

New Constructions of Subspace Codes Using Subsets of MRD codes in Several Blocks

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January 6, 2020

Abstract

A basic problem for the constant dimension subspace coding is to determine the maximal possible size $\mathbf{A}_q(n, d, k)$ of a set of k -dimensional subspaces in \mathbf{F}_q^n such that the subspace distance satisfies $d(U, V) = 2k - 2 \dim(U \cap V) \geq d$ for any two different subspaces U and V in this set. We present two new constructions of constant dimension subspace codes using subsets of maximal rank-distance (MRD) codes in several blocks. This method is firstly applied to the linkage construction and secondly to arbitrary number of blocks of lifting MRD codes. In these two constructions, subsets of MRD codes with bounded ranks play an essential role. The Delsarte theorem of the rank distribution of MRD codes is an important ingredient to count codewords in our constructed constant dimension subspace codes. We give many new lower bounds for $\mathbf{A}_q(n, d, k)$. More than 110 new constant dimension subspace codes better than previously best known codes are constructed.

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1 Introduction

Subspace coding was proposed by R. Koetter and F. R. Kschischang in [17] to correct errors and erasures in random network coding (see [8, 22, 16]). A set \mathbf{C} of M subspaces of the dimension k in \mathbf{F}_q^n is called a $(n, M, d, k)_q$ constant dimension subspace code (CDC) if $d(U, V) = \dim(U + V) - \dim(U \cap V) = 2k - 2\dim(U \cap V) \geq d$ is satisfied for any given two distinct subspaces U, V in \mathbf{C} . A main problem for subspace coding is to determine the maximal possible size $\mathbf{A}_q(n, d, k)$ of such a code for given parameters n, d, k, q .

Maximum rank-distance (MRD) codes have been widely used in the constructions of large constant dimension subspace codes. The rank metric on the space $\mathbf{M}_{m \times n}(\mathbf{F}_q)$ of size $m \times n$ matrices over \mathbf{F}_q is defined by the rank of matrices. That is the distance $d_r(A, B)$ is the rank of the matrix $A - B$. The minimum rank-distance of a code $\mathbf{M} \subset \mathbf{M}_{m \times n}(\mathbf{F}_q)$ is defined as

$$d_r(\mathbf{M}) = \min_{A \neq B} \{d_r(A, B), A \in \mathbf{M}, B \in \mathbf{M}\}$$

For a code \mathbf{M} in $\mathbf{M}_{m \times n}(\mathbf{F}_q)$ with the minimum rank distance $d_r(\mathbf{M}) \geq d$, it is well-known that the number of codewords in \mathbf{M} is upper bounded by $q^{\max\{m,n\}(\min\{m,n\}-d+1)}$ (see [5, 9, 4]). A code attaining this bound is called a maximum rank-distance (MRD) code. The MRD code $\mathbf{Q}_{q,n,t}$ consists of \mathbf{F}_q linear mappings on $\mathbf{F}_q^n \cong \mathbf{F}_{q^n}$ defined by q -polynomials $a_0x + a_1x^q + \cdots + a_ix^{q^i} + \cdots + a_tx^{q^t}$, where $a_t, \dots, a_0 \in \mathbf{F}_{q^n}$ are arbitrary elements in \mathbf{F}_{q^n} . The rank-distance of $\mathbf{Q}_{q,n,t}$ is $n - t$ since there are at most q^t roots in \mathbf{F}_{q^n} for each such q -polynomial. There are $q^{n(t+1)}$ such q -polynomials in $\mathbf{Q}_{q,n,t}$ (see [9, 5]). This kind of MRD codes have been used widely in previous constructions of constant dimension subspace codes (see [6, 7, 14, 15, 23]).

In this paper firstly we give a parallel linkage construction based on the linkage construction proposed by Gluesing-Luerssen and Troha in [10]. The basic idea is to use parallel versions of linkage and to give a suitable sufficient condition such that the subspace distance can be preserved for picking up subsets in these parallel blocks. This lead to some new record lower bounds which are better than the best known lower bounds in [11]. The important ingredient is the Delsarte Theorem on the rank distribution of a MRD code. The idea using matrices having lower and upper bounded ranks first appeared in [12].

Secondly we also give a construction of constant dimension subspace codes from several parallel versions of lifted MRD codes. The basic idea is as follows. If we only use elements A_1, A_2, \dots, A_s in a subset of the MRD code $\mathbf{Q}_{q,n,t}$ to construct dimension n subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by the rows of the $n \times (s+1)n$ matrix $(A_1, \dots, I_n, \dots, A_s)$, then for two such n -dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ the dimension of their intersection is at most t since A_1, \dots, A_s are in the MRD code $\mathbf{Q}_{q,n,t}$. On the other hand some other n -dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by the rows of $(B_1, \dots, I_n, \dots, B_s)$, where B_1, B_2, \dots, B_s are in the MRD code $\mathbf{Q}_{q,n,t}$, can be used to increase the size of constructed constant dimension subspace codes. Here we require that I_n appears at different block positions. We actually take $s+1$ subsets of $s+1$ parallel versions of lifted MRD codes. The key point here is to keep the subspace distances larger than or equal to $2(n-t)$ by a suitable sufficient condition on these $s+1$ subsets. By using this method many new lower bound for $\mathbf{A}_q((s+1)n, 2(n-t), n)$, $2t \geq n$ are given with the help from the Delsarte Theorem about the rank distributions of the MRD code $\mathbf{Q}_{q,n,t}$.

In some cases new constant dimension subspace codes in the second construction are better. More than 110 new constant dimension subspace codes better than [11] are constructed. We give some examples in this paper and refer to the tables of new constant dimension subspace codes to the full version [?] of this paper.

2 Known results

2.1 Lifted MRD code

Let $\binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}$ be the q -ary Gauss coefficient, which is the number of k -dimensional subspaces in \mathbf{F}_q^n . For any given MRD code \mathbf{M} in $\mathbf{M}_{n \times n}(\mathbf{F}_q)$ with the rank distance d , we have an $(2n, q^{n(n-d+1)}, 2d, n)_q$ CDC consisting of $q^{n(n-d+1)}$ subspaces of dimension n in \mathbf{F}_q^{2n} spanned by the rows of (I_n, A) , where A is an element in \mathbf{M} . Here I_n is the $n \times n$ identity matrix. It is clear that for A and B , the subspaces U_A and U_B spanned by rows of (I_n, A) and (I_n, B) are the same if and only if $A = B$. The intersection $U_A \cap U_B$ is the set $\{(\alpha, \alpha A) = (\beta, \beta B) : \alpha(A - B) = 0, \alpha \in \mathbf{F}_q^n\}$. Thus $\dim(U_A \cap U_B) \leq n - d$. The distance of this CDC is $2d$. An CDC constructed as above is called a

lifted MRD code C^{MRD} , we refer to Proposition 4 in [22] for the general form.

2.2 Delsarte Theorem

The rank distribution of a code \mathbf{M} in $\mathbf{M}_{m \times n}(\mathbf{F}_q)$ is defined by $A_i(\mathbf{M}) = |\{M \in \mathbf{M}, \text{rank}(M) = i\}|$ for $i \in \mathbf{Z}^+$ (see [5, 4]). The rank distribution of a MRD code can be determined from its parameters. We refer the following result to Theorem 5.6 in [5] or Corollary 26 in [4]. The Delsarte Theorem is essential in this paper.

Theorem 2.1 (Delsarte 1978) *Assume that $\mathbf{M} \subset \mathbf{M}_{n \times n}(\mathbf{F}_q)$ is an MRD code with rank distance d , then its rank distribution is given by*

$$A_r(\mathbf{M}) = \binom{n}{r}_q \sum_{i=0}^{r-d} (-1)^i q^{\binom{i}{2}} \binom{r}{i}_q \left(\frac{q^{n(n-d+1)}}{q^{n(n+i-r)}} - 1 \right).$$

Corollary 2.1. *Assume that $\mathbf{M} \subset \mathbf{M}_{n \times n}(\mathbf{F}_q)$ is an MRD code with rank distance d , then*

$$A_d(\mathbf{M}) = (q^n - 1) \binom{n}{d}_q .$$

$$A_{d+1}(\mathbf{M}) = \binom{n}{d+1}_q (q^{2n} - 1 - \frac{q^{d+1} - 1}{q - 1} (q^n - 1)).$$

$$A_{d+2}(\mathbf{M}) = \binom{n}{d+2}_q (q^{3n} - 1 - \frac{q^{d+2} - 1}{q - 1} (q^{2n} - 1) + q (\frac{q^{d+2} - 1}{q^2 - 1}) (\frac{q^{d+1} - 1}{q - 1}) (q^n - 1)).$$

Let $\mathbf{Q}_{q,n,t,k} \subset \mathbf{Q}_{q,n,t}$ be the set of all q -polynomials in $\mathbf{Q}_{q,n,t}$ satisfying that the dimensions of kernels of the corresponding \mathbf{F}_q -linear mappings on \mathbf{F}_{q^n} are bigger than or equal to k . It is clear that there is a filtration on $\mathbf{Q}_{q,n,t}$: $\mathbf{Q}_{q,n,t,t} \subset \mathbf{Q}_{q,n,t,t-1} \subset \cdots \subset \mathbf{Q}_{q,n,t,1} \subset \mathbf{Q}_{q,n,t,0} = \mathbf{Q}_{q,n,t}$. Actually the cardinalities of these subsets in $\mathbf{Q}_{q,n,t}$ can be given from the Delsarte Theorem 2.1. We have the explicit formula for the cardinality of the space

$\mathbf{Q}_{q,n,t,j}$ when $j \leq t$.

$$|\mathbf{Q}_{q,n,t,j}| = \Sigma_{i=n-t}^{n-j} A_i(\mathbf{Q}_{q,n,t}).$$

2.3 Previous constructions

One upper bound is the anticode bound (see Theorem 5.2 in [25] or Theorem 1 in [8]) of CDC as follow.

$$\mathbf{A}_q(n, 2\delta, k) \leq \frac{\binom{n}{k-\delta+1}_q}{\binom{k}{k-\delta+1}_q}.$$

This showed that the ratio of this upper bound to the cardinality $|\mathbf{C}^{MRD}|$ depends on q (see Lemma 9 on page 1008 of [7]). Hence a lower bound of $\mathbf{A}_q(n, d, k)$ should be compared with the size $|\mathbf{C}^{MRD}|$ of \mathbf{C}^{MRD} . The Johnson type bound (see Theorem 4 in [8]) is the lower bound

$$\mathbf{A}_q(2n-1, 2(n-t), n-1) \geq \frac{1}{q^n + 1} \mathbf{A}_q(2n, 2(n-t), n).$$

This lower bound can be used to get some better constant dimension subspace codes in our construction.

We refer to some known results about general lower bounds for $\mathbf{A}_q(n, d, k)$ to [6, 7, 21, 23]. Many CDCs from the multilevel construction based on echelon-Ferrers diagrams have been given. For example it was proved in [21] that when $q^2 + q + 1 \geq n - \frac{k^2+k-6}{2}$ and in some other cases (see [21])

$$\mathbf{A}_q(n, 2(k-1), k) \geq q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-\sum_{i=j}^k i)} + \binom{n - \frac{k^2+k-6}{2}}{2}_q.$$

It was also proved in [21] that

$$\mathbf{A}_q(n, 4, k) \geq \sum_{i=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} (q^{(k-1)(n-ik)} + \frac{(q^{2(k-2)} - 1)(q^{2(n-ik-1)} - 1)}{(q^4 - 1)^2} q^{(k-3)(n-ik-2)+4}).$$

If $n \geq 2k + 2$ then

$$\mathbf{A}_q(n, 2, k) \geq \sum_{i=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} (q^{(k-1)(n-ik)} + \frac{(q^{2(k-2)} - 1)(q^{2(n-ik-1)} - 1)}{(q^4 - 1)^2} q^{(k-3)(n-ik-2)+4}).$$

This was proved in [21] Corollary 27.

The linkage construction in [10] and the generalization in [12] were used to give many presently best known lower bounds for constant dimension subspace codes with small parameters, we refer to [11].

3 Parallel Linkage construction

3.1 General construction

We recall some basic notations of linkage in [10]. A set $\mathbf{U} \subset \mathbf{M}_{k \times n}(\mathbf{F}_q)$ of $k \times n$ matrices over \mathbf{F}_q is called a SC-representation of a set of k dimensional subspaces in \mathbf{F}_q^n if the rank of the matrices U is k for all $U \in \mathbf{U}$ and $\mathfrak{S}(U_1) \neq \mathfrak{S}(U_2)$ for all $U_1 \neq U_2$ in \mathbf{U} . Here $\mathfrak{S}(U)$ is the k dimensional subspace spanned by the k rows of U . The following Proposition 3.1 is a weaker version of the linkage construction in [10].

Proposition 3.1. *Let \mathbf{U} be a SC-representation of a $(n_1, N_1, d_1, k)_q$ constant dimension subspace code and $\mathbf{Q} \subset \mathbf{M}_{k \times n_2}(\mathbf{F}_q)$ be a code with rank distance d_2 and N_2 elements. Consider the set of k -dimensional subspaces in $\mathbf{F}_q^{n_1+n_2}$ defined by $\mathbf{C} = \{\mathfrak{S}(U, Q) : U \in \mathbf{U}, Q \in \mathbf{Q}\}$. This is a $(n_1 + n_2, N_1 N_2, \min\{d_1, 2d_2\}, k)_q$ constant dimension subspace code. Here $(U|Q)$ is a $k \times (n_1 + n_2)$ matrix concatenated from U and Q .*

Proof. The intersection of two such k -dimensional subspaces $W_1 = \{x(U_1, Q_1) : x \in \mathbf{F}_q^k\}$ and $W_2 = \{y(U_2, Q_2) : y \in \mathbf{F}_q^k\}$ in $\mathbf{F}_q^{n_1+n_2}$ is

$$W_1 \cap W_2 = \{x(U_1, Q_1) = y(U_2, Q_2) : x \in \mathbf{F}_q^k, y \in \mathbf{F}_q^k\}.$$

If $U_1 \neq U_2$, it is clear that the dimension of this subspace is smaller than or equal to the dimension of $\{xU_1 = yU_2 : x \in \mathbf{F}_q^k, y \in \mathbf{F}_q^k\}$. Then the subspace distance fulfills $d(W_1, W_2) \geq d_1$. If $U_1 = U_2$, then for such x and y we have $x = y$ since $U_1 = U_2$ is full rank. Then $\dim(W_1 \cap W_2) \leq \dim(\{x : x(Q_1 - Q_2) = 0\}) = \dim(\ker(Q_1 - Q_2))$. Then we have $d(W_1, W_2) \geq 2d_2$.

The conclusion follows directly.

The following result is our parallel construction applied to the linkage construction.

Theorem 3.1 (Parallel linkage construction). *Let \mathbf{U} and \mathbf{V} be SC-representations of two $(k+n, N_1, d, k)_q$ and $(n+k, N_2, d, k)_q$ constant dimension subspace codes satisfying that each $k \times (k+n)$ matrix in \mathbf{U} is of the form (U_1, U_2) , where U_1 is a non-singular $k \times k$ matrix. We assume that $d \leq k$. Let $\mathbf{Q}_1 \subset \mathbf{M}_{k \times k}(\mathbf{F}_q)$ be a code with rank distance $\frac{d}{2}$ and N_3 elements. Let $\mathbf{Q}_2 \subset \mathbf{M}_{k \times k}(\mathbf{F}_q)$ be a code with rank distance $\frac{d}{2}$ and N_4 elements such that the rank of each element in \mathbf{Q}_2 is at most $k - \frac{d}{2}$. Then we have a $(k+n+k, N_1N_3 + N_2N_4, d, k)_q$ constant dimension subspace code.*

Proof. The code is defined by

$$\mathbf{C} = \{\mathfrak{S}(U_1, U_2, Q) : (U_1, U_2) \in \mathbf{U}, \mathbf{Q} \in \mathbf{Q}_1\} \cup \{\mathfrak{S}(Q', V_1, V_2) : Q' \in \mathbf{Q}_2, (V_1, V_2) \in \mathbf{V}\}.$$

From the proof of Proposition 3.1, the subspace distances of the two codes

$$\mathbf{W}_1 = \{\mathfrak{S}(U_1, U_2, Q) : (U_1, U_2) \in \mathbf{U}, \mathbf{Q} \in \mathbf{Q}_1\}$$

and

$$\mathbf{W}_2 = \{\mathfrak{S}(Q', V_1, V_2) : Q' \in \mathbf{Q}_2, (V_1, V_2) \in \mathbf{V}\}$$

are at least d . We only need to prove that the subspace distance of $W_1 \in \mathbf{W}_1$ and $W_2 \in \mathbf{W}_2$ is at least d . Thus these two codes are disjoint.

Consider $W_1 \cap W_2 = \{x(U_1, U_2, Q) = y(Q', V_1, V_2) : (U_1, U_2) \in \mathbf{U}, (V_1, V_2) \in \mathbf{V}, Q \in \mathbf{Q}_1, Q' \in \mathbf{Q}_2, x, y \in \mathbf{F}_q^k\}$, then $xU_1 = yQ'$. Since $\text{rank}(Q') \leq k - \frac{d}{2}$, $\dim(W_1 \cap W_2) \leq k - \frac{d}{2}$ because the matrix U_1 is a non-singular matrix and the dimension of the subspace $\{x : \exists y, xU_1 = yQ'\}$ is at most the rank of the matrix Q' , that is $k - \frac{d}{2}$. Then

$$d(W_1, W_2) \geq 2k - 2(k - \frac{d}{2}) = d.$$

The conclusion is proved.

3.2 A new lower bound from parallel linkage

Let d and k be two positive integers satisfying $d \leq k$ and d be an even number. Set $\mathbf{U} = (I_k|Q)$ where I_k is an identity matrix of size $k \times k$, where Q is an arbitrary q -polynomial $a_{k-\frac{d}{2}}x^{q^k-\frac{d}{2}} + \cdots + a_1x^q + a_0x$, $a_i \in \mathbf{F}_{q^k}$. Let \mathbf{V} be any $(2k, N, d, k)_q$ constant dimension subspace code. We set \mathbf{Q}_1 to be the MRD code $\mathbf{Q}_{q,k,k-\frac{d}{2}}$. Let $\mathbf{Q}_2 \subset \mathbf{Q}_{q,k,k-\frac{d}{2}}$ be the set consisting of matrices of ranks $\frac{d}{2}, \frac{d}{2} + 1, \dots, k - \frac{d}{2}$, then

$$|\mathbf{Q}_2| = \sum_{i=\frac{d}{2}}^{k-\frac{d}{2}} A_i(\mathbf{Q}_{q,k,k-\frac{d}{2}}).$$

\mathbf{V} is an arbitrary $(2k, d, k)_q$ code. From Theorem 3.1

$$\mathbf{A}_q(3k, d, k) \geq q^{k(2k-d+2)} + (\sum_{i=\frac{d}{2}}^{k-\frac{d}{2}} A_i(\mathbf{Q}_{q,k,k-\frac{d}{2}}))\mathbf{A}_q(2k, d, k).$$

When $k = 6, d = 6$, then

$$\mathbf{A}_q(18, 6, 6) \geq q^{48} + A_3(\mathbf{Q}_{q,6,3})\mathbf{A}_q(12, 6, 6).$$

For example by using the lower bound $\mathbf{A}_2(12, 6, 6) \geq 16813481$ in [11], then $\mathbf{A}_2(18, 6, 6) \geq 282952629488341$. The previously best known lower bound in [11] is $\mathbf{A}_2(18, 6, 6) \geq 282206169223861$. The new lower bound $\mathbf{A}_2(18, 6, 6) \geq 282957166112041$ by the parallel construction from MRD codes in Section 4 is better. If we use the improved lower bound $\mathbf{A}_2(12, 6, 6) \geq 16865101$ in Section 4 (or see [26]) we get the same lower bound $\mathbf{A}_2(18, 6, 6) \geq 282957166112041$ from our parallel linkage construction.

Let h be a non-negative integer and $\phi : \mathbf{F}_{q^k} \rightarrow \mathbf{F}_{q^{k+h}}$ be a q -linear embedding. Then $a_t\phi(x^{q^t}) + a_{t-1}\phi(x^{q^{t-1}}) + \cdots + a_1\phi(x^q) + a_0\phi(x)$ is a q -linear mapping from \mathbf{F}_{q^k} to $\mathbf{F}_{q^{k+h}}$, where $a_i \in \mathbf{F}_{q^{k+h}}$ for $i = 0, 1, \dots, t$. We denote the set of all such mappings as $\mathbf{Q}_{q,k \times (k+h),t}$. It is clear that the dimension of the kernel of any such mapping is at most t . Then $\mathbf{Q}_{q,k \times (k+h),t} \subset \mathbf{M}_{k \times (k+h)}(\mathbf{F}_q)$ is a MRD code with rank distance $k - t$ and $q^{(k+h)(t+1)}$ elements. When $h = 0$ we have the MRD code $\mathbf{Q}_{q,k,t}$.

Let d and k be two positive integers satisfying $d \leq k$ and d be an even number. In Theorem 3.1 we set $\mathbf{U} = (I_k|Q)$ where Q is an arbitrary element in $\mathbf{Q}_{q,k \times (k+h),k-\frac{d}{2}}$. This is a $(2k+h, q^{(k+h)(k-\frac{d}{2}+1)}, d, k)_q$ code. Let \mathbf{Q}_1 be the

MRD code $\mathbf{Q}_{q,k,k-\frac{d}{2}}$. Let $\mathbf{Q}_2 \subset \mathbf{Q}_{q,k,k-\frac{d}{2}}$ be the set consisting of matrices of ranks $\frac{d}{2}, \frac{d}{2} + 1, \dots, k - \frac{d}{2}$. Thus

$$|\mathbf{Q}_2| = \sum_{i=\frac{d}{2}}^{k-\frac{d}{2}} A_i(\mathbf{Q}_{q,k,k-\frac{d}{2}}).$$

\mathbf{V} is an arbitrary $(2k+h, d, k)_q$ code. From Theorem 3.1 we have the following result.

Corollary 3.1. *Let h be a non-negative integer, d and k be positive integer. We assume that $d \leq k$ and d is even. Then*

$$\mathbf{A}_q(3k+h, d, k) \geq q^{(2k+h)(k-\frac{d}{2}+1)} + (\sum_{i=\frac{d}{2}}^{k-\frac{d}{2}} A_i(\mathbf{Q}_{q,k,k-\frac{d}{2}})) \mathbf{A}_q(2k+h, d, k).$$

When $k = 6, d = 6, h = 1$ we get

$$\mathbf{A}_2(19, 6, 6) \geq 2^{52} + A_3(\mathbf{Q}_{2,6,3}) \mathbf{A}_2(13, 6, 6),$$

we have $A_3(\mathbf{Q}_{2,6,3}) = 87885$ from the Delsarte Theorem. Then a new lower bound $\mathbf{A}_2(19, 6, 6) \geq 4527245732135821$ is proved, where the lower bound $\mathbf{A}_2(13, 6, 6) \geq 269057345$ in [11] is used. The previously known best lower bound in [11] is $\mathbf{A}_2(19, 6, 6) \geq 4515298730748862$.

The 63 new constant dimension subspace codes better than [11] by our parallel linkage construction Theorem 3.1 and Corollary 3.1 are listed in Table 1.

4 Parallel construction using subsets of MRD codes in arbitrary number of blocks

4.1 General construction

Similar to lifting the MRD code $\mathbf{Q}_{q,n,t}$ we have an $((s+1)n, q^{sn(t+1)}, 2(n-t), n)_q$ CDC consisting n dimensional subspace U_{A_1, \dots, A_s}^i in $\mathbf{F}_q^{(s+1)n}$ spanned by the rows of $n \times (s+1)n$ matrices $(A_1, \dots, I_n, \dots, A_s)$ with $q^{sn(t+1)}$ elements, where A_1, A_2, \dots, A_s takes all matrices in the MRD code $\mathbf{Q}_{q,n,t}$ and I_n is at any position $i \in \{1, \dots, s+1\}$. Then we consider other CDCs

consisting of the subspace U_{B_1, \dots, B_s}^j in $\mathbf{F}_q^{(s+1)n}$ spanned by the rows of the $n \times (s+1)n$ matrices $(B_1, \dots, I_n, \dots, B_s)$ where B_1, \dots, B_n are matrices from the MRD code $\mathbf{Q}_{q,n,t}$, I_n is at a position $j \neq i$ in the set $\{1, \dots, s+1\}$.

Proposition 4.1. *The intersection $U_{A_1, \dots, A_s}^i \cap U_{B_1, \dots, B_s}^j$ is*

$$\{(\alpha A_1, \dots, \alpha, \dots, \alpha A_s) = (\beta B_1, \dots, \beta, \dots, \beta B_s) : \exists \alpha \in \mathbf{F}_q^n, \exists \beta \in \mathbf{F}_q^n\}.$$

Then $\dim(U_{A_1, \dots, A_s}^i \cap U_{B_1, \dots, B_s}^j) \leq n - \text{rank}(I_n - A_j B_i) = n - \text{rank}(I_n - B_i A_j)$.

In particular two subspaces in $\mathbf{F}_q^{(s+1)n}$ satisfy $U_{A_1, \dots, A_s}^i = U_{B_1, \dots, B_s}^j$ only if $A_j B_i = B_i A_j = I_n$.

Theorem 4.1 (Parallel MRD construction with arbitrary number of blocks). *If $2t \geq n$, then*

$$\mathbf{A}_q((s+1)n, 2(n-t), n) \geq \sum_{j=0}^s q^{(s-j)n(t+1)} (\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))^j.$$

Proof 1. For the first block position of I_n we take n -dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by rows of $(I_n, A_1^1, \dots, A_s^1)$ where A_1^1, \dots, A_s^1 are from the MRD code $\mathbf{Q}_{q,n,t}$. There are $q^{sn(t+1)}$ such subspaces. For the second block position of I_n we take n -dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by rows of $(A_1^2, I_n, \dots, A_s^2)$ where A_1^2, \dots, A_s^2 are from the MRD code $\mathbf{Q}_{q,n,t}$ and $A_1^2 \in \mathbf{Q}_{q,n,t,n-t}$. There are $q^{(s-1)n(t+1)} |\mathbf{Q}_{q,n,t,n-t}| = q^{(s-1)n(t+1)} (\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))$ such subspaces. For the third block position of I_n we take n -dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by rows of $(A_1^3, A_2^3, I_n, \dots, A_s^3)$ where A_1^3, \dots, A_s^3 are from the MRD code $\mathbf{Q}_{q,n,t}$ and $A_1^3 \in \mathbf{Q}_{q,n,t,n-t}, A_2^3 \in \mathbf{Q}_{q,n,t,n-t}$. There are $q^{(s-2)n(t+1)} |\mathbf{Q}_{q,n,t,n-t}|^2 = q^{(s-2)n(t+1)} (\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))^2$ such subspaces. We continue this process. All these subspaces in $\mathbf{F}_q^{n(s+1)}$ are different from Proposition 4.1.

For any fixed block position j of I_n , the dimension of the intersection of two different subspaces is at most t since A_1^j, \dots, A_s^j are in the MRD code $\mathbf{Q}_{q,n,t}$. For different block positions $j > i$ in the set $\{1, \dots, s\}$, the dimension of the intersection of two different subspaces is at most $\dim(\ker(I_n - A_j^i A_i^j))$ from Proposition 4.1. Since the dimension of the space

$$\ker(I_n - A_j^i A_i^j) = \{x : x = x A_j^i A_i^j\}$$

is at most t from the fact $\dim(\ker(A_i^j)) \geq n - t$, we get the conclusion.

Proof 2. We use the same matrices A_i^j as in the Proof 1. First of all these subspaces are different since A_i^j 's with $j > i$ are singular matrices. For two block position indices $i < j$, the intersection of two n dimensional subspaces in $\mathbf{F}_q^{(s+1)n}$ spanned by rows is of the following form

$$\{\beta A_i^j : \beta \in \mathbf{F}_q^n\}.$$

Since the dimension of the kernel of A_i^j is larger than or equal to $n - t$, the dimension of the space of all possible α 's is at most t . The conclusion is proved.

4.2 Some new lower bounds

In the case $s = 1$ we get the following result.

Corollary 4.1. *If $2t \geq n$, we have $\mathbf{A}_q(2n, 2(n-t), n) \geq q^{n(t+1)} + \sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t})$.*

We list all 42 improvements on [11] in Table 2 of the full version [?] of this paper.

Corollary 4.2 (combining with the Johnson type bound). *If $2t \geq n$, we have $\mathbf{A}_q(2n-1, 2(n-t), n-1) \geq \frac{1}{q^n+1}(q^{n(t+1)} + \sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))$.*

We refer to Table 3 for 7 new constant dimension subspace codes from Corollary 4.2 applied to parameters $n = 9, t = 6$.

From Theorem 4.1 we have the following Corollary 3.2 immediately.

Corollary 4.3. *If $2t \geq n$ then $\mathbf{A}_q(3n, 2(n-t), n) \geq q^{2n(t+1)} + q^{n(t+1)}(\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t})) + (\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))^2$.*

For example in the case $n = 2k, t = k$ we have the following lower bound for $\mathbf{A}_q(6k, 2k, 2k)$.

Corollary 4.4. $\mathbf{A}_q(6k, 2k, 2k) \geq q^{4k(k+1)} + q^{2k(k+1)}(q^{2k}-1)\prod_{i=0}^{k-1} \frac{q^{2k-i}-1}{q^{k-i}-1} + ((q^{2k}-1)\prod_{i=0}^{k-1} \frac{q^{2k-i}-1}{q^{k-i}-1})^2$.

We refer to Table 4 for 21 new better constant dimension subspace codes in the case $s = 2$.

From Theorem 4.1 the following lower bound can be proved for the case $s = 3$.

Corollary 4.5. *If $2t \geq n$ then $\mathbf{A}_q(4n, 2(n-t), n) \geq q^{3n(t+1)} + q^{2n(t+1)}(\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t})) + q^{n(t+1)}(\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))^2 + (\sum_{i=n-t}^t A_i(\mathbf{Q}_{q,n,t}))^3$.*

In the case $n = 2k, t = k$ we have the following lower bound from the Delsarte Theorem.

Corollary 4.6. $\mathbf{A}_q(8k, 2k, 2k) \geq q^{6k(k+1)} + q^{4k(k+1)}(q^{2k}-1) \prod_{i=0}^{k-1} \frac{q^{2k-i}-1}{q^{k-i}-1} + q^{2k(k+1)}((q^{2k}-1) \prod_{i=0}^{k-1} \frac{q^{2k-i}-1}{q^{k-i}-1})^2 + ((q^{2k}-1) \prod_{i=0}^{k-1} \frac{q^{2k-i}-1}{q^{k-i}-1})^3$.

We list some lower bound for $\mathbf{A}_q(20, 4, 5)$ and $\mathbf{A}_q(24, 6, 6)$ in Table 5. No entries in [11] can be compared with these lower bounds.

5 Conclusion

In this paper two parallel constructions of constant dimension subspace codes based on the linkage construction and arbitrary number of lifted MRD codes are given. Many new lower bounds on $\mathbf{A}_q(n, d, k)$ were proved from these parallel constructions using subsets of MRD codes in several blocks. The novelty of this paper is the using subsets counted by the Delsarte Theorem in several parallel blocks of lifted MRD codes. From Tables 1-5 in the full version [?] more than 110 new constant dimension subspace codes better than [11] have been constructed from our new parallel constructions.

Added Notes. Some results in this paper have been extended in [3, 13, 19].

Acknowledgement. We are grateful to the first referee for his/her careful reading of the paper and very helpful comments. We thank the Associate Editor and the second referee for their suggestions.

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A Appendix

Table 1: New subspace codes from parallel linkage

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_2(15, 4, 5)$	1252409384941	1235787711790
$\mathbf{A}_3(15, 4, 5)$	12399152701973746721	12394544365887696067
$\mathbf{A}_4(15, 4, 5)$	1215514411297742359058971	1215478900794081741379237
$\mathbf{A}_5(15, 4, 5)$	9113715532358125452199289569	9113676963739967346201192181
$\mathbf{A}_7(15, 4, 5)$	6369953433032799634940814403 550401	6369951878418978850938882154 998943
$\mathbf{A}_8(15, 4, 5)$	1329603936275508854413747923 192211831	1329603830010446369320349184 800629897
$\mathbf{A}_9(15, 4, 5)$	1478344516592412811422378899 08886770241	1478344472192502033634129606 95716746417
$\mathbf{A}_2(16, 4, 5)$	20021868703021	19772603404689
$\mathbf{A}_3(16, 4, 5)$	1004246333824396831601	1003958093636913086356
$\mathbf{A}_4(16, 4, 5)$	311169775104436392108967291	311162598603284926601722789
$\mathbf{A}_5(16, 4, 5)$	569606782898065136733586366 0369	569604810233747959140019927 4056
$\mathbf{A}_7(16, 4, 5)$	152942576811121287704054973 91944583201	152942544600839682211042601 25199891012
$\mathbf{A}_8(16, 4, 5)$	544605767017671083057925917 9780851932151	544605728772278832873615029 1737261978761
$\mathbf{A}_9(16, 4, 5)$	969941834171302637911606992 269353378624481	969941808205500584267352435 307639908204424
$\mathbf{A}_2(17, 4, 5)$	320365633119931	316361655057323
$\mathbf{A}_3(17, 4, 5)$	81343951054914823057601	81320605584592333256896
$\mathbf{A}_4(17, 4, 5)$	796594623030380982288285895 51	796576252424409420394019074 93

continued table

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_5(17, 4, 5)$	356004239278632417623139424 6966849	356003006396092474470158189 5765556
$\mathbf{A}_7(17, 4, 5)$	367215126923122857711324989 25095511388801	367215049586616076988713969 88494253570488
$\mathbf{A}_8(17, 4, 5)$	223070522170400489301792865 41042123379285951	223070506505125409945032726 04052070322817161
$\mathbf{A}_9(17, 4, 5)$	636378837399770196848173247 3171613640616521601	636378820363628933337809933 8862136580032063628
$\mathbf{A}_2(18, 4, 5)$	5125557140935621	5061786480788587
$\mathbf{A}_3(18, 4, 5)$	6586984882892620375466801	6586969052351977742082856
$\mathbf{A}_4(18, 4, 5)$	203923535631374864844098226 92731	203923520620648811617657149 36261
$\mathbf{A}_5(18, 4, 5)$	222501880086200341340588393 7426160369	222501878997557796543846638 9564044756
$\mathbf{A}_7(18, 4, 5)$	881683334130185794241976851 18451711404973601	881683334057465200849902241 56399442519707072
$\mathbf{A}_8(18, 4, 5)$	913696794659942511668319032 60963819527758453751	913696794644993679134854045 86035285169051407881
$\mathbf{A}_9(18, 4, 5)$	417528144042218931834576543 49776548099387974166561	417528144040576943162937097 62272976069664226251756
$\mathbf{A}_2(19, 4, 5)$	82000714657355896	80988583692738669
$\mathbf{A}_3(19, 4, 5)$	533545775512317389092508801	533544493240510197079493944
$\mathbf{A}_4(19, 4, 5)$	52204425121630728423907635 42302191	5220442127886095774120022 68988497
$\mathbf{A}_5(19, 4, 5)$	13906367505387518067957491 07370809466849	13906367437347362283990414 91814662484526
$\mathbf{A}_7(19, 4, 5)$	21169216852465760915956323 5358302246119908739201	21169216850719739472406152 8199514099366794683044
$\mathbf{A}_8(19, 4, 5)$	37425020709271245277558484 3883548745445452491759551	37425020708658941097363621 7184400516522331058635329
$\mathbf{A}_9(19, 4, 5)$	273940215306099841176451031 332562928972470621968108481	273940215305022532409203029 750272995891203951349923082
$\mathbf{A}_2(18, 4, 6)$	1321065731337118327	1301902384896972957
$\mathbf{A}_3(18, 4, 6)$	43241984500836016475467263377	43225562953761729683056546744
$\mathbf{A}_4(18, 4, 6)$	133649773466469722107967099790 3343231	13364584050324721907495003191 15666769

continued table

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_5(18, 4, 6)$	869154313455571766784982919397 200744721649	86915062398553386450111346455 8715816063570
$\mathbf{A}_7(18, 4, 6)$	508273121397713162379788076978 860454942323352347617	50827299725042503817812207998 9337055565420133852250
$\mathbf{A}_8(18, 4, 6)$	153292907353372012238179216797 2707130036770914661707007	15329289509595962385976540015 68049806785911717931336256
$\mathbf{A}_9(18, 4, 6)$	179732185298838953062296457191 2268841144546741298887430561	17973217989922197607083648785 65965820546280222286535998208
$\mathbf{A}_2(19, 4, 6)$	42208289248791279191	41660876316712223851
$\mathbf{A}_3(19, 4, 6)$	10503812381857770555608261193 203	10503811797764100313173626438 410
$\mathbf{A}_4(19, 4, 6)$	13685334079363182818700343352459 15778619	1368533406753251523327488538756 265820613
$\mathbf{A}_5(19, 4, 6)$	27160957000408320819947997516309 16148882970869	2716095699954793272557435557305 913288978107256
$\mathbf{A}_7(19, 4, 6)$	85425442647896740463687280455997 49570456742066054954023	8542544264787894328644239651424 668612867543534298440406
$\mathbf{A}_8(19, 4, 6)$	50231015865045467079871255733225 534110744159933103898225143	5023101586504404954636792632338 1856072893849422409772176905
$\mathbf{A}_9(19, 4, 6)$	10613005490869209759549527518453 4478863507421929255303118181289	10613005490869158463473211884225 8778340200008478852505231237828
$\mathbf{A}_2(18, 6, 6)$	282952629488341	282206169223861
$\mathbf{A}_3(18, 6, 6)$	79773409456539341408321	7977052899429695519499
$\mathbf{A}_4(18, 6, 6)$	79228596836450068221001288411	79228465213535437618551984193
$\mathbf{A}_5(18, 6, 6)$	355271606149018080912048635023 7569	355271549860537797212597654883 4375
$\mathbf{A}_7(18, 6, 6)$	367033693031672275575986481885 27350390401	367033691269048237620480816438 30813838569
$\mathbf{A}_8(18, 6, 6)$	223007453917574042420344146217 65810585581431	223007453646901902254328287720 81255730905601
$\mathbf{A}_9(18, 6, 6)$	636268545986545103814893081316 5508870317893121	636268545755947049961803989258 2186036406787787
$\mathbf{A}_2(19, 6, 6)$	4527245732135821	4515298730748862
$\mathbf{A}_3(19, 6, 6)$	6461646166374500995275281	6461417369472937542117973
$\mathbf{A}_4(19, 6, 6)$	202825207901329766842749685 10491	20282487415579548140041494 597697

continued table

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_5(19, 6, 6)$	222044753843136427265160012 6040019569	22204471885174522176980972 29003922001
$\mathbf{A}_7(19, 6, 6)$	881247896969045133932229870 51228915467935201	88124789274625593704300569 118173789808207533
$\mathbf{A}_8(19, 6, 6)$	913438531246383277768338625 08434972183748561271	91343853013939625003475366 792854319199699599873
$\mathbf{A}_9(19, 6, 6)$	417455793021772242613440430 29615239593994446168481	41745579287064298367037383 503578317354686374220297

Table 2: New lower bounds on $\mathbf{A}_q(2n, 2(n-t), n)$ in the case
 $s = 1$

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_2(12, 6, 6)$	16865101	16813481
$\mathbf{A}_3(12, 6, 6)$	282454201121	282444003514
$\mathbf{A}_4(12, 6, 6)$	281476519727131	281476052114497
$\mathbf{A}_5(12, 6, 6)$	59604684750269569	59604675306650126
$\mathbf{A}_7(12, 6, 6)$	191581237048517640001	191581236128477745586
$\mathbf{A}_8(12, 6, 6)$	4722366523787007379831	4722366518055302169089
$\mathbf{A}_9(12, 6, 6)$	79766443311676870053761	79766443282767710316742
$\mathbf{A}_2(14, 6, 7)$	34532238023	34432090228
$\mathbf{A}_3(14, 6, 7)$	50035894106387201	50031545103789355
$\mathbf{A}_4(14, 6, 7)$	1180598085852241507903	1180591620717679804753
$\mathbf{A}_5(14, 6, 7)$	2910384996920980634798249	2910383045673376465235151
$\mathbf{A}_7(14, 6, 7)$	378818703472375564718065717033	378818692265664782360946466387
$\mathbf{A}_8(14, 6, 7)$	40564819558769908757687294403071	40564819207303340852292565865025
$\mathbf{A}_9(14, 6, 7)$	250315551236152487860737633625 4513	250315550499324160133844882159 4903
$\mathbf{A}_2(16, 8, 8)$	1099562828461	1099528467457
$\mathbf{A}_3(16, 8, 8)$	12157665957047665121	12157665459056935444
$\mathbf{A}_4(16, 8, 8)$	1208925820022362618115611	1208925819614629174771969
$\mathbf{A}_5(16, 8, 8)$	9094947017807612368002449569	9094947017729282379150781876
$\mathbf{A}_8(16, 8, 8)$	13292279957849213674394203378 73299831	1329227995784915872903807060 297125889
$\mathbf{A}_9(16, 8, 8)$	14780882941434601431198749296 8729104321	1478088294143459233160832102 06426350884

continued table

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_2(16, 6, 8)$	282927683836351	282065502894292
$\mathbf{A}_3(16, 6, 8)$	79773403858211367304001	79766443077154959293127
$\mathbf{A}_4(16, 6, 8)$	79228596795209597286010744831	79228162514264619069883417872
$\mathbf{A}_5(16, 6, 8)$	355271606144635047856413687678 1249	355271367880050098896030301250 3775
$\mathbf{A}_7(16, 6, 8)$	367033693031655064026816246271 51289328001	367033682172941254414217922688 54792155015
$\mathbf{A}_8(16, 6, 8)$	22300745391757287672361562599 998342819479551	2230074519853062314154044063917 0885812523072
$\mathbf{A}_9(16, 6, 8)$	63626854598654462048615260385 54759414900421761	63626854411359423584749085289 81840165075519087
$\mathbf{A}_2(18, 8, 9)$	18015215398068295	18014674602898481
$\mathbf{A}_3(18, 8, 9)$	58149739380417667198523945	58149737003040060077869735
$\mathbf{A}_4(18, 8, 9)$	32451855376784298642321288625 1071	32451855365842672678322474110 1633
$\mathbf{A}_5(18, 8, 9)$	555111512317358783579601168291 64048249	55111512312578270211815872192 48047001
$\mathbf{A}_7(18, 8, 9)$	431811456739659181762301609528 5299264536325745	431811456739643656403529309770 9356501432634891
$\mathbf{A}_8(18, 8, 9)$	584600654932363583793403430292 3933590182378512895	584600654932361167281473933086 5150093023313396225
$\mathbf{A}_9(18, 8, 9)$	3381391913522728424620280247018 514713413256655866641	3381391913522726342930221472392 241320293166235632813
$\mathbf{A}_2(18, 6, 9)$	9271545156551861247	9242714023345881465
$\mathbf{A}_3(18, 6, 9)$	1144661280188113228748844786839	1144561273430987589803690699062
$\mathbf{A}_4(18, 6, 9)$	8507105814618280327650335111984 8669183	8507059173023462058821041620363 9381057
$\mathbf{A}_5(18, 6, 9)$	1084202899657109779066908450170 97020371093749	1084202172485504434152971953749 65698255868876
$\mathbf{A}_7(18, 6, 9)$	1742515033889755513188849225993 69466772818993479502027	1742514982336908143055132035255 56311342652949519875530
$\mathbf{A}_8(18, 6, 9)$	7846377237219197911383816346352 35733830468771226268467199	7846377169233350954794740024195 11965161887731769327059457
$\mathbf{A}_9(18, 6, 9)$	1310020512493866339206870302329 188713348371417431388541027913	1310020508637620352391208118240 901618983186587598009041655712

Table 3: New lower bounds from the Johnson type bound

$\mathbf{A}_q(n, d, k)$	New	Old
$\mathbf{A}_2(17, 6, 8)$	18073187439672244	18052309715589680
$\mathbf{A}_3(17, 6, 8)$	58151863451946414791142287	58149737004893178906982592
$\mathbf{A}_4(17, 6, 8)$	32451909495196476483054550389 9935	324518553658445173598894784069 722
$\mathbf{A}_5(17, 6, 8)$	555111600407300798344248374232 36913732	555111512312578503042579744633 91093912
$\mathbf{A}_7(17, 6, 8)$	431811458814229328190145779776 0474522447137650	431811456739643656513972079940 3479106597531752
$\mathbf{A}_8(17, 6, 8)$	584600655642087187407545566975 9065390165175356426	584600654932361167289396749397 0879175905108820562
$\mathbf{A}_9(17, 6, 8)$	338139191474840770349258063849 2271571254198293516660	338139191352272634293365515622 1409046767887551853552

Table 4: New lower bounds on $\mathbf{A}_q(3n.2(n-t), n)$ in the case $s = 2$

$\mathbf{A}_q(n, d, k)$	New	Code
$\mathbf{A}_2(18, 6, 6)$	282957166112041	282206169223861
$\mathbf{A}_3(18, 6, 6)$	79773409708059646924801	79770528994296955194991
$\mathbf{A}_4(18, 6, 6)$	79228596837171602219181433561	79228465213535437618551984193
$\mathbf{A}_5(18, 6, 6)$	355271606149055831666451347994 5761	355271549860537803173065185548 4501
$\mathbf{A}_7(18, 6, 6)$	367033693031672327723398953389 21195414401	367033691269048247553967909699 87924701979
$\mathbf{A}_8(18, 6, 6)$	223007453917574044765606725592 19358376203601	223007453646901902254328287720 81255730905601
$\mathbf{A}_9(18, 6, 6)$	636268545986545104493692756885 8327487086310721	636268545755947014524069928399 2755392451222033
$\mathbf{A}_2(18, 4, 6)$	1321055665352277121	1301902384896972957
$\mathbf{A}_3(18, 4, 6)$	43241984454039791949376848001	43225562953761729683056546744
$\mathbf{A}_4(18, 4, 6)$	13364977346615645679038498701 19608321	13364584050324721907495003191 15666769
$\mathbf{A}_5(18, 4, 6)$	86915431345555286301049529274 6726010500001	8691506239855338472183793783 22608115640776
$\mathbf{A}_7(18, 4, 6)$	50827312139771315191417379894 7508628845999547723521	508272997250425080540503340954 642021097480629123655
$\mathbf{A}_8(18, 4, 6)$	15329290735337201203431548481 57539946320174365857546241	153292895095959623859765400156 8049806785911717931405888
$\mathbf{A}_9(18, 4, 6)$	17973218529883895304078740000 31880315113074804045244546241	179732179899221976044864635882 5022918933920879979928528654
$\mathbf{A}_2(15, 4, 5)$	1252379805361	1235787711790
$\mathbf{A}_3(15, 4, 5)$	12399152568347096641	12394544365887696067
$\mathbf{A}_4(15, 4, 5)$	1215514411238392851780481	1215478900794081741379237
$\mathbf{A}_5(15, 4, 5)$	9113715532351043940956916001	9113676963739967346201192181
$\mathbf{A}_7(15, 4, 5)$	636995343303278946060145826616 9601	636995187841897885093888215499 8943
$\mathbf{A}_8(15, 4, 5)$	132960393627550866960611827601 3276161	132960383001044636932034918480 0629897
$\mathbf{A}_9(15, 4, 5)$	147834451659241278745558658029 146634561	147834447219250203363412960695 716746417

Table 5: Some lower bounds on $\mathbf{A}_q(4n, 2(n-t), n)$

$\mathbf{A}_q(n, d, k)$	Lower Bounds
$\mathbf{A}_2(20, 4, 5)$	1315398998655356311
$\mathbf{A}_3(20, 4, 5)$	43233485281590911580807321041
$\mathbf{A}_4(20, 4, 5)$	1336472440592799231370494712907901631
$\mathbf{A}_5(20, 4, 5)$	869151650599051646738433375279575594407249
$\mathbf{A}_7(20, 4, 5)$	508273020693237561132754855997185401884597574981601
$\mathbf{A}_8(20, 4, 5)$	1532928970776586688938815376036341347556330253989504511
$\mathbf{A}_9(20, 4, 5)$	1797321806605534646862867182733878159175088330825288747361
$\mathbf{A}_2(24, 6, 6)$	4747234173413401936981
$\mathbf{A}_3(24, 6, 6)$	22530367127371196208130075198509281
$\mathbf{A}_4(24, 6, 6)$	22300867449560834030210344616161360246897891
$\mathbf{A}_5(24, 6, 6)$	211758378832969565256609532806254712000815561347009
$\mathbf{A}_7(24, 6, 6)$	7031676686916460305530685695221278081277908734094305105188801
$\mathbf{A}_8(24, 6, 6)$	105312292581044862467221898491140379101347355113312142905458229671
$\mathbf{A}_9(24, 6, 6)$	507528787550401889216222390017824754036868775577998894410404563393281