

Several classes of minimal linear codes with few weights from weakly regular plateaued functions

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Abstract

Minimal linear codes have significant applications in secret sharing schemes and secure two-party computation. There are several methods to construct linear codes, one of which is based on functions over finite fields. Recently, many construction methods of linear codes based on functions have been proposed in the literature. In this paper, we generalize the recent construction methods given by Tang et al. in [IEEE Transactions on Information Theory, 62(3), 1166-1176, 2016] to weakly regular plateaued functions over finite fields of odd characteristic. We first construct three weight linear codes from weakly regular plateaued functions based on the second generic construction and determine their weight distributions. We next give a subcode with two or three weights of each constructed code as well as its parameter. We finally show that the constructed codes in this paper are minimal, which confirms that the secret sharing schemes based on their dual codes have the nice access structures.

Keywords Linear codes, minimal codes, weight distribution, weakly regular plateaued functions, cyclotomic fields, secret sharing schemes.

1 Introduction

Linear codes have diverse applications in secret sharing schemes, authentication codes, communication, data storage devices, consumer electronics, association schemes, strongly regular graphs and secure two-party computation. Indeed, as a special class of linear codes, minimal linear codes have significant applications in secret sharing schemes and secure two-party computation. Constructing linear codes with perfect parameters, an interesting research topic in cryptography and coding theory, has been widely studied in the literature. There are several methods to construct linear codes, one of which is based on functions over

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finite fields (see *e.g.* [6, 7, 9, 15, 20, 22]). Two generic constructions (say, *first* and *second*) of linear codes from functions have been distinguished from the others in the literature. Recently, several constructions of linear codes based on the second generic construction have been proposed and many linear codes with perfect parameters have been constructed. In fact, Ding has come up with an interesting survey [6] devoted to the construction of binary linear codes from Boolean functions based on the second generic construction. Bent functions (mostly, quadratic and weakly regular bent functions) have been extensively used to construct linear codes with few weights. It was shown in a few papers (see *e.g.* [9, 20, 22]) that bent functions lead to the construction of interesting linear codes with few weights based on the second generic construction. Indeed, Tang et al. (2016) have constructed in [20] two or three weight linear codes from weakly regular bent functions over finite fields of odd characteristic based on the second generic construction. This inspires us to construct linear codes from weakly regular plateaued functions over finite fields of odd characteristic. Within this framework, to construct new linear codes with few weights, we aim to make use of weakly regular plateaued functions for the first time in the second generic construction.

The paper is structured as follows. Section 2 sets main notations and gives background in finite field theory and coding theory. In Section 3, we present some results related to weakly regular plateaued functions, which will be needed to construct linear codes. In Section 4, we construct two or three weight linear codes by using some weakly regular plateaued functions over finite fields of odd characteristic based on the second generic construction. We also determine the weight distributions of the constructed codes. Finally, in Section 5, we analyze the minimality of the constructed codes and hence observe that all nonzero codewords of the constructed codes are minimal for almost all cases.

2 Preliminaries

In this section, after setting basic notations, we mention the connection between linear codes and secret sharing schemes. We end this section by recording some properties of weakly regular plateaued functions.

2.1 Basic notations

Herein after, we fix the following notations unless otherwise stated.

- For any set E , $\#E$ denotes the cardinality of E and $E^* = E \setminus \{0\}$,
- \mathbb{Z} is the ring of integers and \mathbb{Q} is the field of rational numbers,
- $|z|$ denotes the magnitude of $z \in \mathbb{C}$, where \mathbb{C} is the field of complex numbers,
- p is an odd prime and $q = p^n$ is an n -th power of p with n being a positive integer,
- \mathbb{F}_q is the finite field with q elements and $\mathbb{F}_q^* = \langle \zeta \rangle$ is a cyclic group with generator ζ ,

- The trace of $\alpha \in \mathbb{F}_q$ over \mathbb{F}_p is defined by $\text{Tr}^n(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{n-1}}$,
- $\xi_p = e^{2\pi i/p}$ is the complex primitive p -th root of unity, where $i = \sqrt{-1}$ is the complex primitive 4-th root of unity,
- SQ and NSQ denote the set of all squares and non-squares in \mathbb{F}_p^* , respectively,
- η and η_0 are the quadratic characters of \mathbb{F}_q^* and \mathbb{F}_p^* ,
- $p^* = \eta_0(-1)p = (-1)^{\frac{p-1}{2}}p$. Notice that $p^n = \eta_0^n(-1)\sqrt{p^{*2n}}$.

Cyclotomic field $\mathbb{Q}(\xi_p)$. A cyclotomic field $\mathbb{Q}(\xi_p)$ is obtained from the field \mathbb{Q} by adjoining ξ_p . The ring of integers in $\mathbb{Q}(\xi_p)$ is defined as $\mathcal{O}_{\mathbb{Q}(\xi_p)} := \mathbb{Z}(\xi_p)$. An integral basis of $\mathcal{O}_{\mathbb{Q}(\xi_p)}$ is the set $\{\xi_p^i : 1 \leq i \leq p-1\}$. The field extension $\mathbb{Q}(\xi_p)/\mathbb{Q}$ is Galois of degree $p-1$, and the Galois group $\text{Gal}(\mathbb{Q}(\xi_p)/\mathbb{Q}) = \{\sigma_a : a \in \mathbb{F}_p^*\}$, where the automorphism σ_a of $\mathbb{Q}(\xi_p)$ is defined by $\sigma_a(\xi_p) = \xi_p^a$. The cyclotomic field $\mathbb{Q}(\xi_p)$ has a unique quadratic subfield $\mathbb{Q}(\sqrt{p^*})$. For $a \in \mathbb{F}_p^*$, we have $\sigma_a(\sqrt{p^*}) = \eta_0(a)\sqrt{p^*}$. Hence, the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) = \{1, \sigma_\gamma\}$, where $\gamma \in NSQ$. For $a \in \mathbb{F}_p^*$ and $b \in \mathbb{F}_p$, we clearly have $\sigma_a(\xi_p^b) = \xi_p^{ab}$ and $\sigma_a(\sqrt{p^{*n}}) = \eta_0^n(a)\sqrt{p^{*n}}$, which will be needed to prove our subsequent results. The reader is referred to [13] for further reading on cyclotomic fields.

2.2 Linear codes and Secret sharing schemes

A linear code \mathcal{C} of length n and dimension k over \mathbb{F}_p is a k -dimensional linear subspace of an n -dimensional vector space \mathbb{F}_p^n , which can be viewed as an extension field \mathbb{F}_{p^n} . An element of \mathcal{C} is said to be *codeword*. The Hamming weight of a vector $\mathbf{a} = (a_0, \dots, a_{n-1}) \in \mathbb{F}_p^n$, denoted by $wt(\mathbf{a})$, is the cardinality of its support defined as

$$\text{supp}(\mathbf{a}) = \{0 \leq i \leq n-1 : a_i \neq 0\}.$$

The minimum Hamming distance of \mathcal{C} is the minimum Hamming weight of its nonzero codewords. A linear code \mathcal{C} of length n and dimension k over \mathbb{F}_p with minimum Hamming distance d is denoted by $[n, k, d]_p$. Note that d detects the error correcting capability of \mathcal{C} . Let A_w denote the number of codewords with Hamming weight w in \mathcal{C} of length n . Then, $(1, A_1, \dots, A_n)$ is the *weight distribution* of \mathcal{C} and the polynomial $1 + A_1y + \cdots + A_ny^n$ is called the *weight enumerator* of \mathcal{C} . The code \mathcal{C} is called a *t-weight code* if the number of nonzero A_w in the weight distribution is t .

We now state the covering problem of a linear code \mathcal{C} . We say that a codeword \mathbf{a} of \mathcal{C} covers another codeword \mathbf{b} of \mathcal{C} if $\text{supp}(\mathbf{a})$ contains $\text{supp}(\mathbf{b})$. A nonzero codeword \mathbf{a} of \mathcal{C} is said to be *minimal* if \mathbf{a} covers only the codeword $j\mathbf{a}$ for every $j \in \mathbb{F}_p$, but no other nonzero codewords of \mathcal{C} . A linear code \mathcal{C} is said to be *minimal* if every nonzero codeword of \mathcal{C} is minimal. Determining the minimality of a linear code over finite fields has been an attractive research topic in coding theory. The *covering problem* of a linear code \mathcal{C} is to

find all minimal codewords of \mathcal{C} . This problem is rather difficult for general linear codes, but easy for a few special linear codes.

The following lemma says that all nonzero codewords of \mathcal{C} are minimal if the Hamming weights of the nonzero codewords of \mathcal{C} are too close to each other. The reader also notices that a necessary and sufficient condition on minimal linear codes over finite fields has been presented in [11, Theorem 11]. Indeed, it is worth noting that [11] provides an infinite family of minimal linear codes violating the following sufficient condition.

Lemma 1. (*Ashikhmin-Barg*) [1] *All nonzero codewords of a linear code \mathcal{C} over \mathbb{F}_p are minimal if*

$$\frac{p-1}{p} < \frac{w_{\min}}{w_{\max}},$$

where w_{\min} and w_{\max} denote the minimum and maximum weights of nonzero codewords in \mathcal{C} , respectively.

In a secret sharing scheme, a set of participants who can recover the secret value s from their shares is called *an access set*. The set of all access sets is said to be *the access structure* of a secret sharing scheme. An access set is said to be *a minimal access set* if any of its proper subsets cannot recover s from their shares. We take only an interest in the set of all minimal access sets, which is said to be *a nice access structure*.

From [3, Lemma 16], there is a one-to-one correspondence between the set of minimal access sets of the secret sharing scheme based on \mathcal{C} and the set of minimal codewords of the dual code \mathcal{C}^\perp . Hence, to find the minimal access sets of the secret sharing scheme based on \mathcal{C} , it is sufficient to find the minimal codewords of \mathcal{C}^\perp whose first coordinate is 1.

The following proposition describes the access structure of the secret sharing scheme based on a dual code of a minimal linear code.

Proposition 1. [3, 8] *Let \mathcal{C} be a minimal linear $[n, k, d]_p$ code over \mathbb{F}_p with the generator matrix $G = [\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{n-1}]$. We denote by d^\perp the minimum Hamming distance of its dual code \mathcal{C}^\perp . Then in the secret sharing scheme based on \mathcal{C}^\perp , the number of participants is $n-1$, and there exist p^{k-1} minimal access sets.*

- *If $d^\perp = 2$, the access structure is given as follows. If \mathbf{g}_i , $1 \leq i \leq n-1$, is a multiple of \mathbf{g}_0 , then participant P_i must be in all minimal access sets. If \mathbf{g}_i , $1 \leq i \leq n-1$, is not a multiple of \mathbf{g}_0 , then P_i must be in $(p-1)p^{k-2}$ out of p^{k-1} minimal access sets.*
- *If $d^\perp \geq 3$, for any fixed $1 \leq t \leq \min\{k-1, d^\perp-2\}$, every set of t participants is involved in $(p-1)^t p^{k-(t+1)}$ out of p^{k-1} minimal access sets.*

In the following subsection, we introduce some results on weakly regular plateaued functions.

2.3 Weakly regular plateaued functions

Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a p -ary function. The Walsh transform of f is given by:

$$\widehat{\chi}_f(\beta) = \sum_{x \in \mathbb{F}_q} \xi_p^{f(x) - \text{Tr}^n(\beta x)}, \quad \beta \in \mathbb{F}_q.$$

A function f is said to be *balanced* over \mathbb{F}_p if f takes every value of \mathbb{F}_p the same number of p^{n-1} times, in other words, $\widehat{\chi}_f(0) = 0$; otherwise, f is called *unbalanced*. Notice that f can be recovered from $\widehat{\chi}_f$ by the inverse Walsh transform:

$$\xi_p^{f(x)} = \frac{1}{p^n} \sum_{\beta \in \mathbb{F}_q} \widehat{\chi}_f(\beta) \xi_p^{\text{Tr}^n(\beta x)}. \quad (1)$$

Bent functions, introduced first in characteristic 2 by Rothaus [19] in 1976, are the functions whose Walsh coefficients satisfy $|\widehat{\chi}_f(\beta)|^2 = p^n$. A bent function f is called *regular bent* if for every $\beta \in \mathbb{F}_q$, $p^{-\frac{n}{2}} \widehat{\chi}_f(\beta) = \xi_p^{f^*(\beta)}$ for some p -ary function $f^* : \mathbb{F}_q \rightarrow \mathbb{F}_p$, and f is called *weakly regular bent* if there exist a complex number u with $|u| = 1$ and a p -ary function f^* such that $u p^{-\frac{n}{2}} \widehat{\chi}_f(\beta) = \xi_p^{f^*(\beta)}$ for all $\beta \in \mathbb{F}_q$. Notice that f^* is also a weakly regular bent function. As an extension of bent functions, the notion of plateaued functions was introduced first in characteristic 2 by Zheng and Zhang [21] in 1999. A function f is said to be p -ary s -plateaued if $|\widehat{\chi}_f(\beta)|^2 \in \{0, p^{n+s}\}$ for every $\beta \in \mathbb{F}_q$, where s is an integer with $0 \leq s \leq n$. The Walsh support of plateaued f is defined by $\text{Supp}(\widehat{\chi}_f) = \{\beta \in \mathbb{F}_q : |\widehat{\chi}_f(\beta)|^2 = p^{n+s}\}$. The Parseval identity is given by $\sum_{\beta \in \mathbb{F}_q} |\widehat{\chi}_f(\beta)|^2 = p^{2n}$. The absolute Walsh distribution of plateaued functions follows from the Parseval identity.

Lemma 2. [17] *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be an s -plateaued function. Then for $\beta \in \mathbb{F}_q$, $|\widehat{\chi}_f(\beta)|^2$ takes p^{n-s} times the value p^{n+s} and $p^n - p^{n-s}$ times the value 0.*

Definition 1. [18] *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a p -ary s -plateaued function, where s is an integer with $0 \leq s \leq n$. Then, f is called *weakly regular p -ary s -plateaued* if there exists a complex number u having unit magnitude (in fact, $|u| = 1$ and u does not depend on β) such that*

$$\widehat{\chi}_f(\beta) \in \left\{ 0, u p^{\frac{n+s}{2}} \xi_p^{g(\beta)} \right\}$$

for all $\beta \in \mathbb{F}_q$, where g is a p -ary function over \mathbb{F}_q with $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$; otherwise, f is called *non-weakly regular p -ary s -plateaued*. In particular, weakly regular f is said to be *regular* if $u = 1$.

Lemma 3. [18] *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a weakly regular s -plateaued function. For all $\beta \in \text{Supp}(\widehat{\chi}_f)$, we have $\widehat{\chi}_f(\beta) = \epsilon \sqrt{p^{*n+s}} \xi_p^{g(\beta)}$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$ and g is a p -ary function over $\text{Supp}(\widehat{\chi}_f)$.*

The following lemma can be derived from Lemma 3.

Lemma 4. *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a weakly regular s -plateaued function. Then for $x \in \mathbb{F}_q$,*

$$\sum_{\beta \in \text{Supp}(\widehat{\chi}_f)} \xi_p^{g(\beta) + \text{Tr}^n(\beta x)} = \epsilon \eta_0^n(-1) \sqrt{p^{*n-s}} \xi_p^{f(x)},$$

where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$ and g is a p -ary function over \mathbb{F}_q with $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$.

Proof. By the inverse Walsh transform in (1), we have

$$\xi_p^{f(x)} = \frac{1}{p^n} \sum_{\beta \in \mathbb{F}_q} \widehat{\chi}_f(\beta) \xi_p^{\text{Tr}^n(\beta x)} = \frac{1}{p^n} \sum_{\beta \in \text{Supp}(\widehat{\chi}_f)} \epsilon \sqrt{p^{*n+s}} \xi_p^{g(\beta)} \xi_p^{\text{Tr}^n(\beta x)},$$

where we used that $\widehat{\chi}_f(\beta) = 0$ for all $\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$. Hence, we get

$$\sum_{\beta \in \text{Supp}(\widehat{\chi}_f)} \xi_p^{g(\beta) + \text{Tr}^n(\beta x)} = \frac{1}{\sqrt{p^{*n+s}}} \epsilon p^n \xi_p^{f(x)} = \epsilon \eta_0^n(-1) \sqrt{p^{*n-s}} \xi_p^{f(x)},$$

where we used in the last equality that $p^n = \eta_0^n(-1) \sqrt{p^{*2n}}$. □

The following lemma is a direct consequence of Lemmas 2 and 4.

Lemma 5. *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a weakly regular s -plateaued function with $\widehat{\chi}_f(\beta) = \epsilon \sqrt{p^{*n+s}} \xi_p^{g(\beta)}$ for every $\beta \in \text{Supp}(\widehat{\chi}_f)$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$ and g is a p -ary function over \mathbb{F}_q with $g(\beta) = 0$ for all $\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$. Then, we get $\widehat{\chi}_g(0) = p^n - p^{n-s} + \epsilon \eta_0^n(-1) \sqrt{p^{*n-s}} \xi_p^{f(0)}$.*

Remark 1. Notice that Lemma 5 confirms that g can not be balanced.

We now define the subset of the set of weakly regular plateaued functions, which is going to be used to construct linear codes. Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a weakly regular p -ary s -plateaued unbalanced function, where $0 \leq s \leq n$, and we denote by WRP the set of such functions satisfying the following two properties:

P1) $f(0) = 0$, and

P2) there exists a positive even integer h with $\gcd(h-1, p-1) = 1$ such that $f(ax) = a^h f(x)$ for any $a \in \mathbb{F}_p^*$ and $x \in \mathbb{F}_q$.

Lemma 6. *Let $f \in WRP$. Then for any $\beta \in \text{Supp}(\widehat{\chi}_f)$ (resp., $\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$), we have $z\beta \in \text{Supp}(\widehat{\chi}_f)$ (resp., $z\beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f)$) for every $z \in \mathbb{F}_p^*$.*

Proof. For every $z \in \mathbb{F}_p^*$ and $\beta \in \mathbb{F}_q$, we have

$$\widehat{\chi}_f(z\beta) = \sum_{x \in \mathbb{F}_q} \xi_p^{f(x) - \text{Tr}^n(z\beta x)} = \sum_{x \in \mathbb{F}_q} \xi_p^{f(z^k x) - z \text{Tr}^n(\beta z^k x)} = \sum_{x \in \mathbb{F}_q} \xi_p^{z^l f(x) - z^l \text{Tr}^n(\beta x)},$$

where k is a positive odd integer such that $k(h-1) \equiv 1 \pmod{p-1}$ and $l = k+1$, and is

$$\sigma_{z^l} \left(\sum_{x \in \mathbb{F}_q} \xi_p^{f(x) - \text{Tr}^n(\beta x)} \right) = \sigma_{z^l}(\widehat{\chi}_f(\beta)) = \begin{cases} 0, & \text{if } \beta \in \mathbb{F}_q \setminus \text{Supp}(\widehat{\chi}_f), \\ \epsilon \sqrt{p}^{*n+s} \xi_p^{z^l g(\beta)}, & \text{if } \beta \in \text{Supp}(\widehat{\chi}_f), \end{cases}$$

where we used in the last equality that $\eta_0^{n+s}(z^l) = 1$. Hence, the proof is complete. \square

Lemma 6 implies that there exists a subset P_S of the Walsh support of $f \in WRP$ such that $\text{Supp}(\widehat{\chi}_f) = \mathbb{F}_p^* P_S = \{z\beta : z \in \mathbb{F}_p^* \text{ and } \beta \in P_S\}$, where for each pair of distinct elements $\beta_1, \beta_2 \in P_S$ we have $\frac{\beta_1}{\beta_2} \notin \mathbb{F}_p^*$.

We now give a brief introduction to the quadratic functions (see *e.g.* [10]). Recall that any quadratic function from \mathbb{F}_{p^n} to \mathbb{F}_p having no linear term can be represented by

$$Q(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \text{Tr}^n(a_i x^{p^i+1}), \quad (2)$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $a_i \in \mathbb{F}_{p^n}$ for $0 \leq i \leq \lfloor n/2 \rfloor$. Let A be a corresponding $n \times n$ symmetric matrix with $Q(x) = x^T A x$ defined in [10] and L be a corresponding linearized polynomial over \mathbb{F}_{p^n} defined as

$$L(z) = \sum_{i=0}^l (a_i z^{p^i} + a_i^{p^{n-i}} z^{p^{n-i}}).$$

The set of linear structures of quadratic function Q is the kernel of L , defined as

$$\ker_{\mathbb{F}_p}(L) = \{z \in \mathbb{F}_{p^n} : Q(z+y) = Q(z) + Q(y), \forall y \in \mathbb{F}_{p^n}\}, \quad (3)$$

which is an \mathbb{F}_p -linear subspace of \mathbb{F}_{p^n} . Let the dimension of $\ker_{\mathbb{F}_p}(L)$ be s with $0 \leq s \leq n$. Notice that by [12, Proposition 2.1], the rank of A is equal to $n-s$. It was shown in [10] that a quadratic function Q is bent if and only if $s=0$; equivalently, A is nonsingular, i.e., has full rank. Hence, we have the following natural consequence (see *e.g.* [10, Proposition 2] and [17, Example 1]).

Proposition 2. *Any quadratic function Q is s -plateaued if and only if the dimension of the kernel of L defined in (3) is equal to s ; equivalently, the rank of A is $n-s$.*

Indeed, from [10, Proposition 1] and [4, Theorem 4.3] we have also the following reasonable fact.

Proposition 3. *The sign of Walsh transform of quadratic functions does not depend on inputs which means that any quadratic function is weakly regular plateaued. Namely, there is no quadratic non-weakly regular plateaued functions.*

Remark 2. All quadratic unbalanced functions defined in (2) are in the set WRP. Hence, all of these functions can be used to construct linear codes in this paper.

Example 1. The function $f : \mathbb{F}_{3^4} \rightarrow \mathbb{F}_3$ defined as $f(x) = \text{Tr}^4(\zeta x^{10} + \zeta^{51} x^4 + \zeta^{68} x^2)$, where $\mathbb{F}_{3^4}^* = \langle \zeta \rangle$ with $\zeta^4 + 2\zeta^3 + 2 = 0$, is the quadratic 2-plateaued unbalanced function in the set WRP with

$$\widehat{\chi}_f(\beta) \in \{0, \epsilon \eta_0^3(-1) 3^3 \xi_3^{g(\beta)}\} = \{0, -27, -27\xi_3, -27\xi_3^2\}$$

for all $\beta \in \mathbb{F}_{3^4}$, where $\epsilon = 1$, $\eta_0(-1) = -1$ and g is an unbalanced 3-ary function with $g(0) = 0$, $g(\zeta^2) = g(\zeta^{33}) = g(\zeta^{42}) = g(\zeta^{73}) = 1$ and $g(\zeta^5) = g(\zeta^6) = g(\zeta^{45}) = g(\zeta^{46}) = 2$. Clearly, we have $\text{Supp}(\widehat{\chi}_f) = \{0, \zeta^2, \zeta^5, \zeta^6, \zeta^{33}, \zeta^{42}, \zeta^{45}, \zeta^{46}, \zeta^{73}\}$ and $P_S = \{0, \zeta^2 = 2\zeta^{42}, \zeta^5 = 2\zeta^{45}, \zeta^6 = 2\zeta^{46}, \zeta^{33} = 2\zeta^{73}\}$.

We conclude this section by recording the following necessary results.

Lemma 7. [20] *Let $a_i, b_i \in \mathbb{Z}$ for every $i \in \mathbb{F}_p$ such that $\sum_{i=0}^{p-1} a_i \equiv \sum_{i=0}^{p-1} b_i \pmod{2}$ and $\sum_{i=0}^{p-1} a_i \xi_p^i \equiv \sum_{i=0}^{p-1} b_i \xi_p^i \pmod{2}$. Then, $a_i \equiv b_i \pmod{2}$ for every $i \in \mathbb{F}_p$.*

Lemma 8. [14] *For $a \in \mathbb{F}_q^*$, we have $\sum_{x \in \mathbb{F}_q} \xi_p^{\text{Tr}^n(ax^2)} = (-1)^{n-1} \eta(a) \sqrt{p^{*n}}$. In particular (in case of $n = 1$), for $a \in \mathbb{F}_p^*$, we have*

$$\sum_{x \in \mathbb{F}_p} \xi_p^{ax^2} = \begin{cases} \sqrt{p^*}, & a \in SQ, \\ -\sqrt{p^*}, & a \in NSQ. \end{cases}$$

Proposition 4. *Let $f \in WRP$, then $g(0) = 0$.*

Proof. From Lemmas 7 and 8, the proof can be completed by using the same argument used in the proof of [20, Proposition 4]. \square

One can immediately observe the following proposition from the proof of Lemma 6.

Proposition 5. *Let $f \in WRP$, then there exists a positive even integer l with $\gcd(l - 1, p - 1) = 1$ such that $g(a\beta) = a^l g(\beta)$ for any $a \in \mathbb{F}_p^*$ and $\beta \in \text{Supp}(\widehat{\chi}_f)$.*

Lemma 9. [14] *We have*

- i.) $\sum_{a \in \mathbb{F}_p^*} \eta_0(a) = 0$,
- ii.) $\sum_{a \in \mathbb{F}_p^*} \eta_0(a) \xi_p^a = \sqrt{p^*} = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod{4}, \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod{4}, \end{cases}$
- iii.) $\sum_{a \in \mathbb{F}_p^*} \xi_p^{ab} = -1$ for any $b \in \mathbb{F}_p^*$.

3 Exponential sums from weakly regular plateaued functions

In this section, we present some results on exponential sums related to weakly regular plateaued functions, which are going to be needed in Section 4 to construct linear codes. Actually, these results were given in [20] for weakly regular bent functions; however, for the sake of completeness, we state them translated into our framework and give their proofs.

We begin this section with the following lemma, which will be used to find the length of a linear code.

Lemma 10. *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be an unbalanced function with $\widehat{\chi}_f(0) = \epsilon\sqrt{p^{*n+s}}$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. For $j \in \mathbb{F}_p$, define $\mathcal{N}_f(j) = \#\{x \in \mathbb{F}_q : f(x) = j\}$. Then, if $n + s$ is even,*

$$\mathcal{N}_f(j) = \begin{cases} p^{n-1} + \epsilon\eta_0(-1)(p-1)\sqrt{p^{*n+s-2}}, & \text{if } j = 0, \\ p^{n-1} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}, & \text{if } j \in \mathbb{F}_p^*, \end{cases}$$

$$\text{if } n + s \text{ is odd, } \mathcal{N}_f(j) = \begin{cases} p^{n-1}, & \text{if } j = 0, \\ p^{n-1} + \epsilon\sqrt{p^{*n+s-1}}, & \text{if } j \in SQ, \\ p^{n-1} - \epsilon\sqrt{p^{*n+s-1}}, & \text{if } j \in NSQ. \end{cases}$$

Proof. Clearly, we have $\sum_{j=0}^{p-1} \mathcal{N}_f(j)\xi_p^j = \epsilon\sqrt{p^{*n+s}}$; namely, $\sum_{j=0}^{p-1} \mathcal{N}_f(j)\xi_p^j - \epsilon\sqrt{p^{*n+s}} = 0$. If $n + s$ is even, then for all $j \in \mathbb{F}_p^*$ we have $\mathcal{N}_f(j) = a$ and $\mathcal{N}_f(0) = a + \epsilon\sqrt{p^{*n+s}}$ for some constant integer a since $\sum_{j=0}^{p-1} x^j$ is the minimal polynomial of ξ_p over \mathbb{Q} . Hence, since $\sum_{j=0}^{p-1} \mathcal{N}_f(j) = p^n$, we get $a + \epsilon\sqrt{p^{*n+s}} + (p-1)a = p^n$ from which we deduce that $a = p^{n-1} - \epsilon\eta_0^{(n+s)/2}(-1)p^{(n+s-2)/2}$. If $n + s$ is odd, then we get

$$\sum_{j=0}^{p-1} \mathcal{N}_f(j)\xi_p^j - \epsilon\sqrt{p^{*n+s-1}} \sum_{j=1}^{p-1} \eta_0(j)\xi_p^j = 0,$$

where we used the fact that $\sum_{j=1}^{p-1} \eta_0(j)\xi_p^j = \sqrt{p^*}$ by Lemma 9 (ii), equivalently,

$$\mathcal{N}_f(0) + \sum_{j=1}^{p-1} \left(\mathcal{N}_f(j) - \epsilon\eta_0(j)\sqrt{p^{*n+s-1}} \right) \xi_p^j = 0.$$

As in the even case, for all $j \in \mathbb{F}_p^*$ we have $\mathcal{N}_f(j) = \mathcal{N}_f(0) + \epsilon\eta_0(j)\sqrt{p^{*n+s-1}}$ and hence $p\mathcal{N}_f(0) + \epsilon\sqrt{p^{*n+s-1}} \sum_{j=1}^{p-1} \eta_0(j) = p^n$. Then we get $\mathcal{N}_f(0) = p^{n-1}$ by Lemma 9 (i). Thus, the proof is ended. \square

The following lemma has a crucial role in determining the weight distributions of a linear code.

Lemma 11. Let $f \in WRP$ with $\widehat{\chi}_f(\beta) = \epsilon\sqrt{p^{*n+s}}\xi_p^{g(\beta)}$ for every $\beta \in \text{Supp}(\widehat{\chi}_f)$. For $j \in \mathbb{F}_p$, define $\mathcal{N}_g(j) = \#\{\beta \in \text{Supp}(\widehat{\chi}_f) : g(\beta) = j\}$. Then, if $n-s$ is even,

$$\mathcal{N}_g(j) = \begin{cases} p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)(p-1)\sqrt{p^{*n-s-2}}, & \text{if } j = 0, \\ p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}}, & \text{if } j \in \mathbb{F}_p^*, \end{cases}$$

$$\text{if } n-s \text{ is odd, } \mathcal{N}_g(j) = \begin{cases} p^{n-s-1}, & \text{if } j = 0, \\ p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}}, & \text{if } j \in SQ, \\ p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}}, & \text{if } j \in NSQ. \end{cases}$$

Proof. By Lemma 4, for $x = 0$, we have $\sum_{\beta \in \text{Supp}(\widehat{\chi}_f)} \xi_p^{g(\beta)} = \epsilon\eta_0^n(-1)\sqrt{p^{*n-s}}$, equivalently,

$$\sum_{j=0}^{p-1} \mathcal{N}_g(j)\xi_p^j - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s}} = 0.$$

If $n-s$ is even, then for all $j \in \mathbb{F}_p^*$ we have $\mathcal{N}_g(j) = a$ and $\mathcal{N}_g(0) = a + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s}}$ for some constant integer a . Hence, since $\#\text{Supp}(\widehat{\chi}_f) = p^{n-s}$, we get

$$\sum_{j=0}^{p-1} \mathcal{N}_g(j) = a + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s}} + (p-1)a = p^{n-s}.$$

If $n-s$ is odd, with the same argument used in the proof of Lemma 10, we get $\mathcal{N}_g(j) = \mathcal{N}_g(0) + \epsilon\eta_0^n(-1)\eta_0(j)\sqrt{p^{*n-s-1}}$ for all $j \in \mathbb{F}_p^*$ and hence

$$p\mathcal{N}_g(0) + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \sum_{j=1}^{p-1} \eta_0(j) = p^{n-s}.$$

Then we conclude $\mathcal{N}_g(0) = p^{n-s-1}$ from Lemma 9 (i), which completes the proof. \square

Lemma 12. Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be an unbalanced function with $\widehat{\chi}_f(0) = \epsilon\sqrt{p^{*n+s}}$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. Then,

$$\sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{yf(x)} = \begin{cases} \epsilon(p-1)\sqrt{p^{*n+s}}, & \text{if } n+s \text{ is even,} \\ 0, & \text{if } n+s \text{ is odd.} \end{cases}$$

Proof. From the definition, one can immediately observe that

$$\sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{yf(x)} = \sum_{y \in \mathbb{F}_p^*} \sigma_y \left(\sum_{x \in \mathbb{F}_q} \xi_p^{f(x)} \right) = \sum_{y \in \mathbb{F}_p^*} \sigma_y \left(\epsilon\sqrt{p^{*n+s}} \right) = \epsilon\sqrt{p^{*n+s}} \sum_{y \in \mathbb{F}_p^*} \eta_0^{n+s}(y).$$

Hence, the assertion follows clearly from Lemma 9 (i). \square

Lemma 13. Let $f \in WRP$. For $\beta \in \mathbb{F}_q^*$, define $A = \sum_{y,z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{yf(x) - z\text{Tr}^n(\beta x)}$. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$, we have $A = 0$, and for every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, if $n + s$ is even, then

$$A = \begin{cases} \epsilon(p-1)^2 \sqrt{p^{*n+s}}, & \text{if } g(\beta) = 0, \\ -\epsilon(p-1) \sqrt{p^{*n+s}}, & \text{if } g(\beta) \neq 0, \end{cases}$$

if $n + s$ is odd, then $A = \begin{cases} 0, & \text{if } g(\beta) = 0, \\ \epsilon \eta_0(g(\beta))(p-1) \sqrt{p^{*n+s+1}}, & \text{if } g(\beta) \neq 0. \end{cases}$

Proof. By Lemma 6, for every $\beta \in \text{Supp}(\widehat{\chi}_f)$, we have $\widehat{\chi}_f(z\beta) = \epsilon \sqrt{p^{*n+s}} \xi_p^{g(z\beta)}$ for any $z \in \mathbb{F}_p^*$. Then for every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, we have

$$\begin{aligned} A &= \sum_{y,z \in \mathbb{F}_p^*} \sigma_y \left(\sum_{x \in \mathbb{F}_q} \xi_p^{f(x) - \text{Tr}^n(z\beta x)} \right) = \sum_{y,z \in \mathbb{F}_p^*} \sigma_y(\widehat{\chi}_f(z\beta)) \\ &= \sum_{y,z \in \mathbb{F}_p^*} \sigma_y(\epsilon \sqrt{p^{*n+s}} \xi_p^{g(z\beta)}) \\ &= \sum_{y,z \in \mathbb{F}_p^*} \sigma_y(\epsilon \sqrt{p^{*n+s}} \xi_p^{z^l g(\beta)}) \\ &= \epsilon(p-1) \sqrt{p^{*n+s}} \sum_{y \in \mathbb{F}_p^*} \eta_0^{n+s}(y) \xi_p^{yg(\beta)}, \end{aligned}$$

where we used Proposition 5 in the fourth equality and used the fact that z^l is a square in \mathbb{F}_p^* for any $z \in \mathbb{F}_p^*$ in the last equality. When $n + s$ is even, $A = \epsilon(p-1) \sqrt{p^{*n+s}} \sum_{y \in \mathbb{F}_p^*} \xi_p^{yg(\beta)}$, which is $\epsilon(p-1)^2 \sqrt{p^{*n+s}}$ if $g(\beta) = 0$; otherwise, $-\epsilon(p-1) \sqrt{p^{*n+s}}$ from Lemma 9 (iii). When $n + s$ is odd, we clearly have $A = 0$ if $g(\beta) = 0$; otherwise, conclude that

$$A = \epsilon(p-1) \sqrt{p^{*n+s}} \sigma_{g(\beta)} \left(\sum_{y \in \mathbb{F}_p^*} \eta_0(y) \xi_p^y \right) = \epsilon \eta_0(g(\beta))(p-1) \sqrt{p^{*n+s+1}},$$

where we used Lemma 9 (ii). For every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$, we immediately get $A = 0$. Hence, the proof is complete. \square

The following lemma has a significant role in finding the Hamming weights of the codewords of a linear code.

Lemma 14. Let $f \in WRP$ with $\widehat{\chi}_f(0) = \epsilon \sqrt{p^{*n+s}}$. For $\beta \in \mathbb{F}_q^*$, define $\mathcal{N}_{f,\beta} = \#\{x \in \mathbb{F}_q : f(x) = 0 \text{ and } \text{Tr}^n(\beta x) = 0\}$. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$,

$$\mathcal{N}_{f,\beta} = \begin{cases} p^{n-2} + \epsilon(p-1) \sqrt{p^{*n+s-4}}, & \text{if } n + s \text{ is even,} \\ p^{n-2}, & \text{if } n + s \text{ is odd,} \end{cases}$$

and for every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, if $n + s$ is even, then

$$\mathcal{N}_{f,\beta} = \begin{cases} p^{n-2} + \epsilon \eta_0(-1)(p-1)\sqrt{p^{*n+s-2}}, & \text{if } g(\beta) = 0, \\ p^{n-2}, & \text{if } g(\beta) \neq 0, \end{cases}$$

$$\text{if } n + s \text{ is odd, then } \mathcal{N}_{f,\beta} = \begin{cases} p^{n-2}, & \text{if } g(\beta) = 0, \\ p^{n-2} + \epsilon(p-1)\sqrt{p^{*n+s-3}}, & \text{if } g(\beta) \in SQ, \\ p^{n-2} - \epsilon(p-1)\sqrt{p^{*n+s-3}}, & \text{if } g(\beta) \in NSQ. \end{cases}$$

Proof. By the definition of $\mathcal{N}_{f,\beta}$, one can observe that

$$\begin{aligned} \mathcal{N}_{f,\beta} &= p^{-2} \sum_{x \in \mathbb{F}_q} \left(\sum_{y \in \mathbb{F}_p} \xi_p^{yf(x)} \right) \left(\sum_{z \in \mathbb{F}_p} \xi_p^{-z \text{Tr}^n(\beta x)} \right) \\ &= p^{n-2} + p^{-2} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{yf(x)} + p^{-2} \sum_{y, z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{yf(x) - z \text{Tr}^n(\beta x)}. \end{aligned}$$

Hence, the proof is concluded from Lemmas 12 and 13. \square

The following can be immediately observed as in Lemma 12.

Lemma 15. *Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be an unbalanced function with $\widehat{\chi}_f(0) = \epsilon \sqrt{p^{*n+s}}$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. Then, we have*

$$\sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{y^2 f(x)} = \epsilon(p-1)\sqrt{p^{*n+s}}.$$

Lemma 16. *Let $f \in WRP$. For $\beta \in \mathbb{F}_q^*$, define $A = \sum_{y, z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \xi_p^{y^2 f(x) - z \text{Tr}^n(\beta x)}$. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$, we have $A = 0$, and for every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, we have*

$$A = \begin{cases} \epsilon(p-1)^2 \sqrt{p^{*n+s}}, & \text{if } g(\beta) = 0, \\ \epsilon(p-1)\sqrt{p^{*n+s}}(\sqrt{p^*} - 1), & \text{if } g(\beta) \in SQ, \\ -\epsilon(p-1)\sqrt{p^{*n+s}}(\sqrt{p^*} + 1), & \text{if } g(\beta) \in NSQ. \end{cases}$$

Proof. As in the proof of Lemma 13, we get $A = \sum_{y, z \in \mathbb{F}_p^*} \sigma_{y^2}(\widehat{\chi}_f(z\beta))$ for every $\beta \in \mathbb{F}_q^*$. For every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$, we clearly have $A = 0$. For every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, we get

$$\begin{aligned} A &= \sum_{y, z \in \mathbb{F}_p^*} \sigma_{y^2}(\epsilon \sqrt{p^{*n+s}} \xi_p^{g(z\beta)}) = \sum_{y, z \in \mathbb{F}_p^*} \sigma_{y^2}(\epsilon \sqrt{p^{*n+s}} \xi_p^{z^l g(\beta)}) \\ &= \epsilon \sqrt{p^{*n+s}} \sum_{y, z \in \mathbb{F}_p^*} \sigma_{y^2}(\xi_p^{z^l g(\beta)}) \\ &= \epsilon \sqrt{p^{*n+s}} (p-1) \sum_{y \in \mathbb{F}_p^*} \xi_p^{y^2 g(\beta)} \\ &= \epsilon \sqrt{p^{*n+s}} (p-1) \sum_{y \in \mathbb{F}_p} \xi_p^{y^2 g(\beta)} - \epsilon \sqrt{p^{*n+s}} (p-1), \end{aligned}$$

where in the second equality we used Proposition 5 and in the fourth equality we used the fact that $y^2 z^l$ runs through all squares in \mathbb{F}_p^* when y ranges over \mathbb{F}_p^* for any fixed $z \in \mathbb{F}_p^*$. Hence the assertion follows from Lemma 8. \square

The following lemma has a significant role in finding the Hamming weights of the codewords of a linear code.

Lemma 17. *Let $f \in WRP$. For $\beta \in \mathbb{F}_q^*$, define*

$$\begin{aligned}\mathcal{N}_{sq,\beta} &= \#\{x \in \mathbb{F}_q : f(x) \in SQ \text{ and } \text{Tr}^n(\beta x) = 0\}, \\ \mathcal{N}_{nsq,\beta} &= \#\{x \in \mathbb{F}_q : f(x) \in NSQ \text{ and } \text{Tr}^n(\beta x) = 0\}.\end{aligned}$$

Then, for every $\beta \in \mathbb{F}_q^ \setminus \text{Supp}(\widehat{\chi}_f)$, if $n+s$ is even, $\mathcal{N}_{sq,\beta} = \mathcal{N}_{nsq,\beta} = \frac{p-1}{2} (p^{n-2} - \epsilon \sqrt{p^{*n+s-4}})$, if $n+s$ is odd,*

$$\begin{aligned}\mathcal{N}_{sq,\beta} &= \frac{p-1}{2} \left(p^{n-2} + \epsilon \eta_0(-1) \sqrt{p^{*n+s-3}} \right), \\ \mathcal{N}_{nsq,\beta} &= \frac{p-1}{2} \left(p^{n-2} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-3}} \right).\end{aligned}$$

For every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, if $n+s$ is even, then

$$\begin{aligned}\mathcal{N}_{sq,\beta} &= \begin{cases} \frac{p-1}{2} \left(p^{n-2} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right), & \text{if } g(\beta) = 0 \text{ or } g(\beta) \in NSQ, \\ \frac{p-1}{2} \left(p^{n-2} + \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right), & \text{if } g(\beta) \in SQ, \end{cases} \\ \mathcal{N}_{nsq,\beta} &= \begin{cases} \frac{p-1}{2} \left(p^{n-2} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right), & \text{if } g(\beta) = 0 \text{ or } g(\beta) \in SQ, \\ \frac{p-1}{2} \left(p^{n-2} + \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right), & \text{if } g(\beta) \in NSQ, \end{cases}\end{aligned}$$

if $n+s$ is odd, then

$$\begin{aligned}\mathcal{N}_{sq,\beta} &= \begin{cases} \frac{p-1}{2} \left(p^{n-2} + \epsilon \sqrt{p^{*n+s-1}} \right), & \text{if } g(\beta) = 0, \\ \frac{p-1}{2} \left(p^{n-2} - \epsilon \sqrt{p^{*n+s-3}} \right), & \text{if } g(\beta) \in SQ, \\ \frac{p-1}{2} \left(p^{n-2} + \epsilon \sqrt{p^{*n+s-3}} \right), & \text{if } g(\beta) \in NSQ, \end{cases} \\ \mathcal{N}_{nsq,\beta} &= \begin{cases} \frac{p-1}{2} \left(p^{n-2} - \epsilon \sqrt{p^{*n+s-1}} \right), & \text{if } g(\beta) = 0, \\ \frac{p-1}{2} \left(p^{n-2} - \epsilon \sqrt{p^{*n+s-3}} \right), & \text{if } g(\beta) \in SQ, \\ \frac{p-1}{2} \left(p^{n-2} + \epsilon \sqrt{p^{*n+s-3}} \right), & \text{if } g(\beta) \in NSQ. \end{cases}\end{aligned}$$

Proof. In view of Lemmas 14, 15 and 16, the proof can be constructed with the same argument used in the proof of [20, Lemma 14]. \square

4 Linear codes with few weights from weakly regular plateaued functions

In this section, we generalize the recent construction methods of linear codes proposed by Ding et al. [5, 9] and Tang et al. [20] to weakly regular plateaued functions, based on the second generic construction. We also record a subcode of any constructed code.

4.1 Two or three weight linear codes with their weight distributions

In this subsection, to construct new linear codes, we make use of weakly regular plateaued functions in the construction method of linear codes proposed by Ding et al. [9]. Let f be a p -ary function from \mathbb{F}_q to \mathbb{F}_p . Define a set

$$D_f = \{x \in \mathbb{F}_q^* : f(x) = 0\}. \quad (4)$$

Assume $m = \#D_f$ and $D_f = \{d_1, d_2, \dots, d_m\}$. The second generic construction of a linear code from f is obtained from D_f and a linear code involving D_f is defined by

$$\mathcal{C}_{D_f} = \{c_\beta = (\text{Tr}^n(\beta d_1), \text{Tr}^n(\beta d_2), \dots, \text{Tr}^n(\beta d_m)) : \beta \in \mathbb{F}_q^*\}. \quad (5)$$

The set D_f is usually called the *defining set* of the code \mathcal{C}_{D_f} . The code \mathcal{C}_{D_f} has length m and dimension at most n . This construction is generic in the sense that many classes of known codes could be produced by selecting the defining set $D_f \subseteq \mathbb{F}_q$. For a general function f , determining the weight distribution of \mathcal{C}_{D_f} is little hard, but easy for some special functions. For example, the weight distribution of \mathcal{C}_{D_f} was determined by Ding et al. [9] for a quadratic function $f(x) = \text{Tr}^n(x^2)$, by Zhou et al. [22] for quadratic bent functions and by Tang et al. [20] for some weakly regular bent functions. We now solve this problem for some weakly regular plateaued functions.

The Hamming weights of the codewords of the code \mathcal{C}_{D_f} as well as its length are derived from Lemmas 10 and 14, and its weight distribution is determined by Lemmas 2 and 11.

Theorem 1. *Let $n + s$ be an even integer and $f \in WRP$. Then \mathcal{C}_{D_f} is the three-weight linear code with parameters $\left[p^{n-1} - 1 + \epsilon \eta_0(-1)(p-1)\sqrt{p^{*n+s-2}}, n \right]_p$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. The Hamming weights of the codewords and the weight distribution of \mathcal{C}_{D_f} are as in Table 1.*

Proof. Clearly, we get $\#D_f = \mathcal{N}_f(0) - 1 = p^{n-1} - 1 + \epsilon \eta_0(-1)(p-1)\sqrt{p^{*n+s-2}}$ by Lemma 10 and $wt(c_\beta) = \#D_f - \mathcal{N}_{f,\beta} + 1$ for every $\beta \in \mathbb{F}_q^*$ by Lemma 14. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$, we have $wt(c_\beta) = (p-1)(p^{n-2} + \epsilon(p-1)\sqrt{p^{*n+s-4}})$, and the number of such codewords c_β is $p^n - p^{n-s}$ by Lemma 2. For every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$, we obtain

$$wt(c_\beta) = \begin{cases} (p-1)p^{n-2}, & \text{if } g(\beta) = 0, \\ (p-1)\left(p^{n-2} + \epsilon \eta_0(-1)\sqrt{p^{*n+s-2}}\right), & \text{if } g(\beta) \neq 0, \end{cases}$$

Hamming weight w	Multiplicity A_w
0	1
$(p-1) \left(p^{n-2} + \epsilon(p-1)\sqrt{p^{*n+s-4}} \right)$	$p^n - p^{n-s}$
$(p-1)p^{n-2}$	$p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)(p-1)\sqrt{p^{*n-s-2}} - 1$
$(p-1) \left(p^{n-2} + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}} \right)$	$(p-1) \left(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right)$

Table 1: The weight distribution of \mathcal{C}_{D_f} when $n + s$ is even

and the number of such codewords c_β follows from Lemma 11. Hence the proof is ended. \square

Remark 3. In Theorem 1, if $\epsilon\eta_0^{(n+s)/2}(-1) = -1$, then we need the condition $0 \leq s \leq n-4$; otherwise, $0 \leq s \leq n-2$.

Example 2. The function $f : \mathbb{F}_{3^8} \rightarrow \mathbb{F}_3$ defined as $f(x) = \text{Tr}^8(\zeta x^4 + \zeta^{816} x^2)$, where $\mathbb{F}_{3^8}^* = \langle \zeta \rangle$ with $\zeta^8 + 2\zeta^5 + \zeta^4 + 2\zeta^2 + 2\zeta + 2 = 0$, is the quadratic 2-plateaued unbalanced function in the set WRP with

$$\widehat{\chi}_f(\beta) \in \{0, \epsilon\eta_0^5(-1)3^5\xi_3^{g(\beta)}\} = \{0, 243, 243\xi_3, 243\xi_3^2\}$$

for all $\beta \in \mathbb{F}_{3^8}$, where $\epsilon = -1$, $\eta_0(-1) = -1$ and g is an unbalanced 3-ary function with $g(0) = 0$. Then, \mathcal{C}_{D_f} is the three-weight linear code with parameters $[2348, 8, 1458]_3$, weight enumerator $1 + 260y^{1458} + 5832y^{1566} + 468y^{1620}$ and weight distribution $(1, 260, 5832, 468)$, which is verified by MAGMA in [2]. This code is minimal by Lemma 1.

Theorem 2. Let $f \in \text{WRP}$ and $n + s$ be an odd integer with $0 \leq s \leq n-3$. Then, \mathcal{C}_{D_f} is the three-weight linear code with parameters $[p^{n-1} - 1, n, (p-1)(p^{n-2} - p^{(n+s-3)/2})]_p$. The Hamming weights of the codewords and the weight distribution of \mathcal{C}_{D_f} are as in Table 2, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$.

Hamming weight w	Multiplicity A_w
0	1
$(p-1)p^{n-2}$	$p^n + p^{n-s-1} - p^{n-s} - 1$
$(p-1) \left(p^{n-2} - \epsilon\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$(p-1) \left(p^{n-2} + \epsilon\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 2: The weight distribution of \mathcal{C}_{D_f} when $n + s$ is odd

Proof. The proof can be completed in a similar way to the even case in Theorem 1. \square

Example 3. The function $f : \mathbb{F}_{3^3} \rightarrow \mathbb{F}_3$ defined as $f(x) = \text{Tr}^3(x^4 + x^2)$, where $\mathbb{F}_{3^3} = \langle \zeta \rangle$ with $\zeta^3 + 2\zeta + 1 = 0$, is the quadratic bent function in the set WRP with

$$\widehat{\chi}_f(\beta) \in \{i3\sqrt{3}, i3\sqrt{3}\xi_3, i3\sqrt{3}\xi_3^2\} = \{6\xi_3 + 3, -3\xi_3 - 6, -3\xi_3 + 3\}$$

for all $\beta \in \mathbb{F}_{3^3}$, where $\epsilon = -1$ and $\eta_0(-1) = -1$. Then, \mathcal{C}_{D_f} is the three-weight linear code with parameters $[8, 3, 4]_3$, weight enumerator $1 + 8y^6 + 6y^8 + 12y^4$ and weight distribution $(1, 8, 6, 12)$, which is verified by MAGMA in [2].

Since the Hamming weights of all nonzero codewords of \mathcal{C}_{D_f} have a common divisor $p - 1$, we can obtain a shorter linear code from the code \mathcal{C}_{D_f} . Let $f \in \text{WRP}$. For any $x \in \mathbb{F}_q$, $f(x) = 0$ if and only if $f(ax) = 0$, for every $a \in \mathbb{F}_p^*$. Then one can choose a subset \bar{D}_f of the defining set D_f of \mathcal{C}_{D_f} defined by (4) such that $\bigcup_{a \in \mathbb{F}_p^*} a\bar{D}_f$ is a partition of D_f , namely,

$$D_f = \mathbb{F}_p^* \bar{D}_f = \{ab : a \in \mathbb{F}_p^*, b \in \bar{D}_f\},$$

where for each pair of distinct elements $b_1, b_2 \in \bar{D}_f$ we have $\frac{b_1}{b_2} \notin \mathbb{F}_p^*$. This implies that the linear code \mathcal{C}_{D_f} can be punctured into a shorter linear code $\mathcal{C}_{\bar{D}_f}$, where \bar{D}_f is its defining set. Notice that for $\beta \in \mathbb{F}_q^*$,

$$\#\{x \in D_f : f(x) = 0 \text{ and } \text{Tr}^n(\beta x) = 0\} = (p - 1) \#\{x \in \bar{D}_f : f(x) = 0 \text{ and } \text{Tr}^n(\beta x) = 0\}.$$

Hence, the following linear codes in Corollaries 1 and 2 are directly obtained from the constructed ones in Theorems 1 and 2, respectively.

Corollary 1. *The punctured version $\mathcal{C}_{\bar{D}_f}$ of the code \mathcal{C}_{D_f} of Theorem 1 is the three-weight linear code with parameters $\left[(p^{n-1} - 1)/(p - 1) + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}, n \right]_p$ whose weight distribution is listed in Table 3.*

Hamming weight w	Multiplicity A_w
0	1
$p^{n-2} + \epsilon(p-1)\sqrt{p^{*n+s-4}}$	$p^n - p^{n-s}$
p^{n-2}	$p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)(p-1)\sqrt{p^{*n-s-2}} - 1$
$p^{n-2} + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}$	$(p-1) \left(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right)$

Table 3: The weight distribution of $\mathcal{C}_{\bar{D}_f}$ when $n + s$ is even

Example 4. The punctured version $\mathcal{C}_{\bar{D}_f}$ of \mathcal{C}_{D_f} in Example 2 is the three-weight linear code with parameters $[1174, 8, 729]_3$, weight enumerator $1 + 260y^{729} + 5832y^{783} + 468y^{810}$ and weight distribution $(1, 260, 5832, 468)$. This code is minimal by Lemma 1.

Corollary 2. *The punctured version $\mathcal{C}_{\bar{D}_f}$ of the code \mathcal{C}_{D_f} of Theorem 2 is the three-weight linear code with parameters $[(p^{n-1} - 1)/(p - 1), n, p^{n-2} - p^{(n+s-3)/2}]_p$ whose weight distribution is listed in Table 4.*

Hamming weight w	Multiplicity A_w
0	1
p^{n-2}	$p^n + p^{n-s-1} - p^{n-s} - 1$
$p^{n-2} - \epsilon\sqrt{p^{*n+s-3}}$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$p^{n-2} + \epsilon\sqrt{p^{*n+s-3}}$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 4: The weight distribution of $\mathcal{C}_{\bar{D}_f}$ when $n + s$ is odd

Example 5. The punctured version $\mathcal{C}_{\bar{D}_f}$ of \mathcal{C}_{D_f} in Example 3 is the three-weight linear code with parameters $[4, 3, 2]_3$, weight enumerator $1 + 8y^3 + 6y^4 + 12y^2$ and weight distribution $(1, 8, 6, 12)$, which is verified by MAGMA in [2]. This code is optimal owing to the Singleton bound.

In particular, we can work on the Walsh support of a weakly regular plateaued function f to define a subcode of each constructed code above. We consider a linear code involving D_f defined by

$$\bar{\mathcal{C}}_{D_f} = \{c_\beta = (\text{Tr}^n(\beta d_1), \text{Tr}^n(\beta d_2), \dots, \text{Tr}^n(\beta d_m)) : \beta \in \text{Supp}(\widehat{\chi}_f)\}, \quad (6)$$

which is the subcode of \mathcal{C}_{D_f} defined by (5). Hence, the following codes in Corollaries 3, 4, 5 and 6 are the subcodes of the codes of Theorems 1, 2 and Corollaries 1, 2, respectively. Notice that their parameters are directly derived from that of corresponding codes.

Corollary 3. *The subcode $\bar{\mathcal{C}}_{D_f}$ of the code \mathcal{C}_{D_f} of Theorem 1 is the two-weight linear code with parameters $\left[p^{n-1} - 1 + \epsilon\eta_0(-1)(p - 1)\sqrt{p^{*n+s-2}}, n - s \right]_p$ whose weight distribution is listed in Table 5.*

Hamming weight w	Multiplicity A_w
0	1
$(p - 1)p^{n-2}$	$p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)(p - 1)\sqrt{p^{*n-s-2}} - 1$
$(p - 1) \left(p^{n-2} + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}} \right)$	$(p - 1) \left(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right)$

Table 5: The weight distribution of $\bar{\mathcal{C}}_{D_f}$ when $n + s$ is even

Corollary 4. *The subcode $\bar{\mathcal{C}}_{D_f}$ of the code \mathcal{C}_{D_f} of Theorem 2 is the three-weight linear code with parameters $[p^{n-1} - 1, n - s, (p - 1)(p^{n-2} - p^{(n+s-3)/2})]_p$ whose weight distribution is listed in Table 6.*

Hamming weight w	Multiplicity A_w
0	1
$(p - 1)p^{n-2}$	$p^{n-s-1} - 1$
$(p - 1)(p^{n-2} - \epsilon\sqrt{p^{*n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}})$
$(p - 1)(p^{n-2} + \epsilon\sqrt{p^{*n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}})$

Table 6: The weight distribution of $\bar{\mathcal{C}}_{D_f}$ when $n + s$ is odd

Corollary 5. *The subcode $\bar{\mathcal{C}}_{\bar{D}_f}$ of the code $\mathcal{C}_{\bar{D}_f}$ of Corollary 1 is the two-weight linear code with parameters $[(p^{n-1} - 1)/(p - 1) + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}, n - s]_p$ whose weight distribution is listed in Table 7.*

Hamming weight w	Multiplicity A_w
0	1
p^{n-2}	$p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)(p - 1)\sqrt{p^{*n-s-2}} - 1$
$p^{n-2} + \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}$	$(p - 1)(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}})$

Table 7: The weight distribution of $\bar{\mathcal{C}}_{\bar{D}_f}$ when $n + s$ is even

Corollary 6. *The subcode $\bar{\mathcal{C}}_{\bar{D}_f}$ of the code $\mathcal{C}_{\bar{D}_f}$ of Corollary 2 is the three-weight linear code with parameters $[(p^{n-1} - 1)/(p - 1), n - s, p^{n-2} - p^{(n+s-3)/2}]_p$ whose weight distribution is listed in Table 8.*

Hamming weight w	Multiplicity A_w
0	1
p^{n-2}	$p^{n-s-1} - 1$
$p^{n-2} - \epsilon\sqrt{p^{*n+s-3}}$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}})$
$p^{n-2} + \epsilon\sqrt{p^{*n+s-3}}$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}})$

Table 8: The weight distribution of $\bar{\mathcal{C}}_{\bar{D}_f}$ when $n + s$ is odd

Remark 4. When we assume only the quadratic bent-ness (resp., the weakly regular bent-ness) in this subsection, we can obviously recover the linear codes obtained by Zhou et al. [22] (resp., by Tang et al. [20]). Therefore, this subsection can be viewed as an extension of [22] and [20] to the weakly regular plateaued unbalanced functions.

The following section, to construct new linear codes, pushes further the use of weakly regular plateaued functions in the construction methods proposed by Tang et al. [20].

4.2 Two or three weight linear codes with their weight distributions

Let $f : \mathbb{F}_q \rightarrow \mathbb{F}_p$ be a p -ary function. Define the sets

$$D_{f,sq} = \{x \in \mathbb{F}_q : f(x) \in SQ\} \text{ and } D_{f,nsq} = \{x \in \mathbb{F}_q : f(x) \in NSQ\}.$$

With the similar definition of the linear code \mathcal{C}_{D_f} defined by (5), we can define a linear code involving $D_{f,sq} = \{d'_1, d'_2, \dots, d'_m\}$

$$\mathcal{C}_{D_{f,sq}} = \{c_\beta = (\text{Tr}^n(\beta d'_1), \text{Tr}^n(\beta d'_2), \dots, \text{Tr}^n(\beta d'_m)) : \beta \in \mathbb{F}_q\} \quad (7)$$

and a linear code involving $D_{f,nsq} = \{d''_1, d''_2, \dots, d''_m\}$

$$\mathcal{C}_{D_{f,nsq}} = \{c_\beta = (\text{Tr}^n(\beta d''_1), \text{Tr}^n(\beta d''_2), \dots, \text{Tr}^n(\beta d''_m)) : \beta \in \mathbb{F}_q\}. \quad (8)$$

From Lemmas 10 and 17, we find the Hamming weights of the codewords of the linear codes $\mathcal{C}_{D_{f,sq}}$ and $\mathcal{C}_{D_{f,nsq}}$ as well as their length, and we determine their weight distributions from Lemmas 2 and 11.

Theorem 3. *Let $n + s$ be an even integer and $f \in WRP$. Then, $\mathcal{C}_{D_{f,sq}}$ is the three-weight linear code with parameters $\left[\frac{p-1}{2} \left(p^{n-1} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right), n \right]_p$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. The Hamming weights of the codewords and the weight distribution of $\mathcal{C}_{D_{f,sq}}$ are as in Table 9.*

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2} \left(p^{n-2} - \epsilon \sqrt{p^{*n+s-4}} \right)$	$p^n - p^{n-s}$
$\frac{(p-1)^2}{2} p^{n-2}$	$p^{n-s-1} + \frac{p-1}{2} \left(p^{n-s-1} + \epsilon \eta_0^{n+1}(-1) \sqrt{p^{*n-s-2}} \right) - 1$
$(p-1) \left(\frac{p-1}{2} p^{n-2} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-2}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon \eta_0^{n+1}(-1) \sqrt{p^{*n-s-2}} \right)$

Table 9: The weight distribution of $\mathcal{C}_{D_{f,sq}}$ when $n + s$ is even

Proof. We have $\#D_{f,sq} = \frac{p-1}{2}(p^{n-1} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}})$ by Lemma 10 and $wt(c_\beta) = \#D_{f,sq} - \mathcal{N}_{sq,\beta}$ for every $\beta \in \mathbb{F}_q^*$ by Lemma 17. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$,

$$wt(c_\beta) = \frac{(p-1)^2}{2} \left(p^{n-2} - \epsilon\sqrt{p^{*n+s-4}} \right),$$

and the number of such codewords c_β is equal to $p^n - p^{n-s}$ by Lemma 2. For every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$,

$$wt(c_\beta) = \begin{cases} \frac{(p-1)^2}{2} p^{n-2}, & \text{if } g(\beta) = 0 \text{ or } g(\beta) \in NSQ, \\ \frac{(p-1)^2}{2} p^{n-2} - \epsilon\eta_0(-1)(p-1)\sqrt{p^{*n+s-2}}, & \text{if } g(\beta) \in SQ, \end{cases}$$

and the number of such codewords c_β follows from Lemma 11. Hence the proof is ended. \square

Remark 5. In Theorem 3, if $\epsilon\eta_0^{(n+s)/2}(-1) = 1$ and $p = 3$, then we have the condition $0 \leq s \leq n-4$; otherwise, $0 \leq s \leq n-2$ and $n \geq 3$.

Example 6. The function $f : \mathbb{F}_{3^5} \rightarrow \mathbb{F}_3$ defined as $f(x) = \text{Tr}^5(\zeta^{19}x^4 + \zeta^{238}x^2)$, where $\mathbb{F}_{3^5} = \langle \zeta \rangle$ with $\zeta^5 + 2\zeta + 1 = 0$, is the quadratic 1-plateaued unbalanced function in the set WRP with

$$\widehat{\chi}_f(\beta) \in \{0, \epsilon\eta_0^3(-1)3^3\xi_3^{g(\beta)}\} = \{0, -27, -27\xi_3, -27\xi_3^2\}$$

for all $\beta \in \mathbb{F}_{3^5}$, where $\epsilon = 1$, $\eta_0(-1) = -1$ and g is an unbalanced 3-ary function with $g(0) = 0$. Then, $\mathcal{C}_{D_{f,sq}}$ is the three-weight linear code with parameters $[90, 5, 54]_3$, weight enumerator $1 + 50y^{54} + 162y^{60} + 30y^{72}$ and weight distribution $(1, 50, 162, 30)$, which is verified by MAGMA in [2]. This code is minimal by Lemma 1.

Recall that we have the following fact:

$$\eta_0(-1) = \begin{cases} 1 & \text{if and only if } p \equiv 1 \pmod{4}, \\ -1 & \text{if and only if } p \equiv 3 \pmod{4}, \end{cases}$$

which will be needed in Theorems 4 and 5.

Theorem 4. *Let $n + s$ be an odd integer and $f \in \text{WRP}$. Then, $\mathcal{C}_{D_{f,sq}}$ is the three-weight linear code with parameters $\left[\frac{p-1}{2} \left(p^{n-1} + \epsilon\sqrt{p^{*n+s-1}} \right), n \right]_p$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. The Hamming weights of the codewords and the weight distribution of $\mathcal{C}_{D_{f,sq}}$ are as in Table 10 and Table 11 when $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$, respectively.*

Proof. The proof can be completed in a similar way to the even case in Theorem 3. \square

Remark 6. In Theorem 4, if $p \equiv 3 \pmod{4}$ and $\epsilon\eta_0^{(n+s-1)/2}(-1) = -1$ or $p \equiv 1 \pmod{4}$ and $\epsilon = -1$, then we have the condition $0 \leq s \leq n-3$; otherwise, $0 \leq s \leq n-1$ and $n \geq 2$.

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2}((p-1)p^{n-2} + \epsilon(p+1)\sqrt{p^{n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon\sqrt{p^{n-s-1}})$
$\frac{(p-1)^2}{2}(p^{n-2} + \epsilon\sqrt{p^{n+s-3}})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} - \epsilon\sqrt{p^{n-s-1}})$

Table 10: The weight distribution of $\mathcal{C}_{D_{f,sq}}$ when $p \equiv 1 \pmod{4}$ and $n + s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{(p-1)^2}{2}(p^{n-2} - \epsilon\sqrt{p^{*n+s-3}})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} + \epsilon(-1)^n\sqrt{p^{*n-s-1}})$
$\frac{p-1}{2}((p-1)p^{n-2} - \epsilon(p+1)\sqrt{p^{*n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon(-1)^n\sqrt{p^{*n-s-1}})$

Table 11: The weight distribution of $\mathcal{C}_{D_{f,sq}}$ when $p \equiv 3 \pmod{4}$ and $n + s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{(p-1)^2}{2}(p^{n-2} - \epsilon\sqrt{p^{n+s-3}})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} + \epsilon\sqrt{p^{n-s-1}})$
$\frac{p-1}{2}((p-1)p^{n-2} - \epsilon(p+1)\sqrt{p^{n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon\sqrt{p^{n-s-1}})$

Table 12: The weight distribution of $\mathcal{C}_{D_{f,nsq}}$ when $p \equiv 1 \pmod{4}$ and $n + s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2}((p-1)p^{n-2} + \epsilon(p+1)\sqrt{p^{*n+s-3}})$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon(-1)^n\sqrt{p^{*n-s-1}})$
$\frac{(p-1)^2}{2}(p^{n-2} + \epsilon\sqrt{p^{*n+s-3}})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} - \epsilon(-1)^n\sqrt{p^{*n-s-1}})$

Table 13: The weight distribution of $\mathcal{C}_{D_{f,nsq}}$ when $p \equiv 3 \pmod{4}$ and $n + s$ is odd

Theorem 5. Let $n + s$ be an odd integer and $f \in WRP$. Then, $\mathcal{C}_{D_{f,nsq}}$ is the three-weight linear code with parameters $\left[\frac{p-1}{2} \left(p^{n-1} - \epsilon \sqrt{p^{*n+s-1}} \right), n \right]_p$, where $\epsilon = \pm 1$ is the sign of $\widehat{\chi}_f$. The Hamming weights of the codewords and the weight distribution of $\mathcal{C}_{D_{f,nsq}}$ are as in Table 12 and Table 13 when $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$, respectively.

Proof. Obviously, we get $\#D_{f,nsq} = \frac{p-1}{2}(p^{n-1} - \epsilon \sqrt{p^{*n+s-1}})$ by Lemma 10 and $wt(c_\beta) = \#D_{f,nsq} - \mathcal{N}_{nsq,\beta}$ for every $\beta \in \mathbb{F}_q^*$ by Lemma 17. Then, for every $\beta \in \mathbb{F}_q^* \setminus \text{Supp}(\widehat{\chi}_f)$,

$$wt(c_\beta) = \frac{(p-1)^2}{2} \left(p^{n-2} - \epsilon \eta_0(-1) \sqrt{p^{*n+s-3}} \right)$$

and the number of such codewords c_β is equal to $p^n - p^{n-s}$ by Lemma 2. For every $0 \neq \beta \in \text{Supp}(\widehat{\chi}_f)$,

$$wt(c_\beta) = \begin{cases} \frac{p-1}{2}(p-1)p^{n-2}, & \text{if } g(\beta) = 0, \\ \frac{p-1}{2} \left((p-1)p^{n-2} + \epsilon \sqrt{p^{*n+s-3}}(1-p^*) \right), & \text{if } g(\beta) \in SQ, \\ \frac{p-1}{2} \left((p-1)p^{n-2} - \epsilon \sqrt{p^{*n+s-3}}(p^*+1) \right), & \text{if } g(\beta) \in NSQ, \end{cases}$$

and the number of such codewords c_β follows from Lemma 11. Hence the proof is ended. \square

Remark 7. In Theorem 5, if $p \equiv 3 \pmod{4}$ and $\epsilon \eta_0^{(n+s-1)/2}(-1) = 1$ or $p \equiv 1 \pmod{4}$ and $\epsilon = 1$, then we have the condition $0 \leq s \leq n-3$; otherwise, $0 \leq s \leq n-1$ and $n \geq 2$.

Example 7. The function $f : \mathbb{F}_{3^6} \rightarrow \mathbb{F}_3$ defined as $f(x) = \text{Tr}^6(\zeta x^4 + \zeta^{27} x^2)$, where $\mathbb{F}_{3^6}^* = \langle \zeta \rangle$ with $\zeta^6 + 2\zeta^4 + \zeta^2 + 2\zeta + 2 = 0$, is the quadratic 1-plateaued unbalanced function in the set WRP with

$$\widehat{\chi}_f(\beta) \in \{0, i27\sqrt{3}, i27\sqrt{3}\xi_3, i27\sqrt{3}\xi_3^2\} = \{0, 54\xi_3 + 27, -27\xi_3 - 54, -27\xi_3 + 27\}$$

for all $\beta \in \mathbb{F}_{3^6}$, where $\epsilon = -1$, $\eta_0(-1) = -1$ and g is an unbalanced 3-ary function with $g(0) = 0$. Then, $\mathcal{C}_{D_{f,nsq}}$ is the three-weight linear code with parameters $[216, 6, 126]_3$, weight enumerator $1 + 72y^{126} + 576y^{144} + 80y^{162}$ and weight distribution $(1, 72, 576, 80)$, which is verified by MAGMA in [2]. This code is minimal by Lemma 1.

Remark 8. Let $n + s$ be an even and $f \in WRP$. Then, $\mathcal{C}_{D_{f,nsq}}$ is the three-weight linear code with the same parameters and weight distribution of $\mathcal{C}_{D_{f,sq}}$ in Theorem 3.

We now obtain a shorter linear code from the above each constructed code. Let $f \in WRP$. For any $x \in \mathbb{F}_q$, $f(x)$ is a quadratic residue (resp., quadratic non-residue) in \mathbb{F}_p^* if and only if $f(ax)$ is a quadratic residue (resp., quadratic non-residue) in \mathbb{F}_p^* for every $a \in \mathbb{F}_p^*$. Then one can choose a subset $\bar{D}_{f,sq}$ of the defining set $D_{f,sq}$ of $\mathcal{C}_{D_{f,sq}}$ such that

$$D_{f,sq} = \mathbb{F}_p^* \bar{D}_{f,sq} = \{ab : a \in \mathbb{F}_p^*, b \in \bar{D}_{f,sq}\},$$

Hamming weight w	Multiplicity A_w
0	1
$\frac{p-1}{2} \left(p^{n-2} - \epsilon \sqrt{p^{*n+s-4}} \right)$	$p^n - p^{n-s}$
$\frac{(p-1)}{2} p^{n-2}$	$p^{n-s-1} + \frac{p-1}{2} \left(p^{n-s-1} + \epsilon \eta_0^{n+1} (-1) \sqrt{p^{*n-s-2}} \right) - 1$
$\frac{p-1}{2} p^{n-2} - \epsilon \eta_0 (-1) \sqrt{p^{*n+s-2}}$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon \eta_0^{n+1} (-1) \sqrt{p^{*n-s-2}} \right)$

Table 14: The weight distribution of $\mathcal{C}_{\bar{D}_{f,sq}}$ when $n + s$ is even

and a subset $\bar{D}_{f,nsq}$ of the defining set $D_{f,nsq}$ of $\mathcal{C}_{D_{f,nsq}}$ such that $D_{f,nsq} = \{ab : a \in \mathbb{F}_p^*, b \in \bar{D}_{f,nsq}\}$. Hence, one can easily obtain the punctured versions $\mathcal{C}_{\bar{D}_{f,sq}}$ and $\mathcal{C}_{\bar{D}_{f,nsq}}$ of $\mathcal{C}_{D_{f,sq}}$ and $\mathcal{C}_{D_{f,nsq}}$, respectively, whose parameters are derived directly from that of the original codes. Notice that Corollaries 7, 8 and 9 follow directly from Theorems 3, 4 and 5, respectively.

Corollary 7. *The punctured version $\mathcal{C}_{\bar{D}_{f,sq}}$ of the code $\mathcal{C}_{D_{f,sq}}$ of Theorem 3 is the three-weight linear code with parameters $\left[\frac{1}{2}(p^{n-1} - \epsilon \eta_0 (-1) \sqrt{p^{*n+s-2}}), n \right]_p$ whose weight distribution is listed in Table 14.*

Example 8. The punctured version $\mathcal{C}_{\bar{D}_{f,sq}}$ of $\mathcal{C}_{D_{f,sq}}$ in Example 6 is the three-weight linear code with parameters $[45, 5, 27]_3$, weight enumerator $1 + 50y^{27} + 162y^{30} + 30y^{36}$ and weight distribution $(1, 50, 162, 30)$. This code is minimal by Lemma 1 and is almost optimal owing to the Griesmer bound.

Corollary 8. *The punctured version $\mathcal{C}_{\bar{D}_{f,sq}}$ of the code $\mathcal{C}_{D_{f,sq}}$ of Theorem 4 is the three-weight linear code with parameters $\left[\frac{1}{2}(p^{n-1} + \epsilon \sqrt{p^{*n+s-1}}), n \right]_p$ whose weight distribution is listed in Table 15 and Table 16 when $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$, respectively.*

Corollary 9. *The punctured version $\mathcal{C}_{\bar{D}_{f,nsq}}$ of the code $\mathcal{C}_{D_{f,nsq}}$ of Theorem 5 is the three-weight linear code with parameters $\left[\frac{1}{2}(p^{n-1} - \epsilon \sqrt{p^{*n+s-1}}), n \right]_p$ whose weight distribution is listed in Table 17 and Table 18 when $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$, respectively.*

Example 9. The punctured version $\mathcal{C}_{\bar{D}_{f,nsq}}$ of $\mathcal{C}_{D_{f,nsq}}$ in Example 7 is the three-weight linear code with parameters $[108, 6, 63]_3$, weight enumerator $1 + 72y^{63} + 576y^{72} + 80y^{81}$ and weight distribution $(1, 72, 576, 80)$. This code is minimal by Lemma 1 and is almost optimal owing to the Griesmer bound.

With the similar definition of the subcode $\bar{\mathcal{C}}_{D_f}$ defined by (6), we have a linear code involving $D_{f,sq}$ defined by

$$\bar{\mathcal{C}}_{D_{f,sq}} = \{c_\beta = (\text{Tr}^n(\beta d'_1), \text{Tr}^n(\beta d'_2), \dots, \text{Tr}^n(\beta d'_m)) : \beta \in \text{Supp}(\widehat{\chi}_f)\},$$

Hamming weight w	Multiplicity A_w
0	1
$\frac{p-1}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{1}{2}((p-1)p^{n-2} + \epsilon(p+1)\sqrt{p}^{n+s-3})$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon\sqrt{p}^{n-s-1})$
$\frac{(p-1)}{2}(p^{n-2} + \epsilon\sqrt{p}^{n+s-3})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} - \epsilon\sqrt{p}^{n-s-1})$

Table 15: The weight distribution of $\mathcal{C}_{\bar{D}_{f,sq}}$ when $p \equiv 1 \pmod{4}$ and $n+s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{p-1}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2}(p^{n-2} - \epsilon\sqrt{p}^{*n+s-3})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} + \epsilon(-1)^n\sqrt{p}^{*n-s-1})$
$\frac{1}{2}((p-1)p^{n-2} - \epsilon(p+1)\sqrt{p}^{*n+s-3})$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon(-1)^n\sqrt{p}^{*n-s-1})$

Table 16: The weight distribution of $\mathcal{C}_{\bar{D}_{f,sq}}$ when $p \equiv 3 \pmod{4}$ and $n+s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2}(p^{n-2} - \epsilon\sqrt{p}^{n+s-3})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} + \epsilon\sqrt{p}^{n-s-1})$
$\frac{1}{2}((p-1)p^{n-2} - \epsilon(p+1)\sqrt{p}^{n+s-3})$	$\frac{p-1}{2}(p^{n-s-1} - \epsilon\sqrt{p}^{n-s-1})$

Table 17: The weight distribution of $\mathcal{C}_{\bar{D}_{f,nsq}}$ when $p \equiv 1 \pmod{4}$ and $n+s$ is odd

Hamming weight w	Multiplicity A_w
0	1
$\frac{p-1}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{1}{2}((p-1)p^{n-2} + \epsilon(p+1)\sqrt{p}^{*n+s-3})$	$\frac{p-1}{2}(p^{n-s-1} + \epsilon(-1)^n\sqrt{p}^{*n-s-1})$
$\frac{p-1}{2}(p^{n-2} + \epsilon\sqrt{p}^{*n+s-3})$	$p^n - p^{n-s} + \frac{p-1}{2}(p^{n-s-1} - \epsilon(-1)^n\sqrt{p}^{*n-s-1})$

Table 18: The weight distribution of $\mathcal{C}_{\bar{D}_{f,nsq}}$ when $p \equiv 3 \pmod{4}$ and $n+s$ is odd

which is the subcode of $\mathcal{C}_{D_{f,sq}}$ defined by (7). Hence, the following codes in Corollaries 10, 11, 12 and 13 are the subcodes of the codes of Theorems 3, 4 and Corollaries 7, 8, respectively.

Corollary 10. *The subcode $\bar{\mathcal{C}}_{D_{f,sq}}$ of the code $\mathcal{C}_{D_{f,sq}}$ of Theorem 3 is the two-weight linear code with parameters $[\frac{p-1}{2}(p^{n-1} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}), n-s]_p$ whose weight distribution is listed in Table 19.*

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} + \frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right) - 1$
$(p-1) \left(\frac{p-1}{2}p^{n-2} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right)$

Table 19: The weight distribution of $\bar{\mathcal{C}}_{D_{f,sq}}$ when $n+s$ is even

Corollary 11. *The subcode $\bar{\mathcal{C}}_{D_{f,sq}}$ of the code $\mathcal{C}_{D_{f,sq}}$ of Theorem 4 is the three-weight linear code with parameters $[\frac{p-1}{2}(p^{n-1} + \epsilon\sqrt{p^{*n+s-1}}), n-s]_p$ whose weight distribution is listed in Table 20.*

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)^2}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2} \left((p-1)p^{n-2} + \epsilon(p^*+1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$\frac{p-1}{2} \left((p-1)p^{n-2} + \epsilon(p^*-1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 20: The weight distribution of $\bar{\mathcal{C}}_{D_{f,sq}}$ when $n+s$ is odd

Corollary 12. *The subcode $\bar{\mathcal{C}}_{\bar{D}_{f,sq}}$ of the code $\mathcal{C}_{\bar{D}_{f,sq}}$ of Corollary 7 is the two-weight linear code with parameters $[\frac{1}{2}(p^{n-1} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}), n-s]_p$ whose weight distribution is listed in Table 21.*

Corollary 13. *The subcode $\bar{\mathcal{C}}_{\bar{D}_{f,sq}}$ of the code $\mathcal{C}_{\bar{D}_{f,sq}}$ of Corollary 8 is the three-weight linear code with parameters $[\frac{1}{2}(p^{n-1} + \epsilon\sqrt{p^{*n+s-1}}), n-s]_p$ whose weight distribution is listed in Table 22.*

With the similar definition of the subcode $\bar{\mathcal{C}}_{D_f}$ defined by (6), we have a linear code involving $D_{f,nsq}$

$$\bar{\mathcal{C}}_{D_{f,nsq}} = \{c_\beta = (\text{Tr}^n(\beta d''_1), \text{Tr}^n(\beta d''_2), \dots, \text{Tr}^n(\beta d''_m)) : \beta \in \text{Supp}(\widehat{\chi}_f)\},$$

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)}{2}p^{n-2}$	$p^{n-s-1} + \frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right) - 1$
$\frac{p-1}{2}p^{n-2} - \epsilon\eta_0(-1)\sqrt{p^{*n+s-2}}$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^{n+1}(-1)\sqrt{p^{*n-s-2}} \right)$

Table 21: The weight distribution of $\bar{\mathcal{C}}_{\bar{D}_{f,sq}}$ when $n + s$ is even

Hamming weight w	Multiplicity A_w
0	1
$\frac{(p-1)}{2}p^{n-2}$	$p^{n-s-1} - 1$
$\frac{1}{2} \left((p-1)p^{n-2} + \epsilon(p^* + 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$\frac{1}{2} \left((p-1)p^{n-2} + \epsilon(p^* - 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 22: The weight distribution of $\bar{\mathcal{C}}_{\bar{D}_{f,sq}}$ when $n + s$ is odd

which is the subcode of $\mathcal{C}_{D_{f,nsq}}$ defined by (8). Hence, the following codes in Corollaries 14 and 15 are the subcodes of the constructed codes in Theorem 5 and Corollary 9, respectively.

Corollary 14. *The subcode $\bar{\mathcal{C}}_{D_{f,nsq}}$ of the code $\mathcal{C}_{D_{f,nsq}}$ of Theorem 5 is the three-weight linear code with parameters $\left[\frac{p-1}{2}(p^{n-1} - \epsilon\sqrt{p^{*n+s-1}}), n - s \right]_p$ whose weight distribution is listed in Table 23.*

Hamming weight w	Multiplicity A_w
0	1
$\frac{p-1}{2}(p-1)p^{n-2}$	$p^{n-s-1} - 1$
$\frac{p-1}{2} \left((p-1)p^{n-2} - \epsilon(p^* - 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$\frac{p-1}{2} \left((p-1)p^{n-2} - \epsilon(p^* + 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 23: The weight distribution of $\bar{\mathcal{C}}_{D_{f,nsq}}$ when $n + s$ is odd

Corollary 15. *The subcode $\bar{\mathcal{C}}_{\bar{D}_{f,nsq}}$ of the code $\mathcal{C}_{\bar{D}_{f,nsq}}$ of Corollary 9 is the three-weight linear code with parameters $\left[\frac{1}{2}(p^{n-1} - \epsilon\sqrt{p^{*n+s-1}}), n - s \right]_p$ whose weight distribution is listed in Table 24.*

Remark 9. When we assume only the weakly regular bent-ness in this subsection, we can

Hamming weight w	Multiplicity A_w
0	1
$\frac{1}{2}(p-1)p^{n-2}$	$p^{n-s-1} - 1$
$\frac{1}{2} \left((p-1)p^{n-2} - \epsilon(p^* - 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} + \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$
$\frac{1}{2} \left((p-1)p^{n-2} - \epsilon(p^* + 1)\sqrt{p^{*n+s-3}} \right)$	$\frac{p-1}{2} \left(p^{n-s-1} - \epsilon\eta_0^n(-1)\sqrt{p^{*n-s-1}} \right)$

Table 24: The weight distribution of $\bar{\mathcal{C}}_{\bar{D}_f, nsq}$ when $n + s$ is odd

obviously recover the linear codes obtained by Tang et al. [20]. Therefore, this subsection can be viewed as an extension of [20] to the notion of weakly regular plateaued functions.

The following natural question may now spring to mind: Are the constructed codes in this section minimal? The following section investigates the minimality of the constructed codes.

5 The minimality of the constructed linear codes

This section confirms that the constructed codes from weakly regular plateaued functions in Section 4 are minimal. In other words, with the help of Lemma 1, we observe that all nonzero codewords of the constructed codes are minimal for almost all cases. To do this, we consider separately the constructed codes in Theorems 1, 2, 3, 4 and 5.

Theorem 6. *Let $n + s$ be an even integer. If $\epsilon\eta_0^{(n+s)/2}(-1) = 1$, then the linear code \mathcal{C}_{D_f} of Theorem 1 is minimal with parameters*

$$\left[p^{n-1} - 1 + (p-1)p^{(n+s-2)/2}, n, (p-1)p^{n-2} \right]_p$$

when $0 \leq s \leq n-4$; otherwise, $\left[p^{n-1} - 1 - (p-1)p^{(n+s-2)/2}, n, (p-1)(p^{n-2} - p^{(n+s-2)/2}) \right]_p$ when $0 \leq s \leq n-6$.

Proof. If $\epsilon\eta_0^{(n+s)/2}(-1) = 1$, then $w_{\min} = (p-1)p^{n-2}$ and $w_{\max} = (p-1)(p^{n-2} + p^{(n+s-2)/2})$; otherwise, $w_{\min} = (p-1)(p^{n-2} - p^{(n+s-2)/2})$ and $w_{\max} = (p-1)p^{n-2}$. In the first case, we observe that

$$\frac{p-1}{p} < \frac{(p-1)p^{n-2}}{(p-1)(p^{n-2} + p^{(n+s-2)/2})}$$

if $0 \leq s \leq n-4$. Similarly, in the second case, we observe that

$$\frac{p-1}{p} < \frac{(p-1)(p^{n-2} - p^{(n+s-2)/2})}{(p-1)p^{n-2}}$$

if $0 \leq s \leq n-6$. Hence, the proof is completed from Lemma 1. \square

Corollary 16. *The constructed codes in Corollaries 1, 3 and 5 are minimal with the corresponding condition in Theorem 6.*

Theorem 7. *Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the linear code \mathcal{C}_{D_f} of Theorem 2 is minimal with parameters $\left[p^{n-1} - 1, n, (p-1)(p^{n-2} - p^{(n+s-3)/2}) \right]_p$.*

Proof. There are two cases: $\epsilon \eta_0^{(n+s-3)/2}(-1) = \pm 1$. For both cases, we have $w_{\min} = (p-1)(p^{n-2} - p^{(n+s-3)/2})$ and $w_{\max} = (p-1)(p^{n-2} + p^{(n+s-3)/2})$. Then we observe that

$$\frac{p-1}{p} < \frac{w_{\min}}{w_{\max}}$$

if $0 \leq s \leq n - 5$. It then follows from Lemma 1 that all nonzero codewords of \mathcal{C}_{D_f} are minimal if $0 \leq s \leq n - 5$. \square

Corollary 17. *Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the constructed codes in Corollaries 2, 4 and 6 are minimal.*

Theorem 8. *Let $n + s$ be an even integer. If $\epsilon \eta_0^{(n+s)/2}(-1) = 1$, then the linear code $\mathcal{C}_{D_{f,sq}}$ of Theorem 3 is minimal with parameters*

$$\left[\frac{p-1}{2} \left(p^{n-1} - p^{(n+s-2)/2} \right), n, \frac{(p-1)^2}{2} p^{n-2} - (p-1)p^{(n+s-2)/2} \right]_p$$

when $0 \leq s \leq n - 6$; otherwise, $\left[\frac{p-1}{2} \left(p^{n-1} + p^{(n+s-2)/2} \right), n, \frac{(p-1)^2}{2} p^{n-2} \right]_p$ when $0 \leq s \leq n - 4$.

Proof. If $\epsilon \eta_0^{(n+s)/2}(-1) = 1$, then $w_{\min} = \frac{(p-1)^2}{2} p^{n-2} - (p-1)p^{(n+s-2)/2}$ and $w_{\max} = \frac{(p-1)^2}{2} p^{n-2}$; otherwise, $w_{\min} = \frac{(p-1)^2}{2} p^{n-2}$ and $w_{\max} = \frac{(p-1)^2}{2} p^{n-2} + (p-1)p^{(n+s-2)/2}$. In the first case, we have

$$\frac{p-1}{p} < \frac{\frac{(p-1)^2}{2} p^{n-2} - (p-1)p^{(n+s-2)/2}}{\frac{(p-1)^2}{2} p^{n-2}}$$

if $0 \leq s \leq n - 6$, and in the second case, we have

$$\frac{p-1}{p} < \frac{\frac{(p-1)^2}{2} p^{n-2}}{\frac{(p-1)^2}{2} p^{n-2} + (p-1)p^{(n+s-2)/2}}$$

if $0 \leq s \leq n - 4$. Hence, the proof is completed by Lemma 1. \square

Corollary 18. *The constructed codes in Corollaries 7, 10 and 12 are minimal with the corresponding condition in Theorem 8.*

Theorem 9. Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the linear code $\mathcal{C}_{D_f, sq}$ of Theorem 4 is minimal with parameters

$$\left[\frac{p-1}{2}(p^{n-1} + p^{(n+s-1)/2}), n, \frac{(p-1)^2}{2}p^{n-2} \right]_p, \quad \text{if } \epsilon \eta_0^{(n+s-1)/2}(-1) = 1,$$

$$\left[\frac{p-1}{2}(p^{n-1} - p^{(n+s-1)/2}), n, \frac{p-1}{2}((p-1)p^{n-2} - (p+1)p^{(n+s-3)/2}) \right]_p, \quad \text{otherwise.}$$

Proof. When $p \equiv 1 \pmod{4}$, we have

$$\begin{aligned} w_{\min} &= \frac{(p-1)^2}{2}p^{n-2} \text{ and} \\ w_{\max} &= \frac{p-1}{2}((p-1)p^{n-2} + (p+1)p^{(n+s-3)/2}) \end{aligned} \quad (9)$$

if $\epsilon = 1$; otherwise, we have

$$\begin{aligned} w_{\min} &= \frac{p-1}{2}((p-1)p^{n-2} - (p+1)p^{(n+s-3)/2}) \text{ and} \\ w_{\max} &= \frac{(p-1)^2}{2}p^{n-2}. \end{aligned} \quad (10)$$

When $p \equiv 3 \pmod{4}$, we have the Hamming weights in (9) if $\epsilon \eta_0^{(n+s-1)/2}(-1) = 1$; otherwise, in (10). For each case above, we have

$$\frac{p-1}{p} < \frac{w_{\min}}{w_{\max}}$$

if $0 \leq s \leq n - 5$. Hence, Lemma 1 completes the proof. \square

Corollary 19. Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the constructed codes in Corollaries 8, 11 and 13 are minimal.

Theorem 10. Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the linear code $\mathcal{C}_{D_f, nsq}$ of Theorem 5 is minimal with parameters

$$\left[\frac{p-1}{2}(p^{n-1} - p^{(n+s-1)/2}), n, \frac{p-1}{2}((p-1)p^{n-2} - (p+1)p^{(n+s-3)/2}) \right]_p, \quad \text{if } \epsilon \eta_0^{(n+s-1)/2}(-1) = 1,$$

$$\left[\frac{p-1}{2}(p^{n-1} + p^{(n+s-1)/2}), n, \frac{(p-1)^2}{2}p^{n-2} \right]_p, \quad \text{otherwise.}$$

Proof. When $p \equiv 1 \pmod{4}$, we have the Hamming weights in (10) if $\epsilon = 1$; otherwise, in (9). Similarly, in the case of $p \equiv 3 \pmod{4}$, we have the Hamming weights in (10) if $\epsilon \eta_0^{(n+s-1)/2}(-1) = 1$; otherwise, in (9). Hence, the assertion follows directly from Theorem 9. \square

Corollary 20. Let $n + s$ be an odd integer with $0 \leq s \leq n - 5$. Then the constructed codes in Corollaries 9, 14 and 15 are minimal.

Remark 10. We conclude from this section that the constructed codes in this paper are minimal for almost all cases. Hence, the secret sharing schemes based on the dual codes of the constructed minimal linear codes in this paper have the nice access structures described in Proposition 1. This is the motivation why we construct a punctured version and a subcode of each constructed code.

6 Conclusion

In this paper, inspired by the work of [20], we push further the use of weakly regular plateaued functions over finite fields of odd characteristic introduced recently by Mesnager et al. [16]. By generalizing the linear codes constructed from weakly regular bent functions in [20], we obtain new minimal linear codes with more freedom in the choice of the functions involved in the construction of two or three weight linear codes. They contain the (almost) optimal codes with respect to the Singleton and Griesmer bounds. The paper provides the first construction of linear codes with few weights from weakly regular plateaued functions based on the second generic construction. The obtained minimal codes in this paper can be directly used to construct secret sharing schemes with the nice access structures. To the best of our knowledge, they are inequivalent to the known ones (since there is no minimal linear code with these parameters) in the literature.

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