# Shannon meets von Neumann: A Minimax Theorem for Channel Coding in the Presence of a Jammer 

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#### Abstract

We study the setting of channel coding over a family of channels whose state is controlled by an adversarial jammer by viewing it as a zero-sum game between a finite blocklength encoder-decoder team, and the jammer. The encoder-decoder team choose stochastic encoding and decoding strategies to minimize the average probability of error in transmission, while the jammer chooses a distribution on the state-space to maximize this probability. The min-max value of this game is equivalent to channel coding for a compound channel - we call this the Shannon solution of the problem. The max-min value corresponds to finding a mixed channel with the largest value of the minimum achievable probability of error. When the min-max and max-min values are equal, the problem is said to admit a saddle-point or von Neumann solution. While a Shannon solution always exists, a von Neumann solution need not, owing to inherent nonconvexity in the communicating team's problem. Despite this, we show that the min-max and max-min values become equal asymptotically in the large blocklength limit, for all but finitely many rates. We explicitly characterize this limiting value as a function of the rate and obtain tight finite blocklength bounds on the min-max and max-min value. As a corollary we get an explicit expression for the $\epsilon$-capacity of a compound channel under stochastic codes the first such result, to the best of our knowledge. Our results demonstrate a deeper relation between the compound channel and mixed channel than was previously known. They also show that the conventional information-theoretic viewpoint, articulated via the Shannon solution, coincides asymptotically with the game-theoretic one articulated via the von Neumann solution. Key to our results is the derivation of new finite blocklength upper bounds on the min-max value of the game via a novel achievability scheme, and lower bounds on the max-min value obtained via the linear programming relaxation based approach we introduced in [2].


## I. Introduction

Communication theory has traditionally studied the problem of communication in the presence of a jammer, only from the communicater's perspective. The solutions sought have concentrated on what the communication system can achieve in the worst case over all possible scenarios; we call this the Shannon solution. However, alongside one could also ask a dual question, namely, what is the maximum damage that the jammer could achieve in the presence of an intelligent communication system? A natural vantage point for analyzing the system from both points of view together is that of a game between the communication system and the jammer. One could then employ solution concepts from game theory to 'solve' the game and analyze the system; we refer to the resulting solution

[^0]as the von Neumann solution. The focus of this paper is a game of the above kind in the finite blocklength regime and the relation between the Shannon and von Neumann solutions of communication problems in the presence of adversarial action.

We define the following zero-sum game between a finite blocklength communication system and a finite state jammer. The communication system attempts to communicate a discrete uniform source across a discrete memoryless channel by making $n$ uses of the channel. The channel law is determined by its state which is controlled by the jammer. The communication system comprises of a team of two decision makers - an encoder and a decoder - that together attempt to minimize the average probability of error incurred in the transmission of the source. The jammer chooses a state randomly so as to maximize the error incurred in transmission.
Following Fig 1 . let $S$ denote a source message of rate $R$ drawn uniformly at


Fig. 1: A communication system with a finite state jammer random from $\mathcal{S}:=\left\{1, \ldots, 2^{n R}\right\}$. Suppose the jammer chooses a channel state $\theta$ from a set $\Theta$ randomly according to a distribution $q$. The encoder and decoder choose stochastic strategies. A stochastic encoder maps the source $S$ to an $n$ length channel input $X$ randomly according to a conditional distribution $Q_{X \mid S}$. The channel, in state $\theta$, outputs an $n$-length string denoted $Y$, which is decoded by a stochastic decoder that outputs $\widehat{S} \in \mathcal{S}$ according to a conditional distribution $Q_{\widehat{S} \mid Y}$. We assume that the set of states, $\Theta$, is finite, and that the state once chosen remains fixed through the $n$ uses of the channel. The choice of the channel state by the jammer is independent of the source message and the encoded message, and the actual state realized is not known to either encoder or decoder.

The min-max value or upper value of this game is denoted $\bar{\vartheta}(n ; R)$ and is the optimal value of the following problem,

$$
\begin{array}{|ll}
\hline \overline{\mathrm{P}}(n ; R) & \min _{Q_{X \mid S}, Q_{\widehat{S} \mid Y}} \max _{q} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}] \\
\text { s.t } & Q_{X \mid S} \in \mathcal{P}(\mathcal{X} \mid \mathcal{S}), Q_{\widehat{S} \mid Y} \in \mathcal{P}(\widehat{\mathcal{S}} \mid \mathcal{Y}), q \in \mathcal{P}(\Theta) .
\end{array}
$$

where $\mathcal{X}, \mathcal{Y}$ are the sets of $n$-length channel inputs and outputs, respectively and $\mathcal{P}(\cdot)$ is the set of probability distributions on ' $\because$ '. The expectation $\mathbb{E}$ above is under the distribution induced by the code $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}\right)$ and the distribution $q$. Problem $\overline{\mathrm{P}}(n ; R)$ thus corresponds to the minimum probability
of error achievable by the encoder-decoder team in the worst case over all distributions $q$ chosen by the jammer. The maxmin value or lower value of the game, denoted $\underline{\vartheta}(n ; R)$, is the optimal value of the following problem,

$$
\begin{array}{|ll}
\hline \underline{\mathrm{P}}(n ; R) & \max _{q} \min _{Q_{X \mid S}, Q_{\widehat{S} \mid Y}} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}] \\
\text { s.t } & Q_{X \mid S} \in \mathcal{P}(\mathcal{X} \mid \mathcal{S}), Q_{\widehat{S} \mid Y} \in \mathcal{P}(\widehat{\mathcal{S} \mid \mathcal{Y}), q \in \mathcal{P}(\Theta),} \\
\hline
\end{array}
$$

and corresponds to the maximum probability of error achievable by the jammer in the worst case over all stochastic codes employed by the encoder-decoder team. Note that the following relation always holds:

$$
\begin{equation*}
\bar{\vartheta}(n ; R) \geq \underline{\vartheta}(n ; R), \quad \forall n \in \mathbb{N}, R \in[0, \infty) \tag{1}
\end{equation*}
$$

The above zero-sum game is said to admit a saddle point value, or a von Neumann solution, if equality holds, i.e., if $\bar{\vartheta}(n ; R)=\underline{\vartheta}(n ; R)$.

Having defined the game, we ask our first question. Does the game admit a saddle point? For each strategy of the jammer, the team problem of the encoder and decoder has nonclassical information structure [3]. As argued in [4], in the space of stochastic codes, the communicating team's problem is nonconvex for each jammer strategy. This lack of convexity implies that the existence of a saddle point is not guaranteed. Moreover, using the codes that form a saddle point (assuming one exists) may imply capacities, error exponents and suchlike that are distinct from those obtained from the Shannon solution. This leads us to our second question: do the answers obtained from the saddle point, i.e., the von Neumann solution, coincide with those obtained from the Shannon solution?

We find that, despite the skepticism voiced above, answers to both these questions are in the affirmative in the large blocklength limit, for all but finite many values of the rate $R$ of the source. The main result in this paper shows that the asymptotic value of the minimum probability of error achievable by a code of rate $R$, in the worst case over all jammer strategies, equals the asymptotic value of the maximum probability of error that a jammer can induce, in the worst case over all possible codes of rate of $R$, for all rates $R$ barring some finitely many specific values. In other words,

$$
\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R)=\lim _{n \rightarrow \infty} \underline{\vartheta}(n ; R), \quad \forall R \in[0, \infty) \backslash \mathscr{D}
$$

where $\mathscr{D}$ is a finite set. Moreover, $\vartheta(R)$, which is defined as the value of the above limits when they are equal is given by

$$
\vartheta(R)=\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \subseteq \Theta}\left\{1-\sum_{\theta \in \Theta^{\prime}} q(\theta) \mid R<C\left(\Theta^{\prime}\right)\right\}
$$

where $C\left(\Theta^{\prime}\right)$ is the capacity of the compound channel formed by $\Theta^{\prime} \subseteq \Theta$, and $\mathscr{D}$ is precisely the set of points of discontinuity of the right-hand side above when viewed as a function of $R \in[0, \infty)$. At a finite blocklength an approximate minimax theorem holds, i.e., the difference $\bar{\vartheta}(n ; R)-\underline{\vartheta}(n ; R)$ becomes vanishingly small with $n$ for all $R \geq 0$ except in $\mathscr{D}$. Interestingly, the upper value, $\bar{\vartheta}(n ; R)$ also happens to equal the probability of error guaranteed by the Shannon solution. Consequently, we find that as the blocklength becomes large,
the Shannon solution comes in harmony with the von Neumann solution. As a corollary of these results we also obtain the $\epsilon$-capacity of a compound channel under stochastic codes, the first such result, to the best of our knowledge.
Despite the nonconvexity of the communicating team's problem argued in [4], our recent results [2], [5], [6] and [7] have shown that all point-to-point problems and several network problems (without a jammer) admit a near-convexity, in the sense that they can be approximated asymptotically by a linear program. This leads us to conjecture that an 'approximate' minimax theorem may be within reach, whose approximation becomes increasingly accurate as $n \rightarrow \infty$. Our results show that this intuition is correct for almost all rates.
Since the state once chosen is held fixed throughout the $n$ transmissions, problem $\overline{\mathrm{P}}(n ; R)$ corresponds to coding for a finite blocklength compound channel [8] under stochastic codes. Problem $\underline{\mathrm{P}}(n ; R)$ on the other hand amounts to foisting a distribution on the state space so that the resulting finite blocklength mixed channel [9] has the largest probability of error. To the best of our knowledge, ours is the first result showing that these are asymptotically the same. The compound and mixed channels have been known to be intimately related (see [8], [9]); in particular they have the same capacity. Our results show that there is an even deeper relation between them.
Interestingly, this minimax theorem breaks down if one considers the maximum probability of error criterion; there is always an interval of rates where there is a gap between the upper and lower values. Furthermore, to the best of our understanding, local randomization produced by stochastic codes seems essential and we have not been able to show similar results with deterministic codes. Local randomization by the encoder allows the code to randomly communicate with the correct codebook with a positive probability, regardless of the action of the jammer. Correctly choosing this randomization achieves the optimal performance. It is unclear if the same can be achieved with deterministic codes. Indeed, formulating the problem in the space of stochastic codes was also key to our near-convexity results for coding problems found in [2], [5], [6] and [7].
There are two regimes where our result is rather easy to claim. Define,

$$
\begin{equation*}
\underline{C}=\max _{P_{\mathbb{X}}} \min _{\theta \in \Theta} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta), \quad \bar{C}=\min _{\theta \in \Theta} \max _{P_{\mathbb{X}}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta), \tag{2}
\end{equation*}
$$

where $I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)$ is the mutual information between the (single letter) channel input $\mathbb{X}$ and channel output $\mathbb{Y}$ when the state is $\theta$ and the channel input distribution is $P_{\mathbb{X}} . \underline{C}$ is the capacity of the compound channel (under deterministic and stochastic codes). Clearly, by definition of capacity, for $R<\underline{C}$, the upper value and hence the lower value of the game must go to zero as $n \rightarrow \infty . \bar{C}$ is the smallest of the capacities of individual DMCs defined by the state of the channel. For $R>\bar{C}$, the jammer can choose the state with smallest capacity (with probability one), whereby by the strong converse for channel coding for this DMC, the lower, and hence upper values must both approach unity. Thus the asymptotic minimax theorem holds for $R<\underline{C}$ and $R>\bar{C}$. The nontrivial case is


Fig. 2: $\vartheta(R)$ for four-state jammer where $C\left(\Theta^{\prime}\right)^{\prime}$ 's for $\Theta^{\prime} \subset$ $\Theta=\{1,2,3,4\}$ are assumed to satisfy: $\underline{C}<C(a)<$ $C(b)<C(c)<C(d)<C(e)<C(f)<C(g)<$ $C(h)<C(i)<C(j)<\bar{C}=C(\{1\})$ where $a=$ $\{1,2,3\}, b=\{1,2,4\}, c=\{1,3,4\}, d=\{1,2\}, e=$ $\{1,3\}, f=\{1,4\}, g=\{2,3,4\}, h=\{2,3\}, i=\{2,4\}, j=$ $\{3,4\}$. The hollow circle indicates an excluded end-point rate.
$\underline{C}<R<\bar{C}$ where we also show the minimax theorem with a new argument that exploits the stochastic encoding allowed.

The limiting saddle point value $\vartheta(R)$ happens to take a peculiar form. It is a step function wherein the points where jumps occur depend on the capacities of compound channels that correspond to subsets of $\Theta$. For example, in the case where $|\Theta|=2$, we have

$$
\vartheta(R)= \begin{cases}0 & \text { if } R<\underline{C} \\ \frac{1}{2} & \text { if } \underline{C}<R<\bar{C} \\ 1 & \text { if } R>\bar{C}\end{cases}
$$

For $|\Theta|=3$ (say $\Theta=\{1,2,3\}$ ), assuming that the capacities of compound channels corresponding to subsets $\Theta$ satisfy,

$$
\underline{C}<C(\{1,2\})<C(\{1,3\})<C(\{2,3\})<\bar{C}=C(\{1\}),
$$

we get

$$
\vartheta(R)= \begin{cases}0 & \text { if } R<\underline{C} \\ \frac{1}{3} & \text { if } \underline{C}<R<C(\{1,2\}) \\ \frac{1}{2} & \text { if } C(\{1,2\})<R \leq C(\{1,3\}) \\ \frac{1}{2} & \text { if } C(\{1,3\})<R<C(\{2,3\}) \\ \frac{2}{3} & \text { if } C(\{2,3\})<R<\bar{C} \\ 1 & \text { if } R>\bar{C}\end{cases}
$$

The saddle point value for $|\Theta|=4$ is shown in Fig 2 ,

To show our results, we find two categories of bounds: a lower bound on $\underline{\vartheta}(n ; R)$, which is equivalently a finite blocklength converse for the mixed channel, and an upper bound on $\bar{\vartheta}(n ; R)$, which is equivalently a finite blocklength achievability result for the compound channel. The former is obtained via the linear programming (LP) approach we introduced in [2]. For the latter, in the regimes $R<\underline{C}$ and $R>\bar{C}$, we employ a deterministic achievability scheme. In the intermediate region $\underline{C}<R<\bar{C}$, we employ a stochastic construction that randomly chooses between codebooks for compound channels formed by subsets of $\Theta$. This achievability scheme is novel in the sense that it was derived using the converse as a reference, reinterpreting the latter via von Neumann's minimax theorem, then and attempting to match it, rather than the common approach which attempts to find a matching converse for an achievability scheme.

## A. Related Work

The existence of a saddle point solution for a game comprising of a communication system and a jammer has been studied multiple times in the LQG setting (e.g., [10], [11]). For transmitting Gaussian sources over Gaussian channels controlled by a power-constrained jammer with access to the source, the existence of a mixed saddle point solution is shown in [12]. An extension of the above work to general sources over additive noise channel has been considered in [13]. Closely related to our problem setup is the Arbitrarily Varying Channel (AVC) model (see [14]), where a jammer changes the state of the channel at each instant during the transmission of the encoded message. AVC is used to model packet-dropout adversarial channels in [15], where the control problem over an adversarial channel is then posed as a zero-sum game between the jammer and the controller. For gaussian AVC's with power constraints, the problem of evaluating the asymptotic random coding capacity is shown to have an equivalent zero-sum game formulation between the transmitter and the jammer in [16]. To the best of our knowledge ours is the only work that studies the setting we consider, where the channel is the compound channel.

## B. Organization

Section $\Pi$ formulates the zero-sum game between the finite blocklength communication system and the finite state jammer. We also give a background on zero-sum games and compound and mixed channels. In Section III, we present the LPrelaxation based framework to obtain a lower bound on the max-min value of the game. Section IV presents a new upper bound on the min-max value of the game and analyses the asymptotic tightness of this bound to the lower bound obtained via LP for rates $R<\underline{C}$ and $R<\bar{C}$. In Section V , we consider the intermediate rate region $\underline{C}<R<\bar{C}$. We derive a novel finite blocklength upper bound on the min-max value of the game and show that this bound is asymptotically equal to the LP-based lower bound for all but finitely many rates in this range. Finally, we conclude with Section VI.

## C. Notation

Throughout this paper, upper case letters $A, B$ represent random variables taking values in spaces represented by calligraphic letters $\mathcal{A}, \mathcal{B}$ respectively and lower case letters $a, b$ represent the specific values these random variables take. Let $\mathbb{I}\{\bullet\}$ represent the indicator function which is one when • is true and is zero otherwise. $\mathcal{P}(\bullet)$ denotes the set of all probability distributions on ' $\bullet$ '. Let $\mathcal{Z}:=\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \widehat{\mathcal{S}}$ and $z:=(s, x, y, \widehat{s}) \in \mathcal{Z}$. Let $Q_{X \mid S} Q_{\widehat{S} \mid Y} P_{Y \mid X, \theta}(z, \theta) \equiv$ $Q_{X \mid S}(x \mid s) Q_{\widehat{S} \mid Y}(\widehat{s} \mid y) P_{Y \mid X, \theta}(y \mid x, \theta)$. If $P$ represents an optimization problem, $\mathrm{OPT}(P)$ represents its optimal value. If $\mathcal{A}$ represents a set, $\mathcal{A}^{c}$ represents its compliment. Let $\mathrm{L}(\mathrm{R}) \mathrm{HS}$ represent Left (Right) Hand Side. Let $x \in \mathcal{A}^{n}$ be a $n$-length string. Then,

$$
P(a \mid x)=\frac{\sum_{i=1}^{n} \mathbb{I}\left\{x_{i}=a\right\}}{n}, \quad a \in \mathcal{A}
$$

represents the type of $x$ (see [17]). Note that $P(a \mid x) \geq 0$ for all $a \in \mathcal{A}$ and $\sum_{a \in \mathcal{A}} P(a \mid x)=1$, whereby $P \in \mathcal{P}(\mathcal{A})$. Let $\mathcal{T}^{n}$ represent the set of all types in $\mathcal{A}^{n}$. From type counting lemma [17], we then have,

$$
\begin{equation*}
\left|\mathcal{T}^{n}\right| \leq(n+1)^{|\mathcal{A}|} \tag{3}
\end{equation*}
$$

## II. Background and Problem Formulation

## A. Zero-sum games

A zero-sum game comprises of two players $P_{1}, P_{2}$, with strategies denoted $z_{1} \in Z_{1}$ and $z_{2} \in Z_{2}$, respectively, and a function $c: Z_{1} \times Z_{2} \rightarrow \mathbb{R}$ of these strategies. $P_{1}$ attempts to choose $z_{1}^{*} \in Z_{1}$ so as to minimize $c$ and $P_{2}$ attempts to choose $z_{2}^{*} \in Z_{2}$ so as to maximize this function, but the value each gets depends not only what the player chooses but also what the other player chooses. How would these players then play? von Neumann surmised that $P_{1}, P_{2}$ would each choose $z_{1}^{*}, z_{2}^{*}$, respectively, so as to minimize the worst case damage that the other player could do. i.e.,

$$
\begin{equation*}
z_{1}^{*} \in \arg \min _{z_{1} \in Z_{1}} \bar{c}\left(z_{1}\right), \quad \bar{c}\left(z_{1}\right):=\max _{z_{2} \in Z_{2}} c\left(z_{1}, z_{2}\right) \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
z_{2}^{*} \in \arg \max _{z_{2} \in Z_{2}} \underline{c}\left(z_{2}\right), \quad \underline{c}\left(z_{2}\right):=\min _{z_{1} \in Z_{1}} c\left(z_{1}, z_{2}\right) \tag{5}
\end{equation*}
$$

Clearly, these calculations may not match up. Specifically, if $P_{1}$ plays $z_{1}^{*}$ it may not be optimal for $P_{2}$ to play $z_{2}^{*}$, or if $P_{2}$ plays $z_{2}^{*}, P_{1}$ may not want to stick to playing $z_{1}^{*}$. Remarkably, von Neumann showed that in certain classes of games [18], these calculations do match up exactly and we get $\bar{c}\left(z_{1}^{*}\right)=\underline{c}\left(z_{2}^{*}\right)$. In this case, we get the central solution concept in zero-sum games, proposed by von Neumann, namely, a saddle point. $\left(z_{1}^{*}, z_{2}^{*}\right)$ are said to form a saddle point of the game if

$$
\begin{equation*}
c\left(z_{1}^{*}, z_{2}\right) \leq c\left(z_{1}^{*}, z_{2}^{*}\right) \leq c\left(z_{1}, z_{2}^{*}\right) \quad \forall z_{1} \in Z_{1}, z_{2} \in Z_{2} . \tag{6}
\end{equation*}
$$

The existence of a saddle point is equivalent to the following interchangeability of the 'min' and 'max' operations on $c$, i.e.,

$$
\begin{equation*}
\min _{z_{1} \in Z_{1}} \max _{z_{2} \in Z_{2}} c\left(z_{1}, z_{2}\right)=\max _{z_{2} \in Z_{2}} \min _{z_{1} \in Z_{1}} c\left(z_{1}, z_{2}\right) \tag{7}
\end{equation*}
$$

In general, one always has that the left hand side of (7) (called the upper value of the game, and equal to $\bar{c}\left(z_{1}^{*}\right)$ ) is greater than or equal to the right hand side (called the lower value, and equal to $\underline{c}\left(z_{2}^{*}\right)$ ). von Neumann's minimax theorem [18] showed that equality holds when $Z_{1}, Z_{2}$ are finite dimensional probability simplices and $c\left(z_{1}, z_{2}\right) \equiv z_{1}^{\top} A z_{2}$ for a matrix $A$.

## B. Communication in the presence of a jammer as a game

We now present the problem of channel coding in presence of a jammer as a zero-sum game. Let $S$ represent the random source message distributed uniformly on $\mathcal{S}=\{1, \ldots, M\}$ with $M=2^{n R}$ and $R$ is the rate of transmission. Consider a family of channels $P_{\mathbb{Y} \mid \mathbb{X}, \theta} \in \mathcal{P}(\mathcal{B} \mid \mathcal{A})$ with common finite input and output alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively, parametrized by a state $\theta$ taking values in a finite set $\Theta$. Define $\mathcal{X}:=\mathcal{A}^{n}$ and $\mathcal{Y}:=\mathcal{B}^{n}$ as the spaces of $n$-length strings of channel inputs and outputs of any channel from this family. An encoder maps $S$ to an $n$-length $(n<\infty)$ string $X \in \mathcal{X}$ randomly according to the distribution $Q_{X \mid S} \in \mathcal{P}(\mathcal{X} \mid \mathcal{S})$. $X$ is subsequently sent through a channel in state $\theta \in \Theta$, represented by the stochastic kernel $P_{Y \mid X, \theta}$ to get the channel output $Y \in \mathcal{Y}$. For each $\theta \in \Theta$, we assume that the channel is discrete and memoryless, i.e.

$$
\begin{equation*}
P_{Y \mid X, \theta}(y \mid x, \theta)=\prod_{i=1}^{n} P_{\mathbb{Y} \mid \mathbb{X}, \theta}\left(y_{i} \mid x_{i}, \theta\right), \forall x \in \mathcal{X}, y \in \mathcal{Y} \tag{8}
\end{equation*}
$$

Subsequently, a decoder maps the channel output $Y$ randomly accordingly to $Q_{\widehat{S} \mid Y} \in \mathcal{P}(\mathcal{S} \mid \mathcal{Y})$ to get $\widehat{S} \in \mathcal{S}$. Here, $Q_{X \mid S}$ represents a stochastic encoder, $Q_{\widehat{S} \mid Y}$ represents a stochastic decoder and together, they constitute a stochastic code in the sense of Csiszar and Korner [17] or behavioral strategies in the language of game theory [19]. Note that this is distinct from a random code. A deterministic code is a pair of functions $f: \mathcal{S} \rightarrow \mathcal{X}, g: \mathcal{Y} \rightarrow \mathcal{S}$ and a random code is a randomly chosen pair of functions $f, g$. A jammer controls the state of the channel and chooses a state $\theta \in \Theta$ randomly according to some distribution $q \in \mathcal{P}(\Theta)$ independent of $S, X$. State $\theta$ once chosen is held fixed through $n$-uses of the channel and the channel state chosen is not known to both the encoder and decoder.
This setting can be formulated as a zero-sum game by taking $P_{1}$ as a team comprising of the stochastic encoder and decoder $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}\right)$ and its strategy space as $Z_{1}=$ $\mathcal{P}(\mathcal{X} \mid \mathcal{S}) \times \mathcal{P}(\mathcal{S} \mid \mathcal{Y})$, the set of all stochastic codes. The jammer is player $P_{2}$ with its strategy space as $\mathcal{P}(\Theta)$ and the cost function $c$ is the average probability of error over all messages, $\mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]$ evaluated under the probability distribution induced by the strategies of the encoder-decoder team and the jammer. Since for each value of $\theta$, the random variables $S, X, Y, \widehat{S}$ form a Markov chain in that order, we have

$$
\begin{align*}
& \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta] \\
& \quad=\sum_{x, y, \widehat{s}} Q_{X \mid S}(x \mid s) Q_{\widehat{S} \mid Y}(\widehat{s} \mid y) P_{Y \mid X, \theta}(y \mid x, \theta) \mathbb{I}\{s \neq \widehat{s}\} \tag{9}
\end{align*}
$$

and hence

$$
\begin{aligned}
& \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s]=\sum_{\theta \in \Theta} q(\theta) \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta] \\
& \text { and } \quad \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]=\frac{1}{M} \sum_{s \in \mathcal{S}} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s]
\end{aligned}
$$

The central quest in this paper is to investigate whether a minimax theorem can be claimed for this game. However, the answer to this question is in the negative in general when $n$ is finite since while $\mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]$ is concave (in fact, linear) in $q$, it is nonconvex in $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}\right)$, the nonconvexity arising due to the presence of bilinear products $Q_{X \mid S}(x \mid s) Q_{\widehat{S} \mid Y}(\widehat{s} \mid y)$ (see e.g., [4]) in (9]. Consequently, a saddle point value may not exist for this game. In this context, we explore an 'approximate' minmax theorem for the zero-sum game, the approximation becoming increasingly accurate as blocklength $n$ increases. Towards this, we derive new upper bounds on $\bar{\vartheta}(n ; R)$ and lower bounds on $\underline{\vartheta}(n ; R)$ and show that a nearsaddle point holds for the zero-sum game, i.e., for each $n \in \mathbb{N}$, $0 \leq \bar{\vartheta}(n ; R)-\underline{\vartheta}(n ; R) \leq \epsilon_{n}, \epsilon_{n} \in[0,1]$, where for all but finitely many values of the rate $R$, we have $\lim _{n \rightarrow \infty} \epsilon_{n}=0$. Consequently, for such rates we get

$$
\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R)=\lim _{n \rightarrow \infty} \underline{\vartheta}(n ; R)=: \vartheta(R),
$$

where $\vartheta(R)$ is now the limiting saddle-point value of the zerosum game.

## C. Background on mixed and compound channels

Our problem is intimately related to compound and mixed channels. We recall some background about them here and relate them to our problem [14]. A compound channel is defined by a family of channels $\left\{P_{\mathbb{Y} \mid \mathbb{X}, \theta}\right\}_{\theta \in \Theta}$ whose state $\theta$ is held fixed throughout the transmission. The average probability of error incurred by a code $Q_{X \mid S}, Q_{\widehat{S} \mid Y}$ over this compound channel is defined as

$$
\epsilon_{\mathrm{com}}\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y} ; n\right):=\max _{\theta \in \Theta} \sum_{s} \frac{1}{M} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta]
$$

For any $q \in \mathcal{P}(\Theta)$, a mixed channel of blocklength $n$ is defined as

$$
P_{Y \mid X}^{(q)}(y \mid x)=\sum_{\theta \in \Theta} q(\theta) P_{Y \mid X, \theta}(y \mid x, \theta)
$$

where $P_{Y \mid X, \theta}(y \mid x, \theta)$ is as defined in (8). The average probability of error incurred by a code in a mixed channel is defined in the usual sense as

$$
\epsilon_{\operatorname{mix}}\left(q, Q_{X \mid S}, Q_{\widehat{S} \mid Y} ; n\right):=\mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]
$$

From the linearity of $\mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]$ in $q$, it is easy to see that problem $\overline{\mathrm{P}}(n ; R)$ is equivalent to the finite blocklength channel coding of the compound channel under the average probability of error criterion (where the average is taken over all messsages), employing stochastic codes. Moreover, for each strategy $q \in \mathcal{P}(\Theta)$ of the jammer, the inner minimization in problem $\underline{\mathrm{P}}(n ; R)$ corresponds to the finite blocklength channel coding of the mixed channel under the average probability of error criterion. It is then evident that the bound $\epsilon_{n}$
mentioned at the end of the previous section can be established using a combination of a finite blocklength achievability for a compound channel and a finite blocklength converse for a mixed channel. Our main contribution is in showing that these asymptotically the same.
A rate $R$ is said to be achievable for a compound channel (or mixed channel) if there exists a sequence of stochastic codes $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}\right)$ with $M=2^{n R}$ such that $\epsilon_{\mathrm{com}}\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y} ; n\right)$ (or $\epsilon_{\text {mix }}\left(q, Q_{X \mid S}, Q_{\widehat{S} \mid Y} ; n\right)$ ) goes to 0 as $n \rightarrow \infty$. The supremum over all such achievable rates then gives the capacity of a compound channel (or mixed channel) under stochastic codes. Interestingly, the capacity (under both deterministic and stochastic codes) of a compound channel is (see [20])

$$
\underline{C}=\max _{P_{\mathbb{X}}} \min _{\theta \in \Theta} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta),
$$

and the capacity of the mixed channel is (see [14])

$$
\max _{P_{\mathbb{X}}} \min _{\theta: q(\theta)>0} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta),
$$

where the maximization is over $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})$ and $I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta):=$

$$
\sum_{a \in \mathcal{A}, b \in \mathcal{B}} P_{\mathbb{X}}(a) P_{\mathbb{Y} \mid \mathbb{X}, \theta}(b \mid a, \theta) \log \frac{P_{\mathbb{Y} \mid \mathbb{X}, \theta}(b \mid a, \theta)}{\sum_{a \in \mathcal{A}} P_{\mathbb{X}}(a) P_{\mathbb{Y} \mid \mathbb{X}, \theta}(b \mid a, \theta)},
$$

represents the mutual information between random variables $\mathbb{X} \in \mathcal{A}$ and $\mathbb{Y} \in \mathcal{B}$. Notice that the mixed channel capacity depends only on the support of $q$, and when the support is $\Theta$, it equals $\underline{C}$, the capacity of the compound channel. We assume throughout this paper that $\underline{C}>0$.

Under the average probability of error criterion, a strong converse does not hold for mixed and compound channels (see [21], [9]). However, if the average probability of error criterion is replaced with maximum probability of error (over all messages), i.e., $\max _{s \in \mathcal{S}, \theta \in \Theta} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta]$, a strong converse holds for the compound channel ([14]) but not the mixed channel (for the latter the criterion is $\max _{s \in \mathcal{S}} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s]$ ). We will see that this subtle difference plays an important role in our result.

## III. Finite Blocklength Lower Bounds

In this section, we obtain a lower bound on $\underline{\vartheta}(n ; R)$ via a linear programming relaxation of the minimization part of problem $\overline{\mathrm{P}}(n ; R)$. For each $q \in \mathcal{P}(\Theta), \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]$ is nonconvex in $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}\right)$, the nonconvexity resulting from the presence of bilinear products. Consequently, we relax this nonconvexity by a linear programming (LP) relaxation which results in a new min-max problem over relaxed codes.
Towards this, we resort to a lift-and-project-like idea as illustrated in [2]. For sake of completeness, we quickly outline the technique here. In this approach, we introduce new variables, $W(s, x, y, \widehat{s})$ to replace the bilinear product terms $Q_{X \mid S}(x \mid s) Q_{\widehat{S} \mid Y}(\widehat{s} \mid y)$ for each $s \in \mathcal{S}, x \in \mathcal{X}, y \in \mathcal{Y}, \widehat{s} \in \widehat{\mathcal{S}}$, thereby lifting the nonconvex problem to a higher dimensional space of variables, that now includes $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}, W\right)$. New valid inequalities in terms of $W$ are obtained in this space. Towards this, for each $s \in \mathcal{S}$, we multiply both sides of the equation $\sum_{x} Q_{X \mid S}(x \mid s)=1$ with $Q_{\widehat{S} \mid Y}(\widehat{s} \mid y)$ for all $\widehat{s}, y$ and
for each $y \in \mathcal{Y}$, multiply both sides of $\sum_{\widehat{s}} Q_{\widehat{S} \mid Y}(\widehat{s} \mid y)=1$ with $Q_{X \mid S}(x \mid s)$ for all $x, s$. Replace the bilinear products in the resulting set of equations with the new variable $W(s, x, y, \widehat{s})$ and add these to the original constraints. Thus, a lower bound on $\bar{\vartheta}(n ; R)$ is given by the optimal value of the following problem,

$$
\begin{array}{|ll}
\hline \operatorname{LP}(n ; R) & \min _{Q_{X \mid S}, Q_{\widehat{S} \mid Y}, W} \max _{q} \operatorname{err}(q, W) \\
& Q_{X \mid S} \in \mathcal{P}(\mathcal{X} \mid \mathcal{S}), Q_{\widehat{S} \mid Y} \in \mathcal{P}(\widehat{\mathcal{S}} \mid \mathcal{Y}), q \in \mathcal{P}(\Theta) \\
\text { s.t } \quad & \sum_{x} W(z)-Q_{\widehat{S} \mid Y}(\widehat{s} \mid y)=0 \quad \forall s, \widehat{s}, y, \\
& \sum_{\widehat{s}} W(z)-Q_{X \mid S}(x \mid s)=0 \quad \forall s, x, y, \\
\hline
\end{array}
$$

where

$$
\operatorname{err}(q, W)=\sum_{z, \theta} \frac{1}{M} W(z) \mathbb{I}\{s \neq \widehat{s}\} q(\theta) P_{Y \mid X, \theta}(y \mid x, \theta)
$$

A collection $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}, W\right)$ satisfying the constraints above is called a relaxed code. Clearly $\bar{\vartheta}(n ; R) \geq \operatorname{OPT}(\mathrm{LP}(n ; R))$. Note that $\operatorname{err}(q, W)$ is convex in $\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}, W\right)$ and linear in $q$. Moreover, the set of relaxed codes is convex and compact, and so is the set of $\mathcal{P}(\Theta)$. Consequently, a minmax theorem holds, and the order of the minimization and maximization can be interchanged in $\operatorname{LP}(n ; R)$. Thus,

$$
\begin{equation*}
\operatorname{OPT}(\operatorname{LP}(n ; R))=\max _{q} \min _{\left(Q_{X \mid S}, Q_{\widehat{S} \mid Y}, W\right)} \operatorname{err}(q, W) \tag{10}
\end{equation*}
$$

where the minimization is over relaxed codes and the maximization is over $q \in \mathcal{P}(\Theta)$. Now, for each $q \in \mathcal{P}(\Theta)$, the inner minimization over relaxed codes in (10) is a lower bound on the inner minimization $\min _{Q_{X \mid S}, Q_{\widehat{S} \mid Y}} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\}]$ in $\underline{\mathrm{P}}(n ; R)$; it is precisely what one would obtain if one applied the above relaxation directly to this minimization. Consequently, $\operatorname{OPT}(\operatorname{LP}(n ; R))$ is a lower bound also on $\underline{\vartheta}(n ; R)$, whereby,

$$
\begin{equation*}
\bar{\vartheta}(n ; R) \geq \underline{\vartheta}(n ; R) \geq \mathrm{OPT}(\operatorname{LP}(n ; R)) \tag{11}
\end{equation*}
$$

This substantiates our motivation - if $\operatorname{LP}(n ; R)$ is a tight relaxation of $\overline{\mathrm{P}}(n ; R)$, then the upper and lower values of the game ought to approach each other.

To obtain a lower bound from $\operatorname{LP}(n ; R)$, we replace the inner minimization over relaxed codes in (10) by its dual. The dual of the inner minimization in 10 can be written as,

$$
\begin{align*}
& \operatorname{DP}(q, n ; R) \max _{\gamma_{a}^{(q)}, \gamma_{b}^{(q)}, \lambda_{\mathrm{s}}^{(q)}, \lambda_{\mathrm{s}}^{(q)}} \sum_{s} \gamma_{a}^{(q)}(s)+\sum_{y} \gamma_{b}^{(q)}(y) \\
& \gamma_{a}^{(q)}(s)-\sum_{y} \lambda_{c}^{(q)}(s, x, y) \leq 0 \quad \forall x, s  \tag{D1}\\
& \text { s.t } \quad \gamma_{b}^{(q)}(y)-\sum_{s} \lambda_{s}^{(q)}(s, \widehat{s}, y) \leq 0 \quad \forall \hat{s}, y  \tag{D2}\\
& \lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)+\lambda_{\mathrm{c}}^{(q)}(s, x, y) \leq \Lambda^{(q)}(z) \quad \forall z \tag{D3}
\end{align*}
$$

where

$$
\Lambda^{(q)}(z)=\frac{1}{M} \mathbb{I}\{s \neq \widehat{s}\} \sum_{\theta} P_{Y \mid X, \theta}(y \mid x, \theta) q(\theta)
$$

and $\lambda_{\mathrm{s}}^{(q)}: \mathcal{S} \times \widehat{\mathcal{S}} \times \mathcal{Y} \rightarrow \mathbb{R}, \lambda_{c}^{(q)}: \mathcal{S} \times \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}, \gamma_{a}^{(q)}: \mathcal{S} \rightarrow \mathbb{R}$ and $\gamma_{b}^{(q)}: \mathcal{Y} \rightarrow \mathbb{R}$ are Lagrange multipliers (see [2] for details). From duality of linear programming, it then follows that $\operatorname{OPT}(\mathrm{LP}(n ; R))=\max _{q \in \mathcal{P}(\Theta)} \operatorname{OPT}(\mathrm{DP}(q, n ; R))$.

Towards evaluating $\operatorname{OPT}(\operatorname{DP}(q, n ; R))$, note that it is optimal to take $\gamma_{a}^{(q)}(s)=\min _{x} \sum_{y} \lambda_{c}^{(q)}(s, x, y)$ and $\gamma_{b}^{(q)}(y)=\min _{\widehat{s}} \sum_{s} \lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)$. Consequently, we get $\operatorname{OPT}(\operatorname{DP}(q, n ; R))=$

$$
\max _{\lambda_{c}^{(q)}, \lambda_{\mathrm{s}}^{(q)}}\left\{\sum_{s} \min _{x} \sum_{y} \lambda_{c}^{(q)}(s, x, y)+\sum_{y} \min _{\widehat{s}} \sum_{s} \lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)\right\}
$$

$$
\text { s.t } \quad \lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)+\lambda_{\mathrm{c}}^{(q)}(s, x, y) \leq \Lambda^{(q)}(z) \quad \forall z
$$

The following lemma then outlines our framework for deriving lower bound on $\underline{\vartheta}(n ; R)$.
Lemma 3.1: Any choice of functions $\lambda_{\mathrm{s}}^{(q)}: \mathcal{S} \times \widehat{\mathcal{S}} \times \mathcal{Y} \rightarrow \mathbb{R}$, $\lambda_{c}^{(q)}: \mathcal{X} \times \mathcal{S} \times \mathcal{Y} \rightarrow \mathbb{R}$ satisfying constraint (D3) yields the following lower bound on $\underline{\vartheta}(n ; R)$,
$\underline{\vartheta}(n ; R) \geq \max _{q} \operatorname{OPT}(\mathrm{DP}(q, n ; R))$
$\geq \max _{q}\left\{\sum_{s} \min _{x} \sum_{y} \lambda_{c}^{(q)}(s, x, y)+\sum_{y} \min _{\widehat{s}} \sum_{s} \lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)\right\}$.
In fact, a particular choice of functions $\lambda_{s}^{(q)}, \lambda_{c}^{(q)}$ satisfying constraint (D3) of $\operatorname{DP}(q, n ; R)$ for each $q \in \mathcal{P}(\Theta)$ gives the following lower bound on $\underline{\vartheta}(n ; R)$.

Theorem 3.2: The following lower bound on $\underline{\vartheta}(n ; R)$ holds,

$$
\begin{align*}
& \underline{\vartheta}(n ; R) \geq \max _{q} \operatorname{OPT}(\operatorname{DP}(q, n ; R)) \\
& \geq \max _{q} \sup _{P_{\bar{Y} \mid \theta}, \gamma>0}\left\{\operatorname { m i n } _ { x } \sum _ { y } \sum _ { \theta } q ( \theta ) \operatorname { m i n } \left\{P_{Y \mid X, \theta}(y \mid x, \theta)\right.\right. \\
& \left.\left.\quad P_{\bar{Y} \mid \theta}(y \mid \theta) M \exp (-\gamma)\right\}-\exp (-\gamma)\right\} \tag{12}
\end{align*}
$$

where $P_{\bar{Y} \mid \theta}$ is an arbitrary probability distribution in the space of $\mathcal{Y}$ for $\theta \in \Theta$.
Proof : For each $q \in \mathcal{P}(\Theta)$, consider the following choice of functions $\lambda_{\mathrm{s}}^{(q)}, \lambda_{\mathrm{c}}^{(q)}$,
$\lambda_{c}^{(q)}(s, x, y)=\sum_{\theta} \frac{q(\theta)}{M} \min \left\{P_{Y \mid X, \theta}(y \mid x, \theta), M \frac{P_{\bar{Y} \mid \theta}(y \mid \theta)}{\exp (\gamma)}\right\}$,
$\lambda_{\mathrm{s}}^{(q)}(s, \widehat{s}, y)=-\frac{1}{M} \sum_{\theta} q(\theta) P_{\bar{Y} \mid \theta}(y \mid \theta) M \exp (-\gamma) \mathbb{I}\{s=\widehat{s}\}$.
The feasibility of these functions with respect to (D3) of $\mathrm{DP}(q, n ; R)$ can be verified as in the proof of Theorem 2.1 [7] and we skip the proof here. Taking supremum over $P_{\bar{Y} \mid \theta}(y \mid \theta)$ and $\gamma>0$ of the resulting dual cost and applying Lemma 3.1 gives the required lower bound.
Lower bounding $\min \left\{P_{Y \mid X, \theta}(y \mid x, \theta), P_{\bar{Y} \mid \Theta}(y \mid \theta) M \exp (-\gamma)\right\}$ in (12) with $P_{Y \mid X, \theta}(y \mid x, \theta) \mathbb{I}\left\{i_{X ; \bar{Y} \mid \theta}(x ; Y \mid \theta) \leq \log M-\gamma\right\}$, where

$$
i_{X ; \bar{Y} \mid \theta}(x ; y \mid \theta)=\log \frac{P_{Y \mid X, \theta}(y \mid x, \theta)}{P_{\bar{Y} \mid \theta}(y \mid \theta)}
$$

then yields the following bound,

$$
\begin{align*}
& \max _{q} \sup _{P_{\bar{Y} \mid \theta}, \gamma>0} \min _{x}\left\{\sum _ { \theta } q ( \theta ) \left[\mathbb{P}_{\theta}\left[i_{X ; \bar{Y} \mid \theta}(x ; Y \mid \theta) \leq \log M-\gamma\right]\right.\right. \\
& -\exp (-\gamma)\} \tag{13}
\end{align*}
$$

(13) in turn results in the following lower bound on $\underline{\vartheta}(n ; R)$.

Corollary 3.3: The following lower bound on $\underline{\vartheta}(n ; R)$ holds,

$$
\begin{align*}
& \underline{\vartheta}(n ; R) \geq \max _{q} \operatorname{OPT}(\operatorname{DP}(q, n ; R)) \\
& \geq \max _{q \in \mathcal{P}(\Theta)} \sum_{\theta} q(\theta) \mathbb{I}\left\{I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R-2 \xi-\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right\} \\
& \quad-\frac{A(\xi)}{n}-\exp (-n \xi) \tag{14}
\end{align*}
$$

where $\xi>0$ is an arbitrary constant,

$$
\begin{equation*}
P_{\bar{Y} \mid \theta}(y \mid \theta) \equiv \frac{1}{\left|\mathcal{T}^{n}\right|} \sum_{P_{n} \in \mathcal{T}^{n}}\left(P_{n} P_{\mathbb{Y} \mid \mathbb{X}, \theta}\right)^{\times n}(y) \tag{15}
\end{equation*}
$$

and $\bar{P}_{n}(q)$ is the type of $x \in \mathcal{A}^{n}$ that minimizes
$\sum_{\theta} q(\theta) \sum_{y} P_{Y \mid X, \theta}(y \mid x, \theta) \mathbb{I}\left\{\log \frac{P_{Y \mid X, \theta}(y \mid x, \theta)}{P_{\bar{Y} \mid \theta}(y \mid \theta)} \leq n R-n \xi\right\}$.

Proof: The proof is included in Appendix A.
Having obtained lower bounds on the lower value of the game, we now move on to analysing the asymptotic equality of $\bar{\vartheta}(n ; R)$ and $\underline{\vartheta}(n ; R)$ in the limit as $n \rightarrow \infty$. Towards this, we divide the rate region into two different cases $-(a) R>\bar{C}$ or $R<\underline{C}$, (b) $\underline{C}<R<\bar{C}$ - and we obtain new upper bounds on the $\bar{\vartheta}(n ; \bar{R})$ in these regions. Together with the LP-based lower bound, we then show in Sections IV and $V$ that the min-max and max-min values of the game indeed approach each other in the limit as $n \rightarrow \infty$ in these regions.

## IV. Asymptotic EQUALIty of $\bar{\vartheta}(n ; \underline{R})$ AND $\underline{\vartheta}(n ; R)$ WHEN $R<\underline{C}$ AND $R>\bar{C}$

In this section, we analyze the asymptotic equality of $\bar{\vartheta}(n ; R)$ and $\underline{\vartheta}(n ; R)$ in the limit as $n \rightarrow \infty$ when $R<\underline{C}$ and $R>\bar{C}$. Since a lower bound on $\underline{\vartheta}(n ; R)$ follows from the LP relaxation-based bound in (14, we first derive a new finite blocklength upper bound (or achievability bound) on the min-max value $\bar{\vartheta}(n ; R)$.

Towards this, note that the objective value of $\overline{\mathrm{P}}(n ; R)$ corresponding to any choice of distributions $Q_{X \mid S}, Q_{\widehat{S} \mid Y}$ yields an upper bound on $\bar{\vartheta}(n ; R)$. In particular, we construct a deterministic code $(f, g)$ so as to obtain the following upper bound on $\bar{\vartheta}(n ; R)$.

Theorem 4.1: Let $P_{X} \in \mathcal{P}(\mathcal{X})$ be any input distribution. Then, for any stochastic code, the following upper bound holds,

$$
\begin{align*}
\bar{\vartheta}(n ; R) \leq & \max _{\theta \in \Theta, s \in \mathcal{S}} \mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta] \\
\leq & \max _{\theta \in \Theta} \mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq \alpha+\delta\right] \\
& \quad+\exp (-\delta)+\frac{M|\Theta|^{2}}{\exp (\alpha)} \tag{17}
\end{align*}
$$

where $\mathbb{E}[\mathbb{I}\{S \neq \widehat{S}\} \mid S=s, \theta]$ is the probability of error in transmitting message $s$ under this code when the channel state is $\theta, \mathbb{P}_{\theta}$ is the probability with respect to $P_{X} P_{Y \mid X, \theta}, \alpha, \delta>0$ are arbitrary constants and $P_{Y \mid \theta}(y \mid \theta)=$ $\sum_{x \in \mathcal{X}} P_{X}(x) P_{Y \mid X, \theta}(y \mid x, \theta)$.

Proof : Proof is included in Appendix B
In particular, fixing $\alpha=\log M+\frac{n \Delta}{2}, \delta=\frac{n \Delta}{2}$ with $\Delta>0$ an arbitrary positive constant, 17) results in the following upper bound,

$$
\begin{gather*}
\bar{\vartheta}(n ; R) \leq \max _{\theta \in \Theta} \mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq \log M+n \Delta\right] \\
+\exp (-n \Delta / 2)\left[|\Theta|^{2}+1\right] \tag{18}
\end{gather*}
$$

Recall from Section III that evaluating $\bar{\vartheta}(n ; R)$ is equivalent to the problem of minimizing the average probability of error over all stochastic codes in a compound channel. Earliest upper bound known for such compound channels has been obtained by Blackwell et al. [8]. Our new bound in 17) improves on the bound of Blackwell et al. by replacing $\sum_{\theta} \mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq \alpha+\delta\right]$ therein with $\max _{\theta} \mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq \alpha+\delta\right]$.

Employing this newly derived upper bound on $\bar{\vartheta}(n ; R)$ together with the lower bound on $\underline{\vartheta}(n ; R)$ obtained in 14 , the following theorem then shows that the difference between the upper and lower values of the game vanishes asymptotically as $n \rightarrow \infty$ for $R<\underline{C}$ and $R>\bar{C}$.

Theorem 4.2: Consider the zero-sum game with $M=2^{n R}$ and channel conditional distributions as given in 8). Then,

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R) & =\lim _{n \rightarrow \infty} \underline{\vartheta}(n ; R)=0 & \forall R<\underline{C} \\
\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R) & =\lim _{n \rightarrow \infty} \underline{\vartheta}(n ; R)=1 & \forall R>\bar{C}
\end{aligned}
$$

Proof : Proof is included in Appendix C
Recall from the introduction that this result is along expected lines. After all, $\underline{C}$ being the capacity of the compound channel, we must have that for $R<\underline{C}, \bar{\vartheta}(n ; R)$ and hence $\underline{\vartheta}(n ; R)$ approach 0 for large $n$. Similarly $\bar{C}$ being the smallest among the capacities of the individual DMCs corresponding to states $\theta \in \Theta$, by the strong converse of channel coding for $R>\bar{C}$, $\underline{\vartheta}(n ; R)$ and hence $\bar{\vartheta}(n ; R)$ must approach unity. An additional subtlety here is that we have finite blocklength estimates on the "approximate" minimax value, which follow from Theorem 4.1 and Corollary 3.3 Consequently, even though for each finite $n$ the game may not admit a saddle point value, for large $n$ and $R<\underline{C}$ and $R>\bar{C}$, it admits nearsaddle point value (as is evident from (46) in the Appendix). Moreover, for the class of channels where $C=\underline{C}=\bar{C}$, Theorem 4.2 shows that a near-saddle point value exists for all $R$ except when $R=C$. Examples where $\underline{C}=\bar{C}$ holds are families of discrete memoryless channels where the capacity achieving input distribution is independent of $\theta$ [14], that includes the binary symmetric channels and binary erasure channels amongst others.

## V. Asymptotic EQUALITY of $\bar{\vartheta}(n ; R)$ AND $\underline{\vartheta}(n ; R)$ FOR $\underline{C}<R<\bar{C}$

We now come to the asymptotic equality of the min-max and max-min values of the game when the rate of communication $R$ lies in the range $\underline{C}<R<\bar{C}$ and exact characterization of this limiting value. Towards this, we first establish a
fundamental lower bound on the max-min value of the game by employing the LP relaxation from Section III From the lower bound we then inspire a finite blocklength achievability scheme whose performance asymptotically matches this lower bound as $n \rightarrow \infty$ for all but finitely many values of the rate $R$.

## A. An LP-Relaxation Based Fundamental Lower Bound on Max-min problem

We employ the LP relaxation to derive a lower bound on $\liminf _{n \rightarrow \infty} \underline{\vartheta}(n ; R)$. Recall that for each $q \in \mathcal{P}(\Theta)$, the LP relaxation yields that $\underline{\vartheta}(n ; R) \geq \mathrm{OPT}(\mathrm{DP}(q))$ whereby the following limiting bound holds:

$$
\liminf _{n \rightarrow \infty} \underline{\vartheta}(n ; R) \geq \max _{q \in \mathcal{P}(\Theta)} \liminf _{n \rightarrow \infty} \operatorname{OPT}(\mathrm{DP}(q))
$$

Together with the converse obtained in (14), the LP relaxation thus gives the following limiting bound on the max-min value of the game.

Theorem 5.1: The LP-relaxation yields the following fundamental lower bound on the max-min value of the game,

$$
\begin{align*}
& \liminf _{n \rightarrow \infty} \underline{\vartheta}(n ; R) \geq \max _{q \in \mathcal{P}(\Theta)} \liminf _{n \rightarrow \infty} \operatorname{OPT}(\mathrm{DP}(q)) \geq L(R) \\
& \text { where } L(R)=\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \subseteq \Theta}\left\{1-\sum_{\theta \in \Theta^{\prime}} q(\theta) \mid R \leq C\left(\Theta^{\prime}\right)\right\} \tag{20}
\end{align*}
$$

for $R \in[0, \bar{C}]$ and,

$$
C\left(\Theta^{\prime}\right)=\max _{P_{X}} \min _{\theta \in \Theta^{\prime}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)
$$

is the capacity of the compound channel formed by $\Theta^{\prime} \subseteq \Theta$. Proof : For any $q \in \mathcal{P}(\Theta)$, we have from (14),

$$
\underline{\vartheta}(n ; R) \geq \mathrm{OPT}(\mathrm{DP}(q)) \stackrel{(a)}{\geq} \sum_{\theta \in \Theta_{n}^{\prime}} q(\theta)-\frac{A(\xi)}{n}-\exp (-n \xi),
$$

where in $(a), \xi>0$ is an arbitrary constant, $A(\xi)$ is independent of $n$ and $\Theta_{n}^{\prime}:=\left\{\theta \in \Theta \mid R \geq I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta)+\right.$ $\left.2 \xi+\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right\}$. Since

$$
R<I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta)+2 \xi+\frac{\log \left|\mathcal{T}^{n}\right|}{n} \quad \forall \theta \in\left(\Theta_{n}^{\prime}\right)^{c}
$$

we must have

$$
R<\min _{\theta \in\left(\Theta_{n}^{\prime}\right)^{c}} I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta)+2 \xi+\frac{\log \left|\mathcal{T}^{n}\right|}{n}
$$

It is clear that $\Theta_{n}^{\prime}$ satisfies that $R<C\left(\left(\Theta_{n}^{\prime}\right)^{c}\right)+2 \xi+\frac{\log \left|\mathcal{T}^{n}\right|}{n}$, whereby for any $q \in \mathcal{P}(\Theta)$,

$$
\begin{align*}
\underline{\vartheta}(n ; R) \geq & \min _{\widehat{\Theta} \subseteq \Theta}\left\{\sum_{\theta \in \widehat{\Theta}} q(\theta) \left\lvert\, R<C\left(\widehat{\Theta}^{c}\right)+2 \xi+\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right.\right\} \\
& -\frac{A(\xi)}{n}-\exp (-n \xi) \tag{21}
\end{align*}
$$

Subsequently, taking limit $n \rightarrow \infty$ on both sides of (21), noticing that there are only finitely many $\widehat{\Theta} \subseteq \Theta$, and then maximizing over $q \in \mathcal{P}(\Theta)$ yields that

$$
\liminf _{n \rightarrow \infty} \underline{\vartheta}(n ; R) \geq \max _{q \in \mathcal{P}(\Theta)} \min _{\widehat{\Theta} \subseteq \Theta}\left\{\sum_{\theta \in \widehat{\Theta}} q(\theta) \mid R<C\left(\widehat{\Theta}^{c}\right)+2 \xi\right\} .
$$

Our result then follows by taking $\xi \rightarrow 0$.

1) The quantities $L(R)$ and $U(R): L(R)$ defined in 20) plays a key role in our analysis. This section is devoted to understanding its properties. By making a slight change in the definition of $L$, we define the following quantity,

$$
\begin{equation*}
U(R):=\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \subseteq \Theta}\left\{1-\sum_{\theta \in \Theta^{\prime}} q(\theta) \mid R<C\left(\Theta^{\prime}\right)\right\} \tag{22}
\end{equation*}
$$

for $R \in[0, \bar{C}]$. For convenience, we extend the definition of $L$ and $U$ to $R>\bar{C}$ by defining,

$$
L(R)=U(R):=1 \quad \forall R>\bar{C}
$$

Observe that we always have $L(R) \leq U(R)$. To see this, note that for any rate $R$,

$$
\begin{align*}
\underline{\Upsilon}(R) & :=\left\{\Theta^{\prime} \subseteq \Theta \mid R \leq C\left(\Theta^{\prime}\right)\right\} \\
& \supseteq \bar{\Upsilon}(R):=\left\{\Theta^{\prime} \subseteq \Theta \mid R<C\left(\Theta^{\prime}\right)\right\} \tag{23}
\end{align*}
$$

holds in general, whereby for any $q \in \mathcal{P}(\Theta), \min _{\Theta^{\prime} \in \underline{\Upsilon}(R)}[1-$ $\left.\sum_{\theta \in \Theta^{\prime}} q(\theta)\right] \leq \min _{\Theta^{\prime} \in \bar{\Upsilon}(R)}\left[1-\sum_{\theta \in \Theta^{\prime}} q(\theta)\right]$. In particular, for those rates $R$ such that $R=C(\widehat{\Theta})$ for some $\widehat{\Theta} \subseteq \Theta$, it is easy to see that $\widehat{\Theta} \in \Upsilon(R)$ and $\widehat{\Theta} \notin \bar{\Upsilon}(R)$, whereby $\underline{\Upsilon}(R) \supset \bar{\Upsilon}(R)$, a strict inclusion. In some cases this implies $L(R)<U(R)$ as we see in the example below.

Employing $\bar{\Upsilon}(R), U(R)$ can be equivalently written as,

$$
\begin{equation*}
U(R)=\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \in \bar{\Upsilon}(R)}\left\{1-\sum_{\theta \in \Theta^{\prime}} q(\theta)\right\} \tag{24}
\end{equation*}
$$

Interestingly, the following lemma shows that $U(R)$ defined above is in fact equal to the following expression,

$$
\begin{equation*}
\widetilde{U}(R)=\max _{q \in \mathcal{P}(\Theta)} \inf \left\{\epsilon \in[0,1) \mid R<C_{\epsilon}^{(q)}\right\} \tag{25}
\end{equation*}
$$

where $C_{\epsilon}^{(q)}$ is the $\epsilon$-capacity definition of a mixed channel $P_{Y \mid X}^{(q)}$. Under the average probability of error criterion, the $\epsilon$ capacity of this channel has been obtained in [22, Thm 1 and Lem 1(a)] for $\epsilon \in[0,1)$ as

$$
\begin{equation*}
C_{\epsilon}^{(q)}=\max _{P_{\mathbb{X}}} \sup \left\{R \mid \sum_{\theta} q(\theta) \mathbb{I}\left\{I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R\right\} \leq \epsilon\right\}, \tag{26}
\end{equation*}
$$

where $I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)$ is the mutual information between $\mathbb{X} \sim$ $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})$ and $\mathbb{Y} \sim \sum_{x \in \mathcal{A}} P_{\mathbb{Y} \mid \mathbb{X}, \theta}(y \mid x) P_{\mathbb{X}}(x)$.

Lemma 5.2: $\widetilde{U}(R)$ in (25) evaluates to $U(R)$ for $R \in$ $[\underline{C}, \bar{C})$, i.e.

$$
\widetilde{U}(R)=\max _{q \in \mathcal{P}(\Theta)} \inf \left\{\epsilon \in[0,1) \mid R<C_{\epsilon}^{(q)}\right\}=U(R)
$$

Proof : From the definition of $\epsilon$-capacity in (26), it follows that when $R<C_{\epsilon}^{(q)}$ for a given $q \in \mathcal{P}(\Theta)$, there exists $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})$ such that $\sum_{\theta} q(\theta) \mathbb{I}\left\{I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R\right\} \leq \epsilon$. Consequently,

$$
\begin{aligned}
\widetilde{U}(R)= & \max _{q \in \mathcal{P}(\Theta)} \inf \left\{\epsilon \in[0,1) \mid \exists P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})\right. \text { s.t } \\
& \left.\sum_{\theta} q(\theta) \mathbb{I}\left\{I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R\right\} \leq \epsilon\right\} .
\end{aligned}
$$

It is easy to see that for a given $q \in \mathcal{P}(\Theta)$, the inner infimum of $\epsilon$ in the above expression for $\widetilde{U}(R)$ is equivalent to finding the minimum of sums $\sum_{\theta \in \Theta^{\prime}} q(\theta)$ over strict subsets $\Theta^{\prime} \subset \Theta$ corresponding to which there exists a $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})$ such that $R \geq I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)$ for all $\theta \in \Theta^{\prime}$ and $R<$ $\min _{\theta \in \Theta^{\prime c}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)$. Since a minimum is taken over subsets $\Theta^{\prime}$, the first condition is redundant. Consequently, $\widetilde{U}(R)$ can be equivalently written as the following optimization problem,

$$
\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \subset \Theta}\left\{\sum_{\theta \in \Theta^{\prime}} q(\theta) \mid \exists P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A}) \text { s.t } R<\min _{\theta \in \Theta^{\prime} c} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)\right\} .
$$

Now, $R<\min _{\theta \in \Theta^{\prime c}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)$ for some $P_{\mathbb{X}}$ holds if and only if $R<C\left(\Theta^{\prime c}\right)$, which gives the following equivalent expression,

$$
\max _{q \in \mathcal{P}(\Theta)} \min _{\Theta^{\prime} \subset \Theta}\left\{1-\sum_{\theta \in \Theta^{\prime}} q(\theta) \mid R<C\left(\Theta^{\prime}\right)\right\} .
$$

It is then easy to see that the above expression is equivalent to $U(R)$ for $R \geq \underline{C}$, which excludes the feasibility of $\Theta^{\prime}=\Theta$ in 22).

The following proposition describes the form of $L$ and $U$ precisely and gives the condition when $L(R)=U(R)$.

Proposition 5.3:

1) $L$ and $U$ are non-decreasing step functions. $U$ is rightcontinuous and $L$ is left-continuous.
2) $L$ and $U$ are discontinuous at the same points.
3) $L(R)=U(R)$ holds for all rates $R$ where these functions are continuous. In particular, $L(R)=U(R)$ for all rates $R$ such that $R \neq C\left(\Theta^{\prime}\right)$ for any $\Theta^{\prime} \subseteq \Theta$.
Proof : l) We first argue that $U$ is a non-decreasing step function. Note that for rates $R_{1}, R_{2}$ such that $R_{1} \leq R_{2}$, $\bar{\Upsilon}\left(R_{1}\right) \supseteq \bar{\Upsilon}\left(R_{2}\right)$ (where $\bar{\Upsilon}(R)$ is as defined in (23)), which in turn gives that $U\left(R_{1}\right) \leq U\left(R_{2}\right)$. Thus, $U(R)$ is nondecreasing. Moreover, if for a given $R, \Theta_{1}, \Theta_{2} \subseteq \Theta$ are such that $C\left(\Theta_{1}\right)=\max _{\Theta^{\prime} \subseteq \Theta: R \geq C\left(\Theta^{\prime}\right)} C\left(\Theta^{\prime}\right)$ and $C\left(\Theta_{2}\right)=$ $\min _{\Theta^{\prime} \subseteq \Theta: R<C\left(\Theta^{\prime}\right)} C\left(\Theta^{\prime}\right)$, then $U(R)$ is constant in the interval $\left[C\left(\Theta_{1}\right), C\left(\Theta_{2}\right)\right)$. Together, we thus have that $U(R)$ is a non-decreasing step function which is continuous in the range $C\left(\Theta_{1}\right)<R<C\left(\Theta_{2}\right)$ and is right-continuous everywhere. Similarly, it is easy to verify that $L(R)$ is a non-decreasing function. Further, if $\Theta_{3}, \Theta_{4} \subseteq \Theta$ are such that $C\left(\Theta_{3}\right)=\max _{\Theta^{\prime} \subseteq \Theta: R>C\left(\Theta^{\prime}\right)} C\left(\Theta^{\prime}\right)$ and $C\left(\Theta_{4}\right)=$ $\min _{\Theta^{\prime} \subseteq \Theta: R \leq C\left(\Theta^{\prime}\right)} C\left(\Theta^{\prime}\right), L(R)$ is constant in the interval $\left(C\left(\Theta_{3}\right), C\left(\Theta_{4}\right)\right]$. Hence, $L(R)$ is a non-decreasing step function which is continuous in the range $C\left(\Theta_{3}\right)<R<C\left(\Theta_{4}\right)$ and is left-continuous everywhere.

2 and 3) We first show that $L(R)=U(R)$ for $R$ where $U$ is continuous. Consider the rates $R$ where $U$ is continuous. Two cases arise $-(a) R \neq C\left(\Theta^{\prime}\right)$ for any $\Theta^{\prime} \subseteq \Theta$ in which case it is easy to verify that $\underline{\Upsilon}(R)=\bar{\Upsilon}(R)$, whereby $L(R)=U(R)$, and (b) $R=C\left(\Theta^{\prime}\right)$ for some $\Theta^{\prime} \subseteq \Theta$ and $U(R)$ is continuous. In the second case, the continuity of $U(R)$ ensures that $U(R)=U(R-\delta)$ for some small $\delta>0$. However, $\bar{\Upsilon}(R-\delta)=\underline{\Upsilon}(R)$ whereby $U(R-\delta)=L(R)$. Thus $U(R)=L(R)$ for rates $R$ where $U(R)$ is continuous. It follows that $L$ and $U$ have the same points of discontinuity, thereby proving the second claim. Moreover, $U(R)$ and $L(R)$ in turn coincide for all rates $R$ such that $R \neq C\left(\Theta^{\prime}\right)$ for any $\Theta^{\prime} \subseteq \Theta$.
To illustrate this relation between $L(R)$ and $U(R)$, we consider the following example of a three state jammer.

Example: Consider a three state jammer with $\Theta=\{1,2,3\}$. Let $C\left(\Theta^{\prime}\right)$ for $\Theta^{\prime} \subseteq \Theta$ be such that,

$$
\underline{C}<C(\{1,2\})<C(\{1,3\})<C(\{2,3\})<\bar{C}=C(\{1\})
$$

Evaluating $U(R)$ for this three-state jammer then results in,

$$
U(R)= \begin{cases}0 & \text { if } R<\underline{C} \\ \frac{1}{3} & \text { if } C \leq R<C(\{1,2\}) \\ \frac{1}{2} & \text { if } C(\{1,2\}) \leq R<C(\{1,3\}) \\ \frac{1}{2} & \text { if } C(\{1,3\}) \leq R<C(\{2,3\}) \\ \frac{2}{3} & \text { if } C(\{2,3\}) \leq R<\bar{C} \\ 1 & \text { if } R \geq \bar{C}\end{cases}
$$

To see this, let us evaluate $U(R)$ when $C(\{1,3\}) \leq R<$ $C(\{2,3\})$. From the definition in (22), we get that, $U(R)=$

$$
\max _{q \in \mathcal{P}(\Theta)} \min \{q(1), q(2)+q(3), q(1)+q(3), q(1)+q(2)\}=\frac{1}{2}
$$

which follows by taking $q(1)=\frac{1}{2}$ and $q(2)=q(3)=\frac{1}{4}$. We now evaluate $L(R)$ for the considered three-state jammer. This results in

$$
L(R)= \begin{cases}0 & \text { if } R \leq C \\ \frac{1}{3} & \text { if } \underline{C}<R \leq C(\{1,2\}) \\ \frac{1}{2} & \text { if } C(\{1,2\})<R \leq C(\{1,3\}) \\ \frac{1}{2} & \text { if } C(\{1,3\})<R \leq C(\{2,3\}) \\ \frac{2}{3} & \text { if } C(\{2,3\})<R \leq \bar{C} \\ 1 & \text { if } R>\bar{C}\end{cases}
$$

Note that $U(R) \neq L(R)$ at points of discontinuity of $U(R)$ and a strict inequality holds. To see this, consider the case when $R=C(\{2,3\})$. While $U(R)=\frac{2}{3}, L(R)$ yields $\frac{1}{2}$. However, at rates $R$ where $U(R)$ is continuous, say $R=C(\{2,3\})-\Delta$, where $\Delta>0$ is small enough, $U(R)=L(R)=\frac{1}{2}$. Thus, $L(R)$ and $U(R)$ coincide for all rates $R$ where $U(R)$ is continuous. Figure 3 shows the $U(R)$ and Figure 4 shows $L(R)$ for the three-state jammer.

## B. Finite Blocklength Achievability Upper Bound on $\bar{\vartheta}(n ; R)$

In this section, we present a finite blocklength achievability scheme for a compound channel to obtain an upper bound on


Fig. 3: $U(R)$ for three-state jammer. A hollow circle indicates an excluded end-point rate and a filled circle indicates that the end-point rate is included.


Fig. 4: $L(R)$ for three-state jammer. A hollow circle indicates an excluded end-point rate and a filled circle indicates that the end-point rate is included.
the min-max value $\bar{\vartheta}(n ; R)$ of the game when rate $R$ lies in the range $\underline{C}<R<\bar{C}$. While Theorem 5.1 showed a lower bound of $L(R)$ in this range, our scheme will achieve an upper bound of $U(R)$, implying, thanks to Proposition 5.3, that the upper and lower bounds are equal, and equal to $L(R)$, everywhere except at finitely many points.

The proposed scheme has two key features which distinguishes itself from the achievability scheme in Theorem 4.1 employed for the cases when $R<\underline{C}$ and $R>\bar{C}$. Firstly, in contrast to the deterministic encoder employed in Theorem 4.1, the encoder in our new scheme avails a local randomization strategy, with its output depending on the outcome of a random experiment. Precisely stating, for sending each message $s \in \mathcal{S}$, the encoder performs a random experiment independent of $s$, the outcome of which is represented by the random variable $V$ taking values in a known finite space $\mathcal{V}$ and distributed according to a known distribution $P_{V} \in \mathcal{P}(\mathcal{V})$. Corresponding to the input message $s \in \mathcal{S}$ and depending on the outcome $V$ of the random experiment, the encoder then
outputs $x \in \mathcal{X}$ as $x=f(V, s)$ where $f$ is a function that maps $\mathcal{V} \times \mathcal{S}$ to $\mathcal{X}$. Note however that $V$ is not available to the decoder, whereby the randomization is local to the encoder. Thus, the stochastic encoder in our achievability scheme is taken to be,

$$
\begin{equation*}
Q_{X \mid S}(x \mid s)=\sum_{v \in \mathcal{V}} \mathbb{I}\{x=f(v, s)\} P_{V}(v) \tag{27}
\end{equation*}
$$

To develop an achievability scheme we employ a splitachievability technique - the second key feature of the scheme. In the scheme we consider, the space $\mathcal{V}$ is taken as $\bar{\Upsilon}(R)$, i.e., the collection of subsets $\Theta^{\prime} \subseteq \Theta$ that each define a compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta^{\prime}}$ with $C\left(\Theta^{\prime}\right)>R$, and the random experiment chooses a compound channel from the above collection. The value of $V$ specifies the codebook that is used for encoding the messages; when $V=v$, the codebook for compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in v}$ is used to encode messages. $f$ is designed such that it encodes both, the value of $V$ and the message. $V$ is encoded into a string of length $n_{1}$; this string is prefixed before each codeword from the codebook for the compound channel formed by $V$ to get the actual channel input string. From the received output, the decoder attempts to decode $V$, and then decodes the message based on the codebook associated with $V$. We get an error if $(a)$ the channel state $\theta \in \Theta$ does not lie in the choice of the compound channel chosen by the random experiment, i.e., $\theta \notin V$, or $(b)$ the channel state $\theta \in V$, but the decoder fails to correctly decode either the message sent or the choice of the compound channel $V$.
To implement the split-achievability technique outlined above, we split the channel input space as $\mathcal{X}=\mathcal{A}^{n_{1}} \times \mathcal{A}^{n-n_{1}}$ and the channel output space as $\mathcal{Y}=\mathcal{B}^{n_{1}} \times \mathcal{B}^{n-n_{1}}$ where $n_{1}$ also agreed upon by the decoder. We use the following notation. Let $\widehat{\mathcal{X}}=\mathcal{A}^{n_{1}}$ and $\widetilde{\mathcal{X}}=\mathcal{A}^{n-n_{1}}$. A generic channel input $X \in \mathcal{X}$ is written as $X=(\widehat{X}, \widetilde{X})$ where $\widehat{\widetilde{X}} \in \widehat{\mathcal{X}}, \widetilde{X} \in \widetilde{\mathcal{X}}$ and similarly $x=(\widehat{x}, \widetilde{x})$ where $\widehat{x} \in \widehat{\mathcal{X}}, \widetilde{x} \in \widetilde{\mathcal{X}}$. Similarly, let $\widehat{\mathcal{Y}}=\mathcal{B}^{n_{1}}$ represent the channel output space corresponding to $\widehat{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}=\mathcal{B}_{\widetilde{\mathcal{X}}}{ }^{n-n_{1}}$ represent the channel output space corresponding to $\widetilde{\mathcal{X}}$. Let $Y=(\widehat{Y}, \widetilde{Y})$ with $\widehat{Y} \in \widehat{\mathcal{Y}}, \widetilde{Y} \in \widetilde{\mathcal{Y}}$ and $y=(\widehat{y}, \widetilde{y})$ with $\widehat{y} \in \widehat{\mathcal{Y}}, \widetilde{y} \in \mathcal{Y}$. With this notation we have that for each $\theta \in \Theta$,

$$
\begin{align*}
P_{Y \mid X, \theta}(y \mid x, \theta) & =\prod_{i=1}^{n} \mathbb{P}_{\mathbb{Y} \mid \mathbb{X}, \theta}\left(y_{i} \mid x_{i}, \theta\right) \\
& =P_{\widehat{Y} \mid \widehat{X}, \theta}(\widehat{y} \mid \widehat{x}, \theta) P_{\widetilde{Y} \mid \widetilde{X}, \theta}(\widetilde{y} \mid \widetilde{x}, \theta) \tag{28}
\end{align*}
$$

where

$$
\begin{align*}
P_{\widehat{Y} \mid \widehat{X}, \theta}(\widehat{y} \mid \widehat{x}, \theta) & =\prod_{i=1}^{n_{1}} \mathbb{P}_{\mathbb{Y} \mid \mathbb{X}, \theta}\left(\widehat{y}_{i} \mid \widehat{x}_{i}, \theta\right)  \tag{29}\\
\text { and } \quad P_{\widetilde{Y} \mid \widetilde{X}, \theta}(\widetilde{y} \mid \widetilde{x}, \theta) & =\prod_{i=n_{1}+1}^{n} \mathbb{P}_{\mathbb{Y} \mid \mathbb{X}, \theta}\left(y_{i} \mid x_{i}, \theta\right) \\
& =\prod_{i=1}^{n-n_{1}} \mathbb{P}_{\mathbb{Y} \mid \mathbb{X}, \theta}\left(\widetilde{y}_{i} \mid \widetilde{x}_{i}, \theta\right) \tag{30}
\end{align*}
$$

The following theorem gives our finite blocklength achievability result.

Theorem 5.4: For any rate $R$ lying in $\underline{C}<R<\bar{C}$, let $\mathcal{V}=$ $\bar{\Upsilon}(R)$ and $V \in \mathcal{V}$ be a random variable distributed according to

$$
\begin{equation*}
P_{V} \in \arg \min _{\bar{P}_{V} \in \mathcal{P}(\mathcal{V})}\left\{\max _{\theta \in \Theta} \sum_{v \in \mathcal{V}} \bar{P}_{V}(v) \mathbb{I}\{\theta \notin v\}\right\} . \tag{31}
\end{equation*}
$$

Then, for any $\widetilde{\alpha}, \widetilde{\delta}, \widehat{\alpha}, \widehat{\delta}>0$, and any $n_{1}<n$, the following upper bound holds,

$$
\begin{equation*}
\bar{\vartheta}(n ; R) \leq U(R)+2 \lambda \tag{32}
\end{equation*}
$$

where

$$
\begin{align*}
& \lambda=\max \left\{\max _{v \in \mathcal{V}} \operatorname{err}_{S}^{(v)}, \operatorname{err}_{V}\right\},  \tag{33}\\
& \operatorname{err}_{V}=\max _{\theta \in \Theta} \widehat{\mathbb{P}}_{\theta}\left[\log \frac{P_{\widehat{Y} \mid \widehat{X}, \theta}(\widehat{Y} \mid \widehat{X}, \theta)}{P_{\widehat{Y} \mid \theta}(\widehat{Y} \mid \theta)} \leq \widehat{\alpha}+\widehat{\delta}\right] \\
&+|\mathcal{V}||\Theta|^{2} \exp (-\widehat{\alpha})+\exp (-\widehat{\delta})  \tag{34}\\
& \operatorname{err}_{S}^{(v)}=\max _{\theta \in v} \tilde{\mathbb{P}}_{\theta \mid V=v}\left[\log \frac{P_{\widetilde{Y} \mid \widetilde{X}, \theta}\left(\widetilde{Y}_{v} \mid \widetilde{X}_{v}, \theta\right)}{P_{\widetilde{Y_{v}} \mid \theta}\left(\widetilde{Y}_{v} \mid \theta\right)} \leq \widetilde{\alpha}+\widetilde{\delta}\right]+ \\
&|v|^{2} M \exp (-\widetilde{\alpha})+\exp (-\tilde{\delta}) \text { for each } v \in \mathcal{V} \tag{35}
\end{align*}
$$

Here for each $v \in \mathcal{V}, \widetilde{X}_{v} \sim P_{\widetilde{X}_{v}}$ where $P_{\tilde{X}_{v}}$ is any distribution in $\mathcal{P}(\widetilde{\mathcal{X}}), \widetilde{Y}_{v}$ denotes the output of $\widetilde{X}_{v}$ under the channel $P_{\widetilde{Y} \mid \widetilde{X}, \theta}$ defined in (30), and $P_{\widetilde{Y}_{v} \mid \theta}(\widetilde{y} \mid \theta) \equiv$ $\sum_{\widetilde{x} \in \widetilde{\mathcal{X}}} P_{\widetilde{X}_{v}}(\widetilde{x}) P_{\widetilde{Y} \mid \widetilde{X}, \theta}(\widetilde{y} \mid \widetilde{x}, \theta)$. Likewise, $\widehat{X} \sim P_{\widehat{X}} \in \mathcal{P}(\widehat{\mathcal{X}})$ and $\widehat{Y}$ is the output of $\widehat{X}$ under the channel $P_{\widehat{Y} \mid \widehat{X}, \theta}$ defined in (29) and $P_{\widehat{Y} \mid \theta}(\widehat{y} \mid \theta) \equiv \sum_{\widehat{x} \in \widehat{\mathcal{X}}} P_{\widehat{X}}(\widehat{x}) P_{\widehat{Y} \mid \widehat{X}, \theta}(\widehat{y} \mid \widehat{x}, \theta)$. Finally, $\widetilde{\mathbb{P}}_{\theta \mid V=v}$ denotes the probability with respect to $P_{\widehat{Y} \mid \widetilde{X}, \theta} P_{\widetilde{X}_{v}}$ and $\widehat{\mathbb{P}}_{\theta}$ is with respect to $P_{\widehat{Y} \mid \widehat{X}, \theta} P_{\widehat{X}}$.
Proof : To obtain the required bound, we consider the randomized encoder $Q_{X \mid S}$ as in (27) with $f: \mathcal{V} \times \mathcal{S} \rightarrow \widehat{\mathcal{X}} \times \widetilde{\mathcal{X}}$ and $P_{V}$ as defined in 31, and a deterministic decoder, $g: \widehat{\mathcal{Y}} \times \widetilde{\mathcal{Y}} \rightarrow\{1, \ldots, M\}$. We adopt the following splitachievability strategy. For coding the choice of $V$, we consider a deterministic code of size $|\mathcal{V}|$ in the space of $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$, i.e. an encoder $\widehat{f}: \mathcal{V} \rightarrow \widehat{\mathcal{X}}$ and a decoder $\widehat{g}: \widehat{\mathcal{Y}} \rightarrow \mathcal{V}$, such that the maximum probability of erroneous transmission of any $v \in \mathcal{V}$ over the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta}$ is $\lambda$. Since $\lambda \geq \operatorname{err}_{V}$, Theorem 4.1 applied with $M$ replaced by $|\mathcal{V}|$ guarantees the existence of such a code. Subsequently, for each choice of $v \in \mathcal{V}$, we consider another deterministic code of size $M$ in the space of $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$, i.e., an encoder $\widetilde{f}_{v}: \mathcal{S} \rightarrow \widetilde{\mathcal{X}}$, and a decoder $\widetilde{g}_{v}: \widetilde{\mathcal{Y}} \rightarrow \mathcal{S}$, such that the maximum average probability of erroneous transmission over the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in v}$ is $\lambda$. Since $\lambda \geq \operatorname{err}_{S}^{(v)}$, Theorem 4.1 applied with $\Theta$ replaced by $v$, then guarantees the existence of such a code. The encoder $f$ and decoder $g$ are then assembled as follows,

$$
\begin{array}{ll}
f(v, s)=\left(\widehat{f}(v), \widetilde{f}_{v}(s)\right) & \forall v \in \mathcal{V}, s \in \mathcal{S} \\
g(\widehat{y}, \widetilde{y})=\widetilde{g}_{\widehat{g}(\widehat{y})}(\widetilde{y}) & \forall \widehat{y} \in \widehat{\mathcal{Y}}, \widetilde{y} \in \widetilde{\mathcal{Y}}
\end{array}
$$

In other words, the encoder $f$ encodes the value of $V$ using $\widehat{f}$ and the value of $S$ using $\widetilde{f}_{v}$ when $V=v$. The decoder $g$ maps $\widetilde{Y}$ to a message in $\mathcal{S}$ using a function $\widetilde{g}_{v}$ where $v$ is
obtained as $\widehat{g}(\widehat{Y})$. Thus the decoder first decodes $v$ from $\widehat{Y}$ and then uses the resulting value of $v$ to choose $\widetilde{g}_{v}$, which is then used to decode the message from $\widetilde{Y}$.

We now show that this scheme achieves the bound in (32). Recall that $\mathcal{V}=\bar{\Upsilon}(R)$, whereby $\mathcal{V}$ comprises of compound channels formed from those subsets $v$ of $\Theta$ such that $R<$ $C(v)$. Thus when $V=v$, error in transmission of $S$ occurs when either the channel state $\theta \notin v$ or when $\theta \in v$ but either $v$ is not correctly decoded or, $v$ is correctly decoded, but $S$ is not correctly decoded. Concretely, let $\mathbb{P}_{\theta}(S \neq \widehat{S} \mid V=v)$ represent the average probability of error in transmitting $S$ given that $V=v$ under the code $(f, g)$ constructed above when the channel state is $\theta$. Clearly, $\mathbb{P}_{\theta}(S \neq \widehat{S} \mid V=v)$ equals the sum

$$
\begin{equation*}
\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \notin v \mid V=v)+\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v \mid V=v) \tag{36}
\end{equation*}
$$

We upper bound the first term in (36) by $\mathbb{I}\{\theta \notin v\}$. For the second term, note that

$$
\begin{aligned}
\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v \mid V=v) & =\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v, \widehat{g}(\widehat{Y})=v \mid V=v) \\
& +\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v, \widehat{g}(\widehat{Y}) \neq v \mid V=v)
\end{aligned}
$$

which are probabilities corresponding to the event that the decoder correctly decodes $V$ but makes an error in decoding $S$, and the event that decoder incorrectly decodes $V$ and $S$. Consequently,

$$
\begin{aligned}
& \sum_{v} P_{V}(v) \mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v \mid V=v) \leq \\
& \sum_{v} P_{V}(v)\left[\mathbb{P}_{\theta}\left(\widetilde{g}_{v}(\widetilde{Y}) \neq S \mid V=v\right)+\mathbb{P}_{\theta}(\widehat{g}(\widehat{Y}) \neq V \mid V=v)\right] \\
& \leq 2 \lambda
\end{aligned}
$$

where the last inequality follows from the construction of our codes. Consequently,

$$
\begin{aligned}
\bar{\vartheta}(n ; R) \leq & \max _{\theta \in \Theta}\left\{\sum _ { v \in \mathcal { V } } P _ { V } ( v ) \left[\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \in v \mid V=v)+\right.\right. \\
& \left.\left.\mathbb{P}_{\theta}(S \neq \widehat{S}, \theta \notin v \mid V=v)\right]\right\} \\
\leq & 2 \lambda+\max _{\theta \in \Theta}\left\{\sum_{v \in \mathcal{V}} P_{V}(v) \mathbb{I}\{\theta \notin v\}\right\}
\end{aligned}
$$

By the choice of $P_{V}$ in 31, the second term in the above bound can be equivalently written as,

$$
\begin{align*}
\max _{\theta \in \Theta} & \left\{\sum_{v \in \mathcal{V}} P_{V}(v) \mathbb{I}\{\theta \notin v\}\right\}  \tag{37}\\
& =\min _{P_{V}^{\prime} \in \mathcal{P}(\mathcal{V})} \max _{\theta \in \Theta}\left\{\sum_{v \in \mathcal{V}} P_{V}^{\prime}(v) \mathbb{I}\{\theta \notin v\}\right\} \\
& \stackrel{(a)}{=} \min _{P_{V}^{\prime} \in \mathcal{P}(\mathcal{V})} \max _{q \in \mathcal{P}(\Theta)}\left\{\sum_{v \in \mathcal{V}} \sum_{\theta \in \Theta} q(\theta) P_{V}^{\prime}(v) \mathbb{I}\{\theta \notin v\}\right\} \\
& \stackrel{(b)}{=} \max _{q \in \mathcal{P}(\Theta)} \min _{P_{V}^{\prime} \in \mathcal{P}(\mathcal{V})}\left\{\sum_{v \in \mathcal{V}} \sum_{\theta \in \Theta} q(\theta) P_{V}^{\prime}(v) \mathbb{I}\{\theta \notin v\}\right\} \\
& \stackrel{(c)}{=} \max _{q \in \mathcal{P}(\Theta)} \min _{v \in \bar{\Upsilon}(R)}\left\{\sum_{\theta \in \Theta} q(\theta) \mathbb{I}\{\theta \notin v\}\right\} \\
& =\max _{q \in \mathcal{P}(\Theta)} \min _{v \in \bar{\Upsilon}(R)}\left\{1-\sum_{\theta \in v} q(\theta)\right\}=U(R)
\end{align*}
$$

where $(a)$ holds due to the linearity of the expression in the curly braces in $q \in \mathcal{P}(\Theta)$, (b) follows from von Neumann's minimax theorem and $(c)$ follows again from the linearity of the expression in curly braces in $P_{V}^{\prime}$. Comparing the resulting expression with 24, then yields the required bound.

Observe that the bound in Theorem 5.4 is valid for any $n_{1}<n$. We now show that $U(R)$ is achievable asymptotically for rates lying in the range $\underline{C}<R<\bar{C}$ by letting $n_{1}$ grow to infinity such that $n_{1}=o(n)$.

Theorem 5.5: Let $\underline{C}>0$. For $\underline{C}<R<\bar{C}$, the upper bound on the min-max value of the game in (32) yields the following limiting value as $n \rightarrow \infty$, i.e.,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \bar{\vartheta}(n ; R) \leq U(R) \tag{38}
\end{equation*}
$$

Proof : It suffices to argue that for $\mathcal{V}=\bar{\Upsilon}(R)$, we can choose $P_{\widehat{X}}, P_{\widetilde{X}_{v}}, \widehat{\alpha}, \widetilde{\alpha}, \widehat{\delta}, \widetilde{\delta}$ and $n_{1}$ such that for each $v \in \mathcal{V}, \operatorname{err}_{S}^{(v)} \rightarrow$ 0 and $\operatorname{err}_{V} \rightarrow 0$ as $n \rightarrow \infty$. Towards this, as $n$ increases, we let $n_{1}$ grow to infinity at a rate of $o(n)$.
$\operatorname{err}_{V}$ is an upper bound guaranteed by Theorem 4.1 for sending $|\mathcal{V}|$ messages over the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta}$ using a code of blocklength $n_{1}$. Since $n_{1} \rightarrow \infty$ as $n \rightarrow \infty$, and $|\mathcal{V}| \leq 2^{|\Theta|}$, a constant, the rate of this code is asymptotically zero. Since $\underline{C}>0$, arguing as in the proof of Theorem 4.2, we can choose $P_{\widehat{X}}, \widehat{\alpha}, \widehat{\delta}$ such that $\operatorname{err}_{V} \xrightarrow{n} 0$.

For each $v \in \mathcal{V}, \operatorname{err}_{S}^{(v)}$ is an upper bound guaranteed by Theorem 4.1 for sending $M$ messages over the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in v}$ using a code of blocklength $n-n_{1}$. Since $n_{1}=o(n)$, the rate of such a code is asymptotically $R$. Now since $\mathcal{V}=\bar{\Upsilon}(R)$, we have from 23) that $R<C(v)$ for each $v \in \mathcal{V}$. Consequently, arguing again as in the proof of Theorem 4.2. we can choose $\widetilde{X}_{v}, \widetilde{\alpha}, \widetilde{\delta}$ such that $\operatorname{err}_{S}^{(v)} \xrightarrow{n} 0$ for each $v \in \mathcal{V}$.

That gives the upper bound we were looking for.
Remark V.1. We find it rather pleasing to note how, via von Neumann's minimax theorem, the achievable error term for the compound channel in 37), i.e., $\max _{\theta \in \Theta}\left\{\sum_{v \in \mathcal{V}} P_{V}(v) \mathbb{I}\{\theta \notin\right.$ $v\}\}$ becomes exactly the required quantity $U(R)$, which we arrived at from the converse for the mixed channel - a different channel altogether. As outlined in the introduction, our intuitions for this are grounded in the near-convexity of coding problems we discovered in [2]. There may be other operational interpretations of this phenomenon that are probably worthy of further investigation. The above scheme, though natural in hindsight, occurred to us only after first deriving the converse expression and reinterpreting it in these 'dual' terms. It would be illuminating to find an $a b$ initio operational justification for the optimality of this scheme.

## C. Asymptotic Tightness of the Min-max and Max-min Values

We now tie our story together by consolidating the consequences of Theorem 5.5. Theorem 5.1 and Theorem 4.2. The following result then holds.

Theorem 5.6: Let $\underline{C}>0$. For rates $R \geq 0$ such that $U(R)$ is continuous at $R$, the min-max and max-min values of the game approach $\vartheta(R):=U(R)=L(R)$ as $n \rightarrow \infty$, i.e.,

$$
\vartheta(R):=\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R)=\lim _{n \rightarrow \infty} \underline{\vartheta}(n ; R)=L(R)=U(R) .
$$

In particular, for rates $R$ such that $R \neq C\left(\Theta^{\prime}\right)$ for any $\Theta^{\prime} \subseteq \Theta$, the above equation holds.
Proof : Notice that for $R<\underline{C}, L(R)=U(R)=0$. Moreover for $R>\bar{C}$, we defined $L(R)=U(R)=1$. Thus, Theorem4.2 confirms the above claim for $R<\underline{C}$ and $R>\bar{C}$. By combining (19) with (38), we get that for rates $R$ such that $\underline{C}<R<\bar{C}$, the following bound holds in the limit as $n \rightarrow \infty$, i.e.,

$$
\begin{equation*}
U(R) \geq \limsup _{n \rightarrow \infty} \bar{\vartheta}(n ; R) \geq \liminf _{n \rightarrow \infty} \underline{\vartheta}(n ; R) \geq L(R) \tag{39}
\end{equation*}
$$

In particular, for rates $R$ such that $U(R)$ is continuous at $R, L(R)=U(R)$ from Proposition 5.3, whereby the claim holds.

Thus, except for those finitely many rates $R$ where $L$ or $U$ are discontinuous, the min-max and max-min values of the game coincide in the limit as $n \rightarrow \infty$, and they coincide to $\vartheta(R)=U(R)=L(R)$, a value one can explicitly compute. Theorem 5.6 also gives a closed-form expression for the $\epsilon$-capacity of a compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta}$ under stochastic codes as shown in the following corollary.

Corollary 5.7: For any fixed $\epsilon \in[0,1)$, the $\epsilon$-capacity of the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta}$ under stochastic codes and average error probability criterion, denoted $C_{\epsilon}$, is given as,

$$
\begin{equation*}
C_{\epsilon}:=\sup \left\{R \mid \lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R) \leq \epsilon\right\}=\sup \{R \mid L(R) \leq \epsilon\} \tag{40}
\end{equation*}
$$

Proof : Denote $\kappa(\epsilon):=\sup \{R \mid L(R) \leq \epsilon\}$. Since $L$ and $U$ are step functions that are equal everywhere except at points of jump-discontinuity, it follows that $\kappa(\epsilon)=\sup \{R \mid U(R) \leq \epsilon\}$. We first show that if $R$ is $\epsilon$-achievable for the compound channel, i.e. $\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R) \leq \epsilon$, then $R \leq \kappa(\epsilon)$. Towards this, note from 39) that $R$ is $\epsilon$-achievable implies that $L(R) \leq \epsilon$, which in turn gives that $R \leq \kappa(\epsilon)$. Conversely, if $R<\kappa(\epsilon)$, then $U(R) \leq \epsilon$ which shows that $\lim _{n \rightarrow \infty} \bar{\vartheta}(n ; R) \leq \epsilon$.

To the best of our knowledge, ours is the first characterization of the $\epsilon$-capacity of a compound channel under stochastic codes.

We conclude with some final remarks. For those rates $R$ at which $U(R)$ is discontinuous, Theorem 5.6 gives that the limiting value of the difference between min-max and maxmin values of the game amounts to at most $U(R)-L(R)$. The question then arises if we could modify our achievability scheme so as to yield a limiting value of $L(R)$ at the points of discontinuity. Let $R=C\left(\Theta^{\prime}\right), \Theta^{\prime} \subseteq \Theta$ be a point of discontinuity of $U(R)$. Towards modifying the achievability scheme, one can instead take $\mathcal{V}=\underline{\Upsilon}(R)$. Our modified scheme then gives a limiting value of $L(R)$ at the point of discontinuity $R=C\left(\Theta^{\prime}\right)$ if zero probability of error is achievable at
rate $R=C\left(\Theta^{\prime}\right)$ for the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta^{\prime}}$. Whether this is possible depends on the specific channel laws, not on the capacity alone. This is topic of separate research, which is beyond our present scope.

## VI. Conclusion

We considered a game between a team comprising of a finite blocklength encoder and decoder and a finite state jammer where the former team attempts to minimize the probability of error and the jammer attempts to maximize it. The nonclassicality of the information structure renders the team's decision problem nonconvex whereby there may not exist a saddle point value to this game. Despite this, we showed that for all but finitely many rates, an asymptotic saddle-point value exists for this game and derived an exact characterization of this value. Our results demonstrate a deeper relation between compound and mixed channels and provide an new characterization of the $\epsilon$-capacity of a compound channel under stochastic codes.

## Acknowledgment

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## Appendix A <br> Proof of Corollary 3.3

Crucial to proving Corollary 3.3 is the following lemma from [22], which is useful in analyzing the asymptotic tightness of the bound in 13).

Lemma A.1: For any fixed $x \in \mathcal{A}^{n}$, let $P$ denote its type. Let $\delta>0$ be an arbitrary constant and define $C_{\theta \mid x}^{(n)}(\delta):=$

$$
\left\{y \in \mathcal{B}^{n}:\left|\frac{1}{n} \log \frac{P_{Y \mid X, \theta}(y \mid x, \theta)}{\left(P P_{\mathbb{Y} \mid \mathbb{X}, \theta}\right)^{\times n}(y \mid \theta)}-I_{P}(\mathbb{X} ; \mathbb{Y} \mid \theta)\right| \leq \delta\right\}
$$

for all $\theta \in \Theta$, where $\left(P P_{\mathbb{Y} \mid \mathbb{X}, \theta}\right)^{\times n}(y)=$ $\prod_{i=1}^{n} \sum_{a \in \mathcal{A}} P(a) P_{\mathbb{Y} \mid \mathbb{X}=a, \theta}\left(y_{i} \mid a, \theta\right) \quad$ and $\quad I_{P}(\mathbb{X} ; \mathbb{Y} \mid \Theta) \quad$ is the mutual information evaluated when $X$ has the distribution $P$. Then,

$$
\mathbb{P}\left[Y \in C_{\theta \mid x}^{(n)}(\delta)\right] \geq 1-\frac{A(\delta)}{n}
$$

where $\mathbb{P}$ is with respect to $P_{Y \mid X=x, \theta}, A(\delta)$ is a constant independent of $n, P$ and $\theta$.

We now prove Corollary 3.3
Proof of Corollary 3.3. To obtain the bound in (14), lower bound (13) by fixing some $q \in \mathcal{P}(\Theta), \gamma=n \xi, \xi>0$ and $P_{\bar{Y} \mid \theta}$ as in 15). Note that the choice of $\gamma$ and $P_{\bar{Y} \mid \theta}$ are independent of $q$. Let the minimum in the resulting expression be attained by $x_{0} \in \mathcal{X}$ with type $\bar{P}_{n}(q)$. Subsequently, apply Lemma A. 1 as illustrated in [5, Proof of Theorem 2.2] to get the following lower bound on 13,

$$
\begin{align*}
\sum_{\theta} q(\theta) \mathbb{I}\left\{I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R-2 \xi-\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right\} \\
-\frac{A(\xi)}{n}-\exp (-n \xi) \tag{41}
\end{align*}
$$

Subsequently, take supremum over $q \in \mathcal{P}(\Theta)$ to get the required lower bound.

## Appendix B <br> Proof of Theorem 4.1

Proof of Theorem 4.1. To obtain an upper bound on $\bar{\vartheta}(n ; R)$, we design a deterministic code $(f, g)$ for the compound channel $\left\{P_{Y \mid X, \theta}\right\}_{\theta \in \Theta}$. The code will be constructed so that $f(i)=u_{i}, u_{i} \in \mathcal{X}, i \in \mathcal{S}=\{1, \ldots, M\}$ and $g: \mathcal{Y} \rightarrow \mathcal{S}$ partitions $\mathcal{Y}$ into $M$ disjoint decoding sets $\left\{\mathcal{D}_{1}, \ldots, \mathcal{D}_{M}\right\}$ to yield a $(n, M, \lambda)$ code such that

$$
P_{Y \mid X, \theta}\left(Y \in \mathcal{D}_{i} \mid X=u_{i}, \theta\right) \geq 1-\lambda, \quad \forall \theta \in \Theta, i \in[M]
$$

and

$$
\lambda=\max _{\theta} \mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y}(Y)} \leq \alpha\right]+\frac{M|\Theta|^{2}}{\exp (\alpha)}
$$

with $P_{Y}(y) \quad=\quad \frac{1}{|\Theta|} \sum_{\theta} P_{Y \mid \theta}(y \mid \theta) \quad=$ $\sum_{\theta} \sum_{x} P_{X}(x) P_{Y \mid X, \theta}(y \mid x, \theta)$. We will then upper bound $\lambda$ to get the required result. Notice that this strategy ensures a bound on the maximum probability over all messages, as required by the theorem.

Towards this, corresponding to each $x \in \mathcal{X}$, we associate

$$
B_{\theta}(x)=\left\{y \in \mathcal{Y}: \log \frac{P_{Y \mid X, \theta}(y \mid x, \theta)}{P_{Y}(y)} \geq \alpha\right\}
$$

Then, for any $x \in \mathcal{X}$ and $\theta \in \Theta$, the following relation holds:

$$
\begin{align*}
1 & \geq \mathbb{P}_{\theta}\left[Y \in B_{\theta}(x) \mid X=x\right] \\
& \geq \sum_{y} P_{Y}(y) \exp (\alpha) \mathbb{I}\left\{P_{Y \mid X, \theta}(y \mid x, \theta) \geq P_{Y}(y) \exp (\alpha)\right\} \\
& =\exp (\alpha) \mathbb{P}_{Y}\left(B_{\theta}(x)\right) \tag{42}
\end{align*}
$$

We now construct our codebook and decoding sets along the lines of a maximal code construction of Feinstein (see [23]).

1) Choose if possible $u_{1} \in \mathcal{X}$ such that

$$
\min _{\theta} \mathbb{P}_{\theta}\left[B_{\theta}\left(u_{1}\right) \mid X=u_{1}\right] \geq 1-\lambda
$$

Assign $\mathcal{D}_{\theta, 1}=B_{\theta}\left(u_{1}\right)$ as the decoding set of $u_{1}$ with respect to the channel parameter $\theta$. Subsequently, take $\mathcal{D}_{1}=\bigcup_{\theta} \mathcal{D}_{\theta, 1}$ to be the decoding set corresponding to $u_{1}$.
2) Choose $u_{k} \in \mathcal{X} \backslash\left\{u_{1}, \ldots, u_{k-1}\right\}$ such that

$$
\min _{\theta} \mathbb{P}_{\theta}\left[B_{\theta}\left(u_{k}\right) / \bigcup_{j=1}^{k-1} \mathcal{D}_{j} \mid X=u_{k}\right] \geq 1-\lambda
$$

Take $\mathcal{D}_{k}=\bigcup_{\theta} \mathcal{D}_{\theta, k}=\bigcup_{\theta}\left[B_{\theta}\left(u_{k}\right) / \bigcup_{j=1}^{k-1} \mathcal{D}_{j}\right]$ to be the decoding region corresponding to $u_{k}$, where $\mathcal{D}_{\theta, k}$ represents the decoding region for $u_{k}$ with respect to the channel parameter $\theta$. It is easy to verify that the sets $\mathcal{D}_{i}$ are mutually disjoint.
Suppose this process stops after $K$ steps, which results in a codebook $\left\{u_{1}, \ldots, u_{K}\right\}$ with $\left(\mathcal{D}_{1}, \ldots, \mathcal{D}_{K}\right)$ as the corresponding decoding regions. We now show that $K \geq M$. The stopping condition implies that for all $x$, there exists $\theta=\theta_{0}$ such that,

$$
\mathbb{P}_{\theta_{0}}\left[B_{\theta_{0}}(x) / \bigcup_{j=1}^{K} \mathcal{D}_{j} \mid X=x\right]<1-\lambda
$$

which gives that

$$
\begin{aligned}
& \lambda \stackrel{(a)}{<} 1-\mathbb{P}_{\theta_{0}}\left[B_{\theta_{0}}(x) \mid x\right]+\mathbb{P}_{\theta_{0}}\left[\bigcup_{j=1}^{K} \mathcal{D}_{j} \mid X=x\right] \\
& \quad \leq 1-\mathbb{P}_{\theta_{0}}\left[B_{\theta_{0}}(x) \mid x\right]+\sum_{j=1}^{K} \mathbb{P}_{\theta_{0}}\left[\mathcal{D}_{j} \mid X=x\right]
\end{aligned}
$$

where the inequality in $(a)$ follows since $P(A \backslash B) \geq P(A)-$ $P(B)$. Now averaging with respect to $x$, we get that

$$
\begin{aligned}
\lambda & <\mathbb{P}_{\theta_{0}}\left[B_{\theta_{0}}(X)^{c}\right]+\sum_{j=1}^{K} \mathbb{P}_{Y \mid \theta_{0}}\left[\mathcal{D}_{j}\right] \\
& \stackrel{(a)}{\leq} \mathbb{P}_{\theta_{0}}\left[B_{\theta_{0}}(X)^{c}\right]+\sum_{j=1}^{K}|\Theta| \mathbb{P}_{Y}\left[\mathcal{D}_{j}\right] \\
& \stackrel{(b)}{\leq} \mathbb{P}_{\theta_{0}}\left[\log \frac{P_{Y \mid X, \theta_{0}}\left(Y \mid X, \theta_{0}\right)}{P_{Y}(Y)} \leq \alpha\right]+\frac{|\Theta|^{2} K}{\exp (\alpha)}
\end{aligned}
$$

where ( $a$ ) follows since $P_{Y \mid \theta}(y \mid \theta) \leq|\Theta| P_{Y}(y)$ for any $\theta \in \Theta$. To get to (b), upper bound $\mathbb{P}_{Y}\left[\mathcal{D}_{j}\right]$ with $\mathbb{P}_{Y}\left[\bigcup_{\theta}\left[B_{\theta}\left(u_{j}\right)\right]\right]$, which is further bounded by $\sum_{\theta \in \Theta} \mathbb{P}_{Y}\left[B_{\theta}\left(u_{j}\right)\right]$. The inequality in (b) then follows by employing (42). It can now be verified easily that,

$$
\begin{aligned}
& \mathbb{P}_{\theta_{0}}\left[\log \frac{P_{Y \mid X, \theta_{0}}\left(Y \mid X, \theta_{0}\right)}{P_{Y}(Y)} \leq \alpha\right]+M|\Theta|^{2} \frac{1}{\exp (\alpha)} \\
& \leq \lambda \leq \mathbb{P}_{\theta_{0}}\left[\log \frac{P_{Y \mid X, \theta_{0}}\left(Y \mid X, \theta_{0}\right)}{P_{Y}(Y)} \leq \alpha\right]+K|\Theta|^{2} \frac{1}{\exp (\alpha)}
\end{aligned}
$$

which implies that $K \geq M$.
Analyzing the objective value of $\overline{\mathrm{P}}(n ; R)$ over this codebook and decoding sets, we get that

$$
\begin{aligned}
& \max _{q \in \mathcal{P}(\Theta)} \sum_{s, y} \frac{1}{M} \sum_{\theta} q(\theta) P_{Y \mid X, \theta}(y \mid f(s), \theta) \mathbb{I}\{s \neq g(y)\} \\
& \leq \max _{\theta, s \in \mathcal{S}} \mathbb{P}_{\theta}\left(Y \in \mathcal{D}_{s}^{c} \mid X=u_{s}\right) \leq \lambda
\end{aligned}
$$

Further, $\lambda$ can be again upper bounded as follows.

$$
\begin{align*}
& \lambda \leq \max _{\theta \in \Theta}\left\{\mathbb { P } _ { \theta } \left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y}(Y)} \leq \alpha, \log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)}>\right.\right. \\
& \left.\alpha+\delta]+\mathbb{P}_{\theta}\left[\log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq \alpha+\delta\right]\right\}+\frac{M|\Theta|^{2}}{\exp (\alpha)} \tag{43}
\end{align*}
$$

The first term in the RHS can be again upper bounded as,

$$
\begin{aligned}
& \mathbb{P}_{\theta}\left[\exp (\alpha) P_{Y}(Y) \geq P_{Y \mid X, \theta}(Y \mid X, \theta)>\exp (\alpha+\delta) P_{Y \mid \theta}(Y \mid \theta)\right] \\
& =\sum_{x, y} P_{Y \mid X, \theta}(y \mid x, \theta) P_{X}(x) \mathbb{I}\left\{\exp (\alpha) P_{Y}(y) \geq P_{Y \mid X, \theta}(y \mid x, \theta)\right. \\
& \left.\quad \quad \quad \exp (\alpha+\delta) P_{Y \mid \theta}(y \mid \theta)\right\} \\
& \leq \sum_{x, y} P_{Y \mid X, \theta}(y \mid x, \theta) P_{X}(x) \mathbb{I}\left\{\frac{\exp (\alpha)}{\exp (\alpha+\delta)} P_{Y}(y)>P_{Y \mid \theta}(y \mid \theta)\right\} \\
& =\sum_{y} P_{Y \mid \theta}(y \mid \theta) \mathbb{I}\left\{\exp (-\delta) P_{Y}(y)>P_{Y \mid \theta}(y \mid \theta)\right\} \\
& <\exp (-\delta) \sum_{y} P_{Y}(y)=\exp (-\delta),
\end{aligned}
$$

which together with the remaining terms in the RHS of 43), yields the required bound.

## Appendix C <br> Proof of Theorem 4.2

For proving Theorem 4.2, we first employ the following lemma from [22] to derive an upper bound on (18), which is particularly useful in analysing the asymptotic tightness of the bound.

Lemma C.1: Let $P_{X}(x)=\prod_{i=1}^{n} P_{\mathbb{X}}\left(x_{i}\right)$ for all $x \in \mathcal{A}^{n}$, where $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A})$. Then we have that,

$$
\begin{align*}
& \mathbb{P}\left\{\frac{1}{n} \log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq R+\rho_{\theta, n}\right\} \\
& \left.\leq \mathbb{I}\left\{I_{\mathbb{P}_{X}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R+\rho_{\theta, n}+\beta\right)\right\}+\frac{F(\beta)}{n} \quad \forall \theta \tag{44}
\end{align*}
$$

where $\mathbb{P}$ is with respect to $P_{Y \mid X, \theta} P_{X}$ and $F(\beta)$ is a constant independent of $n, \theta,\left\{\rho_{\theta, n}\right\} \geq 0$ is such that $\lim _{n \rightarrow \infty} \rho_{\theta, n}=0$ and $\beta>0$ denotes an arbitrary constant. Taking limsup on both sides yield that

$$
\begin{gather*}
\limsup _{n \rightarrow \infty} \mathbb{P}\left\{\frac{1}{n} \log \frac{P_{Y \mid X, \theta}(Y \mid X, \theta)}{P_{Y \mid \theta}(Y \mid \theta)} \leq R+\rho_{\theta, n}\right\} \\
\left.\leq \mathbb{I}\left\{I_{\mathbb{P}_{X}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R+\beta\right)\right\} \quad \forall \theta \tag{45}
\end{gather*}
$$

Proof of Theorem 4.2. To show the required result, we first obtain an upper bound on the difference between the minmax and max-min values of the game. Towards this, upper bound (18) using Lemma C. 1 with $P_{X}(x)=\prod_{i=1}^{n} P_{\mathbb{X}}\left(x_{i}\right)$ for all $x \in \mathcal{A}^{n}$, where $P_{\mathbb{X}} \in \mathcal{P}(\mathcal{A}), \Delta, \beta>0$ are arbitrary constants and $\left\{\rho_{n, \theta} \geq 0\right\}$ is such that $\lim _{n \rightarrow \infty} \rho_{n, \theta}=0$. Together with the lower bound in (14), we then get that

$$
\begin{align*}
& 0 \leq \bar{\vartheta}(n ; R)-\underline{\vartheta}(n ; R) \\
& \leq \max _{\theta \in \Theta}\left[\mathbb{I}\left\{I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R+\Delta+\rho_{n, \theta}+\beta\right\}\right]+\frac{F(\beta)}{n} \\
&+\exp (-n \Delta / 2)\left[|\Theta|^{2}+1\right] \\
&-\left(\max _{q \in \mathcal{P}(\Theta)} \sum_{\theta} q(\theta) \mathbb{I}\left\{I_{\bar{P}_{n}(q)}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R-2 \xi-\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right\}\right. \\
&\left.-\frac{A(\xi)}{n}-\exp (-n \xi)\right) \tag{46}
\end{align*}
$$

where $\xi>0$ is an arbitrary constant and $\bar{P}_{n}(q)$ is the type of $x \in \mathcal{A}^{n}$ that minimizes (16) with $P_{\bar{Y} \mid \theta}(y \mid \theta)$ as in (15). We show that the difference bound in (46) vanishes asymptotically for the two cases when $R<\underline{C}$ and $R>\bar{C}$.
Case 1: $R>\bar{C}$.
In this case, consider $R=\bar{C}+\epsilon$, where $\epsilon>0$. Let $\bar{\theta} \in \arg \min _{\theta}\left[\max _{P_{\mathbb{X}}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)\right]$ whereby $R=$ $\max _{P_{\mathbb{X}}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \bar{\theta})+\epsilon$. Consequently, upper bounding $\max _{\theta}\left[\mathbb{I}\left\{I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq R+\Delta+\rho_{n, \theta}+\beta\right\}\right]$ with 1 and choosing $q(\theta)=\tilde{q}(\theta)=\mathbb{I}\{\theta=\bar{\theta}\}, \xi=\frac{1}{4} \epsilon$, 46) can be upper bounded as,

$$
\begin{aligned}
& 1+\frac{F(\beta)}{n}+\exp (-n \Delta / 2)\left[|\Theta|^{2}+1\right]+\frac{A(\xi)}{n}+\exp \left(-n \frac{\epsilon}{4}\right) \\
& -\mathbb{I}\left\{I_{\bar{P}_{n}(\tilde{q})}(\mathbb{X} ; \mathbb{Y} \mid \bar{\theta}) \leq \max _{P_{\mathbb{X}}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \bar{\theta})+\frac{1}{2} \epsilon-\frac{\log \left|\mathcal{T}^{n}\right|}{n}\right\}
\end{aligned}
$$

Here the negative term (and thus the lower bound on $\underline{\vartheta}(n ; R)$ ) goes to 1 in the limit since $I_{\bar{P}_{n}(\tilde{q})}(\mathbb{X} ; \mathbb{Y} \mid \bar{\theta}) \leq$ $\max _{P_{\mathbb{X}}} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \bar{\theta})$ and $\frac{\log \left|\mathcal{T}^{n}\right|}{n}$ goes to zero in the limit as seen from (3). Moreover, all other positive terms vanish as $n \rightarrow \infty$. Hence, the difference in (46) goes to zero asymptotically.
Case 2: $R<\underline{C}$.
In this case, we take $R=\underline{C}-\epsilon$, where $0<\epsilon<\underline{C}$. Let $P_{X}^{*} \in \arg \max _{P_{\mathrm{X}}}\left[\min _{\theta} I_{P_{\mathbb{X}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)\right]$ and fix $\Delta=\beta=\frac{\epsilon}{4}$, $\rho_{n, \theta}=\frac{\epsilon}{n}$ in 46. Further, the maximum over $q(\theta)$ term in (46) can be upper bounded by 0 to yield the following upper bound on (46) for any $\xi>0$,
$\max _{\theta}\left[\mathbb{I}\left\{I_{P_{X^{*}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq \underline{C}-\frac{\epsilon}{2}+\frac{\epsilon}{n}\right\}\right]+K(n)$
$\stackrel{(a)}{=} \mathbb{I}\left\{I_{P_{X^{*}}}\left(\mathbb{X} ; \mathbb{Y} \mid \theta^{*}\right) \leq \min _{\theta} I_{P_{X^{*}}}(\mathbb{X} ; \mathbb{Y} \mid \theta)-\frac{\epsilon}{2}+\frac{\epsilon}{n}\right\}+K(n)$ $\leq \mathbb{I}\left\{\frac{\epsilon}{2} \leq \frac{\epsilon}{n}\right\}+K(n)$,
where $K(n)=\frac{F\left(\frac{\epsilon}{4}\right)}{n}+\exp \left(-n \frac{\epsilon}{8}\right)\left[|\Theta|^{2}+1\right]+\frac{A(\xi)}{n}+\exp (-n \xi)$, $\theta^{*} \in \arg \max _{\theta} \mathbb{I}\left\{I_{P_{X^{*}}}(\mathbb{X} ; \mathbb{Y} \mid \theta) \leq \underline{C}-\frac{\epsilon}{2}+\frac{\epsilon}{n}\right\}$ in $(a)$. It is clear that the above bound vanishes as $n \rightarrow \infty$.

## REFERENCES

[1] S. T. Jose and A. A. Kulkarni, in to appear in the Proceedings of the IEEE Conference on Decision and Control. IEEE, 2018.
[2] -, "Linear programming based converses for finite-blocklength jointsource channel coding," IEEE Transactions on Information Theory, vol. 63, no. 11, pp. 7066-7094, 2017.
[3] H. S. Witsenhausen, "A counterexample in stochastic optimum control," SIAM Journal on Control, vol. 6, p. 131, 1968.
[4] A. A. Kulkarni and T. P. Coleman, "An optimizer's approach to stochastic control problems with nonclassical information structures," IEEE Transactions on Automatic Control, vol. 60, no. 4, pp. 937-949, 2015.
[5] S. T. Jose and A. A. Kulkarni, "Linear programming based converses for some network-like problems," in Proceedings of the IEEE Information Theory Workshop, 2017, pp. 544-548.
[6] -, "A linear programming based finite blocklength converse for asymmetric multiple access channels," in to appear in the Proceedings of the International Conference on Signal Processing and Communications (SPCOM). IEEE, 2018.
[7] _, "Improved finite blocklength converses for Slepian-Wolf coding via linear programming," accepted (currently in press), IEEE Transactions on Information Theory, 2018.
[8] D. Blackwell, L. Breiman, and A. Thomasian, "The capacity of a class of channels," The Annals of Mathematical Statistics, pp. 1229-1241, 1959.
[9] R. Ahlswede, "The weak capacity of averaged channels," Probability Theory and Related Fields, vol. 11, no. 1, pp. 61-73, 1968.
[10] T. Basar, "The gaussian test channel with an intelligent jammer," IEEE Transactions on Information Theory, vol. 29, no. 1, pp. 152-157, 1983.
[11] T. Basar et al., "A complete characterization of minimax and maximin encoder-decoder policies for communication channels with incomplete statistical description," IEEE Transactions on Information Theory, vol. 31, no. 4, pp. 482-489, 1985.
[12] T. Basar and Y.-w. Wu, "Solutions to a class of minimax decision problems arising in communication systems," Journal of optimization theory and applications, vol. 51, no. 3, pp. 375-404, 1986.
[13] E. Akyol, K. Rose, and T. Basar, "On optimal jamming over an additive noise channel," in Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on. IEEE, 2013, pp. 3079-3084.
[14] A. Lapidoth and P. Narayan, "Reliable communication under channel uncertainty," IEEE Transactions on Information Theory, vol. 44, no. 6, pp. 2148-2177, 1998.
[15] C. Langbort and V. Ugrinovskii, "One-shot control over an avc-like adversarial channel," in American Control Conference (ACC), 2012. IEEE, 2012, pp. 3528-3533.
[16] B. Hughes and P. Narayan, "Gaussian arbitrarily varying channels," IEEE Transactions on Information Theory, vol. 33, no. 2, pp. 267-284, 1987.
[17] I. Csiszar and J. Körner, Information theory: coding theorems for discrete memoryless systems. Cambridge University Press, 2011.
[18] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior. Princeton: Princeton University Press, 1944.
[19] M. Maschler, E. Solan, and S. Zamir, "Game theory. translated from the hebrew by ziv hellman and edited by mike borns," 2013.
[20] D. Blackwell, L. Breiman, and A. Thomasian, "The capacities of certain channel classes under random coding," The Annals of Mathematical Statistics, vol. 31, no. 3, pp. 558-567, 1960.
[21] R. Ahlswede, "Certain results in coding theory for compound channels," in Proceedings of the Colloquium on Information Theory, 1967.
[22] H. Yagi and R. Nomura, "Single-letter characterization of epsiloncapacity for mixed memoryless channels," in Information Theory (ISIT), 2014 IEEE International Symposium on. IEEE, 2014, pp. 2874-2878.
[23] B. Robert, "Ash. information theory," 1990.


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