Additive quaternary codes related to exceptional linear quaternary codes

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Abstract

We study additive quaternary codes whose parameters are close to those of the extended cyclic $[12, 6, 6]_4$ -code or to the quaternary linear codes generated by the elliptic quadric in PG(3, 4) or its dual. In particular we characterize those codes in the category of additive codes and construct some additive codes whose parameters are better than those of any linear quaternary code.

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1 Introduction

Additive codes are a far-reaching and natural generalization of linear codes, see also Chapter 18 of [1]. Here we restrict to the quaternary case which corresponds to choosing the alphabet as the Klein group $Z_2 \times Z_2$. The quaternary case is of special interest, among others because of a close link to the theory of quantum stabilizer codes and their geometric representations, see [3, 6, 7].

Definition 1. Let k be such that 2k is a positive integer. An additive quaternary $[n, k]_4$ -code C (length n, dimension k) is a 2k-dimensional subspace of \mathbb{F}_2^{2n} , where the coordinates come in pairs. We view the codewords as n-tuples where the coordinate entries are elements of \mathbb{F}_2^2 .

A generator matrix G of C is a binary (2k, 2n)-matrix whose rows form a basis of the binary vector space C.

Definition 2. Let be C an additive $[n, k]_4$ -code. The weight of a codeword is the number of its n coordinates where the entry is different from 00. The minimum weight (equal to minimum distance) d of C is the smallest weight of its nonzero codewords. The parameters are then also written [n, k, d]. A_i denotes the number of codewords of weight i. The strength of C is the largest number t such that all (2k, 2t)-submatrices of a generator matrix whose columns correspond to some t quaternary coordinates have full rank 2t.

The geometric interpretation of an additive $[n, k, d]_4$ -code is a multifamily of n lines in PG(2k - 1, 2) (the codelines) with the property that any hyperplane of PG(2k - 1, 2) contains at most n - d of the codelines. Observe also that codelines may degenerate to points.

Code C has minimum distance d if and only if for each hyperplane H of PG(2k-1,2) we find at least d codelines (in the multiset sense), which are not contained in H. Strength t means that any set of t codelines is in general position.

As a line not intersecting a hyperplane meets it in a point, it is natural to consider also the following generalization which may be called mixed quaternary-binary codes:

Definition 3. An (n,m)-set in PG(k-1,2) is a set of n lines and m points. It has strength t if any t of those codeobjects (lines or points) are in general position.

A basic theorem by Blokhuis-Brouwer [5] characterizes the linear among the additive quaternary codes (see also [1], Theorem 18.48). It states that a family of codelines in PG(2k-1,2) generates an \mathbb{F}_4 -linear code if and only if each subset of codelines generates a vector space of even binary dimension. Among the exceptional linear quaternary codes are the hexacode $[6, 3, 4]_4$ (geometrically the hyperoval in PG(2,4), see[1], Section 17.1 and Exercise (3.7.4), the extended cyclic code $[12, 6, 6]_4$ (see [1], Section 13.4 and Exercise 17.1.10) and the dual pair of codes $[17, 4, 12]_4$ and $[17, 13, 4]_4$ described by the elliptic quadric in PG(3,4) (see Section 17.3 of [1]). In the present paper we study the generalization from linear quaternary to additive codes of code parameters related to those of our exceptional linear codes. The hexacode case is not interesting. In fact, the parameters $[6,3,4]_4$ show immediately that any three of the six codelines describing this additive code are in general position. This implies that the additive code is in fact linear, see |5|. Additive $[12, 6, 6]_4$ -codes are studied in Section 2. We show that the linear $[12, 6, 6]_4$ code is uniquely determined and that each additive $[12, 6, 6]_4$ -code has one of two weight distributions. Either it has the weight distribution of the linear code or it is a quantum $[[12, 0, 6]]_2$ -code. Such a quantum code is uniquely determined. It is known as the dodecacode, see [6]. Section 3 presents characterizations of the elliptic quadric in PG(3, 4). We show that additive $[17, 4, 12]_4$ -codes and additive $[17, 13, 4]_4$ -codes are necessarily \mathbb{F}_4 -linear.

In the final Section 4 we consider additive codes of larger length with parameters related to those of the elliptic quadric. The nonexistence of an additive $[18, 14, 4]_4$ -code is an easily obtained consequence of the characterization of the elliptic quadric. In other words, the additive relaxation of an 18-cap in PG(3, 4) does not exist. Finally we construct $[22, 17.5, 4]_4$ -codes. From them, $[19+i, 14.5+i, 4]_4$ -codes $i = 0, \ldots, 2$ can be obtained by shortening. Such additive codes are new and their parameters cannot be reached by linear codes. For the parameters of optimal additive codes of small dimension see [2] and the references therein. For simplicity we denote PG(i-1, 2) also as V_i .

2 Additive $[12, 6, 6]_4$ -codes

Theorem 1. A linear $[12, 6, 6]_4$ -code is uniquely determined.

Proof. Let us start using geometric arguments. The ambient space is PG(5, 4), each hyperplane contains at most 6 codepoints, each secundum PG(3, 4) con-

tains at most 4 codepoints. This shows that any four codepoints do generate a secundum and any five codepoints generate a hyperplane. In particular the strength is 5, in other words: the dual code has the same parameters $[12, 6, 6]_4$.

Let S be a secundum containing 4 codepoints. There are two possibilities for the distribution of codepoints on the hyperplanes containing S. Either two of those hyperplanes contain one more codepoint, the others contain two more codepoints or one of the 5 hyperplanes containing S has no further codepoint, the four remaining ones contain two more codepoints each. We try to construct a generator matrix G = (I | P) where $P = (p_{ij})$ is a (6, 6)matrix. Denote by z_i the rows of the generator matrix, by $v_i, i = 1, \ldots, 6$ the rows of P.

Let us consider the case $A_7 = 0$, equivalently no hyperplane contains precisely 5 codepoints, again equivalently the hyperplanes containing 6 codepoints form the blocks of a Steiner system S(5, 6, 12).

The usual arguments show that we can choose

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & & & & \\ 1 & 0 & & & & \\ 1 & 0 & & & & \\ 1 & 0 & 0$$

Codewords $\lambda z_1 + z_2$ show that each (nonzero) entry occurring among the $p_{23}, p_{24}, p_{25}, p_{26}$ occurs precisely twice. The same holds for each of the rows $v_i, i > 1$. We can choose

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & a & b & b \\ 1 & 0 & & & \\ 1 & 0 & & & \\ 1 & 0 & & & \\ 1 & 0 & & & \\ 1 & 0 & 0 & \\ 1 & 0 & 0 & \\ \end{pmatrix}$$

where a, b are different nonzero elements. Clearly we can choose notation such that the corresponding nonzero entries in v_3 are the same, a and b. We must have $p_{35} \neq p_{36}$ as otherwise there is a linear relation involving the last two columns of P and the three last columns of the unit matrix. We can choose

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & a & b & b \\ 1 & 0 & a & b & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The codeword $z_2 + z_3$ shows $p_{34} = b$. We have

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & a & b & b \\ 1 & a & 0 & b & a & b \\ 1 & & 0 & & \\ 1 & & 0 & & \\ 1 & & & 0 & \\ 1 & & & 0 & \\ \end{pmatrix}$$

Now

$$\omega z_2 + z_3 = (0, \omega, 1, 0, 0, 0, \overline{\omega}, a, \omega a, \omega a + b, \omega b + a, \overline{\omega} b).$$

Exactly one of $\omega a + b$ and $\omega b + a$ is nonzero. This shows that we have a codeword of weight 7, contradiction.

We have $A_7 \neq 0$, equivalently there is a hyperplane with precisely 5 codepoints, equivalently there is a secundum for which the first of the two possibilities above is satisfied. Let us choose $S = (x_1 = x_2 = 0)$ as this secundum. The two hyperplanes containing S and containing 5 codepoints each are $H_1 = (x_1 = 0)$ and $H_2 = (x_2 = 0)$. This means that v_1, v_2 have no entry 0. Comparing z_1 and z_2 we see that we can choose $v_1 = (1, 1, 1, 1, 1, 1), v_2 = (1, 1, \omega, \omega, \overline{\omega}, \overline{\omega})$. Let ν be the total number of entries 0 in P. As each row of P has at most one such entry it follows $\nu \leq 4$. Assume at first $\nu \leq 3$. Choose notation such that v_3 has no entry 0. Codewords $z_3 + \lambda z_1$ show that each nonzero element occurs precisely twice in v_3 . We can choose $p_{31} = 1$. As $d(v_2, v_3) = 4$ we can choose notation such that $p_{32} \neq \omega$ (as otherwise v_2, v_3 would agree in one of the last two coordinates). It follows $p_{32} = \overline{\omega}$ and $p_{34} = \overline{\omega}$. By symmetry we can choose $p_{35} = 1, p_{36} = \omega$. We have

Observe that rows v_1, v_2, v_3 of P form a generator matrix of the hexacode $[6, 3, 4]_4$. Assume now $\nu \leq 2$ and choose notation such that v_4 has weight $6, p_{41} = 1$. Strength 5 shows $p_{42} \neq 1, p_{45} \neq 1, p_{43} \neq \omega$. Assume $p_{43} = 1$. Then $p_{46} = \overline{\omega}, p_{45} = \omega$ and $v_4 = (1, \omega, 1, \overline{\omega}, \omega, \overline{\omega})$. However $v_1 + v_4 + \omega(v_2 + v_3) = 0$ yields a codeword of weight 4. We have $p_{43} = \overline{\omega}$. Again v_4 is then uniquely determined, $v_4 = (1, \overline{\omega}, \overline{\omega}, \omega, \omega, 1)$ and $\omega(v_1 + v_2) + v_3 + v_4 = 0$, contradiction. It follows $\nu = 3$ or $\nu = 4$.

Assume at first $\nu = 3$. We have the first three rows of P as before (forming a generator matrix G of the hexacode), and each subsequent row of P contains one entry 0. Denote the columns of G by s_1, \ldots, s_6 , where $s_6 = \tilde{s}_4$ and $s_1 = \tilde{s}_1$. Consider transformations $g(\beta, \gamma)$ defined by

$$v_1 \mapsto v_1, v_2 \mapsto \beta v_2, v_3 \mapsto \gamma v_3.$$

Then $g(1,\omega)$ maps

$$(s_1, \ldots, s_6) \mapsto (\tilde{s}_2, s_1, s_4, \tilde{s}_5, s_6, s_4).$$

After taking conjugates and rearranging columns this leads back to G. This maps a candidate row w = (a, b, c, d, e, f) to $(\tilde{b}, \tilde{a}, \tilde{e}, \tilde{f}, \tilde{d}, \tilde{c})$. This describes a permutation of order 4. In the same manner $g(\overline{\omega}, \overline{\omega})$ describes the involution $w = (a, b, c, d, e, f) \mapsto (\tilde{c}, \tilde{d}, \tilde{a}, \tilde{b}, \tilde{f}, \tilde{e})$. Consider $(g(1, \omega)g(\overline{\omega}, \overline{\omega}))^2$. It maps $w \mapsto (d, c, f, e, b, a)$, a permutation of order 6. This shows that we can choose $p_{41} = 0$.

The permutation $\rho : v_1 \mapsto v_2 \mapsto v_3 \mapsto v_1$ maps $w = (a, b, c, d, e, f) \mapsto (a, \overline{\omega}c, \omega e, \overline{\omega}d, b, \omega f)$ and the involution $\sigma : v_1 \mapsto v_1, v_2 \mapsto v_3 \mapsto v_2$ maps $w \mapsto (a, e, c, f, b, d)$. Clearly $\langle \rho, \sigma \rangle \cong S_3$.

A direct check shows that there are only three vectors w with first entry 0, one entry 1 and two entries ω as well as two entries $\overline{\omega}$ which may be used as v_4 (such that the corresponding four-dimensional code has minimum distance 6). Those are

$$w = (0, \omega, 1, \overline{\omega}, \omega, \overline{\omega}), (0, \omega, \overline{\omega}, 1, \overline{\omega}, \omega), (0, \overline{\omega}, \overline{\omega}, \omega, \omega, 1).$$

They form an orbit under the group S_3 above. It follows that we may choose v_4 as the first of those vectors. We have

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \overline{\omega} & \overline{\omega} \\ 1 & \overline{\omega} & \omega & \overline{\omega} & 1 & \omega \\ 0 & \omega & 1 & \overline{\omega} & \omega & \overline{\omega} \\ & & & & & & & & & & & \end{pmatrix}$$

A little computer program shows that this cannot be completed.

Assume finally that $\nu = 4$. The first two rows of P are as before. Clearly we can assume $p_{31} = 0$. Each of the rows $v_i, i > 2$ contains one nonzero field element with frequency 1, each of the others with frequency 2. By an obvious operation we can assume that the field element 1 occurs with frequency 1 in each $v_i, i > 2$. It is impossible that $p_{32} = 1$ as in this case all entries of $v_2 + v_1$ are 0 or 1 which yields a contradiction. Conjugation shows that we may choose $p_{32} = \omega$. Assume $p_{35} = 1$. Then $v_3 = (0, \omega, \omega, \overline{\omega}, 1, \overline{\omega})$ which yields $wt(\omega v_2 + v_3) = 3$, contradiction. We can choose $p_{33} = 1$ and $v_3 = (0, \omega, 1, \overline{\omega}, \omega, \overline{\omega})$. We have

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \overline{\omega} & \overline{\omega} \\ 0 & \omega & 1 & \overline{\omega} & \omega & \overline{\omega} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ \end{array} \right).$$

A computer program shows that this can be completed in precisely 12 ways. The resulting matrices have their zeroes in the last three rows in columns either 2, 3, 4 or 2, 5, 6. The linear mapping $\tau : v_1 \mapsto v_2 \mapsto v_1, v_3 \mapsto v_3$ induces $\tau(w) = (a, b, \omega f, \omega e, \overline{\omega} d, \overline{\omega} c)$. We can therefore assume that the zeroes occur in positions 2, 3, 4 in the last three rows of *P*. Those six matrices generate equivalent codes. We conclude that the linear $[12, 6, 6]_4$ -code is uniquely determined. Here is one example:

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega & \overline{\omega} & \overline{\omega} \\ 0 & \omega & 1 & \overline{\omega} & \omega & \overline{\omega} \\ \overline{\omega} & 0 & \overline{\omega} & \omega & \omega & 1 \\ \overline{\omega} & \omega & 0 & \omega & 1 & \overline{\omega} \\ \omega & 1 & \overline{\omega} & 0 & \omega & \overline{\omega} \end{pmatrix}.$$

The weight distribution of this uniquely determined quaternary linear $[12, 6, 6]_4$ -code is

$$A_0 = 1, A_6 = 330, A_7 = 396, A_8 = 495, A_9 = 1320,$$

 $A_{10} = 990, A_{11} = 396, A_{12} = 168.$

Let us consider additive codes with these parameters. In fact, we work with the dual, a $[12, 6]_4$ -code of strength 5. Projection to some 7 codelines generating the ambient space yields a [7, 6]-code of strength 5. Its dual is a [7, 1, 6]-code. Clearly those codes can be characterized. There may be a 0-coordinate or up to three points. This leads to the following list of [7, 1, 6]codes. In each of these 5 cases we determined the dual and the sets of 5 lines in PG(11, 2) which complete this to a code of strength 5. The first configuration

leads to completions with two different weight distributions:

quantum:
$$A_0 = 1, A_6 = 396, A_8 = 1485, A_{10} = 1980, A_{12} = 234;$$

linear: $A_0 = 1, A_6 = 330, A_7 = 396, A_8 = 495, A_9 = 1320,$
 $A_{10} = 990, A_{11} = 396, A_{12} = 168.$

The latter is the weight distribution of the linear code. Refer to these weight distributions as the quantum case and the quaternary linear case, respectively. The second configuration

and the third configuration

$\int L_1$	L_2	L_3	L_4	L_5	L_6	L_7
10	10	10	10	10	01	00
$\setminus 01$	01	01	01	01	00	01 /

do not produce codes. The fourth configuration

$\int L_1$	L_2	L_3	L_4	L_5	L_6	L_7
10	10	10	10	01	01	00
$\setminus 01$	01	01	01	01	00	01 /

produces a code with the weight distribution of the quantum case, the fifth configuration

(L_1	L_2	L_3	L_4	L_5	L_6	L_7
	10	10	10	10	10	10	10
ĺ	01	01	01	01	01	01	01 /

yields codes with the weight distribution of the linear case. Observe that we do not have a complete classification. Our result is as follows:

Theorem 2. Each additive $[12, 6, 6]_4$ -code has either the weight distribution of the linear code or the weight distribution of the quantum case.

Each additive $[12, 6, 6]_4$ -code having the weight distribution of the quantum case is in fact a $[[12, 0, 6]]_2$ quantum code, equivalently it is self-dual with respect to the symplectic form. It is known that this quantum code is uniquely determined: the dodecacode. It is not clear if all additive $[12, 6, 6]_4$ -codes having the weight distribution of the linear case are necessarily \mathbb{F}_4 -linear.

3 Characterizations of the elliptic quadric

The elliptic quadric in PG(3, 4) is the uniquely determined 17-cap in PG(3, 4). Using representatives of those points as columns of a (4, 17)-matrix with entries in \mathbb{F}_4 , we obtain a generator matrix M of a linear $[17, 4, 12]_4$ -code whose dual is a $[17, 13, 4]_4$ -code. Both those codes have optimal parameters. As 17 is the largest size of a cap in PG(3, 4), there is no linear $[18, 14, 4]_4$ -code. The elliptic quadric has a large sphere of influence. In fact, linear codes $[16, 12, 4]_4$ and $[15, 11, 4]_4$ correspond to caps of sizes 15 and 14 respectively, and it is known that such caps are contained in the elliptic quadric (see [8]). It follows that linear codes with those parameters are uniquely determined. By what we observed above it will be sufficient to prove that such additive codes are necessarily \mathbb{F}_4 -linear. Here the characterization from [5] is useful again. Observe also that concatenation of an additive $[n, k, d]_4$ -code with the binary linear $[3, 2, 2]_2$ -code yields a binary linear $[3n, 2k, 2d]_2$ -code. This fact can be used to obtain non-existence results.

Theorem 3. An additive $[17, 4, 12]_4$ -code is uniquely determined.

Proof. Assume at first that one of the 17 codeobjects is a point. This yields a subcode with parameters $[16, 3.5, 12]_4$. Concatenation yields a binary $[48, 7, 24]_2$ -code which however cannot exist by the Griesmer bound. An analogous argument shows that, because of the non-existence of a $[15, 2.5, 12]_4$ -code, it is impossible that two codelines intersect. It follows that the 17 codelines form a partial spread of lines. Let M be the union of the points on codelines. Denote the elements of M by codepoints.

Observe that the ambient space is a V_8 . The definition of the distance shows that each hyperplane V_7 contains at most $5 \times 2 + 17 = 27$ codepoints. Obvious counting arguments imply that each secundum V_6 has at most 15 codepoints, a V_5 has at most 9 codepoints, a V_4 has at most 6 codepoints and a V_3 has at most 4 codepoints. This implies that our code has strength 3: any three codelines are in general position. In fact, assume some three codelines are in a V_5 . Then the third of those lines meets the V_4 generated by the first two in a point. We obtain 7 codepoints in a V_4 , contradiction.

Let now U be a plane V_3 containing a codeline L_1 and an isolated codepoint on codeline L_2 . Consider the factor space Π , a PG(4, 2). For each point P of Π , let the weight w(P) be the number of codepoints in the preimage of Pwhich are outside U. For each subspace X of Π let $w(X) = \sum_{P \in W} w(P)$. The distribution of codepoints on subspaces shows that $w(P) \leq 2$ for each point P of Π . Also $w(\Pi) = 47$ and each line l has weight $w(l) \leq 5$. Considering a point P of weight 2 and the lines of Π passing through it, we see that all weights of points are 1 or 2 and each line through P contains precisely one point of weight 1. It follows that Π possesses a hyperplane H_0 all of whose points have weight 1, where all affine points (outside H_0) have weight 2. Observe that Π has a special point P_0 of weight 2 whose preimage contains codeline L_2 . Because of strength 2 each of the remaining 15 codelines (different from L_1, L_2) has as image a line of Π . Denote those images by M_1, \ldots, M_{15} . We have that P_0 is not contained in any of those lines of Π , each point of H_0 is on precisely one of those lines and each affine point different from P_0 is on two of those lines.

Choose coordinates: let v_1, \ldots, v_8 be a basis of the ambient space, $U = \langle v_1, v_2, v_3 \rangle$ and use coordinates y_1, \ldots, y_5 for the actor space Π . As hyperplane choose $H_0 = (y_1 = 0)$ and as special point $P_0 = 10000$. An obvious calculation with (5, 5)-matrices shows that the first two lines M_1, M_2 can be chosen as $\langle 01000, 10100 \rangle$ and $\langle 00010, 10001 \rangle$. This leads to the first four lines in PG(7, 2) as follows:

(L_1	L_2	L_3	L_4
	10	00	00	00
	01	00	00	00
	00	10	00	00
	00	01	01	01
	00	00	10	00
	00	00	01	00
	00	00	00	10
$\left(\right)$	00	00	00	01 /

This leads to the problem of finding a family M_1, \ldots, M_{15} of lines in Π such that M_1, M_2 are as above, P_0 is on none of those lines, each of the remaining points of weight 2 is on two of the M_i and each point of the hyperplane H_0 is on precisely one of the M_i . A computer program revealed that this problem in PG(4, 2) has precisely 246 different solutions. The completion to the set of 17 codelines succeeds only in one case, the case of the elliptic quadric.

Theorem 4. An additive $[17, 13, 4]_4$ -code is uniquely determined.

Proof. The dual of our additive $[17, 13, 4]_4$ -code is a [17, 4]-code of strength 3. Because of Theorem 3 we can assume that it has distance ≤ 11 , meaning that at least some 6 codelines are on a hyperplane.

Assume there are 7 lines on a hyperplane. This would yield a (7, 11)-set of strength 3 in PG(6, 2) (see Definition 3). We saw in [4] that such a set does not exist. It follows that we must have a (6, 11)-set of strength 3 on a

hyperplane H. Those sets were determined by a computer program. There are two families of 6 lines each, the hexacode $[6, 3, 4]_4$ and another code. The hexacode yields 6 classes of (6, 11)-sets, the other family of six lines yields a unique (6, 11)-set. An exhaustive search showed that none of those sets could be completed to a set of 17 lines of strength 3. This shows that a $[17, 13, 4]_4$ -code is necessarily the dual of a $[17, 4, 12]_4$ -code.

4 Additive codes of larger length

Theorem 5. There is no additive $[18, 14, 4]_4$ -code.

Proof. Consider the dual, an additive $[18, 4]_4$ -code of strength 3. The first 17 of those lines describe a $[17, 4]_4$ -code of strength 3. By Theorem 4 this is the dual of the elliptic quadric code. It suffices therefore to start from this set of 17 lines and to check by a simple computer program that there is no line in PG(7, 2) which completes it to a $[18, 4]_4$ -code of strength 3.

Finally we construct an example of a $[22, 17.5, 4]_4$ -code. In fact we work with the dual, a $[22, 4.5]_4$ -code of strength 3. We work in PG(8, 2). The underlying vector space is V(9, 2) with basis e_1, \ldots, e_9 . Our code consists of the lines L_1, \ldots, L_{22} . It contains two copies of the hexacode, one consisting of

$$L_{1} = \langle e_{1}, e_{2} \rangle, L_{2} = \langle e_{3}, e_{4} \rangle, L_{3} = \langle e_{5}, e_{6} \rangle, L_{4} = \langle e_{2} + e_{3} + e_{6}, e_{1} + e_{2} + e_{4} + e_{5} \rangle,$$
$$L_{5} = \langle e_{2} + e_{4} + e_{5} + e_{6}, e_{1} + e_{3} + e_{5} \rangle, L_{6} = \langle e_{2} + e_{3} + e_{4} + e_{5}, e_{1} + e_{4} + e_{6} \rangle$$

forming the hexacode in the subspace $\langle e_1, \ldots, e_6 \rangle$, the second consisting of

$$L_1, L_2, L_7 = \langle e_2 + e_8, e_3 + e_4 + e_7 + e_8 \rangle, L_8 = \langle e_3 + e_4 + e_8, e_1 + e_3 + e_7 \rangle,$$

 $L_9 = \langle e_1 + e_2 + e_7, e_1 + e_3 + e_8 \rangle, L_{10} = \langle e_4 + e_7, e_1 + e_2 + e_7 + e_8 \rangle$ in the space $\langle e_1, e_2, e_3, e_4, e_7, e_8 \rangle$. Let *H* be the hyperplane $\langle e_1, \dots, e_8 \rangle$ gener-

ated by the hexacodes. One further codeline $L_{11} = \langle e_2 + e_3 + e_6 + e_8, e_3 + e_4 + e_5 + e_7 \rangle$ is contained in H, the remaining codelines are

1	L_{12}	L_{13}	L_{14}	L_{15}	L_{16}	L_{17}	L_{18}	L_{19}	L_{20}	L_{21}	L_{22}
	01	00	00	10	00	10	10	10	01	10	01
	11	01	01	00	00	00	00	11	10	10	01
	01	01	00	10	01	11	01	10	01	10	10
	10	00	00	10	11	10	10	00	10	00	00
	10	11	10	11	10	11	00	11	10	10	00
	10	10	11	01	00	10	10	11	00	01	10
	00	10	01	01	01	11	00	10	11	10	11
	10	01	10	10	10	10	11	10	00	01	01
	01	01	01	01	01	01	01	01	01	01	01/

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