# Binarization Trees and Random Number Generation 

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#### Abstract

An m-extracting procedure produces unbiased random bits from a loaded dice with $m$ faces. A binarization takes inputs from an $m$-faced dice and produce bit sequences to be fed into a (binary) extracting procedure to obtain random bits. Thus, binary extracting procedures give rise to an $m$-extracting procedure via a binarization. An entropypreserving binarization is to be called complete, and such a procedure has been proposed by Zhou and Bruck. We show that there exist complete binarizations in abundance as naturally arising from binary trees with $m$ leaves. The well-known leaf entropy theorem and a closely related structure lemma play important roles in the arguments.


## Index Terms

Random number generation, binarization, extracting procedures, coin flipping, loaded dice, Peres algorithm, leaf entropy theorem.

## I. Introduction

An m-extracting procedure produces unbiased random bits using a sequence from an i.i.d. source over an alphabet $\{0,1, \ldots, m-1\}$, regardless of its probability distribution $\left\langle p_{0}, p_{1}, \ldots, p_{m-1}\right\rangle$. When $m=2$, the source is a biased coin, and the famous von Neumann trick is 2-extracting: take a pair of coin flips and return random bits by the following rule [1]:

$$
\begin{equation*}
00 \mapsto \lambda, 01 \mapsto 0,10 \mapsto 1,11 \mapsto \lambda \tag{1}
\end{equation*}
$$

where $\lambda$ indicates "no output." Because $\operatorname{Pr}(01)=\operatorname{Pr}(10)=p_{0} p_{1}$, the resulting bit is unbiased, and the output rate, the average number of output per input, is $p_{0} p_{1} \leq 1 / 4$. Elias [2] and Peres [3] extend it by taking inputs of length $n \geq 2$ and returning more than one bit at a time. Both methods are asymptotically optimal; as the input size $n$ increases, the output rate approaches the information-theoretic upper bound $H\left(p_{0}\right)$, the Shannon entropy [4], [5].

Elias's method generalizes naturally from 2-extracting to $m$-extracting procedures for each $m>2$, as discussed in Elias's original paper [2]. However, a similar generalization of Peres's method had been unknown for quite a while and was found only recently [6]. In the meanwhile, Zhou and Bruck proposed a very interesting scheme that transforms any binary extracting procedure into an $m$-extracting procedure [7]. For example, Peres method is turned into an $m$-extracting procedure via a simple process called "binarization." If the above-mentioned generalizations of Elias and Peres are to be called direct generalizations, their scheme is rather a meta-generalization. Moreover, the resulting $m$-extracting procedure is claimed to be asymptotically optimal if the given 2-extracting procedure is asymptotically optimal.

In this paper, such entropy-preserving processes will be called complete binarizations and will be shown to exist in abundance as naturally arising from binary trees with $m$ leaves, and Zhou-Bruck scheme is an instance of them. The main tools in our argument are the well-known leaf entropy theorem and a technical fact which we call the structure lemma.

Consider the following binary tree with 5 nodes and 6 leaves:


[^0]The leaf entropy theorem states that, given a probability distribution $\mathbf{p}=\left\langle p_{0}, \ldots, p_{5}\right\rangle$ on the leaves, the Shannon entropy $H(\mathbf{p})$ is equal to the weighted sum $\sum_{i=1}^{5} P_{i} H\left(\pi_{i}\right)$ of the branching entropies $H\left(\pi_{i}\right)$ of the nodes, where the weight $P_{i}$ of node $i$ is the sum of probabilities of the leaves under it [8], [5], [9]. For example, $P_{3}=p_{0}+p_{1}+p_{3}+p_{4}$, and $\pi_{3}=\left\langle p_{0}+p_{1}+p_{4}, p_{3}\right\rangle$.

As an interpretation of the theorem, consider a loaded dice $X$ with the probability distribution $\mathbf{p}$ of the 6 faces. Each roll of $X$ generates, according to the tree (2), five possible coin tosses $X_{i}$ with biases $\pi_{i}$, and $X_{i}$ has an output with probability $P_{i}$. For example, if the dice roll $X$ is 1 , then coins $X_{1}, X_{3}$, and $X_{4}$ give an output, as the tree is conveniently represented by squares (leaf, dice roll) and circles (node, coin toss). The leaf entropy theorem tells us that the amount of information of the dice roll and the 5 coin tosses are the same. This suggests that $X_{i}$ 's may be used as sources of randomness to generate unbiased and independent random bits, possibly combined together, at a rate as high as the entropy of $X$.

The mapping $X \mapsto\left(X_{1}, \ldots, X_{5}\right)$ is a complete binarization: if $\Psi$ is 2-extracting, then $\Psi^{\prime}(X)=\Psi\left(X_{1}\right) * \cdots *$ $\Psi\left(X_{5}\right)$ is 6-extracting. Note that $X_{i}$ 's are not independent. However, $\Psi\left(X_{i}\right)$ 's are independent and therefore we can concatenate them. Moreover, if $\Psi$ is asymptotically optimal, then $\Psi^{\prime}$ is also asymptotically optimal. If one or more of $X_{i}$ 's are omitted, then the resulting $\Psi^{\prime}$ is still 6-extracting, but not asymptotically optimal anymore. And the same story holds true of any binary tree.

## II. Extracting Procedures and Binarization

## A. Extracting Procedures

Our dice $X$ has $m$ faces with values $0,1, \ldots, m-1$ with probability distribution $\left\langle p_{0}, \ldots, p_{m-1}\right\rangle$. A sequence $x=x_{1} \ldots x_{n} \in\{0,1, \ldots, m-1\}^{n}$ is considered to be taken from $n$ repeated throws of the dice. Summarized below are some necessary facts on extracting procedures. Refer to [10] and [6] for details.

Definition 1 ([3], [10]). A function $f:\{0,1, \ldots, m-1\}^{n} \rightarrow\{0,1\}^{*}$ is $m$-extracting if for each pair $z_{1}, z_{2}$ in $\{0,1\}^{*}$ such that $\left|z_{1}\right|=\left|z_{2}\right|$, we have $\operatorname{Pr}\left(f(x)=z_{1}\right)=\operatorname{Pr}\left(f(x)=z_{2}\right)$, regardless of the distribution $\left\langle p_{0}, \ldots, p_{m-1}\right\rangle$.

Definition 2. A function $\Psi:\{0,1, \ldots, m-1\}^{*} \rightarrow\{0,1\}^{*}$ is called an $m$-extracting procedure if its restriction on $\{0,1, \ldots, m-1\}^{n}$ is extracting, for every $n \geq 0$.

Define $\Psi_{1}$ on $\{0,1\}^{2}$ by the rule (1) and call it von Neumann function. Extend it by, for an empty string,

$$
\Psi_{1}(\lambda)=\lambda
$$

for a nonempty even-length input,

$$
\Psi_{1}\left(x_{1} x_{2} \ldots x_{2 n}\right)=\Psi_{1}\left(x_{1} x_{2}\right) * \cdots * \Psi_{1}\left(x_{2 n-1} x_{2 n}\right)
$$

where $*$ is concatenation, and for an odd-length input, drop the last bit and take the remaining even-length bits. Then the resulting function $\Psi_{1}$ is a 2-extracting procedure. Of course, there are more interesting extracting procedures. Asymptotically optimal 2-extracting procedures like Elias's [2], [11], [10] and Peres's [3], [12], [6] also extend von Neumann function but do not simply repeat it.

Denote by $S_{\left(n_{0}, n_{1}, \ldots, n_{m-1}\right)}$ the subset of $\{0,1, \ldots, m-1\}^{n}$ that consists of sequences with $n_{i} i$ 's. Then

$$
\{0,1, \ldots, m-1\}^{n}=\bigcup_{n_{0}+n_{1}+\cdots+n_{m-1}=n} S_{\left(n_{0}, n_{1}, \ldots, n_{m-1}\right)}
$$

and each $S_{\left(n_{0}, n_{1}, \ldots, n_{m-1}\right)}$ is an equiprobable subset of elements whose probability of occurrence is $p_{0}^{n_{0}} p_{1}^{n_{1}} \cdots p_{m-1}^{n_{m-1}}$. The size of an equiprobable set is given by a multinomial coefficient like

$$
\binom{n}{n_{0}, n_{1}, \ldots, n_{m-1}}=\frac{n!}{n_{0}!n_{1}!\cdots n_{m-1}!}
$$

When $m=2$, an equiprobable set $S_{(l, k)}$ is also written as $S_{n, k}$, where $n=l+k$, and its size can also be written as an equivalent binomial coefficient as well as the multinomial one:

$$
\binom{n}{k}=\binom{n}{l, k}
$$

Extracting functions can be characterized using the concept of multiset. A multiset is a set with repeated elements; formally, a multiset $M$ on a set $S$ is a pair ( $S, \nu$ ), where $\nu: S \rightarrow \mathbf{N}$ is a multiplicity function and $\nu(s)$ is called the multiplicity, or the number of occurrences of $s \in S$. The size $|M|$ of $M=(S, \nu)$ is $\sum_{s \in S} \nu(s)$. For multisets $A$ and $B, A \uplus B$ is the multiset such that an element occurring $a$ times in $A$ and $b$ times in $B$ occurs $a+b$ times in $A \uplus B$. So $|A \uplus B|=|A|+|B|$, and the operation $\uplus$ is associative.

When we write $x \in M=(S, \nu)$, it simply means that $x \in S$. However, when we use the expression " $x \in M$ " as an index, the multiplicity of the elements is taken into account. For example, for multisets $A$ and $B$, the multiset $A \uplus B$ can be redefined as $\{x \mid x \in A$ or $x \in B\}$.

By Definition [1, the image of an extracting function consists of multiple copies of $\{0,1\}^{N}$, the exact full set of binary strings of various lengths $N$ 's. For example, von Neumann procedure defined above sends $\{0,1\}^{6}$ to 12 copies of $\{0,1\}, 6$ copies $\{0,1\}^{2}$, and one copy of $\{0,1\}^{3}$.

Definition 3 ([6]). A multiset $A$ of bit strings is extracting if, for each $z$ that occurs in $A$, all the bit strings of length $|z|$ occur in $A$ the same time as $z$ occurs in $A$.

For multisets $A$ and $B$ of bit strings, define a new multiset $A * B=\{s * t \mid s \in A, t \in B\}$, and this operation is associative, too. If $A$ and $B$ are extracting, both $A * B$ and $A \uplus B$ are extracting. Denote by $f((C))$ the multiset $\{f(x) \mid x \in C\}$, or equivalently, $(f(C), \nu)$ with $\nu(z)=\left|f^{-1}(z) \cap C\right|$ for $z \in f(C)$. Note that $|f((C))|=|C|$. For a disjoint union $C \cup D$, we have $f((C \cup D))=f((C)) \uplus f((D))$. With this notation, $\Psi_{1}\left(\left(\{0,1\}^{6}\right)\right)=$ $12 \cdot\{0,1\} \uplus 6 \cdot\{0,1\}^{2} \uplus 1 \cdot\{0,1\}^{3}$.

The following lemma reinterprets the definition of extracting function in terms of equiprobable sets and their images.
Lemma 4 ([6]). A function $f:\{0,1, \ldots, m-1\}^{n} \rightarrow\{0,1\}^{*}$ is extracting if and only if $f\left(\left(S_{\left(n_{0}, n_{1}, \ldots, n_{m-1}\right)}\right)\right)$ is extracting for each tuple ( $n_{0}, n_{1}, \ldots, n_{m-1}$ ) of nonnegative integers such that $n_{0}+n_{1}+\cdots+n_{m-1}=n$.

## B. Binarization

Given a function $\phi:\{0,1, \ldots, m-1\} \rightarrow\{0,1, \lambda\}, \phi(X)$ is a Bernoulli random variable with distribution $\langle p, q\rangle$, where

$$
p=\sum_{\phi(i)=0} p_{i} / s, q=\sum_{\phi(i)=1} p_{i} / s, \text { and } s=\sum_{\phi(i) \neq \lambda} p_{i} .
$$

Extend $\phi$ to $\{0,1, \ldots, m-1\}^{n}$, by letting, for $x=x_{1} \ldots x_{n}, \phi(x)=\phi\left(x_{1}\right) * \cdots * \phi\left(x_{n}\right)$. Then, for an equiprobable set $S=S_{\left(n_{0}, \ldots, n_{m-1}\right)}$, its image under $\phi$ is also equiprobable, that is,

$$
\phi(S)=S_{(l, k)},
$$

where

$$
l=\sum_{\phi(i)=0} n_{i}, \quad k=\sum_{\phi(i)=1} n_{i} .
$$

A binarization takes a sequence over $\{0,1, \ldots, m-1\}$ and outputs several binary sequences that are to be separately fed into a binary extracting procedure and then concatenated together to obtain random bits.
Definition 5. $A$ collection of functions $\Phi=\left\{\Phi_{i}:\{0,1, \ldots, m-1\} \rightarrow\{0,1, \lambda\} \mid i=1, \ldots, M\right\}$ is called $a$ binarization if, when extended to $\{0,1, \ldots, m-1\}^{n}$, given a 2 -extracting procedure $\Psi$, the mapping $x \mapsto \Psi^{\prime}(x)=$ $\Psi\left(\Phi_{1}(x)\right) * \cdots * \Psi\left(\Phi_{M}(x)\right)$ is an m-extracting function. Here, each $\Phi_{i}$ is called a component of $\Phi$, and we often regard $\Phi$ as a mapping on $\{0,1, \ldots, m-1\}^{*}$ given by $\Phi(x)=\left(\Phi_{1}(x), \ldots, \Phi_{M}(x)\right)$. For an asymptotically optimal 2 -extracting procedure $\Psi$, if the resulting $\Psi^{\prime}$ is asymptotically optimal, then $\Phi$ is called a complete binarization.

Now, for a function $\phi:\{0,1, \ldots, m-1\} \rightarrow\{0,1, \lambda\}$, let

$$
\begin{aligned}
\operatorname{supp}_{0}(\phi) & =\{x \mid \phi(x)=0\} \\
\operatorname{supp}_{1}(\phi) & =\{x \mid \phi(x)=1\}, \\
\operatorname{supp}(\phi) & =\{x \mid \phi(x) \neq \lambda\}=\operatorname{supp}_{0}(\phi) \cup \operatorname{supp}_{1}(\phi),
\end{aligned}
$$

and call them 0 -support, 1 -support, and support of $\phi$, respectively. Call $\phi$ degenerate if its 0 -support or 1 -support is empty so that $\phi(X)$ is a degenerate Bernoulli random variable.

Consider a binary tree with $m$ external nodes labeled uniquely with $0,1, \ldots, m-1$. For an internal node $v$ define a function $\phi_{v}:\{0,1, \ldots, m-1\} \rightarrow\{0,1, \lambda\}$ as follows:

$$
\phi_{v}(x)= \begin{cases}0, & \text { if } x \in \operatorname{leaf}_{0}(v) \\ 1, & \text { if } x \in \operatorname{leaf}_{1}(v) \\ \lambda, & \text { otherwise }\end{cases}
$$

where $\operatorname{leaf}_{0}(v)$ (leaf ${ }_{1}(v)$, respectively) is the set of external nodes on the left (right, respectively) subtree of $v$. Since there are exactly $m-1$ internal nodes, we uniquely name them with $1, \ldots, m-1$, with 1 the root node, and the corresponding functions $\Phi_{1}, \ldots, \Phi_{m-1}$. Call such trees m-binarization trees.

For example, the tree (2) that we considered in the introduction is a 6-binarization tree and defines the following functions:

| $x$ | $\Phi_{1}(x)$ | $\Phi_{2}(x)$ | $\Phi_{3}(x)$ | $\Phi_{4}(x)$ | $\Phi_{5}(x)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | $\lambda$ | 0 | 1 | 1 |
| 1 | 1 | $\lambda$ | 0 | 0 | $\lambda$ |
| 2 | 0 | 0 | $\lambda$ | $\lambda$ | $\lambda$ |
| 3 | 1 | $\lambda$ | 1 | $\lambda$ | $\lambda$ |
| 4 | 1 | $\lambda$ | 0 | 1 | 0 |
| 5 | 0 | 1 | $\lambda$ | $\lambda$ | $\lambda$ |

Theorem 6. For an m-binarization tree, the set of associated functions $\Phi=\left\{\Phi_{1}, \ldots, \Phi_{m-1}\right\}$ is a complete binarization. Also, any nonempty subset of $\Phi$ is a binarization.

For a proof, we use the leaf entropy theorem together with a technical lemma that we call Structure Lemma. The coin $X_{i}=\Phi_{i}(X)$ has an output with probability $P_{i}=\sum_{j \in \operatorname{supp}\left(\Phi_{i}\right)} p_{j}$, and its distribution is $\pi_{i}=\langle p, q\rangle$, where

$$
p=\sum_{j \in \operatorname{supp}_{0}\left(\Phi_{i}\right)} p_{j} / P_{i}, \quad q=\sum_{j \in \operatorname{supp}_{1}\left(\Phi_{i}\right)} p_{j} / P_{i} .
$$

Stated below is the leaf entropy theorem in our context of $m$-binarization trees.
Theorem 7 (Leaf Entropy Theorem). The branching entropies of $\Phi_{i}(X)$ weighted by the probability $P_{i}$ sum up to the entropy of $X$ :

$$
H(X)=\sum_{i=1}^{m-1} P_{i} H\left(\pi_{i}\right) .
$$

The following is the main technical tool of this work and we prove it in Section IV.
Lemma 8 (Structure Lemma). Let $\Phi=\left\{\Phi_{1}, \ldots, \Phi_{m-1}\right\}$ be the set of functions defined by an m-binarization tree. Then the mapping $\Phi: x \mapsto \Phi(x)=\left(\Phi_{1}(x), \ldots, \Phi_{m-1}(x)\right)$ gives a one-to-one correspondence between an equiprobable subset $S=S_{\left(n_{0}, n_{1}, \ldots, n_{m-1}\right)}$ and $\Phi_{1}(S) \times \cdots \times \Phi_{m-1}(S)$.
Proof of Theorem 6 Let $\Psi$ be a 2-extracting procedure. For an equiprobable set $S$, each $S_{i}=\Phi_{i}(S)$ is equiprobable, and thus $\Psi\left(\left(S_{i}\right)\right)$ is extracting, by Lemma 4 Now, by Lemma $8, \Psi^{\prime}((S))=\Psi\left(\left(S_{1}\right)\right) * \cdots * \Psi\left(\left(S_{m-1}\right)\right)$. Since each $\Psi\left(\left(S_{i}\right)\right)$ is extracting, their concatenation $\Psi^{\prime}((S))$ is extracting, by the associativity of concatenation of multisets and the fact that concatenation of extracting multisets is extracting. The same holds true even if we omit some components of $\Phi$.

Since the coin $X_{i}=\Phi_{i}(X)$ has the distribution $\pi_{i}$ and outputs with the probability $P_{i}$, if $\Psi$ is asymptotically optimal, then the output rate of $\Psi\left(X_{i}\right)$ converges to $P_{i} H\left(\pi_{i}\right)$ as the input size $n \rightarrow \infty$. Therefore, the output rate of $\Psi^{\prime}$ approaches to $\sum P_{i} H\left(\pi_{i}\right)$, which equals $H(X)$ by the leaf entropy theorem.

## III. Examples

## A. An Entropy-Preserving Binarization

For a symbol $x \in\{0,1, \ldots, m-1\}$ and $1 \leq i \leq m-1$, consider

$$
x^{(i)}= \begin{cases}0, & x<i, \\ 1, & x=i, \\ \lambda, & x>i\end{cases}
$$

When $m=6$, we have their values as follow:

| $x$ | $\operatorname{Pr}(x)$ | $x^{(1)}$ | $x^{(2)}$ | $x^{(3)}$ | $x^{(4)}$ | $x^{(5)}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $p_{0}$ | 0 | 0 | 0 | 0 | 0 |
| 1 | $p_{1}$ | 1 | 0 | 0 | 0 | 0 |
| 2 | $p_{2}$ | $\lambda$ | 1 | 0 | 0 | 0 |
| 3 | $p_{3}$ | $\lambda$ | $\lambda$ | 1 | 0 | 0 |
| 4 | $p_{4}$ | $\lambda$ | $\lambda$ | $\lambda$ | 1 | 0 |
| 5 | $p_{5}$ | $\lambda$ | $\lambda$ | $\lambda$ | $\lambda$ | 1 |

These functions are associated with the following 6-binarization tree:


For $x=x_{1} \ldots x_{n} \in\{0,1, \ldots, m-1\}^{n}$, define $x^{(i)}=x_{1}^{(i)} * \cdots * x_{n}^{(i)}$. So for a sequence $x$ of length $n, x^{(i)}$ is a binary sequence of length at most $n$. For a binary extracting procedure $\Psi$, the function $\Psi^{\prime}:\{0,1, \ldots, m-1\}^{n} \rightarrow\{0,1\}^{*}$, defined by

$$
\Psi^{\prime}(x)=\Psi\left(x^{(1)}\right) * \cdots * \Psi\left(x^{(m-1)}\right),
$$

is $m$-extracting, and if $\Psi$ is asymptotically optimal, then so is $\Psi^{\prime}$.
To illustrate the structure lemma, for $m=4$, consider an equiprobable subset $S=S_{(1,2,1)} \subset\{0,1,2\}^{4}$, and let $S^{(i)}=\left\{x^{(i)} \mid x \in S\right\}$. Then, $S^{(i)}$ is another equiprobable set in $\{0,1\}^{n^{\prime}}$. For example, for $S=S_{(1,2,1)}$, observe that

| $x$ | $x^{(2)}$ | $x^{(1)}$ |
| :---: | :---: | :---: |
| 0112 | 0001 | 011 |
| 0121 | 0010 | 011 |
| 0211 | 0100 | 011 |
| 1012 | 0001 | 101 |
| 1021 | 0010 | 101 |
| 1102 | 0001 | 110 |
| 1120 | 0010 | 110 |
| 1201 | 0100 | 101 |
| 1210 | 0100 | 110 |
| 2011 | 1000 | 011 |
| 2101 | 1000 | 101 |
| 2110 | 1000 | 110 |

and we can see that, as multiset images of $x^{(1)}$ and $x^{(2)}$,

$$
\begin{aligned}
& S^{(1))}=4 \cdot S_{(1,2)}, \\
& S^{(2))}=3 \cdot S_{(3,1)} .
\end{aligned}
$$

Note that

$$
\left|S_{(1,2,1)}\right|=\frac{4!}{1!2!1!}=\frac{3!}{1!2!} \times \frac{4!}{3!1!}=\left|S_{(1,2)}\right| \times\left|S_{(3,1)}\right| .
$$

Of course, by the structure lemma, $S$ is in one-to-one correspondence with $S^{(1)} \times S^{(2)}$.

## B. Zhou-Bruck Binarization

The following method was proposed by Zhou and Bruck [7]. For $x \in\{0,1, \ldots, m-1\}$, let $x^{\prime}$ be the $\lceil\lg m\rceil$-bit binary expansion of $x$, and also for $\alpha \in\{0,1\}^{*}$, let

$$
x^{\alpha}= \begin{cases}a, & \text { if } \alpha a \text { is a prefix of } x^{\prime} \\ \lambda, & \text { otherwise }\end{cases}
$$

That is, $x^{\alpha}$ is the bit that immediately follows $\alpha$ in the standard binary expansion of $x$. For example, when $m=6$, we have the following functions:

| $x$ | $x^{\prime}$ | $x^{\lambda}$ | $x^{0}$ | $x^{1}$ | $x^{00}$ | $x^{01}$ | $x^{10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 000 | 0 | 0 | $\lambda$ | 0 | $\lambda$ | $\lambda$ |
| 1 | 001 | 0 | 0 | $\lambda$ | 1 | $\lambda$ | $\lambda$ |
| 2 | 010 | 0 | 1 | $\lambda$ | $\lambda$ | 0 | $\lambda$ |
| 3 | 011 | 0 | 1 | $\lambda$ | $\lambda$ | 1 | $\lambda$ |
| 4 | 100 | 1 | $\lambda$ | 0 | $\lambda$ | $\lambda$ | 0 |
| 5 | 101 | 1 | $\lambda$ | 0 | $\lambda$ | $\lambda$ | 1 |

After the degenerate $x^{1}$ is removed, they are associated with the following 6-binarization tree:


The mapping $x \mapsto \Psi^{\prime}(x)=\Psi\left(x^{\lambda}\right) * \cdots * \Psi\left(x^{1 \cdots 1}\right)$ is an asymptotically optimal $m$-extracting procedure if $\Psi$ is asymptotically optimal.

## IV. The Structure Lemma

Given a binarization tree and its subtree $T$, let $X_{T}$ be the restriction of $X$ on the leaf set of $T$. The leaf entropy theorem is proved by induction using the following recursion $\sqrt{\square}$

$$
H\left(X_{T}\right)= \begin{cases}0, & \text { if } T \text { is a leaf, }  \tag{3}\\ H(\pi)+p H\left(X_{T_{1}}\right)+q H\left(X_{T_{2}}\right), & \text { otherwise }\end{cases}
$$

where, for nonempty $T, T_{1}$ and $T_{2}$ are the left and right subtrees and $\pi=\langle p, q\rangle$ is the branching distribution of the root of $T$. The structure lemma holds for a similar reason.

Proof of Structure Lemma. For an equiprobable subset $S=S_{\left(n_{0}, \ldots, n_{m-1}\right)}$ and a subtree $T$ of the given binarization tree, let $S_{T}$ be the restriction of $S$ on the leaf set of $T$. Then we have a similar recursion

$$
S_{T} \cong \begin{cases}\{0\}, & \text { if } T \text { is a leaf },  \tag{4}\\ S_{(l, k)} \times S_{T_{1}} \times S_{T_{2}}, & \text { otherwise },\end{cases}
$$

where, for nonempty $T$ and $\phi$ the branching function associated with the root of $T, T_{1}$ and $T_{2}$ are the left and right subtrees and

$$
l=\sum_{\phi(i)=0} n_{i}, \quad k=\sum_{\phi(i)=1} n_{i} .
$$

[^1]First, if $T$ is a leaf with label $i$, then $S_{T}$ is a singleton set that consists of a single string of $n_{i} i$ 's, hence the first part of (4). When $T$ is nonempty, the correspondence $S_{T} \rightarrow S_{(l, k)} \times S_{T_{1}} \times S_{T_{2}}$ is given by $x \mapsto\left(\phi(x), x_{T_{1}}, x_{T_{2}}\right)$, where $x_{T_{1}}$ and $x_{T_{2}}$ are restrictions of $x$. This correspondence is one-to-one because $\phi(x)$ encodes the branching with which $x$ is recovered from $x_{T_{1}}$ and $x_{T_{2}}$, giving an inverse mapping $S_{(l, k)} \times S_{T_{1}} \times S_{T_{2}} \rightarrow S_{T}$. For example, consider tree (2) and suppose that $T$ is the subtree rooted at the node 3. For $x=102235315401$, the following shows the restrictions of $x$ and $\Phi_{i}(x)$ 's.


By taking symbols one by one from $x_{T_{1}}=101401$ and $x_{T_{2}}=33$, according to $\Phi_{3}(x)=00110000=\left(b_{i}\right)_{i=1}^{8}$, if $b_{i}$ is 0 , from $x_{T_{1}}$, otherwise, from $x_{T_{2}}$, we recover $x_{T}=10331401$.

Induction on subtrees proves the lemma.
See [13] for an alternative proof.

## V. Remarks

## A. Leaf Entropy Theorem and Structure Lemma

The leaf entropy theorem is well known in the information theory, and it follows from the grouping rule of entropy (see, e.g., the defining property 3 of entropy in Shannon's original work [4, p. 49], or Problem 2.27 of [5]), which is essentially the recursion (3) in Section IV] As we saw, the structure lemma is proved similarly, hinting that they are closely related. In fact, using the asymptotic equipartition property (AEP) [5], the structure lemma implies the leaf entropy theorem.

For a large $n$, the typical set $A^{(n)}$ consists of $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ that contains about $n_{0}=p_{0} n 0$ 's, $n_{1}=p_{1} n$ 1's, $\ldots, n_{m-1}=p_{m-1} n(m-1)$ 's. Let $S=S_{\left(n_{0}, \ldots, n_{m-1}\right)}$. The asymptotic equipartition property implies that $\lim _{n \rightarrow \infty} \frac{1}{n} \log |S|=H(X)$. On the other hand, by Structure Lemma, $S=S_{1} \times \cdots \times S_{m-1}$, where $S_{i}=\Phi_{i}(S)$. Note that $S_{i}=S_{\left(l_{i}, k_{i}\right)}$, where

$$
l_{i}=\sum_{j \in \operatorname{supp}_{0}\left(\Phi_{i}\right)} n_{j}, \quad k_{i}=\sum_{j \in \operatorname{supp}_{1}\left(\Phi_{i}\right)} n_{j}
$$

and $\left(l_{i}+k_{i}\right) / n \rightarrow P_{i}$ and $\left\langle l_{i} / n, k_{i} / n\right\rangle \rightarrow \pi_{i}$ as $n \rightarrow \infty$. Since $\frac{1}{\left(l_{i}+k_{i}\right)} \log \left|S_{i}\right| \rightarrow H\left(\pi_{i}\right)$, we have

$$
\frac{1}{n} \log \left|S_{i}\right| \rightarrow P_{i} H\left(\pi_{i}\right),
$$

and

$$
\frac{1}{n} \log |S|=\frac{1}{n} \sum_{i=1}^{m-1} \log \left|S_{i}\right| \rightarrow \sum_{i=1}^{m-1} P_{i} H\left(\pi_{i}\right)
$$

as $n \rightarrow \infty$.

## B. Generalization of Structure Lemma to Non-Binary Trees

The leaf entropy theorem holds for general trees. The structure lemma also can be generalized to trees whose nodes are not necessarily of degree 2 and whose leaves have unique labels, although in that case, the naming "binarization tree" might not be appropriate.

## C. m-ary Asymptotically Optimal Extracting Algorithm

As an immediate application, take the original binary Peres procedure $\Psi$ and apply Theorem6 The resulting $\Psi^{\prime}$ is an $m$-ary asymptotically optimal extracting procedure. As with the original Peres algorithm and its generalization, $\Psi^{\prime}$ runs in $O(n \log n)$ time, for a fixed $m$, because $\Phi_{i}(x)$ is computed in linear time and $\left|\Phi_{i}(x)\right| \leq n$ for each $i$.

## D. Other Applications of Binarization Trees

Peres algorithm is a simple extracting algorithm defined recursively using the famous von Neumann trick as a base, whose output rate approaches the information-theoretic upper bound [3]. However, it is relatively hard to explain why it works, and it appears partly due to this difficulty that its generalization to many-valued source was discovered only recently [6]. Binarization tree provides a new unified way to understand the original Peres algorithm and its generalizations and facilitates finding many new Peres-style recursive algorithms [14]. By coming up with an appropriate binarization tree (not necessarily based on binary tree but possibly a general tree), a Peres-style recursion follows. As with our main result, Theorem 6, the Peres-style recursive algorithms are extracting by the corresponding structure lemma, and asymptotically optimal by the leaf entropy theorem.
The structure lemma gives many different ways to factorize a set of $m$-combinations into sets of binary combinations. We can use this idea to give a ranking on $m$-combinations, which can be seen as a mixed-radix number system whose radices are binomial numbers [15].

## E. Binarization Trees and DDG-trees

DDG-trees (discrete distribution generation trees) work in the opposite way of binarization trees [16], [17], [11], [10]. With a binarization tree, the leaves correspond to the source and various coins are produced. With DDG trees, the nodes correspond to the source and target symbols of the leaves are produced. However, the essential difference is that DDG has the same branching distribution for every node and that the leaves don't have to have unique labels. If the various source coins with distributions $\pi_{i}$ 's are provided, and the coins are tossed starting from the root in the fashion of DDG-trees, then we arrive at leaves with the target probability distribution $\left\langle p_{0}, \ldots, p_{m-1}\right\rangle$. Therefore, binarization tree can be regarded as a generalization of DDG-tree with more than one source and unique labels on leaves.

## References

[1] J. von Neumann, "Various techniques for use in connection with random digits. Notes by G. E. Forsythe," in Monte Carlo Method, Applied Mathematics Series. U.S. National Bureau of Standards, Washington D.C., 1951, vol. 12, pp. 36-38, reprinted in von Neumann's Collected Works 5 (Pergammon Press, 1963), 768-770.
[2] P. Elias, "The efficient construction of an unbiased random sequence," The Annals of Mathematical Statistics, vol. 43, no. 3, pp. 865-870, 1972.
[3] Y. Peres, "Iterating von Neumann's procedure for extracting random bits," Annals of Statistics, vol. 20, no. 1, pp. 590-597, 1992.
[4] C. E. Shannon and W. Weaver, The Mathematical Theory of Communication. Urbana: The University of Illinois Press, 1964.
[5] T. M. Cover and J. A. Thomas, Elements of information theory (2. ed.). Wiley, 2006.
[6] S. Pae, "A generalization of Peres's algorithm for generating random bits from loaded dice," IEEE Transactions on Information Theory, vol. 61, no. 2, 2015.
[7] H. Zhou and J. Bruck, "A universal scheme for transforming binary algorithms to generate random bits from loaded dice," CoRR, vol. abs/1209.0726, 2012. [Online]. Available: http://arxiv.org/abs/1209.0726
[8] J. L. Massey, "The entropy of a rooted tree with probabilities," in Proceedings of the 1983 IEEE International Symposium on Information Theory, 1983.
[9] D. E. Knuth, The Art of Computer Programming, Sorting and Searching, 2nd ed. Addison-Wesley, 1998, vol. 3.
[10] S. Pae and M. C. Loui, "Randomizing functions: Simulation of discrete probability distribution using a source of unknown distribution," IEEE Transactions on Information Theory, vol. 52, no. 11, pp. 4965-4976, November 2006.
[11] ——, "Optimal random number generation from a biased coin," in Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, January 2005, pp. 1079-1088.
[12] S. Pae, "Exact output rate of Peres's algorithm for random number generation," Inf. Process. Lett., vol. 113, no. 5-6, pp. 160-164, 2013.
[13] -_, "Binarizations in random number generation," in IEEE International Symposium on Information Theory, ISIT 2016, Barcelona, Spain, July 10-15, 2016, 2016, pp. 2923-2927. [Online]. Available: https://doi.org/10.1109/ISIT.2016.7541834
[14] ——, "Peres-style recursive algorithms," 2018, submitted.
[15] --, "Recursive enumerations of combinations," 2018, in preparation.
[16] D. E. Knuth and A. C.-C. Yao, "The complexity of nonuniform random number generation," in Algorithms and Complexity: New Directions and Recent Results. Proceedings of a Symposium, J. F. Traub, Ed., Carnegie-Mellon University, Computer Science Department. New York, NY: Academic Press, 1976, pp. 357-428, reprinted in Knuth's Selected Papers on Analysis of Algorithms (CSLI, 2000).
[17] T. S. Han and M. Hoshi, "Interval algorithm for random number generation," IEEE Transactions on Information Theory, vol. 43, no. 2, pp. 599-611, 1997.


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[^1]:    ${ }^{1}$ Recall that a binary tree is recursively defined to be a set of nodes that is either an empty set (a terminal node), or consists of a root node, a left subtree and a right subtree, both of which are binary trees.

