# On the list decodability of Rank Metric codes 

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Let $k, n, m \in \mathbb{Z}^{+}$be integers such that $k \leq n \leq m$, let $\mathrm{G}_{n, k} \in$ $\mathbb{F}_{q^{m}}^{n}$ be a Delsarte-Gabidulin code ([8],[10]).

In [37], Wachter-Zeh proved that codes belonging to this family cannot be efficiently list decoded for any radius $\tau$, providing $\tau$ is large enough. This achievement essentially relies on proving a lower bound for the list size of some specific words in $\mathbb{F}_{q^{m}}^{n} \backslash \mathrm{G}_{n, k}$; [37, Theorem 1].

In [31], Raviv and Wachter-Zeh improved this bound in a special case, i.e. when $n \mid m$. As a consequence, they were able to detect infinite families of Delsarte-Gabidulin codes that cannot be efficiently list decoded at all.

In this article we determine similar lower bounds for Maximum Rank Distance codes belonging to a wider class of examples, containing Generalized Gabidulin codes, Generalized Twisted Gabidulin codes, and examples recently described by the first author and Yue Zhou in [35]. By exploiting arguments such like those used in [37] and [31], when $n \mid m$, we also show infinite families of generalized Gabidulin codes that cannot be list decoded efficiently at any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+1$, where $d$ is its minimum distance. Nonetheless, in all the examples belonging to above mentioned class, we detect infinite families that cannot be list decoded efficiently at any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+2$, where $d$ is its minimum distance. In particular, this leads to show infinite families of Gabidulin codes, with underlying parameters not covered by [31, Theorem 4], having this decodability defect.

Finally, relying on the properties of a set of subspace trinomials recently presented in [23], we are able to prove our main result, that is any rank-metric code of $\mathbb{F}_{q^{m}}^{n}$ of order $q^{k n}$ with $n$ dividing $m$, such that $4 n-3$ is a square in $\mathbb{Z}$ and containing $\mathrm{G}_{n, 2}$, is not efficiently list decodable at some values of the radius $\tau$.

## I. Introduction

Let $q$ be a prime power. The set $\mathbb{F}_{q}^{m \times n}$ of all $m \times n$ matrices over $\mathbb{F}_{q}$, is an $\mathbb{F}_{q}$-vector space. The rank metric distance on $\mathbb{F}_{q}^{m \times n}$ is defined by

$$
d(A, B)=\operatorname{rk}(A-B) \text { for } A, B \in \mathbb{F}_{q}^{m \times n}
$$

where $\operatorname{rk}(C)$ stands for the rank of $C$.
A subset $\mathrm{C} \subseteq \mathbb{F}_{q}^{m \times n}$ is called a rank-metric code. The minimum distance of C is

$$
d(\mathrm{C})=\min _{A, B \in \mathrm{C}, A \neq B}\{d(A, B)\}
$$

[^0]When C is a subspace of $\mathbb{F}_{q}^{m \times n}$, we say that C is an $\mathbb{F}_{q^{-}}$ linear code and its dimension $\operatorname{dim}_{\mathbb{F}_{q}}(\mathrm{C})$ is defined to be the dimension of C as a subspace over $\mathbb{F}_{q}$. It is well-known that

$$
\begin{equation*}
|\mathrm{C}| \leq q^{\max \{m, n\}(\min \{m, n\}-d(\mathrm{C})+1)} \tag{1}
\end{equation*}
$$

When the equality holds, we call C a maximum rank distance (MRD for short) code.

For MRD codes with minimum distance less than $\min \{m, n\}$, there are a few known constructions. The first and most famous family is due to Delsarte [8] and Gabidulin [10] who found it independently. This family is later generalized by Kshevetskiy and Gabidulin in [16], and we often call them generalized Gabidulin codes. More recently in [34], the author exhibited two infinite families of linear MRD codes which are not equivalent to generalized Gabidulin codes. We call them twisted Gabidulin codes and generalized twisted Gabidulin codes. In [20] it was shown that the latter family contains both generalized Gabidulin codes and twisted Gabidulin codes as proper subsets, and in [27] this family was further generalized. Finally in [35] the authors presented a new family of MRD codes in the case when $m$ is even. Further families of MRD codes are known for some particular values of the parameters [1], [4], [6], [7], [22].

Consider the set of $q$-polynomials with coefficients in $\mathbb{F}_{q^{m}}$; i.e., the set of polynomials defined as follows:

$$
\mathcal{L}_{m, q}[x]=\left\{\sum_{i=0}^{l} c_{i} x^{q^{i}}: c_{i} \in \mathbb{F}_{q^{m}} \ell \in \mathbb{N}_{0}\right\}
$$

Any polynomial $f$ in $\mathcal{L}_{m, q}[x]$ gives rise to an $\mathbb{F}_{q}$-linear map $x \in \mathbb{F}_{q^{m}} \mapsto f(x) \in \mathbb{F}_{q^{m}}$. If $c_{l} \neq 0$ we will refer to $l$ as to the $q$-degree of $f$.

It is well known that

$$
\left(\mathcal{L}_{m, q}[x] /\left(x^{q^{m}}-x\right),+, \circ, \cdot\right)
$$

where + is the addition of maps, $\circ$ is the composition of maps and • is the scalar multiplication by elements of $\mathbb{F}_{q}$, is isomorphic to the algebra of $m \times m$ matrices over $\mathbb{F}_{q}$, and hence to $\operatorname{End}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{m}}\right)$; i.e., the set of endomorphisms on $\mathbb{F}_{q^{m}}$ seen as an $\mathbb{F}_{q}$-algebra. In the following we will denote this algebra by the symbol $\tilde{\mathcal{L}}_{\tilde{2}, q}[x]$ and we will always silently identify the elements of $\tilde{\mathcal{L}}_{m, q}[x]$ with the endomorphisms of $\mathbb{F}_{q^{m}}$ they represent. Consequently, we will speak also of kernel and rank of a polynomial meaning by this the kernel and rank of the corresponding endomorphism. Clearly, the kernel of
$f \in \tilde{\mathcal{L}}_{m, q}[x]$ coincides with the set of the roots of $f$ over $\mathbb{F}_{q^{m}}$ and as usual $\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{Im}(f)+\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{ker}(f)=m$.

Let $\sigma$ be a generator of the Galois group $\operatorname{Gal}\left(\mathbb{F}_{q^{m}}: \mathbb{F}_{q}\right)$. A $\sigma$-linearized polynomial over $\mathbb{F}_{q^{m}}$, is a polynomial of type $f(x)=c_{0} x+c_{1} x^{\sigma}+\cdots+c_{\ell} x^{\sigma^{\ell}}$ with $c_{j} \in \mathbb{F}_{q^{m}}$. If $c_{\ell} \neq 0$, we say that $\ell$ is the $\sigma$-degree of the polynomial and we will denote it by $\operatorname{deg}_{\sigma} f$. It is clear that a $\sigma$-linearized polynomial can always be seen as an element of $\mathcal{L}_{m, q}[x]$.

From now on suppose $n \leq m$. All above mentioned examples produce MRD codes that can be presented in terms of so called puncturing operation applied to a subset of $\mathcal{L}_{m, q}[x]$.

Precisely, let $f_{1}$ and $f_{2}$ be two additive functions of $\mathbb{F}_{q^{m}}$ and let $k \leq m-1$. Following [33, Proposition 1], the subset of $\mathcal{L}_{m, q}[x]$
$\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)=\left\{f_{1}(a) x+\sum_{i=1}^{k-1} a_{i} x^{\sigma^{i}}+f_{2}(a) x^{\sigma^{k}}: a, a_{i} \in \mathbb{F}_{q^{m}}\right\}$
with $N\left(f_{1}(a)\right) \neq(-1)^{m k} N\left(f_{2}(a)\right)$ for all $a \in \mathbb{F}_{q^{m}}^{*}$, where $N(f(a))=\mathrm{N}_{q^{m} / q}(f(a))=f(a)^{1+q+\ldots+q^{m-1}}$, defines an MRD code of $\mathbb{F}_{q}^{m \times m}$ with minimum distance $d=m-k+1$. For instance, if $f_{1}(a)=a$ and $f_{2}=0$, then $\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)$ is a generalized Gabidulin code (commonly indicated with the symbol $\mathcal{G}_{m, k, \sigma}$ ).

More in detail, Table $\mathbb{\square}$ reports the examples of the so far known MRD codes of $\tilde{\mathcal{L}}_{m, q}[x]$ that can be represented as in (2).

All of the examples above contain a subcode equivalent to $\mathcal{G}_{m, k-1, \sigma}$ and if $\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)$ is not equivalent to any generalized Gabidulin code, then $k-1$ is the maximum dimension of a subcode in it which is equivalent to a generalized Gabidulin code, i.e. $\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)$ has Gabidulin index $k-1$, (see [11]).

There are other equivalent ways of representing a rankmetric code of $\mathbb{F}_{q}^{m \times n}$. For our purpose, we will see such codes also as subsets of $\mathbb{F}_{q^{m}}^{n}$.

For a vector $\mathbf{v}=\left(v_{1}, \ldots, v_{n}\right) \in \mathbb{F}_{q^{m}}^{n}$, we define its rank weight as follows

$$
\operatorname{rk}(\mathbf{v})=\operatorname{dim}_{\mathbb{F}_{q}}\left\langle v_{1}, \ldots, v_{n}\right\rangle_{\mathbb{F}_{q}} .
$$

The rank distance between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_{q^{m}}^{n}$ is defined as $d(\mathbf{u}, \mathbf{v})=\operatorname{rk}(\mathbf{u}-\mathbf{v})$. A rank-metric code of $\mathbb{F}_{q^{m}}^{\underline{q}}$ is a subset of $\mathbb{F}_{q^{m}}^{n}$ equipped with the aforementioned metric. The same bound (1) holds and hence we can define again an MRD code C as the code whose parameters attain the equality in (1), i.e. $|\mathrm{C}|=q^{m k}$ and for each $\mathbf{u}, \mathbf{v} \in \mathrm{C}$ with $\mathbf{u} \neq \mathbf{v}$ we have that $\operatorname{rk}(\mathbf{u}-\mathbf{v}) \geq n-k+1$.

Of course we may always jump from the model of linearized polynomials to $\mathbb{F}_{q^{m}}^{m}$. More in general we have the following
Lemma 1. Let $\mathcal{C}$ be an MRD code of $\tilde{\mathcal{L}}_{m, q}$ with $|\mathcal{C}|=q^{m k}$ and $d(\mathcal{C})=m-k+1$. Let $n$ be a positive integer greater than or equal to $k$ and the $\mathcal{S}=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ be an $n$-set in $\mathbb{F}_{q^{m}}$ (i.e. $n \mathbb{F}_{q^{-}}$-linearly independent elements). Then the rankmetric code

$$
\mathrm{C}=\left\{\left(g\left(\alpha_{1}\right), \ldots, g\left(\alpha_{n}\right)\right): g \in \mathcal{C}\right\} \subseteq \mathbb{F}_{q^{m}}^{n}
$$

is an MRD code of $\mathbb{F}_{q^{m}}^{n}$ with $|\mathcal{C}|=q^{m k}$ and $d(\mathrm{C})=n-k+1$.
As a consequence of this lemma, we have that
$\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)[\underline{\alpha}]=\left\{\left(g\left(\alpha_{1}\right), \ldots, g\left(\alpha_{n}\right)\right): g \in \mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)\right\}$,
where $\underline{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ with $\operatorname{dim}_{\mathbb{F}_{q^{m}}}\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle_{\mathbb{F}_{q^{m}}}=n$, gives rise to an MRD code of $\mathbb{F}_{q^{m}}^{n}$ with minimum distance $d=n-k+1$.

In the following, when $\sigma, k$ and $\underline{\alpha}$ are clear from the context, we denote $\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)[\underline{\alpha}]$ by $\mathrm{H}_{n}\left(f_{1}, f_{2}\right)$.

Codes $H_{n, k, 1}(a, 0)=\left\{\left(g\left(\alpha_{1}\right), \ldots, g\left(\alpha_{n}\right)\right): g \in \mathcal{G}_{m, k}\right\}$, correspond to those presented in [8] and [10], and are known as Delsarte-Gabidulin codes. We will denote them in the following by the symbol $\mathrm{G}_{n, k}$. In particular, $\mathrm{G}_{n, k, \sigma}[\underline{\alpha}]$ denotes the rank-metric code obtained by evaluating each element of $\mathcal{G}_{m, k, \sigma}$ in $\underline{\alpha} \in \mathbb{F}_{q^{m}}^{n}$.

For each element $\omega \in \mathbb{F}_{q^{m}}^{n}$ and $\tau \in \mathbb{Z}^{+}$, we define

$$
B_{\tau}(\omega):=\left\{c \in \mathbb{F}_{q^{m}}^{n} \mid \operatorname{rk}(\omega-c) \leq \tau\right\}
$$

Elias in [9] and Wozencraft in [41] introduced, in the Hamming metric, the problem of list decoding a given code. Such a problem may be stated in the following very general fashion. Let $C$ be any code of lenght $n$ in the metric space $\mathbb{F}_{q^{m}}^{n}$, and let $\tau$ be a positive number (a radius). Given a received word, output the list of all codewords of the code within distance $\tau$ from it. A list decoding algorithm returns the list of all codewords with distance at most $\tau$ from any given word.

We say that $\mathrm{C} \subseteq \mathbb{F}_{q^{m}}^{n}$ is efficiently list decodable at the radius $\tau$, if there exists a polynomial-time (in the length of the code, i.e. $n$ ) list decoding algorithm. Of course, if there exists a word $\omega \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{C}$ for which $B_{\tau}(\omega) \cap \mathrm{C}$ has exponential size in $n$, such an algorithm cannot exist since writing down the list already has exponential complexity. When such an algorithm does not exist we say that $C$ is not efficiently list decodable at the radius $\tau$. See [13] for further details on the list decodability issue.

It is well known that many of the codes in (3) can be efficiently decoded whenever up to $\left\lfloor\frac{d-1}{2}\right\rfloor$ rank errors occur, where $d$ is its minimum distance. Several decoding algorithms exist for Gabidulin codes as shown by Gabidulin in [10], by Richter and Plass in [32] and by Loidreau in [19]. These methods were speeded up by Afanassiev, Bossert, Sidorenko and Wachter-Zeh in [38], [39], [40] and more recently by Puchinger and Wachter-Zeh in [28], see also [36]. First Randrianarisoa and Rosenthal in [30] and then, completing missing cases, Randrianarisoa in [29] proposed a decoding algorithm for generalized twisted Gabidulin codes, see also [17] for the additive case and for fields of characteristic two.

However, in general it is not clear whether these codes, as well as others in Class (3), can be efficiently list decoded from a larger number of errors.

In [37], by adapting to the rank metric setting techniques appeared in [2], [15], the author proved that Delsarte-Gabidulin

TABLE I
KNown examples of MrD Codes in $\tilde{\mathcal{L}}_{m, q}[x]$

| Symbol | $\sigma$ | $f_{1}(a)$ | $f_{2}(a)$ | Conditions | References |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathcal{G}_{m, k}$ | $q$ | $a$ | 0 | - | [8], [10] |
| $\mathcal{G}^{\text {m,k, }}$, | $q^{s}$ | $a$ | 0 | - | [16] |
| $\mathcal{H}_{m, k, \sigma}(\eta, h)$ | $q^{s}$ | $a$ | $\eta a^{q^{h}}$ | $N_{q^{m} / q}(\eta) \neq(-1)^{m k}$ | [34], [20] |
| $\overline{\mathcal{H}}_{m, k, \sigma}(\eta, h)$ | $q^{s}$ | $a$ | $\eta a^{p^{h}}$ | $N_{q^{m} / p}(\eta) \neq(-1)^{m k}$ | [27] |
| $\mathcal{D}_{m, k, \sigma}(\eta)$ | $q^{s}$ | $a$ | $\eta b$ | $m$ even, $N_{q^{m} / q}(\eta) \notin \square, a, b \in \mathbb{F}_{q^{m / 2}}$ | [35] |

codes $\mathrm{G}_{n, k} \subset \mathbb{F}_{q^{m}}^{n}$ with minimum distance $d$ cannot be efficiently list decoded at any radius $\tau$ such that

$$
\begin{equation*}
\tau \geq \frac{m+n}{2}-\sqrt{\frac{(m+n)^{2}}{2}-m(d-\epsilon)} \tag{4}
\end{equation*}
$$

where $0 \leq \epsilon<1$.
More precisely, as a corollary of [37, Theorem 1], an exponential lower bound for the size of $\mathrm{G}_{n, k} \cap B_{\tau}(\omega)$ was proven, for suitable elements $\omega$ in $\mathbb{F}_{q^{m}}^{n}$, providing (4) holds true.

In [31] the authors improved this result under specific restrictions for the involved parameters. As a consequence, infinite families of Delsarte-Gabidulin codes are explicitly exhibited, which are not efficiently list decodable for each $\tau$ exceeding the unique decoding radius by one. In other words codes in these latter families, cannot be list decoded efficiently at all.

In this article we prove a similar limit in list decoding behavior for all others examples in the Class described in (3).

Precisely, elaborating on the techniques used in [37] and in [31], we generalize to these latter examples, results contained in [37] and [31]. As a consequence of this, providing that $n$ divides $m$, infinite families of generalized Gabidulin codes that cannot be efficiently list decoded at all, are detected. Also, we exhibit infinite families of codes in $\mathrm{H}_{n}\left(f_{1}, f_{2}\right)$, that cannot be list decoded efficiently at any radius greater than or equal to $\tau=\left\lfloor\frac{d-1}{2}\right\rfloor+2$.

Finally, relying on the properties of a set of subspace of $q$-trinomials recently presented in [23], we are able to prove that any rank-metric code of $\mathbb{F}_{q^{m}}^{n}$ of order $q^{k n}$ with $n$ dividing $m$ containing $\mathrm{G}_{n, 2}$ (which implies that its Gabidulin index is at least two), and such that $4 n-3$ is a square in $\mathbb{Z}$, is not efficiently list decodable for any radius greater than or equal to $\frac{2 n-1-\sqrt{4 n-3}}{2}$.

## II. PRELIMINARY RESULTS

Let $\mathcal{S}$ be an $n$-subset of $\mathbb{F}_{q^{m}}$, and let $U_{\mathcal{S}}$ be the $\mathbb{F}_{q^{-} \text {-subspace of }}$ $\mathbb{F}_{q^{m}}$, seen as $m$-dimensional vector space over $\mathbb{F}_{q}$, generated by the elements of $\mathcal{S}$.

In order to investigate maximum rank-metric codes in $\mathbb{F}_{q^{m}}^{n}$ where $n<m$ in terms of polynomials, we will need the following results concerning with $q$-polynomials over $\mathbb{F}_{q^{m}}$.

Lemma 2. Let $n$, $m$ be in $\mathbb{Z}^{+}$satisfying that $n \leqslant m$, and let $q$ be a prime power. Let $\mathcal{S}$ be a subset consisting of $n$ arbitrary
$\mathbb{F}_{q^{-}}$-linearly independent elements $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{F}_{q^{m}}$. Define $\theta_{\mathcal{S}}:=\prod_{u \in U_{\mathcal{S}}}(x-u)$. Then we have

$$
\mathcal{L}_{m, q}[x] /\left(\theta_{\mathcal{S}}\right) \cong\left\{\left(f\left(\alpha_{1}\right), \ldots, f\left(\alpha_{n}\right)\right): f \in \mathcal{L}_{m, q}[x]\right\}
$$

Proof. The map given by

$$
\varphi: f \in \mathcal{L}_{m, q}[x] \mapsto\left(f\left(\alpha_{1}\right), \ldots, f\left(\alpha_{n}\right)\right) \in \mathbb{F}_{q^{m}}^{n}
$$

is clearly surjective and $\mathbb{F}_{q}$-linear. By noting that $\varphi(f)$ is the zero vector if and only if $f(x)=0$ for every $x \in U_{\mathcal{S}}$, we see that $\operatorname{ker}(\varphi)=\left\{f \in \mathcal{L}_{m, q}[x]: f \equiv 0 \bmod \theta_{\mathcal{S}}\right\}$. This concludes the proof.

For the subset $\mathcal{S}$ made up of $n$ arbitrary $\mathbb{F}_{q}$-linearly independent elements in $\mathbb{F}_{q^{m}}$, we define

$$
\begin{array}{rlr}
\pi_{\mathcal{S}}: \quad \mathcal{L}_{m, q}[x] & \rightarrow \quad \mathcal{L}_{m, q}[x] /\left(\theta_{\mathcal{S}}\right) \\
f & \mapsto & f \bmod \theta_{\mathcal{S}}
\end{array}
$$

In particular, as already observed, when $U_{\mathcal{S}}=\mathbb{F}_{q^{m}}$, by Lemma 2 we have

$$
\operatorname{End}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{m}}\right) \cong \mathcal{L}_{m, q}[x] /\left(x^{q^{m}}-x\right)
$$

Lemma 3. Let $\mathcal{S}$ be an $n$-subset of $\mathbb{F}_{q}$-linearly independent elements in $\mathbb{F}_{q^{m}}$. Let $\mathcal{C}$ be a subset of $\mathcal{L}_{m, q}[x]$. Assume that for any distinct $f$ and $g \in \mathcal{C}$, the number of solutions of $f=g$ in $U_{\mathcal{S}}$ is strictly smaller than $q^{n}$. Then $\pi_{\mathcal{S}}$ is injective on $\mathcal{C}$.
Proof. It follows directly from the assumption $\mid\left\{x \in \mathbb{F}_{q^{m}}\right.$ : $f(x)=0\}\left|<q^{n}=\left|U_{\mathcal{S}}\right|\right.$ and the fact that $f \equiv 0 \bmod \theta_{\mathcal{S}}$ if and only if $f(u)=0$ for every $u \in U_{\mathcal{S}}$.

By Lemma 3 and the fact that $\mathcal{G}_{n, k, \sigma}$ is an MRD code, the following result can readily be verified.

Corollary 4. Let $\mathcal{S}$ be an $n$-subset of $\mathbb{F}_{q}$-linearly independent elements in $\mathbb{F}_{q^{m}}$. Let $\sigma$ be a generator of $\operatorname{Gal}\left(\mathbb{F}_{q^{m}}: \mathbb{F}_{q}\right)$. Then the set

$$
\begin{equation*}
\operatorname{Tran}=\left\{a_{0} x+a_{1} x^{\sigma}+\cdots+a_{n-1} x^{\sigma^{n-1}}: a_{i} \in \mathbb{F}_{q^{m}}\right\} \tag{5}
\end{equation*}
$$

is a transversal, namely a system of distinct representatives, for the ideal $\left(\theta_{\mathcal{S}}\right)$ in $\mathcal{L}_{m, q}[x]$.

This implies that if $f(x) \in \mathcal{L}_{m, q}[x]$ is a $\sigma$-polynomial with $\sigma$-degree less or equal to $n-1$, then

$$
f(x) \equiv 0 \quad\left(\bmod \theta_{\mathcal{S}}\right) \Leftrightarrow f(x)=0
$$

and if $f(x)$ and $g(x)$ are two $\sigma$-polynomials in Tran then

$$
f(x) \equiv g(x) \quad\left(\bmod \theta_{\mathcal{S}}\right) \Leftrightarrow f(x)=g(x)
$$

Clearly, if an $\mathbb{F}_{q}$-linear subset $\mathcal{C} \subseteq \mathcal{L}_{m, q}[x]$ is of size $q^{n k}$ and each nonzero polynomial in $\mathcal{C}$ has at most $q^{k-1}$ roots over $U_{\mathcal{S}}$, for instance $\mathcal{C}=\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)$, then the assumption on $\mathcal{C}$ in Lemma 3 is satisfied.

By Lemmas 2 and 3, it follows that codes described in (3) can be equivalently written as
$\pi_{\mathcal{S}}\left(\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)\right)=\left\{f \bmod \theta_{\mathcal{S}}: f \in \mathcal{H}_{n}\left(f_{1}, f_{2}\right)\right\} \subseteq \mathcal{L}_{m, q}[x] /\left(\theta_{\mathcal{S}}\right)$
In particular, when $n=m$, it becomes

$$
\left\{f \bmod \left(x^{q^{m}}-x\right): f \in \mathcal{H}_{k, m, \sigma}\left(f_{1}, f_{2}\right)\right\} \subseteq \tilde{\mathcal{L}}_{m, q}[x] .
$$

## III. Bounds on list decodability of MRD codes in <br> $$
\mathrm{H}_{n}\left(f_{1}, f_{2}\right)
$$

Let $\sigma$ be a generator of $\operatorname{Gal}\left(\mathbb{F}_{q^{m}}: \mathbb{F}_{q}\right)$. In what follows the concept of $\sigma$-subspace polynomial will be of some importance. A $\sigma$-subspace polynomial with respect to $\mathbb{F}_{q^{m}}$, is a monic linearized polynomial, say $s(x)$, satisfying the property that, if $r=\operatorname{deg}_{\sigma} s$, there exists an $r$-dimensional subspace $U$ of $\mathbb{F}_{q^{m}}$, seen as a vector space over $\mathbb{F}_{q}$, such that
$s(x)=(-1)^{r+1} \frac{1}{\left|\left(\begin{array}{cccc}u_{1} & u_{1}^{\sigma} & \cdots & u_{1}^{\sigma^{r-1}} \\ \vdots & & & \\ u_{r} & u_{r}^{\sigma} & \cdots & u_{r}^{\sigma^{r-1}}\end{array}\right)\right|\left(\begin{array}{cccc}x & x^{\sigma} & \cdots & x^{\sigma^{r}} \\ u_{1} & u_{1}^{\sigma} & \cdots & u_{1}^{\sigma^{r}} \\ \vdots & & & \\ u_{r} & u_{r}^{\sigma} & \cdots & u_{r}^{\sigma^{r}}\end{array}\right), ~}$
where $u_{1}, \ldots, u_{r}$ is an $\mathbb{F}_{q}$-basis of $U$.
Clearly, $s(x)=a_{0} x+a_{1} x^{\sigma}+\cdots+a_{r-1} x^{\sigma^{r-1}}+x^{\sigma^{r}}$ for some $a_{0}, a_{1}, \ldots, a_{r-1} \in \mathbb{F}_{q^{m}}$. Also, we have that each subspace of $\mathbb{F}_{q^{m}}$ corresponds to a unique $\sigma$-subspace polynomial.

Denote by $\mathcal{P}_{r, \sigma} \subset \mathcal{L}_{m, q}[x]$ the set of all $\sigma$-subspace polynomials in $\mathcal{L}_{m, q}[x]$ associated with $r$-dimensional subspaces of $\mathbb{F}_{q}^{m}$. Clearly, we have

$$
\left|\mathcal{P}_{r, \sigma}\right|=\left[\begin{array}{c}
m \\
r
\end{array}\right]_{q}
$$

Let $\mathcal{S}$ be an $n$-subset of $\mathbb{F}_{q}$-linearly independent elements in $\mathbb{F}_{q^{m}}$ and suppose that $r<n$. As a consequence of Corollary 4 and since $r<n$, it is also plain that $\left|\pi_{\mathcal{S}}\left(\mathcal{P}_{r, \sigma}\right)\right|=\left[\begin{array}{c}m \\ r\end{array}\right]_{q}$.

Arguing as in [31, Theorem 1], we may now show the existence of a large set of $\sigma$-subspace polynomials in $\mathcal{L}_{m, q}[x]$ agreeing on their top-most $\sigma$-coefficients, whose kernels are contained in a fixed $n$-dimensional subspace. More precisely,

Lemma 5. Let $g, r, n$ and $m \in \mathbb{Z}^{+}$be positive integers such that $g \leq r<n \leq m$. Let $\mathcal{S}$ be an $n$-subset of $\mathbb{F}_{q^{m}}$ and let denote by $\tilde{\mathcal{P}}_{r, \sigma}$ the subset of $\mathcal{P}_{r, \sigma}$ whose polynomials have kernel contained in $U_{\mathcal{S}}$. There exists a subset $\mathcal{F} \subset \tilde{\mathcal{P}}_{r, \sigma}$ of $\sigma$ subspace polynomials whose elements have $\sigma$-degree $r$, and agree on the last $g \sigma$-coefficients, such that

$$
|\mathcal{F}| \geq \frac{\left[\begin{array}{c}
n \\
{ }_{n}
\end{array}\right]_{q}}{q^{m(g-1)}}
$$

Proof. Clearly, $\left|\tilde{\mathcal{P}}_{r, \sigma}\right|=\left[\begin{array}{c}n \\ r\end{array}\right]_{q}$. We can partition $\tilde{\mathcal{P}}_{r, \sigma}$ into $q^{m(g-1)}$ subsets according to their last $g \sigma$-coefficients. Then, by applying the pigeonhole principle, there exists $\mathcal{F} \subseteq \tilde{\mathcal{P}}_{r, \sigma}$ as in the assertion.

In particular, when $n \mid m$, we can take as $U_{\mathcal{S}}$ the subfield $\mathbb{F}_{q^{n}}$ of $\mathbb{F}_{q^{m}}$. In this case we can explicitly exhibit a set of $\sigma$-subspace polynomials agreeing on their top-most $\sigma$ coefficients with exponential size in the value $n$. Toward this aim we briefest the following
Lemma 6. Let $t, n$ and $m \in \mathbb{Z}^{+}$be positive integers such that $\cdot t \mid n$ and $n \mid m$. Consider the $\sigma$-polynomial

$$
f(x)=\sum_{i=0}^{\frac{n}{t}-1} \beta^{\sigma^{i t}} x^{\sigma^{i t}} \in \mathcal{L}_{m, \sigma}[X]
$$

with $\beta \in \mathbb{F}_{q^{n}}^{*}$. The number of roots of $f$ over $\mathbb{F}_{q^{m}}$ is $q^{n-t}$.
Proof. Since $f(x)$ has coefficients in $\mathbb{F}_{q^{n}}$, we may look to the $\mathbb{F}_{q^{-}}$-linear transformation $F: \mathbb{F}_{q^{n}} \rightarrow \mathbb{F}_{q^{n}}$ defined by $F(x)=$ $f(x)$. Clearly, $F$ is also $\mathbb{F}_{q^{t}}$-linear (because of the $\sigma$-powers that appear in the expression of $f$ ), $\operatorname{dim}_{\mathbb{F}_{q^{t}}} \operatorname{Im} F \geq 1$ and ker $F$ corresponds to the roots of $f$ over $\frac{q^{q^{I}}}{\mathbb{F}_{q^{n}}}$. Note that, if $x_{0} \in \mathbb{F}_{q^{n}}$ then

$$
\begin{gathered}
F\left(x_{0}\right)^{\sigma^{t}}=\left(\sum_{i=0}^{\frac{n}{t}-1} \beta^{\sigma^{i t}} x_{0}^{\sigma^{i t}}\right)^{\sigma^{t}}=\sum_{i=0}^{\frac{n}{t}-1} \beta^{\sigma^{(i+1) t}} x_{0}^{\sigma^{(i+1) t}}= \\
=\sum_{j=1}^{\frac{n}{t}} \beta^{\sigma^{j t}} x_{0}^{\sigma^{j t}}=\beta x_{0}+\sum_{j=1}^{\frac{n}{t}-1} \beta^{\sigma^{j t}} x_{0}^{\sigma^{j t}}=F\left(x_{0}\right)
\end{gathered}
$$

and hence $\operatorname{Im} F=\mathbb{F}_{q^{t}}$. It follows that the number of roots of $f$ over $\mathbb{F}_{q^{n}}$ is equal to $q^{n-t}$, since $\operatorname{dim}_{\mathbb{F}_{q^{t}}} \operatorname{Im} F=1$. Moreover, since the $\sigma$-degree of $f$ is $n-t$, by [12, Theorem 5] the number of roots of $f$ over $\mathbb{F}_{q^{m}}$ is at most $q^{n-t}$. Then, since $\mathbb{F}_{q^{n}} \subseteq \mathbb{F}_{q^{m}}$, the assertion follows.

The following construction extends [31, Construction 2].
Proposition 7. Let $t, n$ and $m \in \mathbb{Z}^{+}$be positive integers such that $t \mid n$ and $n \mid m$. The set

$$
\mathcal{T}=\left\{f_{\beta}:=\sum_{i=0}^{\frac{n}{t}-1} \beta^{\sigma^{i t}-\sigma^{n-t}} x^{\sigma^{i t}}: \beta \in \mathbb{F}_{q^{n}}^{*}\right\} \subset \mathcal{L}_{m, \sigma}[x]
$$

is a set of $\sigma$-subspace polynomials whose elements have $\sigma$ degree $n-t$, agree on their last $t \sigma$-coefficients and

$$
|\mathcal{T}|=\frac{q^{n}-1}{q^{t}-1}
$$

Proof. Since for each non zero $\beta \in \mathbb{F}_{q^{n}}$, the polynomial $f_{\beta}$ may be viewed as a linearized polynomial of the form discussed in previous Lemma 6 (up to multiplying by a suitable power of $\beta$ ), we have that $\operatorname{dim}_{\mathbb{F}_{q}}$ ker $f_{\beta}=n-t$ and because of the definition of $f_{\beta}$ it is also monic. The second part follows by applying Corollary 4 and from the fact that $f_{\alpha}=f_{\beta}$ for some $\alpha, \beta \in \mathbb{F}_{q^{n}}^{*}$ if and only if $(\alpha / \beta)^{\sigma^{t}-1}=1$.
Remark 1. If $P \in \mathcal{L}_{m, q}[x]$ and $\mathcal{S}=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$ is an $n$ subset of $\mathbb{F}_{q^{m}}$ then, denoting by $c_{P}=\left(P\left(\alpha_{1}\right), \ldots, P\left(\alpha_{n}\right)\right)$ we have that

$$
\operatorname{rk}\left(c_{P}\right)=n-\operatorname{dim}_{\mathbb{F}_{q}}\left((\operatorname{ker} P) \cap U_{\mathcal{S}}\right)
$$

viewing $P$ as an $\mathbb{F}_{q}$-linear application from $U_{\mathcal{S}}=$ $\left\langle\alpha_{1}, \ldots, \alpha_{n}\right\rangle_{\mathbb{F}_{q}}$ to $\mathbb{F}_{q^{m}}$. If ker $P \subseteq U_{\mathcal{S}}$, it follows that

$$
\operatorname{rk}\left(c_{P}\right)=n-\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{ker} P .
$$

We can now state a slightly more general version of [37, Theorem 1], which in fact may be applied to generalized Gabidulin codes. The relevan result may be derived from Lemma 5 and arguing as in the proof of Theorem 1 of [37].
Theorem 8. Let $k, n$ and $m \in \mathbb{Z}^{+}$such that $k \leq n \leq m$. Let $\mathrm{G}_{n, k, \sigma}$ be a generalized Gabidulin code with minimum distance $d=n-k+1$. Let $\tau$ be an integer such that $\left\lfloor\frac{d-1}{2}\right\rfloor+$ $1 \leq \tau \leq d-1$. Then, there exists a word $c \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{G}_{n, k, \sigma}$ such that

$$
\left|\mathrm{G}_{n, k, \sigma} \cap B_{\tau}(c)\right| \geq \frac{\left[\begin{array}{c}
n \\
n-\tau
\end{array}\right]_{q}}{q^{m(n-\tau-k)}}
$$

Of course, we are interested in the smallest values of $\tau$ for which the expression for these lower bounds grows exponentially in the integer $n$. For the generalized Gabidulin code $\mathrm{G}_{n, k, \sigma}$ we have that

$$
\begin{aligned}
& \frac{\left[\begin{array}{c}
n \\
n-\tau
\end{array}\right]_{q}}{q^{m(n-\tau-k)}} \geq \frac{q^{\tau(n-\tau)}}{q^{m(n-\tau-k)}} \geq \\
\geq & q^{m(1-\epsilon)} \cdot q^{\tau(m+n)-\tau^{2}-m(d-\epsilon)}
\end{aligned}
$$

where $0 \leq \epsilon<1$.
Hence, as already computed in [37, Section III], if

$$
\tau \geq \frac{m+n}{2}-\sqrt{\frac{(m+n)^{2}}{4}-m(d-\epsilon)}
$$

where $0 \leq \epsilon<1$; then the code $\mathrm{G}_{n, k, \sigma}$ cannot be list decoded efficiently, since we find a word $c$ for which $\mathrm{G}_{n, k, \sigma} \cap B_{\tau}(c)$ has exponential size in $n$.

By using Lemma[5] we can now extend the previous result to other MRD codes in (3). Precisely we have

Theorem 9. Let $k, n$ and $m \in \mathbb{Z}^{+}$such that $k \leq n \leq m$. Let $\mathrm{C}=\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)$ be an MRD code as in (3) with minimum distance $d=n-k+1$ and $\mathrm{C} \notin\left\{\mathrm{G}_{n, k}, \mathrm{G}_{n, k, \sigma}\right\}$. Let $\tau$ be an integer such that $\left\lfloor\frac{d-1}{2}\right\rfloor+1 \leq \tau \leq d-1$. Then, there exists a word $c \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{C}$ such that

$$
\left|\mathrm{C} \cap B_{\tau}(c)\right| \geq \frac{\left[\begin{array}{c}
n \\
n-\tau
\end{array}\right]_{q}}{q^{m(d-\tau)}}
$$

Proof. Let $\tilde{\mathcal{P}}_{n-\tau, \sigma} \subset \mathcal{L}_{m, q}[x]$ be the set of $\sigma$-subspace polynomials of $\sigma$-degree $n-\tau$ associated with $(n-\tau)$ dimensional subspaces of $\mathbb{F}_{q_{\tilde{\mathcal{P}}}^{m}}$ contained in $U_{\mathcal{S}}$. By Lemma 5] there exists a subset $\mathcal{F}$ of $\tilde{\mathcal{P}}_{n-\tau, \sigma}$ whose elements agree on the last $n-k-\tau+1=d-\tau \sigma$-coefficients, with cardinality at least

$$
\frac{\left[\begin{array}{c}
n \\
n-\tau
\end{array}\right]_{q}}{q^{m(d-\tau)}}
$$

Precisely,

$$
\begin{gathered}
\mathcal{F}=\left\{\sum_{i=0}^{l-1} a_{i} x^{\sigma^{i}}+b_{l} x^{\sigma^{l}}+\cdots+b_{n-\tau-1} x^{\sigma^{n-\tau-1}}+x^{\sigma^{n-\tau}}:\right. \\
\left.\left(a_{0}, a_{1}, \ldots, a_{l-1}\right) \in \mathcal{A}\right\},
\end{gathered}
$$

where $\mathcal{A}$ is a subset of $\mathbb{F}_{q^{m}}^{l-1}$ such that $|\mathcal{A}| \geq \frac{\left.{ }_{n-\tau}^{n}\right]_{q}}{q^{m(d-\tau)}}$, and the $b_{j}$ are fixed elements of $\mathbb{F}_{q^{m}}$, where $l=n-\tau-(n-k-\tau+1)=$ $k-1$.

Define $\mathcal{F}^{\prime}:=\mathcal{F} \circ x^{\sigma}$ and note that the $\sigma$-polynomials of $\mathcal{F}^{\prime}$ are not $\sigma$-subspace polynomials, although they still have $q^{n-\tau}$ roots. Clearly they are of type
$a_{0} x^{\sigma}+a_{1} x^{\sigma^{2}}+\ldots+a_{l-1} x^{\sigma^{l}}+b_{l} x^{\sigma^{l+1}}+\ldots+b_{n-\tau-1} x^{\sigma^{n-\tau}}+x^{\sigma^{n-\tau+1}}$.
Let $R$ be a polynomial in $\mathcal{F}^{\prime}$. Note that $c_{R} \notin \mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)$, since

$$
\operatorname{rk}\left(c_{R}\right)=n-\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{ker} R=\tau<d
$$

Arguing as in the proof of [37, Theorem 1], we have that $c_{R-P} \in \mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)$, for each $P \in \mathcal{F}^{\prime}$, and so in $B_{\tau}\left(c_{R}\right)$ is contains a subset of codewords of $\mathrm{H}_{n}\left(f_{1}, f_{2}\right)$ of size at least

$$
\frac{\left[\begin{array}{c}
n  \tag{7}\\
n-\tau
\end{array}\right]_{q}}{q^{m(d-\tau)}},
$$

and the assertion follows.
It is straightforward to show that in this case a code C of type $\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)$ with $\mathrm{C} \notin\left\{\mathrm{G}_{n, k}, \mathrm{G}_{n, k, \sigma}\right\}$ cannot be list decoded efficiently at the radius $\tau$ if

$$
\tau \geq \frac{m+n}{2}-\sqrt{\frac{(m+n)^{2}}{4}-m(d+1-\epsilon)}
$$

where $0 \leq \epsilon<1$.

## IV. More MRD codes not list decodable EFFICIENTLY AT ALL

Arguing as in Section IV of [31], and exploiting results of the previous section, we exhibit here infinite families of generalized Gabidulin codes with minimum distance $d$, that cannot be list decoded efficiently at all. More precisely, for any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor$ we prove the existence of a word $c$ for which $\mathrm{G}_{n, k, \sigma} \cap B_{\tau}(c)$ has exponential size in $n$. Also, we show infinite families of the other types of MRD codes in (3) which are not efficiently list decodable for any radius larger than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+2$, where $d$ represents their minimum distance. In what follows we will consider $\underline{\alpha}=$ $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, to be an ordered $\mathbb{F}_{q^{-}}$-basis of $\mathbb{F}_{q^{n}} \subseteq \mathbb{F}_{q^{m}}$.

Also, we remind here that the symbol $\mathrm{G}_{n, k, \sigma}[\underline{\alpha}]$, denotes the rank-metric code obtained by evaluating each element of $\mathcal{G}_{m, k, \sigma}$ in $\underline{\alpha} \in \mathbb{F}_{q^{m}}^{n}$.

## A. Infinite families of Generalized Gabidulin codes not list decodable efficiently at all

In this section we provide some refining of Theorems 4 of [31] and of subsequent results in [31] directly ensuing from it.

Theorem 10. Let $k, \tau, n$ and $m \in \mathbb{Z}^{+}$positive integers such that $k \leq n, \tau \mid n$ and $n \mid m$.

If $\tau>\left\lfloor\frac{d-1}{2}\right\rfloor$, then there exists a word $c \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{G}_{n, k, \sigma}[\underline{\alpha}]$ such that

$$
\begin{equation*}
\left|\mathrm{G}_{n, k, \sigma}[\underline{\alpha}] \cap B_{\tau}(c)\right| \geq \frac{q^{n}-1}{q^{\tau}-1} . \tag{8}
\end{equation*}
$$

Proof. By Proposition [7, there exists a set $\mathcal{T} \subset \mathcal{L}_{m, q}[x]$ of $\sigma$-subspace polynomials whose elements have $\sigma$-degree $n-\tau$, agreeing on the last $\tau \sigma$-coefficients and

$$
|\mathcal{T}|=\frac{q^{n}-1}{q^{\tau}-1}
$$

As in the proof of [37, Theorem 1], we can choose a polynomial $R \in \mathcal{T}$ and prove that for each $P \in \mathcal{T}$ we have that

$$
c_{R-P} \in \mathrm{G}_{k, n, \sigma}[\underline{\alpha}] \cap B_{\tau}\left(c_{R}\right)
$$

Hence the assertion.
Consequently, we have the following result.
Corollary 11. Let $\tau, n, m \in \mathbb{Z}^{+}$such that $\tau \mid n$ and $n \mid m$. Then any generalized Gabidulin code $\mathrm{G}_{n, k, \sigma}[\underline{\alpha}]$ with minimum distance $d=2 \tau$, cannot be list decoded efficiently at all.

Proof. The unique decoding radius of any code $\mathrm{G}_{n, k, \sigma}[\underline{\alpha}]$ indicated in the statement, is

$$
\left\lfloor\frac{d-1}{2}\right\rfloor
$$

Also, we have proved in Theorem 10 that we may find at least $\left(q^{n}-1\right) /\left(q^{\tau}-1\right) \sim q^{n / 2}$ codewords in a ball of radius $\tau=\left\lfloor\frac{d-1}{2}\right\rfloor+1$, centered in a suitable word. Hence the list size of such a word is exponential in $n$. It follows that these codes cannot be list decoded efficiently at all.
B. Infinite families of codes in $\mathrm{H}_{n}\left(f_{1}, f_{2}\right)$ almost not list decodable efficiently at all

We now prove similar results for other examples of MRD codes in (3).
Theorem 12. Let $k, \tau, n$ and $m \in \mathbb{Z}^{+}$such that $k \leq n$, $\tau+1 \mid n$ and $n \mid m$. Let $\mathrm{C}=\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)[\underline{\alpha}]$ be an MRD code defined as in (3) with minimum distance $d=n-k+1$ and C .

If $\tau>\left\lfloor\frac{d-1}{2}\right\rfloor$, then there exists a word $c \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{C}$ such that

$$
\left|\mathrm{C} \cap B_{\tau+1}(c)\right| \geq \frac{q^{n}-1}{q^{\tau+1}-1}
$$

Proof. By Proposition 7
$\mathcal{T}=\left\{\sum_{i=0}^{\frac{n}{\tau+1}-1} \beta^{\sigma^{n-\tau-1}-\sigma^{i(\tau+1)}} x^{\sigma^{i(\tau+1)}}: \beta \in \mathbb{F}_{q^{n}}^{*}\right\} \subset \mathcal{L}_{m, \sigma}[x]$
is a set of $\sigma$-subspace polynomials whose elements have $\sigma$ degree $n-\tau-1$, agreeing on the last $\tau+1 \sigma$-coefficients and

$$
|\mathcal{T}|=\frac{q^{n}-1}{q^{\tau+1}-1}
$$

Then we consider

$$
\mathcal{T}^{\prime}:=\mathcal{T} \circ x^{\sigma}
$$

which is a set of $\sigma$-polynomials of degree $n-\tau$, agreeing on the last $\tau+1 \sigma$-coefficients and

$$
\left|\mathcal{T}^{\prime}\right|=|\mathcal{T}|=\frac{q^{n}-1}{q^{\tau+1}-1}
$$

Note that the polynomials in $\mathcal{T}^{\prime}$ have still $q^{n-\tau-1}$ roots.
Let $R$ be a polynomial in $\mathcal{T}^{\prime}$. Note that $c_{R} \notin \mathrm{C}$, since $\operatorname{rk}\left(c_{P}\right)=\tau<d$. For any $P \in \mathcal{T}^{\prime}$ we have that $c_{R-P} \in \mathrm{C}$. Indeed,
$\operatorname{deg}_{\sigma}(R-P) \leq n-\tau-(\tau+1)=n-d-1=k-2<k-1$ and the coefficient of $\sigma$-degree 0 is zero, hence $R-P \in$ $\mathcal{H}_{m, k, \sigma}\left(f_{1}, f_{2}\right)$ and $c_{R-P} \in \mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)[\underline{\alpha}]$. Now, we show that $c_{R-P} \in B_{\tau+1}\left(c_{R}\right)$ for each $P \in \mathcal{F}^{\prime}$. In fact, by Remark (1) since ker $P \subseteq \mathbb{F}_{q^{n}}$ we have that
$\operatorname{rk}\left(c_{R}-c_{R-P}\right)=\operatorname{rk}\left(c_{P}\right) \leq n-\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{ker} P=n-(n-\tau-1)=\tau+1$.
Therefore, arguing as in Theorem 8, we have shown that $B_{\tau_{n}+1}\left(c_{R}\right)$ contains a subset of codewords of C of size at least $\frac{q^{n}-1}{q^{\tau+1}-1}$.

Corollary 13. Let $\tau, n, m \in \mathbb{Z}^{+}$such that $\tau+1 \mid n$ and $n \mid m$. Let $\mathrm{C}=\mathrm{H}_{n, k, \sigma}\left(f_{1}, f_{2}\right)[\underline{\alpha}]$ be an MRD code defined as in (3) with minimum distance $d=2 \tau$. Then C cannot be list decoded efficiently at any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+2$.
Proof. The unique decoding radius of any code described in the statement clearly is

$$
\left\lfloor\frac{d-1}{2}\right\rfloor=\tau-1
$$

We have just proved in Theorem 12 that we may find at least $\left(q^{n}-1\right) /\left(q^{\tau+1}-1\right) \sim q^{n / 2}$ codewords in a ball of radius $\tau+1$; hence, this code cannot be list decoded efficiently for any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+2$.

We end the section by underlining that, as a direct consequence of Corollary 13, we get families of Gabidulin codes that cannot be list decoded efficiently at any radius greater than or equal to $\left\lfloor\frac{d-1}{2}\right\rfloor+2$, with parameters not covered by [31, Theorem 4].

## V. List decodability of RANK-METRIC CODES of $\mathbb{F}_{q^{n}}^{m}$ CONTAINING $G_{2, n}$

As a consequence of results proven in [26] and [3], MRD codes are dense in the set of all rank-metric codes; although, very few families of such codes are currently known up to equivalence (for details on the equivalence relation defined for rank-metric codes, we refer the reader to [25] and [34]).

In order to distinguish rank-metric codes, in [11], the authors introduced an invariant called the Gabidulin index of a rank-metric code $\mathcal{C}$; precisely, it is the maximum dimension of a subcode of $\mathcal{C}$ equivalent to a generalized Gabidulin code.

The fact that almost all MRD codes known so far contain a generalized Gabidulin code of large dimension, together with aforementioned density results motivate the study of the list decodability of rank-metric codes having Gabidulin index greater or equal than two. Indeed, it is clear that any result in this direction would have a deep impact on a really wide class of codes. In order to do this we first recall the following result very recently obtained by McGuire and Mueller in [23] relying on the results in [5] and [24].

Theorem 14. [23] Theorem 1.1] Let $n=(t-1) t+1$ and $f(x)=x^{q^{t}}-b x^{q}-a x \in \mathcal{L}_{n, q}[x]$. If

- $\mathrm{N}_{q^{n} / q}(a)=(-1)^{t-1}$;
- $b=-a^{\frac{q^{n}-q}{q^{t}-1}}$;
- $t-1$ is a power of the characteristic of $\mathbb{F}_{q^{n}}$,
then $f$ has $q^{t}$ roots in $\mathbb{F}_{q^{n}}$.
Therefore, we can derive the following construction.
Corollary 15. Let $t$ and $n \in \mathbb{Z}^{+}$be positive integers such that $n=(t-1) t+1$ and $t-1$ is a power of the characteristic of $\mathbb{F}_{q^{n}}$. The set

$$
\begin{gathered}
\operatorname{Tri}=\left\{x^{q^{t}}-b x^{q}-a x: a, b \in \mathbb{F}_{q^{n}}, \mathrm{~N}_{q^{n} / q}(a)=(-1)^{t-1}\right. \\
\text { and } \left.b=-a^{\frac{q^{n}-q}{q^{t-1}}}\right\} \subset \mathcal{L}_{n, q}[x]
\end{gathered}
$$

is a set of $\frac{q^{n}-1}{q-1}$ subspace polynomials whose elements have $q$ degree $t$. In particular, if $n \mid m$, then Tri can be also seen as a set of $\frac{q^{n}-1}{q-1}$ subspace polynomials of $\mathcal{L}_{m, q}[x]$ whose elements have q-degree $t$ over $\mathbb{F}_{q^{m}}$.

Proof. By the previous result it follows that the polynomials in Tri are subspace polynomials and the cardinality of Tri exactly coincides with the number of elements of $\mathbb{F}_{q^{m}}$ with norm $(-1)^{t-1}$. The second part follows from the fact that $\mathbb{F}_{q^{n}} \subseteq \mathbb{F}_{q^{m}}$ and by [12, Theorem 5].

By using the previous techniques we are able to prove the following main result.

Theorem 16. Let $n, m \in \mathbb{Z}^{+}$be positive integers such that $n \mid m$. Let $\mathcal{C}$ be a rank-metric code of $\mathcal{L}_{m, q}[x]$ and let C be the associated evaluation code over an $\mathbb{F}_{q}$-basis of $\mathbb{F}_{q^{n}}$, with minimum distance d. Let $\tau \in \mathbb{Z}^{+}$such that:

1. $\left\lfloor\frac{d-1}{2}\right\rfloor+1 \leq \tau \leq d-1$
2. $n-\tau-1$ is a power of the characteristic of $\mathbb{F}_{q^{n}}$;
3. $n=(n-\tau)(n-\tau-1)+1$.

Assume that $\mathcal{C}$ contains $\mathcal{G}_{m, 2}$. Then, there exists a word $c \in$ $\mathbb{F}_{q^{m}}^{n} \backslash \mathrm{C}$ such that

$$
\left|\mathrm{C} \cap B_{\tau}(c)\right| \geq \frac{q^{n}-1}{q-1}
$$

Proof. First note that since $\mathcal{C} \supseteq \mathcal{G}_{n, 2}$ then $\mathrm{C} \supseteq \mathrm{G}_{m, 2}$. By Corollary 15, the set
$\operatorname{Tri}=\left\{x^{q^{n-\tau}}-b x^{q}-a x: a, b \in \mathbb{F}_{q^{n}}, \mathrm{~N}_{q^{n} / q}(a)=(-1)^{n-\tau-1}\right.$

$$
\text { and } \left.b=-a^{\frac{q^{n}-q}{q^{n-\tau}-1}}\right\}
$$

is a set of $\frac{q^{n}-1}{q-1}$ subspace polynomials over $\mathbb{F}_{q^{m}}$ with $q$-degree $n-\tau$. Let $R \in$ Tri. Since $\operatorname{deg}_{q} R=n-\tau$ and ker $R \subseteq \mathbb{F}_{q^{n}}$, by Remark 1 we have that $\operatorname{rk}\left(c_{R}\right)=\tau<d$ and hence $c_{R} \notin \mathrm{C}$.

Let $P \in$ Tri, then

$$
\operatorname{deg}_{q}(R-P) \leq 1
$$

and so $R-P \in \mathcal{G}_{m, 2} \subseteq \mathcal{C}$ and $c_{R-P} \in \mathrm{C}$. Also, for each $P \in \operatorname{Tri}$, by Remark 1 , since $\operatorname{ker} P \subseteq \mathbb{F}_{q^{n}}$ it follows

$$
\begin{gathered}
\operatorname{rk}\left(c_{R}-c_{R-P}\right)=\operatorname{rk}\left(c_{P}\right)= \\
=n-\operatorname{dim}_{\mathbb{F}_{q}} \operatorname{ker} P=n-(n-\tau)=\tau .
\end{gathered}
$$

Therefore, for each $P \in$ Tri we have that

$$
c_{R-P} \in \mathrm{G}_{n, 2} \cap B_{\tau}\left(c_{R}\right) \subseteq \mathrm{C} \cap B_{\tau}\left(c_{R}\right)
$$

Finally, we have to prove that different choices of $P \in \operatorname{Tri}$ produces different codewords $c_{R-P}$ of C. Indeed, suppose that $P, P^{\prime} \in$ Tri with $P \neq P^{\prime}$ and $c_{R-P}=c_{R-P^{\prime}}$, then it follows that

$$
c_{P^{\prime}-P}=\mathbf{0}
$$

but since $P-P^{\prime} \in$ Tran, this can not be the case because of Corollary 4 This completely proves the assertion.

Remark 2. Once we fix the integer $n$, providing $4 n-3$ is a square in $\mathbb{Z}^{+}$, we may always find an integer $\tau$ such that

$$
n=(n-\tau)(n-\tau-1)+1
$$

in fact, $\tau=\frac{2 n-1-\sqrt{4 n-3}}{2}$.
Remark 3. We observe that if $\mathrm{C}=\mathrm{G}_{n, k}$ with $k \geq 2$ and with constraints on the involved parameters as prescribed in Theorem 16, then there exists a word $c \in \mathbb{F}_{q^{m}}^{n} \backslash \mathrm{C}$ such that

$$
\left|\mathrm{C} \cap B_{\tau}(c)\right| \geq \frac{q^{n}-1}{q-1} \sim q^{n-1}
$$

which improves the list size provided in [31, Theorem 3], for any value of $\tau$ greater than or equal to $\frac{2 n-1-\sqrt{4 n-3}}{2}$.

Hence, as a corollary of Theorem 16 and by applying Remark 2] we have the following result.

Corollary 17. Let $n, m \in \mathbb{Z}^{+}$such that

- $n \mid m$;
- $4 n-3$ is a square in $\mathbb{Z}$ and $\tau=\frac{2 n-1-\sqrt{4 n-3}}{2}$;
- $n-\tau-1$ is a power of the characteristic of $\mathbb{F}_{q^{n}}$.

Let $\mathcal{C}$ be an rank-metric code of $\mathcal{L}_{m, q}[x]$ containing $\mathcal{G}_{m, 2}$ and let C be the associated evaluation code over a basis of $\mathbb{F}_{q^{n}}$ with minimum distance $d$. Then the code C is not $\tau$-list decodable efficiently. In particular, it is not t-list decodable efficiently for each $t \geq \tau$.

We conclude the section by showing an example.
Example 18. Suppose that $n=7$ and $q$ is even. By Remark 2 we have that $\tau=4$. Let $\mathcal{C}$ be an MRD code of $\mathcal{L}_{m, q}$, with $m=7 \cdot \ell$ and $|\mathcal{C}|=q^{3 m}$ and so C is an MRD code of $\mathbb{F}_{q^{m}}^{7}$ with minimum distance $d=5$. By the previous corollary, such an MRD code is not 4-list decodable efficiently.

## VI. Conclusions and final remarks

Applying the puncturing operation to a Delsarte-Gabidulin code $\mathrm{G}_{n, k}$, under some strict constraints on involved parameters, in [31], the authors succeeded in showing the existence of a Delsarte-Gabidulin code $\mathrm{G}_{n-1, k}$ not list decodable efficiently at any value beyond the unique decoding radius. As a consequence of Theorems 8 and 9 of Section 历II, same approach as that used in Lemma 7 and subsequent Corollary 3 of [31], leads to similar achievements for generalized Gabidulin codes and for the other examples in Class (3). This strengthens the belief that divisibility condition between $n$ and $m$ may be actually ruled out.

Nonetheless, Theorems 8 and 9 have a counterpart for subspace codes naturally associated with the MRD codes in the Class (2) by means of the so called lifting procedure. Precisely, providing the radius is large enough, the list associated to certain subspaces of the vector space $V(m+n, q)$ turns out to be exponential, which makes these subspace codes not efficiently list decodable, as well. Also, values of the parameters can be introduced in order to get examples that can not be list decoded efficiently at all.
The behavior of the codes in (3) from the list decodability point of view does not rule out the possibility to find out subcodes of relevant codes, for which efficient algorithms for list decoding exists, whenever a reasonable amount of rank error beyond the unique decoding radius occur. For instance in [14], under the hypothesis that $n \mid m$, the authors provide a subcode of a Delsarte-Gabidulin code $\mathrm{G}_{n, k}$ that in fact can be list decoded efficiently up to $\frac{s(n-k)}{s+1}$ errors, where $s$ is any integer such that $1 \leq s \leq m$.

We point out that techniques developed in Section $I V$ of [14], specifically Lemmas 14 and 15 and 16 , which are key tools towards the determination of relevant subcodes and related list decoding algorithm, may be adapted to codes in (3), still providing divisibility condition.

Finally, one interesting problem to be addressed for future research is studying in which circumstances, if there are, it is possible to generalize results of [23] to $\sigma$-polynomials. In fact, this will yield to a generalization of Theorem 16 and Corollary 17 to any rank-metric code of Gabidulin index two.

## REFERENCES

[1] D. Bartoli, C. Zanella and F. Zullo: A new family of maximum scattered linear sets in $\operatorname{PG}\left(1, q^{6}\right)$, arXiv:1910.02278
[2] E. Ben-Sasson, S. Kopparty, and J. Radhakrishnan: Subspace polynomials and limits to list decoding of Reed-Solomon codes, IEEE Trans. Inf. Theory, 56(1) (2010), 113-120.
[3] E. Byrne and A. Ravagnani: Partition-balanced families of codes and asymptotic enumeration in coding theory, https://arxiv.org/abs/1805.02049
[4] B. Csajbók, G. Marino, O. Polverino and C. Zanella: A new family of MRD-codes, Linear Algebra Appl. 548 (2018), 203-220.
[5] B. Csajbók, G. Marino, O. Polverino and F. Zullo: A characterization of linearized polynomials with maximum kernel, Finite Fields Appl. 56 (2019), 109-130.
[6] B. Csajbók, G. Marino, O. Polverino and Y. Zhou: Maximum rank-distance codes with maximum left and right idealisers, https://arxiv.org/abs/1807.08774
[7] B. CsAJBÓK, G. MARINO AND F. Zullo: New maximum scattered linear sets of the projective line, Finite Fields Appl. 54 (2018), 133150.
[8] P. Delsarte: Bilinear forms over a finite field, with applications to coding theory, J. Combin. Theory Ser. A, 25 (1978), 226-241.
[9] P. Elias: List decoding for noisy channels, Massachusetts Inst. Technol. Cambridge, MA, USA, 1957, Tech. Rep. 335.
[10] E. Gabidulin: Theory of codes with maximum rank distance, Problems of information transmission, 21(3) (1985), 3-16.
[11] L. GiUZZI And F. Zullo: Identifiers for MRD-codes, Linear Algebra Appl. 575 (2019), 66-86.
[12] R. Gow and R. Quinlan: Galois extensions and subspaces of alterning bilinear forms with special rank properties, Linear Algebra Appl. 430 (2009), 2212-2224.
[13] V. Guruswami: Algorithmic results in list decoding, Boston, MA, USA: Now Publishers Inc., 2006.
[14] V. Guruswami, C. Wang and C. Xing: Explicit list-decodable rank metric and subspace codes via subspace designs, IEEE Trans. Inf. Theory, 62(5) (2016), 2707-2717.
[15] J. Justesen and T. Hoholdt: Bounds on list decoding of MDS codes, IEEE Trans. Inf. Theory, 47(4) (2001), 1604-1609.
[16] A. Kshevetskiy and E. Gabidulin: The new construction of rank codes, International Symposium on Information Theory, 2005. ISIT 2005. Proceedings, pages 2105-2108, Sept. 2005.
[17] C. LI AND W. KADIR: On decoding additive generalised twisted Gabidulin codes, In International Workshop on Coding and Cryptography, 2019
[18] D. Liebhold and G. Nebe: Automorphism groups of Gabidulin-like codes, Arch. Math. 107(4) (2016), 355-366.
[19] P. Loidreau: A Welch-Berlekamp like algorithm for decoding Gabidulin codes, In Coding and cryptography, Springer, Berlin, Heidelberg (2006), 36-45.
[20] G. Lunardon, R. Trombetti and Y. Zhou: Generalized twisted Gabidulin codes, J. Combin. Theory Ser. A 159 (2018), 79-106.
[21] G. Lunardon, R. Trombetti and Y. Zhou: On kernels and nuclei of rank metric codes, J. Algebraic Combin. 46 (2017), 313-340.
[22] G. Marino, M. Montanucci and F. Zullo: MRD-codes arising from the trinomial $x^{q}+x^{q^{3}}+c x^{q^{5}} \in \mathbb{F}_{q^{6}}[x]$, $\operatorname{arXiv:1907.08122}$ to appear in Linear Algebra and its Applications.
[23] G. McGuire and D. Mueller: Results on linearized trinomials having certain rank, https://arxiv.org/abs/1905.11755
[24] G. McGuire and J. Sheekey: A Characterization of the number of roots of linearized and projective polynomials in the field of coefficients, Finite Fields Appl. 57 (2019), 68-91.
[25] K. Morrison: Equivalence for rank-metric and matrix codes and automorphism groups of Gabidulin codes, IEEE Trans. Inform. Theoryl. 60 (2014), 7035-7046.
[26] A. Neri, A. Horlemann-Trautmann, T. Randrianarisoa and J. Rosenthal: On the genericity of maximum rank distance and Gabidulin codes, Des. Codes Cryptogr. 86(2) (2018), 1-23.
[27] K. Otal and F. Ozbudak: Additive rank-metric codes, IEEE Trans. Inform. Theory 63 (2017), 164-168.
[28] S. Puchinger and A. Wachter-Zeh: Fast operations on linearized polynomials and their applications in coding theory, J. Symbolic Comput. 89 (2018), 194-215.
[29] T. H. RANDRIANARISOA: A decoding algorithm for rank metric codes, https://arxiv.org/abs/1712.07060
[30] T. H. Randrianarisoa and J. Rosenthal: A decoding algorithm for twisted Gabidulin codes, IEEE International Symposium on Information Theory (ISIT) (2017), 2771-2774.
[31] N. Raviv, A. Wachter-Zeh: Some Gabidulin codes cannot be list decoded efficiently at any radius, IEEE Trans. Inform. Theory 62(4) (2016), 1605-1615.
[32] G. Richter and S. Plass: Error and erasure decoding of rank-codes with a modified Berlekamp-Massey algorithm, In 5th International ITG Conference on Source and Channel Coding (2004), 249-256.
[33] J. Sheekey: MRD codes: constructions and connections, Combinatorics and finite fields: Difference sets, polynomials, pseudorandomness and applications, Radon Series on Computational and Applied Mathematics, K.-U. Schmidt and A. Winterhof (eds.).
[34] J. Sheekey: A new family of linear maximum rank distance codes, Adv. Math. Commun. 10(3) (2016), 475-488.
[35] R. Trombetti and Y. Zhou: A new family of MRD codes in $\mathbb{F}_{q}^{2 n \times 2 n}$ with right and middle nuclei $\mathbb{F}_{q^{n}}$, IEEE Trans. Inform. Theory 65(2) (2019), 1054-1062.
[36] A. WACHTER-ZEH: Decoding of block and convolutional codes in rank metric, Ph.D Thesis, Ulm University and University of Rennes 1 (2013).
[37] A. Wachter-Zeh: Bounds on list decoding of rank-metric codes. IEEE Trans. Inform. Theory 59(11) (2013), 7268-7276.
[38] A. Wachter-Zeh, V.B. Afanassiev and V.R. Sidorenko: Fast decoding of Gabidulin codes, in Int. Workshop Coding Cryptogr. (WCC) Paris France, Apr. 2011, 433-442.
[39] A. Wachter-Zeh, V.B. Afanassiev and V.R. Sidorenko: Fast decoding of Gabidulin codes, Des. Codes Cryptogr. 66(1) (2013), 5773.
[40] A. Wachter-Zeh, V.R. Sidorenko and M. Bossert: A fast linearized euclidean algorithm for decoding Gabidulin codes, in Int. Workshop Alg. Combin. Coding Theory (ACCT) Novosibirsk, Russia (2010) 298-303.
[41] J. M. Wozencraft: List decoding, Massachusetts Inst. Technol., Cambridge, MA, USA, 1958, Tech. Rep.

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