# Properties of a Generalized Divergence Related to Tsallis Relative Entropy

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#### Abstract

In this paper, we investigate the partition inequality, joint convexity, and Pinsker's inequality, for a divergence that generalizes the Tsallis Relative Entropy and Kullback–Leibler divergence. The generalized divergence is defined in terms of a deformed exponential function, which replaces the Tsallis *q*-exponential. We also constructed a family of probability distributions related to the generalized divergence. We found necessary and sufficient conditions for the partition inequality to be satisfied. A sufficient condition for the joint convexity was established. We proved that the generalized divergence satisfies the partition inequality, and is jointly convex, if, and only if, it coincides with the Tsallis relative entropy. As an application of partition inequality, a criterion for the Pinsker's inequality was found.

## Index Terms

Kullback-Leibler divergence, Tsallis relative entropy, generalized divergence, family of probability distributions, partition inequality, joint convexity, Pinsker's inequality.

#### I. INTRODUCTION

Statistical divergences play an essential role in Information Theory [1]. Divergence can be interpreted as a measure of dissimilarity between two probability distributions. Applications that use it span from areas such as communications to econometric and other physical systems [1]. Entropy can be derived from the notion of divergence. Numerous definitions of divergence can be found in the literature. The interest in different statistical divergences is motivated by applications related to optimization and statistical learning, since more flexible functions and expressions may be suitable for larger classes of data and signals, leading to more efficient information recovery methods [2], [3], [4]. The divergence usefulness depends on its properties, such as non negativity, monotonicity and joint convexity, among others.

The counterpart of Shannon entropy is the well-known Kullback–Leibler (KL) divergence [5], denoted by  $D_{\text{KL}}(\cdot || \cdot)$ , which is extensively used in Information Theory. Tsallis relative entropy  $D_q(\cdot || \cdot)$ , which generalizes KL divergence, is defined in terms of the q-logarithm [6], [7]. Both KL divergence and Tsallis relative entropy satisfy some important properties, such as non negativity, joint convexity, and Pinsker's inequality [8], [9], [10]. A generalized divergence  $D_{\varphi}(\cdot || \cdot)$  can be defined in terms of a deformed exponential function  $\varphi$ , which plays the role of q-logarithm in Tsallis relative entropy. The generalized divergence appeared before in the literature, as a specific case in a broader class of divergences. Zhang in [11] introduced a divergence denoted by  $D_{f,\rho}^{(\alpha)}(\cdot || \cdot)$ , where  $\alpha \in [-1,1]$ , and f and  $\rho$  are functions. The generalized divergence corresponds to Zhang's divergence with  $\alpha = -1$ , and  $\rho = f^{-1} = \varphi^{-1}$  for a deformed exponential function  $\varphi$ . In [12], another class of divergences was investigated. The divergences  $D_{\beta}^{c}(\cdot || \cdot)$  in this class are given in terms of parameters  $\beta = (\phi, M_1, M_2, M_3, \lambda)$ . Expression (1) in [12], which defines  $D_{\beta}^{c}(\cdot || \cdot)$ , reduces to the generalized divergence, with  $\phi = -\varphi^{-1}$ ,  $M_1 = 1$ ,  $M_2 = 1$ ,  $M_3 = (\varphi^{-1})'(q)$ , and  $\lambda = \lambda_{\#}$  is the counting measure.

In [11], [12], the proposed divergences were investigated from a geometric and minimization perspectives. Some properties, which are useful in Information Theory, have not been analyzed for these divergences. In this work, we investigate the partition inequality, joint convexity, and Pinsker's inequality. We also consider the family of probability distributions associated with the generalized divergence  $D_{\varphi}(\cdot || \cdot)$ . We showed necessary and sufficient conditions for the generalized divergence to satisfy the partition inequality. A sufficient condition for the joint convexity of  $D_{\varphi}(\cdot || \cdot)$  was found. We proved that  $D_{\varphi}(\cdot || \cdot)$  satisfies the partition inequality, and is jointly convex, if, and only if, it coincides with the Tsallis relative entropy  $D_q(\cdot || \cdot)$ . Ours results for Pinsker's inequality are in accordance to previous works [13], [14].

The rest of paper is organized as follows. In Section II-A we provide the definition of generalized divergence. Section II-B is devoted to the construction of a family of probability distributions. Properties of the generalized divergence are studied in Section III. Finally, conclusions and perspectives are stated in Section IV.

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## II. GENERALIZED DIVERGENCE

The generalized divergence is defined in terms of a deformed exponential function  $\varphi(\cdot)$ . Writing the KL divergence or Tsallis relative entropy in appropriate form, we can obtain the generalized divergence by replacing  $\ln(\cdot)$  or  $\ln_q(\cdot)$  by the inverse of a deformed exponential  $\varphi^{-1}(\cdot)$ . We also provide a construction of a family of probability distributions related the generalized divergence.

#### A. Definitions

For simplicity we denote the set of all probability distributions on  $I_n = \{1, \ldots, n\}$  by

$$\Delta_n = \bigg\{ (p_1, \dots, p_n) : \sum_{i=1}^n p_i = 1 \text{ and } p_i \ge 0 \text{ for all } i \bigg\}.$$

The generalized divergence is defined for probability distribution in the interior of  $\Delta_n$ , which is denoted by  $\Delta_n^{\circ}$ . A probability distribution  $\mathbf{p} = (p_i)$  belongs to  $\Delta_n^{\circ}$  if and only if  $p_i > 0$  for each i,

A deformed exponential function is a convex function  $\varphi \colon \mathbb{R} \to [0, \infty)$  such that  $\lim_{u \to -\infty} \varphi(u) = 0$  and  $\lim_{u \to \infty} \varphi(u) = \infty$ . It is easy to verify that the ordinary exponential and Tsallis q-exponential are deformed exponential functions. The *Tsallis* q-exponential  $\exp_q \colon \mathbb{R} \to [0, \infty)$  is given by

$$\exp_q(x) = \begin{cases} [1 + (1 - q)x]_+^{1/(1 - q)}, & \text{if } q \in (0, 1], \\ \exp(x), & \text{if } q = 1, \end{cases}$$

where  $[x]_+ = x$  for  $x \ge 0$ , and = 0 otherwise. The *Tsallis q-logarithm*  $\ln_q : (0, \infty) \to \mathbb{R}$  is defined as the inverse of  $\exp_q(\cdot)$ , which is given by  $\ln_q(x) = \frac{1}{1-q}(x^{1-q}-1)$  if  $q \in (0,1]$ .

Fixed a deformed exponential function  $\varphi \colon \mathbb{R} \to [0, \infty)$ , the generalized divergence (or generalized relative entropy) between two probability distributions  $\mathbf{p} = (p_i)$  and  $\mathbf{q} = (q_i)$  in  $\Delta_n^\circ$  is defined as

$$D_{\varphi}(\boldsymbol{p} || \boldsymbol{q}) = \sum_{i=1}^{n} \frac{\varphi^{-1}(p_i) - \varphi^{-1}(q_i)}{(\varphi^{-1})'(p_i)}.$$
(1)

Clearly, expression (1) reduces to the KL divergence  $D_{\text{KL}}(\boldsymbol{p} || \boldsymbol{q}) = -\sum_{i=1}^{n} p_i \ln\left(\frac{q_i}{p_i}\right)$  if  $\varphi$  is the exponential function. Tsallis relative entropy in its standard form is given by  $D_q(\boldsymbol{p} || \boldsymbol{q}) = -\sum_{i=1}^{n} p_i \ln_q\left(\frac{q_i}{p_i}\right)$ . The equality

$$-p_i \frac{(q_i/p_i)^{1-q} - 1}{1-q} = \frac{1}{p_i^q} \left(\frac{p_i^{1-q} - 1}{1-q} - \frac{q_i^{1-q} - 1}{1-q}\right)$$

shows that  $D_q(\cdot \| \cdot)$  can be written as in (1) if  $\varphi$  is the Tsallis q-exponential.

The non-negativity of  $D_{\varphi}(\cdot \| \cdot)$  is a consequence of the concavity of  $\varphi^{-1}(\cdot)$ . Because  $\varphi^{-1}(\cdot)$  is concave, it follows that

$$(y-x)(\varphi^{-1})'(y) \le \varphi^{-1}(y) - \varphi^{-1}(x), \quad \text{for all } x, y > 0.$$
 (2)

Using this inequality with  $y = p_i$  and  $x = q_i$ , we can write

$$D_{\varphi}(\boldsymbol{p} \| \boldsymbol{q}) = \sum_{i=1}^{n} \frac{\varphi^{-1}(p_i) - \varphi^{-1}(q_i)}{(\varphi^{-1})'(p_i)} \ge \sum_{i=1}^{n} (p_i - q_i) = 0$$

Its is clear that  $D_{\varphi}(\boldsymbol{p} \parallel \boldsymbol{q}) = 0$  if  $\boldsymbol{p} = \boldsymbol{q}$ . The converse depends on whether  $\varphi^{-1}(x)$  is strictly concave. Indeed, if we suppose that  $\varphi^{-1}(x)$  is strictly concave, then an equality in equation (2) is attained if and only if x = y. Therefore, when  $\varphi^{-1}(x)$  is strictly concave, the equality  $D_{\varphi}(\boldsymbol{p} \parallel \boldsymbol{q}) = 0$  is satisfied if and only if  $\boldsymbol{p} = \boldsymbol{q}$ .

In addition to similarities between the generalized divergence, KL divergence, and Tsallis relative entropy, there exists another motivation for the choice of expression given as in (1). We can associate with the generalized relative entropy  $D_{\varphi}(\cdot \| \cdot)$  a  $\varphi$ -family of probability distributions, just as the KL divergence is related to the moment-generating function in a exponential family of probability distributions.

#### B. Families of probability distributions

For each probability distribution  $p = (p_i) \in \Delta_n^\circ$ , we can define a *deformed exponential family* (of probability distributions) *centered at* p. A deformed exponential family consists of a parameterization for the set  $\Delta_n^\circ$ . We remark that a deformed exponential family depends on the centered probability distribution p. We can associate with each probability distribution  $p \in \Delta_n^\circ$  a deformed exponential family centered at p.

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Assume that  $\varphi \colon \mathbb{R} \to [0, \infty)$  is a positive, deformed exponential function with continuous derivative. Fixed  $\mathbf{p} = (p_i) \in \Delta_n^\circ$ , let  $\mathbf{c} = (c_i)$  be a vector such that  $p_i = \varphi(c_i)$  for each *i*. We also fix a vector  $\mathbf{u}_0 = (u_{0i})$  such that  $u_{0i} > 0$  for each *i*, and

$$\sum_{i=1}^{n} u_{0i} \varphi'(c_i) = 1.$$
(3)

A deformed exponential family (of probability distributions) centered at p is a parameterization of  $\Delta_n^\circ$ , which maps each vector  $u = (u_i)$  in the subspace

$$B_{\boldsymbol{c}}^{\varphi} = \left\{ (u_1, \dots, u_n) : \sum_{i=1}^n u_i \varphi'(c_i) = 0 \right\}$$

to a probability distribution  $\boldsymbol{q} = (q_i) \in \Delta_n^\circ$  by the expression

$$q_i = \varphi(c_i + u_i - \psi_c(\boldsymbol{u})u_{0i}), \tag{4}$$

where  $\psi_c \colon B_c^{\varphi} \to [0,\infty)$  is the *normalizing function*, which is introduced so that (4) defines a probability density in  $\Delta_n^{\circ}$ .

The choice for  $u \in B_c^{\varphi}$  is not arbitrary. Thanks to this choice, it is possible to find  $\psi_c(u) \ge 0$  for which expression (4) is a probability density in  $\Delta_n^{\circ}$ . We will justify this claim. Because  $\varphi(\cdot)$  is convex, it follows that

$$y\varphi'(x) \le \varphi(x+y) - \varphi(x), \quad \text{for all } x, y \in \mathbb{R}.$$
 (5)

Using (5) with  $x = c_i$  and  $y = u_i$ , we can write, for any  $u \in B_c^{\varphi}$ ,

$$1 = \sum_{i=1}^{n} u_i \varphi'(c_i) + \sum_{i=1}^{n} \varphi(c_i) \le \sum_{i=1}^{n} \varphi(c_i + u_i).$$

By the definition of  $\varphi(\cdot)$ , the map

$$g(\lambda) = \sum_{i=1}^{n} \varphi(c_i + u_i - \lambda u_{0i})$$

is continuous, approaches 0 as  $\lambda \to \infty$ , and tends to  $\infty$  as  $\alpha \to \infty$ . Since  $\varphi(\cdot)$  is strictly increasing, it follows that  $g(\cdot)$  is strictly decreasing. Then we can conclude that there exists a unique  $\lambda_0 = \psi_c(u) \ge 0$  for which  $q_i = \varphi(c_i + u_i - \lambda_0 u_{0i})$  is a probability distribution in  $\Delta_n^\circ$ .

The generalized divergence  $D_{\varphi}(\cdot \| \cdot)$  is associated with the deformed exponential family (4) by the equality

$$\psi_{\boldsymbol{c}}(\boldsymbol{u}) = D_{\varphi}(\boldsymbol{p} \| \boldsymbol{q}) = \sum_{i=1}^{n} \frac{\varphi^{-1}(p_i) - \varphi^{-1}(q_i)}{(\varphi^{-1})'(p_i)}.$$
(6)

Using  $\sum_{i=1}^{n} u_i \varphi'(c_i) = 0$ , together with the constraint (3), we can write

$$\psi_{\boldsymbol{c}}(\boldsymbol{u}) = \sum_{i=1}^{n} (-u_i + \psi_{\boldsymbol{c}}(\boldsymbol{u})u_{0i})\varphi'(c_i).$$
(7)

It is clear that

$$-u_i + \psi_{\boldsymbol{c}}(\boldsymbol{u})u_{0i} = \varphi^{-1}(p_i) - \varphi^{-1}(q_i),$$
(8)

and

$$\varphi'(c_i) = \frac{1}{(\varphi^{-1})'(p_i)}.$$
(9)

Inserting (8) and (9) into (7), we obtain (6).

If  $\varphi$  is the exponential function, and  $u_{0i} = 1$ , the deformed exponential family reduces to the well known *exponential family*:

$$q_i = \exp(u_i - K_p(\boldsymbol{u})) \cdot p_i, \tag{10}$$

where  $K_p(u)$  is the *cumulant-generating function*, which equals the normalizing function  $\psi_c(u)$ .

## III. PROPERTIES OF THE GENERALIZED DIVERGENCE

The KL divergence and Tsallis relative entropy satisfy the partition inequality, and are jointly convex. They also satisfy Pinsker's inequality. We will investigate under what conditions these properties hold for the generalized divergence. Throughout this section we assume that  $(\varphi^{-1})''(x)$  is continuous and > 0.

# A. Partition inequality

Partition inequality, which is a case of the data processing inequality, will be used in the proof of Pinsker's inequality. Let  $\mathcal{A} = \{A_1, \ldots, A_k\}$  be a partition of  $I_n = \{1, \ldots, n\}$ , i.e.,  $\mathcal{A}$  is a collection of subsets  $A_j \subseteq I_n$  such that  $A_i \cap A_j = \emptyset$ , for  $i \neq j$ , and  $\bigcup_{j=1}^k A_j = I_n$ . For any probability distribution  $\mathbf{p} = (p_i)$ , we define the probability distribution  $\mathbf{p}^{\mathcal{A}} = (p_j^{\mathcal{A}})$  as

$$p_j^{\mathcal{A}} = \sum_{i \in A_j} p_i, \quad \text{for each } j = 1, \dots, k.$$

The next result gives a necessary and sufficient condition for the partition inequality to be satisfied.

**Proposition 1.** For the divergence  $D_{\varphi}(\cdot || \cdot)$  to satisfy the partition inequality

$$D_{\varphi}(\boldsymbol{p} \mid\mid \boldsymbol{q}) \ge D_{\varphi}(\boldsymbol{p}^{\mathcal{A}} \mid\mid \boldsymbol{q}^{\mathcal{A}}), \tag{11}$$

for all probability distributions  $\mathbf{p} = (p_i)$  and  $\mathbf{q} = (q_i)$ , and any partition  $\mathcal{A}$  of  $I_n$ , it is necessary and sufficient that the function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  be superadditive, i.e., the inequality

$$g(x+y) \ge g(x) + g(y), \tag{12}$$

be satisfied for all  $x, y \in (0, 1)$  such that  $x + y \in (0, 1)$ .

The proof of Proposition 1 requires some preliminary results which are presented in the sequel.

**Lemma 2.** Fix any  $\alpha \in (0, 1)$ . The mapping

$$F_{\alpha}(x,y) = \varphi((1-\alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y))$$

is superadditive in  $(0,1) \times (0,1)$  if, and only if,

$$G(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y))$$

is convex in  $\{(x, y) \in \mathbb{R}^2 : \varphi(x) + \varphi(y) \in (0, 1)\}.$ 

*Proof:* Let  $x_i, y_i \in (0, 1)$  be such that  $x_1 + x_2 \in (0, 1)$  and  $y_1 + y_2 \in (0, 1)$ . The superadditivity of  $F_{\alpha}$  implies that

$$\varphi((1-\alpha)\varphi^{-1}(x_1+x_2) + \alpha\varphi^{-1}(y_1+y_2)) \\ \ge \varphi((1-\alpha)\varphi^{-1}(x_1) + \alpha\varphi^{-1}(y_1)) \\ + \varphi((1-\alpha)\varphi^{-1}(x_2) + \alpha\varphi^{-1}(y_2)).$$
(13)

Denote  $s_i = \varphi^{-1}(x_i)$  and  $t_i = \varphi^{-1}(y_i)$  for i = 1, 2. Thus inequality equation (13) is equivalent to

$$(1-\alpha)\varphi^{-1}(\varphi(s_1)+\varphi(s_2))+\alpha\varphi^{-1}(\varphi(t_1)+\varphi(t_2))$$

$$\geq \varphi^{-1}[\varphi((1-\alpha)s_1+\alpha t_1)+\varphi((1-\alpha)s_2+\alpha t_2)],$$

which shows the desired result.

**Lemma 3.** The function G, as defined in Lemma 2, is convex if and only if  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  is superadditive in (0, 1).

*Proof:* For the function G to be convex, it is necessary and sufficient that its Hessian  $H_G$  be positive semi-definitive, which is equivalent to  $\operatorname{tr}(H_G) \ge 0$  and  $J_G = \det(H_G) \ge 0$ , where  $\operatorname{tr}(\cdot)$  denotes the trace of a matrix and  $\det(\cdot)$  is the determinant of a matrix (see [15]). Letting  $z = \varphi(x) + \varphi(y)$ , we can express

$$\frac{\partial^2 G}{\partial x^2}(x,y) = \varphi''(x)(\varphi^{-1})'(z) + [\varphi'(x)]^2(\varphi^{-1})''(z),$$
(14)

$$\frac{\partial^2 G}{\partial y^2}(x,y) = \varphi''(y)(\varphi^{-1})'(z) + [\varphi'(y)]^2(\varphi^{-1})''(z),$$
(15)

and

$$\frac{\partial^2 G}{\partial x \partial y}(x,y) = \varphi'(x)\varphi'(y)(\varphi^{-1})''(z).$$
(16)

If we divide the right-hand side of (14) by  $-\varphi''(x)(\varphi^{-1})''(z) \ge 0$ , and we use

$$\frac{[\varphi(x)']^2}{\varphi(x)''} = -\frac{(\varphi^{-1})'(\varphi(x))}{(\varphi^{-1})''(\varphi(x))}$$
(17)

into the resulting expression, we obtain

$$-\frac{(\varphi^{-1})'(z)}{(\varphi^{-1})''(z)} + \frac{(\varphi^{-1})'(\varphi(x))}{(\varphi^{-1})''(\varphi(x))} = g(z) - g(\varphi(x)).$$

As a result, we conclude that  $\partial^2 G/\partial x^2 \ge 0$  (and similarly  $\partial^2 G/\partial y^2 \ge 0$ ) if g is superadditive. Using expressions (14)–(16) for the partial derivatives of G, we find

$$J_G(x,y) = (\varphi^{-1})'(z)(\varphi^{-1})''(z)\varphi''(x)\varphi''(y)$$

 $\cdot \bigg\{ \frac{(\varphi^{-1})'(z)}{(\varphi^{-1})''(z)} + \frac{[\varphi'(y)]^2}{\varphi''(y)} + \frac{[\varphi'(x)]^2}{\varphi''(x)} \bigg\}.$ 

In view of (17), it follows that  $J_G(x, y) \ge 0$  is equivalent to  $g(z) \ge g(\varphi(x)) + g(\varphi(y))$ . Thus G is convex if and only if g is superadditive in (0, 1).

*Remark* 4. Similar versions of these lemmas appeared previously in the literature (see [16, sec. 3.16] and [17]). The hypothesis in these versions was weaker, or just one direction was proved.

Now, we may proceed to the proof of the main result in this section.

Proof of Proposition 1: Sufficiency. Lemmas 2 and 3 imply that

$$F_{\alpha}(x,y) = \varphi((1-\alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y)),$$

is superadditive in  $(0,1) \times (0,1)$ , for each  $\alpha \in (0,1)$ . Considering  $\mathcal{A} = \{A_1, \ldots, A_k\}$ , we denote  $p_j^{\mathcal{A}} = \sum_{i \in A_j} p_i$  and  $q_j^{\mathcal{A}} = \sum_{i \in A_j} q_i$ . By the superadditivity of  $F_{\alpha}(x, y)$ , we can write

$$\frac{1}{1-\alpha}\sum_{i=1}^{n}[p_i - F_{\alpha}(q_i, p_i)]$$

$$\geq \frac{1}{1-\alpha} \sum_{j=1}^{k} [p_j^{\mathcal{A}} - F_{\alpha}(q_j^{\mathcal{A}}, p_j^{\mathcal{A}})]. \quad (18)$$

An application of L'Hôpital's rule on the limit below provides

$$\lim_{\alpha \uparrow 1} \frac{y - F_{\alpha}(x, y)}{1 - \alpha} = \lim_{\alpha \uparrow 1} \frac{y - \varphi((1 - \alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y))}{1 - \alpha}$$
$$= \varphi'(\varphi^{-1}(y))[-\varphi^{-1}(x) + \varphi^{-1}(y)]$$
$$= \frac{\varphi^{-1}(y) - \varphi^{-1}(x)}{(\varphi^{-1})'(y)}.$$

Thus, in the limit  $\alpha \uparrow 1$ , expression (18) becomes

$$D_{\varphi}(\boldsymbol{p} \mid\mid \boldsymbol{q}) \geq D_{\varphi}(\boldsymbol{p}^{\mathcal{A}} \mid\mid \boldsymbol{q}^{\mathcal{A}}),$$

which is the asserted inequality.

*Necessity.* It is clear that if (11) holds for all  $p = (p_i)$ ,  $q = (q_i)$ , and A, then

$$\frac{\varphi^{-1}(p_1) - \varphi^{-1}(q_1)}{(\varphi^{-1})'(p_1)} + \frac{\varphi^{-1}(p_2) - \varphi^{-1}(q_2)}{(\varphi^{-1})'(p_2)} \ge \frac{\varphi^{-1}(p_1 + p_2) - \varphi^{-1}(q_1 + q_2)}{(\varphi^{-1})'(p_1 + p_2)} \quad (19)$$

is satisfied for all  $p_1, p_2$  and  $q_1, q_2$  in (0, 1) such that the sums  $p_1 + p_2$  and  $q_1 + q_2$  are in (0, 1). Let us fix  $p_1, p_2 \in (0, 1)$ . We rewrite (19) as

$$\frac{\varphi^{-1}(p_1)}{(\varphi^{-1})'(p_1)} + \frac{\varphi^{-1}(p_2)}{(\varphi^{-1})'(p_2)} - \frac{\varphi^{-1}(p_1 + p_2)}{(\varphi^{-1})'(p_1 + p_2)} \ge \frac{\varphi^{-1}(q_1)}{(\varphi^{-1})'(p_1)} + \frac{\varphi^{-1}(q_2)}{(\varphi^{-1})'(p_2)} - \frac{\varphi^{-1}(q_1 + q_2)}{(\varphi^{-1})'(p_1 + p_2)},$$

which is satisfied if and only if the function

$$F(q_1, q_2) = \frac{\varphi^{-1}(q_1)}{(\varphi^{-1})'(p_1)} + \frac{\varphi^{-1}(q_2)}{(\varphi^{-1})'(p_2)} - \frac{\varphi^{-1}(q_1 + q_2)}{(\varphi^{-1})'(p_1 + p_2)}$$

attains a global maximum at  $(q_1, q_2) = (p_1, p_2)$ . By a simple calculation, it can be verified that  $\nabla F(p_1, p_2) = 0$ . Moreover, we express the determinant of the Hessian of F at  $(p_1, p_2)$  as

$$J_F(p_1, p_2) = \frac{(\varphi^{-1})''(p_1)}{(\varphi^{-1})'(p_1)} \frac{(\varphi^{-1})''(p_2)}{(\varphi^{-1})'(p_2)} - \frac{(\varphi^{-1})''(p_1 + p_2)}{(\varphi^{-1})'(p_1 + p_2)} \frac{(\varphi^{-1})''(p_2)}{(\varphi^{-1})'(p_2)} - \frac{(\varphi^{-1})''(p_1)}{(\varphi^{-1})'(p_1)} \frac{(\varphi^{-1})''(p_1 + p_2)}{(\varphi^{-1})'(p_1 + p_2)} = \frac{1}{g(p_1)} \frac{1}{g(p_2)} - \frac{1}{g(p_1 + p_2)} \left[ \frac{1}{g(p_1)} + \frac{1}{g(p_2)} \right].$$

Because  $J_F(p_1, p_2) \ge 0$ , it follows that  $g(p_1 + p_2) \ge g(p_1) + g(p_2)$ . *Remark* 5. If  $\varphi(x) = \exp(x)$  the function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  is the identity function which is additive, therefore superadditive.

## B. Joint convexity

In this section, we find a sufficient condition for the joint convexity of  $D_{\varphi}(\cdot || \cdot)$ . We also show that  $D_{\varphi}(\cdot || \cdot)$  satisfies the partition inequality, and is jointly convex, if, and only if, the deformed exponential function is a scaled and translated version of the Tsallis exponential.

The generalized divergence  $D_{\varphi}(\cdot || \cdot)$  is said to be jointly convex if the inequality

$$D_{\varphi}(\lambda \boldsymbol{p}_{1} + (1-\lambda)\boldsymbol{p}_{2} || \lambda \boldsymbol{q}_{1} + (1-\lambda)\boldsymbol{q}_{2}) \leq \lambda D_{\varphi}(\boldsymbol{p}_{1} || \boldsymbol{q}_{1}) + (1-\lambda)D_{\varphi}(\boldsymbol{p}_{2} || \boldsymbol{q}_{2}) \quad (20)$$

is satisfied for all probability distributions  $p_1, p_2$  and  $q_1, q_2$  in  $\Delta_n^\circ$ , and each  $\lambda \in [0, 1]$ . Before we find a sufficient condition for the joint convexity of  $D_{\varphi}(\cdot || \cdot)$ , we show some preliminary results.

**Lemma 6.** The function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  is (strictly) concave if and only if  $h = \frac{\varphi'}{\varphi''}$  is (strictly) concave.

*Proof:* Inserting the expressions  $(\varphi^{-1})' = 1/\varphi'(\varphi^{-1})$  and

$$(\varphi^{-1})'' = -\frac{\varphi''(\varphi^{-1}) \cdot (\varphi^{-1})'}{[\varphi'(\varphi^{-1})]^2}$$

into the definition of g, we can write

$$g = \varphi'(\varphi^{-1}) \frac{\varphi'(\varphi^{-1})}{\varphi''(\varphi^{-1})} = \varphi'(\varphi^{-1})h(\varphi^{-1})$$

Some calculations show that

$$g'_{+} = 1 + h'_{+}(\varphi^{-1}),$$

where  $(\cdot)'_+$  denotes the right derivative. By the fact of  $\varphi^{-1}$  is strictly increasing, we conclude that  $g'_+$  is (strictly) decreasing if and only if  $h'_+$  is (strictly) decreasing. As a result, for g to be (strictly) concave, it is necessary and sufficient that h be (strictly) concave.

**Lemma 7.** The function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  is concave if and only if the mapping  $F_{\alpha}(x,y) = \varphi((1-\alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y)), \quad (x,y) \in \mathbb{R}^2,$ 

is concave for each  $\alpha \in (0, 1)$ .

*Proof:* Let us denote  $z_{\alpha} = (1 - \alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y)$ . Some calculations show that

$$\begin{aligned} \frac{\partial^2 F_\alpha}{\partial x^2}(x,y) &= (1-\alpha)(\varphi^{-1})''(x)\varphi'(z_\alpha) \\ &+ [(1-\alpha)(\varphi^{-1})'(x)]^2\varphi''(z_\alpha) \\ \frac{\partial^2 F_\alpha}{\partial y^2}(x,y) &= \alpha(\varphi^{-1})''(y)\varphi'(z_\alpha) \\ &+ [\alpha(\varphi^{-1})'(y)]^2\varphi''(z_\alpha), \end{aligned}$$

and

$$\frac{\partial^2 F_{\alpha}}{\partial x \partial y}(x,y) = \alpha (1-\alpha)(\varphi^{-1})'(x)(\varphi^{-1})'(y)\varphi''(z_{\alpha}),$$

which we use to find the following expression for the determinant of the Hessian of  $F_{\alpha}$  at (x, y):

$$J_{F_{\alpha}}(x,y) = \alpha(1-\alpha)\varphi'(z_{\alpha})\varphi''(z_{\alpha})(\varphi^{-1})''(x)(\varphi^{-1})''(y) \\ \cdot \left\{ \frac{\varphi'(z_{\alpha})}{\varphi''(z_{\alpha})} + \alpha \frac{[(\varphi^{-1})'(y)]^2}{(\varphi^{-1})''(y)} + (1-\alpha)\frac{[(\varphi^{-1})'(x)]^2}{(\varphi^{-1})''(x)} \right\}.$$

Denote  $h = \varphi'/\varphi''$ . Noticing that

$$\frac{[(\varphi^{-1})']^2}{(\varphi^{-1})''} = -\frac{\varphi'(\varphi^{-1})}{\varphi''(\varphi^{-1})},$$
(21)

we conclude that  $J_{F_{\alpha}}(x,y) \ge 0$  is equivalent to  $h(z_{\alpha}) \ge (1-\alpha)h(\varphi^{-1}(x)) + \alpha h(\varphi^{-1}(y))$ .

To show that the Hessian of  $F_{\alpha}$  is negative semi-definitive, we have to verify, in addition, that its trace is non-positive. Since h is concave and non-negative, we have

$$\frac{\varphi'(z_{\alpha})}{\varphi''(z_{\alpha})} - (1-\alpha)\frac{\varphi'(\varphi^{-1}(x))}{\varphi''(\varphi^{-1}(x))} \ge 0.$$
(22)

If we insert (21) into (22), and multiply the resulting expression by  $(1 - \alpha)\varphi''(z_{\alpha})(\varphi^{-1})''(x) \leq 0$ , we get

$$\frac{\partial^2 F_{\alpha}}{\partial x^2}(x,y) = (1-\alpha)(\varphi^{-1})''(x)\varphi'(z_{\alpha})$$

+  $[(1 - \alpha)(\varphi^{-1})'(x)]^2 \varphi''(z_\alpha) \le 0.$ 

Analogously, we also have  $(\partial^2 F_{\alpha}/\partial x^2)(x,y) \leq 0$ . Consequently, the Hessian of  $F_{\alpha}$  has a negative trace.

From Lemma 6, it follows that g is concave if and only if  $F_{\alpha}$  is concave for each  $\alpha \in (0, 1)$ .

**Proposition 8.** If the function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  is concave, then the divergence  $D_{\varphi}(\cdot || \cdot)$  is jointly convex.

Proof: According to Lemma 7, the mapping

$$F_{\alpha}(x,y) = \varphi((1-\alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y)), \quad (x,y) \in \mathbb{R}^2,$$

is concave for each  $\alpha \in (0,1)$ . Fixed an arbitrary  $p_j = (p_{ji})$  and  $q_j = (q_{ji})$  in  $\Delta_n^{\circ}$  for j = 0, 1, define

$$\begin{aligned} \boldsymbol{p}_{\lambda} &= (1-\lambda)\boldsymbol{p}_0 + \lambda \boldsymbol{p}_1, \\ \boldsymbol{q}_{\lambda} &= (1-\lambda)\boldsymbol{q}_0 + \lambda \boldsymbol{q}_1, \end{aligned}$$

for each  $\lambda \in (0, 1)$ . Hence we can write

$$\frac{1}{1-\alpha} \sum_{i=1}^{n} [p_{\lambda i} - F_{\alpha}(q_{\lambda i}, p_{\lambda i})] \\ \leq (1-\lambda) \frac{1}{1-\alpha} \sum_{i=1}^{n} [p_{0i} - F_{\alpha}(q_{0i}, p_{0i})] \\ + \lambda \frac{1}{1-\alpha} \sum_{i=1}^{n} [p_{1i} - F_{\alpha}(q_{1i}, p_{1i})].$$
(23)

Using L'Hôpital's rule in the limit below, we obtain

$$\lim_{\alpha \uparrow 1} \frac{y - F_{\alpha}(x, y)}{1 - \alpha} = \lim_{\alpha \uparrow 1} \frac{y - \varphi((1 - \alpha)\varphi^{-1}(x) + \alpha\varphi^{-1}(y))}{1 - \alpha}$$
$$= \varphi'(\varphi^{-1}(y))[-\varphi^{-1}(x) + \varphi^{-1}(y)]$$
$$= \frac{\varphi^{-1}(y) - \varphi^{-1}(x)}{(\varphi^{-1})'(y)}.$$

Thus, in the limit  $\alpha \uparrow 1$ , expression (23) becomes

$$D_{\varphi}(\boldsymbol{p}_{\lambda} \mid\mid \boldsymbol{q}_{\lambda}) \leq (1 - \lambda) D_{\varphi}(\boldsymbol{p}_{0} \mid\mid \boldsymbol{q}_{0}) + \lambda D_{\varphi}(\boldsymbol{p}_{1} \mid\mid \boldsymbol{q}_{1})$$

which is the desired result.

The next result is a partial converse of Proposition 8.

**Lemma 9.** If the divergence  $D_{\varphi}(\cdot || \cdot)$  is jointly convex for some  $n \ge 3$ , then the function  $g = -\frac{(\varphi^{-1})'}{(\varphi^{-1})''}$  satisfies the inequality

$$g\left(\frac{x+y}{2}\right) \ge \frac{g(x)+g(y)}{2},\tag{24}$$

for all  $x, y \in (0, 1)$  such that  $x + y \in (0, 1)$ .

*Proof:* If  $p_1 = (p_{1i})$ ,  $p_2 = (p_{2i})$  and  $q_1 = (q_{1i})$ ,  $q_2 = (q_{2i})$  then inequality (20) is equivalent to

$$\sum_{i=1}^{n} \left[ \lambda \frac{\varphi^{-1}(p_{1i})}{(\varphi^{-1})'(p_{1i})} + (1-\lambda) \frac{\varphi^{-1}(p_{2i})}{(\varphi^{-1})'(p_{2i})} - \frac{\varphi^{-1}(\lambda p_{1i} + (1-\lambda)p_{2i})}{(\varphi^{-1})'(\lambda p_{1i} + (1-\lambda)p_{2i})} \right] \\ \ge \sum_{i=1}^{n} \left[ \lambda \frac{\varphi^{-1}(q_{1i})}{(\varphi^{-1})'(p_{1i})} + (1-\lambda) \frac{\varphi^{-1}(q_{2i})}{(\varphi^{-1})'(p_{2i})} - \frac{\varphi^{-1}(\lambda q_{1i} + (1-\lambda)q_{2i})}{(\varphi^{-1})'(\lambda p_{1i} + (1-\lambda)p_{2i})} \right].$$
(25)

For the fixed probability distributions

$$\boldsymbol{p}_1 = (p_1, p_2, p, p_{13}, \dots, p_{1n}),$$
 (26)

$$\boldsymbol{p}_2 = (p_2, p_1, p, p_{23}, \dots, p_{2n}),$$
 (27)

in  $\Delta_n^\circ$ , we consider

$$\boldsymbol{q}_1 = (p_1 + x, p_2 + y, p - x - y, p_{13}, \dots, p_{1n}), \tag{28}$$

$$\boldsymbol{q}_2 = (p_2 + y, p_1 + x, p - x - y, p_{23}, \dots, p_{2n}), \tag{29}$$

where x and y are taken so that  $q_1$  and  $q_2$  are in  $\Delta_n^{\circ}$ . Inserting these probability distributions into (25) with  $\lambda = 1/2$ , we can infer that the function

$$\begin{split} F(x,y) &= \frac{1}{2} \frac{\varphi^{-1}(p_1+x)}{(\varphi^{-1})'(p_1)} + \frac{1}{2} \frac{\varphi^{-1}(p_2+y)}{(\varphi^{-1})'(p_2)} \\ &\quad - \frac{\varphi^{-1}(\frac{1}{2}(p_1+x) + \frac{1}{2}(p_2+y))}{(\varphi^{-1})'(\frac{1}{2}p_1 + \frac{1}{2}p_2)} + \frac{1}{2} \frac{\varphi^{-1}(p_2+y)}{(\varphi^{-1})'(p_2)} \\ &\quad + \frac{1}{2} \frac{\varphi^{-1}(p_1+x)}{(\varphi^{-1})'(p_1)} - \frac{\varphi^{-1}(\frac{1}{2}(p_2+y) + \frac{1}{2}(p_1+x))}{(\varphi^{-1})'(\frac{1}{2}p_2 + \frac{1}{2}p_1)} \end{split}$$

attains a global maximum at (x, y) = (0, 0). Further, we can also write

$$J_F(0,0) = \left[\frac{1}{g(p_1)} - \frac{1}{2}\frac{1}{g(\frac{1}{2}p_1 + \frac{1}{2}p_2)}\right] \\ \cdot \left[\frac{1}{g(p_2)} - \frac{1}{2}\frac{1}{g(\frac{1}{2}p_1 + \frac{1}{2}p_2)}\right] - \left[\frac{1}{2}\frac{1}{g(\frac{1}{2}p_1 + \frac{1}{2}p_2)}\right]^2 \\ = \frac{1}{g(p_1)}\frac{1}{g(p_2)} - \frac{1}{g(\frac{1}{2}p_1 + \frac{1}{2}p_2)}\left[\frac{1}{2}\frac{1}{g(p_2)} + \frac{1}{2}\frac{1}{g(p_1)}\right],$$

where  $J_F(0,0)$  is the determinant of the Hessian of F at (0,0). Since F(x,y) attains a maximum at (0,0), inequality  $J_F(0,0) \ge 0$  implies  $g(\frac{1}{2}p_1 + \frac{1}{2}p_2) \ge \frac{1}{2}g(p_1) + \frac{1}{2}g(p_2)$ .

**Proposition 10.** Assume that  $n \ge 3$ . Then the generalized divergence  $D_{\varphi}(\cdot || \cdot)$  satisfies the partition inequality, and is jointly convex, if, and only if,

$$\varphi^{-1}(x) = b \ln_q(x) - a, \quad \text{for } x \in (0, 1),$$

for some q > 0 and b > 0,  $a \in \mathbb{R}$ .

*Proof:* Clearly, inequalities (12) and (24) are satisfied for all  $x, y \in (0, 1]$ . Therefore, the function g(x) is superadditive and concave for  $x \in (0, 1/2)$ . It is easy to verify that g(0+) = 0. To see this, we apply the limit  $x \downarrow 0$  in  $0 \le g(x) \le g(x+y) - g(y)$ , and use the continuity of g at y. In addition, because g(x) is concave with g(0+) = 0, the function g(x) is also subadditive

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for  $x \in (0, 1/2)$ . Making  $y \downarrow 0$  in  $g(\lambda x + (1 - \lambda)y) \ge \lambda g(x) + (1 - \lambda)g(y)$ , we obtain that  $g(\lambda x) \ge \lambda g(x)$  for  $\lambda \in [0, 1]$ . From the inequalities  $g(x) \ge \frac{x}{x+y}g(x+y)$  and  $g(y) \ge \frac{y}{x+y}g(x+y)$ , it follows that  $g(x) + g(y) \ge g(x+y)$ . Hence we conclude that g(x) is additive for  $x \in (0, 1/2)$ .

By [18, Theorem 13.5.2], there exists q > 0 such that g(x) = x/q for  $x \in (0, 1/2)$ . Using (12), and letting  $y \downarrow 0$  in (24), we get

$$g(x) \ge g\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right), \quad \text{and} \quad g\left(\frac{x}{2}\right) \ge \frac{g(x)}{2}$$

which imply g(x) = 2g(x/2) for all  $x \in (0,1)$ . Hence, expression g(x) = x/q is also verified for  $x \in (0,1)$ . Solving

$$g(x) = -\frac{(\varphi^{-1})'(x)}{(\varphi^{-1})''(x)} = \frac{x}{q}$$

with respect to  $\varphi^{-1}(x)$ , we find b > 0 and  $a \in \mathbb{R}$  such that

$$\varphi^{-1}(x) = b \frac{x^{1-q}}{1-q} - a$$
$$= b \ln_q(x) - a, \qquad \text{for } q \neq 1,$$

and

$$\varphi^{-1}(x) = b \ln(x) - a,$$
 for  $q = 1,$ 

for every  $x \in (0, 1)$ .

The converse direction follows from Propositions 1 and 8.

# C. Pinsker's inequality

Pinsker's inequality relates the divergence with the  $\ell_1$ -distance. This inequality implies that convergence in divergence is stronger than convergence in the  $\ell_1$ -distance For the KL divergence, Pinsker's inequality is given by

$$D_{\rm KL}(\boldsymbol{p} || \boldsymbol{q}) \ge \frac{1}{2} || \boldsymbol{p} - \boldsymbol{q} ||_1^2,$$
 (30)

where  $\|\boldsymbol{p} - \boldsymbol{q}\|_1 = \sum_{i=1}^n |p_i - q_i|$  is the  $\ell_1$ -distance between probability distributions  $\boldsymbol{p} = (p_i)$  and  $\boldsymbol{q} = (q_i)$  in  $\Delta_n^\circ$ . The next result shows Pinsker's inequality for the generalized divergence.

**Theorem 11** (Pinsker's Inequality). Suppose that the partition inequality (11) holds. In addition, assume that

$$c = \inf_{0 0.$$
(31)

Then, for any probability distributions  $\mathbf{p} = (p_i)$  and  $\mathbf{q} = (q_i)$  in  $\Delta_n^{\circ}$ , the generalized divergence satisfies the inequality

$$D_{\varphi}(\boldsymbol{p} \mid\mid \boldsymbol{q}) \ge c \|\boldsymbol{p} - \boldsymbol{q}\|_{1}^{2}.$$
(32)

*Proof:* Let  $\mathcal{A} = \{A_1, A_2\}$  be a partition of  $I_n$ , where  $A_1 = \{i : p_i \ge q_i\}$  and  $A_2 = \{i : p_i < q_i\}$ . Hence we can write

$$|\boldsymbol{p} - \boldsymbol{q}||_1 = \sum_{i=1}^n |p_i - q_i|$$
  
=  $\sum_{i \in A_1} (p_i - q_i) + \sum_{i \in A_2} (q_i - p_i)$   
=  $(p_1^{\mathcal{A}} - q_1^{\mathcal{A}}) + (q_2^{\mathcal{A}} - p_2^{\mathcal{A}})$   
=  $\|\boldsymbol{p}^{\mathcal{A}} - \boldsymbol{q}^{\mathcal{A}}\|_1.$ 

By the partition inequality

$$D_{\varphi}(\boldsymbol{p} \mid\mid \boldsymbol{q}) \geq D_{\varphi}(\boldsymbol{p}^{\mathcal{A}} \mid\mid \boldsymbol{q}^{\mathcal{A}}),$$

we see that it suffices to show

$$D_{\varphi}(\boldsymbol{p}^{\mathcal{A}} \mid\mid \boldsymbol{q}^{\mathcal{A}}) \ge c \mid\mid \boldsymbol{p}^{\mathcal{A}} - \boldsymbol{q}^{\mathcal{A}} \mid\mid_{1}^{2}.$$
(33)

Let us denote  $p_1^{\mathcal{A}} = p$  and  $q_1^{\mathcal{A}} = q$ . Then inequality (33) can be rewritten as

$$\frac{\varphi^{-1}(p) - \varphi^{-1}(q)}{(\varphi^{-1})'(p)} + \frac{\varphi^{-1}(1-p) - \varphi^{-1}(1-q)}{(\varphi^{-1})'(1-p)} \ge 4c(p-q)^2,$$

since  $\|\boldsymbol{p}^{\mathcal{A}} - \boldsymbol{q}^{\mathcal{A}}\|_1 = 2(p-q)$ . For a fixed  $p \in (0,1)$ , we define the function

$$F(q) = \frac{\varphi^{-1}(p) - \varphi^{-1}(q)}{(\varphi^{-1})'(p)}$$

+ 
$$\frac{\varphi^{-1}(1-p)-\varphi^{-1}(1-q)}{(\varphi^{-1})'(1-p)} - 4c(p-q)^2$$
,

for  $q \in (0, 1)$ . By the symmetry of the terms p and q in (31), it is clear that

$$c = \inf_{0 < q < p < 1} \frac{1}{8} \frac{1}{q - p} \left[ -\frac{(\varphi^{-1})'(q)}{(\varphi^{-1})'(p)} + \frac{(\varphi^{-1})'(1 - q)}{(\varphi^{-1})'(1 - p)} \right] > 0.$$

As a result, the derivative

$$F'(q) = (q-p) \left\{ \frac{1}{q-p} \left[ -\frac{(\varphi^{-1})'(q)}{(\varphi^{-1})'(p)} + \frac{(\varphi^{-1})'(1-q)}{(\varphi^{-1})'(1-p)} \right] - 8c \right\}$$

is  $\geq 0$  for q > p, and  $\leq 0$  for q < p. We conclude that F(q) attains a minimum at q = p. Therefore,

$$D_{\varphi}(\boldsymbol{p}^{\mathcal{A}} \mid\mid \boldsymbol{q}^{\mathcal{A}}) - c \mid\mid \boldsymbol{p}^{\mathcal{A}} - \boldsymbol{q}^{\mathcal{A}} \mid\mid_{1}^{2} = F(q) \ge F(p) = 0,$$

and inequality (32) follows.

If we assume  $\varphi^{-1}(x) = \log(x)$ , then expression (31) results in c = 1/2, which is the constant in Pinsker's inequality for the KL divergence. For the Tsallis exponential, an easy computation shows that c = q/2 in equation (31) with  $\varphi^{-1}(x) = \ln_q(x)$ . This result is in accordance to the work of Gilardoni [13], which investigated the Pinsker's inequality for f-divergences. Gilardoni showed that the f-divergence  $D_f(\mathbf{p} || \mathbf{q}) = \sum_{i=1}^n p_i f(\frac{q_i}{p_i})$  satisfies the inequality  $D_f(\mathbf{p} || \mathbf{q}) \ge \frac{f''(1)}{2} ||\mathbf{p} - \mathbf{q}||_1^2$ , supposing that f is convex and three times differentiable at x = 1 with f''(1) > 0. Tsallis relative entropy is an f-divergence with  $f(x) = -\ln_q(x)$ . In this case, we have f''(1) = q.

# **IV. CONCLUSIONS**

In this work, we found necessary and sufficient conditions for the generalized divergence  $D_{\varphi}(\cdot || \cdot)$  to satisfy the partition inequality. We also showed a condition that implies the joint convexity of  $D_{\varphi}(\cdot || \cdot)$ . It was proved that, for the generalized divergence  $D_{\varphi}(\cdot || \cdot)$  to coincide with the Tsallis relative entropy  $D_q(\cdot || \cdot)$ , it is necessary and sufficient that  $D_{\varphi}(\cdot || \cdot)$  satisfy the partition inequality, and be jointly convex. As an application of partition inequality, a criterion for the Pinsker's inequality was found. We also constructed a family of probability distributions associated with the generalized divergence.

This work can be extended in many aspects. The data processing inequality was not proved. Comparisons between generalized divergences, as investigated in [19] for f-divergences, have the potential of being a prosperous topic of research. In [20], a generalization of Rényi divergence was defined in terms of a deformed exponential. As future work, we aim to investigate the properties of this generalized Rényi divergence.

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