# Coding Theorems for Asynchronous Slepian-Wolf Coding Systems

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#### Abstract

The Slepian-Wolf (SW) coding system is a source coding system with two encoders and a decoder, where these encoders independently encode source sequences from two correlated sources into codewords, and the decoder reconstructs both source sequences from the codewords. In this paper, we consider the situation in which the SW coding system is asynchronous, i.e., each encoder samples a source sequence with some unknown delay. We assume that delays are unknown but maximum and minimum values of possible delays are known to encoders and the decoder. We also assume that sources are discrete stationary memoryless and the probability mass function (PMF) of the sources is unknown but the system knows that it belongs to a certain set of PMFs. For this asynchronous SW coding system, we clarify the achievable rate region which is the set of rate pairs of encoders such that the decoding error probability vanishes as the blocklength tends to infinity. We show that this region does not always coincide with that of the synchronous SW coding system in which each encoder samples a source sequence without any delay.

#### **Index Terms**

Achievable rate region, asynchronous, Slepian-Wolf coding, universal coding scheme.

#### I. INTRODUCTION

The Slepian-Wolf (SW) coding system [3] is one of famous source coding systems with many terminals. In this coding system (see Fig. 1), two encoders independently encode source sequences from two correlated sources into codewords, and the decoder reconstructs both source sequences from the codewords. For this coding system, Slepian and Wolf [3] characterized the achievable rate region for discrete stationary memoryless sources (DMSs), where the achievable rate region is the set of rate pairs of encoders such that the decoding error probability vanishes as the blocklength tends to infinity.

Discrete-time source symbols are regarded as discrete-time samples of a discrete-time process such as coin flips or a continuous-time process such as a wave. In the above SW coding system, it is assumed that these processes are sampled at the encoders without delay. Thus, encoders can encode a pair of source sequences with an expected correlation. In other words, two encoders are *synchronous*. However, in practice, the encoders are not always

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Source 1	$X^n$	Encoder 1		
Source 2	$Y^n$	Encoder 2	Decoder	$\rightarrow (\hat{X}^n, \hat{Y}^n)$

Fig. 1: Slepian-Wolf coding system

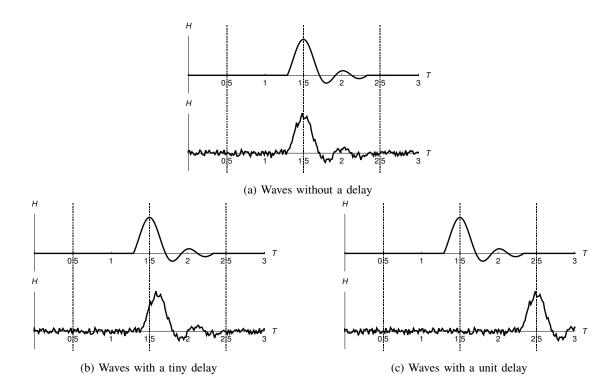


Fig. 2: Asynchronicity of sampling

synchronous. It is natural to assume that these processes are sampled at encoders with some unknown delays. In other words, the encoders are *asynchronous*. There are two reasons (i) and (ii) to justify this assumption: (i) Since the encoders are independent in the system and cannot exchange any information, it is difficult to completely adjust the time to start sampling among the encoders. Thus, in general, uncontrollable unknown delays occur. (ii) Even if the encoders can adjust the time to start sampling by exchanging information or referring a shared clock, the encoders become asynchronous when processes arrive late to the encoders. An example of this case is as follows: Observatories (encoders) on islands sample wave heights (source sequences) per unit time caused by breeze, an earthquake, a typhoon, etc. Since islands are separated, a wave reaches an island later than it reaches the other island. The observatories (and also the weather center (decoder) on a coast city distant from there. In this example, the observatories (and also the weather center) do not know the actual delay of the wave in advance because there are many uncertainties such as the direction of breeze, the point of the earthquake center, shielding on the sea, etc. Thus, even if observatories can adjust the time to start sampling, they sample the wave with some unknown delay.

To make matters worse, unknown delays may cause uncertainty of statistical properties of sources. To justify

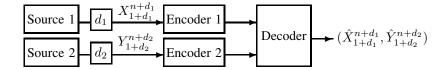


Fig. 3: Slepian-Wolf coding system with delays  $d_1$  and  $d_2$ , where  $d_1, d_2 \in \{\cdots, -2, -1, 0, 1, 2\cdots\}$ 

this, we give an example where discrete-time source symbols are quantized and sampled version of a continuous signal. Let us consider correlated wave signals shown in Fig. 2 (a). The upper wave is merely a constant wave in which sometimes large changes occur. The lower wave is its noisy version where a Gaussian process is added. In this example, we assume that each wave is sampled at each encoder per unit time at dotted lines and quantized with some resolution. We also assume that a large change does not affect two unit times. These sampled and quantized signals can be regarded as discrete-time and discrete-valued source symbols. If waves are sampled without any delay (see Fig. 2 (a)), the encoders are synchronous. In that case, the correlation between two symbols is characterized by only one channel induced by a Gaussian process. On the other hand, if waves are sampled with a tiny delay as shown in Fig. 2 (b), the correlation between two symbols no longer corresponds to that of the case without delay. When the delay is unknown, this causes uncertainty of statistical properties of sources, and now the properties are characterized by a set of channels rather than a singleton i.e., only one channel. We also note that when the delay corresponds to a unit time as shown in Fig. 2 (c), the encoders encode source sequences with integer-valued delays as shown in Fig. 3.

As a consequence of the discussion so far, the asynchronous SW coding system can be represented as the SW coding system with integer-valued delays (Fig. 3), where the delays and statistical properties of sources are unknown. Thus, in what follows, a delay refers to an integer-valued delay unless otherwise stated. We note that when a continuous signal is not assumed behind source symbols, the case where delays are unknown but statistical properties are known is also worth considering. We also note that, in general, uncertainty of statistical properties does not only come from unknown (real-valued) delays.

There are some related studies to the asynchronous SW coding system in the case where statistical properties of sources are *known* in advance. Willems [4] considered the situation in which delays are unknown to encoders but known to the decoder. He showed that the achievable rate region for DMSs coincides with that of the synchronous SW coding system. In the same assumption as [4], Rimoldi and Urbanke [5] and Sun et al. [6] gave coding schemes based on source splitting. In these studies [4]–[6], it is implicitly assumed that for a given finite blocklength, the encoders continue to transmit codewords infinitely and the decoder has infinitely large memory to receive those infinitely long time until receiving correlated codewords even if the blocklength is finite. However, in practice, the decoder cannot have infinitely large memory and wait for decoding infinitely long time. Moreover, when delays are very large, the decoding delay is also very large regardless of the blocklength. This justifies considering the coding system encoding only one pair of source sequences for a given blocklength as the system shown in Fig. 3. In this coding system, the decoder outputs an estimation from a pair of codewords and does not wait for the next

pair. Obviously, this system includes the case where a codeword consists of sub-codewords. Thus, the decoder waits until it receives a pair of codewords or finite pairs of sub-codewords, and does not wait infinitely long time for a given finite blocklength. Here, for fairness among encoders and simplicity of the system, the blocklength of source sequences are assumed to be the same. Oki and Oohama [7] considered this coding system, i.e., the system shown in Fig. 3, where statistical properties of sources are known in advance. They assumed that delays are unknown to encoders and also the decoder, but maximum and minimum values of possible delays are bounded and known to the decoder. They showed that the achievable rate region for DMSs coincides with that of the synchronous SW coding system.

In this paper, we also consider the asynchronous SW coding system. Specifically, we consider the SW coding system with delays under the following two assumptions (i) and (ii): (i) Delays are unknown but maximum and minimum values of possible delays are known to encoders and the decoder. In the above example of islands, the maximum delay depends on the distance between the islands and is naturally known to the observatories and the weather center because the distance is known in advance. (ii) Sources are DMSs and the probability mass function (PMF) of the sources is unknown but a set of PMFs including it is known to encoders and the decoder. Unlike the assumption in [7], we allow delays to be unbounded and maximum and minimum values of possible delays to be subject to change by the blocklength. This allows us more detailed analyses such as the case where delays affect a half of source sequences, the case where a delay always occurs, etc. This can also be seen as the following situation: Each encoder has a FIFO (or LIFO) memory (i.e., a source sequence is always new or old). Since the encoding and decoding delay of a preceding sequence has an order depending on the blocklength, which sequence is stored in the memory depends on the blocklength. Consequently, possible delays also depend on the blocklength.

For this asynchronous SW coding system, we clarify the achievable rate region and show that the region does not always coincide with that of the synchronous SW coding system. This result is completely different from results of the above related studies. We use a usual information-theoretic technique as in [8] to the proof of the converse part. On the other hand, the direct part is the challenging part, and its proof is somewhat different from the usual one. Instead of directly dealing with coding for sources with delays, we deal with coding for a *mixed* source given by the mixture of all sources with possible delays and employ Gallager's random coding techniques [9] and [10] to the mixed source. We note that given encoders and a decoder are *universal* in the sense that they can encode and decode (asymptotically) correctly even if statistical parameters such as the delays and the PMF of DMSs are unknown. We used an analogous technique using a mixed source in [11] to prove the existence of a code. However, since we could not use Gallager's techniques to the mixed source in [11], we did not give an exponential bound in it. We also give an extension of our coding scheme: If possible delays are bounded, our coding scheme can be extended to a scheme which does not require knowledge of the actual bound of delays. We note that this is also an extension of the result by Oki and Oohama [7].

The rest of this paper is organized as follows. In Section II, we give some notations and the formal definition of the asynchronous SW coding system and the achievable rate region. In Section III, we show the achievable rate region and some properties of it. In Sections IV and V, we show the converse part and the direct part to clarify the achievable rate region, respectively. In Section VI, we give the extension of our coding scheme. In Section VII, we

conclude the paper.

# II. PRELIMINARIES

In this section, we provide some notations and the precise definition of the asynchronous SW coding system.

We will denote an *n*-length sequence of symbols  $(a_1, a_2, \dots, a_n)$  by  $a^n$ , a sequence of symbols  $(a_m, a_{m+1}, \dots, a_{m'})$  by  $a_m^{m'}$ , and a pair of sequences of symbols  $((a_m, b_l), (a_{m+1}, b_{l+1}), \dots, (a_{m'}, b_{l'}))$  by  $(a_m^{m'}, b_l^{l'})$ . If m > m', we assume  $a_m^{m'} = \emptyset$ . For any finite sets  $\mathcal{X}$  and  $\mathcal{Y}$ , we will denote the set of all PMFs over  $\mathcal{X}$  by  $\mathcal{P}(\mathcal{X})$ , and the set of all conditional PMFs over  $\mathcal{Y}$  given elements of  $\mathcal{X}$  by  $\mathcal{P}(\mathcal{Y}|\mathcal{X})$ . Unless otherwise stated, the PMF of a random variable (RV) X on  $\mathcal{X}$  will be denoted by  $P_X \in \mathcal{P}(\mathcal{X})$ , and the conditional PMF of Y on  $\mathcal{Y}$  given X will be denoted by  $P_{Y|X} \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$ . We will denote the *n*th power of a PMF  $P_X$  by  $P_X^n$ , i.e.,  $P_X^n(x^n) = \prod_{i=1}^n P_X(x_i)$ , and the *n*th power of a conditional PMF  $P_{Y|X}$  by  $P_{Y|X}^n$ , i.e.,  $P_{Y|X}^n(y^n|x^n) = \prod_{i=1}^n P_{Y|X}(y_i|x_i)$ . In what follows, all logarithms and exponentials are taken to the base 2.

We assume that  $\mathcal{X}$  and  $\mathcal{Y}$  are finite sets. We will denote a general source  $\{(X^n, Y^n)\}_{n=1}^{\infty}$  (i.e., a sequence of *n*-length RVs) by the corresponding boldface letter  $(\mathbf{X}, \mathbf{Y})$  (cf. [12]). Since a pair of DMSs is represented by a sequence of independent copies of a pair of RVs (X, Y), we simply write it as (X, Y).

In the asynchronous SW coding system, two *n*-length sequences from DMSs (X, Y) are independently encoded by encoder 1 and encoder 2, respectively. Hence, for positive integers  $M_n^{(1)}$  and  $M_n^{(2)}$ , encoder 1 and encoder 2 are defined by the mappings

$$f_n^{(1)} : \mathcal{X}^n \to \mathcal{M}_n^{(1)} = \{1, \cdots, M_n^{(1)}\},\$$
  
$$f_n^{(2)} : \mathcal{Y}^n \to \mathcal{M}_n^{(2)} = \{1, \cdots, M_n^{(2)}\},\$$

and the rates of these encoders are defined as

$$R_n^{(1)} \triangleq \frac{1}{n} \log M_n^{(1)},$$
$$R_n^{(2)} \triangleq \frac{1}{n} \log M_n^{(2)},$$

respectively. Since the encoders are asynchronous, encoder 1 might encode a source sequence  $X^n = (X_1, X_2, \dots, X_n)$ while encoder 2 might encode a source sequence  $Y_{-2}^{n-3} = (Y_{-2}, Y_{-1}, \dots, Y_{n-3})$ . In general, encoder 1 and encoder 2 encode sequences  $X^n$  and  $Y_{1+d}^{n+d} = (Y_{1+d}, Y_{2+d}, \dots, Y_{n+d})$ , respectively, where d is an integer which represents a *relative*<sup>†</sup> delay (see Tables I and II).

Without loss of generality, we assume that  $-n \le d \le n$  because, for any  $d \ge n$  or  $d \le -n$ ,  $X^n$  is independent of  $Y_{1+d}^{n+d}$ . We denote  $Y_{1+d}^{n+d}$  by  $Y_{(d)}^n$  for the sake of brevity. Note that

$$P_{X^{n}Y_{(d)}^{n}}(x^{n}, y^{n}) = P_{X}^{|d|}(x_{\mathrm{I}}^{|d|})P_{XY}^{n-|d|}(x_{\mathrm{C}}^{n-|d|}, y_{\mathrm{C}}^{n-|d|})P_{Y}^{|d|}(y_{\mathrm{I}}^{|d|}),$$
(1)

<sup>†</sup>Suppose that encoder 1 and encoder 2 run with delays  $d_1$  and  $d_2$ , respectively. Then, for a source sequence  $(X_{1+d_1}^{n+d_1}, Y_{1+d_2}^{n+d_2})$  encoded by encoders, it holds that  $P_{X_{1+d_1}^{n+d_1}Y_{1+d_2}^{n+d_2}}(x^n, y^n) = P_{X_1^nY_{1+d_2}^{n+d_2-d_1}}(x^n, y^n)$ . Hence, we may only consider a relative delay.

TABLE I: Correlation in the case where  $0 \le d$ 

	$X_1$		$X_{1+d}$		$X_n$			
		$Y_{1+d}$	•••	$Y_n$	$Y_{n+1}$	•••	$Y_{n+d}$	

TABLE II: Correlation in the case where  $d \leq 0$ 

		$X_1$	 $X_{n+d}$	$X_{1+n+d}$		$X_n$
$Y_{1+d}$	• • • •	$Y_1$	 $Y_{n+d}$			

where  $P_{XY}$  is the PMF of the pair of DMSs (X, Y),  $P_X$  and  $P_Y$  are the marginal PMFs of  $P_{XY}$ , and

$$\begin{split} x_{\mathrm{I}}^{|d|} &= \begin{cases} x_{1}^{d} & \text{if } 0 \leq d \leq n, \\ x_{1+n+d}^{n} & \text{if } -n \leq d \leq 0, \end{cases} \\ (x_{\mathrm{C}}^{n-|d|}, y_{\mathrm{C}}^{n-|d|}) &= \begin{cases} (x_{1+d}^{n}, y_{1}^{n-d}) & \text{if } 0 \leq d \leq n, \\ (x_{1}^{n+d}, y_{1-d}^{n}) & \text{if } -n \leq d \leq 0, \end{cases} \\ y_{\mathrm{I}}^{|d|} &= \begin{cases} y_{1+n-d}^{n} & \text{if } 0 \leq d \leq n, \\ y_{1}^{-d} & \text{if } -n \leq d \leq 0. \end{cases} \end{split}$$

We denote DMSs (X, Y) with a delay d simply by  $(X, Y_{(d)})$  which implies the sequence of RVs  $\{(X^n, Y_{(d)}^n)\}_{n=1}^{\infty}$ . We introduce the maximum  $\overline{d}_n \ (\leq n)$  and the minimum  $\underline{d}_n \ (\geq -n)$  of possible delays, and denote the sequence  $\{(\underline{d}_n, \overline{d}_n)\}_{n=1}^{\infty}$  by d. Hence, any possible delay d satisfies  $\underline{d}_n \leq d \leq \overline{d}_n$  for any blocklength n. We allow the maximum and the minimum of delays to be changed with the blocklength.

The decoder receives two codewords  $f_n^{(1)}(X^n)$  and  $f_n^{(2)}(Y_{(d)}^n)$ , and outputs an estimate of the pair of sequences  $(X^n, Y_{(d)}^n)$ . Hence, the decoder is defined by the mapping

$$\varphi_n: \mathcal{M}_n^{(1)} \times \mathcal{M}_n^{(2)} \to \mathcal{X}^n \times \mathcal{Y}^n$$

Then, for DMSs (X, Y) and a delay d, the error probability is defined as

$$\varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \triangleq \Pr\{\varphi_n(f_n^{(1)}(X^n), f_n^{(2)}(Y_{(d)}^n)) \neq (X^n, Y_{(d)}^n)\}.$$
(2)

More generally, we will denote the error probability for a general source  $(\mathbf{X}, \mathbf{Y})$  by  $\varepsilon_{\mathbf{XY}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ , i.e.,

$$\varepsilon_{\mathbf{XY}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \triangleq \Pr\{\varphi_n(f_n^{(1)}(X^n), f_n^{(2)}(Y^n)) \neq (X^n, Y^n)\}$$

We will sometimes omit  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  in the notation of  $\varepsilon_{\mathbf{XY}}^{(n)}$  when it is clear from the context.

In this coding system, we assume that the actual delay d is unknown but the bound d of delays is known to the encoders and the decoder. Furthermore, we assume that the PMF  $P_{XY}$  of the pair of DMSs is an element of a given set  $S \subseteq \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  of PMFs and that the PMF  $P_{XY}$  is unknown but the set S of PMFs is known to the encoders and the decoder. More precisely, the code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  is independent of d and  $P_{XY} \in S$ , but is allowed to be

dependent on d and S. Hence, as we mentioned earlier, the code is universal in the sense that we require that they can encode and decode (asymptotically) correctly even if statistical parameters d and  $P_{XY}$  are unknown.

We now define *achievability* and *achievable rate region* for the asynchronous SW coding system for a given set S of PMFs of sources and a bound d of delays.

**Definition 1** (Achievability). A pair  $(R_1, R_2)$  is called *achievable* for a set S of PMFs and a bound d of delays if and only if there exists a sequence of codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  satisfying

$$\limsup_{n \to \infty} R_n^{(1)} \le R_1,$$
$$\limsup_{n \to \infty} R_n^{(2)} \le R_2,$$

and

$$\lim_{n \to \infty} \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) = 0,$$
(3)

where  $\mathcal{D}_n \triangleq \{\underline{d}_n, \underline{d}_n + 1, \cdots, \overline{d}_n - 1, \overline{d}_n\}$  is the set of possible delays.

**Definition 2** (Achievable rate region). For a set S of PMFs and a bound d of delays, the achievable rate region  $\mathcal{R}_{d}(S)$  is defined by

 $\mathcal{R}_{\mathbf{d}}(\mathcal{S}) \triangleq \{ (R_1, R_2) : (R_1, R_2) \text{ is achievable for the set } \mathcal{S} \text{ and the bound } \mathbf{d} \}.$ 

When  $S = \{P_{XY}\}$  is a singleton, we simply write  $\mathcal{R}_{\mathbf{d}}(S)$  as  $\mathcal{R}_{\mathbf{d}}(X,Y)$ . Moreover, when there is no delay, i.e,  $\overline{d}_n = \underline{d}_n = 0$ , we simply write  $\mathcal{R}_{\mathbf{d}}(X,Y)$  as  $\mathcal{R}(X,Y)$ .

**Remark 1.** When S is a singleton and  $\overline{d}_n = \underline{d}_n = 0$ , our coding system corresponds to the usual synchronous SW coding system. Hence,  $\mathcal{R}(X, Y)$  denotes the achievable rate region of the synchronous SW coding system.

Oki and Oohama [7] considered the case where S is a singleton and delays are bounded in the sense that for a constant  $c \in [0, \infty)$ ,  $-c = \underline{d}_n \leq d \leq \overline{d}_n = c$ ,  $\forall n > 0$ . They assume that the delay is unknown to encoders and the decoder but c is known to the decoder. For this special case, they showed that the achievable rate region coincides with that of the synchronous SW coding system, i.e.,

$$\mathcal{R}_{\mathbf{d}}(X,Y) = \mathcal{R}(X,Y)$$
  
= {(R<sub>1</sub>, R<sub>2</sub>) : R<sub>1</sub> ≥ H(X|Y), R<sub>2</sub> ≥ H(Y|X), R<sub>1</sub> + R<sub>2</sub> ≥ H(X,Y)}, (4)

where H(X) denotes the entropy of X, and H(X|Y) denotes the conditional entropy of X given Y. We will also denote H(X) by  $H(P_X)$ , and H(X|Y) by  $H(P_{X|Y}|P_Y)$  to clear their PMFs.

In what follows, we sometimes use the following notations for the sake of simplicity:  $H_1(X, Y) = H(X|Y)$ ,  $H_2(X, Y) = H(Y|X)$ ,  $H_3(X, Y) = H(X, Y)$ , and  $R_3 = R_1 + R_2$ . Due to these notations, (4) is also written as

$$\{(R_1, R_2) : R_i \ge H_i(X, Y), \ \forall i \in \{1, 2, 3\}\}.$$

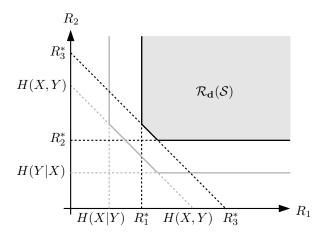


Fig. 4: An image of  $\mathcal{R}_{\mathbf{d}}(\mathcal{S})$ , where  $P_{XY} \in \mathcal{S}$ ,  $R_1^* = \sup_{P_{XY} \in \mathcal{S}} (H(X|Y) + \Delta_{\mathbf{d}}I(X;Y))$ ,  $R_2^* = \sup_{P_{XY} \in \mathcal{S}} (H(Y|X) + \Delta_{\mathbf{d}}I(X;Y))$ , and  $R_3^* = \sup_{P_{XY} \in \mathcal{S}} (H(X,Y) + \Delta_{\mathbf{d}}I(X;Y))$ .

#### **III. ACHIEVABLE RATE REGION**

In this section, we show the achievable rate region of the asynchronous SW coding system. We also show that the obtained region does not always coincide with that of the synchronous SW coding system.

The next theorem clarifies the achievable rate region (see also Fig. 4).

**Theorem 1.** For a set S of PMFs of DMSs and a bound d of delays, we have

$$\mathcal{R}_{\mathbf{d}}(\mathcal{S}) = \left\{ \left( R_1, R_2 \right) : R_i \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y) \right), \ \forall i \in \{1, 2, 3\} \right\}$$

where

$$\Delta_{\mathbf{d}} \triangleq \limsup_{n \to \infty} \frac{\max\left\{ |\overline{d}_n|, |\underline{d}_n| \right\}}{n}$$

**Remark 2.** By the definition of d, it holds that  $0 \le \Delta_d \le 1$ .

The proof of this theorem is given later in Sections IV and V. In this theorem,  $\Delta_{\mathbf{d}}$  denotes the ratio of the maximum delay to the blocklength. For example, for an arbitrarily fixed constant  $\alpha \in (0, 1]$ , suppose that  $\underline{d}_n = -\alpha n$  and  $\overline{d}_n = \alpha n$ . Then  $\Delta_{\mathbf{d}} = \alpha$ , and hence we have for any subset  $S \subseteq \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ ,

$$\mathcal{R}_{\mathbf{d}}(\mathcal{S}) = \left\{ (R_1, R_2) : R_i \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \alpha I(X; Y) \right), \ \forall i \in \{1, 2, 3\} \right\}.$$
(5)

Since  $\alpha > 0$ , the rate region does not coincide with  $\mathcal{R}(X, Y)$ , i.e., that of the synchronous SW coding system, where we assume that  $P_{XY} \in S$  and I(X;Y) > 0. This means that if delays have a significant influence in terms of the blocklength *n* like this example, we cannot achieve a pair of rates of the synchronous SW coding system. As shown in a later section, this is because delays affect the *first-order* of rates. Although this fact is very important, it is not clear from previous studies [4] and [7].

More generally, we have the following corollaries.

**Corollary 1.** Let  $P_{XY} \in S$ . Then,  $\mathcal{R}_{\mathbf{d}}(S) \neq \mathcal{R}(X,Y)$  if and only if  $\sup_{P_{XY} \in S} (H_i(X,Y) + \Delta_{\mathbf{d}}I(X;Y)) \neq H_i(X,Y)$  for some  $i \in \{1, 2, 3\}$ .

*Proof:* If it holds that  $\sup_{P_{XY} \in S} (H_i(X,Y) + \Delta_d I(X;Y)) \neq H_i(X,Y)$  for some  $i \in \{1,2,3\}$ , we have

$$\sup_{P_{XY}\in\mathcal{S}}\left(H_i(X,Y)+\Delta_{\mathbf{d}}I(X;Y)\right)>H_i(X,Y).$$

For such  $i \in \{1, 2, 3\}$ , we choose a boundary point  $(R'_1, R'_2) \in \mathcal{R}(X, Y)$  such that  $R'_i = H_i(X, Y)$ . Then, according to Theorem 1, we have

$$\inf\{R_i: (R_1, R_2) \in \mathcal{R}_{\mathbf{d}}(\mathcal{S})\} \ge \sup_{P_{XY} \in \mathcal{S}} (H_i(X, Y) + \Delta_{\mathbf{d}}I(X; Y)) > H_i(X, Y) = R'_i.$$

This means that  $(R'_1, R'_2) \notin \mathcal{R}_d(\mathcal{S})$ , and hence  $\mathcal{R}_d(\mathcal{S}) \neq \mathcal{R}(X, Y)$ . This gives the *if* part.

On the other hand, if it holds that  $\sup_{P_{XY} \in \mathcal{S}} (H_i(X, Y) + \Delta_d I(X; Y)) = H_i(X, Y)$  for all  $i \in \{1, 2, 3\}$ , it immediately holds that  $\mathcal{R}_d(\mathcal{S}) = \mathcal{R}(X, Y)$ . This completes the proof.

**Corollary 2.** Let  $S = \{P_{XY}\}$  be a singleton. Then,  $\mathcal{R}_{\mathbf{d}}(X, Y) \neq \mathcal{R}(X, Y)$  if and only if  $\Delta_{\mathbf{d}} \neq 0$  and  $I(X; Y) \neq 0$ .

*Proof:* According to Corollary 1,  $\mathcal{R}_{\mathbf{d}}(X, Y) \neq \mathcal{R}(X, Y)$  if and only if

$$H_i(X,Y) + \Delta_{\mathbf{d}}I(X;Y) \neq H_i(X,Y), \ \exists i \in \{1,2,3\}.$$

This holds if and only if  $\Delta_{\mathbf{d}} \neq 0$  and  $I(X;Y) \neq 0$ .

#### IV. CONVERSE PART

In this section, we give a proof of the converse part of Theorem 1.

Before the proof of the converse part, we show the next fundamental lemma.

**Lemma 1.** For any  $P_{XY} \in S$  and  $d \in D_n$ , we have

$$H_i(X^n, Y^n_{(d)}) = nH_i(X, Y) + |d|I(X; Y), \ \forall i \in \{1, 2, 3\}.$$
(6)

*Proof:* According to (1), for any  $P_{XY} \in S$  and delay  $d \in D_n$ , we have

$$\begin{split} H(X^{n}|Y_{(d)}^{n}) &= -\sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{X}^{|d|}(x_{\mathrm{I}}^{|d|})P_{XY}^{n-|d|}(x_{\mathrm{C}}^{n-|d|},y_{\mathrm{C}}^{n-|d|})P_{Y}^{|d|}(y_{\mathrm{I}}^{|d|})\log P_{X}^{|d|}(x_{\mathrm{I}}^{|d|})P_{X|Y}^{n-|d|}(x_{\mathrm{C}}^{n-|d|}|y_{\mathrm{C}}^{n-|d|}) \\ &= -\sum_{x^{|d|}\in\mathcal{X}^{|d|}} P_{X}^{|d|}(x^{|d|})\log P_{X}^{|d|}(x^{|d|}) \\ &- \sum_{(x^{n-|d|},y^{n-|d|})\in\mathcal{X}^{n-|d|}\times\mathcal{Y}^{n-|d|}} P_{XY}^{n-|d|}(x^{n-|d|},y^{n-|d|})\log P_{X|Y}^{n-|d|}(x^{n-|d|}|y^{n-|d|}) \\ &= |d|H(X) + (n-|d|)H(X|Y) \\ &= nH(X|Y) + |d|I(X;Y). \end{split}$$

Similarly, we have

$$H(Y_{(d)}^n|X^n) = nH(Y|X) + |d|I(X;Y)$$

and

$$H(X^{n}, Y^{n}_{(d)}) = |d|H(X) + (n - |d|)H(X, Y) + |d|H(Y)$$
$$= nH(X, Y) + |d|I(X; Y).$$

In order to emphasize the affect of delays, we give a converse bound for a given finite blocklength in the next theorem.

**Theorem 2.** For any n > 0 and any code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ , we have

$$R_n^{(i)} \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}}^{(n)} I(X; Y) \right) - \epsilon_n, \ \forall i \in \{1, 2, 3\}$$

where  $\Delta_{\mathbf{d}}^{(n)} = \frac{\max\{|\overline{d}_n|, |\underline{d}_n|\}}{n}$ ,  $R_n^{(3)} = R_n^{(1)} + R_n^{(2)}$ , and

$$\epsilon_n = \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}| |\mathcal{Y}| + \frac{1}{n}$$

*Proof:* By using a usual converse technique (see Appendix A), for any n > 0, code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ , delay  $d \in \mathcal{D}_n$ , and  $P_{XY} \in \mathcal{S}$ , we have

$$nR_n^{(i)} \ge H_i(X^n, Y_{(d)}^n) - n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| - 1, \ \forall i \in \{1, 2, 3\}.$$
(7)

Hence, we have

$$\begin{split} R_n^{(i)} &\geq \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \left( \frac{1}{n} H_i(X^n, Y_{(d)}^n) - \varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}| |\mathcal{Y}| - \frac{1}{n} \right) \\ &\stackrel{(a)}{=} \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \left( H_i(X, Y) + \frac{|d|}{n} I(X; Y) - \varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}| |\mathcal{Y}| - \frac{1}{n} \right) \\ &\geq \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \left( H_i(X, Y) + \frac{|d|}{n} I(X; Y) \right) - \left( \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}| |\mathcal{Y}| + \frac{1}{n} \right) \\ &= \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \frac{\max_{d \in \mathcal{D}_n} |d|}{n} I(X; Y) \right) - \epsilon_n \\ &= \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}}^{(n)} I(X; Y) \right) - \epsilon_n, \ \forall i \in \{1, 2, 3\}, \end{split}$$

where (a) comes from Lemma 1.

According to this theorem, delay affects the first-order of rates as the term  $\Delta_{\mathbf{d}}^{(n)}I(X;Y)$ . Hence, the larger  $\Delta_{\mathbf{d}}^{(n)}$  becomes, the greater the difference between rates of synchronous SW coding systems and that of asynchronous SW coding systems. As an instance, we consider the case were delay occurs a ratio of  $\alpha > 0$  for a given blocklength n, i.e.,  $\underline{d}_n = -\alpha n$  and  $\overline{d}_n = \alpha n$ . Since the error probability must be small, this theorem implies that any pair of rates  $(R_n^{(1)}, R_n^{(2)})$  must satisfy

$$R_n^{(i)} \gtrsim \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \alpha I(X; Y) \right), \ \forall i \in \{1, 2, 3\}.$$

This also justifies the achievable rate region in Theorem 1 (cf. (5)).

Now, we prove the converse part.

Proof of the converse part: Suppose that  $(R_1, R_2) \in \mathcal{R}_{\mathbf{d}}(\mathcal{S})$ . By definition of the achievability, there exists a sequence of codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  such that

$$\limsup_{n \to \infty} R_n^{(i)} \le R_i, \ \forall i \in \{1, 2\},$$
$$\lim_{n \to \infty} \sup_{X_Y \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) = 0.$$

According to Theorem 2, for this sequence of codes and any  $P_{XY} \in S$ , we have

$$\begin{aligned} R_i &\geq \limsup_{n \to \infty} R_n^{(i)} \\ &\geq H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y), \ \forall i \in \{1, 2, 3\}, \end{aligned}$$

where we use the fact that

$$\limsup_{n \to \infty} \Delta_{\mathbf{d}}^{(n)} = \Delta_{\mathbf{d}}.$$

Since this inequality holds for any  $P_{XY} \in S$ , we have

$$R_i \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y) \right), \ \forall i \in \{1, 2, 3\}$$

Since this holds for any  $(R_1, R_2) \in \mathcal{R}_{\mathbf{d}}(\mathcal{S})$ , we have

$$\mathcal{R}_{\mathbf{d}}(\mathcal{S}) \subseteq \left\{ (R_1, R_2) : R_i \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y) \right), \ \forall i \in \{1, 2, 3\} \right\}.$$

This completes the proof of the converse part of Theorem 1.

#### V. DIRECT PART AND A UNIVERSAL CODING SCHEME

In this section, we give a proof of the direct part of Theorem 1. To this end, we will show a universal coding scheme for asynchronous SW coding systems.

The universal coding scheme using the minimum entropy decoder [13] is well known. However, since it is too much specialized to DMSs without delay, we cannot use it to our coding system straightforwardly. A natural extension of the minimum entropy decoder may be as follows:

$$\varphi_{n}(m_{1},m_{2}) = \begin{cases} (x^{n},y^{n}) & \text{ if } (f_{n}^{(1)}(x^{n}),f_{n}^{(2)}(y^{n})) = (m_{1},m_{2}), \\ \exists d \in \mathcal{D}_{n}, \ \forall \hat{d} \in \mathcal{D}_{n}, \ \forall (\tilde{x}^{n},\tilde{y}^{n}) \in \mathcal{X}^{n} \times \mathcal{Y}^{n}, \text{ s.t.} \\ (f_{n}^{(1)}(\tilde{x}^{n}),f_{n}^{(2)}(\tilde{y}^{n})) = (m_{1},m_{2}), \\ |d|H(P_{x_{1}^{\lfloor d \rfloor}}) + (n-|d|)H(P_{(x_{C}^{n-\lfloor d \rfloor},y_{C}^{n-\lfloor d \rfloor})}) + |d|H(P_{y_{1}^{\lfloor d \rfloor}}) \\ < |\hat{d}|H(P_{\tilde{x}_{1}^{\lfloor d \rfloor}}) + (n-|\hat{d}|)H(P_{(\tilde{x}_{C}^{n-\lfloor d \rfloor},\tilde{y}_{C}^{n-\lfloor d \rfloor})}) + |\hat{d}|H(P_{\tilde{y}_{1}^{\lfloor d \rfloor}}), \end{cases}$$

declare the error otherwise,

where  $P_{a^k}$  is the type (cf. [14]) of a sequence  $a^k$  defined by  $P_{a^k}(a) \triangleq |\{i \in \{1, \dots, k\} : a_i = a\}|/k$ . However, it may not be able to evaluate well the error probability of this decoder by using a usual method of the types. This is because, for sequences  $y^n$  and  $\tilde{y}^n$  with the same type (or even for  $y^n = \tilde{y}^n$ ), the type of  $y^{n-d}$  does not always

coincide with the type of  $\tilde{y}^{n-\hat{d}}$  when  $d \neq \hat{d}$ . Thus, when we use the minimum entropy decoder, we will probably need a more sophisticated and somewhat complicated technique.

Instead of using the minimum entropy decoder, we employ a decoder for a mixed source and analyze the error probability by using Gallager's random coding techniques [9] and [10]. To this end, we introduce  $\mathcal{P}_n(\mathcal{X})$  that is the set of PMFs  $P_{\bar{X}} \in \mathcal{P}(\mathcal{X})$  taking rational numbers such that for each  $x \in \mathcal{X}$ ,

$$P_{\bar{X}}(x) = \frac{m}{n}$$

where  $m \in \{0, 1, \dots, n\}$ . This type of PMF is referred to as *n*-type in this paper. The next bound is well known (see e.g., [14, Lemma 2.2]).

$$|\mathcal{P}_n(\mathcal{X})| \le (n+1)^{|\mathcal{X}|}.\tag{8}$$

Let  $d_v$  be the variational distance between two PMFs  $P, Q \in \mathcal{P}(\mathcal{X})$  defined as

$$d_{\mathbf{v}}(P,Q) \triangleq \sum_{x \in \mathcal{X}} |P(x) - Q(x)|.$$

Then, we have the next lemma which gives an approximation of a given distribution by an n-type. The proof is given in Appendix B.

Lemma 2. Let  $\alpha = |\mathcal{X}||\mathcal{Y}|(|\mathcal{X}||\mathcal{Y}|-1)$  and  $\varepsilon_n = \alpha \ln \left(1 + \frac{\alpha}{n-\alpha}\right)$ . Then, for any  $n \ge \alpha$  and  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ , there exists an *n*-type  $P_{\bar{X}\bar{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$  such that

$$d_{\mathbf{v}}(P_{XY}, P_{\bar{X}\bar{Y}}) \le \frac{2(|\mathcal{X}||\mathcal{Y}| - 1)}{n},\tag{9}$$

and

$$P_{X^{n}Y_{(d)}^{n}}(x^{n}, y^{n}) \leq P_{\bar{X}^{n}\bar{Y}_{(d)}^{n}}(x^{n}, y^{n})e^{3(\alpha+\varepsilon_{n})},$$
  
$$\forall (x^{n}, y^{n}) \in \mathcal{X}^{n} \times \mathcal{Y}^{n}, \ \forall d \in \{-n, \cdots, n\},$$
(10)

where e is the base of the natural logarithm.

For each  $P_{XY} \in S$ , we consider the corresponding *n*-type  $P_{\bar{X}\bar{Y}}$  in Lemma 2. We denote the set of those *n*-types for elements of S by  $\bar{S}_n (\subseteq \mathcal{P}_n(\mathcal{X} \times \mathcal{Y}))$ , where  $\bar{S}_n$  is arbitrary for  $n < \alpha$ . Hence by definition, for any  $P_{XY} \in S$ (and  $n \ge \alpha$ ), there exists an *n*-type  $P_{\bar{X}\bar{Y}} \in \bar{S}_n$  satisfying (9) and (10). Roughly speaking,  $\bar{S}_n$  is an approximation of S by *n*-types. By using this set  $\bar{S}_n$ , we define the *mixed source*  $(\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}) = \{(\tilde{X}^n, \tilde{Y}^n)\}_{n=1}^{\infty}$ , where the pair of RVs  $(\tilde{X}^n, \tilde{Y}^n)$  is drawn according to the joint PMF  $P_{\tilde{X}^n\tilde{Y}^n}$  defined as

$$P_{\widetilde{X}^{n}\widetilde{Y}^{n}}(x^{n}, y^{n}) \triangleq \sum_{P_{\widetilde{X}\widetilde{Y}} \in \overline{\mathcal{S}}_{n}} \sum_{d \in \mathcal{D}_{n}} \frac{1}{|\overline{\mathcal{S}}_{n}||\mathcal{D}_{n}|} P_{\widetilde{X}^{n}\overline{Y}^{n}_{(d)}}(x^{n}, y^{n}).$$
(11)

This mixed source gives a bound on the maximum error probability for DMSs with delay as shown in the next theorem.

**Theorem 3.** Let  $\alpha = |\mathcal{X}||\mathcal{Y}|(|\mathcal{X}||\mathcal{Y}| - 1)$  and  $\varepsilon_n = \alpha \ln \left(1 + \frac{\alpha}{n-\alpha}\right)$ . Then, for any  $n \ge \alpha$  and any code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ , we have

$$\sup_{P_{XY}\in\mathcal{S}}\max_{d\in\mathcal{D}_{n}}\varepsilon_{XY_{(d)}}^{(n)}(f_{n}^{(1)},f_{n}^{(2)},\varphi_{n}) \leq e^{3(\alpha+\varepsilon_{n})}(2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}\varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)}(f_{n}^{(1)},f_{n}^{(2)},\varphi_{n}).$$
(12)

*Proof:* We prove this theorem in two parts. In the first part, we show that for any n > 0 and any code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ ,

$$\max_{P_{\bar{X}\bar{Y}}\in\bar{\mathcal{S}}_{n}}\max_{d\in\mathcal{D}_{n}}\varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_{n}^{(1)},f_{n}^{(2)},\varphi_{n}) \leq (2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}\varepsilon_{\tilde{X}\tilde{\mathbf{Y}}}^{(n)}(f_{n}^{(1)},f_{n}^{(2)},\varphi_{n}),$$
(13)

where  $(\bar{X}, \bar{Y}_{(d)})$  denotes DMSs  $(\bar{X}, \bar{Y})$  induced by the *n*-type  $P_{\bar{X}\bar{Y}}$  with the delay *d*. In the second part, we show that for any  $n \ge \alpha$  and any code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ ,

$$\sup_{P_{XY}\in\mathcal{S}}\max_{d\in\mathcal{D}_n}\varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)},f_n^{(2)},\varphi_n) \le e^{3(\alpha+\varepsilon_n)}\max_{P_{\bar{X}\bar{Y}}\in\bar{\mathcal{S}}_n}\max_{d\in\mathcal{D}_n}\varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_n^{(1)},f_n^{(2)},\varphi_n).$$
(14)

The theorem follows immediately from (13) and (14).

First part: Since the decoder  $\varphi_n$  is a deterministic function, for any source  $(\mathbf{X}, \mathbf{Y})$ , the error probability  $\varepsilon_{\mathbf{XY}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  can be written as

$$\varepsilon_{\mathbf{XY}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) = \sum_{(x^n, y^n) \in \mathcal{E}_n(f_n^{(1)}, f_n^{(2)}, \varphi_n)} P_{X^n Y^n}(x^n, y^n),$$

where  $\mathcal{E}_n(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  is the set of pairs of sequences which cannot be decoded correctly, i.e.,

$$\mathcal{E}_n(f_n^{(1)}, f_n^{(2)}, \varphi_n) \triangleq \{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \varphi_n(f_n^{(1)}(x^n), f_n^{(2)}(y^n)) \neq (x^n, y^n) \}.$$

Thus, we have

$$\begin{split} \max_{P_{\bar{X}\bar{Y}}\in\bar{S}_{n}} \max_{d\in\mathcal{D}_{n}} \varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n}) &\leq \sum_{P_{\bar{X}\bar{Y}}\in\bar{S}_{n}} \sum_{d\in\mathcal{D}_{n}} \sum_{e_{\bar{X}\bar{Y}_{(d)}}} \varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n}) \\ &= \sum_{P_{\bar{X}\bar{Y}}\in\bar{S}_{n}} \sum_{d\in\mathcal{D}_{n}} \sum_{(x^{n}, y^{n})\in\mathcal{E}_{n}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n})} P_{\bar{X}^{n}\bar{Y}_{(d)}}^{(n)}(x^{n}, y^{n}) \\ &= |\mathcal{D}_{n}||\bar{S}_{n}| \sum_{(x^{n}, y^{n})\in\mathcal{E}_{n}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n})} P_{\bar{X}^{n}\bar{Y}^{n}}(x^{n}, y^{n}) \\ &= |\mathcal{D}_{n}||\bar{S}_{n}|\varepsilon_{\tilde{X}\bar{Y}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n}) \\ &\leq (2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}\varepsilon_{\tilde{X}\bar{Y}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n}), \end{split}$$

where the last inequality comes from the fact that  $-n \leq \underline{d}_n \leq \overline{d}_n \leq n$  and (8). Hence, we have (13).

Second part: For an arbitrarily fixed  $P_{XY} \in S$ , we consider a corresponding *n*-type  $P_{\bar{X}\bar{Y}} \in \bar{S}_n$  satisfying (10) in Lemma 2. Then, for any  $d \in \mathcal{D}_n$ ,  $\varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  can be bounded by

$$\varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) = \sum_{(x^n, y^n) \in \mathcal{E}_n(f_n^{(1)}, f_n^{(2)}, \varphi_n)} P_{X^n Y_{(d)}^n}(x^n, y^n)$$
  
$$\leq e^{3(\alpha + \varepsilon_n)} \sum_{(x^n, y^n) \in \mathcal{E}_n(f_n^{(1)}, f_n^{(2)}, \varphi_n)} P_{\bar{X}^n \bar{Y}_{(d)}^n}(x^n, y^n)$$
  
$$= e^{3(\alpha + \varepsilon_n)} \varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n)$$

$$\leq e^{3(\alpha+\varepsilon_n)} \max_{P_{\bar{X}\bar{Y}}\in\bar{\mathcal{S}}_n} \max_{d\in\mathcal{D}_n} \varepsilon_{\bar{X}\bar{Y}_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n)$$

where the first inequality follows since  $P_{\bar{X}\bar{Y}}$  satisfies (10). This inequality holds for any  $P_{XY} \in S$  and  $d \in D_n$ , we have (14).

This theorem implies that if the error probability for the mixed source vanishes at the order  $o(n^{-|\mathcal{X}||\mathcal{Y}|-1})$ , the maximum error probability for DMSs with delay also vanishes. In the next theorem, we show the existence of a code of which the error probability vanishes exponentially rather than  $o(n^{-|\mathcal{X}||\mathcal{Y}|-1})$ .

To this end, we introduce some notations: For  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  and a real number  $\rho \ge 0$ , let  $P_X^{(\rho)} \in \mathcal{P}(\mathcal{X})$  be a PMF given by

$$P_X^{(\rho)}(x) = \frac{P_X(x)^{\frac{1}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}},$$
(15)

 $P_{X|Y}^{(\rho)} \in \mathcal{P}(\mathcal{X}|\mathcal{Y})$  be a conditional PMF given by

$$P_{X|Y}^{(\rho)}(x|y) = \frac{P_{XY}(x,y)^{\frac{1}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}},$$
(16)

and  $\bar{P}_X^{(\rho)} \in \mathcal{P}(\mathcal{X})$  be a PMF given by

$$\bar{P}_{X}^{(\rho)}(x) = \frac{\left(\sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}.$$
(17)

We note that these PMFs are continuous with respect to  $\rho \in [0, 1]$ . We also note that when  $\rho = 0$ , it holds that  $P_X^{(\rho)} = \bar{P}_X^{(\rho)} = P_X$  and  $P_{X|Y}^{(\rho)} = P_{X|Y}$ .

Now, we show an exponential bound on the error probability.

**Theorem 4.** For any  $R_1, R_2 > 0$  and  $n \ge \alpha$ , there exists a code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  such that

$$M_n^{(1)} = [2^{nR_1}], \ M_n^{(2)} = [2^{nR_2}],$$

and

$$-\frac{1}{n}\log \varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)} \geq \min_{i \in \{1,2,3\}} \sup_{0 \leq \rho \leq 1} \min_{P_{XY} \in \overline{\mathcal{S}}_n} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) - \epsilon_n,$$

where  $\epsilon_n = \frac{\log 3(2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}}{n}$ ,

$$F_{1}(\rho, R_{1}, P_{XY}, \Delta) = \rho \left( R_{1} - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right|_{+} \right) + D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - \Delta \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - D(P_{X}^{(\rho)} | P_{X}) \right|_{+}, \quad (18)$$

$$F_{2}(\rho, R_{2}, P_{XY}, \Delta) = \rho \left( R_{2} - H(P_{Y|X}^{(\rho)} | \bar{P}_{X}^{(\rho)}) - \Delta \left| H(P_{Y}^{(\rho)}) - H(P_{Y|X}^{(\rho)} | \bar{P}_{X}^{(\rho)}) \right|_{+} \right) + D(\bar{P}_{X}^{(\rho)} \times P_{Y|X}^{(\rho)} | P_{XY}) - \Delta \left| D(\bar{P}_{X}^{(\rho)} \times P_{Y|X}^{(\rho)} | P_{XY}) - D(P_{Y}^{(\rho)} | P_{Y}) \right|_{+},$$
(19)

$$F_{3}(\rho, R_{3}, P_{XY}, \Delta) = \rho \left( R_{3} - H(P_{XY}^{(\rho)}) - \Delta \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) + D(P_{XY}^{(\rho)} \| P_{XY}) - \Delta \left| D(P_{XY}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) - D(P_{Y}^{(\rho)} \| P_{Y}) \right|_{+}, \quad (20)$$

and  $|x|_+ \triangleq \max\{0, x\}.$ 

*Proof:* By employing Gallager's random coding techniques [9] and [10] to the mixed source  $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}})$ , we can show the existence of a code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$  such that  $M_n^{(1)} = \lceil 2^{nR_1} \rceil$ ,  $M_n^{(2)} = \lceil 2^{nR_2} \rceil$ , and

$$\varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \le 3 \exp\left(-n \min_{i \in \{1, 2, 3\}} \max_{0 \le \rho \le 1} \left(\rho R_i - E_n^{(i)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n})\right)\right),\tag{21}$$

where

$$E_n^{(1)}(\rho, P_{X^n Y^n}) = \frac{1}{n} \log \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_{X^n Y^n}(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho},$$
(22)

$$E_n^{(2)}(\rho, P_{X^n Y^n}) = \frac{1}{n} \log \sum_{x^n \in \mathcal{X}^n} \left( \sum_{y^n \in \mathcal{Y}^n} P_{X^n Y^n}(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho},$$
(23)

$$E_n^{(3)}(\rho, P_{X^n Y^n}) = \frac{1}{n} \log \left( \sum_{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n} P_{X^n Y^n}(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{-1} .$$
(24)

This bound is also given by Nomura and Han [15, Appendix C] more explicitly.

On the other hand, according to Appendix C, we have

$$\max_{0 \le \rho \le 1} \left( \rho R_i - E_n^{(i)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n}) \right) \ge \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{\mathcal{S}}_n} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) - \epsilon'_n, \ \forall i \in \{1, 2, 3\},$$
(25)

where

$$\epsilon'_n = \frac{\log(2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}}{n}.$$

By combining (21) and (25), we have the theorem.

By using Theorems 3 and 4, we obtain the next theorem which gives a bound on the error exponent for the asynchronous SW coding system.

**Theorem 5.** For any  $R_1, R_2 > 0$ , there exists a sequence of codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  such that

$$M_n^{(1)} = \left\lceil 2^{nR_1} \right\rceil, \ M_n^{(2)} = \left\lceil 2^{nR_2} \right\rceil, \ \forall n > 0,$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)} \ge \sup_{\delta > 0} \min_{i \in \{1,2,3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}),$$

where

$$\mathcal{S}_{\delta} = \bigcup_{P_{XY} \in \mathcal{S}} \left\{ P_{X'Y'} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : d_{v}(P_{XY}, P_{X'Y'}) \le \delta \right\}.$$

*Proof:* For any *n*-type  $P_{\bar{X}\bar{Y}} \in \bar{S}_n$ , there exists  $P_{XY} \in S$  satisfying (9) by definition. Thus, for any  $\delta > 0$  and  $n \geq \frac{2(|\mathcal{X}||\mathcal{Y}|-1)}{\delta}$ , we have

$$d_{\mathbf{v}}(P_{XY}, P_{\bar{X}\bar{Y}}) \leq \frac{2(|\mathcal{X}||\mathcal{Y}| - 1)}{n} \leq \delta.$$

This implies that

$$P_{\bar{X}\bar{Y}} \in \{P_{X'Y'} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}) : d_{v}(P_{XY}, P_{X'Y'}) \le \delta\} \subseteq \mathcal{S}_{\delta}$$

Since this holds for any *n*-type  $P_{\bar{X}\bar{Y}} \in \bar{S}_n$ , we have  $\bar{S}_n \subseteq S_\delta$  for any  $\delta > 0$  and  $n > \frac{2(|\mathcal{X}||\mathcal{Y}|-1)}{\delta}$ .

According to this property and Theorem 4, for any  $R_1, R_2 > 0$ , there exists a sequence of codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  such that

$$M_n^{(1)} = \left[2^{nR_1}\right], \ M_n^{(2)} = \left[2^{nR_2}\right], \ \forall n > 0,$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log \varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \\
\geq \liminf_{n \to \infty} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{\mathcal{S}}_n} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) \\
\geq \liminf_{n \to \infty} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}^{(n)}), \, \forall \delta > 0,$$
(26)

where the last inequality comes from the fact that  $\bar{S}_n \subseteq S_{\delta}$  for any  $\delta > 0$  and sufficiently large n > 0.

Furthermore, for this sequence of codes, we have

$$\lim_{n \to \infty} \inf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_{n}} \varepsilon_{XY_{(d)}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n})$$

$$\stackrel{(a)}{\geq} \liminf_{n \to \infty} -\frac{1}{n} \log e^{3(\alpha + \varepsilon_{n})} (2n + 1)(n + 1)^{|\mathcal{X}||\mathcal{Y}|} \varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n})$$

$$= \liminf_{n \to \infty} -\frac{1}{n} \log \varepsilon_{\widetilde{\mathbf{X}}\widetilde{\mathbf{Y}}}^{(n)}(f_{n}^{(1)}, f_{n}^{(2)}, \varphi_{n})$$

$$\stackrel{(b)}{\geq} \sup_{\delta > 0} \liminf_{n \to \infty} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_{i}(\rho, R_{i}, P_{XY}, \Delta_{\mathbf{d}}^{(n)}), \qquad (27)$$

where (a) comes from Theorem 3 and (b) comes from (26).

On the other hand, we have

$$\left| H(P_X^{(\rho)}) - H(P_{X|Y}^{(\rho)}|\bar{P}_Y^{(\rho)}) \right|_+ \stackrel{(a)}{\leq} \left| H(P_X^{(\rho)}) \right|_+ \\ \leq \log |\mathcal{X}|,$$

and

$$\left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) \right|_{+} \leq \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} \| P_{XY}) \right|_{+}$$

$$\stackrel{(b)}{\leq} \log |\mathcal{X}|,$$

where (a) follows since  $|\cdot|_+$  is a monotonically increasing function, and (b) comes from Corollary 3 in Appendix C. Similarly, we have

$$\begin{aligned} \left| H(P_Y^{(\rho)}) - H(P_{Y|X}^{(\rho)} | \bar{P}_X^{(\rho)}) \right|_+ &\leq \log |\mathcal{Y}|, \\ \left| D(\bar{P}_X^{(\rho)} \times P_{Y|X}^{(\rho)} \| P_{XY}) - D(P_Y^{(\rho)} \| P_Y) \right|_+ &\leq \log |\mathcal{Y}|, \\ \left| H(P_X^{(\rho)}) + H(P_Y^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_+ &\leq \log |\mathcal{X}||\mathcal{Y}|, \\ \left| D(P_{XY}^{(\rho)} \| P_{XY}) - D(P_X^{(\rho)} \| P_X) - D(P_Y^{(\rho)} \| P_Y) \right|_+ &\leq \log |\mathcal{X}||\mathcal{Y}|. \end{aligned}$$

Since there exists a sequence  $\{\epsilon_n\}$  such that  $\epsilon_n > 0$ ,  $\lim_{n \to \infty} \epsilon_n = 0$ , and

$$\Delta_{\mathbf{d}}^{(n)} \le \Delta_{\mathbf{d}} + \epsilon_n, \ \forall n > 0,$$

we have

$$F_{1}(\rho, R_{1}, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) = \rho \left( R_{1} - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}}^{(n)} \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right|_{+} \right) + D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - \Delta_{\mathbf{d}}^{(n)} \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - D(P_{X}^{(\rho)} | P_{X}) \right|_{+} \geq \rho \left( R_{1} - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}} \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right|_{+} \right) - \epsilon_{n} \log |\mathcal{X}| + D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - \Delta_{\mathbf{d}} \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - D(P_{X}^{(\rho)} | P_{X}) \right|_{+} - \epsilon_{n} \log |\mathcal{X}| = F_{1}(\rho, R_{1}, P_{XY}, \Delta_{\mathbf{d}}) - 2\epsilon_{n} \log |\mathcal{X}|.$$
(28)

Similarly, we have

$$F_2(\rho, R_2, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) \ge F_2(\rho, R_2, P_{XY}, \Delta_{\mathbf{d}}) - 2\epsilon_n \log |\mathcal{Y}|,$$
<sup>(29)</sup>

$$F_3(\rho, R_3, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) \ge F_3(\rho, R_3, P_{XY}, \Delta_{\mathbf{d}}) - 2\epsilon_n \log |\mathcal{X}||\mathcal{Y}|.$$
(30)

Substituting (28)–(30) into (27), we have the desired bound.

**Remark 3.** In the proof of Theorem 4, the existence of a code is shown by using random binning and the maximum likelihood (ML) decoder (see, e.g., [15, Appendix C]). Thus, obtained encoders of Theorem 5 do not use a set S of PMFs and a bound d of delays. On the other hand, the ML decoder must use S and d (or the PMF of the mixed source) to calculate probabilities of source sequences. This claim is the same as [7]. However, it is not guaranteed that the error exponent for these encoders is bounded by the same exponent in Theorem 5 for other parameters of S and d. Thus, in this theorem, we do not state that the encoders are independent of S and d. In Section VI, we give a sequence of codes that are independent of d and whose error exponent is bounded by the same exponent in Theorem 5 if delays are bounded.

We are interested in the condition that the error exponent of Theorem 5 is positive. The next theorem shows that it is positive whenever  $(R_1, R_2)$  is in the achievable rate region. The proof of the theorem is given in Appendix D.

**Theorem 6.** For any  $S \subseteq \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ ,  $R_1, R_2 > 0$ , and  $\Delta \in [0, 1]$  such that

$$R_i > \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta I(X; Y) \right), \ \forall i \in \{1, 2, 3\},$$

we have

$$\sup_{\delta > 0} \min_{i \in \{1,2,3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, \Delta) > 0.$$
(31)

Now, we prove the direct part.

*Proof of the direct part:* We show that any pair  $(r_1, r_2)$  satisfying

$$(r_1, r_2) \in \left\{ (R_1, R_2) : R_i \ge \sup_{P_{XY} \in \mathcal{S}} (H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y)), \ \forall i \in \{1, 2, 3\} \right\}$$
(32)

is achievable.

For an arbitrarily fixed  $\gamma > 0$  and  $(r_1, r_2)$  satisfying (32), let

$$R_i = r_i + \gamma, \ \forall i \in \{1, 2\}$$

Then, according to Theorem 5, there exists a sequence of codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  such that

$$M_n^{(1)} = \left[2^{nR_1}\right], \ M_n^{(2)} = \left[2^{nR_2}\right], \ \forall n > 0,$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \ge \sup_{\delta > 0} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}})$$

On the other hand, by noticing that

$$R_i \ge \sup_{P_{XY} \in \mathcal{S}} \left( H_i(X, Y) + \Delta_{\mathbf{d}} I(X; Y) \right) + \gamma, \ \forall i \in \{1, 2, 3\}$$

Theorem 6 tells us that

$$\sup_{\delta>0} \min_{i \in \{1,2,3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, \Delta_{\mathbf{d}}) > 0$$

Thus, we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) > 0.$$

This impels that there exists an integer  $N \ge 1$  such that for any  $n \ge N$ ,

$$\sup_{Y_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \le \gamma$$

Now, we choose a sequence  $\{\gamma_k\}_{k=1}^{\infty}$  such that  $\gamma_1 > \gamma_2 > \cdots > 0$  and  $\gamma_k \to 0$  as  $k \to \infty$ . Then, by repeating the above argument for each  $k \ge 1$ , we can show the existence of a sequence of codes  $\{(f_{n,k}^{(1)}, f_{n,k}^{(2)}, \varphi_{n,k})\}$  and an integer  $N_k \ge 1$  such that for any  $n \ge N_k$ ,

$$M_n^{(i)} = \left[2^{n(r_i + \gamma_k)}\right], \ \forall i \in \{1, 2\},$$
$$\sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_{n,k}^{(1)}, f_{n,k}^{(2)}, \varphi_{n,k}) \le \gamma_k.$$

Without loss of generality, we assume that  $N_1 < N_2 < \cdots \rightarrow \infty$ . We denote by  $k_n$  the integer k satisfying  $N_k \leq n < N_{k+1}$  and define a code as  $(f_n^{(1)}, f_n^{(2)}, \varphi_n) \triangleq (f_{n,k_n}^{(1)}, f_{n,k_n}^{(2)}, \varphi_{n,k_n})$ . Then, for the sequence of these codes  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  and any  $n \geq N_1$ , we have

$$M_n^{(i)} = \left[2^{n(r_i + \gamma_{k_n})}\right], \ \forall i \in \{1, 2\}$$
$$\sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \le \gamma_{k_n}.$$

Since  $\gamma_{k_n} \to 0$  as  $n \to \infty$ , we have

$$\limsup_{n \to \infty} R_n^{(i)} = r_i, \ \forall i \in \{1, 2\},$$

$$\lim_{n \to \infty} \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) = 0.$$

This means that  $(r_1, r_2)$  is achievable and completes the proof.

# VI. ELIMINATION OF KNOWLEDGE OF THE BOUND OF DELAYS

In this section, we give an extension of our coding scheme which does not require knowledge of the bound of delays if possible delays are bounded. We note that this scheme is inspired by [16].

We assume that delays are bounded, i.e., for the bound  $\mathbf{d} = \{(\underline{d}_n, \overline{d}_n)\}_{n=1}^{\infty}$ , there are some constants  $\overline{d}$  and  $\underline{d}$  such that

$$\underline{d} \le \underline{d}_n \le \overline{d}_n \le \overline{d}, \ \forall n > 0.$$
(33)

In this case, according to Theorem 5, there exists a sequence  $\{(f_n^{(1)}, f_n^{(2)}, \varphi_n)\}$  of codes such that the error exponent is bounded as

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(f_n^{(1)}, f_n^{(2)}, \varphi_n) \ge \sup_{\delta > 0} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, 0).$$

We can show the existence of a sequence of codes achieving the same bound even if there is no knowledge of the bound d.

To this end, we introduce the *dummy* bound  $\tilde{\mathbf{d}} = \{(\underline{b}_n, \overline{b}_n)\}$  of delays, where  $\underline{b}_n = -\lceil \sqrt{n} \rceil$  and  $\overline{b}_n = \lceil \sqrt{n} \rceil$ . For this bound  $\tilde{\mathbf{d}}$ , according to Theorem 5, there exists a sequence  $\{(\tilde{f}_n^{(1)}, \tilde{f}_n^{(2)}, \tilde{\varphi}_n)\}$  of codes such that the error exponent is bounded as, by the same exponent,

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \widetilde{\mathcal{D}}_n} \varepsilon_{XY_{(d)}}^{(n)}(\widetilde{f}_n^{(1)}, \widetilde{f}_n^{(2)}, \widetilde{\varphi}_n) \ge \sup_{\delta > 0} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, 0),$$

where

$$\widetilde{\mathcal{D}}_n \triangleq \left\{ \underline{b}_n, \underline{b}_n + 1, \cdots, \overline{b}_n - 1, \overline{b}_n \right\}.$$

Since for sufficiently large n, dummy bounds  $\overline{b}_n$  and  $\underline{b}_n$  exceed  $\overline{d}$  and  $\underline{d}$ , i.e.,  $\mathcal{D}_n \subseteq \widetilde{\mathcal{D}}_n$ , we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(\widetilde{f}_n^{(1)}, \widetilde{f}_n^{(2)}, \widetilde{\varphi}_n) \ge \liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \widetilde{\mathcal{D}}_n} \varepsilon_{XY_{(d)}}^{(n)}(\widetilde{f}_n^{(1)}, \widetilde{f}_n^{(2)}, \widetilde{\varphi}_n).$$

We note that this inequality holds for any bound d satisfying (33), and the sequence of codes  $\{(\tilde{f}_n^{(1)}, \tilde{f}_n^{(2)}, \tilde{\varphi}_n)\}$  depends on the dummy bound  $\tilde{d}$  and is independent of the true bound d. Thus, we have the following theorem.

**Theorem 7.** For any  $R_1, R_2 > 0$ , there exists a sequence  $\{(\tilde{f}_n^{(1)}, \tilde{f}_n^{(2)}, \tilde{\varphi}_n)\}$  of codes such that

$$M_n^{(1)} = \begin{bmatrix} 2^{nR_1} \end{bmatrix}, \ M_n^{(2)} = \begin{bmatrix} 2^{nR_2} \end{bmatrix}, \ \forall n > 0,$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log \sup_{P_{XY} \in \mathcal{S}} \max_{d \in \mathcal{D}_n} \varepsilon_{XY_{(d)}}^{(n)}(\widetilde{f}_n^{(1)}, \widetilde{f}_n^{(2)}, \widetilde{\varphi}_n) \ge \sup_{\delta > 0} \min_{i \in \{1, 2, 3\}} \sup_{0 \le \rho \le 1} \inf_{P_{XY} \in \mathcal{S}_{\delta}} F_i(\rho, R_i, P_{XY}, 0),$$

where the sequence  $\{(\tilde{f}_n^{(1)}, \tilde{f}_n^{(2)}, \tilde{\varphi}_n)\}$  is independent of **d**, and the inequality holds for any bound  $\mathbf{d} = \{(\underline{d}_n, \overline{d}_n)\}_{n=1}^{\infty}$  such that there are some constants  $\overline{d}$  and  $\underline{d}$  satisfying

$$\underline{d} \leq \underline{d}_n \leq \overline{d}_n \leq \overline{d}, \ \forall n > 0.$$

# VII. CONCLUSION

In this paper, we have dealt with the asynchronous SW coding system and clarified the achievable rate region. According to the achievable rate region, it does not always coincide with that of the synchronous SW coding system. In the converse part, we used a usual information-theoretic technique. In the direct part, we employed a universal coding scheme using the mixed source. We extended our coding scheme to a scheme which does not require knowledge of the bound of delays.

#### APPENDIX A

#### **CONVERSE BOUND**

In this appendix, we prove the bound (7) in a similar way as in [8, Sec. 15.4.2].

Due to Fano's inequality [8, Theorem 2.10.1], for any n > 0, code  $(f_n^{(1)}, f_n^{(2)}, \varphi_n)$ , delay  $d \in \mathcal{D}_n$ , and  $P_{XY} \in \mathcal{S}$ , we have

$$H(X^{n}, Y^{n}_{(d)}|f_{n}^{(1)}(X^{n}), f_{n}^{(2)}(Y^{n}_{(d)})) \leq n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| + 1.$$
(34)

We also have

$$H(X^{n}|f_{n}^{(1)}(X^{n}), Y_{(d)}^{n}) = H(X^{n}|f_{n}^{(1)}(X^{n}), f_{n}^{(2)}(Y_{(d)}^{n}), Y_{(d)}^{n})$$

$$\leq H(X^{n}, Y_{(d)}^{n}|f_{n}^{(1)}(X^{n}), f_{n}^{(2)}(Y_{(d)}^{n}))$$

$$\leq n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| + 1, \qquad (35)$$

and similarly

$$H(Y_{(d)}^{n}|f_{n}^{(2)}(Y_{(d)}^{n}), X^{n}) \le n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| + 1.$$
(36)

Thus, we have

$$nR_{n}^{(1)} \geq H(f_{n}^{(1)}(X^{n}))$$

$$\geq H(f_{n}^{(1)}(X^{n})|Y_{(d)}^{n})$$

$$= I(X^{n}; f_{n}^{(1)}(X^{n})|Y_{(d)}^{n})$$

$$= H(X^{n}|Y_{(d)}^{n}) - H(X^{n}|f_{n}^{(1)}(X^{n}), Y_{(d)}^{n})$$

$$\geq H(X^{n}|Y_{(d)}^{n}) - n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| - 1, \qquad (37)$$

where the last inequality comes from (35). Similarly, due to (36), we have

$$nR_n^{(2)} \ge H(Y_{(d)}^n | X^n) - n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}| |\mathcal{Y}| - 1.$$

$$(38)$$

We also have

$$\begin{split} n(R_n^{(1)} + R_n^{(2)}) &\geq H(f_n^{(1)}(X^n), f_n^{(2)}(Y_{(d)}^n)) \\ &= I(X^n, Y_{(d)}^n; f_n^{(1)}(X^n), f_n^{(2)}(Y_{(d)}^n)) \\ &= H(X^n, Y_{(d)}^n) - H(X^n, Y_{(d)}^n|f_n^{(1)}(X^n), f_n^{(2)}(Y_{(d)}^n)) \end{split}$$

$$\geq H(X^n, Y^n_{(d)}) - n\varepsilon_{XY_{(d)}}^{(n)} \log |\mathcal{X}||\mathcal{Y}| - 1,$$
(39)

where the last inequality comes from (34).

Combining (37)–(39), we have (7).

# APPENDIX B

# PROOF OF LEMMA 2

We first give the next lemma which is an extension of [11, Lemma 3] to non-binary alphabets.

**Lemma 3.** Let  $\alpha = |\mathcal{X}|(|\mathcal{X}| - 1)$  and  $\varepsilon_n = \alpha \ln \left(1 + \frac{\alpha}{n - \alpha}\right)$ . Then, for any  $n \ge \alpha$  and any  $P_X \in \mathcal{P}(\mathcal{X})$ , there exists an *n*-type  $P_{\bar{X}} \in \mathcal{P}_n(\mathcal{X})$  such that

$$d_{\mathbf{v}}(P_X, P_{\bar{X}}) \le \frac{2(|\mathcal{X}| - 1)}{n},$$

and

$$P_X^n(x^n) \le P_{\bar{X}}^n(x^n) e^{(\alpha + \varepsilon_n)}, \ \forall x^n \in \mathcal{X}^n.$$

$$\tag{40}$$

*Proof:* For any fixed  $P_X \in \mathcal{P}(\mathcal{X})$ , let  $x^* \in \mathcal{X}$  be a symbol with the highest probability, i.e.  $P_X(x^*) \ge P_X(x)$  for any  $x \in \mathcal{X}$ . Let N(x) be an integer defined as

$$N(x) \triangleq \begin{cases} \lceil nP_X(x) \rceil & \text{if } x \neq x^*, \\ n - \sum_{x \in \mathcal{X}: x \neq x^*} \lceil nP_X(x) \rceil & \text{if } x = x^*. \end{cases}$$
(41)

Then, we have

$$0 \le N(x) \le n, \ \forall x \ne x^*,$$

and

$$n \ge N(x^*)$$

$$= n - \sum_{x \in \mathcal{X}: x \neq x^*} \lceil n P_X(x) \rceil$$

$$\ge n - \sum_{x \in \mathcal{X}: x \neq x^*} (n P_X(x) + 1)$$

$$= n \left( 1 - \sum_{x \in \mathcal{X}: x \neq x^*} P_X(x) \right) - (|\mathcal{X}| - 1)$$

$$= n P_X(x^*) - (|\mathcal{X}| - 1)$$

$$\ge n \frac{1}{|\mathcal{X}|} - (|\mathcal{X}| - 1)$$

$$= \frac{n - |\mathcal{X}|(|\mathcal{X}| - 1)}{|\mathcal{X}|}.$$

Thus, for any  $n \ge |\mathcal{X}|(|\mathcal{X}| - 1) = \alpha$ , we have

$$0 \le N(x) \le n, \ \forall x \in \mathcal{X},$$

and also

$$\sum_{x \in \mathcal{X}} N(x) = n$$

For this N(x), we define an  $n\text{-type}\ P_{\bar{X}}\in\mathcal{P}_n(\mathcal{X})$  as

$$P_{\bar{X}}(x) \triangleq \frac{N(x)}{n}.$$

Then, according to (41), we have

$$0 \le P_{\bar{X}}(x) - P_X(x) < \frac{1}{n}, \ \forall x \ne x^*,$$
(42)

and

$$0 \le P_X(x^*) - P_{\bar{X}}(x^*) < \frac{|\mathcal{X}| - 1}{n}.$$
(43)

Hence, we have

$$d_{\mathbf{v}}(P_X, P_{\bar{X}}) = \sum_{x \in \mathcal{X}} |P_{\bar{X}}(x) - P_X(x)| < \frac{2(|\mathcal{X}| - 1)}{n}.$$

We also show that the *n*-type  $P_{\bar{X}}$  satisfies (40). For any  $x^n \in \mathcal{X}^n$  and  $x \in \mathcal{X}$ , define  $N(x|x^n)$  as

$$N(x|x^n) \triangleq |\{i \in \{1, \cdots, n\} : x_i = x\}|.$$

Then, for any  $x^n \in \mathcal{X}^n$ , we have

$$\begin{split} P_{X}^{n}(x^{n}) &= \prod_{x \in \mathcal{X}} P_{X}(x)^{N(x|x^{n})} \\ &= P_{X}(x^{*})^{N(x^{*}|x^{n})} \prod_{x \in \mathcal{X}: x \neq x^{*}} P_{X}(x)^{N(x|x^{n})} \\ &\leq \left( P_{\bar{X}}(x^{*}) + \frac{|\mathcal{X}| - 1}{n} \right)^{N(x^{*}|x^{n})} \prod_{x \in \mathcal{X}: x \neq x^{*}} P_{\bar{X}}(x)^{N(x|x^{n})} \\ &= P_{\bar{X}}(x^{*})^{N(x^{*}|x^{n})} \left( 1 + \frac{|\mathcal{X}| - 1}{nP_{\bar{X}}(x^{*})} \right)^{N(x^{*}|x^{n})} \prod_{x \in \mathcal{X}: x \neq x^{*}} P_{\bar{X}}(x)^{N(x|x^{n})} \\ &= P_{\bar{X}}(x^{n}) \left( 1 + \frac{|\mathcal{X}| - 1}{nP_{\bar{X}}(x^{*})} \right)^{N(x^{*}|x^{n})}, \end{split}$$

where the inequality comes from (42) and (43). According to (43) and the fact that  $P_X(x^*) \ge \frac{1}{|\mathcal{X}|}$ , we have

$$\frac{|\mathcal{X}| - 1}{nP_{\bar{X}}(x^*)} < \frac{|\mathcal{X}| - 1}{nP_X(x^*) - (|\mathcal{X}| - 1)} \le \frac{\alpha}{n - \alpha}.$$

Since  $N(x^*|x^n) \leq n$ , we have

$$P_X^n(x^n) \le P_{\bar{X}}^n(x^n) \left(1 + \frac{|\mathcal{X}| - 1}{nP_{\bar{X}}(x^*)}\right)^{N(x^*|x^n)}$$
$$\le P_{\bar{X}}^n(x^n) \left(1 + \frac{\alpha}{n - \alpha}\right)^{N(x^*|x^n)}$$

where  $n' = n/\alpha - 1$ . Since  $\left(1 + \frac{1}{n'}\right)^{n'}$  is monotonically increasing function of n' and converges to e, we have  $P_X^n(x^n) \le P_{\bar{X}}^n(x^n) \left(1 + \frac{\alpha}{n-\alpha}\right)^{\alpha} e^{\alpha}$ 

$$= P_{\bar{X}}^{n}(x^{n})e^{\alpha+\varepsilon_{n}}.$$

This completes the proof.

By using the above lemma, we prove Lemma 2.

Proof of Lemma 2: According to Lemma 3, for any  $n \ge \alpha$   $(= |\mathcal{X}||\mathcal{Y}|(|\mathcal{X}||\mathcal{Y}| - 1))$ , there exists an *n*-type  $P_{\bar{X}\bar{Y}} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$  such that

$$P_{XY}^n(x^n, y^n) \le P_{\bar{X}\bar{Y}}^n(x^n, y^n) e^{\alpha + \varepsilon_n},\tag{44}$$

,

and

$$d_{\mathbf{v}}(P_{XY}, P_{\bar{X}\bar{Y}}) \leq \frac{2(|\mathcal{X}||\mathcal{Y}| - 1)}{n}.$$

Note that, for any  $k \leq n$ , marginal distributions  $P_{XY}^k(x^k, y^k)$ ,  $P_X^k(x^k)$ , and  $P_Y^k(y^k)$  of  $P_{XY}^n(x^n, y^n)$  satisfy

$$P_{XY}^{k}(x^{k}, y^{k}) \leq P_{\bar{X}\bar{Y}}^{k}(x^{k}, y^{k})e^{\alpha+\varepsilon_{n}}$$
$$P_{X}^{k}(x^{k}) \leq P_{\bar{X}}^{k}(x^{k})e^{\alpha+\varepsilon_{n}},$$
$$P_{Y}^{k}(y^{k}) \leq P_{\bar{Y}}^{k}(y^{k})e^{\alpha+\varepsilon_{n}}.$$

Thus, for any  $d \in \{-n, -n+1, \cdots, n\}$ , we have

$$P_{X^{n}Y_{(d)}^{n}}(x^{n}, y^{n}) = P_{X}^{|d|}(x_{\mathrm{I}}^{|d|})P_{XY}^{n-|d|}(x_{\mathrm{C}}^{n-|d|}, y_{\mathrm{C}}^{n-|d|})P_{Y}^{|d|}(y_{\mathrm{I}}^{|d|})$$

$$\leq P_{\bar{X}}^{|d|}(x_{\mathrm{I}}^{|d|})P_{\bar{X}\bar{Y}}^{n-|d|}(x_{\mathrm{C}}^{n-|d|}, y_{\mathrm{C}}^{n-|d|})P_{\bar{Y}}^{|d|}(y_{\mathrm{I}}^{|d|})e^{3(\alpha+\varepsilon_{n})}$$

$$= P_{\bar{X}^{n}\bar{Y}_{(d)}^{n}}(x^{n}, y^{n})e^{3(\alpha+\varepsilon_{n})}.$$

This completes the proof.

#### APPENDIX C

### GALLAGER-TYPE BOUNDS

In this appendix, we show the bound (25) in the proof of Theorem 4. First of all, we give the next two lemmas. Lemma 4. For any finite set  $S_n \subseteq \mathcal{P}(\mathcal{X}^n \times \mathcal{Y}^n)$  of PMFs and the mixture  $Q_n$  of all PMFs in  $S_n$ , i.e.,

$$Q_n(x^n, y^n) = \sum_{P_n \in \mathcal{S}_n} \frac{1}{|\mathcal{S}_n|} P_n(x^n, y^n),$$

we have for any  $\rho \in [0,1]$ ,

$$E_n^{(i)}(\rho, Q_n) \le \frac{1}{n} \log |\mathcal{S}_n| + \max_{P_n \in \mathcal{S}_n} E_n^{(i)}(\rho, P_n), \ \forall i \in \{1, 2, 3\}.$$
(45)

*Proof:* For any  $\rho \in [0, 1]$ , we have

$$\begin{split} E_n^{(1)}(\rho, Q_n) &= \frac{1}{n} \log \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} Q_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} \left( \sum_{P_n \in \mathcal{S}_n} \frac{1}{|\mathcal{S}_n|} P_n(x^n, y^n) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \frac{1}{|\mathcal{S}_n|} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} \left( \sum_{P_n \in \mathcal{S}_n} P_n(x^n, y^n) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &\stackrel{(a)}{\leq} \frac{1}{n} \log \frac{1}{|\mathcal{S}_n|} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} \sum_{P_n \in \mathcal{S}_n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &\stackrel{(b)}{\leq} \frac{1}{n} \log \frac{1}{|\mathcal{S}_n|} \sum_{y^n \in \mathcal{Y}^n} \sum_{P_n \in \mathcal{S}_n} \frac{1}{|\mathcal{S}_n|} \left( |\mathcal{S}_n| \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \frac{|\mathcal{S}_n|^{1+\rho}}{|\mathcal{S}_n|^2} \sum_{P_n \in \mathcal{S}_n} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &\leq \frac{1}{n} \log \frac{|\mathcal{S}_n|^{1+\rho}}{|\mathcal{S}_n|^2} |\mathcal{S}_n| \sum_{P_n \in \mathcal{S}_n} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log |\mathcal{S}_n|^\rho \max_{P_n \in \mathcal{S}_n} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &\leq \frac{1}{n} \log |\mathcal{S}_n| \max_{P_n \in \mathcal{S}_n} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log |\mathcal{S}_n| + \max_{P_n \in \mathcal{S}_n} \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} P_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho} \end{aligned}$$

where (a) follows since  $(a + b)^r \le a^r + b^r$  for  $0 \le r \le 1$  and  $a, b \ge 0$  (see Lemma 8 in Appendix E), and (b) comes from Jensen's inequality (see, e.g., [8]). Similarly, we have

$$E_n^{(2)}(\rho, Q_n) \le \frac{1}{n} \log |\mathcal{S}_n| + \max_{P_n \in \mathcal{S}_n} E_n^{(2)}(\rho, P_n).$$
(47)

We also have

$$E_n^{(3)}(\rho, Q_n) = \frac{1}{n} \log \left( \sum_{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n} Q_n(x^n, y^n)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$
$$= \frac{1}{n} \log \left( \sum_{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n} \left( \sum_{P_n \in \mathcal{S}_n} \frac{1}{|\mathcal{S}_n|} P_n(x^n, y^n) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$
$$= \frac{1}{n} \log \frac{1}{|\mathcal{S}_n|} \left( \sum_{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n} \left( \sum_{P_n \in \mathcal{S}_n} P_n(x^n, y^n) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$

$$\overset{(a)}{\leq} \frac{1}{n} \log \frac{1}{|\mathcal{S}_{n}|} \left( \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} \sum_{P_{n}\in\mathcal{S}_{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
\overset{(b)}{\leq} \frac{1}{n} \log \frac{1}{|\mathcal{S}_{n}|} \sum_{P_{n}\in\mathcal{S}_{n}} \frac{1}{|\mathcal{S}_{n}|} \left( |\mathcal{S}_{n}| \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
= \frac{1}{n} \log \frac{|\mathcal{S}_{n}|^{1+\rho}}{|\mathcal{S}_{n}|^{2}} \sum_{P_{n}\in\mathcal{S}_{n}} \left( \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
\leq \frac{1}{n} \log \frac{|\mathcal{S}_{n}|^{1+\rho}}{|\mathcal{S}_{n}|^{2}} |\mathcal{S}_{n}| \max_{P_{n}\in\mathcal{S}_{n}} \left( \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
= \frac{1}{n} \log |\mathcal{S}_{n}|^{\rho} \max_{P_{n}\in\mathcal{S}_{n}} \left( \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
\leq \frac{1}{n} \log |\mathcal{S}_{n}| \max_{P_{n}\in\mathcal{S}_{n}} \left( \sum_{(x^{n},y^{n})\in\mathcal{X}^{n}\times\mathcal{Y}^{n}} P_{n}(x^{n},y^{n})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\
= \frac{1}{n} \log |\mathcal{S}_{n}| + \max_{P_{n}\in\mathcal{S}_{n}} E_{n}^{(3)}(\rho,P_{n}), \tag{48}$$

where (a) and (b) come from the same reason as above.

Combining (46)–(48), we have (45).

The next lemma gives identities by Gallager (cf. [10, Section II]).

**Lemma 5.** For any  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  and  $\rho \in [0, 1]$ , it holds that

$$\rho H(P_X^{(\rho)}) - E^{(3)}(\rho, P_X) = D(P_X^{(\rho)} || P_X), \tag{49}$$

$$\rho H(P_{X|Y}^{(\rho)}|\bar{P}_Y^{(\rho)}) - E^{(1)}(\rho, P_{XY}) = D(\bar{P}_Y^{(\rho)} \times P_{X|Y}^{(\rho)} \| P_{XY}),$$
(50)

$$\rho H(P_{Y|X}^{(\rho)}|\bar{P}_X^{(\rho)}) - E^{(2)}(\rho, P_{XY}) = D(\bar{P}_X^{(\rho)} \times P_{Y|X}^{(\rho)} || P_{XY}),$$
(51)

where

$$E^{(3)}(\rho, P_X) \triangleq E_1^{(3)}(\rho, P_X) = \log\left(\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho},$$
  

$$E^{(1)}(\rho, P_{XY}) \triangleq E_1^{(1)}(\rho, P_{XY}) = \log\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho},$$
  

$$E^{(2)}(\rho, P_{XY}) \triangleq E_1^{(2)}(\rho, P_{XY}) = \log\sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho},$$

Proof: Since  $P_X^{(\rho)}(x) = \frac{P_X(x)^{\frac{1}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}}$ , we have for any  $x \in \mathcal{X}$  such that  $P_X(x) > 0$ ,  $\left(\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho} = \left(\frac{P_X(x)^{\frac{1}{1+\rho}}}{P_X^{(\rho)}(x)}\right)^{1+\rho}$   $= \frac{P_X(x)}{P_X^{(\rho)}(x)^{1+\rho}}.$  
$$-\log\left(\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho} = -\sum_{x\in\mathcal{X}:P_X(x)>0} P_X^{(\rho)}(x)\log\frac{P_X(x)}{P_X^{(\rho)}(x)^{1+\rho}}$$
$$=\sum_{x\in\mathcal{X}:P_X^{(\rho)}(x)>0} P_X^{(\rho)}(x)\log\frac{P_X^{(\rho)}(x)P_X^{(\rho)}(x)^{\rho}}{P_X(x)}$$
$$= D(P_X^{(\rho)} || P_X) - \rho H(P_X^{(\rho)}).$$

This identity gives (49).

On the other hand, since

$$P_{X|Y}^{(\rho)}(x|y) = \frac{P_{XY}(x,y)^{\frac{1}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}}$$

and

$$\bar{P}_{Y}^{(\rho)}(y) = \frac{\left(\sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho}},$$

we have for any  $(x,y) \in \mathcal{X} \times \mathcal{Y}$  such that  $P_{XY}(x,y) > 0$ ,

$$\sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} = \frac{P_{XY}(x, y)^{\frac{1}{1+\rho}}}{P_{X|Y}^{(\rho)}(x|y)},$$

and

$$\sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} = \frac{\left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho}}{\bar{P}_{Y}^{(\rho)}(y)}$$
$$= \frac{\left( \frac{P_{XY}(x, y)^{\frac{1}{1+\rho}}}{P_{X|Y}^{(\rho)}(x|y)} \right)^{1+\rho}}{\bar{P}_{Y}^{(\rho)}(y)}$$
$$= \frac{P_{XY}(x, y)}{\bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)^{1+\rho}}.$$

Thus, we have

$$-\log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$

$$= \sum_{(x,y) \in \mathcal{X} \times \mathcal{Y}: P_{XY}(x,y) > 0} \bar{P}_{Y}^{(\rho)}(y) P_{X|Y}^{(\rho)}(x|y) \log \frac{\bar{P}_{Y}^{(\rho)}(y) P_{X|Y}^{(\rho)}(x|y)^{1+\rho}}{P_{XY}(x, y)}$$

$$= D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} || P_{XY}) - \rho H(P_{X|Y}^{(\rho)} |\bar{P}_{Y}^{(\rho)}).$$
(52)

Similarly, we have

$$-\log \sum_{x \in \mathcal{X}} \left( \sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho}$$
$$= D(\bar{P}_X^{(\rho)} \times P_{Y|X}^{(\rho)} || P_{XY}) - \rho H(P_{Y|X}^{(\rho)} |\bar{P}_X^{(\rho)}).$$
(53)

Identities (52) and (53) give (50) and (51), respectively.

**Corollary 3.** For any  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  and  $\rho \in [0, 1]$ , it holds that

$$D(P_X^{(\rho)} || P_X) \le \log |\mathcal{X}|,$$
$$D(\bar{P}_Y^{(\rho)} \times P_{X|Y}^{(\rho)} || P_{XY}) \le \log |\mathcal{X}|.$$

*Proof:* The corollary immediately follows since  $E^{(1)}(\rho, P_X) \ge 0$  and  $E^{(3)}(\rho, P_{XY}) \ge 0$ . Now, we show the bound (25). For any  $\rho \in [0, 1]$  and  $i \in \{1, 2, 3\}$ , we have

$$E_n^{(i)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n}) \stackrel{\text{(a)}}{\leq} \frac{1}{n} \log |\bar{\mathcal{S}}_n| |\mathcal{D}_n| + \max_{P_{XY} \in \bar{\mathcal{S}}_n} \max_{d \in \mathcal{D}_n} E_n^{(i)}(\rho, P_{X^n Y_{(d)}^n})$$

$$\stackrel{\text{(b)}}{\leq} \frac{\log(2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}}{n} + \max_{P_{XY} \in \bar{\mathcal{S}}_n} \max_{d \in \mathcal{D}_n} E_n^{(i)}(\rho, P_{X^n Y_{(d)}^n})$$

$$= \epsilon'_n + \max_{P_{XY} \in \bar{\mathcal{S}}_n} \max_{d \in \mathcal{D}_n} E_n^{(i)}(\rho, P_{X^n Y_{(d)}^n}), \qquad (54)$$

where (a) comes form Lemma 4, (b) comes from the fact that  $|\bar{\mathcal{S}}_n| \leq (n+1)^{|\mathcal{X}||\mathcal{Y}|}$  and  $|\mathcal{D}_n| \leq 2n+1$ , and

$$\epsilon'_n = \frac{\log(2n+1)(n+1)^{|\mathcal{X}||\mathcal{Y}|}}{n}.$$

Thus, we have

$$\sup_{0 \le \rho \le 1} \left( \rho R_i - E_n^{(i)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n}) \right) \ge \sup_{0 \le \rho \le 1} \left( \rho R_i - \max_{P_{XY} \in \overline{S}_n} \max_{d \in \mathcal{D}_n} E_n^{(i)}(\rho, P_{X^n Y_{(d)}^n}) \right) - \epsilon'_n$$
$$= \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{S}_n} \left( \rho R_i - \max_{d \in \mathcal{D}_n} E_n^{(i)}(\rho, P_{X^n Y_{(d)}^n}) \right) - \epsilon'_n.$$
(55)

First, we show the bound (25) for the case where i = 1. For any  $\rho \in [0, 1]$ , we have

$$\begin{split} E_n^{(1)}(\rho, P_{X^n Y_{(d)}^n}) \\ &= \frac{1}{n} \log \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x^n \in \mathcal{X}^n} \left( P_X^{[d]}(x_1^{[d]}) P_{XY}^{n-|d|}(x_C^{n-|d|}, y_C^{n-|d|}) P_Y^{[d]}(y_1^{[d]}) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \sum_{y^n \in \mathcal{Y}^n} \left( \sum_{x_1^{[d]} \in \mathcal{X}^{|d|}} P_X^{[d]}(x_1^{[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \left( \sum_{x_C^{n-|d|} \in \mathcal{X}^{n-|d|}} P_{XY}^{n-|d|}(x_C^{n-|d|}, y_C^{n-|d|})^{\frac{1}{1+\rho}} \right)^{1+\rho} \left( P_Y^{[d]}(y_1^{[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \left( \sum_{x_1^{[d]} \in \mathcal{X}^{|d|}} P_X^{[d]}(x_1^{[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \sum_{y_C^{n-|d|} \in \mathcal{Y}^{n-|d|}} \left( \sum_{x_C^{n-|d|} \in \mathcal{X}^{n-|d|}} P_{XY}^{n-|d|}(x_C^{n-|d|}, y_C^{n-|d|})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \left( \sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \frac{n-|d|}{n} \log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \frac{|d|}{n} \left( \log \left( \sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}} \right)^{1+\rho} - \log \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} \right) \\ &= E^{(1)}(\rho, P_{XY}) + \frac{|d|}{n} \left( E^{(3)}(\rho, P_X) - E^{(1)}(\rho, P_{XY}) \right). \end{split}$$

27

On the other hand, due to (reverse) Minkowski's inequality (see [17, Theorem 25]), we have

$$\log\left(\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho} \ge \log\sum_{y\in\mathcal{Y}} \left(\sum_{x\in\mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho},$$

i.e.,

$$E^{(3)}(\rho, P_X) \ge E^{(1)}(\rho, P_{XY}).$$

Hence, for any n > 0 and  $\rho \in [0, 1]$ , we have

$$\max_{d \in \mathcal{D}_n} E_n^{(1)}(\rho, P_{X^n Y_{(d)}^n}) = E^{(1)}(\rho, P_{XY}) + \frac{\max\{|\overline{d}_n|, |\underline{d}_n|\}}{n} \left( E^{(3)}(\rho, P_X) - E^{(1)}(\rho, P_{XY}) \right)$$
$$= E^{(1)}(\rho, P_{XY}) + \Delta_{\mathbf{d}}^{(n)} \left( E^{(3)}(\rho, P_X) - E^{(1)}(\rho, P_{XY}) \right)$$
$$= (1 - \Delta_{\mathbf{d}}^{(n)}) E^{(1)}(\rho, P_{XY}) + \Delta_{\mathbf{d}}^{(n)} E^{(3)}(\rho, P_X).$$
(56)

Then, we have

$$\begin{split} \rho R_{1} &- \max_{d \in \mathcal{D}_{n}} E_{n}^{(1)}(\rho, P_{X^{n}Y_{(d)}^{n}}) \\ \stackrel{(a)}{=} \rho R_{1} - (1 - \Delta_{\mathbf{d}}^{(n)}) E^{(1)}(\rho, P_{XY}) - \Delta_{\mathbf{d}}^{(n)} E^{(3)}(\rho, P_{X}) \\ &= \rho \left( R_{1} - (1 - \Delta_{\mathbf{d}}^{(n)}) H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}}^{(n)} H(P_{X}^{(\rho)}) + (1 - \Delta_{\mathbf{d}}^{(n)}) H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} H(P_{X}^{(\rho)}) \right) \\ &- (1 - \Delta_{\mathbf{d}}^{(n)}) E^{(1)}(\rho, P_{XY}) - \Delta_{\mathbf{d}}^{(n)} E^{(3)}(\rho, P_{X}) \\ \stackrel{(b)}{=} \rho \left( R_{1} - (1 - \Delta_{\mathbf{d}}^{(n)}) H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}}^{(n)} H(P_{X}^{(\rho)}) \right) \\ &+ (1 - \Delta_{\mathbf{d}}^{(n)}) D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) + \Delta_{\mathbf{d}}^{(n)} D(P_{X}^{(\rho)} | P_{X}) \\ &= \rho \left( R_{1} - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}}^{(n)} \left( H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right) \right) \\ &+ D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - \Delta_{\mathbf{d}}^{(n)} \left( D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - D(P_{X}^{(\rho)} | P_{X}) \right) \\ &\geq \rho \left( R_{1} - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) - \Delta_{\mathbf{d}}^{(n)} \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right|_{+} \right) \\ &+ D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - \Delta_{\mathbf{d}}^{(n)} \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} | P_{XY}) - D(P_{X}^{(\rho)} | P_{X}) \right|_{+} \\ &= F_{1}(\rho, R_{1}, P_{XY}, \Delta_{\mathbf{d}}^{(n)}), \end{split}$$
(57)

where (a) comes from (56) and (b) comes from Lemma 5. By combining (55) and (57), we have (25) for the case where i = 1 as follows:

$$\sup_{0 \le \rho \le 1} \left( \rho R_1 - E_n^{(1)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n}) \right) \ge \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{\mathcal{S}}_n} \left( \rho R_1 - \max_{d \in \mathcal{D}_n} E_n^{(1)}(\rho, P_{X^n Y_{(d)}^n}) \right) - \epsilon'_n$$
$$\ge \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{\mathcal{S}}_n} F_1(\rho, R_1, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) - \epsilon'_n.$$

Similarly, we have (25) for the case where i = 2.

Next, we show the bound (25) for the case where i = 3. We have

 $E_n^{(3)}(\rho, P_{X^n Y_{(d)}^n})$ 

$$\begin{split} &= \frac{1}{n} \log \left( \sum_{x^{n} \in \mathcal{X}^{n}} \sum_{y^{n} \in \mathcal{Y}^{n}} \left( P_{X}^{[d]}(x_{\mathrm{I}}^{[d]}) P_{XY}^{n-[d]}(x_{\mathrm{C}}^{n-[d]}, y_{\mathrm{C}}^{n-[d]}) P_{Y}^{[d]}(y_{\mathrm{I}}^{[d]}) \right)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{1}{n} \log \left( \sum_{x_{\mathrm{I}}^{[d]} \in \mathcal{X}^{[d]}} P_{X}^{[d]}(x_{\mathrm{I}}^{[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \left( \sum_{x_{\mathrm{C}}^{n-[d]} \in \mathcal{X}^{n-[d]}} \sum_{y_{\mathrm{C}}^{n-[d]} \in \mathcal{Y}^{n-[d]}} P_{XY}^{n-[d]}(x_{\mathrm{C}}^{n-[d]}, y_{\mathrm{C}}^{n-[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &\times \left( \sum_{y_{\mathrm{I}}^{[d]} \in \mathcal{Y}^{[d]}} P_{Y}^{[d]}(y_{\mathrm{I}}^{[d]})^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \frac{|d|}{n} \log \left( \sum_{x \in \mathcal{X}} P_{X}(x)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \frac{n-|d|}{n} \log \left( \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \frac{|d|}{n} \log \left( \sum_{y \in \mathcal{Y}} P_{Y}(y)^{\frac{1}{1+\rho}} \right)^{1+\rho} \\ &= \left( 1 - \frac{|d|}{n} \right) \log \left( \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \frac{|d|}{n} \left( \log \left( \sum_{x \in \mathcal{X}} P_{X}(x)^{\frac{1}{1+\rho}} \right)^{1+\rho} + \log \left( \sum_{y \in \mathcal{Y}} P_{Y}(y)^{\frac{1}{1+\rho}} \right)^{1+\rho} \right) \\ &= \left( 1 - \frac{|d|}{n} \right) E^{(3)}(\rho, P_{XY}) + \frac{|d|}{n} \left( E^{(3)}(\rho, P_{X}) + E^{(3)}(\rho, P_{Y}) \right). \end{split}$$
(58)

On the other hand, for any  $\rho \in [0,1]$  and  $d \in \mathcal{D}_n,$  we have,

$$\begin{split} H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left| H(P_X^{(\rho)}) + H(P_Y^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_+ \\ \geq H(P_{XY}^{(\rho)}) + \frac{|d|}{n} \left| H(P_X^{(\rho)}) + H(P_Y^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_+ \\ \geq \left( 1 - \frac{|d|}{n} \right) H(P_{XY}^{(\rho)}) + \frac{|d|}{n} \left( H(P_X^{(\rho)}) + H(P_Y^{(\rho)}) \right). \end{split}$$

Thus, we have

$$\begin{split} \rho(R_{1}+R_{2}) &= E_{n}^{(3)}(\rho, P_{X^{n}Y_{(d)}^{n}}) \\ &= \rho\left(R_{1}+R_{2} - \left(H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left|H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &+ \rho\left(H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left|H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right) - E_{n}^{(3)}(\rho, P_{X^{n}Y_{(d)}^{n}}) \\ &\geq \rho\left(R_{1}+R_{2} - \left(H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left|H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &+ \rho\left(\left(1 - \frac{|d|}{n}\right) H(P_{XY}^{(\rho)}) + \frac{|d|}{n} \left(H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &= \rho\left(R_{1}+R_{2} - \left(H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left|H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &+ \rho\left(\left(1 - \frac{|d|}{n}\right) H(P_{XY}^{(\rho)}) + \frac{|d|}{n} \left(H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &= \rho\left(R_{1}+R_{2} - \left(H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left|H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)})\right|_{+}\right)\right) \\ &+ \left(1 - \frac{|d|}{n}\right) D(P_{XY}^{(\rho)} \|P_{XY}) + \frac{|d|}{n} \left(D(P_{X}^{(\rho)} \|P_{X}) + D(P_{Y}^{(\rho)}) \|P_{Y})\right) \end{split}$$

$$\geq \rho \left( R_{1} + R_{2} - \left( H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) \right) + D(P_{XY}^{(\rho)} \| P_{XY}) - \frac{|d|}{n} \left| D(P_{XY}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) - D(P_{Y}^{(\rho)} \| P_{Y}) \right|_{+} \geq \rho \left( R_{1} + R_{2} - \left( H(P_{XY}^{(\rho)}) + \Delta_{\mathbf{d}}^{(n)} \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) \right) + D(P_{XY}^{(\rho)} \| P_{XY}) - \Delta_{\mathbf{d}}^{(n)} \left| D(P_{XY}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) - D(P_{Y}^{(\rho)} \| P_{Y}) \right|_{+} = F_{3}(\rho, R_{1} + R_{2}, P_{XY}, \Delta_{\mathbf{d}}^{(n)}),$$
(59)

where (a) comes from (58) and (b) comes from Lemma 5. By combining (55) and (59), we have (25) for the case where i = 3 as follows:

$$\sup_{0 \le \rho \le 1} \left( \rho(R_1 + R_2) - E_n^{(3)}(\rho, P_{\widetilde{X}^n \widetilde{Y}^n}) \right) \ge \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{S}_n} \min_{d \in \mathcal{D}_n} \left( \rho(R_1 + R_2) - E_n^{(3)}(\rho, P_{X^n Y_{(d)}^n}) \right) - \epsilon'_n$$
$$\ge \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{S}_n} \min_{d \in \mathcal{D}_n} F_3(\rho, R_1 + R_2, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) - \epsilon'_n$$
$$= \sup_{0 \le \rho \le 1} \min_{P_{XY} \in \overline{S}_n} F_3(\rho, R_1 + R_2, P_{XY}, \Delta_{\mathbf{d}}^{(n)}) - \epsilon'_n.$$

#### APPENDIX D

#### **PROOF OF THEOREM 6**

In this appendix, we prove Theorem 6. To this end, we introduce some necessary lemmas.

**Lemma 6.** For any  $\rho \in [0, 1]$  and any  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ , we have

$$d_{\rm v}(P_X, P_X^{(\rho)}) \le 2\left(1 - |\mathcal{X}|^{-\rho}\right),$$
(60)

$$d_{v}(P_{XY}, \bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)}) \leq 2\left(1 - |\mathcal{X}|^{-\rho}\right).$$
(61)

This lemma shows that each variational distance uniformly converges to 0 as  $\rho \rightarrow 0$ . The proof is given in Appendix E.

The next lemma follows immediately from [14, Problem 3.10].

**Lemma 7.** For any  $\delta \in [0, 1]$ , and  $P, Q \in \mathcal{P}(\mathcal{X})$  satisfying  $d_v(P, Q) \leq \delta$ , we have

$$|H(P) - H(Q)| \le \overline{\delta}(|\mathcal{X}|, \delta),$$

where  $\bar{\delta}(N,\delta) = \frac{1}{2}\delta \log(N-1) + h\left(\frac{\delta}{2}\right)$  if  $N \ge 2$ ,  $\bar{\delta}(1,\delta) = 0$ ,  $h(p) = -(1-p)\log(1-p) - p\log p$ , and note that  $\bar{\delta}(N,\delta) \to 0$  as  $\delta \to 0$ . Furthermore, for any  $\delta \in [0,1]$ ,  $P,Q \in \mathcal{P}(\mathcal{X})$ , and  $V,W \in \mathcal{P}(\mathcal{Y}|\mathcal{X})$  satisfying  $d_{v}(P \times V, Q \times W) \le \delta$ , we have

$$|H(V|P) - H(W|Q)| \le 2\overline{\delta}(|\mathcal{X}||\mathcal{Y}|, \delta).$$

Now, we prove Theorem 6. Let  $\gamma > 0$  be a constant such that

$$R_{i} \ge \sup_{P_{XY} \in \mathcal{S}} (H_{i}(X, Y) + \Delta I(X; Y)) + 3\gamma, \ \forall i \in \{1, 2, 3\}.$$
(62)

$$(1+\Delta)2\bar{\delta}(|\mathcal{X}||\mathcal{Y}|,\delta) + \Delta\bar{\delta}(|\mathcal{X}|,\delta) \le \gamma,$$
(63)

$$(1+\Delta)2\bar{\delta}(|\mathcal{X}||\mathcal{Y}|,\delta) + \Delta\bar{\delta}(|\mathcal{Y}|,\delta) \le \gamma,$$
(64)

$$\bar{\delta}(|\mathcal{X}||\mathcal{Y}|,\delta) + \Delta\left(\bar{\delta}(|\mathcal{X}|,\delta) + \bar{\delta}(|\mathcal{Y}|,\delta) + \bar{\delta}(|\mathcal{X}||\mathcal{Y}|,\delta)\right) \le \gamma.$$
(65)

We also fix  $\rho \in (0, 1]$  so that

$$2\left(1-|\mathcal{X}|^{-\rho}\right) \le \delta,\tag{66}$$

$$2\left(1-|\mathcal{Y}|^{-\rho}\right) \le \delta,\tag{67}$$

$$2\left(1 - (|\mathcal{X}||\mathcal{Y}|)^{-\rho}\right) \le \delta.$$
(68)

We note that  $\gamma > 0$  and  $\delta, \rho \in (0, 1]$  are determined depending only on  $(R_1, R_2, \mathcal{S}, \Delta, |\mathcal{X}|, |\mathcal{Y}|)$ . For these constants  $\gamma, \delta$  and  $\rho$ , we will show that

$$R_{1} \geq \sup_{P_{XY} \in \mathcal{S}_{\delta}} \left( H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)} | \bar{P}_{Y}^{(\rho)}) \right|_{+} \right) + \gamma, \tag{69}$$

$$R_{2} \geq \sup_{P_{XY} \in \mathcal{S}_{\delta}} \left( H(P_{Y|X}^{(\rho)} | \bar{P}_{X}^{(\rho)}) + \Delta \left| H(P_{Y}^{(\rho)}) - H(P_{Y|X}^{(\rho)} | \bar{P}_{X}^{(\rho)}) \right|_{+} \right) + \gamma,$$
(70)

$$R_{1} + R_{2} \ge \sup_{P_{XY} \in \mathcal{S}_{\delta}} \left( H(P_{XY}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) + \gamma.$$
(71)

Then, by using these inequalities, we can easily show that  $F_i(\rho, R_i, P_{XY}, \Delta) \ge \rho\gamma > 0$  for any  $P_{XY} \in S_\delta$  and  $i \in \{1, 2, 3\}$ . This implies (31). Here, we use the fact that for (18)–(20),

$$D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} \| P_{XY}) - \Delta \left| D(\bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) \right|_{+} \ge 0,$$
  
$$D(\bar{P}_{X}^{(\rho)} \times P_{Y|X}^{(\rho)} \| P_{XY}) - \Delta \left| D(\bar{P}_{X}^{(\rho)} \times P_{Y|X}^{(\rho)} \| P_{XY}) - D(P_{Y}^{(\rho)} \| P_{Y}) \right|_{+} \ge 0,$$
  
$$D(P_{XY}^{(\rho)} \| P_{XY}) - \Delta \left| D(P_{XY}^{(\rho)} \| P_{XY}) - D(P_{X}^{(\rho)} \| P_{X}) - D(P_{Y}^{(\rho)} \| P_{Y}) \right|_{+} \ge 0.$$

Hence, all we need is to show (69)-(71).

From the definition of  $S_{\delta}$ , for any  $P_{XY} \in S_{\delta}$ , there exists  $P_{X'Y'} \in S$  such that  $d_{v}(P_{XY}, P_{X'Y'}) \leq \delta$ . Hence, according to Lemma 7, we have for any  $i \in \{1, 2, 3\}$ ,

$$H_i(X',Y') + \Delta I(X';Y') \ge H_i(X,Y) + \Delta I(X;Y) - \left((1+\Delta)2\bar{\delta}(|\mathcal{X}||\mathcal{Y}|,\delta) + \Delta\bar{\delta}(|\mathcal{X}|,\delta)\right).$$

By recalling that  $\delta \in (0,1]$  satisfies (63), we have for any  $i \in \{1,2,3\}$ ,

$$\sup_{P_{XY}\in\mathcal{S}} \left(H_i(X,Y) + \Delta I(X;Y)\right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left(H_i(X,Y) + \Delta I(X;Y)\right) - \gamma.$$
(72)

On the other hand, according to Lemma 6 and (66)–(68), for any  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ , we have

$$d_{\mathbf{v}}(P_X, P_X^{(\rho)}) \le 2\left(1 - |\mathcal{X}|^{-\rho}\right) \le \delta,$$
  
$$d_{\mathbf{v}}(P_Y, P_Y^{(\rho)}) \le 2\left(1 - |\mathcal{Y}|^{-\rho}\right) \le \delta,$$
  
$$d_{\mathbf{v}}(P_{XY}, P_{XY}^{(\rho)}) \le 2\left(1 - (|\mathcal{X}||\mathcal{Y}|)^{-\rho}\right) \le \delta,$$

$$d_{v}(P_{XY}, \bar{P}_{Y}^{(\rho)} \times P_{X|Y}^{(\rho)}) \leq 2 \left(1 - |\mathcal{X}|^{-\rho}\right) \leq \delta, d_{v}(P_{XY}, \bar{P}_{X}^{(\rho)} \times P_{Y|X}^{(\rho)}) \leq 2 \left(1 - |\mathcal{Y}|^{-\rho}\right) \leq \delta.$$

Hence, according to Lemma 7, we have

$$\begin{split} H(X|Y) + \Delta I(X;Y) &= H(P_{X|Y}|P_Y) + \Delta \left| H(P_X) - H(P_{X|Y}|P_Y) \right|_+ \\ &\geq H(P_{X|Y}^{(\rho)}|\bar{P}_Y^{(\rho)}) + \Delta \left| H(P_X^{(\rho)}) - H(P_{X|Y}^{(\rho)}|\bar{P}_Y^{(\rho)}) \right|_+ \\ &- \left( (1 + \Delta) 2\bar{\delta}(|\mathcal{X}||\mathcal{Y}|, \delta) + \Delta \bar{\delta}(|\mathcal{X}|, \delta) \right), \\ H(Y|X) + \Delta I(X;Y) &\geq H(P_{Y|X}^{(\rho)}|\bar{P}_X^{(\rho)}) + \Delta \left| H(P_Y^{(\rho)}) - H(P_{Y|X}^{(\rho)}|\bar{P}_X^{(\rho)}) \right|_+ \\ &- \left( (1 + \Delta) 2\bar{\delta}(|\mathcal{X}||\mathcal{Y}|, \delta) + \Delta \bar{\delta}(|\mathcal{Y}|, \delta) \right), \\ H(X,Y) + \Delta I(X;Y) &= H(P_{XY}) + \Delta \left| H(P_X) + H(P_Y) - H(P_{XY}) \right|_+ \\ &\geq H(P_{XY}^{(\rho)}) + \Delta \left| H(P_X^{(\rho)}) + H(P_Y^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_+ \\ &- \left( \bar{\delta}(|\mathcal{X}||\mathcal{Y}|, \delta) + \Delta \left( \bar{\delta}(|\mathcal{X}|, \delta) + \bar{\delta}(|\mathcal{Y}|, \delta) + \bar{\delta}(|\mathcal{X}||\mathcal{Y}|, \delta) \right) \right). \end{split}$$

By recalling that  $\delta > 0$  satisfies (63)–(65), we have

$$\sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(X|Y) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{X|Y}^{(\rho)}|\bar{P}_{Y}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)}|\bar{P}_{Y}^{(\rho)}) \right|_{+} \right) - \gamma, \quad (73)$$

$$\sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(Y|X) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{Y|X}^{(\rho)}|\bar{P}_{X}^{(\rho)}) + \Delta \left| H(P_{Y}^{(\rho)}) - H(P_{Y|X}^{(\rho)}|\bar{P}_{X}^{(\rho)}) \right|_{+} \right) - \gamma, \quad (74)$$

$$\sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(X,Y) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{XY}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) - \gamma.$$
(75)

As a consequence of inequalities (72)-(75), we have

$$\sup_{P_{XY}\in\mathcal{S}} \left( H(X|Y) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{X|Y}^{(\rho)}|\bar{P}_{Y}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) - H(P_{X|Y}^{(\rho)}|\bar{P}_{Y}^{(\rho)}) \right|_{+} \right) - 2\gamma,$$

$$\sup_{P_{XY}\in\mathcal{S}} \left( H(Y|X) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{Y|X}^{(\rho)}|\bar{P}_{X}^{(\rho)}) + \Delta \left| H(P_{Y}^{(\rho)}) - H(P_{Y|X}^{(\rho)}|\bar{P}_{X}^{(\rho)}) \right|_{+} \right) - 2\gamma,$$

$$\sup_{P_{XY}\in\mathcal{S}} \left( H(X,Y) + \Delta I(X;Y) \right) \ge \sup_{P_{XY}\in\mathcal{S}_{\delta}} \left( H(P_{XY}^{(\rho)}) + \Delta \left| H(P_{X}^{(\rho)}) + H(P_{Y}^{(\rho)}) - H(P_{XY}^{(\rho)}) \right|_{+} \right) - 2\gamma.$$

Hence, due to these inequalities and (62), we have (69)-(71).

# APPENDIX E

#### PROOF OF LEMMA 6

In order to prove Lemma 6, we give some inequalities in the following lemmas and corollary.

Lemma 8 ( [17, Theorem 27]). For any  $x, y \ge 0$ , and  $r \in [0, 1]$ , we have

$$(x+y)^r \le x^r + y^r.$$

Lemma 9. For any  $P_X \in \mathcal{P}(\mathcal{X})$  and any  $\rho \in [0,1]$ , we have

$$1 \le \left(\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho} \le |\mathcal{X}|^{\rho}.$$

33

*Proof:* The first inequality comes from Lemma 8 and the monotonicity of the function  $f(a) = a^{1+\rho}$  for  $a \ge 0$ . The second inequality comes from Jensen's inequality as follows:

$$\left(\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho} = |\mathcal{X}|^{1+\rho} \left(\sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho}$$
$$\leq |\mathcal{X}|^{1+\rho} \sum_{x\in\mathcal{X}} \frac{1}{|\mathcal{X}|} P_X(x) = |\mathcal{X}|^{\rho}.$$

**Corollary 4.** For any  $P_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  and any  $\rho \in [0, 1]$ , we have

$$P_Y(y) \le \left(\sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}}\right)^{1+\rho} \le P_Y(y) |\mathcal{X}|^{\rho}, \ \forall y \in \mathcal{Y},$$

and hence

$$1 \le \sum_{y \in \mathcal{Y}} \left( \sum_{x \in \mathcal{X}} P_{XY}(x, y)^{\frac{1}{1+\rho}} \right)^{1+\rho} \le |\mathcal{X}|^{\rho}.$$

Now, we show (60) and (61). First, we show (60). For any  $\rho \in [0, 1]$  and any  $x \in \mathcal{X}$ , we have

$$P_X^{(\rho)}(x) = \frac{P_X(x)^{\frac{1+\rho}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}}$$

$$\stackrel{(a)}{\geq} \frac{P_X(x)}{\left(\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho}}$$

$$\stackrel{(b)}{\geq} P_X(x)|\mathcal{X}|^{-\rho},$$
(76)

where (a) comes from elementary inequalities  $P_X(x)^{\frac{1}{1+\rho}} \ge P_X(x)$  and  $\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}} \le \left(\sum_{x \in \mathcal{X}} P_X(x)^{\frac{1}{1+\rho}}\right)^{1+\rho}$ , and (b) comes from Lemma 9. Thus, we have (60) as follows:

$$d_{v}(P_{X}, P_{X}^{(\rho)}) = \sum_{x \in \mathcal{X}} |P_{X}(x) - P_{X}^{(\rho)}(x)|$$
  
=  $2 \sum_{x \in \mathcal{X}: P_{X}(x) \ge P_{X}^{(\rho)}(x)} \left(P_{X}(x) - P_{X}^{(\rho)}(x)\right)$   
 $\stackrel{(a)}{\le} 2 \sum_{x \in \mathcal{X}: P_{X}(x) \ge P_{X}^{(\rho)}(x)} \left(P_{X}(x) - P_{X}(x)|\mathcal{X}|^{-\rho}\right)$   
=  $2 \left(1 - |\mathcal{X}|^{-\rho}\right) \sum_{x \in \mathcal{X}: P_{X}(x) \ge P_{X}^{(\rho)}(x)} P_{X}(x)$   
 $\le 2 \left(1 - |\mathcal{X}|^{-\rho}\right),$ 

where (a) comes from (76).

Next, we show (61). For any  $\rho \in [0,1]$  and any  $(x,y) \in \mathcal{X} \times \mathcal{Y}$ , we have

$$\bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y) = \frac{\left(\sum_{x\in\mathcal{X}}P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}}P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}\frac{P_{XY}(x,y)^{\frac{1}{1+\rho}}}{\sum_{x\in\mathcal{X}}P_{XY}(x,y)^{\frac{1}{1+\rho}}}$$
$$= \frac{P_{Y}(y)\left(\sum_{x\in\mathcal{X}}P_{X|Y}(x|y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{y\in\mathcal{Y}}\left(\sum_{x\in\mathcal{X}}P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}\frac{P_{Y}(y)^{\frac{1}{1+\rho}}P_{X|Y}(x|y)^{\frac{1}{1+\rho}}}{P_{Y}(y)^{\frac{1}{1+\rho}}\sum_{x\in\mathcal{X}}P_{X|Y}(x|y)^{\frac{1}{1+\rho}}}$$

$$= \frac{P_{Y}(y) \left(\sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}} \frac{P_{X|Y}(x|y)^{\frac{1}{1+\rho}}}{\sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\frac{1}{1+\rho}}}$$

$$\stackrel{(a)}{=} \frac{P_{Y}(y) \left(\sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}{\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}} \frac{P_{X|Y}(x|y)}{\left(\sum_{x \in \mathcal{X}} P_{X|Y}(x|y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}$$

$$= \frac{P_{XY}(x,y)}{\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{X}} P_{XY}(x,y)^{\frac{1}{1+\rho}}\right)^{1+\rho}}$$

$$\stackrel{(b)}{=} P_{XY}(x,y)|\mathcal{X}|^{-\rho}, \qquad (77)$$

where (a) again comes from the same type of elementary inequalities as before, and (b) comes from Corollary 4. Thus, we have (61) as follows:

$$\begin{aligned} d_{v}(P_{XY},\bar{P}_{Y}^{(\rho)}\times P_{X|Y}^{(\rho)}) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |P_{XY}(x,y) - \bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)| \\ &= 2\sum_{\substack{(x,y)\in\mathcal{X}\times\mathcal{Y}:\\P_{XY}(x,y)\geq\bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)} \left(P_{XY}(x,y) - \bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)\right) \\ &\stackrel{(a)}{\leq} 2\sum_{\substack{(x,y)\in\mathcal{X}\times\mathcal{Y}:\\P_{XY}(x,y)\geq\bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)} \left(P_{XY}(x,y) - P_{XY}(x,y)|\mathcal{X}|^{-\rho}\right) \\ &= 2\left(1 - |\mathcal{X}|^{-\rho}\right)\sum_{\substack{(x,y)\in\mathcal{X}\times\mathcal{Y}:\\P_{XY}(x,y)\geq\bar{P}_{Y}^{(\rho)}(y)P_{X|Y}^{(\rho)}(x|y)}} P_{XY}(x,y) \\ &\leq 2\left(1 - |\mathcal{X}|^{-\rho}\right), \end{aligned}$$

where (a) comes from (77).

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