B$ is $\mathcal{B}=\{e\} \cup \mathcal{A}$, and for all $a \in \mathcal{A}$,

$$
p_{B \mid A}(\mathrm{e} \mid a)=\epsilon, \quad p_{B \mid A}(a \mid a)=1-\epsilon,
$$

and $p_{B \mid A}\left(a^{\prime} \mid a\right)=0$ if $a^{\prime} \neq a$. Then a direct calculation shows that

$$
\begin{aligned}
W_{\alpha}(A \mid B) & =(1-\epsilon)\left[\left(1-\frac{1}{|\mathcal{A}|}\right)^{\alpha}+(|\mathcal{A}|-1) \frac{1}{|\mathcal{A}|^{\alpha}}\right]^{\frac{1}{\alpha}} \\
V_{\alpha}(A ; B) & =|\mathcal{A}|^{\frac{1}{\alpha^{\prime}}} \cdot W_{\alpha}(A \mid B) .
\end{aligned}
$$

Furthermore, for any $|\mathcal{C}|$-to- 1 function $f: \mathcal{A} \rightarrow \mathcal{A}_{0}$ with $A_{0}=f(A)$ we have

$$
\begin{aligned}
W_{\alpha}\left(A_{0} \mid B\right) & =(1-\epsilon)\left[\left(1-\frac{1}{\left|\mathcal{A}_{0}\right|}\right)^{\alpha}+\left(\left|\mathcal{A}_{0}\right|-1\right) \frac{1}{\left|\mathcal{A}_{0}\right|^{\alpha}}\right]^{\frac{1}{\alpha}} \\
V_{\alpha}\left(A_{0} ; B\right) & =\left|\mathcal{A}_{0}\right|^{\frac{1}{\alpha^{\prime}}} \cdot W_{\alpha}\left(A_{0} \mid B\right)
\end{aligned}
$$

Hence, for fixed $|\mathcal{C}|=\frac{|\mathcal{A}|}{\left|\mathcal{A}_{0}\right|}$ when $|\mathcal{A}|$ tends to infinity we have

$$
\begin{gathered}
\lim _{|\mathcal{A}| \rightarrow \infty} \min _{f:|\mathcal{C}| \text {-to- } 1} \frac{W_{\alpha}\left(A_{0} \mid B\right)}{W_{\alpha}(A \mid B)}=1, \\
\lim _{|\mathcal{A}| \rightarrow \infty} \min _{f:|\mathcal{C}| \text {-to- } 1} \frac{V_{\alpha}\left(A_{0} ; B\right)}{V_{\alpha}(A ; B)}=|\mathcal{C}|^{-\frac{1}{\alpha^{\prime}}},
\end{gathered}
$$

for any $\alpha>1$. Thus Theorem 17 is asymptotically tight up to a constant.

## 5 Applications in secrecy

The common practice in the information theoretic security literature is to use mutual information and conditional entropy to measure the amount of leakage to an adversary. In particular, for a message $A$ and adversary's side information $B$, the conditional entropy $H(A \mid B)=H(A)-I(A ; B)$, called the equivocation, is the most favorite measure. When $I(A ; B)$ is small, or equivalently $H(A \mid B)$ is close to $H(A)$, by Pinsker's inequality ${ }^{6}$ the trace distance between $\rho_{A B}$ and $\rho_{A} \otimes \rho_{B}$ is small too.

[^5]Nevertheless as will be shown later in this section, mutual information has some disadvantages as a secrecy parameter.

Here we suggest the use of $V_{\alpha}(A ; B)$ or $I_{\alpha}(A ; B)$ as a replacement of mutual information for measuring secrecy $]^{7}$ The point is that, by Proposition 8 when $V_{\alpha}(A ; B)$ is small, again $\rho_{A B}$ and $\rho_{A} \otimes \rho_{B}$ are close in trace distance. Moreover, our decoupling theorems in the previous section can be used to prove more effective exponentially small bounds on $V_{\alpha}$.

There have been a few recent works that provide further justifications for our suggestion. Controlling the Rényi mutual information of order infinity $I_{\infty}(A ; B)$ finds an operational justification in [6]. Since the Rényi mutual information is non-decreasing as a function of its order, any upper bound on Rényi mutual information of order infinity yields a bound on Rényi mutual information of other orders. Moreover, authors in [12] study the secure capacity of the wiretap channel when the security is measured by the Rényi divergence 8

In the following, we study the problem of privacy amplification and present an application of our new correlation measure and decoupling theorems there. Moreover, we discuss the advantages of $V_{\alpha}$ as a secrecy parameter over mutual information in this problem. Also, in Appendix $\mathbb{C}$ we show a connection between $V_{\alpha}$ for $\alpha=\infty$ and semantic security.

### 5.1 Privacy amplification

Suppose that a party has a secret key $A$ of $k$ uniform random bits, i.e., $\mathcal{A}=\{0,1\}^{k}$ and $p_{A}$ is the uniform distribution over $\mathcal{A}$. However, the key has partially leaked to an adversary who has access to a quantum register $B$ which is correlated with the secret key $A$ according to some known $\rho_{A B}$ with

$$
\rho_{A B}=\frac{1}{2^{k}} \sum_{a \in\{0,1\}^{k}}|a\rangle\langle a| \otimes \rho_{a} .
$$

Level of security of the key depends on the amount of information obtainable by the eavesdropper and may be measured by a correlation metric between the secret key $A$ and adversary's subsystem $B$.

Suppose that we want to decrease the correlation between $B$ and the secret key at the cost of reducing the length of the key (privacy amplification). More precisely, suppose that we have a function $f: \mathcal{A}=\{0,1\}^{k} \rightarrow \mathcal{A}_{0}=\{0,1\}^{k-1}$ such that $A_{0}=f(A)$ is uniform over $\{0,1\}^{k-1}$. Then by replacing $A$ with $A_{0}=f(A)$, and reducing the number of bits in the key, we expect to reduce the amount of correlation between the key and $B$. Indeed, we are interested in finding a suitable function $f$ such that the correlation between the distilled secret key $A_{0}$ and $B$ is minimized.

Measuring the correlation between the key and $B$ in terms of $V_{\alpha}$ for $\alpha \in(1,2]$ and using Theorem 17, if we are willing to reduce the length of key from $k$ to $k-\ell$ bits, there exists a $2^{\ell}$-to- 1 function $f:\{0,1\}^{k} \rightarrow\{0,1\}^{k-\ell}$ such that

$$
\begin{equation*}
V_{\alpha}\left(A_{0} ; B\right) \leq 2^{\frac{2}{\alpha}-1} 2^{-\frac{\ell}{\alpha^{\prime}}} V_{\alpha}(A ; B) \tag{27}
\end{equation*}
$$

where $A_{0}=f(A)$ is still uniform. Therefore, the correlation between the key and $B$ reduces exponentially in $\ell$. Observe that from Proposition 8 and Proposition 10 we obtain that there exists

[^6]a $2^{\ell}$-to- 1 function such that for $A_{0}=f(A)$ we have
\[

$$
\begin{equation*}
\left\|\rho_{A_{0} B}-\frac{I_{A_{0}}}{2^{k-\ell}} \otimes \rho_{B}\right\|_{1} \leq 2^{-\frac{\ell}{\alpha^{\prime}}}\left(2^{\frac{1}{\alpha^{\prime}} I_{\alpha}(A ; B)}+1\right) \tag{28}
\end{equation*}
$$

\]

When $\alpha=2$, using (19) the above bound can be improved to

$$
\begin{equation*}
\left\|\rho_{A_{0} B}-\frac{I_{A_{0}}}{2^{k-\ell}} \otimes \rho_{B}\right\|_{1} \leq 2^{-\ell / 2} 2^{\frac{1}{2} I_{2}(A ; B)} \tag{29}
\end{equation*}
$$

when $B$ is classical.
Inequality (29) can also be obtained by the result of Renner on privacy amplification [24, Theorem 5.5.1] for classical-quantum systems (see also [25]). Nevertheless, (27) is stronger than Renner's result, at least in the fully classical case. While Renner's result works only for $\alpha=2$, equation (27) allows for all orders $\alpha \in(1,2]$. On the other hand, Renner's result is more general because it does not assume uniform distribution on the random variable $A$.

A closely related result is in Hayashi's work on privacy amplification [26, Theorem 1]. Even though this result is stated in terms of mutual information, the key step in its proof is the following theorem (see equation (29) of [26]). This theorem should be compared with part (i) of Theorem [17]
Theorem 22 ( [26]). Let $\mathcal{A}=\mathcal{A}_{0} \times \mathcal{C}$ and $\mathcal{B}$ be arbitrary finite sets, and let $p_{A B}$ be an arbitrary bipartite distribution. Then for any $\alpha \in(1,2]$ we have

$$
\begin{equation*}
\mathbb{E}_{f}\left[2^{-\tilde{H}_{\alpha}\left(A_{0} \mid B\right)}\right] \leq 2^{-\tilde{H}_{\alpha}(A \mid B)}+\frac{1}{\left|\mathcal{A}_{0}\right|^{\alpha-1}}, \tag{30}
\end{equation*}
$$

where $A_{0}=f(A)$ and the expectation is taken with respect to the uniform distribution over all $|\mathcal{C}|$-to- 1 functions $f$ (or over a class of two-universal $|\mathcal{C}|$-to- 1 hash functions). Here, the following definition of the conditional Rényi entropy is utilized:

$$
\tilde{H}_{\alpha}(A \mid B)=-\log \sum_{a, b} p_{B}(b) p_{A \mid B}(a \mid b)^{\alpha} .
$$

Hayashi uses a different definition of conditional Rényi entropy than the one used in this paper; Comparing to (91) there is no minimization in the definition of $\tilde{H}_{\alpha}(A \mid B)$. Furthermore, our theorem does not have an additive term like $\frac{1}{\left|\mathcal{A}_{0}\right|^{\alpha-1}}$ as in (30). We should also remark that, in our results, similar to Hayashi's, the uniform distribution over all $|\mathcal{C}|$-to- 1 functions can be replaced with the uniform distribution over a class of two-universal hash functions simply because in the proofs we only use the first and second moments of the underlying distribution on the functions.

### 5.2 Mutual information versus $V_{\alpha}$

We mentioned above that Shannon used mutual information as a secrecy parameter, while we propose to use $V_{\alpha}$ for $\alpha \in(1,2]$ instead. Here we discuss this in more details. Let us start with a result similar to Theorem 17 for mutual information.
Theorem 23. Let $\mathcal{A}=\{0,1\}^{k}$, and let $p_{A B}$ be such that $p_{A}$ is the uniform distribution over $\{0,1\}^{k}$. Then there exists a 2-to-1 function $f:\{0,1\}^{k} \rightarrow\{0,1\}^{k-1}$ such that for $A_{0}=f(A)$ we have

$$
I\left(A_{0} ; B\right) \leq \frac{k-1}{k} I(A ; B) .
$$

Furthermore, the ratio $(k-1) / k$ in the above statement is optimal and cannot be replaced with a smaller constant that depends only on $k$ (and not on $p_{A B}$ ).

Proof. Let us denote the $i$-th bit of $A$ by $A_{i}$, so that $A=\left(A_{1}, \ldots, A_{k}\right)$. We let $f$ to be the function that drops one bit of $A$. Indeed, we let $A_{0}=A_{S}=f_{S}(A)$ where $S$ is some $(k-1)$-element subset of $\{1, \ldots, k\}$, and $A_{S}$ is the subsequence of its associated bits. By Shearer's lemma [27] we have

$$
\frac{1}{k} \sum_{S:|S|=k-1} H\left(A_{S} \mid B\right) \geq \frac{k-1}{k} H(A \mid B) .
$$

Since $I(A ; B)=k-H(A \mid B)$ and $I\left(A_{S} ; B\right)=(k-1)-H\left(A_{S} \mid B\right)$ for any subset $S$ of size $k-1$, we obtain

$$
\frac{1}{k} \sum_{S:|S|=k-1} I\left(A_{S} ; B\right) \leq \frac{k-1}{k} I(A ; B) .
$$

Therefore, there exists a subset $S$ satisfying $I\left(A_{S} ; B\right) \leq \frac{k-1}{k} I(A ; B)$.
To verify the optimality of $(k-1) / k$, consider the case of $B=A$. In this case we have $I(A ; B)=k$ and $I\left(A_{0} ; B\right)=k-1$ for any 2 -to-1 function $f$. As another example we can also consider the erasure channel of Example 21. In this case, $I(A ; B)=k(1-\epsilon)$ and $I\left(A_{0} ; B\right)=(k-1)(1-\epsilon)$ for any such $f$.

The ratio $(k-1) / k$ in the above theorem, is not desirable since it is close to 1 for large values of $k$. Furthermore, if we repeatedly use the above theorem to reduce the message-length from $k$ to $k-\ell$, the product $\prod_{i=k-\ell+1}^{k}(i-1) / i$ equals $(k-\ell) / k$, which is linear in $\ell$. As a result, if we convert the bound on mutual information to a bound on the total variation distance between $p_{A_{0} B}$ and $p_{A_{0}} \times p_{B}$ (by expressing mutual information in terms of the Kullback-Leibler divergence and applying Pinsker's inequality), we do not get an exponential decrease of the total variation distance in terms of $\ell$. This comparison illustrates the advantage of utilizing the proposed new measure of correlation $V_{\alpha}$ for privacy amplification.

## 6 Bounding the random coding exponent

Decoupling type theorems are widely used in quantum information theory for proving achievability results, e.g., in state merging, the mother protocol, and channel coding, see [1 and reference therein. Since Theorem 14 works for all $1 \leq \alpha \leq 2$ and not just $\alpha=2$, as in [22] we can use our decoupling theorems not only for proving achievability type results but also for proving bounds on the error exponents. In the following, we illustrate this application via the problem of entanglement generation over a noisy quantum channel and refer to [22] for other such examples.

While decoupling is a quantum phenomenon, decoupling-type theorems have also been proven useful in classical information theory. The OSRB method of [2] provides some techniques for proving achievability type results based on decoupling. Thus our decoupling theorems can be used to prove achievability results in classical network information theory as well. Moreover, as discussed above, we can state effective bounds on the error exponents of such achievability results. In the following, we take this path for the problem of secure communication over wiretap channels and establish an interesting connection between the secrecy exponent for this problem and Rényi mutual information according to Csiszár's proposal.

### 6.1 Entanglement generation

Entanglement generation via a noisy quantum channel is the problem of generating a maximally entangled state of the highest possible dimension between two parties Alice and Bob who are connected by a noisy quantum channel $\mathcal{N}_{A \rightarrow B}$ from Alice to Bob. To this end, Alice prepares a bipartite state $\rho_{R A}$ send the subsystems $A$ via the channel to Bob. Thus Bob receives the subsystem $B$ of $\mathcal{I}_{R} \otimes \mathcal{N}\left(\rho_{R A}\right)$. He then applies a decoding map $\mathcal{D}_{B \rightarrow R^{\prime}}$ and prepares $\mathcal{I}_{R} \otimes(\mathcal{D} \circ \mathcal{N})\left(\rho_{R A}\right)$. The goal of the protocol is that the latter state to be close to a maximally entangled state. A $(\log m, \epsilon)$ code for this problem, with rate $\log m$ and error $\epsilon$, is a choice of the starting state $\rho_{R A}$ and the decoding map $\mathcal{D}_{B \rightarrow R^{\prime}}$ such that

$$
F\left(\Phi_{R R^{\prime}}^{m}, \mathcal{I}_{R} \otimes(\mathcal{D} \circ \mathcal{N})\left(\rho_{R A}\right)\right) \geq 1-\epsilon
$$

where $\Phi_{R R^{\prime}}^{m}$ is a maximally entangled state of local dimension $m$ and $F$ denotes the fidelity function given by $F(\sigma, \tau)=\|\sqrt{\sigma} \cdot \sqrt{\tau}\|_{1}$. It is well-known that the entanglement generation problem is closely related to quantum commutation over the channel $\mathcal{N}_{A \rightarrow B}$. More precisely, the asymptotic rate of entanglement generation with asymptotically vanishing error equals the capacity of $\mathcal{N}_{A \rightarrow B}$, for which maximum coherent information is a lower bound, see e.g., [28].

Theorem 24. Let $\mathcal{N}_{A \rightarrow B}$ be an arbitrary quantum channel. Then for any bipartite pure state $|\psi\rangle_{R A}$ and $\alpha \in(1,2]$ there exists an entanglement generation $(\log m, \epsilon)$ code over $\mathcal{N}$ if

$$
\begin{equation*}
H_{\alpha}(R \mid E)-\alpha^{\prime} \log (1 / \epsilon)+3-\alpha^{\prime} \geq \log m \tag{31}
\end{equation*}
$$

where $\rho_{R E}=\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(|\psi\rangle\left\langle\left.\psi\right|_{R A}\right)\right.$ and $\mathcal{N}_{A \rightarrow E}^{c}$ is the complementary channel to $\mathcal{N}$.
Before getting to the proof of this theorem (that is quite standard) let us first state the asymptotic version of the above one-shot bound.

Corollary 25. Let $\mathcal{N}_{A \rightarrow B}$ be an arbitrary quantum channel. Then for any bipartite pure state $|\psi\rangle_{R A}$ and $\alpha \in(1,2]$ there exists an entanglement generation code over $\mathcal{N}$ with rate $r$ and error rate at most

$$
2^{-\frac{n}{\alpha^{\prime}}\left(H_{\alpha}(R \mid E)_{\rho}-r+o(n)\right)},
$$

where $\rho_{R E}=\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(|\psi\rangle\left\langle\left.\psi\right|_{R A}\right)\right.$ and $\mathcal{N}_{A \rightarrow E}^{c}$ is the complementary channel to $\mathcal{N}$.
Proof of Theorem 24. Since $|\psi\rangle_{R A}$ is a pure state, we may assume without no of generality that $\operatorname{dim} \mathcal{H}_{R}=\operatorname{dim} \mathcal{H}_{A}=d$. Let $\{|1\rangle, \ldots,|d\rangle\}$ be an orthonormal basis for $\mathcal{H}_{R}$, and let $\mathcal{H}_{R^{\prime}}$ be isomorphic to $\mathcal{H}_{R}$. Let

$$
\left|\Phi^{m}\right\rangle_{R R^{\prime}}=\frac{1}{m} \sum_{i=1}^{m}|i\rangle_{R} \otimes|i\rangle_{R^{\prime}},
$$

be a maximally entangled state of local dimension $m$, and $\Phi_{R R^{\prime}}^{m}=\left|\Phi^{m}\right\rangle\left\langle\left.\Phi^{m}\right|_{R R^{\prime}}\right.$ be its associated density matrix. Let $P_{R}$ be the following rank $m$ projection

$$
P_{R}=\sum_{i=1}^{m}|i\rangle\left\langle\left. i\right|_{R} .\right.
$$

Let $W_{\mathcal{N}}: \mathcal{H}_{A} \rightarrow \mathcal{H}_{B} \otimes \mathcal{H}_{E}$ be the Stinespring isometry associated to $\mathcal{N}$ so that $\mathcal{N}(X)=$ $\operatorname{tr}_{E}\left(W X W^{\dagger}\right)$. Then the complementary channel $\mathcal{N}_{A \rightarrow E}^{c}$ is given by $\mathcal{N}^{c}(X)=\operatorname{tr}_{B}\left(W_{\mathcal{N}} X W_{\mathcal{N}}^{\dagger}\right)$.

Let

$$
|\rho\rangle_{R B E}=\left(I_{R} \otimes W_{\mathcal{N}}\right)|\psi\rangle_{R A},
$$

and $\rho_{R B E}=|\rho\rangle\left\langle\left.\rho\right|_{R B E}\right.$ be its associated density matrix. By Corollary 16 for every $\alpha \in(1,2]$ there exists a unitary $U_{R}$ such that

$$
\begin{equation*}
\left\|\frac{d}{m}\left(P_{R} U_{R} \otimes I_{E}\right) \rho_{R E}\left(U_{R}^{\dagger} P_{R} \otimes I_{E}\right)-\frac{1}{m} P_{R} \otimes \rho_{E}\right\|_{1} \leq 2^{\frac{2}{\alpha}-1} m^{\frac{1}{\alpha^{\alpha}}} W_{\alpha}(R \mid E)_{\rho} \tag{32}
\end{equation*}
$$

Now define

$$
\left|\xi^{\prime}\right\rangle_{R A}=\sqrt{\frac{d}{m}}\left(P_{R} U_{R} \otimes I_{A}\right)|\psi\rangle_{R A},
$$

and let $|\xi\rangle=\frac{1}{\theta}\left|\xi^{\prime}\right\rangle$ where $\theta=\|\left|\xi^{\prime}\right\rangle \|$ is a normalization factor. Also let $\xi_{R A}=|\xi\rangle\left\langle\left.\xi\right|_{R A}\right.$ be the corresponding density matrix. Observe that

$$
\begin{aligned}
\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(\xi_{R A}\right) & =\frac{d}{\theta^{2} m} \operatorname{tr}_{B}\left(\left(P_{R} U_{R} \otimes W_{\mathcal{N}}\right)|\psi\rangle\left\langle\left.\psi\right|_{R A}\left(U_{R}^{\dagger} P_{R} \otimes W_{\mathcal{N}}^{\dagger}\right)\right)\right. \\
& =\frac{d}{\theta^{2} m}\left(P_{R} U_{R} \otimes I_{A}\right) \rho_{R E}\left(U_{R}^{\dagger} P_{R} \otimes I_{A}\right)
\end{aligned}
$$

Then using the Fuchs-van de Graaf inequality and letting $\delta$ to be the right hand side of (32) we obtain

$$
\begin{aligned}
F\left(\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(\xi_{R A}\right), \frac{1}{m} P_{R} \otimes \rho_{E}\right) & \geq 1-\frac{1}{2}\left\|\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(\xi_{R A}\right)-\frac{1}{m} P_{R} \otimes \rho_{E}\right\|_{1} \\
& =1-\frac{1}{2}\left\|\frac{d}{\theta^{2} m}\left(P_{R} U_{R} \otimes I_{A}\right) \rho_{R E}\left(U_{R}^{\dagger} P_{R} \otimes I_{A}\right)-\frac{1}{m} P_{R} \otimes \rho_{E}\right\|_{1} \\
& \geq 1-\left\|\frac{d}{m}\left(P_{R} U_{R} \otimes I_{A}\right) \rho_{R E}\left(U_{R}^{\dagger} P_{R} \otimes I_{A}\right)-\frac{1}{m} P_{R} \otimes \rho_{E}\right\|_{1} \\
& \geq 1-\delta,
\end{aligned}
$$

where the third line follows from the fact that for any two density matrices $\sigma, \sigma^{\prime}$ and $c \in \mathbb{R}$ we have $\left\|\sigma-\sigma^{\prime}\right\|_{1} \leq 2\left\|c \sigma-\sigma^{\prime}\right\|_{1}$ whose proof can be found in [29].

Observe that $I_{R} \otimes W_{\mathcal{N}}|\xi\rangle_{R A}$ is a purification of $\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(\xi_{R A}\right)$ and $\left|\Phi^{m}\right\rangle_{R R^{\prime}}$ is a purification of $\frac{1}{m} P_{R}$. Fix some purification $|\tau\rangle_{E E^{\prime}}$ of $\rho_{E}$. Then by Uhlmann's theorem there exists an isometry $Z: \mathcal{H}_{B} \rightarrow \mathcal{H}_{R^{\prime}} \otimes \mathcal{H}_{E^{\prime}}$ such that

$$
F\left(\mathcal{I}_{R} \otimes \mathcal{N}^{c}\left(\xi_{R A}\right), \frac{1}{m} P_{R} \otimes \rho_{E}\right)=\mid\left\langle\left.\Phi^{m}\right|_{R R^{\prime}} \otimes\left\langle\left.\tau\right|_{E E^{\prime}}\left(I_{R} \otimes Z W_{\mathcal{N}}\right) \mid \xi\right\rangle_{R A}\right|
$$

and then by the monotonicity of fidelity

$$
\begin{aligned}
1-\delta & \leq F\left(\left|\Phi^{m}\right\rangle_{R R^{\prime}} \otimes|\tau\rangle_{E E^{\prime}},\left(I_{R} \otimes Z W_{\mathcal{N}}\right)|\xi\rangle_{R A}\right) \\
& \leq F\left(\Phi_{R R^{\prime}}^{m}, \mathcal{I}_{R} \otimes(\mathcal{D} \circ \mathcal{N})\left(\xi_{R A}\right)\right),
\end{aligned}
$$

where $\mathcal{D}: \mathbf{L}(B) \rightarrow \mathbf{L}\left(R^{\prime}\right)$ is given by $\mathcal{D}(X)=\operatorname{tr}_{E^{\prime}}\left(Z X Z^{\dagger}\right)$. Thus the only remaining step is to show that $\epsilon \geq \delta$. That is, we need to verify that

$$
2^{\frac{2}{\alpha}-1} m^{\frac{1}{\alpha^{\prime}}} W_{\alpha}(R \mid E)_{\rho} \leq \epsilon
$$

Using Proposition 10 and the fact that $H_{\alpha}(R \mid E)_{\rho} \leq \log d$, the above inequality is implied once we have

$$
2^{-\frac{1}{\alpha^{\prime}}\left(H_{\alpha}(R \mid E)_{\rho}+1-\log m-\alpha^{\prime}(2 / \alpha-1)\right)} \leq \epsilon,
$$

which is equivalent to our assumption (31).

### 6.2 Statistics of random binning

Decoupling-type theorems are also utilized in classical information theory for proving achievability results via the method of OSRB [2]. Moreover, as in the quantum case for the problem of entanglement generation, our decoupling theorems can be used for proving bounds on the error exponents in such achievability results. Yet in the classical case we are able to prove even stronger error exponents, comparing to that of Corollary 25, by replacing Rényi information measures according to the proposal of Sibson, by those of Csiszár. Thus here we prove an asymptotic version of our decoupling theorem in the classical case in which surprisingly Csiszár's proposal of $\alpha$-Rényi mutual information appears. Next, we will apply this result to the problem of the capacity of the wiretap channel.

Let $\left(A^{n}, B^{n}\right)$ be i.i.d. classical random variables distributed according to $p_{A B}$ :

$$
p\left(a^{n} b^{n}\right)=\prod_{i=1}^{n} p\left(a_{i} b_{i}\right) .
$$

Suppose that we randomly (and uniformly) bin the set $\mathcal{A}^{n}$ into $2^{n R}$ bins and let $A_{0}$ to denote the bin index. Finding the correlation between the bin index $A_{0}$ and $B^{n}$ (averaged over all random bin mappings) is of interest, see [2]. It is known that if the binning rate $R$ is below the Slepian-Wolf rate, i.e., $R<H(A \mid B)$, the average total variation distance $\left\|p_{A_{0} B^{n}}-p_{A_{0}} \times p_{B^{n}}\right\|_{1}=V_{1}\left(A_{0} ; B^{n}\right)$ vanishes asymptotically as $n$ tends to infinity.

Here we are interested in the same question as above when we replace $V_{1}\left(A_{0} ; B^{n}\right)$ with the correlation measure $V_{\alpha}\left(A_{0} ; B^{n}\right)$ for some $\alpha \in(1,2]$. Our tool for answering this question is Theorem [17, yet this theorem is applicable only if the first variable is distributed uniformly. For this reason, we do not assume that $A^{n}$ is i.i.d., but is completely uniform on a type set.

Let $p_{A B}$ be a bipartite distribution such that $p_{A}(a)$ is a rational number for all $a \in \mathcal{A}$. In the following, let $n$ be some natural number such that $n p(a)$ is an integer for all $a \in \mathcal{A}$. For such $n$, let $\mathcal{T}_{n}\left(p_{A}\right) \subseteq \mathcal{A}^{n}$ be the set of all sequences $a^{n}$ of length $n$ whose empirical distribution (type) is equal to $p_{A}$, i.e., each symbol $a^{\prime} \in \mathcal{A}$ occurs exactly $n p\left(a^{\prime}\right)$ times in sequence $a^{n}$. Instead of the i.i.d. distribution on $A^{n}$, let $A^{n}$ be uniformly distributed over $\mathcal{T}_{n}\left(p_{A}\right)$. The conditional distribution of $B^{n}$ given $A^{n}$ is still assumed to be

$$
p\left(b^{n} \mid a^{n}\right)=\prod_{i=1}^{n} p\left(b_{i} \mid a_{i}\right) .
$$

For random binning, we use a randomly chosen $k$-to- 1 function $f$ on $\mathcal{T}_{n}\left(p_{A}\right) \subseteq \mathcal{A}^{n}$ and let $A_{0}=$ $f\left(A^{n}\right)$. We call this a regular random binning. This corresponds to a binning procedure with rate

$$
\begin{equation*}
R=\frac{1}{n} \log \left(\frac{\left|\mathcal{T}_{n}\left(p_{A}\right)\right|}{k}\right) . \tag{33}
\end{equation*}
$$

Theorem 26. Let $A^{n}$ be uniformly distributed over $\mathcal{T}_{n}\left(p_{A}\right)$ and

$$
p_{B^{n} \mid A^{n}}=\prod_{i=1}^{n} p_{B_{i} \mid A_{i}}
$$

Also let $k$ be an integer that divides $\left|\mathcal{T}_{n}\left(p_{A}\right)\right|$ and define $R$ by (33). Then for every $\alpha \in(1,2]$ we have

$$
\begin{equation*}
\mathbb{E}\left[V_{\alpha}\left(A_{0} ; B^{n}\right)\right] \leq 2^{-\frac{n}{\alpha^{\prime}}\left(H(A)-I_{\alpha}^{c}(A ; B)-R+o(n)\right)} \tag{34}
\end{equation*}
$$

where $A_{0}=f\left(A^{n}\right)$, the average is taken over all $k$-to- 1 functions $f: \mathcal{T}_{n}\left(p_{A}\right) \rightarrow \mathcal{A}_{0}$ (i.e., over all regular random bin mappings $f$ ) and $I_{\alpha}^{\mathrm{c}}(A ; B)$ is the $\alpha$-Rényi mutual information according to Csiszár's proposal [17, Eq. 29] defined by

$$
I_{\alpha}^{\mathrm{c}}(A ; B)=\min _{q_{B}} \sum_{a} p(a) D_{\alpha}\left(p_{B \mid a} \| q_{B}\right)
$$

In particular, the average correlation $\mathbb{E}\left[V_{\alpha}\left(A_{0} ; B^{n}\right)\right]$ vanishes as $n$ tends to infinity if

$$
R<H(A)-I_{\alpha}^{\mathrm{c}}(A ; B)
$$

Furthermore, we have

$$
\begin{align*}
\mathbb{E}\left[\left\|p_{A_{0} B^{n}}-p_{A_{0}} \times p_{B^{n}}\right\|_{1}\right] & \leq 2^{-\max _{1 \leq \alpha \leq 2}\left\{\frac{n}{\alpha^{\prime}}\left(H(A)-I_{\alpha}^{c}(A ; B)-R+o(n)\right)\right\}} \\
& =2^{-n\left(\min _{q_{A B}: q_{A}=p_{A}} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)+\left[\frac{1}{2} H(A \mid B)_{q}-R\right]_{+}+o(n)\right)} \tag{35}
\end{align*}
$$

From [30, Eq. 24], we have $H(A) \geq I_{\alpha}^{\mathrm{c}}(A ; B)$ with equality when $B=A$. Thus, the above bound $H(A)-I_{\alpha}^{\mathrm{c}}(A ; B)$ on the binning rate is always non-negative. Moreover, since $I_{\alpha}^{\mathrm{c}}(A ; B) \geq I(A ; B)$, we have $H(A)-I_{\alpha}^{\mathrm{c}}(A ; B) \leq H(A)-I(A ; B)=H(A \mid B)$. Hence, the bound given in the statement of the theorem on $R$ does not exceed $H(A \mid B)$, the conditional Slepian-Wolf rate, as expected.

Remark 27. To the best of our knowledge, the generalized cut-off rates of Csiszár for the dependencies of random bin indices are not defined or studied in the literature. However, we point out that resolvability exponents are studied in [4,31]33]. In particular, [4] finds the following resolvability exponent for i.i.d. codewords:

$$
\begin{equation*}
\alpha\left(R^{\prime}, P_{X}, P_{Y \mid X}\right)=\max _{\lambda \in[0,1]}\left\{\frac{\lambda}{2} R^{\prime}-\log \mathbb{E}\left[\left(\mathbb{E}\left[\left.\exp \left(\frac{\lambda}{2-\lambda} \imath_{X ; Y}(X ; Y)\right) \right\rvert\, Y\right]\right)^{\frac{2-\lambda}{2}}\right]\right\} \tag{36}
\end{equation*}
$$

With the change of variable $1 / \alpha^{\prime}=\lambda / 2$, the above expression equals

$$
\max _{\alpha \in[1,2]} \frac{1}{\alpha^{\prime}}\left(R^{\prime}-I_{\alpha}^{s}(A ; B)\right)
$$

where $I_{\alpha}^{s}(A ; B)$ is the $\alpha$-Rényi mutual information according to Sibson's proposal. To relate the resolvability problem and our problem, let $R^{\prime}=H(A)-R$. Then, we see that the exponent of [4] has the same form as our exponent, except that our $\alpha$-Rényi mutual information is computed according to Csiszar's proposal which result in stronger bounds.

Proof of Theorem [26. From Theorem 17, with a randomly chosen $k$-to-1 function $f$ acting on $\mathcal{T}_{n}\left(p_{A}\right)$, we have

$$
\begin{equation*}
\mathbb{E}\left[V_{\alpha}\left(A_{0} ; B^{n}\right)\right] \leq 2^{\frac{2}{\alpha}-1} k^{-\frac{1}{\alpha^{\prime}}} V_{\alpha}\left(A^{n} ; B^{n}\right) \leq 2^{\frac{2}{\alpha}-1} k^{-\frac{1}{\alpha^{\prime}}}\left(2^{\frac{1}{\alpha^{\prime}} I_{\alpha}\left(A^{n} ; B^{n}\right)}+1\right) \tag{37}
\end{equation*}
$$

where for the second inequality we use Propositin 10 .
Note that the distribution of $\left(A^{n}, B^{n}\right)$ is not i.i.d., so $I_{\alpha}\left(A^{n} ; B^{n}\right)$ is not equal to $n I_{\alpha}(A ; B)$. It is shown in Lemma 28 below that

$$
\begin{equation*}
2^{\frac{1}{\alpha^{\prime}} I_{\alpha}\left(A^{n} ; B^{n}\right)}=2^{\frac{n}{\alpha^{\prime}}}\left(I_{\alpha}^{\mathrm{c}}(A ; B)+o(n)\right) \tag{38}
\end{equation*}
$$

Then, from (37) we have

$$
\begin{align*}
\mathbb{E}\left[V_{\alpha}\left(A_{0} ; B^{n}\right)\right] & \leq k^{-\frac{1}{\alpha^{\prime}}} 2^{\frac{n}{\alpha^{\prime}}\left(I_{\alpha}^{\mathrm{c}}(A ; B)+o(n)\right)} \\
& =2^{-\frac{n}{\alpha^{\prime}}\left(\frac{1}{n} \log \left|\mathcal{T}_{n}\left(p_{A}\right)\right|-I_{\alpha}^{\mathrm{c}}(A ; B)-R+o(n)\right)} \\
& =2^{-\frac{n}{\alpha^{\prime}}\left(H(A)-I_{\alpha}^{\mathrm{c}}(A ; B)-R+o(n)\right)} . \tag{39}
\end{align*}
$$

To prove equation (35), applying Proposition 8, it suffices to verify that

$$
\begin{equation*}
\max _{1 \leq \alpha \leq 2} \frac{1}{\alpha^{\prime}}\left(H(A)-I_{\alpha}^{c}(A ; B)-R\right)=\min _{q_{A B}: q_{A}=p_{A}} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)+\left[\frac{1}{2} H(A \mid B)_{q}-R\right]_{+} \tag{40}
\end{equation*}
$$

To see this, we use [34, Eq. 7]

$$
\begin{equation*}
I_{\alpha}^{\mathrm{c}}(A ; B)=\max _{q_{A B}: q_{A}=p_{A}}\left(I(A ; B)_{q}-\alpha^{\prime} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)\right) \tag{41}
\end{equation*}
$$

Therefore,

$$
\begin{aligned}
& \max _{1 \leq \alpha \leq 2} \frac{1}{\alpha^{\prime}}\left(H(A)-I_{\alpha}^{c}(A ; B)-R\right) \\
& =\max _{1 \leq \alpha \leq 2} \min _{q_{A B}: q_{A}=p_{A}} \frac{1}{\alpha^{\prime}}\left(H(A)_{p}-I(A ; B)_{q}+\alpha^{\prime} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)-R\right) \\
& =\max _{1 \leq \alpha \leq 2} \min _{q_{A B}: q_{A}=p_{A}} \frac{1}{\alpha^{\prime}}\left(H(A \mid B)_{q}+\alpha^{\prime} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)-R\right) \\
& =\max _{0 \leq \zeta \leq \frac{1}{2}} \min _{q_{A B}: q_{A}=p_{A}} \zeta\left(H(A \mid B)_{q}-R\right)+D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right) .
\end{aligned}
$$

Then (40) follows once we exchange the maximum and minimum in the above equation. This exchange is possible since the expression is easily seen to be convex in $q_{B \mid A}$ and linear in $\zeta$ since for $q_{A B}=p_{A} \times q_{B \mid A}$ we have
$\zeta H(A \mid B)_{q}+D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)=\xi H(A)_{p}-(1-\zeta) H(B \mid A)_{q}-\zeta H(B)_{q}-\sum_{a, b} p(a) q(b \mid a) \log p(b \mid a)$.

It remains to verify (38) to complete the above proof.

Lemma 28. Let $p_{A B}$ be an arbitrary joint probability distribution. Let $A^{n}$ be uniform over $\mathcal{T}_{n}\left(p_{A}\right)$ and

$$
p\left(b^{n} \mid a^{n}\right)=\prod_{i=1}^{n} p\left(b_{i} \mid a_{i}\right)
$$

Then, for any $\alpha>1$ we have

$$
\lim _{n \rightarrow \infty} \frac{1}{n} I_{\alpha}\left(A^{n} ; B^{n}\right)=I_{\alpha}^{\mathrm{c}}(A ; B)
$$

Proof. We use standard arguments from the method of types. For simplicity of notation, for two sequences $\left\{x_{n}: n \geq 1\right\}$ and $\left\{y_{n}: n \geq 1\right\}$, we use $x_{n} \stackrel{\circ}{=} y_{n}$ to denote

$$
\lim _{n \rightarrow \infty} \frac{1}{n} x_{n}=\lim _{n \rightarrow \infty} \frac{1}{n} y_{n}
$$

Then we have $\log \left|\mathcal{T}_{n}\left(p_{A}\right)\right| \doteq n H(A)_{p}$. Using (8) we have

$$
\begin{align*}
\frac{1}{\alpha^{\prime}} I_{\alpha}\left(A^{n} ; B^{n}\right) & =\log \left(\sum_{b^{n}}\left(\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} \frac{1}{\left|\mathcal{T}_{n}\left(p_{A}\right)\right|} p\left(b^{n} \mid a^{n}\right)^{\alpha}\right)^{1 / \alpha}\right) \\
& =-\frac{1}{\alpha} \log \left|\mathcal{T}_{n}\left(p_{A}\right)\right|+\log \left(\sum_{b^{n}}\left(\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}\right)^{1 / \alpha}\right) \\
& \doteq-\frac{n}{\alpha} H(A)_{p}+\log \left(\sum_{b^{n}}\left(\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}\right)^{1 / \alpha}\right) \tag{42}
\end{align*}
$$

Observe that for any $b^{n} \in \mathcal{B}^{n}$, the expression $\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}$ depends only on the type of $b^{n}$ (since $\mathcal{T}_{n}\left(p_{A}\right)$ is permutation invariant). Thus letting $b_{0}^{n} \in \mathcal{T}_{n}\left(q_{B}\right)$ to be of type $q_{B}$ we define

$$
\begin{equation*}
F\left(q_{B}\right)=\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b_{0}^{n} \mid a^{n}\right)^{\alpha} \tag{43}
\end{equation*}
$$

Then denoting the set of all types in $\mathcal{B}^{n}$ by $\Upsilon_{n}(\mathcal{B})$, the second term on the right hand side of (42) can be expressed as

$$
\begin{aligned}
\log \left(\sum_{b^{n}}\left(\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}\right)^{1 / \alpha}\right) & =\log \left(\sum_{q_{B} \in \Upsilon_{n}(\mathcal{B})}\left|\mathcal{T}_{n}\left(q_{B}\right)\right| \cdot F\left(q_{B}\right)^{1 / \alpha}\right) \\
& \stackrel{\circ}{\max _{q_{B} \in \Upsilon_{n}(\mathcal{B})} \log \left(\left|\mathcal{T}_{n}\left(q_{B}\right)\right| \cdot F\left(q_{B}\right)^{1 / \alpha}\right)} \\
& \stackrel{\circ}{\max _{q_{B} \in \Upsilon_{n}(\mathcal{B})} n H(B)_{q}+\frac{1}{\alpha} \log F\left(q_{B}\right),}
\end{aligned}
$$

where in the second line we use the fact that there are polynomially many types in $\Upsilon_{n}(\mathcal{B})$.
The next step is to compute $F\left(q_{B}\right)$. Since (43) depends only on the type of $b_{0}^{n} \in \mathcal{T}_{n}\left(q_{B}\right)$ we have

$$
\begin{aligned}
F\left(q_{B}\right) & =\sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b_{0}^{n} \mid a^{n}\right)^{\alpha} \\
& =\frac{1}{\left|\mathcal{T}_{n}\left(q_{B}\right)\right|} \sum_{b^{n} \in \mathcal{T}_{n}\left(q_{B}\right)} \sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}
\end{aligned}
$$

Let us denote the joint type of $\left(a^{n}, b^{n}\right) \in \mathcal{T}_{n}\left(p_{A}\right) \times \mathcal{T}_{n}\left(q_{B}\right)$ by $q_{A B}$. Note that the marginal type of $a^{n}$ is $p_{A}=q_{A}$ and $q_{A B}$ is an "extension" of $q_{B}$. Denoting the set of all such joint types by $\widetilde{\Upsilon}_{n}\left(q_{B}\right)=\Upsilon_{n}\left(\mathcal{A} \times \mathcal{B} \mid p_{A}, q_{B}\right)$, for any sequence $\left(a^{n}, b^{n}\right)$ of joint type $q_{A B} \in \widetilde{\Upsilon}_{n}\left(q_{B}\right)$ the value of $p\left(b^{n} \mid a^{n}\right)^{\alpha}$ equals $\prod_{a, b} p(b \mid a)^{n \alpha q(a, b)}$. Therefore, we can compute $F\left(q_{B}\right)$ by splitting the sum over different joint types. By a similar argument as before, to compute the exponential growth of the summation, we should only consider the "dominant" type. Therefore,

$$
\begin{aligned}
\log F\left(q_{B}\right) & =-\log \left|\mathcal{T}_{n}\left(q_{B}\right)\right|+\log \left(\sum_{b^{n} \in \mathcal{T}_{n}\left(q_{B}\right)} \sum_{a^{n} \in \mathcal{T}_{n}\left(p_{A}\right)} p\left(b^{n} \mid a^{n}\right)^{\alpha}\right) \\
& =-\log \left|\mathcal{T}_{n}\left(q_{B}\right)\right|+\log \left(\sum_{q_{A B} \in \widetilde{\Upsilon}_{n}\left(q_{B}\right)}\left|\mathcal{T}_{n}\left(q_{A B}\right)\right| \cdot \prod_{a, b} p(b \mid a)^{n \alpha q(a, b)}\right) \\
& \circ-n H(B)_{q}+\max _{q_{A B} \in \widetilde{\Upsilon}_{n}\left(q_{B}\right)} \log \left(\left|\mathcal{T}_{n}\left(q_{A B}\right)\right| \cdot \prod_{a, b} p(b \mid a)^{n \alpha q(a, b)}\right) \\
& \circ-n H(B)_{q}+\max _{q_{A B} \in \widetilde{\Upsilon}_{n}\left(q_{B}\right)} n H(A B)_{q}+n \alpha \sum_{a, b} q(a b) \log p(b \mid a) .
\end{aligned}
$$

Putting everything together, we have

$$
\begin{aligned}
\frac{1}{\alpha^{\prime}} I_{\alpha}\left(A^{n} ; B^{n}\right) & \stackrel{\circ}{=}-\frac{n}{\alpha} H(A)_{p}+\max _{q_{B} \in \Upsilon_{n}(\mathcal{B})} \max _{q_{A B} \in \widetilde{\Upsilon}_{n}\left(q_{B}\right)}\left(n H(B)_{q}+\frac{n}{\alpha} H(A \mid B)_{q}+n \sum_{a, b} q(a b) \log p(b \mid a)\right) \\
& =\max _{q_{A B}: q_{A}=p_{A}}\left(-\frac{n}{\alpha} H(A)_{p}+n H(B)_{q}+\frac{n}{\alpha} H(A \mid B)_{q}+n \sum_{a, b} q(a b) \log p(b \mid a)\right) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \frac{1}{n} I_{\alpha}\left(A^{n} ; B^{n}\right) & =\alpha^{\prime} \max _{q_{A B}: q_{A}=p_{A}}\left(-\frac{1}{\alpha} H(A)_{q}+H(B)_{q}+\frac{1}{\alpha} H(A \mid B)_{q}+\sum_{a, b} q(a b) \log p(b \mid a)\right) \\
& =\alpha^{\prime} \max _{q_{A B}: q_{A}=p_{A}}\left(\frac{1}{\alpha^{\prime}} I(A ; B)_{q}+H(B \mid A)_{q}+\sum_{a, b} q(a b) \log p(b \mid a)\right) \\
& =\alpha^{\prime} \max _{q_{A B}: q_{A}=p_{A}}\left(\frac{1}{\alpha^{\prime}} I(A ; B)_{q}+\sum_{a, b} q(a b) \log \frac{p(b \mid a)}{q(b \mid a)}\right) \\
& =\max _{q_{A B}: q_{A}=p_{A}}\left(I(A ; B)_{q}-\alpha^{\prime} D\left(q_{B \mid A} \| p_{B \mid A} \mid p_{A}\right)\right) .
\end{aligned}
$$

The last expression, as mentioned in (41), equals $I_{\alpha}^{\mathrm{c}}(A ; B)$.

### 6.3 The wiretap channel

A wiretap channel is determined by a bipartite conditional distribution $p_{Y Z \mid X}$ in which $X$ is the input of the channel, output $Y$ is received by the legitimate receiver and output $Z$ is received by an eavesdropper. The goal of communication over a wiretap channel is to securely send information to the legitimate receiver. It is well-known that for any input distribution $p_{X}$, the rate $I(X ; Y)-$
$I(X ; Z)$ is achievable. Our goal here is to establish a bound on the secrecy exponent of random coding over a wiretap channel.

Theorem 29. Let $p_{Y Z \mid X}$ be an arbitrary wiretap channel and take $\alpha \in(1,2]$. Then for any input distribution $p_{X}$ there exists a code for reliably sending message $M$ of rate $R$ over the channel (with asymptotically vanishing error) such that

$$
\begin{equation*}
V_{\alpha}\left(M ; Z^{n}\right) \leq 2^{-\frac{n}{\alpha^{\prime}}\left(I(X ; Y)-I_{\alpha}^{c}(X ; Z)-R+o(n)\right)} . \tag{44}
\end{equation*}
$$

In particular, for such a code we have

$$
\begin{equation*}
\left\|p_{M Z^{n}}-p_{M} \times p_{Z^{n}}\right\|_{1} \leq 2^{-\frac{n}{\alpha^{\prime}}\left(I(X ; Y)-I_{\alpha}^{c}(X ; Z)-R+o(n)\right)} . \tag{45}
\end{equation*}
$$

Proof. By a continuity type argument we can assume with no loss of generality that $p(x)$ for any $x \in \mathcal{X}$ is a rational number, and in the following, we take $n$ to be a sufficiently large number such that $n p(x)$ is a natural number for all $x$. Let $\mathcal{T}_{n}\left(p_{X}\right) \subseteq \mathcal{X}^{n}$ be the set of sequences of type $p_{X}$, and let $X^{n}$ be uniformly distributed over $\mathcal{T}_{n}\left(p_{X}\right)$.

Choose positive reals $R_{1}, R_{2}, R_{3}$, which may depend on $n$, such that

- $R_{1}=R+o(n)$,
- $R_{3}>H(X \mid Y)$,
- $R_{1}+R_{3}<H(X)-I_{\alpha}^{c}(X ; Z)$
- $2^{n R_{i}}$ is an integer for $i=1,2,3$ and

$$
\left|\mathcal{T}_{n}\left(p_{X}\right)\right|=\prod_{i=1}^{3} 2^{n R_{i}} .
$$

Observe that if $R<I(X ; Y)-I_{\alpha}^{c}(X ; Z)$ such a triple $\left(R_{1}, R_{2}, R_{3}\right)$ exists.
Let $f=(m, g, u): \mathcal{T}_{n}\left(p_{X}\right) \rightarrow\left[2^{n R_{1}}\right] \times\left[2^{n R_{2}}\right] \times\left[2^{n R_{3}}\right]$ be a random 1-to-1 function (relabeling), and define $M=m\left(X^{n}\right), G=g\left(X^{n}\right), U=u\left(X^{n}\right)$. Note that, for example, $(m, g): \mathcal{T}_{n}\left(p_{X}\right) \rightarrow$ $\left[2^{n R_{1}}\right] \times\left[2^{n R_{2}}\right]$ is a random $2^{\left[n R_{3}\right]}$-to- 1 function. Moreover, since $X^{n}$ is distributed uniformly over $\mathcal{T}_{n}\left(p_{X}\right)$, random variables $M, G$ and $U$ will be uniform and mutually independent.

If $R_{3}>H(X \mid Y)$, having access to $\left(U, Y^{n}\right)$, the legitimate receiver can decode $X^{n}$ with a vanishing average error probability:

$$
\begin{equation*}
\mathbb{E}[\operatorname{Pr}(\text { error })] \rightarrow 0, \tag{46}
\end{equation*}
$$

as $n$ goes to infinity, where the average is taken over the random choice of $f$. Next, by Theorem 26 since $R_{1}+R_{3}<H(X)-I_{\alpha}^{\mathrm{c}}(X ; Z)$, we have

$$
\begin{equation*}
\mathbb{E}\left[V_{\alpha}\left(M, U ; Z^{n}\right)\right] \leq 2^{-\frac{n}{\alpha^{\prime}}\left(H(X)-I_{\alpha}^{c}(X ; Z)-R_{1}-R_{3}+o(n)\right)} . \tag{47}
\end{equation*}
$$

On the other hand, by Theorem 11 we obtain

$$
\begin{align*}
\mathbb{E}\left[V_{\alpha}\left(M ; Z^{n} \mid U\right)\right] & \leq 2^{\frac{2}{\alpha}-1} \mathbb{E}\left[V_{\alpha}\left(M, U ; Z^{n}\right)\right] \\
& \leq 2^{-\frac{n}{\alpha^{\prime}}\left(H(X)-I_{\alpha}^{c}(X ; Z)-R_{1}-R_{3}+o(n)\right)} . \tag{48}
\end{align*}
$$

Therefore, using (46) and (48), and Markov's inequality together with a union bound, for any $\epsilon>0$ and sufficiently large $n$, there exists $u \in\left[2^{n R_{3}}\right]$ and a random labeling $f_{0}$ such that

$$
\begin{equation*}
\operatorname{Pr}\left(\text { error } \mid f_{0}, U=u\right) \leq \epsilon \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{\alpha}\left(M ; Z^{n} \mid f_{0}, U=u\right) \leq 2^{-\frac{n}{\alpha^{\prime}}\left(H(X)-I_{\alpha}^{\mathrm{c}}(X ; Z)-R_{1}-R_{3}+o(n)\right)} \tag{50}
\end{equation*}
$$

Now, as in [2], the code can be constructed as follows. We treat $M$ as the message (which is distributed uniformly), select $G$ uniformly at random and independent of $M$ and transmit the codeword $X^{n}=f_{0}^{-1}(M, G, u)$. The legitimate receiver can decode $M$ with an asymptotically vanishing error because of (49), and the eavesdropper would gain no information about $M$ due to (50).

## Appendix

## A Riesz-Thorin interpolation theorem

In this appendix, we provide a very brief simplified overview of the theory of interpolation spaces and the Riesz-Thorin theorem. For a detailed introduction to the subject, we refer to [35].

Let $X$ be a finite dimensional complex vector space which can be equipped with different norms. Let us denote this vector space with two different such norms on it by $X_{0}, X_{1}$. Thus $X_{0}$ and $X_{1}$ are Banach spaces. Then the theory of complex interpolation provides us with a method for constructing intermediate Banach spaces $X_{\theta}$ for all $\theta \in[0,1]$. A typical example for such an interpolation family is the $\ell_{p}$ spaces. For $i=0,1$, letting $X_{i}=\ell_{p_{i}}(A)$ be the vector space $X=\ell(A)$ equipped with the $p_{i}$-norm, then the interpolating space $X_{\theta}$, for $\theta \in[0,1]$, is equal to $\ell_{p_{\theta}}(A)$ where $p_{\theta}$ is given by

$$
\begin{equation*}
\frac{1}{p_{\theta}}=\frac{1-\theta}{p_{0}}+\frac{\theta}{p_{1}} \tag{51}
\end{equation*}
$$

Similarly, the non-commutative spaces $L_{p_{\theta}}(A)$ form an interpolation family for $\theta \in[0,1]$ if $p_{\theta}$ 's satisfy the above equation. A more sophisticated example is the family of vector-valued spaces; For example, the interpolation of the $\left(p_{0}, q_{0}\right)$-norm and the $\left(p_{1}, q_{1}\right)$-norm is the $\left(p_{\theta}, q_{\theta}\right)$-norm where both $p_{\theta}$ and $q_{\theta}$ satisfy (51).

We can now state a version of the Riesz-Thorin interpolation theorem.
Theorem 30 (Riesz-Thorin theorem). Let $\left\{X_{\theta}: \theta \in[0,1]\right\}$ and $\left\{Y_{\theta}: \theta \in[0,1]\right\}$ be two families of interpolation spaces over finite dimensional vector spaces $X$ and $Y$ respectively. Assume that $T: X \rightarrow Y$ is a linear map which can be regarded as a continuous map from $X_{\theta}$ to $Y_{\theta}$. Then for any $\theta \in[0,1]$ we have

$$
\|T\|_{X_{\theta} \rightarrow Y_{\theta}} \leq\|T\|_{X_{0} \rightarrow Y_{0}}^{1-\theta} \cdot\|T\|_{X_{1} \rightarrow Y_{1}}^{\theta}
$$

where $\|T\|_{X_{\theta} \rightarrow Y_{\theta}}$ is the operator norm given by

$$
\|T\|_{X_{\theta} \rightarrow Y_{\theta}}=\sup _{x \in X} \frac{\|T(x)\|_{Y_{\theta}}}{\|x\|_{X_{\theta}}}
$$

Restricting to the example of vector-valued $p$-norms, we obtain the following.
Corollary 31. Let $\Xi: \mathbf{L}(B A) \rightarrow \mathbf{L}(C B A)$ be a linear map. Suppose that

$$
\|\Xi\|_{(1, \alpha) \rightarrow(1,1, \alpha)}=t_{\alpha}, \quad \alpha=1,2 .
$$

Then for every $\alpha \in[1,2]$ we have

$$
\|\Xi\|_{(1, \alpha) \rightarrow(1,1, \alpha)} \leq t_{1}^{1-\theta} \cdot t_{2}^{\theta}
$$

where $\theta=2 / \alpha^{\prime}$.

## B A new Tsallis mutual information

Given a convex function f satisfying $\mathrm{f}(1)=0$ and two distributions $p(x)$ and $q(x)$ on a discrete space $\mathcal{X}$, the f-divergence between $p$ and $q$ is defined as

$$
D_{\mathfrak{f}}(p \| q)=\sum_{x} q(x) \mathrm{f}\left(\frac{p(x)}{q(x)}\right)
$$

There are two proposals for defining a mutual information in terms of such a divergence. The first one given in [36, Eq. 3.10.1] is

$$
I_{\mathrm{f}}^{C K Z}(A ; B):=D_{\mathrm{f}}\left(p_{A B} \| p_{A} \times p_{B}\right)=\sum_{a, b} p(a) p(b) \mathrm{f}\left(\frac{p(a b)}{p(a) p(b)}\right)=\sum_{a} p(a) D_{\mathrm{f}}\left(p_{B \mid a} \| p_{B}\right)
$$

and has been studied in the literature (e.g. see [37, Theorem 5.2], [38]). Another definition is given in [39, Eq. 79]:

$$
I_{\mathrm{f}}^{P V}(A ; B):=\min _{q_{B}} D_{\mathrm{f}}\left(p_{A B} \| p_{A} \times q_{B}\right)
$$

Herein, we propose yet a new definition of mutual f-information. Given a convex function $f$, we define its mutual f-information by

$$
\begin{align*}
I_{\mathrm{f}}(A ; B) & :=\min _{q_{B}} D_{\mathrm{f}}\left(p_{A B} \| p_{A} \times q_{B}\right)-D_{\mathrm{f}}\left(p_{B} \| q_{B}\right) \\
& =\min _{q_{B}} \sum_{a} p(a) D_{\mathrm{f}}\left(p\left({ }_{B \mid a} \| q_{B}\right)-D_{\mathrm{f}}\left(p_{B} \| q_{B}\right)\right. \tag{52}
\end{align*}
$$

Observe that our mutual f-information is smaller then the previous ones:

$$
I_{\mathrm{f}}(A ; B) \leq I_{\mathrm{f}}^{P V}(A ; B) \leq I_{\mathrm{f}}^{C K Z}(A ; B)
$$

Moreover, when $D_{\mathrm{f}}(\cdot \| \cdot)$ is the KL divergence, $I_{\mathrm{f}}(A ; B)$ reduces to Shannon's mutual information.
Since we expect mutual f-information to satisfy the data processing inequality, we impose a further assumption on the convex function $f$. Interestingly, this assumption is the same as the one that gives the subadditivity of the $\Phi$-entropy.

Definition 2. Define $\mathscr{F}$ be the class of convex functions $f(t)$ on $[0, \infty]$ that are not affine (not of the form $t \mapsto a t+b$ for some constants $a$ and $b), f(1)=0$, and $1 / \mathrm{f}^{\prime \prime}$ is concave.

An important property of the class $\mathscr{F}$ is the following.
Lemma 32. For any function $\mathrm{f} \in \mathcal{F}$ and non-negative weights $\lambda_{i}, i=1,2, \cdots, n$, adding up to one, the function

$$
G\left(x_{1}, x_{2}, \cdots, x_{n}\right)=\sum_{i} \lambda_{i} \mathrm{f}\left(x_{i}\right)-\mathrm{f}\left(\sum_{i} \lambda_{i} x_{i}\right)
$$

is jointly convex.
Proof. From [40, Exercise 14.2], concavity of $1 / \mathrm{f}^{\prime \prime}$ implies that the function $(s, t) \mapsto \lambda f(s)+(1-$ $\lambda) \mathrm{f}(t)-\mathrm{f}(\lambda s+(1-\lambda) t)$, for any $\lambda \in[0,1]$, is jointly convex. One can use induction to obtain the claim of this lemma.

Examples of functions in $\mathscr{F}$ include $f(t)=t \log t$ and $f(t)=\frac{1}{\alpha-1}\left(t^{\alpha}-1\right)$ for $\alpha \in(1,2]$.
Theorem 33. For any function $\mathrm{f} \in \mathscr{F}$, the mutual $f$-information $I_{\mathrm{f}}(A ; B)$ satisfies the followings:
(i) $I_{\mathrm{f}}(A ; B)=0$ if and only if $A$ and $B$ are independent.
(ii) If $C-A-B-D$ forms a Markov chain, then $I_{\mathrm{f}}(A ; B) \geq I_{\mathrm{f}}(C ; D)$.

Proof. The proof of (i) is easy, so only present the proof of (ii). Observe that

$$
I_{\mathrm{f}}(A ; B)=\min _{q_{B}} \sum_{b} q(b)\left(\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right)-\mathrm{f}\left(\frac{p(b)}{q(b)}\right)\right) .
$$

Take some Markov chain $C-A-B$. Since f is convex, by Jensen's inequality we have

$$
\begin{aligned}
\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right) & =\sum_{a, c} p(a, c) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right) \\
& \geq \sum_{c} p(c) \mathrm{f}\left(\sum_{a} p(a \mid c) \frac{p(b \mid a)}{q(b)}\right) \\
& =\sum_{c} p(c) \mathrm{f}\left(\frac{p(b \mid c)}{q(b)}\right)
\end{aligned}
$$

Therefore, $I_{\mathrm{f}}(C ; B) \leq I_{\mathrm{f}}(A ; B)$.
Next, take some Markov chain $A-B-D$. To prove

$$
I_{\mathrm{f}}(A ; B) \geq I_{\mathrm{f}}(A ; D)
$$

it suffices to take some $q(b)$ and introduce some $q(d)$ such that

$$
\begin{equation*}
\sum_{b} q(b)\left[\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right)-\mathrm{f}\left(\frac{p(b)}{q(b)}\right)\right] \geq \sum_{d} q(d)\left[\sum_{a} p(a) \mathrm{f}\left(\frac{p(d \mid a)}{q(d)}\right)-\mathrm{f}\left(\frac{p(d)}{q(d)}\right)\right] \tag{53}
\end{equation*}
$$

Let $q(d)=\sum_{b} q(b) p(d \mid b)$. Then, we can write

$$
\sum_{b} q(b)\left[\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right)-\mathrm{f}\left(\frac{p(b)}{q(b)}\right)\right]=\sum_{b, d} q(d) p(d \mid b)\left[\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right)-\mathrm{f}\left(\frac{p(b)}{q(b)}\right)\right]
$$

To prove (53), it suffices to show that for every $d$, we have

$$
\begin{equation*}
\sum_{b} p(d \mid b)\left[\sum_{a} p(a) \mathrm{f}\left(\frac{p(b \mid a)}{q(b)}\right)-\mathrm{f}\left(\frac{p(b)}{q(b)}\right)\right] \geq \sum_{a} p(a) \mathrm{f}\left(\frac{p(d \mid a)}{q(d)}\right)-\mathrm{f}\left(\frac{p(d)}{q(d)}\right) \tag{54}
\end{equation*}
$$

For a fixed and given $p(a)$, consider the function

$$
g \in \ell(\mathcal{A}) \mapsto \sum_{a} p(a) \mathrm{f}(g(a))-\mathrm{f}\left(\sum_{a} p(a) g(a)\right) .
$$

According to Lemma 32, this function is jointly convex in $g$. Therefore, (54) follows from Jensen's inequality on this jointly convex function since

$$
\frac{p(d \mid a)}{q(d)}=\sum_{b} p(d \mid b) \frac{p(b \mid a)}{q(d)}
$$

Let $\mathrm{f}_{\alpha}(t)=\frac{1}{\alpha-1}\left(t^{\alpha}-1\right)$ for $\alpha \in(1,2]$. As mentioned above this function belongs to $\mathscr{F}$. Then, following (52) we can define the Tsallis mutual information of order $\alpha$ by

$$
\begin{equation*}
I_{\mathrm{f}_{\alpha}}(A ; B)=\frac{1}{\alpha-1} \min _{q_{B}}\left\{\sum_{a} p(a)\left(\sum_{b} q(b)^{1-\alpha} p(b \mid a)^{\alpha}\right)-\sum_{b} q(b)^{1-\alpha} p(b)^{\alpha}\right\} . \tag{55}
\end{equation*}
$$

The reason that we call it Tsallis mutual information is that the Tsallis relative entropy can be defined in terms of the function $f_{\alpha}$.

Theorem 34. The Tsallis mutual information defined in (55) equals

$$
I_{\mathrm{f}_{\alpha}}(A ; B)=\frac{1}{\alpha-1}\left(\sum_{b}\left(\sum_{a} p(a) p(b \mid a)^{\alpha}-p(b)^{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\alpha}
$$

In particular, we have

$$
\begin{aligned}
\sqrt{I_{\mathrm{f}_{2}}(A ; B)} & =\sum_{b}\left(\sum_{a} p(a) p(b \mid a)^{2}-p(b)^{2}\right)^{\frac{1}{2}} \\
& =\sum_{b}\left(\sum_{a} p(a)(p(b \mid a)-p(b))^{2}\right)^{\frac{1}{2}} \\
& =V_{2}(A ; B) .
\end{aligned}
$$

Proof. We use the Lagrange multipliers method for the optimal $q_{B}$ in (55). The Lagrangian function of the optimization problem equals

$$
\begin{aligned}
\mathcal{L}\left(q_{B}, \lambda\right) & =\frac{1}{\alpha-1}\left(\sum_{a, b} p(a) q(b)^{1-\alpha} p(b \mid a)^{\alpha}-\sum_{b} q(b)^{1-\alpha} p(b)^{\alpha}\right)-\lambda\left(\sum_{b} q(b)-1\right) \\
& =\frac{1}{\alpha-1}\left(\sum_{a, b} p(a) q(b)^{1-\alpha}\left(p(b \mid a)^{\alpha}-p(b)^{\alpha}\right)\right)-\lambda\left(\sum_{b} q(b)-1\right) \\
& =\frac{1}{\alpha-1}\left(\sum_{b} q(b)^{1-\alpha} g(b)\right)-\lambda\left(\sum_{b} q(b)-1\right),
\end{aligned}
$$

where $g(b)=\sum_{a} p(a) p(b \mid a)^{\alpha}-p(b)^{\alpha}$. Note that by Jensen's inequality we have $g(b) \geq 0$ for all $b \in \mathcal{B}$. The function $\mathcal{L}\left(q_{B}, \lambda\right)$ is convex in $q_{B}$. Then to find its minimum with respect to $q_{B}$, we take the derivative:

$$
\frac{\partial \mathcal{L}\left(q_{B}, \lambda\right)}{\partial q(b)}=-q(b)^{-\alpha} g(b)-\lambda=0 .
$$

This shows that $q(b)$ must be proportional to $g(b)^{\frac{1}{\alpha}}$. Then the optimal $q_{B}$ is given by

$$
q^{*}(b)=\frac{g(b)^{\frac{1}{\alpha}}}{\sum_{\bar{b}} g(\bar{b})^{\frac{1}{\alpha}}} .
$$

Substituting $q(b)=q^{*}(b)$ in (55) yields the desired result.

## C Semantic security and $V_{\infty}$

Most existing works in information theoretic security literature assume a message that is random and uniformly distributed. However, as pointed out in [41 this assumption may not be valid for many real-life messages such as files or votes. The semantic security is a cryptographic requirement that addresses this point. It was shown in [10] that semantic security is equivalent with a negligible mutual information between the message and the adversary's observations for all message distributions.

For a bipartite probability distribution $p_{A B}$ we have

$$
V_{\infty}(A ; B)=\sum_{b} \max _{a: p(a)>0}|p(b \mid a)-p(b)| .
$$

This expression is similar to $V_{1}(A ; B)$, except that the average over $a \in \mathcal{A}$ is replaced by a maximum over $a$. We show that $V_{\infty}(A ; B)$ is related to the semantic security. If $A$ is the message and $B$ is an eavesdropper's information, $V_{1}(A ; B)$ can be understood as the average leakage (over all messages), whereas $V_{\infty}(A ; B)$ controls the worst-case leakage. That is, if $V_{\infty}(A ; B)$ is small, any two distinct message symbols $a, a^{\prime}$ cannot be distinguished by the eavesdropper. However, if $V_{1}(A ; B)$ is small, it may be still the case that few of the message symbols are perfectly distinguishable.

Given $p_{A B}$, the authors in [10] show that semantic security holds if and only if $I(A ; B)_{q}$ is small for all $q_{A B}$ of the form $q_{A B}=q_{A} \times p_{B \mid A}$ where $q_{A}$ is an arbitrary input distribution on $\mathcal{A}$. We claim that for any $q_{A B}=q_{A} \times p_{B \mid A}$ we have

$$
\begin{equation*}
I(A ; B)_{q} \leq 2 \log (e) V_{\infty}(A ; B) . \tag{56}
\end{equation*}
$$

Therefore, if $V_{\infty}(A ; B)$ is small, semantic security is guaranteed. This establishes the connection
between our measure of correlation and semantic security. Note that

$$
\begin{align*}
I(A ; B)_{q} & =\sum_{a, b} q(a, b) \log \frac{p(b \mid a)}{q(b)} \\
& =\sum_{a, b} q(a, b) \log \left(1+\frac{p(b \mid a)-q(b)}{q(b)}\right) \\
& \leq \sum_{a, b} q(a, b) \log (e) \frac{|p(b \mid a)-q(b)|}{q(b)}  \tag{57}\\
& =\log (e) \sum_{a, b} q(a \mid b)|p(b \mid a)-q(b)| \\
& \leq \log (e) \sum_{a, b} q(a \mid b) \max _{a^{\prime}}\left|p\left(b \mid a^{\prime}\right)-q(b)\right| \\
& =\log (e) \sum_{b} \max _{a^{\prime}}\left|p\left(b \mid a^{\prime}\right)-q(b)\right|,
\end{align*}
$$

where in (57) we used the inequality $\log (1+x) \leq \log (e)|x|$ for $x>-1$. Using the triangle inequality we continue

$$
\begin{aligned}
I(A ; B)_{q} & \leq \log (e) \sum_{b} \max _{a^{\prime}}\left|p\left(b \mid a^{\prime}\right)-\sum_{a} q(a) p(b \mid a)\right| \\
& \leq \log (e) \sum_{b} \max _{a^{\prime}} \sum_{a} q(a)\left|p\left(b \mid a^{\prime}\right)-p(b \mid a)\right| \\
& \leq \log (e) \sum_{b} \max _{a, a^{\prime}}\left|p\left(b \mid a^{\prime}\right)-p(b \mid a)\right| \\
& \leq \log (e) \sum_{b} \max _{a, a^{\prime}}\left|p\left(b \mid a^{\prime}\right)-p(b)\right|+|p(b \mid a)-p(b)| \\
& =2 \log (e) V_{\infty}(A ; B) .
\end{aligned}
$$

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[^1]:    ${ }^{1} \mathrm{~A}$ density matrix is a positive semidefinite operator with trace one.

[^2]:    ${ }^{2}$ All the logarithms in this paper are in base two.
    ${ }^{3}$ See 17 for different definitions and properties of Rényi mutual information.

[^3]:    ${ }^{4}$ When $\alpha=1$ we have $\alpha^{\prime}=+\infty$.

[^4]:    ${ }^{5}$ A function $f$ is $k$-to- 1 if $\left|f^{-1}\left(a_{0}\right)\right|=k$ for all $a_{0}$.

[^5]:    ${ }^{6}$ Here we use $I(A ; B)=D\left(\rho_{A B} \| \rho_{A} \otimes \rho_{B}\right)$.

[^6]:    ${ }^{7}$ Note that by Proposition 10 the $\alpha$-Rényi mutual information and $V_{\alpha}(A ; B)$ are related quantities.
    ${ }^{8}$ There are also other approaches for defining security metrics (e.g. see [5]) based on different correlation measures.

