# Efficient and Explicit Balanced Primer Codes

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Abstract—To equip DNA-based data storage with random-access capabilities, Yazdi et al. (2018) prepended DNA strands with specially chosen address sequences called primers and provided certain design criteria for these primers. We provide explicit constructions of errorcorrecting codes that are suitable as primer addresses and equip these constructions with efficient encoding algorithms.

Specifically, our constructions take cyclic or linear codes as inputs and produce sets of primers with similar error-correcting apabilities. Using certain classes of BCH codes, we obtain infinite families of primer sets of length n, minimum distance d with  $(d+1)\log_4 n + O(1)$  redundant symbols. Our techniques involve reversible cyclic codes (1964), an encoding method of Tavares et al. (1971) and Knuth's balancing technique (1986). In our investigation, we also construct efficient and explicit binary balanced error-correcting codes and codes for DNA computing.

#### I. Introduction

Advances in synthesis and sequencing technologies have made DNA macromolecules an attractive medium for digital information storage. Besides being biochemically robust, DNA strands offer ultrahigh storage densities of  $10^{15} - 10^{20}$  bytes per gram of DNA, as demonstrated in recent experiments (see [1, Table 1]). Therefore, in recent years, new error models were proposed and novel coding schemes were constructed by various authors (see [2] for a survey).

In this paper, we study the problem of *primer design*. To introduce random-access and rewriting capabilities into DNA-based data storage, Yazdi *et al.* developed an architecture that allows selective access to encoded DNA strands through the process of *hybridization*. Their technique involves prepending information-carrying DNA strands with specially chosen address sequences called primers. Yazdi *et al.* provided certain design considerations for these primers [3] and also, verified the feasibility of their architecture in a series of experiments [2], [4].

We continue this investigation and provide efficient and explicit constructions of error-correcting codes that are suitable as primer addresses. Our techniques include novel modifications of *Knuth's* 

addresses. Our techniques include novel modifications of Knuth's balancing technique [5] and involve the use of reversible cyclic codes [6]. We also revisit the work of Tavares et al. [7] that efficient encodes messages into cyclic classes of a cyclic code and adapt their method for our codes. We note that reversible cyclic codes have been studied in another coding application for DNA computing. It turns out our techniques can be also modified to improve code constructions in the latter application.

### II. PRELIMINARY AND CONTRIBUTIONS

Let  $\mathbb{F}_q$  denote the finite field of size q. Two cases of special interest are q=2 and q=4. In the latter case, we let  $\omega$  denote a primitive element of  $\mathbb{F}_4$  and identify the elements of  $\mathbb{F}_4$  with the four DNA bases  $\Sigma = \{A, C, T, G\}$ . Specifically,

$$0 \leftrightarrow \mathtt{A}, \quad 1 \leftrightarrow \mathtt{T}, \quad \omega \to \mathtt{C}, \quad \omega + 1 \leftrightarrow \mathtt{G}.$$

Hence, for an element  $x \in \mathbb{F}_4$ , its Watson-Crick complement corresponds to x + 1.

Let n be a positive integer. Let [n] denote the set  $\{1, 2, \ldots, n\}$ , while [n] denotes the set  $\{0, 1, \dots, n-1\}$ . For a word a = $(a_1,\ldots,a_n)\in\mathbb{F}_q^n$ , let  $\boldsymbol{a}[i]$  denote the *i*th symbol  $a_i$  and  $\boldsymbol{a}[i,j]$ denote the subword of a starting at position i and ending at position j. In other words,

$$\mathbf{a}[i,j] = \begin{cases} (a_i, a_{i+1}, \dots, a_j), & \text{if } i \leq j; \\ (a_j, a_{j-1}, \dots, a_i), & \text{if } i > j. \end{cases}$$

Moreover, the *reverse* of  $\boldsymbol{a}$ , denoted as  $\boldsymbol{a}^r$ , is  $(a_n, a_{n-1}, \dots, a_1)$ ; the complement  $\overline{a}$  of a is  $(\overline{a}_1, \overline{a}_2, \dots, \overline{a}_n)$ , where  $\overline{x} = x + 1$  for  $x \in \mathbb{F}_2$  or  $x \in \mathbb{F}_4$ ; and the reverse-complement  $\mathbf{a}^{rc}$  of  $\mathbf{a}$  is  $\overline{\mathbf{a}^r}$ .

For two words a and b, we use ab to denote the concatenation of a and b, and  $a^{\ell}$  to denote the sequence of length  $\ell n$  comprising  $\ell$  copies of a.

A q-ary code  $\mathcal{C}$  of length n is a collection of words from  $\mathbb{F}_q^n$ . For two words **a** and **b** of the same length, we use  $d(\mathbf{a}, \mathbf{b})$ to denote the Hamming distance between them. A code C has minimum Hamming distance d if any two distinct codewords in  $\mathcal{C}$  is at least distance d apart. Such a code is denoted as an  $(n,d)_q$ code. Its size is given by  $|\mathcal{C}|$ , while its redundancy is given by  $n - \log_q |\mathcal{C}|$ . An  $[n, k, d]_q$ -linear code is an  $(n, d)_q$ -code that is also an k-dimension vector subspace of  $\mathbb{F}_q^n$ . Hence, an  $[n, k, d]_q$ linear code has redundancy n-k.

#### A. Cyclic and Reversible Codes

For a vector  $\mathbf{a} \in \mathbb{F}_q^n$ , let  $\mathbf{\sigma}^i(\mathbf{a})$  be the vector obtained by cyclically shifting the components of a to right i times. So,  $\sigma^1(a)=(a_n,a_1,a_2,\ldots,a_{n-1}).$  An  $[n,k,d]_q$ -cyclic code  ${\mathfrak C}$  is an  $[n, k, d]_q$ -linear code that is closed under cyclic shifts. In other words,  $a \in \mathcal{C}$  implies  $\sigma^1(a) \in \mathcal{C}$ .

Cyclic codes are well-studied because of their rich algebraic structure. In the theory of cyclic codes (see for example, MacWilliams and Sloane [8, Chapter 7]), we identify a word  $c = (c_i)_{i \in \llbracket n \rrbracket}$  of length n with the polynomial  $\sum_{i=0}^{n-1} c_i X^i$ . Given a cyclic code  $\mathcal{C}$  of length n and dimension k, there exists a unique monic polynomial g(X) of degree n-k such that  $\mathcal{C}$  is given by the set  $\{m(X)g(X): \deg m < k\}$ . The polynomial q(X) is referred to as the generator polynomial of  $\mathcal{C}$  and we write  $\mathcal{C} = \langle q(X) \rangle$ . We continue this discussion on this algebraic structure in Section VI, where we exploit certain polynomial properties for efficient encoding.

When d is fixed, there exists a class of Bose-Chaudhuri-Hocquenghem (BCH) codes that are cyclic codes whose redundancy is asymptotically optimal.

**Theorem 1** (Primitive narrow-sense BCH codes [9, Theorem 10]). Fix m > 1 and  $2 < d < 2^m - 1$ . Set  $n = 2^m - 1$  and  $t = 2^m - 1$  $\lceil (d-1)/2 \rceil$ . There exists an  $\lceil n, k, d \rceil_2$ -cyclic code  $\mathfrak C$  with  $k \geq 1$ n-tm. In other words,  $\mathfrak{C}$  has redundancy at most  $t \log_2(n+1)$ .

A cyclic code  $\mathcal{C}$  is called *reversible* if  $a \in \mathcal{C}$  implies  $a^r \in \mathcal{C}$ . A reversible cyclic code is also known as an LCD cyclic code and has been studied extensively [6], [10]-[12]. In this paper, reversible cyclic codes containing the all-one vector  $1^n$  are of particular interest. Suppose that  $\mathcal{C}$  is one such code. Then for any codeword  $a \in \mathcal{C}$ , both its complement  $\overline{a} = a + 1^n$  and its reverse-complement  $a^{rc} = a^r + 1^n$  belong to  $\mathcal{C}$ .

Recently, Li *et al.* [11] explored two other classes of BCH codes and determined their minimum distances and dimensions. These codes are reversible cyclic and contain the all-one vector.

**Theorem 2** (Li et al. [11]). Let  $m \ge 2$ ,  $m \ne 3$  and  $1 \le \tau \le \lceil m/2 \rceil$ . Let q be even and set  $n = q^m - 1$  and  $d = q^\tau - 1$ . There exists an  $[n, k, d]_q$ -reversible cyclic code that contains  $1^n$  and has dimension

$$k = \begin{cases} n - (d-q+1)m, & \text{if } m \geq 5 \text{ is odd and } \tau = \frac{m+1}{2}; \\ n - (d-1)m, & \text{otherwise}. \end{cases}$$

In other words,  $\mathcal{C}$  has redundancy at most  $(d-1)\log_a(n+1)$ .

## B. Balanced Codes

A binary word of length n is balanced if  $\lfloor n/2 \rfloor$  or  $\lceil n/2 \rceil$  bits are zero, while a quaternary word of length n is GC-balanced if  $\lfloor n/2 \rfloor$  or  $\lceil n/2 \rceil$  symbols are either G or C. A binary (or quaternary) code is balanced (resp. GC-balanced) if all its codewords are balanced (resp. GC-balanced).

Motivated by applications in laser disks, Knuth [5] studied balanced binary codes and proposed an efficient method to encode an arbitrary binary message to a binary balanced codeword by introducing  $\log_2 n$  redundant bits. Recently, Weber  $et\ al.$  [13] extended Knuth's scheme to include error-correcting capabilities. Specifically, their construction takes two input codes of distance d: a linear code of length n and a short balanced code  $\mathcal{C}_p$ ; and outputs a long balanced code of distance d. Even though the balanced code  $\mathcal{C}_p$  is only required to be size n, it is unclear how to find one efficiently, especially when d grows with n.

On the other hand, GC-balanced codes have been extensively studied in the context of DNA computing and DNA-based storage (see [2], [14], [15] for a survey). However, most constructions are based on search heuristics or apply to a restricted set of parameters. Recently, Yazdi *et al.* [3] introduced the coupling construction (Lemma 6) that takes two binary error-correcting codes, one of which is balanced, as inputs and outputs a GC-balanced error-correcting code. As with the construction of Weber *et al.* [13], it is unclear how to find the balanced binary error-correcting code efficiently.

In this work, we avoid these requirements of additional balanced codes. Specifically, we provide construction that takes a binary cyclic code (or two binary linear codes) and outputs a binary balanced code (resp. a GC-balanced code) with errorcorrecting capabilities.

#### C. Primer Codes

In order to introduce random access to DNA-based data storage systems, Yazdi *et al.* [3] proposed the following criteria for the design of primer addresses.

**Definition 3.** A code  $\mathcal C$  of length n is  $\kappa$ -weakly mutually uncorrelated  $(\kappa$ -WMU) if for all  $\ell \geq \kappa$ , no proper prefix of length  $\ell$  of a codeword appears as a suffix of another codeword (including itself). In other words, for any two codewords  $a,b\in \mathcal C$ , not necessarily distinct, and  $\kappa \leq \ell \leq n$ ,

$$a[1,\ell] \neq b[n-\ell+1,n].$$

When  $\mathcal{C}$  is 1-WMU, we say that  $\mathcal{C}$  is mutually uncorrelated (MU).

**Definition 4.** A code  $\mathcal C$  of length n is said to avoid primer dimer byproducts of effective length f (f-APD) if the reverse complement and the complement of any substring of length f in a codeword does not appear in as a substring of another codeword (including itself). In other words, for any two codewords  $a, b \in \mathcal C$ , not necessarily distinct, and  $1 \le i, j \le n+1-f$ , we have

$$\overline{a}[i, i+f-1] \notin \{b[j, j+f-1], b[j+f-1, j]\}.$$

For primer design in DNA-based storage, WMU codes are desired to be GC-balanced, have large Hamming distance and avoid primer dimer byproducts.

**Definition 5.** A code  $\mathcal{C} \in \mathbb{F}_q^n$  is an  $(n, d; \kappa, f)_q$ -primer code if the following are satisfied:

- (P1)  $\mathcal{C}$  is an  $(n,d)_q$ -code;
- (P2)  $\mathcal{C}$  is  $\kappa$ -WMU;
- (P3)  $\mathcal{C}$  is an f-APD code.

Furthermore, if  ${\mathfrak C}$  is balanced or GC-balanced, then  ${\mathfrak C}$  is an  $(n,d;\kappa,f)_q$ -balanced primer code.

Yazdi *et al.* [3] provided a number of constructions for WMU codes which satisfy some combinations of the constraints (P1), (P2) and (P3). In particular, Yazdi *et al.* provided the following coupling construction.

**Lemma 6** (Coupling Construction - Yazdi *et al.* [3]). For  $i \in [2]$ , let  $\mathcal{C}_i$  be an  $(n, d_i)_2$ -code of size  $M_i$ . Define the map  $\Psi : \mathbb{F}_2^n \times \mathbb{F}_2^n \to \Sigma^n$  such that  $\Psi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$  where for  $i \in [n]$ ,

$$c_i = \begin{cases} \mathtt{A}, & \textit{if } a_ib_i = 00; \\ \mathtt{T}, & \textit{if } a_ib_i = 01; \end{cases} \quad c_i = \begin{cases} \mathtt{C}, & \textit{if } a_ib_i = 10; \\ \mathtt{G}, & \textit{if } a_ib_i = 11. \end{cases}$$

Then the code  $\mathbb{C} \triangleq \{\Psi(\boldsymbol{a}, \boldsymbol{b}) : \boldsymbol{a} \in \mathbb{C}_1, \boldsymbol{b} \in \mathbb{C}_2\}$  is an  $(n, d)_4$ -code of size  $M_1M_2$ , where  $d = \min\{d_1, d_2\}$ . Furthermore,

- (i) if  $C_1$  is balanced, C is GC-balanced;
- (ii) if  $C_2$  is  $\kappa$ -WMU, then C is also  $\kappa$ -WMU;
- (iii) if  $C_2$  is an f-APD code, then C is also an f-APD code.

Yazdi *et al.* also provided an iterative construction for primer codes satisfying all the constraints, *i.e.* balanced primer codes. However, the construction requires a short balanced primer code and a collection of subcodes, some of which disjoint. Hence, it is unclear whether the code can be constructed efficiently and whether efficient encoding is possible.

In this work, we provide constructions that take cyclic, reversible cyclic or linear codes as inputs and produce primer or balanced primer codes as outputs. Using known families of cyclic codes given by Theorems 1 and 2, we obtain infinite families of primer codes and provide explicit upper bounds on the redundancy. We also describe methods that efficiently encode into these codewords.

#### D. Our Contributions

In this paper, we study balanced codes, primer codes and other related coding problems. Our contributions are as follow:

A. In Section III, we propose efficient methods to construct both balanced and GC-balanced error-correcting codes. Unlike previous methods that require short balanced error-correcting codes, our method uses only cyclic and linear codes as inputs. Furthermore, our method always increases the redundancy only by  $\log_2 n + 1$  (where n is the block length), regardless of the value of the minimum distance.

- B. In Section IV, we provide three constructions of primer codes. For general parameters, the first construction produces a class of  $(n,d;\kappa,f)_4$ -balanced primer codes whose redundancy is  $(d+1)\log_4 n + O(1)$ , while the other two rely on cyclic codes and use less redundancy albeit for a specific set of parameters. In particular, we have a class of  $(n,d;\kappa,\kappa)_4$ -balanced primer codes with redundancy  $(d+1)\log_4(n+1)$ .
- C. In Section V, we construct codes for DNA computing. In particular, we provide a class of GC-balanced  $(n, d)_4$ -DNA computing codes with redundancy  $(d+1)\log_4(n+1)$ .
- D. In Section VI, we adapt the technique of Tavares *et al.* to efficiently encode messages into codes constructed in this paper.

#### III. BALANCED ERROR-CORRECTING CODES

The celebrated Knuth's balancing technique [5] is a linear-time algorithm that maps a binary message of length m to a balanced word of length approximately  $m + \log m$ . The technique first finds an index z such that flipping the first z bits yields a balanced word c. Then Knuth appends a short balanced word p that represents the index z. Hence, cp is the resulting codeword and the redundancy of the code is equal to the length of p which is approximately  $\log m$ . The crucial observation demonstrated by Knuth is that such an index z always exists and z is commonly referred to as the balancing index.

Recently, Weber et~al.~[13] modified Knuth's balancing technique to endow the code with error-correcting capabilities. Their method requires two error-correcting codes as inputs: an  $(m,d)_2$  code  $\mathcal{C}_m$  and a short  $(p,d)_2$  balanced code  $\mathcal{C}_p$  where  $|\mathcal{C}_p| \geq m$ . Given a message, they first encode it into a codeword  $m \in \mathcal{C}_m$ . Then they find the balancing index z of m and flip the first z bits to obtain a balanced c. Using  $\mathcal{C}_p$ , they encode z into a balanced word p and the resulting codeword is cp. Since both  $\mathcal{C}_m$  and  $\mathcal{C}_p$  has distance d, the resulting code has minimum distance d.

Now, this method introduces p additional redundant bits and since p is necessarily at least d, the method introduces more than  $\log_2 n$  bits of redundancy when d is big. Furthermore, the method requires the existence of a short balanced code  $\mathcal{C}_p$ . We overcome this obstacle in our next two constructions. Specifically, Construction A and B require only a cyclic code and a linear codes, respectively. Both constructions do not require short balanced codes and introduces only  $\log_2 n + 1$  additional bits of redundancy, regardless the value of d.

# A. Binary Balanced Error-Correcting Codes

Let n be odd. In contrast with Knuth's balancing technique, we always flip the first (n+1)/2 bits of a word a. However, this does not guarantee a balanced word. Nevertheless, if we consider all cyclic shifts of a, i.e.  $\sigma^i(a)$  for  $i \in \llbracket n \rrbracket$ , then flipping the first (n+1)/2 bits of one of these shifts must yield a balanced word. Formally, let  $\phi: \mathbb{F}_2^n \to \mathbb{F}_2^n$  be the map where  $\phi(a) = a + 1^{(n+1)/2}0^{(n-1/2)}$ . In other words, the map  $\phi$  flips the first (n+1)/2 bits of a. For  $a \in \mathbb{F}_2^n$ , denote its Hamming weight as  $\operatorname{wt}(a)$ . Let  $\operatorname{wt}_1(a)$  be the Hamming weight of the first (n+1)/2 bits and  $\operatorname{wt}_2(a)$  be the Hamming weight of the last (n-1)/2 bits.

So, we have  $\operatorname{wt}(a) = \operatorname{wt}_1(a) + \operatorname{wt}_2(a)$ . We have the following crucial lemma.

**Lemma 7.** Let n be odd. For  $\mathbf{a} \in \mathbb{F}_2^n$ , we can find  $i \in [n]$  such that  $\phi(\sigma^i(\mathbf{a}))$  has weight either (n-1)/2 or (n+1)/2.

*Proof.* Let  $a' = \sigma^{(n+1)/2}(a)$ . Then the first (n-1)/2 bits of a' are exactly the last (n-1)/2 bits of a and so  $\operatorname{wt}_2(a) \leq \operatorname{wt}_1(a') \leq \operatorname{wt}_2(a) + 1$ .

We first consider the case when  $\operatorname{wt}(a)$  is even. Assume that  $\operatorname{wt}(a) = 2w$ . If  $\operatorname{wt}_1(a) \leq w$ , then  $\operatorname{wt}_1(a') \geq \operatorname{wt}_2(a) = 2w - \operatorname{wt}_1(a) \geq w$ . Note that shifting the components of a once only increases or decreases the value of  $\operatorname{wt}_1(a)$  by at most one. It follows that we can find an integer i such that  $\operatorname{wt}_1(\sigma^i(a)) = w$ , and so

$$wt(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a}))) = wt_{1}(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a}))) + wt_{2}(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a})))$$
$$= ((n+1)/2 - w) + w = (n+1)/2.$$

Similarly, if  $\operatorname{wt}_1(a) > w$ , since  $\operatorname{wt}_1(a') \le \operatorname{wt}_2(a) + 1 = 2w - \operatorname{wt}_1(a) + 1 \le w$ , we can still find i such that  $\operatorname{wt}_1(\sigma^i(a)) = w$  and  $\operatorname{wt}(\phi(\sigma^i(a))) = (n+1)/2$ .

Next, we assume that the weight is odd, or,  $\operatorname{wt}(a) = 2w + 1$ . If  $\operatorname{wt}_1(a) < w + 1$ , then  $\operatorname{wt}_1(a') \ge \operatorname{wt}_2(a) = 2w + 1 - \operatorname{wt}_1(a) \ge w + 1$ ; if  $\operatorname{wt}_1(a) \ge w + 1$ , then  $\operatorname{wt}_1(a') \le \operatorname{wt}_2(a) + 1 = 2w + 1 - \operatorname{wt}_1(a) + 1 \le w + 1$ . In both cases we can always find i such that  $\operatorname{wt}_1(\sigma^i(a)) = w + 1$ , and so

$$\operatorname{wt}(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a}))) = \operatorname{wt}_{1}(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a}))) + \operatorname{wt}_{2}(\phi(\boldsymbol{\sigma}^{i}(\boldsymbol{a})))$$
$$= ((n+1)/2 - w - 1) + w = (n-1)/2. \quad \square$$

**Remark.** In Lemma 7, we show that we can balance some shift of a by flipping its first (n+1)/2 bits. In fact, we can also balance a shift of a (not necessary the same shift) by flipping its first (n-1)/2 bits. This observation is used in the construction of DNA computing codes.

Before we describe our construction, we introduce the notion of cyclic equivalence classes. Given a cyclic code  $\mathcal{B}$  of length n, we define the following equivalence relation:  $\mathbf{a} \sim \mathbf{b}$  if and only if  $\mathbf{a} = \mathbf{\sigma}^i(\mathbf{b})$  for some  $i \in [n]$ ; and partition the codewords  $\mathcal{B}$  into classes. We use  $\mathcal{B}/\sim$  to denote a set of representatives.

Construction A. Let n be an odd integer.

INPUT: An  $[n, k, d]_2$ -cyclic code  $\mathfrak{B}$ .

OUTPUT: A balanced  $(n+1,d')_2$ -code  ${\mathcal C}$  of size at least  $2^k/n$  where  $d'=2\lceil d/2 \rceil$ .

- Let  $u_1, u_2, \ldots, u_m$  be the set of representatives  $\mathcal{B}/\underset{\mathrm{cyc}}{\sim}$ .
- For each  $u_i$ , find  $j_i \in [n]$  such that  $\phi(\sigma^{j_i}(u_i))$  has weight (n-1)/2 or (n+1)/2.
- For  $i \in [m]$ , append a check bit to  $\phi(\sigma^{j_i}(u_i))$  so that its weight is (n+1)/2 and denote the modified vector as  $v_i$ . In other words,

$$\boldsymbol{v}_i = \begin{cases} \phi(\boldsymbol{\sigma}^{j_i}(\boldsymbol{u}_i))0, & \text{if } \operatorname{wt}(\phi(\boldsymbol{\sigma}^{j_i}(\boldsymbol{u}_i))) = \frac{n+1}{2}; \\ \phi(\boldsymbol{\sigma}^{j_i}(\boldsymbol{u}_i))1, & \text{if } \operatorname{wt}(\phi(\boldsymbol{\sigma}^{j_i}(\boldsymbol{u}_i))) = \frac{n-1}{2}. \end{cases}$$

• Set  $C = \{v_i : 1 \le i \le m\}$ .

**Theorem 8.** Construction A is correct. In other words,  $\mathfrak{C}$  is a balanced  $(n+1,2\lceil d/2 \rceil)_2$ -code of size at least  $2^k/n$ .

*Proof.* It is easy to see that  $\mathcal C$  is a balanced code of length n+1 and size m. Since the m cyclic classes are pairwise disjoint and each of them consists of at most n codewords, we have that  $m \geq |\mathcal B|/n = 2^k/n$ .

Since  $\mathcal B$  is an  $[n,k,d]_2$ -cyclic code and the map  $\phi$  does not change the distance between any two vectors, the minimum distance of  $\mathcal C$  is at least d. Moreover, when d is odd, the minimum distance is at least d+1, as the distance between any two binary balanced words is even.  $\square$ 

Let d be even and set t=d/2-1. If we apply Construction A to the family of primitive narrow-sense BCH  $[n',k,d]_2$ -cyclic codes, where  $n'=2^m-1=n-1$ . we obtain a family of balanced codes with redundancy at most  $(t+1)\log_2 n+1$ .

**Corollary 9.** Let d be even. There exists a family of  $(n, d)_2$ -balanced codes with redundancy at most  $(t+1) \log_2 n + 1$ , where t = d/2 - 1.

In contrast, if we apply the technique of Weber *et al.* [13] to the same family of codes, the balanced  $(n,d)_2$ -codes have redundancy approximately  $(t+1)\log_2 n + (t+1/2)\log_2\log_2 n$ . Hence, we reduce the redundancy by  $(t+1/2)\log_2\log_2 n$  bits.

Finally, we consider the encoding complexity for our construction. Given a vector  $\mathbf{u}$ , we can find in linear time the index i such that  $\operatorname{wt}(\phi(\sigma^i(\mathbf{u}))) \in \{(n-1)/2, (n+1)/2\}$ . Thus, it remains to provide an efficient method to enumerate a set of representatives for the cyclic classes. This problem was solved completely by Tavares  $et\ al.\ [7],\ [16]$  and the solution uses the polynomial representation of cyclic codewords. Furthermore, the encoding method can be adapted for Constructions E and F in the later sections. Hence, we review Tavares' method in detail and discuss our modifications in Section VI.

## B. GC-Balanced Error-Correcting Codes

A direct application of the coupling construction in Lemma 6 and Corollary 9 yields a family of GC-balanced  $(n,d)_4$ -codes with redundancy at most  $d\log_4 n$ . However, this construction requires cyclic codes of length n-1.

The following construction removes the need for cyclic codes.

#### Construction B.

INPUT: An  $[n+p,n,d]_2$ -linear code  $\mathcal{A}$  and an  $(n,d)_2$ -code  $\mathcal{B}$  of size  $2^p nM$ .

OUTPUT: A balanced  $(n, d)_4$ -code  $\mathcal{C}$  code of size  $2^n M$ .

- Given  $m \in \mathbb{F}_2^n$ , let  $j_m$  be the balancing index of m and  $a_m$  be the corresponding balanced word of length n.
- Consider a systematic encoder for  $\mathcal{A}$ . For  $a_m \in \mathbb{F}_2^n$ , let  $a_m p_m$  be the corresponding codeword in  $\mathcal{A}$ .
- Finally, since  $\mathcal{B}$  is of size  $2^p nM$ , we may assume without loss of generality an encoder  $\phi_{\mathcal{B}} : [M] \times [n] \times \mathbb{F}_2^p \to \mathcal{B}$ . We set  $b_m = \phi(i, j_m, p_m)$ .
- Set  $\mathcal{C} \triangleq \{ \Psi(a_m, b_m) : m \in \mathbb{F}_2^n, i \in [M] \}.$

**Theorem 10.** Construction B is correct. In other words,  $\mathbb{C}$  is a GC-balanced  $(n, d)_4$ -code of size at least  $2^n M$ .

*Proof.* The size of  $\mathcal{C}$  follows from its definition.

For all words  $c = \Psi(a, b)$  in  $\mathbb{C}$ , since a is balanced, we have that c is GC-balanced. Hence,  $\mathbb{C}$  is GC-balanced.

Finally, to prove that  $\mathcal C$  has distance d, we show that  $\mathcal C$  can always correct  $t=\lfloor (d-1)/2 \rfloor$  errors. Specifically, let  $c\in \mathcal C$ 

and let  $\hat{c}$  be a word over  $\Sigma$  such that  $d(c,\hat{c}) \leq t$ . Suppose that  $c = \Psi(a,b)$  and  $\hat{c} = \Psi(\hat{a},\hat{b})$ . Then  $d(a,\hat{a}) \leq t$  and  $d(b,\hat{b}) \leq t$ . Since b belongs to  $\mathcal{B}$  an  $(n,d)_2$ -code, we correct the errors in  $\hat{b}$  to recover b.

Suppose that  $b = \phi(i, j, p)$ . Then we have that ap is a codeword in  $\mathcal{A}$ . Since  $\mathcal{A}$  an  $[n + p, n, d]_2$ -code, we correct the errors in  $\hat{a}p$  to recover ap and hence, recover a. Therefore,  $\mathcal{C}$  is an  $(n, d)_2$ -code.

**Corollary 11.** Fix d and set  $t = \lceil (d-1)/2 \rceil$ . There exists an GC-balanced  $(n,d)_4$ -code with redundancy at most  $(2t+1)\lceil \log_4 n \rceil + 2t$  symbols for sufficiently large n.

*Proof.* For sufficiently large n, we choose an  $[n+p,n,d]_2$ -and an  $[n,k,d]_2$ -linear code so that  $p \leq t \lceil log_2 n \rceil + t$  and  $n-k \leq t \lceil log_2 n \rceil + t$ . Then applying Construction B, we obtain a GC-balanced code with at most  $(2t+1)\lceil \log_4 n \rceil + 2t$  redundant symbols.

#### IV. PRIMER CODES

In this section, we provide three constructions of primer codes: one direct modification of Yazdi *et al.* that yields primer codes for general parameters and the other two that rely on cyclic codes and have lower redundancy for a specific set of parameters.

A.  $\kappa$ -Mutually Uncorrelated Codes that Avoid Primer Dimer Byproducts of Length f

Yazdi et al. [3] constructed a set of mutually uncorrelated primers that avoids primer dimer byproducts.

**Definition 12.** A code  $A \subseteq \mathbb{F}_2^n$  is  $\ell$ -APD-constrained if for each  $a \in A$ ,

- a ends with one,
- a contains  $01^{\ell}0$  as a substring exactly once,
- a does not contain  $0^{\ell}$  as a substring.

**Lemma 13** (Yazdi et al. [3, Lemma 5]). Let  $n, f, \ell, r$  be positive integers such that  $n = rf + \ell + 1$  and  $\ell + 3 \le f$ . Suppose that A is an  $\ell$ -APD-constrained code of length f. Then the code

$$C = \{0^{\ell} 1 a_1 a_2 \dots a_r : a \in A^r\}$$

is both MU and (2f)-APD and its size is  $|\mathcal{A}|^r$ .

The following construction equips the primer code in Lemma 13 with error-correcting capabilities.

**Construction C.** Let f, r, d and  $\ell$  be positive integers where  $\ell + 3 \le f$  and  $p + \lfloor p/(\ell - 1) \rfloor + 1 \le f$ .

INPUT: An  $[rf + p, rf, d]_2$ -linear code  $\mathcal{B}$  and an  $\ell$ -APD-constrained code  $\mathcal{A}$  of length f.

OUTPUT: An (n, d; 1, 2f)-primer code  $\mathcal C$  of length  $n = rf + p + \lfloor p/(\ell-1) \rfloor + \ell + 2$  and size  $|\mathcal A|^r$ .

- Consider a systematic encoder for B.
- For every message  $a \in \mathbb{F}_2^{rf}$ , let  $ap_a$  be the corresponding codeword in  $\mathfrak{B}$ .
- For the vector  $p_a$ , we insert a one after every  $(\ell-1)$  bits and append a one. In other words, we insert  $\lfloor p/(\ell-1) \rfloor + 1$  ones and we call the resulting vector  $p'_a$ .
- Set  $\mathcal{C} \triangleq \{01^{\ell} a p_a^{\prime} : a \in \mathcal{A}^r\}.$

Next, for fixed values of r and d, we describe a family of  $(n, d; 1, f)_2$ -primer codes with n = rf + o(f) and redundancy

at most  $t \log_2 n + O(1)$ , where  $t = \lceil (d-1)/2 \rceil$ . Specifically, we provide the constructions for the input codes  $\mathcal B$  and  $\mathcal A$  in Construction C.

**Lemma 14.** For  $\ell \geq 8$ , set  $f = 2^{\ell-4}$ . Then there exists an  $\ell$ -APD-constrained code  $\mathcal A$  of size  $(f-\ell-2)2^{f-\ell-4}$ . Furthermore, there is a linear-time encoding algorithm that maps  $[f-\ell-2] \times \mathbb F_2^{f-\ell-4}$  to  $\mathcal A$ .

*Proof.* We first construct a code  $\mathcal{A}_0$  of length  $f-\ell-3$ , where all codewords do not contain either  $0^{\ell-1}$  or  $1^{\ell-1}$  as substrings. Then  $\mathcal{A}$  can be constructed by inserting  $01^{\ell}0$  to the codewords in  $\mathcal{A}_0$  and appending a symbol 1. Since there are  $f-\ell-2$  possible positions to insert  $01^{\ell}0$ , we have  $|\mathcal{A}|=(f-\ell-2)|\mathcal{A}_0|$ .

To construct the code  $\mathcal{A}_0$ , we use the encoding algorithm  $\phi$  proposed by Schoeny *et al.* [17] that maps a binary sequence of length  $(f - \ell - 4)$  to a binary sequence of length  $(f - \ell - 3)$  that avoids  $0^{\ell-1}$  and  $1^{\ell-1}$  as substrings. Furthermore, the encoding map  $\phi$  has running time O(f).

Hence, for  $\ell \geq 8$ , we choose  $f=2^{\ell-4}$ . For the input code  $\mathcal{B}$ , we shorten an appropriate BCH code given in Theorem 1 to obtain an [rf+p,rf,d]-linear code with redundancy  $p\leq t\log_2 n+t$ . Hence, applying Construction C, we obtain a primer code with  $(n,d;1,f)_2$ -primer codes with  $n=rf+p+\lfloor p/(\ell-1)\rfloor+\ell+2$ .

Observe that for sufficiently large  $\ell$ , we have that rf < n < (r+1)f. By choice of  $\ell$ , we have that  $\log_2 n + C_1 \le \ell \le \log_2 n + C_2$  for some constants  $C_1$ ,  $C_2$  dependent only on r.

To analyse the redundancy of the construction, we have that

$$\log_2 |\mathcal{A}|^r = r(f - \ell - 4) + r \log_2(f - \ell - 2)$$
  
 
$$\geq r(f - \ell - 4) + r \log_2(f/2)$$
  
=  $r(f - \ell - 4) + r(\ell - 5) = rf - 9r$ .

Therefore, the redundancy is given by  $n - \log_2 |\mathcal{A}|^r$ , which is at most

$$p + \lfloor p/(\ell - 1) \rfloor + \ell + 2 + 9r$$

$$\leq (t \log_2 n + t) + \frac{t \log_2 n + t}{\log_2 n + C_1 - 1} + (\log_2 n + C_2) + 2 + 9r$$

$$= (t + 1) \log_2 n + O(1).$$

In summary, we have the following theorem.

**Theorem 15.** Fix r and d. Then there exists a family of  $(n,d;1,f)_2$ -primer codes with n=rf+o(f) and redundancy at most  $(t+1)\log_2 n + O(1)$ , where  $t=\lceil (d-1)/2 \rceil$ . Furthermore, there exists a linear-time encoding algorithm for these primer codes.

Applying Lemma 6, we obtain primer codes over {A, T, C, G}.

**Corollary 16.** Fix r and d, and set  $t = \lceil (d-1)/2 \rceil$ .

- (i) There exists a family of  $(n, d; 1, f)_4$ -primer codes with n = rf + o(f) and redundancy at most  $(2t + 1) \log_4 n + O(1)$ .
- (ii) There exists a family of balanced  $(n, d; 1, f)_4$ -primer codes with n = rf + o(f) and redundancy at most  $(d+1)\log_4 n + O(1)$ .

#### B. Almost GC-Balanced κ-Mutually Uncorrelated Only

Using cyclic codes and modifying Construction A, we obtain *almost balanced* primer codes that satisfy conditions (P1) and (P2) only. Here, a code is *almost balanced* if the

weight (or GC-content) of every word belongs to  $\{\lfloor n/2 \rfloor - 1, \lfloor n/2 \rfloor, \lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ .

Let n be odd and we abuse notation by using  $\phi$  to also denote the map  $\phi: \mathbb{F}_4^n \to \mathbb{F}_4^n$  where  $\phi(a) = a + \omega^{(n+1)/2} 0^{(n-1/2)}$ . In other words,  $\phi$  switches A with C and T with G, and vice versa, in the first (n+1)/2 coordinates of a. We have the following analogue of Lemma 7.

**Lemma 17.** For  $\mathbf{a} \in \mathbb{F}_4^n$ , we can find  $i \in [n]$  such that  $\phi(\mathbf{\sigma}^i(\mathbf{a}))$  is GC-balanced.

**Construction D.** Let n be odd,  $k \leq \lceil (n+1)/4 \rceil$  and  $q \in \{2,4\}$  INPUT: An  $[n,k,d]_q$ -cyclic code  $\mathcal B$  containing  $1^n$ .

OUTPUT: An almost balanced  $(n,d;k+1,n)_q$ -primer code  $\mathcal{B}$  of size at least  $q^k/n$ .

- Let  $u_1, u_2, \ldots, u_m$  be the set of representatives  $\mathcal{C}/\underset{\text{cyc}}{\sim}$ .
- For each  $u_i$ , find  $j_i \in [n]$  such that  $\phi(\sigma^{j_i}(u_i))$  is either balanced or GC-balanced.
- Let  $\mu = (n-1)/2$ . For each  $u_i$ , set

$$v_i = \begin{cases} \sigma^{j_i}(u_i) + 1^{\mu+1}0^{\mu-1}1, & \text{if } q = 2, \\ \sigma^{j_i}(u_i) + \omega^{\mu+1}0^{\mu-1}\omega, & \text{if } q = 4. \end{cases}$$

• Set  $C = \{v_i : 1 \le i \le m\}$ .

**Theorem 18.** Construction D is correct. In other words,  $\mathfrak{C}$  is an almost balanced  $(n,d;k+1,n)_q$ -primer code of size at least  $q^k/n$ .

To prove Theorem 18, we require the following technical lemma modified from Yazdi et al. [3].

**Lemma 19.** Let C be a cyclic code of dimension k containing  $1^n$ . Then the run of any symbols in any non-constant codeword is at most k-1.

Proof of Theorem 18. Since  $\mathcal C$  is coset of  $\mathcal B$ , we have that  $\mathcal C$  is an  $(n,d)_q$ -code. For  $i\in [m]$ , since  $\phi(\pmb\sigma^{j_i}(\pmb u_i))$  is balanced and  $\pmb v_i$  differs from the former in one symbol, we have that  $\pmb v_i$  is almost balanced.

Now, we demonstrated weakly mutually uncorrelatedness for the case of q=2. The case of q=4 can be proceeded in the same way. Suppose on the contrary that  $\mathcal C$  is not k-WMU. Then there is a proper prefix p of length  $\ell, \ \ell \geq k+1$  such that both pa and bp belong to  $\mathcal C$ . In other words,  $\mathcal B$  contains the words

$$pa + 1^{\mu+1}0^{\mu-1}1$$
 and  $bp + 1^{\mu+1}0^{\mu-1}1$ ,

where  $\mu=(n-1)/2$ . Consequently, since  $\mathcal{B}$  is cyclic, we have that  $pb+\sigma^{\ell}(1^{\mu+1}0^{\mu-1}1)$  belongs to  $\mathcal{B}$ . Hence, by linearity of  $\mathcal{B}$ , the word

$$c \triangleq 0^{\ell}(a - b) + 1^{\mu + 1}0^{\mu - 1}1 + \sigma^{\ell}(1^{\mu + 1}0^{\mu - 1}1)$$

belongs to  $\mathcal{B}$ . We look at prefix of length  $\ell$  of c.

- When  $\ell \leq \mu$ , the word c has prefix  $1^{\ell-1}0$ . Hence, c is a non-constant codeword of  $\mathcal C$  and since  $\ell-1\geq k$ , this contradicts Lemma 19.
- When  $\ell = \mu + 1$ , the word c has prefix  $01^{\mu-1}$ . Hence, c is a non-constant codeword of  $\mathfrak C$  and since  $\mu 1 \ge k$ , this contradicts Lemma 19.
- When  $\ell \geq \mu + 2$ , the word c has prefix  $0^{\ell-\mu}1^{2\mu+1-\ell}$ . Since either  $\ell - \mu$  or  $2\mu + 1 - \ell$  is at least  $\lceil \mu + 1/2 \rceil =$

 $\lceil (n+1)/4 \rceil \ge k$ , the word c contains a run of ones or zeros of length k, contradicting Lemma 19.

C.  $\kappa$ -Mutually Uncorrelated Codes that Avoid Primer Dimer Byproducts of Length  $\kappa$ 

Using reversible cyclic codes, we further reduce the redundancy for primer codes in the case when  $\kappa = f$ .

**Definition 20.** Let g(X) be the generator polynomial of a reversible cyclic code  $\mathcal{B}$  of length n and dimension k that contains  $1^n$ . Set  $h(X) = (X^n - 1)/g(X)$ . The set  $\{h^*(X), p_1(X), p_2(X), \dots, p_P(X)\}$  of polynomials is (g, k)-regenerating if the following hold:

- (R1)  $h^*(X)$  divides h(X);
- (R2)  $h^*(1) \neq 0$ ;
- (R3)  $h^*(X) = X^{d^*}h^*(X^{-1})/h^*(0)$ , where  $d^* = \deg h^*$ ;
- (R4)  $h^*(X)$  does not divide  $X^s p_i(X) p_j(X)$  for all  $i, j \in [P]$  and  $s \in [n-1]$ .
- (R5)  $h^*(X)$  does not divide  $X^s p_i(X) X^{k-1} p_j(X^{-1})$  for all  $i, j \in [P]$  and  $0 \le s \le n-k$ .
- (R6)  $h^*(X)$  does not divide  $X^{s+k-1}p_i(X^{-1})-p_j(X)$  for all  $i,j\in [P]$  and  $0\leq s\leq n-k$ .
- (R7)  $\deg p_i(X) < \deg h^* \text{ for } i \in [P].$

#### Construction E.

INPUT: An  $[n,k,d]_q$ -reversible cyclic code  ${\mathbb B}$  containing  $1^n$  with generator polynomial g(X) and a (g,k)-rc-generating set of polynomials  $\{h^*(X),p_1(X),p_2(X),\ldots,p_P(X)\}$ . OUTPUT: An  $(n,d;k,k)_q$ -primer code  ${\mathbb C}$  of size  $q^{k^*}P$ , where  $k^*=k-\deg h^*$ .

• Set

$$\mathfrak{C} \triangleq \{ (m(X)h^*(X) + p_i(X))g(X) : \deg m < k^*, i \in [P] \}.$$

**Theorem 21.** Construction E is correct. In other words,  $\mathbb{C}$  is an  $(n, d; k, k)_q$ -primer code.

We illustrate Construction E via an example.

**Example 22.** Set n=15 and q=4. Let  $g(x)=x^6+x^5+(\omega+1)x^4+x^3+(\omega+1)x^2+x+1$  be the generator polynomial of an  $[15,9,5]_4$ -reversible cyclic code that contains  $1^n$ . Consider  $h^*(X)=X^4+\omega X^3+\omega X^2+\omega X+1$  and

$$\begin{array}{lll} p_1 = \omega, & p_{10} = \omega x^3 + (\omega + 1)x^2 + x + \omega + 1, \\ p_2 = \omega + 1, & p_{11} = \omega x^3 + (\omega + 1)x^2 + x + 1, \\ p_3 = 1, & p_{12} = \omega x^3 + x^2, \\ p_4 = \omega x + \omega, & p_{13} = \omega x^3 + x^2 + x + 1, \\ p_5 = (\omega + 1)x + \omega + 1, & p_{14} = (\omega + 1)x^3 + \omega x^2 + \omega x + \omega, \\ p_6 = x + 1, & p_{15} = (\omega + 1)x^3 + \omega x^2 + (\omega + 1)x + \omega + 1, \\ p_7 = \omega x^2 + \omega x + \omega + 1, & p_{16} = (\omega + 1)x^3 + x^2 + \omega x + 1, \\ p_8 = \omega x^3 + (\omega + 1)x^2 + 1, & p_{17} = x^3 + \omega x^2 + (\omega + 1)x + \omega + 1. \\ p_9 = \omega x^3 + (\omega + 1)x^2 + x, & p_{17} = x^3 + \omega x^2 + (\omega + 1)x + \omega + 1. \end{array}$$

We can verify that the set  $\{h^*(X), p_1(X), \ldots, p_{17}(X)\}$  is (g,9)-rc-generating. Therefore,  $k^*=15-6-4=5$  and the size of the  $(15,5;9,9)_4$ -primer code have size  $17(4^5)\geq 2^{14}$ .

In contrast, for their experiment, Yazdi *et al.* constructed a set of weakly mutually uncorrelated primers of length 16, distance four and size four. Specifically, they set  $\mathcal{C}_1 = \{01^701^7, 10^710^7\}$  and  $\mathcal{C}_2$  to be an extended BCH [16,11,4]-cyclic code. Then they applied the coupling construction to obtain an (16,4;9,16)-primer code of size  $2^{12}$ .

Therefore, Construction E provides a larger set of primers using less bases, while improving the minimum distance and avoiding primer dimer products at the same time.

We outline our steps in establishing Theorem 21. First, we demonstrate Lemma 23. The lemma provides certain *combinatorial* sufficiency conditions for a subcode of a reversible cyclic code to be a primer code. Next, using the *algebraic* properties of the polynomials in Construction E, we then show that  $\mathcal{C}$  satisfy the combinatorial conditions in Lemma 23. The second step is deferred to Section VI.

**Lemma 23.** Let  $\mathcal{B}$  be an  $(n, k, d)_q$  reversible cyclic code containing  $1^n$ . Let  $\mathcal{C} \subseteq \mathcal{B}$  be a subcode such that for any two codewords u, v in  $\mathcal{C}$ , not necessarily distinct, the following holds.

(S1)  $\sigma^i(\mathbf{u}) \neq \mathbf{v}$  for  $k \leq i < n$ ;

(S2)  $\sigma^{i}(\mathbf{u}) \neq \overline{\mathbf{v}} \text{ for } 0 \leq i \leq n-k;$ 

(S3) 
$$\sigma^i(\mathbf{u}) \neq \mathbf{v}^{rc}$$
 and  $\sigma^i(\mathbf{u}^{rc}) \neq \mathbf{v}$  for  $0 \leq i \leq n - k$ .

Then C is an  $(n, d; k, k)_q$ -primer code.

*Proof.* Since  $\mathcal C$  is a subcode of  $\mathcal B$ , we have that  $\mathcal C$  is an  $(n,d)_q$ -code. It remains to show the WMU and APD properties.

We first show that  $\mathcal C$  is k-WMU. Suppose to the contrary that there is a proper sequence p of length  $\ell$ , where  $k \leq \ell < n$ , such that both pa and bp belong to  $\mathcal C$ . Since  $\mathcal C \subseteq \mathcal B$  and  $\mathcal B$  is a cyclic code, the word  $pa-pb=0^\ell(a-b)$  belongs to  $\mathcal B$ . Since  $\ell \geq k$ , Lemma 19 implies that a=b and so,

$$\sigma^{\ell}(pa)=ap=bp.$$

Since  $k \leq \ell < n$  and pa and bp belong to C, we obtain a contradiction for condition (S1).

Now we show that  $\mathcal C$  is a k-APD code. Towards a contradiction, we suppose that there is a proper sequence p of length  $\ell$ , where  $k \leq \ell < n$ , such that both  $a_1pb_1$  and  $a_2\overline{p}b_2$  belong to  $\mathcal C$ . Since  $\mathcal C \subseteq \mathcal B$  and  $\mathcal B$  is a cyclic code containing  $1^n$ , the word  $pb_1a_1 - p\overline{b_2}\overline{a_2} = \mathbf 0(b_1a_1 - \overline{b_2}\overline{a_2})$  also belongs to  $\mathcal B$ . It follows from Lemma 19 that

$$pb_1a_1=p\overline{b_2}\overline{a_2}.$$

Without loss of generality, we assume that  $|a_2| \ge |a_1|$ . Then

$$\sigma^{\ell'}(a_1pb_1) = \overline{a_2}p\overline{b_2} = \overline{a_2}\overline{p}b_2,$$

where  $\ell' = |a_2| - |a_1|$ . Since the length of p is no less than k, we have that  $\ell' \le n - k$ , which contradicts condition (S2).

Finally, suppose that  $a_1pb_1$  and  $a_2p^{rc}b_2$  belong to  $\mathbb{C}$ , where p is a proper sequence of length  $\ell$  and  $k \leq \ell < n$ . Proceeding as before, we can show that

$$pb_1a_1=pa_2^{rc}b_2^{rc},$$

or equivalently,

$$b_1a_1 = a_2^{rc}b_2^{rc}$$
 and  $a_1^{rc}b_1^{rc} = b_2a_2$ .

If  $|\boldsymbol{b}_2| > |\boldsymbol{a}_1|$ , we have that

$$\sigma^{\ell'}(a_1pb_1) = b_2^{rc}pa_2^{rc} = (a_2p^{rc}b_2)^{rc},$$

where  $\ell' = |\boldsymbol{b}_2| - |\boldsymbol{a}_1| \le n - k$ , contradicting the first inequality of Condition (S3); if  $|\boldsymbol{b}_2| < |\boldsymbol{a}_1|$ , then  $|\boldsymbol{a}_2| > |\boldsymbol{b}_1|$  and we have

$$oldsymbol{\sigma}^{\ell'}\left((oldsymbol{a}_1oldsymbol{p}oldsymbol{b}_1)^{rc}
ight) = oldsymbol{\sigma}^{\ell'}\left(oldsymbol{b}_1^{rc}oldsymbol{p}^{rc}oldsymbol{a}_1^{rc}
ight) = oldsymbol{a}_2oldsymbol{p}^{rc}oldsymbol{b}_2,$$

where  $\ell' = |a_2| - |b_1| \le n - k$ , contradicting the second inequality of Condition (S3).

Finally, applying Construction E to the class of reversible cyclic codes in Theorem 2, we obtain a family of primer codes that has efficient encoding algorithms. The detailed proof is deferred to Section VI.

**Corollary 24.** Let  $m \ge 6$  and  $1 \le \tau \le \lceil m/2 \rceil$  Set  $n = 4^m - 1$  and  $d = 4^{\tau} - 1$ . There exists an  $(n, d; k, k)_4$ -primer code of size  $4^{k-2m}$ , where

$$k = \begin{cases} n - (d - 3)m, & \text{if } m \text{ is odd and } \tau = \frac{m+1}{2}; \\ n - (d - 1)m, & \text{otherwise.} \end{cases}$$

Therefore, there is a family of  $(n,d;k,k)_4$ -primer codes with  $d \approx \sqrt{n}$ ,  $k \approx n - \sqrt{n} \log_4 n$ , and redundancy at most  $(d+1) \log_4 (n+1)$ .

#### V. CODES FOR DNA COMPUTING

Since Adleman demonstrated the use of DNA hybridization to solve a specific instance of the directed Hamiltonian path problem [18], the coding community have investigated the possibility of error control via code design [19], [20]. In this paper, we focus on designing codes with the following constraints.

**Definition 25.** A GC-balanced  $(n, d)_4$ -code is a balanced (n, d)-DNA computing code if the following hold.

- (C1)  $d(\mathbf{a}, \mathbf{b}^r) \geq d$  for all  $\mathbf{a}, \mathbf{b} \in \mathcal{C}$ .
- (C2)  $d(\mathbf{a}, \mathbf{b}^{rc}) \geq d$  for all  $\mathbf{a}, \mathbf{b} \in \mathcal{C}$ .

More generally, DNA computing codes require that the GC-content, the number of symbols that correspond to either G or C, of all codewords to be the same or approximately the same. As always, the fundamental problem for DNA computing codes is to find the largest possible codes satisfying the constraints above. Many approaches have been considered for this problem. These include search algorithms, template-based constructions and constructions over certain algebraic rings (see, Limbachiya et al. [14] for a survey).

There are few explicit families of DNA computing codes satisfying all constraints for large n. In this section we propose a class of balanced DNA computing codes that satisfies both the constraints (C1) and (C2).

We modify our balancing techniques in Sections III and IV. Recall that by flipping, we mean exchanging A with C and T with G. Then Lemma 17 states that for  $a \in \mathbb{F}_4^n$ , we can balance one of its cyclic shifts by flipping its first  $\lceil n/2 \rceil$  components. However, in order to accommodate the reverse and reverse-complement distance constraints, we do the following.

Let n be odd and set s be the integer nearest to n/4. In other words, s is the unique integer in the set  $\{(n-1)/4,(n+1)/4\}$ . Let  $\pi: \mathbb{F}_4^n \to \mathbb{F}_4^n$  be the map such that  $\pi(a) = a + \omega^s 0^{n-2s} \omega^s$  for any  $a \in \mathbb{F}_4^n$ . In other words,  $\pi$  flips the first s and the last s symbols of a. The following lemma follows directly from Lemma 17.

**Lemma 26.** Let n be odd. For any  $\mathbf{a} \in \mathbb{F}_4^n$ , there exists  $i \in [n]$  such that  $\pi(\sigma^i(\mathbf{a}))$  is GC-balanced.

As before, we next define a set of polynomials that enables us to generate our code efficiently.

**Definition 27.** Let g(X) be the generator polynomial of a reversible cyclic code  $\mathcal{B}$  of length n and dimension k that contains  $1^n$ . Set  $h(X) = (X^n - 1)/g(X)$ . The set  $\{h^*(X), p_1(X), p_2(X), \ldots, p_P(X)\}$  of polynomials is (g, k)-rc2-generating if the set obeys conditions (R1) to (R4), (R7) in Definition 20 and

(R5') 
$$h^*(X)$$
 does not divide  $X^s p_i(X) - X^{k-1} p_j(X^{-1})$  for all  $i, j \in [P]$  and  $s \in [n]$ .

It is immediate from definition that an (g, k)-rc-generating set is also an (g, k)-rc2-generating set.

Construction F. Let n be odd.

INPUT: An  $[n,k,d]_q$ -reversible cyclic code  ${\mathfrak B}$  containing  $1^n$  with generator polynomial g(X) and a (g,k)-rc2-generating set of polynomials  $\{h^*(X),p_1(X),p_2(X),\ldots,p_P(X)\}$ . OUTPUT: A balanced  $(n,d)_4$ -DNA computing code  ${\mathfrak C}$  of size  $4^{k^*}P$ , where  $k^*=k-\deg h^*$ .

• Set

$$A \triangleq \{(m(X)h^*(X) + p_i(X))g(X) : \deg m < k^*, i \in [P]\}.$$

- For  $u \in \mathcal{A}$ , find  $i_u \in [n-1]$  such that  $v_u = \pi(\sigma^{i_u}(u))$  is GC-balanced.
- Set  $\mathcal{C} = \{v_u : u \in \mathcal{A}\}.$

**Theorem 28.** Construction F is correct. In other words, C is a balanced  $(n, d)_4$ -DNA computing code of size  $4^{k^*}$ .

As in Section IV, to provide Theorem 28, we first provide certain combinatorial sufficiency conditions for a subcode of a reversible cyclic code to be a DNA computing code, and then show that  $\mathcal C$  satisfy these combinatorial conditions. As before, we defer the second step to Section VI.

**Lemma 29.** Let  $\mathcal{B}$  be an  $(n, k, d)_q$  reversible cyclic code containing  $1^n$ . Let  $\mathcal{A} \subseteq \mathcal{B}$  be a subcode such that for any two codewords u, v in  $\mathcal{A}$ , not necessarily distinct, the following holds.

- (S1')  $\sigma^{i}(u) \neq v \text{ for } i \in [n-1];$
- (S2')  $\sigma^i(\mathbf{u}) \neq \mathbf{v}^r$  for  $i \in [n]$ ;
- (S3')  $\sigma^i(\mathbf{u}) \neq \mathbf{v}^{rc}$  for  $i \in [n]$ .

If we define  $\mathbb{C}$  as in Construction F, then  $\mathbb{C}$  is a balanced  $(n,d)_4$ -DNA computing code of size  $|\mathcal{A}|$ .

*Proof.* First, condition (S1') ensures that the codewords  $v_u$  and  $v_u'$  are distinct whenever  $u \neq u'$ . Therefore, the size of  $\mathcal{C}$  is given by  $|\mathcal{A}|$ .

Next, by choice of  $i_u$ , we have that all codewords in  $\mathcal{C}$  are GC-balanced. Since  $\mathcal{C}$  belongs to a coset of  $\mathcal{B}$ , we have that  $\mathcal{C}$  is an  $(n,d)_4$ -code.

Therefore, it remains to demonstrate constraints (C1) and (C2). For any  $a, b \in \mathcal{C}$ , let u, v be the corresponding vectors in  $\mathcal{A}$ . In other words,

$$a = \pi(\sigma^{i_u}(u))$$
 and  $b = \pi(\sigma^{i_v}(v))$ .

We first show that  $d(\boldsymbol{a}, \boldsymbol{b}^r) \geq d$ . Since  $\mathcal{A}$  satisfies condition (S2'), we have that  $\boldsymbol{\sigma}^{i_u}(\boldsymbol{u}) \neq \boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r$ . Since  $\boldsymbol{\sigma}^{i_u}(\boldsymbol{u})$ ,  $\boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r$  belongs to  $\mathcal{B}$ , we have that  $d(\boldsymbol{\sigma}^{i_u}(\boldsymbol{u}), \boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r) \geq d$ . Now,  $\omega^s 0^{n-2s} \omega^s = (\omega^s 0^{n-2s} \omega^s)^r$ , and so,  $\boldsymbol{b}^r = \pi(\boldsymbol{\sigma}^{i_v}(\boldsymbol{v}))^r = \pi(\boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r)$ . Therefore,

$$d(\boldsymbol{a}, \boldsymbol{b}^r) = d(\pi(\boldsymbol{\sigma}^{i_u}(\boldsymbol{u})), \pi(\boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r)) = d((\boldsymbol{\sigma}^{i_u}(\boldsymbol{u}), \boldsymbol{\sigma}^{i_v}(\boldsymbol{v})^r)) \ge d.$$

Constraint (C2) can be similarly demonstrated.

As before, we apply Construction F to the reversible cyclic codes in Theorem 2 to obtain a family of balanced DNA computing codes. The proof is deferred to Section VI.

**Corollary 30.** Let  $m \ge 6$  and  $1 \le \tau \le \lceil m/2 \rceil$  Set  $n = 4^m - 1$ ,  $d = 4^{\tau} - 1$ , and

$$k = \begin{cases} n - (d - 3)m, & \text{if } m \text{ is odd and } \tau = \frac{m+1}{2}; \\ n - (d - 1)m, & \text{otherwise.} \end{cases}$$

Then there exists a GC-balanced  $(n,d)_4$ -DNA computing code of size at least  $4^{k-2m}$ . Therefore, there exists a family of GC-balanced  $(n,d)_4$ -primer codes with  $d \approx \sqrt{n}$  and redundancy at most  $(d+1)\log_4(n+1)$ . Furthermore, these codes have efficient encoding algorithms.

#### VI. EFFICIENT ENCODING INTO CYCLIC CLASSES

In this section, unless stated otherwise, all words are of length n and we index them using [n]. Recall that a word  $c \in \mathbb{F}_q^n$  is identified with the polynomial  $c(X) = \sum_{i=0}^{n-1} c_i X^i$ . We further set  $X^n = 1$  and hence, all polynomials reside in the quotient ring  $\mathbb{F}_q[X]/\langle X^n - 1 \rangle$ .

Hence, in this quotient ring, we have the following properties. Let  $c(X) \in \mathbb{F}_q[X]/\langle X^n-1 \rangle$  be the polynomial corresponding to the word c.

- For  $s \in [n]$ , the polynomial  $X^s c(X)$  corresponds to the word  $\sigma^i(c)$ .
- $X^{n-1}c(X^{-1})$  corresponds to the word  $c^r$ . Given c(X), we further define the *reciprocal polynomial* of c(X) to be  $c^{\dagger}(X) = X^{\deg c}c(X^{-1})$  and we say c(X) is self-reciprocal if  $c(0) \neq 0$  and  $c(X) = c^{\dagger}(X)/c(0)$ .
- $(X^n-1)/(X-1)$  corresponds to  $1^n$ , and so,  $c(X)+(X^n-1)/(X-1)$  corresponds to  $\overline{c}$ .
- $X^{n-1}c(X^{-1}) + (X^n 1)/(X 1)$  corresponds to  $c^{rc}$ .

From these observations, we can then easily characterise when a cyclic code contains  $1^n$  or when a cyclic code is reversible.

**Proposition 31.** Let C be a cyclic code with generator polynomial g(X). Then

- (i)  $\mathbb{C}$  contains  $1^n$  if and only if (X-1) does not divide g(X), i.e.  $g(1) \neq 0$ .
- (ii)  $\mathbb{C}$  is reversible if and only if q(X) is self-reciprocal.

Next, we review the method of Tavares *et al.* that efficiently encodes into distinct cyclic classes. We restate a special case of their method and reproduce the proof here as the proof is instructive for the subsequent encoding methods.

**Theorem 32** (Tavares *et al.* [7]). Let  $\mathbb B$  be a cyclic code of dimension k with generator polynomial g(X) and define  $h(x) = (X^n - 1)/g(X)$ . Suppose  $h^*(X)$  divides h(X) and h(X) does not divide  $X^s - 1$  for  $s \in [n-1]$ . Set  $k^* = k - \deg h^*(X)$ 

$$\mathcal{B}^* = \{ (m(X)h^*(X) + 1)g(X) : \deg m < k^* \}.$$

Then  $\mathfrak{B}^* \subseteq \mathfrak{B}/\underset{cyc}{\sim}$ .

*Proof.* It suffices to show for distinct polynomials m(X) and m'(X) with  $\deg m, \deg m' < k^*$  and  $s \in [n-1]$ , we have that

$$X^{s}(m(X)h^{*}(X)+1)g(X) \neq (m'(X)h^{*}(X)+1)g(X) \pmod{X^{n}-1}.$$

To do so, we prove by contradiction and suppose that equality holds. In other words, there exists a polynomial f(X) such that

$$X^{s}(m(X)h^{*}(X)+1)g(X) = (m'(X)h^{*}(X)+1)g(X)+f(X)(X^{n}-1).$$

Dividing throughout by g(X) and rearranging the terms, we have that

$$(X^{s} - 1) + (X^{s}m(X) - m'(X))h^{*}(X) = f(X)h(X).$$

Since  $h^*(X)$  divides h(X), then  $h^*(X)$  must divide  $X^s - 1$ , yielding a contradiction.

Suppose  $n=2^m-1$  in Theorem 32. It is not difficult to show that the degree of  $h^*(X)$  is m. Thus, Theorem 32 encodes into  $2^{k-m}=2^k/(n+1)$  cyclic classes. Since the size of the cyclic code is  $2^k$  and each class contains at most n words, the theorem in fact encodes most cyclic classes.

The method of Tavares *et al.* [7] encodes more classes by considering more factors of h(X) that satisfy the conditions of the theorem. In some special cases, like n is a prime, this iterative process can encode all the cyclic classes.

# A. Detailed Proofs for Section IV

Borrowing ideas from Tavares *et al.*, we complete the proof of Theorem 21. Specifically, we demonstrate the following lemma.

**Lemma 33.** Let g(X) be the generator polynomial of a reversible cyclic code  $\mathbb B$  of length n and dimension k that contains  $1^n$ . If  $\{h(X), p_1(X), p_2(X), \ldots, p_P(X)\}$  is (g, k)-rc-generating and  $k^* = k - \deg h^*$ , then the subcode  $\mathbb C = \{(m(X)h^*(X) + p_i(X))g(X) : \deg m < k^*, i \in [P]\}$  satisfies conditions (S1) to (S3) in Lemma 23.

*Proof.* Here we only prove condition (S3). The other two conditions can be proved similarly. In particular, we demonstrate that the violation of condition (S3) contradicts either condition (R5) or condition (R6) in Definition 20.

Suppose to the contrary of (S3). We first assume there are two codewords  $u, v \in \mathcal{C}$  such that  $\sigma^s(u) = v^{rc}$  for some  $s \in [n-k]$ . The other case can be treated similarly. Hence, there exist two polynomials m(X) and m'(X) with degrees strictly less than  $k^*$ , two polynomials  $p_i(X)$  and  $p_j(X)$  with  $i, j \in [P]$  such that the following equality holds with some polynomial f(X).

$$X^{s} (m(X)h^{*}(X) + p_{i}(X)) g(X)$$

$$= X^{n-1} (m'(X^{-1})h^{*}(X^{-1}) + p_{j}(X^{-1})) g(X^{-1})$$

$$+ \frac{X^{n} - 1}{X - 1} + f(X)(X^{n} - 1).$$

Since  $\mathcal{B}$  is a reversible code, g(x) is self-reciprocal, i.e.,  $X^{n-k}g(X^{-1})=g(X)$ . Similarly, we have  $h^*(X)=X^{\deg h^*}h^*(X^{-1})/h^*(0)$ . Dividing the equation by g(X) and rearranging the terms, we have the following equality.

$$(X^{s}p_{i}(X) - X^{k-1}p_{j}(X^{-1}))$$

$$+ (X^{s}m(X) - X^{k^{*}-1}m'(X^{-1})h(0))h^{*}(X)$$

$$= \frac{h(X)}{X-1} + f(X)h(X).$$

Since  $h^*(1) \neq 0$ , we have that  $h^*(X)$  divides h(X)/(X-1). Therefore,

$$h^*(X)$$
 divides  $X^s p_i(X) - X^{k-1} p_j(X^{-1})$ ,

Constr.	Input	Output	Redundancy for Infinite Family
A	binary cyclic code	balanced binary code	$(t+1)\log_2 n + 1$ , where $t = d/2 - 1$
			(c.f. Corollary 9)
В	two binary linear codes	GC-balanced code	$(2t+1)\log_4 n + 2t$ , where $t = \lceil (d-1)/2 \rceil$
			(c.f. Corollary 11)
С	binary linear code, and	primer code	$(2t+1)\log_4 n + O(1)$ , where $t = \lceil (d-1)/2 \rceil$
	$\ell$ -APD-constrained code		(no GC-balanced constraint)
			$(d+1)\log_4 n + O(1)$ (GC-balanced)
			(c.f. Corollary 16)
D	cyclic code containing 1 <sup>n</sup>	almost GC-balanced $(n, d; \kappa, n)$ -primer	N.A.
		code	
Е	reversible cyclic code containing $1^n$ ,	primer code with $\kappa = f$	$(d+1)\log_4(n+1)$ (c.f. Corollary 24)
	and rc-generating set of polynomials		
F	reversible cyclic code containing $1^n$ ,	GC-balanced DNA computing codes	$(d+1)\log_4(n+1)$ (c.f. Corollary 30)
	and rc2-generating set of polynomials		

TABLE I

Summary of Constructions for Codes of Length n and Distance d

contradicting condition (R5) in Definition 20.

Next, we complete the proof of Corollary 24. To do so, we recall some concepts in finite field theory.

For  $m \geq 2$ , we consider the finite field  $F \triangleq \mathbb{F}_{4^m}$ . A nonzero element  $\alpha \in F$  is said to be primitive if  $\alpha^i \neq 1$  for  $i \in [4^m -$ 2]. For  $\alpha \in F$ , we let  $M(\alpha)$  denote the minimal polynomial of  $\alpha$  in the base field  $\mathbb{F}_4$ . Then the following facts are useful in establishing our results.

**Lemma 34.** Let F be a field with  $4^m$  elements.

- (a) For nonzero  $\alpha \in F$ , the polynomial  $M(\alpha)M(\alpha^{-1})$  is selfreciprocal.
- (b) If  $\alpha \in F$  is primitive, then  $M(\alpha)$  does not divide  $X^s 1$  for  $s \in [4^m - 2].$
- (c) There are  $\varphi(4^m-1)$  primitive elements in F.

Next, we provide a set of polynomials that satisfies Definition 20.

**Lemma 35.** Let q(X) be the generator polynomial of a reversible cyclic code  $\mathbb{B}$  of length n and dimension k that contains  $1^n$ . Let  $\alpha$ be a primitive element of F such that  $g(\alpha) \neq 0$  and  $g(\alpha^{-1}) \neq 0$ . If  $h^*(X) = M(\alpha)M(\alpha^{-1})$  and n-k < k-1, then the set  $\{h^*(X),1\}$  is (g,k)-rc-generating.

*Proof.* We verify conditions (R1) to (R6) in Definition 20.

(R1) follows from the fact that both  $\alpha$  and  $\alpha^{-1}$  are not roots of g. Since  $h^*$  is the product of two minimal polynomials of primitive elements,  $h^*(1)$  is not zero and so (R2) holds. (R3) follows from Lemma 34(a).

Next, observe that P = 1 with  $p_1(X) = 1$ . Hence, (R7) trivially holds. Also, (R4) to (R6) reduces to verifying that

- (i)  $h^*(X)$  does not divide  $X^s-1$  for  $s\in [n-1]$ ; and (ii)  $h^*(X)$  does not divide  $X^s-X^{k-1}$  for  $0\leq s\leq n-k$ .

Since n-k < k-1, we have that  $X^s - X^{k-1}$  is nonzero for  $0 \le k-1$  $s \le n - k$ . Then both (i) and (ii) follows from Lemma 34(b).  $\square$ 

*Proof of Corollary 24.* Let g(X) be the generator polynomial of the reversible cyclic code C constructed in Theorem 2.

Consider the set  $\Lambda = \{ \alpha \in F : g(\alpha) = 0 \text{ or } g(\alpha^{-1}) = 0 \}.$ Since the degree of g is  $n-k \leq (d-1)m$ , we have that  $|\Lambda| \le 2(d-1)m$ . Since  $\varphi(4^m-1) > 2(d-1)m$  for  $m \ge 6$ , there exists a primitive element  $\alpha \in F$  that does not belong to Λ. In other words,  $g(\alpha) \neq 0$  and  $g(\alpha^{-1}) \neq 0$ . By Lemma 35, the set  $\{h^*(X) \triangleq M(\alpha)M(\alpha^{-1}), 1\}$  is (q, k)-re-generating and therefore, Construction E yields an (n, d; k, k)-primer code of size  $4^{k-2m}$ 

## B. Detailed Proofs for Section V

We complete the proof of Theorem 28 by establishing the following lemma.

**Lemma 36.** Let g(X) be the generator polynomial of a reversible cyclic code  $\mathbb{B}$  of length n and dimension k that contains  $1^n$ . If  $\{h(X), p_1(X), p_2(X), \dots, p_P(X)\}\$  is (g, k)-rc2-generating and  $k^* = k - \deg h^*$ , then the subcode  $\mathcal{A} = \{(m(X)h^*(X) + 1)g(X) :$  $\deg m < k^*$  satisfies conditions (S1') to (S3') in Lemma 29.

*Proof.* Here we only prove condition (S2'). The other two conditions can be proved similarly. In particular, we demonstrate that the violation of condition (S2') contradicts condition (R5') in Definition 27.

Suppose to the contrary of (S2') that we have two codewords  $u, v \in \mathcal{A}$  such that  $\sigma^s(u) = v^r$  for some  $s \in [n]$ . Hence, there exists two polynomials m(X) and m'(X) with degrees strictly less than  $k^*$ , two polynomials  $p_i(X)$  and  $p_j(X)$  with  $i, j \in [P]$ such that the following equality holds with some polynomial f(X).

$$X^{s}(m(X)h^{*}(X) + p_{i}(X)) g(X)$$

$$= X^{n-1} (m'(X^{-1})h^{*}(X^{-1}) + p_{j}(X^{-1})) g(X^{-1})$$

$$+ f(X)(X^{n} - 1).$$

As before, by choice of g and h, we have that  $X^{n-k}g(X^{-1}) =$ g(X) and  $h^*(X) = X^{\deg h^*} h^*(X^{-1})/h^*(0)$ . Dividing the equation by q(X) and rearranging the terms, we have the following equality.

$$(X^{s}p_{i}(X) - X^{k-1}p_{j}(X^{-1}))$$

$$+ (m(X) - X^{k^{*}}m'(X^{-1})h(0))h^{*}(X) = f(X)h(X)$$

Therefore,

$$h^*(X)$$
 divides  $p_i(X) - X^{k-1}p_j(X^{-1})$ ,

contradicting condition (R5') in Definition 27.

To complete the proof of Corollary 30, we provide a set of polynomials that satisfies Definition 27.

**Lemma 37.** Let g(X) be the generator polynomial of a reversible cyclic code  $\mathcal{B}$  of length n and dimension k that contains  $1^n$ . Let  $\alpha$  be a primitive element of F such that  $g(\alpha) \neq 0$  and  $g(\alpha^{-1}) \neq 0$ . If  $h^*(X) = M(\alpha)M(\alpha^{-1})$  and  $p(X) = M(\alpha)$ , then the set  $\{h^*(X), p(X)\}$  is (g, k)-rc2-generating.

*Proof.* We verify conditions in Definition 27. Conditions (R1) to (R4) and (R6) follows directly from the proof of Lemma 35. Hence, we verify (R5') which reduces to verifying that

 $h^*(X)$  does not divide  $X^s p(X) - X^{k-1} p(X^{-1})$  for  $s \in [n]$ .

This is equivalent to showing that  $r(X) \triangleq X^s p(X) - X^{k-1} p(X^{-1})$  is nonzero for some root of  $h^*(X)$ . Observe that since  $\alpha$  is primitive, we have that  $\alpha^{-1}$  is not a root of p(X). In other words,  $p(\alpha^{-1}) \neq 0$ . Since  $h^*(\alpha) = p(\alpha) = 0$  and  $p(\alpha^{-1}) \neq 0$ , we have that  $r(\alpha) = \alpha^{k-1} p(\alpha^{-1}) \neq 0$ .

#### VII. CONCLUSION

We provide efficient and explicit methods to construct balanced codes, primer codes and DNA computing codes with error-correcting capabilities. Using certain classes of BCH codes as inputs, we obtain infinite families of  $(n,d)_q$ -codes satisfying our constraints with redundancy  $C_d \log n + O(1)$ . Here,  $C_d$  is a constant dependent only on d and we provide a summary of our constructions and the corresponding value of  $C_d$  in Table I. Note that in all our constructions, we have  $C_d \leq d+1$ . On the other hand, the sphere-packing bound requires  $C_d \geq \lfloor (d-1)/2 \rfloor$ . Therefore, it remains open to provide efficient and explicit constructions that reduce the value of  $C_d$  further.

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