# Bounding and Estimating the Classical Information Rate of Quantum Channels with Memory

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Abstract—We consider the scenario of classical communication over a finite-dimensional quantum channel with memory using a separable-state input ensemble and local output measurements. We propose algorithms for estimating the information rate of such communication setups, along with algorithms for bounding the information rate based on so-called auxiliary channels. Some of the algorithms are extensions of their counterparts for (classical) finite-state-machine channels. Notably, we discuss suitable graphical models for doing the relevant computations. Moreover, the auxiliary channels are learned in a data-driven approach; *i.e.*, only input/output sequences of the true channel are needed, but not the channel model of the true channel.

Index Terms—Quantum Channel, Memory, Information Rate, Bounds

### I. INTRODUCTION

E consider the transmission rate of classical information over a finite-dimensional quantum channel with memory [3], [4], [5]. Recall that in the memoryless case, given an input system A and an output system B, described by some Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively, a memoryless quantum channel can be modeled as a completely positive trace-preserving (CPTP) map from the set of density operators on  $\mathcal{H}_A$  to the set of density operators on  $\mathcal{H}_B$  [6], [7]; such a quantum channel is said to be finite-dimensional if both  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are of finite dimension. A *quantum channel with* memory is a quantum channel equipped with a memory system S; namely it is a CPTP map from the set of density operators on  $\mathcal{H}_A \otimes \mathcal{H}_S$  to the set of density operators on  $\mathcal{H}_B \otimes \mathcal{H}_{S'}$ , where  $\mathcal{H}_{S} \ (\equiv \mathcal{H}_{S'})$  is the Hilbert space describing S, and  $\otimes$  stands for the tensor product. The system S can be understood either as a state of the channel (as illustrated in Fig. 1(a)), or as a part of the environment that does not decay between consecutive channel uses (as illustrated in Fig. 1(b)). Interesting examples of quantum channels with memory include spin chains [8] and fiber optic links [9].

Classical communication over such channels is accomplished by encoding classical data into some density operators before the transmission and applying measurements at the

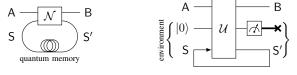
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The work described in this paper was partially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project Nos. CUHK 14209317 and CUHK 14207518).

This paper was presented in part at the IEEE International Symposium on Information Theory (ISIT), Aachen, Germany, July 2017 [1], and the IEEE International Symposium on Information Theory (ISIT), Paris, France, July 2019 [2].

Submitted. Date of current version: February 18, 2020.



(a) Memory as the state of the channel (b) Memory as undecayed partial environment

Fig. 1. Interpretations of quantum channels with memory.

outputs of the channel [6], [7]. In the most generic case, a joint input ensemble and a joint output measurement across multiple channels can be used for encoding and decoding, respectively. The scenario involving a *k*-channel joint ensemble and a *k*-channel joint measurement is depicted in Fig. 2(b), where

- the encoding process *E* is described by some ensemble {*P*<sub>X</sub>(*x*), ρ<sup>(x)</sup><sub>A<sup>1</sup><sub>1</sub></sub>}<sub>*x*∈*X*</sub> on the joint input system (A<sub>1</sub>,..., A<sub>k</sub>), with *X* being the input alphabet, *P*<sub>X</sub>(*x*) being the input distribution, and ρ<sup>(x)</sup><sub>A<sup>1</sup><sub>1</sub></sub> being the density operator on the input systems A<sup>k</sup><sub>1</sub> corresponding to the classical input *x*;
- the decoding process D is described by some positiveoperator valued measure (POVM) {Λ<sup>(y)</sup><sub>B<sup>1</sup><sub>1</sub></sub>}<sub>y∈V</sub> on the joint output system (B<sub>1</sub>,...,B<sub>k</sub>), with Y being the output alphabet;
- the classical input and output are represented by some random variables X and Y, respectively.

For comparison, Fig. 2(a) shows the corresponding memoryless setup. The above arrangement results in a (classical) channel from X to Y, whose rate of transmission is given by

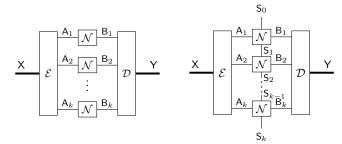
$$\mathbf{I}(\mathcal{E}, \mathcal{N}^{\boxtimes k}, \mathcal{D}) = \limsup_{n \to \infty} \frac{1}{n} \mathbf{I}(\mathsf{X}_1^n; \mathsf{Y}_1^n), \tag{1}$$

where we use the above transmission scheme n times consecutively (as depicted in Fig. 2(c)), and where

$$\begin{split} \mathcal{N}^{\boxtimes k} &\triangleq \Bigl(\mathcal{N}_{\mathsf{A}_k\mathsf{S}_{k-1}\to\mathsf{B}_k\mathsf{S}_k}\otimes\mathcal{I}_{\mathsf{B}_1^{k-1}\to\mathsf{B}_1^{k-1}}\Bigr) \circ \\ & \left(\mathcal{I}_{\mathsf{A}_k\to\mathsf{A}_k}\otimes\mathcal{N}_{\mathsf{A}_{k-1}}\mathsf{s}_{k-2}\to\mathsf{B}_{k-1}}\mathsf{s}_{k-1}\otimes\mathcal{I}_{\mathsf{B}_1^{k-2}\to\mathsf{B}_1^{k-2}}\Bigr) \circ \\ & \cdots \circ \Bigl(\mathcal{I}_{\mathsf{A}_2^k\to\mathsf{A}_2^k}\otimes\mathcal{N}_{\mathsf{A}_1}\mathsf{s}_0\to\mathsf{B}_1}\mathsf{s}_1\Bigr). \end{split}$$

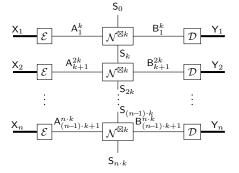
Here, I stands for the mutual information. As a fundamental result, this quantity can be simplified to I(X;Y) for the memoryless case [10], [11]. Optimizing  $I(\mathcal{E}, \mathcal{N}^{\boxtimes k}, \mathcal{D})$  over  $\mathcal{E}$  and  $\mathcal{D}$  (with  $k \to \infty$ ) yields the classical capacity of the quantum channel  $\mathcal{N}$ , namely

$$C(\mathcal{N}) = \limsup_{k} \frac{1}{k} \sup_{\mathcal{E}, \mathcal{D}} I(\mathcal{E}, \mathcal{N}^{\boxtimes k}, \mathcal{D}).$$
(2)



(a) Memoryless channel  $(\mathcal{N}^{\otimes k})$ 

(b) Channel with memory  $(\mathcal{N}^{\boxtimes k})$ 



(c) Generic classical communication corresponding to Eq. (1)

Fig. 2. Classical communications over quantum channels

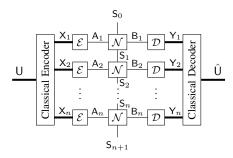


Fig. 3. Classical communication over a quantum channel with memory using a separable ensemble and local measurements.

In this paper, we are interested in computing and bounding the information rate as in (1) for finite-dimensional quantum channels with memory using only separable input ensembles and local output measurements, *i.e.*, the case k = 1, which is depicted in Fig. 3. This restriction is equivalent to the scenario where no quantum computing device is present at the sending or receiving end; or the scenario where our manipulation of the channel is limited to a single-channel use. The difficulty of the problem lies with the presence of the quantum memory. In the simplest situation, the memory system exhibits classical properties under certain ensembles and measurements. In this case, the resulting classical communication setup is equivalent to a finite-state-machine channel (FSMC) [12]. Though the evaluation of the information rate of an FSMC is nontrivial in general, efficient stochastic methods for estimating and bounding this quantity have been developed [13], [14].

Our work is highly inspired by [13], where the authors considered the information rate of FSMCs. In particular, for an indecomposable FSMC [12] with channel law W, its information rate, which is independent from the initial channel

state, is given by

$$\mathbf{I}_{W}(Q) = \lim_{n \to \infty} \frac{1}{n} \mathbf{I}(\mathsf{X}_{1}^{n}; \mathsf{Y}_{1}^{n}), \tag{3}$$

where  $X_1^n = (X_1, \dots, X_n)$  is the channel input process characterized by some sequence of distributions  $\{Q^{(n)}\}_n$ , and where  $Y_1^n = (Y_1, \dots, Y_n)$  is the channel output process. Although, except for very special cases, there are no single-letter or other simple expressions for information rates available, efficient stochastic techniques have been developed for estimating the information rate for stationary and ergodic input processes  $\{Q^{(n)}\}_n$  [13], [15], [16]. (For these techniques, under mild conditions, the numerical estimate of the information rate converges with probability one to the true value when the length of the channel input sequence goes to infinity.) In this paper, we extend such techniques to quantum channels with memory; in particular, we use similar (but extended) graphical models, namely factor graphs for quantum probabilities [17] for estimating quantities of interest. These graphical models are useful for visualizing the relevant computations and for providing a clear comparison between the setup considered in this paper and its classical counterparts in [13] and [14].<sup>1</sup>

Our work is also partially inspired by [14], where the authors proposed upper and lower bounds based on some so-called auxiliary FSMCs, which are often lower-complexity approximations of the original FSMC. They also provided efficient methods for optimizing these bounds. Such techniques have been proven useful for FSMCs with large state spaces, when the above-mentioned information rate estimation techniques can be overly time-consuming. Interestingly enough, the lower bounds represent achievable rates under mismatched decoding, where the decoder bases its computations not on the true FSMC but on the auxiliary FSMC [18]. (See the paper [14] for a more detailed discussion of this topic and for further references.) In this paper, we also consider auxiliary channels and their induced bounds. However, the auxiliary channels of our interest are chosen from a larger set of channels called quantum-state channels, which will be defined in Section III. We also propose a method for optimizing these bounds. In particular, our method for optimizing the lower bound is "data-driven" in the sense that only the input/output sequences of the original channel are needed, but not the mathematical model of the original channel.

One must note that even if we can efficiently compute or bound the information rate, it is still a long way to go to compute the classical capacity of a quantum channel with memory. On the one hand, maximizing  $I(\mathcal{E}, \mathcal{N}, \mathcal{D})$  is a difficult problem. (The analogous classical problems have been addressed in [19], [20], and [21].) On the other hand, due to the superadditivity property [22] of quantum channels, which happens to be more common for quantum channels with memory [23], [24], [25] (compared with memoryless quantum channels), it is inevitable to consider joint ensembles

<sup>&</sup>lt;sup>1</sup>Clearly, the graphical models that we use are very similar to tensor networks (see, for example, the discussion in Appendix A of [17]). A benefit of the graphical models that we use (including the corresponding terminology), is that they are compatible with the graphical models that are being used in classical information processing.

on input systems and joint measurements on output systems across multiple channel uses.

The rest of this paper is organized as follows. Section II reviews the method of estimating the information rate of an FSMC. Section III models the classical communication scheme over a quantum channel with memory, and defines the notion of quantum-state channels as an equivalent description. A graphical notation for representing such channels is also presented in this section. Section IV estimates the information rate of such channels. Section V considers the upper and lower bounds induced by auxiliary quantum-state channels, and presents methods for optimizing them. Section VI contains numerical examples. Section VII concludes the paper.

## A. Further references

In the following, we assume that the reader is familiar with the basic elements of quantum information theory (see [6] or [7] for an introduction). For a general introduction to quantum channels with memory, we refer to the papers by Kretschmann and Werner [4] and by Caruso *et al.* [5].

Moreover, some familiarity with graphical models (like factor graphs) [26], [27], [28] and with techniques for estimating the information rate of an FSMC as presented in [13], [14] will be beneficial. Recall that graphical models are a popular approach for representing multivariate functions with nontrivial factorizations and for doing computations like marginalization [26], [27], [28]. In particular, graphical models can be used to represent joint probability mass functions (pmfs) / probability density functions (pdfs). In the present paper we will heavily rely on the paper [17], which discussed an approach for using normal factor graphs (NFGs) for representing functions that typically appear when doing computations w.r.t. some quantum systems. Alternatively, we could also have used the slightly more compact double-edge normal factor graphs (DE-NFGs) [29]. Probabilities of interest are then obtained by suitably applying the sum-product algorithm or closing-thebox operations.

#### **B.** Notations

We use the following conventions throughout the paper:

- Vectors are denoted using boldface letters.
- Sans-serif letters are being used to denote either random variables or quantum systems.
- Lower and upper indices are used as the starting and ending indices, respectively, of the elements in a vector or an ordered collections of random variables or quantum systems. For example,
  - $-x_1^n \equiv (x_1, x_2, \dots, x_n)$  denotes an *n*-tuple with elements  $x_1$  up to  $x_n$ ;
  - $X_1^n \equiv (X_1, X_2, \dots, X_n)$  denotes a sequence of random variables;
  - $S_0^n \equiv (S_0, S_1, \dots, S_n)$  denotes the collective quantum system consisting of subsystems  $S_0$  up to  $S_n$ .
- The set of all density operators over a Hilbert space  $\mathcal{H}$  is denoted by  $\mathfrak{D}(\mathcal{H})$ ; its elements are represented using Greek letters, *e.g.*,  $\rho_{\mathsf{S}}$  denotes a density operator of some quantum system S.

As it should be clear from the context, we also overload the symbol  $\mathbf{H}$  to denote either the Shannon entropy or the von Neumann entropy, and the symbol  $\mathbf{I}$  to denote either the classical or quantum mutual information.

# II. REVIEW OF (CLASSICAL) FINITE-STATE MACHINE CHANNELS: INFORMATION RATE, ITS ESTIMATION, AND BOUNDS

In this section, we review the methods developed in [13] for estimating the information rate of a (classical) FSMC, and the auxiliary-channel-induced upper and lower bounds studied in [14]. As we will see, the development in later sections about quantum channels will have many similarities, but also some important differences. We emphasize that this section is a *brief review* of [13] and [14] for the purpose of introducing necessary tools and ideas for later sections.

# A. Finite-State Machine Channels (FSMCs) and their Graphical Representation

A (time-invariant) finite-state machine channel (FSMC) consists of an input alphabet  $\mathcal{X}$ , an output alphabet  $\mathcal{Y}$ , a state alphabet  $\mathcal{S}$ , all of which are finite, and a channel law W(y, s'|x, s), where the latter equals the probability of receiving  $y \in \mathcal{Y}$  and ending up in state  $s' \in \mathcal{S}$  given channel input  $x \in \mathcal{X}$  and previous channel state  $s \in \mathcal{S}$ . The relationship among the input, output, and state processes  $X_1^n, Y_1^n, S_0^n$  of *n*-channel uses can be described by the conditional pmf

$$W(\boldsymbol{y}_{1}^{n}, \boldsymbol{s}_{1}^{n} | \boldsymbol{x}_{1}^{n}, s_{0}) \triangleq P_{\mathbf{Y}_{1}^{n}, \mathbf{S}_{1}^{n} | \mathbf{X}_{1}^{n}, \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n}, \boldsymbol{s}_{1}^{n} | \boldsymbol{x}_{1}^{n}, s_{0})$$
  
= 
$$\prod_{\ell=1}^{n} W(y_{\ell}, s_{\ell} | x_{\ell}, s_{\ell-1}), \qquad (4)$$

where  $x_{\ell} \in \mathcal{X}, y_{\ell} \in \mathcal{Y}$ , and  $s_{\ell} \in \mathcal{S}$  for each  $\ell$ .

**Example 1** (Gilbert–Elliott channels). A notable class of examples of FSMCs are the Gilbert–Elliott channels [30], which behave like a binary symmetric channel (BSC) with cross-over probability  $p_s$  controlled by the channel state  $s \in \{\text{"b", "g"}\}$ , where usually  $|p_b - \frac{1}{2}| < |p_g - \frac{1}{2}|$ . The state process itself is a first-order stationary ergodic Markov process that is independent of the input process.<sup>2</sup> (For more details, see, *e.g.*, the discussions in [14].)

Given an input process  $\{Q^{(n)}\}_n$  and an initial state pmf  $P_{S_0}(s_0)$ , we can write down the joint pmf of  $(X_1^n, Y_1^n, S_0^n)$  as

$$g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n) \triangleq P_{\mathsf{X}_1^n, \mathsf{Y}_1^n, \mathsf{S}_0^n}(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$$
(5)

$$= P_{\mathsf{S}_0}(s_0) \cdot Q^{(n)}(\boldsymbol{x}_1^n) \cdot \prod_{\ell=1} W(y_\ell, s_\ell | x_\ell, s_{\ell-1}).$$
(6)

The factorization of  $g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$  as shown in (5) can be visualized with the help of a normal factor graph (NFG) as in Fig. 4. In this context,  $g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$  is called the global function of the NFG. In particular:

<sup>&</sup>lt;sup>2</sup>The independence of the state process on the input process is a particular feature of the Gilbert–Elliott channel. In general, the state process of a finite-state channel can depend on the input process.

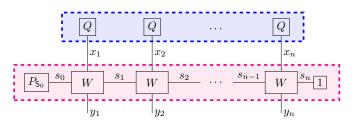


Fig. 4. Channel with a classical state: closing the top box yields the input process  $Q^{(n)}$ , closing the bottom box yields the joint channel law  $W(\mathbf{y}_1^n | \mathbf{x}_1^n)$ .

- a) The part of the NFG inside the bottom box represents  $W(\boldsymbol{y}_1^n, \boldsymbol{s}_1^n | \boldsymbol{x}_1^n, s_0)$ , *i.e.*, the probability of obtaining  $\boldsymbol{y}_1^n$  and  $\boldsymbol{s}_1^n$  given  $\boldsymbol{x}_1^n$  and  $s_0$ . After applying the *closing*-the-box operation, *i.e.*, after summing over all the variables associated with edges completely inside the bottom box, we obtain the joint channel law  $W(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n) \triangleq \sum_{\boldsymbol{s}_1^n} P_{\mathsf{S}_0}(s_0) \cdot W(\boldsymbol{y}_1^n, \boldsymbol{s}_1^n | \boldsymbol{x}_1^n, s_0)$ .
- b) The part of the NFG inside the top box represents the input process Q<sup>(n)</sup>(x<sub>1</sub><sup>n</sup>). Here, for simplicity, the input process is an i.i.d. process characterized by the pmf Q, *i.e.*, Q<sup>(n)</sup>(x<sub>1</sub><sup>n</sup>) = ∏<sub>l=1</sub><sup>n</sup> Q(x<sub>l</sub>).
  c) The function g(x<sub>1</sub><sup>n</sup>, y<sub>1</sub><sup>n</sup>) ≜ ∑<sub>s<sub>0</sub><sup>n</sup></sub> g(x<sub>1</sub><sup>n</sup>, y<sub>1</sub><sup>n</sup>, s<sub>0</sub><sup>n</sup>), which is
- c) The function  $g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n) \triangleq \sum_{\boldsymbol{s}_0^n} g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$ , which is obtained by summing the global function  $g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$  over  $\boldsymbol{s}_0^n$ , represents the corresponding marginal pmf over  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$ . The function  $g(\boldsymbol{s}_0^n) \triangleq \sum_{\boldsymbol{x}_1^n, \boldsymbol{y}_1^n} g(\boldsymbol{x}_1^n, \boldsymbol{y}_1^n, \boldsymbol{s}_0^n)$ , which is obtained by summing the global function over  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$ , represents the corresponding marginal pmf over  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$ , represents the corresponding marginal pmf over  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$ , represents the corresponding marginal pmf over  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$ , represents the corresponding marginal pmf over  $\boldsymbol{s}_0^n$ . Other marginal pmfs can be obtained similarly.

Equipped with the notion of the closing-the-box operation (see item a) above), such NFG representations can be useful in computing a number of quantities of interests. For example, to prove that (4) is indeed a valid conditional pmf, it suffices to show that

$$\sum_{\boldsymbol{s}_1^n, \boldsymbol{y}_1^n} W(\boldsymbol{y}_1^n, \boldsymbol{s}_1^n | \check{\boldsymbol{x}}_1^n, \check{s}_0) = 1 \quad \forall \check{\boldsymbol{x}}_1^n \in \mathcal{X}^n, \, \check{s}_0 \in \mathcal{S}, \quad (7)$$

which can be verified via a sequence of closing-the-box operations as shown in Fig. 12 in the Appendix. Such techniques are at the heart of the information-rate-estimation methods as in [13]. The details are reviewed in the next subsection.

## B. Information Rate Estimation

The approach of [13] for estimating information rates of FSMCs, as reviewed in this section, is based on the Shannon–McMillan–Breiman theorem (see *e.g.*, [11]) and suitable generalizations. We make the following assumptions.

- As already mentioned, the derivations in this paper are for the case where the input process  $X = (X_1, X_2, ...)$  is an i.i.d. process. The results can be generalized to other stationary ergodic input processes that can be represented by a finite-state-machine source (FSMS). Technically, this is done by defining a new state that combines the source state and the channel state.
- We assume that the FSMC is indecomposable, which roughly means that in the long term the behavior of the channel is independent of the initial channel state

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distribution  $P_{S_0}$  (see [12, Section 4.6] for the exact definition). For such channels and stationary ergodic input processes, the information rate  $I_W$  in (3) is well defined. Let  $W(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n)$  be the joint channel law of an FSMC satisfying the assumptions above. As aforementioned, the information rate of such a channel using the i.i.d. input distribution  $\{Q^{(n)} \triangleq Q^{\otimes n}\}_n$  is given by (3), *i.e.*, by

$$I_W(Q) = \lim_{n \to \infty} \frac{1}{n} \mathbf{I}(\mathsf{X}_1^n; \mathsf{Y}_1^n), \tag{3'}$$

where the input process  $X_1^n$  and the output process  $Y_1^n$  are jointly distributed according to

$$P_{\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}}\left(\boldsymbol{x}_{1}^{n},\boldsymbol{y}_{1}^{n}\right) = \prod_{\ell=1}^{n} Q(x_{\ell}) \cdot W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}).$$
(8)

One can rewrite (3) as

$$I_W(Q) = H(X) + H(Y) - H(X, Y),$$
(9)

where the *entropic rates* H(X), H(Y) and H(X, Y) are defined as

$$\mathbf{H}(\mathbf{X}) \triangleq \lim_{n \to \infty} \frac{1}{n} \mathbf{H}(\mathbf{X}_1^n), \tag{10}$$

$$\mathbf{H}(\mathbf{Y}) \triangleq \lim_{n \to \infty} \frac{1}{n} \mathbf{H}(\mathbf{Y}_1^n), \tag{11}$$

$$\mathbf{H}(\mathsf{X},\mathsf{Y}) \triangleq \lim_{n \to \infty} \frac{1}{n} \mathbf{H}(\mathsf{X}_1^n,\mathsf{Y}_1^n).$$
(12)

We proceed as in [13]. (For more background information, see the references in [13], in particular [31].) Namely, because of (9) and

$$-\frac{1}{n}\log P_{\mathsf{X}_{1}^{n}}(\mathsf{X}_{1}^{n}) \xrightarrow{n \to \infty} \mathrm{H}(\mathsf{X}) \qquad \text{w.p. 1}, \qquad (13)$$

$$-\frac{1}{n}\log P_{\mathsf{Y}_1^n}(\mathsf{Y}_1^n) \xrightarrow{n \to \infty} \mathrm{H}(\mathsf{Y}) \qquad \text{w.p. 1}, \qquad (14)$$

$$-\frac{1}{n}\log P_{\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}}(\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}) \xrightarrow{n \to \infty} \mathrm{H}(\mathsf{X},\mathsf{Y}) \quad \text{w.p. 1}, \qquad (15)$$

by choosing some large number n, we have the approximation

$$I_{W}(Q) \approx -\frac{1}{n} \log P_{\mathsf{X}_{1}^{n}}(\check{\mathbf{x}}_{1}^{n}) - \frac{1}{n} \log P_{\mathsf{Y}_{1}^{n}}(\check{\mathbf{y}}_{1}^{n}) \qquad (16) + \frac{1}{n} \log P_{\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}}(\check{\mathbf{x}}_{1}^{n},\check{\mathbf{y}}_{1}^{n}),$$

where  $\check{x}_1^n$  and  $\check{y}_1^n$  are some input and output sequences, respectively, randomly generated according to

$$P_{\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}}(\check{\boldsymbol{x}}_{1}^{n},\check{\boldsymbol{y}}_{1}^{n}) = \sum_{\boldsymbol{s}_{0}^{n}} P_{\mathsf{S}_{0}}(s_{0}) \cdot Q^{(n)}(\check{\boldsymbol{x}}_{1}^{n}) \cdot W(\check{\boldsymbol{y}}_{1}^{n},\boldsymbol{s}_{1}^{n}|\check{\boldsymbol{x}}_{1}^{n},s_{0}), (17)$$

where  $W(\check{\mathbf{y}}_1^n, s_1^n | \check{\mathbf{x}}_1^n, s_0)$  is defined in (4). Note that  $\check{\mathbf{x}}_1^n$  can be obtained by simulating the input process, and  $\check{\mathbf{y}}_1^n$  can be obtained by simulating the channel for the given input string  $\check{\mathbf{x}}_1^n$ . The latter can be done by keeping track of  $P_{\mathsf{Y}_\ell | \mathsf{X}_1^\ell, \mathsf{Y}_1^{\ell-1}}(y_\ell | \check{\mathbf{x}}_1^\ell, \check{\mathbf{y}}_1^{\ell-1})$ , which is proportional to  $P_{\mathsf{Y}_\ell, \mathsf{Y}_1^{\ell-1} | \mathsf{X}_1^\ell}(y_\ell, \check{\mathbf{y}}_1^{\ell-1} | \check{\mathbf{x}}_1^\ell)$ , and can be efficiently calculated by applying suitable closing-the-box operations as in Fig. 14 in the Appendix.

We continue by showing how the three terms appearing on the right-hand side of (16) can be computed efficiently. We show it explicitly for the second term, and then outline it for the first and the third term. In order to efficiently compute the second term on the righthand side of (16), *i.e.*,  $-\frac{1}{n} \log P_{\mathbf{Y}_1^n}(\check{\boldsymbol{y}}_1^n)$ , we consider the *state metric* defined in [13] as

$$\mu_{\ell}^{\mathsf{Y}}(s_{\ell}) \triangleq \sum_{\boldsymbol{x}_{1}^{\ell}} \sum_{\boldsymbol{s}_{0}^{\ell-1}} P_{\mathsf{S}_{0}}(s_{0}) \cdot Q^{(\ell)}(\boldsymbol{x}_{1}^{\ell}) \cdot W(\check{\boldsymbol{y}}_{1}^{\ell}, \boldsymbol{s}_{1}^{\ell} | \boldsymbol{x}_{1}^{\ell}, s_{0}).$$
(18)

In this case,

$$P_{\mathsf{Y}_1^n}(\check{\boldsymbol{y}}_1^n) = \sum_{s_n} \mu_n^{\mathsf{Y}}(s_n), \tag{19}$$

and the calculation of  $\mu_{\ell}^{\mathsf{Y}}(s_{\ell})$  can be done iteratively as

$$\mu_{\ell}^{\mathsf{Y}}(s_{\ell}) = \sum_{x_{\ell}} \sum_{s_{\ell-1}} \mu_{\ell-1}^{\mathsf{Y}}(s_{\ell-1}) \cdot Q(x_{\ell} | \boldsymbol{x}_{1}^{\ell-1}) \cdot W(\check{y}_{\ell}, s_{\ell} | x_{\ell}, s_{\ell-1})$$
$$= \sum_{x_{\ell}} \sum_{s_{\ell-1}} \mu_{\ell-1}^{\mathsf{Y}}(s_{\ell-1}) \cdot Q(x_{\ell}) \cdot W(\check{y}_{\ell}, s_{\ell} | x_{\ell}, s_{\ell-1}).$$
(20)

Eq. (20) is visualized in Fig. 16 as applying suitable closingthe-box operations to the NFG in Fig. 4.

However, since the value of  $\mu_{\ell}^{\mathsf{Y}}(s_{\ell})$  tends to zero as  $\ell$  grows, such recursive calculations are numerically inconvenient. A solution is to *normalize*  $\mu_{\ell}^{\mathsf{Y}}(s_{\ell})$  after each use of (20) and to keep track of the scaling coefficients. Namely,

$$\bar{\mu}_{\ell}^{\mathsf{Y}}(s_{\ell}) \triangleq \frac{1}{\lambda_{\ell}^{\mathsf{Y}}} \sum_{x_{\ell}} \sum_{s_{\ell-1}} \bar{\mu}_{\ell-1}^{\mathsf{Y}}(s_{\ell}) \cdot Q(x_{\ell}) \cdot W(\check{y}_{\ell}, s_{\ell} | x_{\ell}, s_{\ell-1}),$$
(21)

where the scaling factor  $\lambda_{\ell}^{\mathsf{Y}} > 0$  is chosen such that  $\sum_{s_{\ell}} \bar{\mu}_{\ell}^{\mathsf{Y}}(s_{\ell}) = 1$ . With this, Eq. (19) can be rewritten as

$$P_{\mathsf{Y}_1^n}(\check{\boldsymbol{y}}_1^n) = \prod_{\ell=1}^n \lambda_\ell^{\mathsf{Y}}.$$
 (22)

Finally, we arrive at the following efficient procedure for computing  $-\frac{1}{n}\log P_{\mathsf{Y}_1^n}(\boldsymbol{y}_1^n)$ :

- For ℓ = 1,..., n, iteratively compute the normalized state metric and with that the scaling factors λ<sup>Y</sup><sub>ℓ</sub>.
- Conclude with the result

$$-\frac{1}{n}\log P_{\mathsf{Y}_{1}^{n}}(\boldsymbol{y}_{1}^{n}) = \frac{1}{n}\sum_{\ell=1}^{n}\log(\lambda_{\ell}^{\mathsf{Y}}).$$
 (23)

The third term on the right-hand side of (16) can be evaluated by an analogous procedure, where the state metric  $\mu_{\ell}^{Y}(s_{\ell})$  is replaced by the state metric

$$\mu_{\ell}^{\mathsf{X}\mathsf{Y}}(s_{\ell}) \triangleq \sum_{\boldsymbol{s}_{0}^{\ell-1}} P_{\mathsf{S}_{0}}(s_{0}) \cdot Q^{(\ell)}(\boldsymbol{\check{x}}_{1}^{\ell}) \cdot W(\boldsymbol{\check{y}}_{1}^{\ell}, \boldsymbol{s}_{1}^{\ell} | \boldsymbol{\check{x}}_{1}^{\ell}).$$
(24)

The iterative calculation of  $\mu_{\ell}^{XY}(s_{\ell})$  is visualized in Fig. 18.

Finally, the first term on the right-hand side of (16) can be trivially evaluated if X is an i.i.d. process, and with a similar approach as above if it is described by an FSMS.

The above discussion is summarized as Algorithm 2. On the side, note that for each  $\ell = 2, \ldots, n$ , the quantities  $\lambda_{\ell}^{\mathsf{Y}}$  and  $\lambda_{\ell}^{\mathsf{X}\mathsf{Y}}$  in the algorithm are the conditional probabilities  $P_{\mathsf{Y}_{\ell}|\mathsf{Y}_{1}^{\ell-1}}(\check{y}_{\ell}|\check{y}_{1}^{\ell-1})$  and  $P_{\mathsf{X}_{\ell}\mathsf{Y}_{\ell}|\mathsf{X}_{1}^{\ell-1}}\check{\mathsf{Y}_{1}^{\ell-1}}(\check{x}_{\ell},\check{y}_{\ell}|\check{x}_{1}^{\ell-1},\check{y}_{1}^{\ell-1})$ , respectively.

# C. Auxiliary Channels and Bounds on the Information Rate

As already mentioned in Section I, auxiliary channels<sup>3</sup> are introduced when the state space of the FSMC is too large, making the calculation in Algorithm 2 (pratically) intractable. More precisely, given an auxiliary forward FSMC (AF-FSMC)  $\hat{W}(y_{\ell}, \hat{s}_{\ell} | x_{\ell}, \hat{s}_{\ell-1})$  and an auxiliary backward FSMC (AB-FSMC)  $\hat{V}(x_{\ell}, \hat{s}_{\ell} | y_{\ell}, \hat{s}_{\ell-1})$ , a pair of upper and lower bounds of the information rate is given in [13], [14] as

$$\bar{\mathbf{I}}_{W}^{(n)}(\hat{W}) \triangleq \frac{1}{n} \sum_{\boldsymbol{x}_{1}^{n}, \boldsymbol{y}_{1}^{n}} Q(\boldsymbol{x}_{1}^{n}) W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}) \log \frac{W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n})}{(Q\hat{W})(\boldsymbol{y}_{1}^{n})},$$
(25)

$$\underline{\mathbf{I}}_{W}^{(n)}(\hat{V}) \triangleq \frac{1}{n} \sum_{\boldsymbol{x}_{1}^{n}, \boldsymbol{y}_{1}^{n}} Q(\boldsymbol{x}_{1}^{n}) W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}) \log \frac{V(\boldsymbol{x}_{1}^{n} | \boldsymbol{y}_{1}^{n})}{Q(\boldsymbol{x}_{1}^{n})}, \quad (26)$$

where  $(Q\hat{W})(\boldsymbol{y}_1^n) \triangleq \sum_{\boldsymbol{x}_1^n} Q(\boldsymbol{x}_1^n) \cdot \hat{W}(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n)$ . To see that (25) and (26) are, respectively, upper and lower bounds, one can verify the following two equalities,

$$\bar{\mathbf{I}}_{W}(\hat{W}) - \mathbf{I}_{W} = \frac{1}{n} \mathbf{D}_{\mathrm{KL}} \Big( (QW)(\mathbf{Y}_{1}^{n}) \mid || (Q\hat{W})(\mathbf{y}_{1}^{n}) \Big), \quad (27)$$
$$\mathbf{I}_{W} - \underline{\mathbf{I}}_{W}(\hat{V}) = \frac{1}{n} \sum_{\boldsymbol{y}_{1}^{n}} (QW)(\boldsymbol{y}_{1}^{n}) \cdot \tag{28}$$

$$D_{\mathrm{KL}}\Big(V(\mathsf{X}_1^n|\boldsymbol{y}_1^n) \| \hat{V}(\mathsf{X}_1^n|\boldsymbol{y}_1^n)\Big),$$

where  $D_{KL}(\cdot \| \cdot)$  stands for the Kullback–Leibler (KL) divergence, and where the backward channel  $V(\boldsymbol{x}|\boldsymbol{y})$  is defined as  $V(\boldsymbol{x}|\boldsymbol{y}) \triangleq Q(\boldsymbol{x})W(\boldsymbol{y}|\boldsymbol{x})/(QW)(\boldsymbol{y})$ . In particular, given an AF-FSMC  $\hat{W}$ , the paper [14] considered the induced AB-FSMC  $\hat{V}(\boldsymbol{x}|\boldsymbol{y}) \triangleq Q(\boldsymbol{x})\hat{W}(\boldsymbol{y}|\boldsymbol{x})/(Q\hat{W})(\boldsymbol{y})$ . In this case,

$$\underline{\mathbf{I}}_{W}^{(n)}(\hat{V}) = \frac{1}{n} \sum_{\boldsymbol{x}_{1}^{n}, \boldsymbol{y}_{1}^{n}} Q(\boldsymbol{x}_{1}^{n}) W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}) \log \frac{W(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n})}{(Q\hat{W})(\boldsymbol{y}_{1}^{n})}.$$
 (29)

The difference function  $\Delta^{(n)}_W(\hat{W})$  is defined as

$$\begin{split} \Delta_W^{(n)}(\hat{W}) &\triangleq \bar{\mathbf{I}}_W^{(n)}(\hat{W}) - \underline{\mathbf{I}}_W^{(n)}(\hat{V}) \quad (30) \\ &= \frac{1}{n} \sum_{\boldsymbol{x}_1^n, \boldsymbol{y}_1^n} Q(\boldsymbol{x}_1^n) W(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n) \log\left(\frac{W(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n)}{\hat{W}(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n)}\right) \\ &= \frac{1}{n} \mathcal{D}_{\mathrm{KL}} \Big( Q(\mathsf{X}_1^n) W(\mathsf{Y}_1^n | \mathsf{X}_1^n) \, \Big\| \, Q(\mathsf{X}_1^n) \hat{W}(\mathsf{Y}_1^n | \mathsf{X}_1^n) \Big) \end{split}$$

Apparently,  $\Delta_W^{(n)}(\hat{W}) \ge 0$ , and equality holds if and only if  $\hat{W}(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n) = W(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n)$  for all  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$  with positive support w.r.t.  $P_{X_1^n,Y_1^n}$  defined in (8). An efficient algorithm for finding a local minimum of the difference function was proposed in [14]; we refer to [14] for further details.

## III. QUANTUM CHANNEL WITH MEMORY AND THEIR GRAPHICAL REPRESENTATION

In this section, we formalize our notations and modeling of quantum channels with memory [3], [4], [5] and of classical communications over such channels. In particular, we will define a class of channels named *quantum-state channels*, which

<sup>&</sup>lt;sup>3</sup>Technically speaking, an auxiliary channel can be defined as *any* channel with the same input/output alphabet. For example, an auxiliary channel for an FSMC can be just another FSMC with smaller state space; whereas in Section V, an auxiliary channel can also be a quantum-state channel.

# Algorithm 2 Estimating the information rate of an FSMC

**Input:** indecomposable FSMC channel law W, input distribution Q, positive integer n large enough.

- **Output:**  $I_W(Q) \approx \mathbf{H}(\mathsf{X}) + \hat{\mathrm{H}}(\mathsf{Y}) \hat{\mathrm{H}}(\mathsf{X},\mathsf{Y}).$
- 1: Initialize the channel state distribution  $P_{S_0}$  as a uniform distribution over S
- 2: Generate an input sequence  $\check{\boldsymbol{x}}_1^n \sim Q^{\otimes n}$
- 3: Generate a corresponding output sequence  $\check{\boldsymbol{y}}_1^n$

4:  $\bar{\mu}_{\ell}^{\mathbf{Y}} \leftarrow P_{\mathbf{S}_{0}}$ 5: for each  $\ell = 1, ..., n$  do 6:  $\mu_{\ell}^{\mathbf{Y}}(s_{\ell}) \leftarrow \sum_{x_{\ell}, s_{\ell-1}} \bar{\mu}_{\ell-1}^{\mathbf{Y}}(s_{\ell-1}) \cdot Q(x_{\ell}) \cdot W(\check{y}_{\ell}, s_{\ell} | x_{\ell}, s_{\ell-1})$ 7:  $\lambda_{\ell}^{\mathbf{Y}} \leftarrow \sum_{s_{\ell}} \mu_{\ell}^{\mathbf{Y}}(s_{\ell});$ 8:  $\bar{\mu}_{\ell}^{\mathbf{Y}} \leftarrow \mu_{\ell}^{\mathbf{Y}} / \lambda_{\ell}^{\mathbf{Y}}$ 9: end for 10:  $\hat{\mathbf{H}}(\mathbf{Y}) \leftarrow -\frac{1}{n} \sum_{\ell=1}^{n} \log(\lambda_{\ell}^{\mathbf{Y}})$ 11:  $\bar{\mu}_{0}^{\mathbf{XY}} \leftarrow P_{\mathbf{S}_{0}}$ 12: for each  $\ell = 1, ..., n$  do 13:  $\mu_{\ell}^{\mathbf{XY}}(s_{\ell}) \leftarrow \sum_{s_{\ell-1}} \bar{\mu}_{\ell-1}^{\mathbf{XY}}(s_{\ell-1}) \cdot Q(\check{x}_{\ell}) \cdot W(\check{y}_{\ell}, s_{\ell} | \check{x}_{\ell}, s_{\ell-1})$ 14:  $\lambda_{\ell}^{\mathbf{YY}} \leftarrow \sum_{s_{\ell}} \mu_{\ell}^{\mathbf{XY}}(s_{\ell})$ 15:  $\bar{\mu}_{\ell}^{\mathbf{XY}} \leftarrow \mu_{\ell}^{\mathbf{XY}} / \lambda_{\ell}^{\mathbf{XY}}$ 16: end for 17:  $\hat{\mathbf{H}}(\mathbf{X}, \mathbf{Y}) \leftarrow -\frac{1}{n} \sum_{\ell=1}^{n} \log(\lambda_{\ell}^{\mathbf{XY}})$ 18:  $\mathbf{H}(\mathbf{X}) \leftarrow -\sum_{x} Q(x) \log Q(x)$ 19: Estimate  $\mathbf{I}_{W}(Q)$  as  $\mathbf{H}(\mathbf{X}) + \hat{\mathbf{H}}(\mathbf{Y}) - \hat{\mathbf{H}}(\mathbf{X}, \mathbf{Y}).$ 

is an alternative description of the classical communications over quantum channels with memory. In addition, we will introduce several NFGs for representing these channels and processes.

# A. Classical Communication over a Quantum Channel with Memory

As already mentioned in Section I, a quantum channel with memory is a completely positive trace-preserving (CPTP) map

$$\mathcal{N}:\mathfrak{D}(\mathcal{H}_{\mathsf{A}}\otimes\mathcal{H}_{\mathsf{S}})\to\mathfrak{D}(\mathcal{H}_{\mathsf{B}}\otimes\mathcal{H}_{\mathsf{S}'}),\tag{31}$$

where A is the input system, B is the output system, S and S' are, respectively, the memory systems before and after the channel use. The Hilbert spaces  $\mathcal{H}_A$ ,  $\mathcal{H}_B$ , and  $\mathcal{H}_S \equiv \mathcal{H}_{S'}$  are the state spaces corresponding to those systems.

In the present paper, we consider classical communication over such channels using some separable input ensemble and local output measurements; namely, the encoder and decoder are, respectively, some classical-to-quantum and quantum-to-classical channels involving a single input or output system. In particular, given an ensemble  $\{\rho_A^{(x)}\}_{x\in\mathcal{X}}$  and a measurement  $\{\Lambda_B^{(y)}\}_{y\in\mathcal{Y}}$ , we define the encoding and decoding function, respectively, as

Encoding 
$$\mathcal{E}: p_{\mathsf{X}} \mapsto \sum_{x \in \mathcal{X}} p_{\mathsf{X}}(x) \rho_{\mathsf{A}}^{(x)} \quad \forall p_{\mathsf{X}} \text{ over } \mathcal{X}, \quad (32)$$

Decoding 
$$\mathcal{D}: \sigma_{\mathsf{B}} \mapsto \left\{ \operatorname{tr}(\Lambda_{\mathsf{B}}^{(y)} \cdot \sigma_{\mathsf{B}}) \right\}_{y \in \mathcal{Y}} \forall \sigma_{\mathsf{B}} \text{ over } \mathcal{H}_{\mathsf{B}}.$$
 (33)

We emphasize that in our setup, the ensemble  $\{\rho_A^{(x)}\}_{x\in\mathcal{X}}$  and measurements  $\{\Lambda_B^{(y)}\}_{y\in\mathcal{Y}}$  are given and fixed. Furthermore,

we assume that one does not have access to the memory systems of the channel. For the case of i.i.d. inputs, the memory system S before each channel use shall be independent of the input system A, namely, the joint memory-input operator shall take the form of  $\rho_A \otimes \rho_S$  at each channel input.<sup>4</sup>

With this, the probability of receiving  $y \in \mathcal{Y}$ , given that  $x \in \mathcal{X}$  was sent and given that the density operator of the memory system *before* the usage of the channel was  $\rho_{S}$ , equals

$$P_{\mathsf{Y}|\mathsf{X};\mathsf{S}}(y|x;\rho_{\mathsf{S}}) = \operatorname{tr}\left(\Lambda_{\mathsf{B}}^{(y)} \cdot \operatorname{tr}_{\mathsf{S}'}\left(\mathcal{N}(\rho_{\mathsf{A}}^{(x)} \otimes \rho_{\mathsf{S}})\right)\right), \quad (34)$$

which can also be written as

$$P_{\mathsf{Y}|\mathsf{X};\mathsf{S}}(y|x;\rho_{\mathsf{S}}) = \operatorname{tr}\left(\left(\Lambda_{\mathsf{B}}^{(y)}\otimes I_{\mathsf{S}}\right)\cdot\mathcal{N}(\rho_{\mathsf{A}}^{(x)}\otimes\rho_{\mathsf{S}})\right),\quad(35)$$

where  $tr_{S'}$  stands for the *partial trace* operator (see, *e.g.* [6, Section 2.4.3]) that extracts the subsystem B from the joint system (BS'). Moreover, assuming that *y* was observed, the density operator of the memory system *after* the channel use is given by

$$\rho_{\mathsf{S}'} = \frac{\operatorname{tr}_{\mathsf{B}}\left(\left(\Lambda_{\mathsf{B}}^{(y)} \otimes I_{\mathsf{S}}\right) \cdot \mathcal{N}(\rho_{\mathsf{A}}^{(x)} \otimes \rho_{\mathsf{S}})\right)}{\operatorname{tr}\left(\left(\Lambda_{\mathsf{B}}^{(y)} \otimes I_{\mathsf{S}}\right) \cdot \mathcal{N}(\rho_{\mathsf{A}}^{(x)} \otimes \rho_{\mathsf{S}})\right)}.$$
(36)

Notice that the denominator in (36) equals the expressions in (34) and (35). One should note that, though the input and the memory systems are independent before each channel use (given i.i.d. inputs), the output and the memory systems after each channel use can be correlated or even entangled. In particular, this translates to the fact that the measurement outcome y can have an influence on the memory system as indicated in (36).

Consider using the channel *n* times consecutively with the above scheme. The joint channel law, namely the conditional pmf of the channel outputs  $Y_1^n$  given the channel inputs  $X_1^n$  and the initial channel state  $\rho_{S_0}$ , can be computed iteratively using (35) and (36). In particular, the joint conditional pmf can be computed as

$$P_{\mathsf{Y}_{1}^{n}|\mathsf{X}_{1}^{n};\mathsf{S}_{0}}(\boldsymbol{y}_{1}^{n}|\boldsymbol{x}_{1}^{n};\rho_{\mathsf{S}_{0}}) = \prod_{\ell=1}^{n} P_{\mathsf{Y}_{\ell}|\mathsf{X}_{\ell};\mathsf{S}_{\ell-1}}(y_{\ell}|x_{\ell};\rho_{\mathsf{S}_{\ell-1}}), \quad (37)$$

where we compute the density operators  $\{\rho_{S_{\ell}}\}_{\ell=1}^{n}$  iteratively using (36) as

$$\rho_{\mathsf{S}_{\ell}} = \frac{\operatorname{tr}_{\mathsf{B}}\left(\left(\Lambda_{\mathsf{B}}^{(y_{\ell})} \otimes I_{\mathsf{S}}\right) \cdot \mathcal{N}(\rho_{\mathsf{A}}^{(x_{\ell})} \otimes \rho_{\mathsf{S}_{\ell-1}})\right)}{\operatorname{tr}\left(\left(\Lambda_{\mathsf{B}}^{(y_{\ell})} \otimes I_{\mathsf{S}}\right) \cdot \mathcal{N}(\rho_{\mathsf{A}}^{(x_{\ell})} \otimes \rho_{\mathsf{S}_{\ell-1}})\right)}.$$
(38)

#### B. Quantum-State Channels

For each channel-ensemble-measurement configuration  $(\mathcal{N}, \{\rho_A^{(x)}\}_{x \in \mathcal{X}}, \{\Lambda_B^{(y)}\}_{y \in \mathcal{Y}})$  as introduced above, one ends up with a joint conditional pmf, as in (37). However, this relationship is not bijective. In particular, consider unitary operators  $U_A$ 

<sup>4</sup> More generally, for FSMSs, this statement also holds by conditioning on all previous inputs.

and  $U_B$  acting on  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. The following Such an  $\mathcal{N}$  can be constructed as setup induces exactly the same joint conditional pmf:

$$\begin{split} \tilde{\mathcal{N}} &: \tilde{\rho}_{\mathsf{AS}} \mapsto (U_{\mathsf{B}} \otimes I_{\mathsf{S}}) \cdot \mathcal{N} \left( (U_{\mathsf{A}} \otimes I_{\mathsf{S}}) \tilde{\rho}_{\mathsf{AS}} (U_{\mathsf{A}}^{\dagger} \otimes I_{\mathsf{S}}) \right) \cdot (U_{\mathsf{B}}^{\dagger} \otimes I_{\mathsf{S}}), \\ \tilde{\rho}_{\mathsf{A}}^{(x)} &\triangleq U_{\mathsf{A}}^{\dagger} \cdot \rho_{\mathsf{A}}^{(x)} \cdot U_{\mathsf{A}} \quad \forall x \in \mathcal{X}, \\ \tilde{\Lambda}_{\mathsf{B}}^{(y)} &\triangleq U_{\mathsf{B}}^{\dagger} \cdot \Lambda_{\mathsf{B}}^{(y)} \cdot U_{\mathsf{B}} \quad \forall y \in \mathcal{Y}. \end{split}$$

Such redundancy is not only tedious, but also detrimental when we try to compare different channels; in particular, when we try to introduce proper auxiliary channels to approximate the original communication scheme.

In this subsection, we introduce a class of channels called quantum-state channels to eliminate such redundancies. In particular, notice that the statistical behavior of the aforementioned communication scheme is fully specified via (35) and (36); which are in turn determined by the set of completely positive mappings  $\{\mathcal{N}^{y|x}\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$  defined as

$$\mathcal{N}^{y|x}: \rho_{\mathsf{S}} \mapsto \operatorname{tr}_{\mathsf{B}} \left( (\Lambda_{\mathsf{B}}^{(y)} \otimes I_{\mathsf{S}}) \cdot \mathcal{N}(\rho_{\mathsf{A}}^{(x)} \otimes \rho_{\mathsf{S}}) \right).$$
(39)

In this case, (35), (36), and (37) can be rewritten, respectively, as

$$P_{\mathsf{Y}|\mathsf{X};\mathsf{S}}(y|x;\rho_{\mathsf{S}}) = \operatorname{tr}\left(\mathcal{N}^{y|x}(\rho_{\mathsf{S}})\right),\tag{40}$$

$$\rho_{\mathsf{S}'} = \mathcal{N}^{y|x}(\rho_{\mathsf{S}}) / \operatorname{tr}\left(\mathcal{N}^{y|x}(\rho_{\mathsf{S}})\right), \quad (41)$$

$$P_{\mathsf{Y}_1^n|\mathsf{X}_1^n;\mathsf{S}_0}(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n;\rho_{\mathsf{S}_0}) = \operatorname{tr}\left(\mathcal{N}^{y_n|x_n} \circ \cdots \circ \mathcal{N}^{y_1|x_1}(\rho_{\mathsf{S}_0})\right). (42)$$

Thus, the operators  $\{\mathcal{N}^{y|x}\}_{x\in\mathcal{X},y\in\mathcal{Y}}$  fully specify the joint conditional pmf as in (42). Moreover, such specification is also unique; namely, any two sets of channel-ensemblemeasurement configuration shall end up with the same joint channel law if and only if the mappings defined in (39) are identical. This inspires us to make the following definition.

Definition 3 (Quantum-State Channel). A (finite indexed) set of completely positive operators  $\{\mathcal{N}^{y|x}\}_{x\in\mathcal{X},y\in\mathcal{Y}}$  (acting on the same Hilbert space) is said to be a (classicalinput classical-output) quantum-state channel (CC-QSC) if  $\sum_{y \in \mathcal{Y}} \mathcal{N}^{y|x}$  is trace-preserving for each  $x \in \mathcal{X}$ . 

Given any channel-ensemble-measurement configuration as described in Section III-A, one can always define a corresponding CC-QSC by (39). On the other hand, as stated in the proposition below, the converse is also true.

**Proposition 4.** For any CC-QSC  $\{\mathcal{N}^{y|x}\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$ , there exists some quantum channel with memory  $\mathcal{N}$  as in (31) such that (39) holds with ensemble  $\{\rho_{\mathsf{A}}^{(x)} = |x\rangle\langle x|\}_{x\in\mathcal{X}}$  and measurement  $\{\Lambda_{\mathsf{B}}^{(y)} = |y\rangle\langle y|\}_{y\in\mathcal{Y}}$ . Here,  $\mathcal{H}_{\mathsf{A}}$  and  $\mathcal{H}_{\mathsf{B}}$  are defined such that  $\{|x\rangle\}_x$  and  $\{|y\rangle\}_y$  are orthonormal bases of  $\mathcal{H}_{\mathsf{A}}$  and  $\mathcal{H}_{\mathsf{B}}$ , respectively.

*Proof.* It suffices to show that there exists a CPTP map  $\mathcal{N}$ :  $\mathfrak{D}(\mathcal{H}_{\mathsf{A}} \otimes \mathcal{H}_{\mathsf{S}}) \to \mathfrak{D}(\mathcal{H}_{\mathsf{B}} \otimes \mathcal{H}_{\mathsf{S}})$  such that for all  $\rho_{\mathsf{S}} \in \mathfrak{D}(\mathcal{H}_{\mathsf{S}})$ , and  $x \in \mathcal{X}$ ,

$$\mathcal{N}: |x\rangle\langle x|\otimes 
ho_{\mathsf{S}}\mapsto \sum_{y\in\mathcal{Y}}|y\rangle\langle y|\otimes \mathcal{N}^{y|x}(
ho_{\mathsf{S}}).$$

$$\mathcal{N}: \rho \mapsto \sum_{x,y,k} \left( |y\rangle \langle x| \otimes E_k^{y|x} \right) \cdot \rho \cdot \left( |y\rangle \langle x| \otimes E_k^{y|x} \right)^{\dagger},$$

where  $\left\{ E_{k}^{y|x} \right\}_{\iota}$  is a Kraus representation of  $\mathcal{N}^{y|x}$ , namely,

$$\mathcal{N}^{y|x}(\rho_{\mathsf{S}}) \equiv \sum_{k} E_{k}^{y|x} \cdot \rho_{\mathsf{S}} \cdot (E_{k}^{y|x})^{\dagger} \qquad \forall \rho_{\mathsf{S}} \in \mathfrak{D}(\mathcal{H}_{\mathsf{S}}).$$

It remains to check if  $\mathcal{N}$  is a CPTP, which is indeed the case:

$$\sum_{x,y,k} \left( |y\rangle \langle x| \otimes E_k^{y|x} \right)^{\dagger} \cdot \left( |y\rangle \langle x| \otimes E_k^{y|x} \right)$$
$$= \sum_x \sum_{y,k} |x\rangle \langle x| \otimes (E_k^{y|x})^{\dagger} E_k^{y|x} = \sum_x |x\rangle \langle x| \otimes I = I. \quad \Box$$

# C. Visualization using Normal Factor Graphs

In this subsection, we focus on the computations of (40), (41), and (42) for the situation where the involved channel  $\mathcal N$  is of finite dimension. In analogy to the FSMCs, we demonstrate how to use NFGs to facilitate and visualize the relevant computations. Our use of NFGs for describing quantum systems follows [17].

By Proposition 4, let us consider a CC-QSC  $\{\mathcal{N}^{y|x}\}_{x\in\mathcal{X},y\in\mathcal{Y}}$  acting on  $\mathcal{H}_{S}$ , where  $d = \dim(\mathcal{H}_{S})$ is finite, and  $\{|s\rangle\}_{s\in\mathcal{S}}$  is an orthonormal basis of  $\mathcal{H}_{S}$ . (Apparently,  $|\mathcal{S}| = d$ .) Since for each x and y,  $\mathcal{N}^{y|x}$  is a completely positive map, there must exist finitely many (not necessarily unique) matrices  $\{F_k^{y|x} \in \mathbb{C}^{S \times S}\}_k$  such that

$$\left[\mathcal{N}^{y|x}(\rho_{\mathsf{S}})\right] \equiv \sum_{k} F_{k}^{y|x} \cdot \left[\rho_{\mathsf{S}}\right] \cdot (F_{k}^{y|x})^{\dagger} \quad \forall \rho_{\mathsf{S}} \in \mathfrak{D}(\mathcal{H}_{\mathsf{S}}), \ (43)$$

where  $\left[\mathcal{N}^{y|x}(\rho_{\mathsf{S}})\right]$  and  $\left[\rho_{\mathsf{S}}\right]$  are, respectively, the matrix representation of the operator  $\mathcal{N}^{y|x}(\rho_{\mathsf{S}})$  and  $\rho_{\mathsf{S}}$  under  $\{|s\rangle\}_{s\in\mathcal{S}}$ . The reason for such matrices  $\{F_k^{y|x}\}_k$  to exist is the same as for the Kraus operators of CPTP maps (see [6, Theorems 8.1 and 8.3]). Also note that  $\sum_{y \in \mathcal{Y}} \mathcal{E}^{y|x}$  is trace-preserving, thus it must hold that

$$\sum_{y \in \mathcal{Y}} \sum_{k} (F_k^{y|x})^{\dagger} F_k^{y|x} = I \quad \forall x \in \mathcal{X}.$$
(44)

Now, define a set of functions  $\{W^{y|x}\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$  as

$$W^{y|x}: (s', s, \tilde{s}', \tilde{s}) \mapsto \sum_{k} F_{k}^{y|x}(s', s) \overline{F_{k}^{y|x}(\tilde{s}', \tilde{s})}, \qquad (45)$$

where  $s', s, \tilde{s}', \tilde{s} \in \mathcal{S}$  are indices of the corresponding matrices, namely,  $F_k^{y|x}(s',s)$  is the (s',s)-th entry of matrix  $F_k^{y|x}$ . In this case, one can rewrite (40), (41) and (42), respectively, into

$$P_{\mathsf{Y}|\mathsf{X};\mathsf{S}}(y|x;\rho_{\mathsf{S}}) = \sum_{\substack{s',\tilde{s}':\\s'=\tilde{s}'}} \sum_{s,\tilde{s}} W^{y|x}(s',s,\tilde{s}',\tilde{s}) \cdot [\rho_{\mathsf{S}}]_{s,\tilde{s}}, \quad (46)$$

$$[\rho_{\mathsf{S}'}]_{s',\tilde{s}'} = \frac{\sum_{s,\tilde{s}} W^{y|x}(s',s,\tilde{s}',\tilde{s}) \cdot [\rho_{\mathsf{S}}]_{s,\tilde{s}}}{\sum_{s'=\tilde{s}'} \sum_{s,\tilde{s}} W^{y|x}(s',s,\tilde{s}',\tilde{s}) \cdot [\rho_{\mathsf{S}}]_{s,\tilde{s}}}, \quad (47)$$

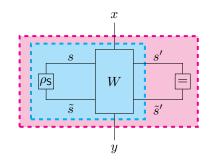


Fig. 5. Representation of  $\{W^{y|x}\}_{x,y}$  using an NFG.

$$P_{\mathsf{Y}_{1}^{n}|\mathsf{X}_{1}^{n};\mathsf{S}_{0}}(\boldsymbol{y}_{1}^{n}|\boldsymbol{x}_{1}^{n};\rho_{\mathsf{S}_{0}}) = \sum_{\substack{s_{n},\tilde{s}_{n}:\\s_{n}=\tilde{s}_{n}}} \sum_{\boldsymbol{s}_{0}^{n-1},\tilde{\boldsymbol{s}}_{0}^{n-1}} [\rho_{\mathsf{S}_{0}}]_{s_{0},\tilde{s}_{0}} \cdot \prod_{\ell=1}^{n} W^{y_{\ell}|x_{\ell}}(s_{\ell},s_{\ell-1},\tilde{s}_{\ell},\tilde{s}_{\ell-1}).$$

$$(48)$$

By rearranging the entries of  $W^{y|x}$  (for each x, y) into a matrix  $[W^{y|x}] \in \mathbb{C}^{S^2 \times S^2}$  as

$$[W^{y|x}]_{(s',\tilde{s}'),(s,\tilde{s})} \triangleq W^{y|x}(s',s,\tilde{s}',\tilde{s}), \tag{49}$$

where  $(s', \tilde{s}') \in S^2$  is the first index, and  $(s, \tilde{s}) \in S^2$  is the second index of  $[W^{y|x}]$ , we can simplify (46), (47), and (48) as

$$P_{\mathsf{Y}|\mathsf{X};\mathsf{S}}(y|x;\rho_{\mathsf{S}}) = \operatorname{tr}([W^{y|x}] \cdot [\rho_{\mathsf{S}}]),$$
(50)

$$[\rho_{\mathsf{S}'}] = \frac{[W^{y|x}] \cdot [\rho_{\mathsf{S}}]}{\operatorname{tr}([W^{y|x}] \cdot [\rho_{\mathsf{S}}])},\tag{51}$$

$$P_{\mathsf{Y}_{1}^{n}|\mathsf{X}_{1}^{n};\mathsf{S}_{0}}(\boldsymbol{y}_{1}^{n}|\boldsymbol{x}_{1}^{n};\rho_{\mathsf{S}_{0}}) = \mathrm{tr}\Big([W^{y_{n}|x_{n}}]\cdots[W^{y_{1}|x_{1}}]\cdot[\rho_{\mathsf{S}_{0}}]\Big),(52)$$

respectively. Here we treat  $[\rho_{S}]$  as a length- $d^{2}$  vector indexed by  $(s, \tilde{s}) \in S^2$  in the above equations.

By considering  $\{W^{y|x}\}_{x,y}$  as a function of six variables, we can represent it using a factor node of degree six in an NFG as in Fig. 5. In this case, Eqs. (46) and (50) can be visualized as "closing the outer box" in the NFG, *i.e.*, summing over all the variables represented by the edges interior to the box. Similarly, (47) and (51) can be visualized as "closing the inner box". The NFG corresponding to using the channel n times consecutively is depicted in Fig. 6, where (48) and (52) are visualized as closing the outermost box. Interestingly, this closing-the-box operation can be carried out by a sequence of simpler closing-the-box operations as shown in the figure.

A number of statistical quantities and density operators of interest can be computed and visualized as closing-the-box operations on suitable NFGs similar to that of Fig. 6. The following example highlights how quantities of this kind can be computed in such a manner.

Example 5 (BCJR [32] decoding for CC-QSCs). For fixed  $\check{m{y}}_1^n \in \mathcal{Y}^n$  and a given initial density operator  $ho_{\mathsf{S}_0}$ , the conditional probability  $P_{X_{\ell}|Y_1^n;S_0}(x_{\ell}|\check{y}_1^n;\rho_{S_0})$  can be computed via

$$P_{\mathsf{X}_{\ell}|\mathsf{Y}_{1}^{n};\mathsf{S}_{0}}(\cdot|\check{\boldsymbol{y}}_{1}^{n};\rho_{\mathsf{S}_{0}}) \propto P_{\mathsf{X}_{\ell},\mathsf{Y}_{1}^{n}|\mathsf{S}_{0}}(\cdot,\check{\boldsymbol{y}}_{1}^{n}|\rho_{\mathsf{S}_{0}}), \qquad (53)$$

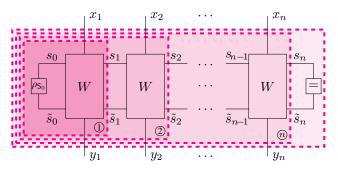


Fig. 6. The joint channel law (48) and (52) can be visualized as the result of the "closing of the outermost box" above, which can in turn be carried out by a sequence of "closing-the-box" operations as indicated.

where the right-hand side of (53) is a marginal pmf defined as

$$P_{\mathsf{X}_{\ell},\mathsf{Y}_{1}^{n}|\mathsf{S}_{0}}(x_{\ell},\check{\boldsymbol{y}}_{1}^{n}|\rho_{\mathsf{S}_{0}}) = \sum_{\boldsymbol{x}_{1}^{\ell-1},\boldsymbol{x}_{\ell+1}^{n}} \sum_{\boldsymbol{s}_{0}^{n},\check{\boldsymbol{s}}_{0}^{n}} [\rho_{\mathsf{S}_{0}}]_{\boldsymbol{s}_{0},\check{\boldsymbol{s}}_{0}} \cdot \prod_{i=1}^{n} Q(x_{i}) \cdot \prod_{j=1}^{n} W^{\check{y}_{j}|x_{j}}(s_{j},s_{j-1},\tilde{s}_{j},\tilde{s}_{j-1}),$$
(54)

where we have assumed that the input process  $X_1^n$  is i.i.d. characterized by some pmf Q. The evaluation of (54) can be carried out efficiently using a sequence of closing-the-box operations as visualized in Fig. 7. These operations can be roughly divided into the following three steps.

- 1) Closing the left inner box: this results in an operator
- crossing and σ<sup>(y<sub>l</sub>-1)</sup> on H<sub>S<sub>l</sub>-1</sub>.
   Closing the right inner box: this results in another operator σ<sup>(y<sub>l</sub>+1)</sup> on H<sub>S<sub>l</sub></sub>.
- box: the result is the marginal pmf  $P_{X_{\ell},Y_1^n|S_0}(x_{\ell},\check{y}_1^n|\rho_{S_0})$ , from which the desired conditional probability can be easily obtained by  $P_{\mathsf{X}_{\ell}|\mathsf{Y}_{1}^{n};\mathsf{S}_{0}}(x_{\ell}|\check{\boldsymbol{y}}_{1}^{n};\rho_{\mathsf{S}_{0}})$ normalization.

The operators mentioned in 1) and 2) can be computed recursively, using a sequence of closing-the-box operations. Namely, one can carry out the computations in 1) consecutively with  $\ell = 1, 2, ..., n$ ; or the computations in 2) consecutively with  $\ell = n, n-1, \dots, 1$ . This provides an efficient way to evaluate  $P_{X_{\ell}|Y_1^n;S_0}(x_{\ell}|\check{y}_1^n;\rho_{S_0})$  for each  $\ell = 1,\ldots,n$ ; and thus provides an efficient symbol-wise decoding algorithm. The idea in this example is conceptually identical to that of the BCJR decoding algorithm for an FSMC. 

As shown in the above example, very often the desired functions or quantities are based on the same partial results. The NFG framework is very helpful to visualize these partial results and to show how they can be combined to obtain the desired functions and quantities.

We emphasize that the functions  $\{W^{y|x}\}_{x,y}$  defined in (45) are unique for a given finite-dimensional CC-QSC  $\{\mathcal{N}^{y|x}\}_{x,y}$ ; even though such uniqueness does not apply to the Kraus operators  $\{F^{y|x}\}_k$  being used to define  $\{W^{y|x}\}_{x,y}$ . This can be proven by making the identification that

$$\left[\mathcal{N}^{y|x}(\rho_{\mathsf{S}})\right] \equiv \left[W^{y|x}\right] \cdot \left[\rho_{\mathsf{S}}\right] \qquad \forall \rho_{\mathsf{S}} \in \mathfrak{D}(\mathcal{H}_{\mathsf{S}}), \tag{55}$$

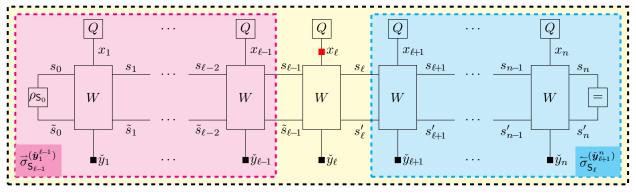


Fig. 7. Computation of the marginal pmf  $P_{X_{\ell}|Y_1^n;S_0}$  using a sequence of closing-the-box operations.

for all x and y. Moreover, we argue that the functions  $\{W^{y|x}\}_{x,y}$ , are an *equivalent* way to specify a CC-QSC, or classical communication over a quantum channel with memory as described at the beginning of this section. Namely, for any set of complex-valued functions  $\{W^{y|x}\}_{x,y}$  on  $S^4$  satisfying some constraints to be clarified later, there must exist a unique CC-QSC  $\{\mathcal{N}^{y|x}\}_{x,y}$  such that (55) holds; and thus, there must exist some corresponding channel-ensemble-measurement configuration, unique up to its channel law. As for such constraints, we rearrange the entries of  $W^{y|x}$  (for each x, y) into another matrix  $[W^{y|x}] \in \mathbb{C}^{S^2 \times S^2}$  (a.k.a. Choi–Jamiołkowski matrix [33]), whose entries are defined as

$$\llbracket W^{y|x} \rrbracket_{(s',s),(\tilde{s}',\tilde{s})} \triangleq W^{y|x}(s',s,\tilde{s}',\tilde{s}), \tag{56}$$

where  $(s',s) \in S^2$  is the first index, and  $(\tilde{s}',\tilde{s}) \in S^2$  is the second index of  $[\![W^{y|x}]\!]$ . Notice that,  $[\![W^{y|x}]\!]$  is a positive semi-definite (p.s.d.) matrix, and it satisfies the following equation

$$\sum_{y \in \mathcal{Y}} \sum_{s', \tilde{s}': s' = \tilde{s}'} \llbracket W^{y|x} \rrbracket_{(s',s), (\tilde{s}', \tilde{s})} = \delta_{s, \tilde{s}} \quad \forall x \in \mathcal{X},$$
(57)

where  $\delta_{s,\tilde{s}}$  is the Kronecker-delta function. In this case, the "equivalence" can be shown by the following proposition.

**Proposition 6.** Let  $\mathcal{X}$ ,  $\mathcal{Y}$  be finite sets, and  $\mathcal{H}_{S}$  be a finite-dimensional Hilbert space with an orthonormal basis  $\{|s\rangle\}_{s\in\mathcal{S}}$ . For any set of functions

$$\{W^{y|x}: \mathcal{S} \times \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to \mathbb{C}\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$$

such that their matrix form  $\{\llbracket W^{y|x} \rrbracket\}_{x,y}$  consists of p.s.d. matrices and satisfies (57), there must exist a unique CC-QSC  $\{\mathcal{N}^{y|x}\}_{x,y}$  acting on  $\mathcal{H}_{\mathsf{S}}$  such that (55) holds.

*Proof.* The idea of the proof is to consider the eigenvalue decomposition of  $[W^{y|x}]$ , and reconstruct  $\mathcal{N}^{y|x}$  by following the equations (45) and (43) backwardly. We omit the details here.

Let us conclude this section by pointing out that the functions  $\{W^{y|x}\}_{x,y}$ , particularly the corresponding NFG, can be constructed from the channel-ensemble-measurement configuration  $(\mathcal{N}, \{\rho_A^{(x)}\}_{x \in \mathcal{X}}, \{\Lambda_B^{(y)}\}_{y \in \mathcal{Y}})$  as in Fig. 8. This can be justified by checking (39) and (55).

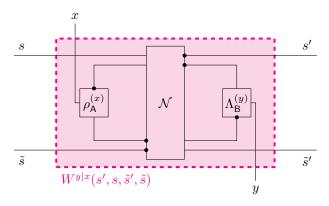


Fig. 8. NFG representation of the channel-ensemble-measurement configuration  $(\mathcal{N}, \{\rho_A^{(x)}\}_{x \in \mathcal{X}}, \{\Lambda_B^{(y)}\}_{y \in \mathcal{Y}}).$ 

## IV. INFORMATION RATE AND ITS ESTIMATION

In this section, we focus on the information rate of the communication scheme described in Section III. As defined in (1), the information rate is the limit superior of the average mutual information  $\frac{1}{n}\mathbf{I}(X_1^n; \mathbf{Y}_1^n)$  between the input and output processes  $X_1^n$  and  $\mathbf{Y}_1^n$  as *n* tends to infinity. We assume that  $X_1^n$  is distributed according to some i.i.d. process<sup>5</sup> characterized by the pmf Q, *i.e.*,  $Q^{(n)}(\boldsymbol{x}_1^n) = \prod_{\ell=1}^n Q(x_\ell)$ . In this case, the joint distribution of  $(X_1^n, \mathbf{Y}_1^n)$  is given by

$$P_{\mathsf{X}_{1}^{n},\mathsf{Y}_{1}^{n}|\mathsf{S}_{0}}(\boldsymbol{x}_{1}^{n},\boldsymbol{y}_{1}^{n}|\rho_{\mathsf{S}_{0}}) = \prod_{\ell=1}^{n} Q(x_{\ell}) P_{\mathsf{Y}_{1}^{n}|\mathsf{X}_{1}^{n};\mathsf{S}_{0}}(\boldsymbol{y}_{1}^{n}|\boldsymbol{x}_{1}^{n};\rho_{\mathsf{S}_{0}}) (58)$$

where  $P_{\mathbf{Y}_1^n|\mathbf{X}_1^n;\mathbf{S}_0}$  is specified in (37), (42), (48) or (52), depending on which notation we use to specify the channel (see Propositions 4 and 6). It is obvious that the value of (58), and thus the information rate, depends on the initial density operator  $\rho_{\mathbf{S}_0}$ . In this sense, we denote the information rate as a function of the input pmf Q, the CC-QSC  $\{\mathcal{N}^{y|x}\}_{x,y}$ describing the channel, and the initial density operator  $\rho_{\mathbf{S}_0}$ , namely

$$I(Q, \{\mathcal{N}^{y|x}\}_{x,y}, \rho_{\mathsf{S}_0}) \triangleq \limsup_{n \to \infty} I^{(n)}(Q, \{\mathcal{N}^{y|x}\}_{x,y}, \rho_{\mathsf{S}_0}), (59)$$
$$I^{(n)}(Q, \{\mathcal{N}^{y|x}\}_{x,y}, \rho_{\mathsf{S}_0}) \triangleq \frac{1}{-1} I(\mathsf{X}_1^n; \mathsf{Y}_1^n)(\rho_{\mathsf{S}_0}). \tag{60}$$

<sup>5</sup>For more general type of sources, like a finite-state-machine source (FSMS), one can consider "merging" the memory of the source into that of the channel, and thus obtaining an equivalent memoryless input process.

Here,  $\mathbf{I}(X_1^n; Y_1^n)(\rho_{S_0})$  is the mutual information between  $X_1^n$  and  $Y_1^n$ ; and the latter are jointly distributed according to (58). The argument  $\rho_{S_0}$  emphasizes the dependency of  $\frac{1}{n}\mathbf{I}(X_1^n; Y_1^n)$  on  $\rho_{S_0}$ .

Similar to the case of an FSMC, the dependency of the information rate on the initial density operator usually cannot be ignored. However, as already mentioned in Section II-B, for a class of FSMCs, namely the indecomposable FSMCs, it is known that the information rate is independent from the initial channel state [12]. An indecomposable FSMC, intuitively speaking, is an FSMC whose state distribution, given different initial states, tends to be indistinguishable as  $n \rightarrow \infty$ , independently of the input sequence realized. A quantum analogy was proposed by Bowen, Devetak, and Mancini [34], where they defined the indecomposable quantum channels with memory, and proved that the quantum entropic bound for such channels is independent from the initial density operator.

In the remainder of this section we firstly define the indecomposability of CC-QSCs, and prove the independence of the information rate as in (59) from the initial density operator. Secondly, we generalize the methods in Algorithm 2 for estimating such information rates efficiently.

The definition of an indecomposable CC-QSC in our paper is similar (but different) and closely related to that of an indecomposable (quantum) channel with memory in [34]. Namely, an indecomposable channel with memory equipped with separable input ensemble and local output measurement will always induce an indecomposable CC-QSC, but not necessarily vice versa. Moreover, in [34] the classical capacity of quantum channels with finite memory was considered, where the capacity is essentially the Holevo bound, and where the latter was proven to be achievable [3]. However, in our work, we focus on the situation where the ensemble and the measurement are fixed.

### A. Indecomposable Quantum-State Channel

**Definition 7.** A CC-QSC  $\{\mathcal{N}^{y|x}\}_{x,y}$  is said to be *indecomposable* if for any initial density operators  $\alpha_{S_0}$  and  $\beta_{S_0}$ , the following statement holds: for any  $\epsilon > 0$ , there exists some positive integer N s.t.

$$\left\| \alpha_{\mathsf{S}_n}^{(\boldsymbol{x}_1^n)} - \beta_{\mathsf{S}_n}^{(\boldsymbol{x}_1^n)} \right\|_1 < \varepsilon \quad \forall n \ge N, \ \forall \boldsymbol{x}_1^n \in \mathcal{X}^n, \tag{61}$$

where

$$\alpha_{\mathsf{S}_n}^{(\boldsymbol{x}_1^n)} \triangleq \sum_{\boldsymbol{y}_1^n} \mathcal{N}^{y_n | x_n} \circ \cdots \circ \mathcal{N}^{y_1 | x_1}(\alpha_{\mathsf{S}_0}), \qquad (62)$$

$$\beta_{\mathsf{S}_{n}}^{(\boldsymbol{x}_{1}^{n})} \triangleq \sum_{\boldsymbol{y}_{1}^{n}} \mathcal{N}^{y_{n}|x_{n}} \circ \cdots \circ \mathcal{N}^{y_{1}|x_{1}}(\beta_{\mathsf{S}_{0}}), \tag{63}$$

and where  $||A||_1$  is the trace distance for an operator A on  $\mathcal{H}_{\mathsf{S}}$ , *i.e.*,  $||A||_1 \triangleq \frac{1}{2} \operatorname{tr} \sqrt{A^{\dagger} A}$ .

**Theorem 8.**<sup>6</sup> The information rate of an indecomposable CC-QSC with an i.i.d. input process is independent of the initial density operator. Namely, if  $\{\mathcal{N}^{y|x}\}_{x,y}$  is indecomposable, then

$$\mathbf{I}^{(n)}(Q, \{\mathcal{N}^{y|x}\}_{x,y}, \alpha_{\mathsf{S}_0}) - \mathbf{I}^{(n)}(Q, \{\mathcal{N}^{y|x}\}_{x,y}, \beta_{\mathsf{S}_0}) \xrightarrow{n \to \infty} 0,$$

for any initial density operators  $\alpha_{S_0}, \ \beta_{S_0} \in \mathfrak{D}(\mathcal{H}_{S_0}).$ 

In the proof below, we follow a similar idea as in [12] for indecomposable FSMCs, and as that in [34] for indecomposable quantum channels with memory.

*Proof.* Let A and B be quantum systems described by Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively, where  $\{|x\rangle\}_{x\in\mathcal{X}}$  and  $\{|y\rangle\}_{y\in\mathcal{Y}}$  are orthonormal bases of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. Let  $A_1^n$  and  $B_1^n$  be *n* copies of A and B, respectively. Let  $\rho_{S_0}$  be some initial density operator; and let the joint density operator on system  $A_1^n B_1^n$  be

$$ho_{\mathsf{A}_1^n\mathsf{B}_1^n} \stackrel{ riangle}{=} \sum_{oldsymbol{x}_1^n} Q(oldsymbol{x}_1^n) \cdot |oldsymbol{x}_1^n| \otimes \sum_{oldsymbol{y}_1^n} \operatorname{tr}\left(\mathcal{N}^{oldsymbol{y}_1^n}|oldsymbol{x}_1^n(
ho_{\mathsf{S}_0})
ight) \cdot |oldsymbol{y}_1^n\rangle \langle oldsymbol{y}_1^n|$$

where  $\mathcal{N} \mathbf{y}_1^n | \mathbf{x}_1^n \triangleq \mathcal{N}^{y_n | x_n} \circ \cdots \circ \mathcal{N}^{y_1 | x_1}$ . In this case, it is not hard to see that

$$\mathbf{I}(\mathsf{X}_1^n;\mathsf{Y}_1^n)[\rho_{\mathsf{S}_0}] = \mathbf{I}(\mathsf{A}_1^n;\mathsf{B}_1^n)[\rho_{\mathsf{S}_0}].$$

In fact, one can easily check that

$$\begin{split} \mathbf{H}(\mathsf{A}_1^n) &= \mathbf{H}(\mathsf{X}_1^n),\\ \mathbf{H}(\mathsf{B}_1^n) &= \mathbf{H}(\mathsf{Y}_1^n),\\ \mathbf{H}(\mathsf{A}_1^n,\mathsf{B}_1^n) &= \mathbf{H}(\mathsf{X}_1^n,\mathsf{Y}_1^n). \end{split}$$

In particular,  $\mathbf{H}(\mathsf{A}_1^n)$  is independent of the initial density operator  $\rho_{\mathsf{S}_0}$ . We also claim that, for each  $\rho_{\mathsf{S}_0} \in \mathfrak{D}(\mathcal{H}_{\mathsf{S}_0})$ and positive integer N < n,

$$\mathbf{I}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N};\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}) \leqslant 2\mathbf{H}(\mathsf{S}_{N}), \tag{64}$$

$$\mathbf{I}(\mathsf{B}_1^N;\mathsf{B}_{N+1}^n) \leqslant 2\mathbf{H}(\mathsf{S}_N), \tag{65}$$

where the density operator for  $S_N$  is defined as (depending on  $\rho_{S_0}$ )

$$ho_{\mathsf{S}_N} riangleq \sum_{m{x}_1^N} Q(m{x}_1^N) \cdot \sum_{m{y}_1^N} \mathcal{N}^{m{y}_1^N | m{x}_1^N}(
ho_{\mathsf{S}_0}).$$

**Proof of** (64): We define a class of CPTP maps  $\{\Phi_a^b : \mathfrak{D}(\mathcal{H}_{S_a}) \to \mathfrak{D}(\mathcal{H}_{A_a^b} B_a^b)\}_{a < b \in \mathbb{N}}$  as

$$\Phi_{a}^{b}:\rho_{\mathsf{S}_{a}}\mapsto\sum_{\boldsymbol{x}_{a}^{b}}Q(\boldsymbol{x}_{a}^{b})\cdot\big|\boldsymbol{x}_{a}^{b}\big\rangle\!\big\langle\boldsymbol{x}_{a}^{b}\big|\!\otimes\!\sum_{\boldsymbol{y}_{a}^{b}}\mathrm{tr}\left(\!\mathcal{N}^{\boldsymbol{y}_{a}^{b}|\boldsymbol{x}_{a}^{b}}(\rho_{\mathsf{S}_{a}})\right)\cdot\big|\boldsymbol{y}_{a}^{b}\big\rangle\!\big\langle\boldsymbol{y}_{a}^{b}\big|$$

Since the input process Q is i.i.d., we can rewrite  $\rho_{A_1^n B_1^n}$ , for each positive integer N < n, as

$$\rho_{\mathsf{A}_1^n\mathsf{B}_1^n} = \left(I_{\mathsf{A}_1^N\mathsf{B}_1^N}\otimes\Phi_{N+1}^n\right)\left(\rho_{\mathsf{A}_1^N\mathsf{B}_1^N}\mathsf{S}_N\right)$$

where

$$\rho_{\mathsf{A}_1^N\mathsf{B}_1^N\mathsf{S}_N} \stackrel{\text{\tiny{dest}}}{=} \sum_{\boldsymbol{x}_1^N} Q(\boldsymbol{x}_1^N) \left| \boldsymbol{x}_1^N \right\rangle \!\! \left\langle \boldsymbol{x}_1^N \right| \! \otimes \! \sum_{\boldsymbol{y}_1^N} \mathcal{N}^{\boldsymbol{y}_1^N \left| \boldsymbol{x}_1^N}(\rho_{\mathsf{S}_0}) \left| \boldsymbol{y}_1^N \right\rangle \!\! \left\langle \boldsymbol{y}_1^N \right| \!$$

Hence, by data processing inequality for quantum mutual information (see *e.g.*, [7, Theorem 11.9.4]), one must have

$$\mathbf{I}(\mathsf{A}_1^N\mathsf{B}_1^N;\mathsf{A}_{N+1}^n\mathsf{B}_{N+1}^n) \leqslant \mathbf{I}(\mathsf{A}_1^N\mathsf{B}_1^N;\mathsf{S}_N).$$

<sup>&</sup>lt;sup>6</sup> A similar result regarding indecomposable/forgetful quantum channel with memory can be found in [4] and [34].

Additionally, by subadditivity of joint entropy, we have

$$\begin{split} \mathbf{I}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N};\mathsf{S}_{N}) &\triangleq \mathbf{H}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N}) + \mathbf{H}(\mathsf{S}_{N}) - \mathbf{H}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N}\mathsf{S}_{N}) \\ &\leqslant \mathbf{H}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N}) + \mathbf{H}(\mathsf{S}_{N}) - \left|\mathbf{H}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N}) - \mathbf{H}(\mathsf{S}_{N})\right| \\ &\leqslant 2\mathbf{H}(\mathsf{S}_{N}). \end{split}$$

Combining the above two inequalities, we have proven (64).

**Proof of (65):** We follow the same approach as above for proving (64) by defining another class of CPTP maps  $\{\Psi_a^b : \mathfrak{D}(\mathcal{H}_{S_a}) \to \mathfrak{D}(\mathcal{H}_{B_a^b})\}_{a < b \in \mathbb{N}}$  as

$$\Psi_a^b: \rho_{\mathsf{S}_a} \mapsto \sum_{\boldsymbol{x}_a^b} Q(\boldsymbol{x}_a^b) \cdot \sum_{\boldsymbol{y}_a^b} \operatorname{tr} \left( \mathcal{N}^{\boldsymbol{y}_a^b | \boldsymbol{x}_a^b}(\rho_{\mathsf{S}_a}) \right) \cdot | \boldsymbol{y}_a^b \rangle \langle \boldsymbol{y}_a^b | \,.$$

We omit the detailed derivation of (65).

We now return to the main proof. Given the initial density operators  $\alpha_{S_0}$ , and  $\beta_{S_0}$ , we define  $\alpha_{A_1^n B_1^n}$ ,  $\beta_{A_1^n B_1^n}$  and  $\alpha_{S_N}$ ,  $\beta_{S_N}$  in a similar fashion as we have defined  $\rho_{A_1^n B_1^n}$  and  $\rho_{S_N}$ based on  $\rho_{S_0}$ . In this case, one obtains

$$\begin{aligned} \left| \mathbf{H}(\alpha_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}) - \mathbf{H}(\beta_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}) \right| - \left| \mathbf{H}(\alpha_{\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}}) - \mathbf{H}(\beta_{\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}}) \right| \\ & \leqslant \\ \left| \mathbf{H}(\alpha_{\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N}}) - \mathbf{H}(\beta_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}) \right| + \\ \left| \mathbf{I}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N};\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}) [\alpha_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}] \\ - \mathbf{I}(\mathsf{A}_{1}^{N}\mathsf{B}_{1}^{N};\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}) [\beta_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}] \right| \\ & \leqslant \\ N \cdot \log(\dim \mathcal{H}_{\mathsf{A}\mathsf{B}}) + 2 \cdot \max\left\{ \mathbf{H}(\alpha_{\mathsf{S}_{N}}), \mathbf{H}(\beta_{\mathsf{S}_{N}}) \right\}, \ (66) \end{aligned}$$

where we have used the triangle inequality in step (a), and a basic property of von Neumann entropy [6, Theorem 11.8] and (64) in step (b). Similarly, using (65), one can prove

$$\frac{\left|\mathbf{H}(\alpha_{\mathsf{B}_{1}^{n}})-\mathbf{H}(\beta_{\mathsf{B}_{1}^{n}})\right|-\left|\mathbf{H}(\alpha_{\mathsf{B}_{N+1}^{n}})-\mathbf{H}(\beta_{\mathsf{B}_{N+1}^{n}})\right|}{\leqslant N \cdot \log(\dim \mathcal{H}_{\mathsf{B}})+2 \cdot \max\left\{\mathbf{H}(\alpha_{\mathsf{S}_{N}}),\mathbf{H}(\beta_{\mathsf{S}_{N}})\right\}}.$$
(67)

By assumption, there exists some positive integer d such that  $\max\{\dim \mathcal{H}_A, \dim \mathcal{H}_B, \dim \mathcal{H}_S\} \leq d$ . Thus, we have

$$\begin{aligned} \frac{1}{n} |\mathbf{I}(\mathbf{X}_{1}^{n};\mathbf{Y}_{1}^{n})[\alpha_{\mathsf{S}_{0}}] - \mathbf{I}(\mathbf{X}_{1}^{n};\mathbf{Y}_{1}^{n})[\beta_{\mathsf{S}_{0}}]| \\ &= \frac{1}{n} |\mathbf{I}(\mathsf{A}_{1}^{n};\mathsf{B}_{1}^{n})[\alpha_{\mathsf{S}_{0}}] - \mathbf{I}(\mathsf{A}_{1}^{n};\mathsf{B}_{1}^{n})[\beta_{\mathsf{S}_{0}}]| \\ &= \frac{1}{n} |\left(\mathbf{H}(\alpha_{\mathsf{B}_{1}^{n}}) - \mathbf{H}(\alpha_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}})\right) - \left(\mathbf{H}(\beta_{\mathsf{B}_{1}^{n}}) - \mathbf{H}(\beta_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}})\right)| \\ &\leq \frac{1}{n} |\mathbf{H}(\alpha_{\mathsf{B}_{1}^{n}}) - \mathbf{H}(\beta_{\mathsf{B}_{1}^{n}})| + \frac{1}{n} |\mathbf{H}(\alpha_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}}) - \mathbf{H}(\beta_{\mathsf{A}_{1}^{n}\mathsf{B}_{1}^{n}})| \\ &\leq \frac{3N+4}{n} \cdot \log d + \frac{1}{n} |\mathbf{H}(\alpha_{\mathsf{B}_{N+1}^{n}}) - \mathbf{H}(\beta_{\mathsf{B}_{N+1}^{n}})| \\ &\quad + \frac{1}{n} |\mathbf{H}(\alpha_{\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}}) - \mathbf{H}(\beta_{\mathsf{A}_{N+1}^{n}\mathsf{B}_{N+1}^{n}})| \\ &= \frac{3N+4}{n} \cdot \log d + \frac{1}{n} |\mathbf{H}(\Psi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}})) - \mathbf{H}(\Psi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}))| \\ &\quad + \frac{1}{n} |\mathbf{H}(\Phi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}})) - \mathbf{H}(\Phi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}))| \end{aligned}$$

where we have used the triangle inequality in step (c), and [6, Theorem 11.8], (66), (67) in step (d). Using a loose variant of Fannes' inequality  $[35]^7$ , we have

$$\begin{aligned} \left| \mathbf{H}(\Psi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}})) - \mathbf{H}(\Psi_{N+1}^{n}(\beta_{\mathsf{S}_{N}})) \right| &\leq (n-N) \cdot \log d \cdot \\ & \left\| \Psi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}}) - \Psi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}) \right\|_{1} + e^{-1}, \\ \left| \mathbf{H}(\Phi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}})) - \mathbf{H}(\Phi_{N+1}^{n}(\beta_{\mathsf{S}_{N}})) \right| &\leq 2 \cdot (n-N) \cdot \log d \cdot \\ & \left\| \Phi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}}) - \Phi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}) \right\|_{1} + e^{-1}. \end{aligned}$$

Moreover, by the contractivity of the trace distance, we have,

$$\begin{aligned} \left\| \Psi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}}) - \Psi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}) \right\|_{1} &\leq \left\| \alpha_{\mathsf{S}_{N}} - \beta_{\mathsf{S}_{N}} \right\|_{1}, \\ \left\| \Phi_{N+1}^{n}(\alpha_{\mathsf{S}_{N}}) - \Phi_{N+1}^{n}(\beta_{\mathsf{S}_{N}}) \right\|_{1} &\leq \left\| \alpha_{\mathsf{S}_{N}} - \beta_{\mathsf{S}_{N}} \right\|_{1}. \end{aligned}$$

This allows us to bound the difference of the information rates by

$$\frac{1}{n} \left| \mathbf{I}(\mathsf{X}_1^n; \mathsf{Y}_1^n)[\alpha_{\mathsf{S}_0}] - \mathbf{I}(\mathsf{X}_1^n; \mathsf{Y}_1^n)[\beta_{\mathsf{S}_0}] \right| \leq \frac{3N+4}{n} \cdot \log d \\ + \frac{3(n-N)}{n} \cdot \log d \cdot \|\alpha_{\mathsf{S}_n} - \beta_{\mathsf{S}_n}\|_1 + \frac{2}{n \cdot e}.$$

Finally, because the CC-QSC is indecomposable, for any  $\varepsilon > 0$ , we can choose N large enough such that

$$\|\alpha_{\mathsf{S}_N} - \beta_{\mathsf{S}_N}\|_1 < \frac{\varepsilon}{6 \cdot \log d},$$

and then choose an integer M > N such that

$$\frac{3N+4}{M} \cdot \log d + \frac{2}{M \cdot e} < \frac{\varepsilon}{2}$$

This will ensure that for any n > M, we have

$$\frac{3N+4}{n} \cdot \log d + \frac{3(n-N)}{n} \cdot \log d \cdot \|\alpha_{\mathsf{S}_n} - \beta_{\mathsf{S}_n}\|_1 + \frac{2}{n \cdot e} < \varepsilon,$$
which concludes the proof of the theorem

which concludes the proof of the theorem.

## B. Estimation of the Information Rate

The development in this section is very similar to the development in Section II-B. In particular, we follow the same approach as in Eqs. (9)–(16). This similarity stems from the similarity of the NFGs in Figs. 4 and 7, and highlights one of the benefits of the factor-graph approach that we take to estimate information rates of quantum channels with memory. We make the following assumptions.

- As already mentioned, the derivations in this paper are for the case where the input process  $X_1^n = (X_1, ..., X_n)$  is an i.i.d. process. The results can be generalized to other stationary ergodic input processes that can be represented by a finite-state-machine source (FSMS). Technically, this is done by defining a new state that combines the FSMS state and the channel state.
- We assume that the corresponding quantum-state channel {N<sup>y|x</sup>}<sub>x∈X,y∈Y</sub> is finite-dimensional and indecomposable. We also assume it can be represented by some functions {W<sup>y|x</sup>}<sub>x,y</sub> as defined in (45).

The major difference compared with Section II-B is the conditional pmf  $P_{Y_1^n|X_1^n|S_0}$ , and thus the joint pmf  $P_{Y_1^n,X_1^n|S_0}$  as

<sup>&</sup>lt;sup>7</sup> Namely, we used the inequality  $|\mathbf{H}(\rho) - \mathbf{H}(\sigma)| \leq \log \dim \|\rho - \sigma\|_1 + e^{-1}$ . Note that tighter variants of Fannes' inequality exist, but the above inequality is good enough to prove the desired result.

specified in (52) and (58), respectively. In this case, in order to compute  $-\frac{1}{n}\log P_{Y_1^n}(\check{y}_1^n)$  and  $-\frac{1}{n}\log P_{X_1^nY_1^n}(\check{x}_1^n,\check{y}_1^n)$  using a similar method as in Section II-B, we consider the state metrics  $\{\sigma_{\ell}^{\mathsf{Y}}\}_{\ell=1}^n$  and  $\{\sigma_{\ell}^{\mathsf{XY}}\}_{\ell=1}^n$  (which are operators on  $\mathcal{H}_{\mathsf{S}_{\ell}}$  for each  $\ell$ ) defined w.r.t.  $\check{y}_1^n$  and w.r.t.  $\check{x}_1^n$  and  $\check{y}_1^n$ , respectively, as

$$\sigma_{\ell}^{\mathsf{Y}} \triangleq \sum_{\boldsymbol{x}_{1}^{\ell}} Q^{(\ell)}(\boldsymbol{x}_{1}^{\ell}) \cdot \mathcal{N}^{\check{y}_{n}|x_{n}} \circ \cdots \circ \mathcal{N}^{\check{y}_{1}|x_{1}}(\rho_{\mathsf{S}_{0}}), \quad (68)$$

$$\sigma_{\ell}^{\mathsf{XY}} \triangleq \mathcal{N}^{\check{y}_n | \check{x}_n} \circ \dots \circ \mathcal{N}^{\check{y}_1 | \check{x}_1}(\rho_{\mathsf{S}_0}).$$
(69)

In this case, we have  $P_{\mathsf{Y}_1^n}(\check{\boldsymbol{y}}_1^n) = \operatorname{tr}(\sigma_n^{\mathsf{Y}})$ , and  $P_{\mathsf{X}_1^n\mathsf{Y}_1^n}(\check{\boldsymbol{x}}_1^n,\check{\boldsymbol{y}}_1^n) = \operatorname{tr}(\sigma_n^{\mathsf{X}\mathsf{Y}})$ . Notice that  $\{\sigma_\ell^{\mathsf{Y}}\}_\ell$  and  $\{\sigma_\ell^{\mathsf{X}\mathsf{Y}}\}_\ell$  can be computed iteratively as

$$[\sigma_{\ell}^{\mathsf{Y}}] = \sum_{x_{\ell}} Q(x_{\ell}) \cdot [W^{y|x}] \cdot [\sigma_{\ell-1}^{\mathsf{Y}}], \tag{70}$$

$$[\sigma_{\ell}^{\mathsf{X}\mathsf{Y}}] = [W^{y|x}] \cdot [\sigma_{\ell-1}^{\mathsf{X}\mathsf{Y}}],\tag{71}$$

where we treat  $[\sigma_{\ell}^{\mathsf{Y}}]$  and  $[\sigma_{\ell}^{\mathsf{X}\mathsf{Y}}]$  as length- $d^2$  vectors indexed by  $(s, \tilde{s}) \in S^2$  in the above two equations. (See (49) and (52) for notations.) Moreover, we can also introduce normalizing coefficients  $\{\lambda_{\ell}^{\mathsf{Y}}\}_{\ell}$  and  $\{\lambda_{\ell}^{\mathsf{X}\mathsf{Y}}\}_{\ell}$ , similar to (21), for the sake of numerical stability. In the latter case, we have iterative updating rules

$$[\bar{\sigma}_{\ell}^{\mathsf{Y}}] = \frac{1}{\lambda_{\ell}^{\mathsf{Y}}} \cdot \sum_{x_{\ell}} Q(x_{\ell}) \cdot [W^{y|x}] \cdot [\bar{\sigma}_{\ell-1}^{\mathsf{Y}}], \qquad (72)$$

$$[\bar{\sigma}_{\ell}^{\mathsf{X}\mathsf{Y}}] = \frac{1}{\lambda_{\ell}^{\mathsf{X}\mathsf{Y}}} \cdot [W^{y|x}] \cdot [\bar{\sigma}_{\ell-1}^{\mathsf{X}\mathsf{Y}}], \tag{73}$$

where the scaling factors  $\lambda_{\ell}^{\mathsf{Y}} > 0$  and  $\lambda_{\ell}^{\mathsf{X}\mathsf{Y}} > 0$  are chosen such that  $\operatorname{tr}(\bar{\sigma}_{\ell}^{\mathsf{Y}}) = 1$  and  $\operatorname{tr}(\bar{\sigma}_{\ell}^{\mathsf{X}\mathsf{Y}}) = 1$ , respectively. In addition, one can verify that  $P_{\mathsf{Y}_{1}^{n}}(\check{\boldsymbol{y}}_{1}^{n}) = \prod_{\ell=1}^{n} \lambda_{\ell}^{\mathsf{Y}}$ , and  $P_{\mathsf{X}_{1}^{n}\mathsf{Y}_{1}^{n}}(\check{\boldsymbol{x}}_{1}^{n},\check{\boldsymbol{y}}_{1}^{n}) = \prod_{\ell=1}^{n} \lambda_{\ell}^{\mathsf{Y}}$ .

The above discussion is summarized as Algorithm 9. The computations corresponding to Line 3, 5–9 and 12–16 are visualized in Figs. 15, 17, and 19 in the Appendix, respectively.

# V. INFORMATION RATE UPPER/LOWER BOUNDS AND THEIR OPTIMIZATION

In this section, we consider auxiliary channels and their induced upper and lower bounds on the information rate. As already mentioned in the introduction, auxiliary channels are often introduced as a low-complexity approximation of the original channel, which are useful in mismatch decoding. The techniques developed in this section only require the channel input/output data, but not the channel model itself. This is particularly useful when the channel is only made physically, but not mathematically, available. In this case, the task of minimizing the difference between the upper and lower bound is equivalent to finding the channel model (within a specified class of channel models) best fitting the empirical channel law. Similarly, minimizing the upper bound corresponds to finding the channel model best fitting the *empirical* channel output distribution, and maximizing the lower bound corresponds to finding the channel model best fitting the *empirical* reverse channel law. Motivated by the above scenarios, we particularly consider the auxiliary channels chosen from the domain of

Algorithm 9 Estimating the information rate of a CC-QSC

**Input:** indecomposable CC-QSC  $\{\mathcal{N}^{y|x}\}_{x \in \mathcal{X}, y \in \mathcal{Y}}$ , which can be represented by functions  $\{W^{y|x}\}_{x,y}$ , input distribution Q, positive integer n large enough.

**Output:**  $I^{(n)}(Q, {\mathcal{N}^{y|x}}_{x,y}) \approx \mathbf{H}(\mathsf{X}) + \hat{\mathrm{H}}(\mathsf{Y}) - \hat{\mathrm{H}}(\mathsf{X},\mathsf{Y}).$ 

1: Initialize the memory density operator  $\rho_{S_0} \leftarrow |0_S\rangle\langle 0_S|$ 

- 2: Generate an input sequence  $\check{\boldsymbol{x}}_1^n \sim Q^{\otimes n}$
- 3: Generate a corresponding output sequence  $\check{m{y}}_1^n$
- 4:  $\bar{\sigma}_{0}^{\mathbf{Y}} \leftarrow \rho_{\mathbf{S}_{0}}$ 5: **for each**  $\ell = 1, ..., n$  **do** 6:  $[\sigma_{\ell}^{\mathbf{Y}}] \leftarrow \sum_{x_{\ell}} Q(x_{\ell}) \cdot [W^{\check{y}_{\ell}|x}] \cdot [\bar{\sigma}_{\ell-1}^{\mathbf{Y}}]$ 7:  $\lambda_{\ell}^{\mathbf{Y}} \leftarrow \operatorname{tr}(\sigma_{\ell}^{\mathbf{Y}})$ 8:  $\bar{\sigma}_{\ell}^{\mathbf{Y}} \leftarrow \sigma_{\ell}^{\mathbf{Y}} / \lambda_{\ell}^{\mathbf{Y}}$ 9: **end for** 10:  $\hat{\mathbf{H}}(\mathbf{Y}) \leftarrow -\frac{1}{n} \sum_{\ell=1}^{n} \log(\lambda_{\ell}^{\mathbf{Y}})$ 11:  $\bar{\sigma}_{0}^{\mathbf{XY}} \leftarrow \rho_{\mathbf{S}_{0}}$ 12: **for each**  $\ell = 1, ..., n$  **do** 13:  $[\sigma_{\ell}^{\mathbf{XY}}] \leftarrow [W^{\check{y}_{\ell}|\check{x}_{\ell}}] \cdot [\bar{\sigma}_{\ell-1}^{\mathbf{XY}}]$ 14:  $\lambda_{\ell}^{\mathbf{XY}} \leftarrow \operatorname{tr}(\sigma_{\ell}^{\mathbf{XY}})$ 15:  $\bar{\sigma}_{\ell}^{\mathbf{XY}} \leftarrow \sigma_{\ell}^{\mathbf{XY}} / \lambda_{\ell}^{\mathbf{XY}}$
- 15:  $\bar{\sigma}_{\ell}^{XY} \leftarrow \sigma_{\ell}^{XY} / \lambda_{\ell}^{XY}$ 16: end for 17:  $\hat{\mathrm{H}}(\mathrm{X}, \mathrm{Y}) \leftarrow -\frac{1}{n} \sum_{\ell=1}^{n} \log(\lambda_{\ell}^{XY})$
- 10. Club for 17:  $\hat{\mathrm{H}}(\mathrm{X}, \mathrm{Y}) \leftarrow -\frac{1}{n} \sum_{\ell=1}^{n} \log(\lambda_{\ell}^{\mathrm{XY}})$ 18:  $\mathbf{H}(\mathrm{X}) \leftarrow -\sum_{x} Q(x) \log Q(x)$ 19: Estimate  $\mathrm{I}^{(n)}(Q, \{\mathcal{N}^{y|x}\}_{x,y})$  as  $\mathbf{H}(\mathrm{X}) + \hat{\mathrm{H}}(\mathrm{Y}) - \hat{\mathrm{H}}(\mathrm{X}, \mathrm{Y}).$

all CC-QSCs with the same input and output alphabet as the original channel, and acting on a memory system of a certain dimension (which can be different from the memory dimension of the original channel). Throughout this section, we assume the original channel as described in Section III is indecomposable, and that all the involved Hilbert spaces are of finite dimension, and that the alphabets  $\mathcal{X}$  and  $\mathcal{Y}$  are finite.

Suppose we have some auxiliary CC-QSC  $\{\hat{\mathcal{N}}^{y|x}\}_{x,y}$ , describable by some functions  $\{\hat{W}^{y|x}\}_{x,y}$  as in (45). Let  $\hat{P}_{\mathbf{Y}_1^n|\mathbf{X}_1^n,\hat{\mathbf{S}}_0}$  denote its joint channel law, similar to (42), (48), or (52). Namely,

$$\hat{P}_{\mathsf{Y}_1^n|\mathsf{X}_1^n,\hat{\mathsf{S}}_0}(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n;\hat{\rho}_{\mathsf{S}_0}) \triangleq \operatorname{tr}\left([\hat{W}^{y_n|x_n}]\cdots[\hat{W}^{y_1|x_1}]\cdot[\hat{\rho}_{\mathsf{S}_0}]\right) 74)$$

We follow a similar approach as in [13], [14], and define the quantities

$$\bar{\mathbf{I}}_{W}^{(n)}(\hat{W}) \triangleq \frac{1}{n} \sum_{\boldsymbol{x}_{1}^{n}, \boldsymbol{y}_{1}^{n}} Q^{(n)}(\boldsymbol{x}_{1}^{n}) \cdot P_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}}) \quad (75)$$

$$\cdot \log \frac{P_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}})}{\sum_{\tilde{\boldsymbol{x}}_{1}^{n}} Q^{(n)}(\tilde{\boldsymbol{x}}_{1}^{n}) \hat{P}_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \tilde{\boldsymbol{x}}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}})},$$

$$\underline{\mathbf{I}}_{W}^{(n)}(\hat{W}) \triangleq \frac{1}{n} \sum_{\boldsymbol{x}_{1}^{n}, \boldsymbol{y}_{1}^{n}} Q^{(n)}(\boldsymbol{x}_{1}^{n}) \cdot P_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}}) \quad (76)$$

$$\cdot \log \frac{\hat{P}_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \boldsymbol{x}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}})}{\sum_{\tilde{\boldsymbol{x}}_{1}^{n}} Q^{(n)}(\tilde{\boldsymbol{x}}_{1}^{n}) \hat{P}_{\mathbf{Y}_{1}^{n} | \mathbf{X}_{1}^{n}; \mathbf{S}_{0}}(\boldsymbol{y}_{1}^{n} | \tilde{\boldsymbol{x}}_{1}^{n}; \boldsymbol{\rho}_{\mathbf{S}_{0}})},$$

where  $P_{Y_1^n|X_1^n;S_0}$  is defined in (42), (48) or (52). By following similar arguments like those in (27) and (28), one can verify that

$$\underline{\mathbf{I}}_{W}^{(n)}(\hat{W}) \leqslant \mathbf{I}_{W}^{(n)} \leqslant \overline{\mathbf{I}}_{W}^{(n)}(\hat{W}), \tag{77}$$

$$\bar{\mathbf{I}}_{W}^{(n)}(\hat{W}) = \frac{1}{n} \left\langle \log \frac{\operatorname{tr}\left( [W^{\mathbf{Y}_{n}|\mathbf{X}_{n}}] \cdots [W^{\mathbf{Y}_{1}|\mathbf{X}_{1}}] \cdot [\rho_{\mathbf{S}_{0}}] \right)}{\sum_{\boldsymbol{x}_{1}^{n}} Q^{(n)}(\boldsymbol{x}_{1}^{n}) \cdot \operatorname{tr}\left( [\hat{W}^{\mathbf{Y}_{n}|\boldsymbol{x}_{n}}] \cdots [\hat{W}^{\mathbf{Y}_{1}|\boldsymbol{x}_{1}}] \cdot [\rho_{\hat{\mathbf{S}}_{0}}] \right)} \right\rangle_{\mathbf{X}_{1}^{n} \mathbf{Y}_{1}^{n}},$$
(78)

$$\underline{I}_{W}^{(n)}(\hat{W}) = \frac{1}{n} \left\langle \log \frac{\operatorname{tr}\left( [W^{\mathsf{Y}_{n}|\mathsf{X}_{n}}] \cdots [W^{\mathsf{Y}_{1}|\mathsf{X}_{1}}] \cdot [\rho_{\hat{\mathsf{S}}_{0}}] \right)}{\sum_{\boldsymbol{x}_{1}^{n}} Q^{(n)}(\boldsymbol{x}_{1}^{n}) \cdot \operatorname{tr}\left( [\hat{W}^{\mathsf{Y}_{n}|\boldsymbol{x}_{n}}] \cdots [\hat{W}^{\mathsf{Y}_{1}|\boldsymbol{x}_{1}}] \cdot [\rho_{\hat{\mathsf{S}}_{0}}] \right)} \right\rangle_{\mathsf{X}_{1}^{n} \mathsf{Y}_{1}^{n}},$$
(79)

$$\Delta_{W}^{(n)}(\hat{W}) = \frac{1}{n} \left\langle \log \frac{\operatorname{tr}\left([W^{\mathsf{Y}_{n}|\mathsf{X}_{n}}] \cdots [W^{\mathsf{Y}_{1}|\mathsf{X}_{1}}] \cdot [\rho_{\mathsf{S}_{0}}]\right)}{\operatorname{tr}\left([\hat{W}^{\mathsf{Y}_{n}|\mathsf{X}_{n}}] \cdots [\hat{W}^{\mathsf{Y}_{1}|\mathsf{X}_{1}}] \cdot [\rho_{\hat{\mathsf{S}}_{0}}]\right)} \right\rangle_{\mathsf{X}_{1}^{n}\mathsf{Y}_{1}^{n}},\tag{80}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \Big| \bar{\mathrm{I}}_{W}^{(n)}(\hat{W} + tH) &\propto -\frac{1}{n} \left\langle \sum_{k=1}^{n} \sum_{\mathbf{x}_{1}^{n}} Q^{(n)}(\mathbf{x}_{1}^{n}) \cdot \mathrm{tr} \left( [\hat{W}^{\mathbf{Y}_{n}|x_{n}}] \cdots [\hat{W}^{\mathbf{Y}_{k+1}|x_{k+1}}] [H^{\mathbf{Y}_{k}|x_{k}}] [\hat{W}^{\mathbf{Y}_{k-1}|x_{k-1}}] \cdots [\hat{W}^{\mathbf{Y}_{1}|x_{1}}] \cdot [\rho_{\mathbf{S}_{0}}] \right) \right\rangle_{\mathbf{Y}_{1}^{n}} \\ &= -\frac{1}{n} \sum_{\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}} P_{\mathbf{X}_{1}^{n}, \mathbf{Y}_{1}^{n}|\mathbf{S}_{0}}(\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\rho_{\mathbf{S}_{0}}) \cdot \sum_{k} \sum_{s', s, s', \bar{s}} \tilde{\varrho}_{\mathbf{S}_{k-1}}^{(\mathbf{y}_{k-1})}(s, \bar{s}) \cdot H^{y_{k}|x_{k}}(s', s, \bar{s}', \bar{s}) \cdot \tilde{\varrho}_{\mathbf{S}_{k}}^{(\mathbf{y}_{k-1})}(s', \bar{s}'), \quad (81) \\ \frac{\mathrm{d}}{\mathrm{d}t} \Big| \mathbf{I}_{W}^{(n)}(\hat{W} + tH) &\propto -\frac{1}{n} \left\langle \sum_{k=1}^{n} \sum_{\mathbf{x}_{1}^{n}} Q^{(n)}(\mathbf{x}_{1}^{n}) \cdot \mathrm{tr} \left( [\hat{W}^{\mathbf{Y}_{n}|x_{n}}] \cdots [\hat{W}^{\mathbf{Y}_{k+1}|x_{k+1}}] [H^{\mathbf{Y}_{k}|x_{k}}] [\hat{W}^{\mathbf{Y}_{k-1}|x_{k-1}}] \cdots [\hat{W}^{\mathbf{Y}_{1}|x_{1}}] \cdot [\rho_{\mathbf{S}_{0}}] \right) \right\rangle_{\mathbf{Y}_{1}^{n}} \\ &+ \frac{1}{n} \left\langle \sum_{k=1}^{n} \mathrm{tr} \left( [\hat{W}^{\mathbf{Y}_{n}|\mathbf{X}_{n}] \cdots [\hat{W}^{\mathbf{Y}_{k+1}|\mathbf{X}_{k+1}}] [H^{\mathbf{Y}_{k}|\mathbf{X}_{k}}] [\hat{W}^{\mathbf{Y}_{k-1}|\mathbf{X}_{k-1}}] \cdots [\hat{W}^{\mathbf{Y}_{1}|\mathbf{X}_{1}}] \cdot [\rho_{\mathbf{S}_{0}}] \right) \right\rangle_{\mathbf{X}_{1}^{n} \mathbf{Y}_{1}^{n}} \\ &= -\frac{1}{n} \sum_{\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}} P_{\mathbf{X}_{1}^{n}, \mathbf{Y}_{1}^{n}|\mathbf{S}_{0}}(\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\rho_{\mathbf{S}_{0}}) \cdot \sum_{k} \sum_{s', s, \bar{s}', \bar{s}} \tilde{\varrho}_{\mathbf{S}_{k-1}}^{(\mathbf{y}_{k-1}^{k-1}|\mathbf{X}_{k-1}^{k-1}]} (s, \bar{s}) \cdot H^{y_{k}|x_{k}}(s', s, \bar{s}', \bar{s}) \cdot \tilde{\varrho}_{\mathbf{S}_{k}}^{(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}^{k})}(s', \bar{s}'), \\ &+ \frac{1}{n} \sum_{\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\mathbf{S}_{0}}(\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\rho_{\mathbf{S}_{0}}) \cdot \sum_{k} \sum_{s', s, \bar{s}', \bar{s}} \tilde{\varrho}_{\mathbf{S}_{k-1}}^{(\mathbf{x}_{k-1}^{k-1})}(s, \bar{s}) \cdot H^{y_{k}|x_{k}}(s', s, \bar{s}', \bar{s}) \cdot \tilde{\varrho}_{\mathbf{S}_{k}}^{(\mathbf{x}_{k+1}, \mathbf{y}_{k+1}^{k})}(s', \bar{s}'), \\ &+ \frac{1}{n} \sum_{\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\mathbf{S}_{0}}(\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\rho_{\mathbf{S}_{0}}) \cdot \sum_{k} \sum_{s', s, \bar{s}', \bar{s}} \tilde{\varrho}_{\mathbf{S}_{k-1}}^{(\mathbf{x}_{k-1}^{k-1})}(s, \bar{s}) \cdot H^{y_{k}|x_{k}}(s', s, \bar{s}', \bar{s}) \cdot \tilde{\varrho}_{\mathbf{S}_{k}}^{(\mathbf{x}_{k+1}, \mathbf{y}_{k+1})}(s', \bar{s}'), \\ &= -\frac{1}{n} \sum_{\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\mathbf{S}_{0}}(\mathbf{x}_{1}^{n}, \mathbf{y}_{1}^{n}|\rho$$

$$\left(\nabla \bar{\mathbf{I}}_{W,\text{ext}}^{(n)}(\hat{W})\right)^{y|x} \propto -\frac{1}{n} \left\langle \sum_{k=1}^{n} \delta_{\mathsf{X}_{k},x} \cdot \delta_{\mathsf{Y}_{k},y} \cdot \bar{\varrho}_{\hat{\mathsf{S}}_{k-1}}^{(\mathsf{Y}_{1}^{k-1})} \otimes \bar{\varrho}_{\hat{\mathsf{S}}_{k}}^{(\mathsf{Y}_{k+1}^{n})} \right\rangle_{\mathsf{X}_{1}^{n}\mathsf{Y}_{1}^{n}},\tag{84}$$

$$\left(\nabla \underline{\mathbf{I}}_{W,\mathrm{ext}}^{(n)}(\hat{W})\right)^{y|x} \propto -\frac{1}{n} \left\langle \sum_{k=1}^{n} \delta_{\mathsf{X}_{k,x}} \cdot \delta_{\mathsf{Y}_{k,y}} \cdot \left( \vec{\varrho}_{\hat{\mathsf{S}}_{k-1}}^{(\mathsf{Y}_{1}^{k-1})} \otimes \vec{\varrho}_{\hat{\mathsf{S}}_{k}}^{(\mathsf{Y}_{k+1}^{n})} - \vec{\varrho}_{\hat{\mathsf{S}}_{k-1}}^{(\mathsf{X}_{1}^{k-1},\mathsf{Y}_{1}^{k-1})} \otimes \vec{\varrho}_{\hat{\mathsf{S}}_{k}}^{(\mathsf{X}_{k+1}^{n},\mathsf{Y}_{k+1}^{n})} \right) \right\rangle_{\mathsf{X}_{1}^{n}\mathsf{Y}_{1}^{n}}, \tag{85}$$

$$\left(\nabla\Delta_{W,\text{ext}}^{(n)}(\hat{W})\right)^{y|x} \propto -\frac{1}{n} \left\langle \sum_{k=1}^{n} \delta_{\mathsf{X}_{k},x} \cdot \delta_{\mathsf{Y}_{k},y} \cdot \vec{\varrho}_{\hat{\mathsf{S}}_{k-1}}^{(\mathsf{X}_{1}^{k-1},\mathsf{Y}_{1}^{k-1})} \otimes \vec{\varrho}_{\hat{\mathsf{S}}_{k}}^{(\mathsf{X}_{k+1}^{n},\mathsf{Y}_{k+1}^{n})} \right\rangle_{\mathsf{X}_{1}^{n}\mathsf{Y}_{1}^{n}}.$$
(86)

where the first inequality holds with equality if and only if  $\hat{P}_{Y_1^n|X_1^n,\hat{S}_0}(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n;\rho_{\hat{S}_0})$  and  $P_{Y_1^n|X_1^n;S_0}(\boldsymbol{y}_1^n|\boldsymbol{x}_1^n;\rho_{S_0})$  coincide for all  $\boldsymbol{x}_1^n$  and  $\boldsymbol{y}_1^n$  with positive support of  $P_{Y_1^n|X_1^n;S_0}$ , and where the second inequalities holds with equality if and only if  $\hat{P}_{Y_1^n|\hat{S}_0}(\boldsymbol{y}_1^n|\rho_{\hat{S}_0})$  and  $P_{Y_1^n|S_0}(\boldsymbol{y}_1^n|\rho_{S_0})$  coincide for all  $\boldsymbol{y}_1^n$  with positive support of  $P_{Y_1^n|S_0}$ . Another quantity of interest is the *difference function* defined as

$$\Delta_W^{(n)}(\hat{W}) \triangleq \bar{\mathbf{I}}_W^{(n)}(\hat{W}) - \underline{\mathbf{I}}_W^{(n)}(\hat{W}).$$
(87)

Explicit expressions of (75), (76), and (87) are given by (78), (79), and (80), respectively, at the top of this page, where  $X_1^n$  and  $Y_1^n$  are random variables distributed according to the joint distribution  $Q^{(n)}(\boldsymbol{x}_1^n) \cdot P_{\mathbf{Y}_1^n | \mathbf{X}_1^n; \mathbf{S}_0}(\boldsymbol{y}_1^n | \boldsymbol{x}_1^n; \rho_{\mathbf{S}_0})$ , and where  $\langle \cdot \rangle$  stands for the expectation function.

In the remainder of this section, we propose an algorithm based on the gradient-descent method and the techniques described in Section III-C and IV for optimizing the quantities in (75), (76), and (87). In particular, we consider  $\{\hat{W}^{y|x}\}_{x,y}$  to be an *interior* point in the domain of CC-QSCs, namely

The Choi–Jamiołkowski matrices [[Ŵ<sup>y|x</sup>]], defined similarly as (56), are strictly positive definite for each x and

• Eq. (57) holds by replacing  $W^{y|x}$  with  $\hat{W}^{y|x}$ , namely  $\sum_{y \in \mathcal{Y}} \sum_{s', \tilde{s}': s' = \tilde{s}'} \llbracket \hat{W}^{y|x} \rrbracket_{(s',s),(\tilde{s}',\tilde{s})} = \delta_{s,\tilde{s}}$  for all  $x \in \mathcal{X}$ .

For any set of functions  $\{H^{y|x}: S^4 \to \mathbb{C}\}_{x,y}$  such that  $\llbracket H^{y|x} \rrbracket$  (again, defined similarly as (56)) is Hermitian for each x and y, and such that

$$\sum_{y \in \mathcal{Y}} \sum_{s', \tilde{s}': s' = \tilde{s}'} \llbracket H^{y|x} \rrbracket_{(s',s),(\tilde{s}',\tilde{s})} = 0 \quad \forall x \in \mathcal{X},$$
(88)

the functions  $\{\hat{W}^{y|x} + t \cdot H^{y|x}\}_{x,y}$  describe a valid CC-QSC, for all t in some neighborhood of 0. In this case, the directional derivatives of functions  $\underline{I}_{W}^{(n)}$ ,  $\overline{I}_{W}^{(n)}$ , and  $\Delta_{W}^{(n)}$  at  $\{\hat{W}^{y|x}\}_{x,y}$  along  $\{H^{y|x}\}_{x,y}$  is well defined, and can be expressed as (81), (82), and (83) at the top of the last page, where we define the messages  $\{\overline{\varrho}_{S_{\ell}}^{(\tilde{y}_{\ell}^{1})}\}_{\ell}, \{\overline{\varrho}_{S_{\ell}}^{(\tilde{x}_{\ell}^{n}, \tilde{y}_{\ell+1}^{n})}\}_{\ell}$  in a recursive manner as

$$[\vec{\varrho}_{\mathsf{S}_{\ell}}^{(\check{\boldsymbol{y}}_{1}^{\ell})}] \triangleq \sum_{\boldsymbol{x}_{1}^{\ell}} Q(\boldsymbol{x}_{1}^{\ell}) \cdot [\hat{W}^{\check{y}_{\ell}|x_{\ell}}] \cdots [\hat{W}^{\check{y}_{1}|x_{1}}] \cdot [\rho_{\mathsf{S}_{0}}], \tag{89}$$

$$\begin{bmatrix} \mathcal{L}(\check{\boldsymbol{y}}_{\ell+1}^{n}) \\ \mathcal{L}_{\mathsf{S}_{\ell}} \end{bmatrix} \triangleq \sum_{\boldsymbol{x}_{\ell+1}^{n}} Q(\boldsymbol{x}_{\ell+1}^{n}) \cdot [I_{\mathsf{S}_{n}}] \cdot [\hat{W}^{\check{y}_{n}|x_{n}}] \cdots [\hat{W}^{\check{y}_{\ell+1}|x_{\ell+1}}], (90)$$

$$[\vec{\varrho}_{\mathsf{S}_{\ell}}^{(\check{\boldsymbol{x}}_{1}^{\ell},\check{\boldsymbol{y}}_{1}^{\ell})}] \triangleq [\hat{W}^{\check{y}_{\ell}|\check{x}_{\ell}}] \cdots [\hat{W}^{\check{y}_{1}|\check{x}_{1}}] \cdot [\rho_{\mathsf{S}_{0}}],\tag{91}$$

$$\begin{bmatrix} \hat{\boldsymbol{x}}_{\ell+1}^{\tilde{\boldsymbol{x}}}, \tilde{\boldsymbol{y}}_{\ell+1}^{n} \end{bmatrix} \triangleq \begin{bmatrix} I_{\mathsf{S}_n} \end{bmatrix} \cdot \begin{bmatrix} \hat{W}^{\check{\boldsymbol{y}}_n | \check{\boldsymbol{x}}_n} \end{bmatrix} \cdots \begin{bmatrix} \hat{W}^{\check{\boldsymbol{y}}_{\ell+1} | \check{\boldsymbol{x}}_{\ell+1}} \end{bmatrix}.$$
(92)

Recall that, in above equations,  $[I_{S_n}]$  is a row vector, whereas  $[\rho_{S_0}]$  is a column vector.

By extending the domain of the functions  $\underline{I}_{W}^{(n)}$ ,  $\overline{I}_{W}^{(n)}$ , and  $\Delta_{W}^{(n)}$  to include *all* p.s.d. matrices  $[\![\hat{W}^{y|x}]\!]$ , one can omit the linear constraint (88). Namely, the "direction"  $\{[\![H^{y|x}]\!]\}_{x,y}$  can take any Hermitian matrices. Using some linear algebra, the gradient w.r.t.  $\hat{W}$  of these functions on this *extended* domain can be expressed as (84), (85), and (86), respectively, at the top of the last page. For stationary and ergodic input and output processes  $(X_1^n, Y_1^n)$ , we can *estimate* (84) and (86), respectively, as

$$\left(\nabla \bar{\mathbf{I}}_{W,\text{ext}}^{(n)}(\hat{W})\right)^{y|x} \propto -\frac{1}{n} \sum_{\substack{k: \stackrel{\check{x}_{k}=x\\\check{y}_{k}=y}}} \bar{\varrho}_{\hat{\mathbf{S}}_{k-1}}^{(\check{\mathbf{y}}_{1}^{k-1})} \otimes \bar{\varrho}_{\hat{\mathbf{S}}_{k}}^{(\check{\mathbf{y}}_{k+1}^{n})}, \tag{93}$$

$$\left(\nabla\Delta_{W,\mathrm{ext}}^{(n)}(\hat{W})\right)^{y|x} \stackrel{\cdot}{\propto} -\frac{1}{n} \sum_{\substack{k: \stackrel{x_k=x}{\hat{y}_k=y}}} \overline{\varrho}_{\hat{\mathsf{S}}_{k-1}}^{(\check{\mathbf{x}}_1^{k-1}\check{\mathbf{y}}_1^{k-1})} \otimes \overline{\varrho}_{\hat{\mathsf{S}}_k}^{(\check{\mathbf{x}}_{k+1}^n,\check{\mathbf{y}}_{k+1}^n)}, (94)$$

where  $(\check{x}_1^n, \check{y}_1^n)$  is a realization of the channel input/output processes generated by the original channel model. The dot in (93) and (94) stands for "approximation". Notice that the messages  $\bar{\varrho}_{S_{k-1}}^{(\check{y}_1^{k-1})}$ ,  $\bar{\varrho}_{S_{k-1}}^{(\check{x}_{k-1}^{k-1},\check{y}_1^{k-1})}$ , and  $\bar{\varrho}_{S_k}^{(\check{x}_{k+1}^{k-1},\check{y}_{k+1}^{n})}$  can be computed iteratively. Thus, (93) and (94) provide efficient means to estimate the gradient. However, due to the extension of the domain, the gradients computed above may not satisfy constraint (88). This can be compensated using a projection w.r.t. the linear constraint, which can be solved using linear programming. On the other hand, the above gradient method may lead to a violation of the p.s.d. condition required by CC-QSCs. However, since the feasible domain of CC-QSCs is convex and bounded, this can be corrected using convex programming at each step.

Algorithm 10 Optimizing the difference function							
Input: indecomposable CC-QSC, input distribution Q, pos-							
itive	integer	n	large	enough.	initial	auxiliarv	CC-

QSC  $\{\hat{W}^{y|x}\}_{x,y}$ , step size  $\gamma > 0$ . Output:  $\{\hat{W}^{y|x}\}_{x,y}$ , an estimated local minimum point of

- **Output:**  $\{W^{|y|x}\}_{x,y}$ , an estimated local minimum point of  $\Delta_W^{(n)}$ .
- 1: Initialize the memory density operator  $\rho_{\hat{S}_0} \leftarrow |0\rangle \langle 0|$
- 2: Generate an input sequence  $\check{x}_1^n \sim Q^{\otimes n}$
- 3: Generate a corresponding output sequence  $\check{\boldsymbol{y}}_1^n$
- 4: repeat

8: 9:

10: 11:

12: 13: 14: 15:

16:

18:

23: 24:

5: 
$$\vec{\varrho}_{\hat{\mathsf{S}}_0} \leftarrow \rho_{\hat{\mathsf{S}}_0}$$

6: **for each**  $\ell = 1, ..., n$  **do** 7:  $[\vec{\rho}_{\hat{\epsilon}}] \leftarrow [\hat{W}^{\check{y}_{\ell}|\check{x}_{\ell}}] \cdot [\vec{\rho}_{\hat{\epsilon}}]$ 

$$[\mathcal{Q}_{\hat{\mathsf{S}}_{\ell}}] \leftarrow [\mathcal{W}^{\circ,\mathsf{s}_{\ell}}] \cdot [\mathcal{Q}_{\hat{\mathsf{S}}_{\ell-1}}]$$
$$\lambda_{\ell} \leftarrow \operatorname{tr}(\vec{\rho}_{\hat{c}})$$

$$\vec{\varrho}_{\hat{\mathsf{S}}_{\ell}} \leftarrow \lambda_{\ell}^{-1} \cdot \vec{\varrho}_{\hat{\mathsf{S}}_{\ell}}$$
  
end for

 $\overline{\varrho}_{\hat{\mathsf{S}}_n} \leftarrow I_{\hat{\mathsf{S}}_n}$ 

for each 
$$\ell = n, ..., 1$$
 do  
 $[\bar{\varrho}_{\hat{\mathsf{S}}_{\ell-1}}] \leftarrow [\bar{\varrho}_{\hat{\mathsf{S}}_{\ell}}] \cdot [\hat{W}^{\check{y}_{\ell}|\check{x}_{\ell}}]$ 

$$\overline{\varrho}_{\hat{\mathsf{S}}_{\ell-1}} \leftarrow \left( \operatorname{tr}(\overline{\varrho}_{\hat{\mathsf{S}}_{\ell-1}}) \right) \quad \cdot \overline{\varrho}$$

end for

for each x, y, let  $\left(\nabla \Delta_{W, \text{ext}}^{(n)}(\hat{W})\right)^{y|x} \leftarrow \mathbf{0}$ 

17: for each 
$$k = 1, \ldots, n$$
 do

$$\left(\nabla \Delta_{W,\text{ext}}^{(n)}(\hat{W})\right)^{g_{k},x_{k}} += \frac{1}{n} \cdot \frac{\varrho_{\hat{\mathbf{\xi}}_{k-1}} \otimes \varrho_{\hat{\mathbf{\xi}}_{k}}}{\lambda_{k} \cdot \operatorname{tr}(\bar{\varrho}_{\hat{\mathbf{\xi}}_{k}} \cdot \bar{\varrho}_{\hat{\mathbf{\xi}}_{k}})}$$

19: end for

20: Project  $\{\left(\nabla\Delta_{W,\text{ext}}^{(n)}(\hat{W})\right)^{y|x}\}_{x,y}$  onto the subspace satisfying (88); denoting the result by  $\left\{\left(\nabla\Delta_{W}^{(n)}(\hat{W})\right)^{y|x}\right\}_{x,y}$ 

21: 
$$\{\hat{W}^{y|x}\}_{x,y} \leftarrow \{\hat{W}^{y|x}\}_{x,y} - \gamma \cdot \left\{ \left(\nabla \Delta_W^{(n)}(\hat{W})\right)^{y|x} \right\}_{x,y}$$

22: Solve the following convex program w.r.t. 
$$\{W^{y|x}\}_{x,y}$$
:

$$\begin{array}{ll} \min & \sum_{x,y} \operatorname{tr} \left( \left( \left[ \tilde{W}^{y|x} \right] \right] - \left[ \left[ \hat{W}^{y|x} \right] \right] \right) \cdot \left( \left[ \left[ \tilde{W}^{y|x} \right] \right] - \left[ \left[ \hat{W}^{y|x} \right] \right] \right)^{\dagger} \right) \\ \text{s.t.} & \left[ \left[ \tilde{W}^{y|x} \right] \right] \in \mathbb{C}^{S^2 \times S^2} \text{ is p.s.d. for each } x, y \\ & \sum_{y \in \mathcal{Y}} \sum_{s', \tilde{s}': \ s' = \tilde{s}'} \left[ \left[ \tilde{W}^{y|x} \right] \right]_{(s',s),(\tilde{s}',\tilde{s})} = \delta_{s,\tilde{s}} \quad \forall x \\ \left\{ \hat{W}^{y|x} \right\} \leftarrow \left\{ \tilde{W}^{y|x} \right\} \\ \text{until } \left\{ \hat{W}^{y|x} \right\}_{x,y} \text{ has converged.} \end{array}$$

We summarize the above discussion as Algorithm 10, which is an iterative gradient-descent method for minimizing  $\Delta_W^{(n)}$ . Notice that the quantity  $\lambda_\ell$  in this case is the conditional probability  $P_{\mathbf{X}_\ell \mathbf{Y}_\ell | \mathbf{X}_1^{\ell-1} \mathbf{Y}_1^{\ell-1}}(\check{x}_\ell, \check{y}_\ell | \check{\mathbf{x}}_1^{\ell-1}, \check{\mathbf{y}}_1^{\ell-1})$ . The algorithm for minimizing the upper and lower bounds are similar, and we omit the details.

#### VI. EXAMPLE: QUANTUM GILBERT-ELLIOTT CHANNELS

In this section we present some numerical results as a demonstration of the algorithms introduced in this paper. In particular, as a generalization of Example 1, we consider a

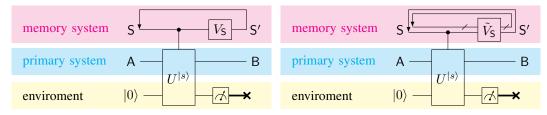
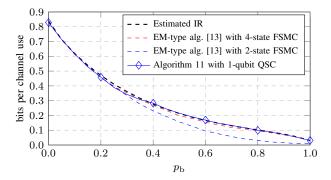
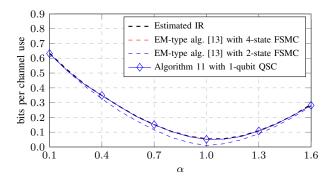


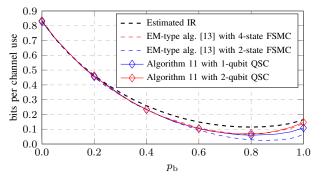
Fig. 9. A quantum Gilbert–Elliott channel (LHS), and a variant where the memory system consists of multiple qubits with only one of them controlling  $U^{|s\rangle}$  (RHS).



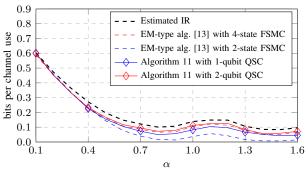
(a) Quantum Gilbert–Elliott Channel:  $p_{\rm g} = 0.05$  is fixed;  $p_{\rm b}$  varies from 0 to 1;  $V_{\rm S} = \exp(-j\alpha H)$ , where H is some fixed 2-by-2 Hermitian matrix and where  $\alpha = 1$  is fixed;  $n = 10^5$ .



(c) Quantum Gilbert–Elliott Channel:  $p_{\rm g} = 0.05$  is fixed;  $p_{\rm b} = 0.95$  is fixed;  $V_{\rm S} = \exp(-j\alpha H)$ , where H is the same 2-by-2 Hermitian matrix as in Fig. 10(a) and where  $\alpha$  varies from 0.1 to +1.5;  $n = 10^5$ .



(b) Variant of the Quantum Gilbert–Elliott Channel described in the RHS of Fig. 9. Here, the memory system S consists of two qubits, with only the first one interacting with the primary system by serving as the controlling qubit of the controlled bit-flip channel. Parameters:  $p_{\rm g} = 0.05$ ;  $p_{\rm b} \in [0, 1]$ ;  $\tilde{V}_{\rm S} = \exp(-j\alpha H)$ , where H is some fixed 4-by-4 Hermitian matrix and where  $\alpha = 1$  is fixed;  $n = 10^5$ .



(d) Same variant of the Quantum Gilbert–Elliott Channel as in Fig. 10(b) with different parameters:  $p_{\rm g} = 0.05$ ;  $p_{\rm b} = 0.95$ ;  $V_{\rm S} = \exp(-j\alpha H)$ , where *H* is the same 4-by-4 Hermitian matrix as in Fig. 10(b) and where  $\alpha$  varies from 0.1 to +1.5;  $n = 10^5$ .

Fig. 10. Some numerical information rate lower bounds estimated for a QGEC and a variant of a QGEC, equipped with "trivial" orthonormal ensemble and projective measurements. The estimated information rates were obtained using Alg. 9.

class of quantum channels with memory named the quantum Gilbert–Elliott channels (QGECs), which were introduced in [1], and consider their information rates using some separable input ensemble and local output measurement.

A QGEC is a quantum channel with memory defined by<sup>8</sup>

$$\mathcal{N}: \mathfrak{D}(\mathcal{H}_{\mathsf{S}} \otimes \mathcal{H}_{\mathsf{A}}) \to \mathfrak{D}(\mathcal{H}_{\mathsf{S}'} \otimes \mathcal{H}_{\mathsf{B}})$$
  
$$\rho_{\mathsf{S}\mathsf{A}} \mapsto (V_{\mathsf{S}} \otimes I_{\mathsf{B}}) \cdot \Phi^{\mathrm{CBF}}(\rho_{\mathsf{S}\mathsf{A}}) \cdot (V_{\mathsf{S}}^{\dagger} \otimes I_{\mathsf{B}}),$$

where  $\mathcal{H}_A$ ,  $\mathcal{H}_B$ , and  $\mathcal{H}_S = \mathcal{H}_{S'}$  are of dimension 2, namely each of them is made up of one qubit; and where  $\Phi^{CBF}$ 

<sup>8</sup>We put the system S ahead of A and B in this example to emphasize the role of S as a *control* qubit, and also for simplicity reasons.

is the *controlled bit-flip channel* defined by  $\Phi^{\text{CBF}}(\rho_{\text{SA}}) \triangleq E_0 \rho^{\text{SA}} E_0^{\dagger} + E_1 \rho^{\text{SA}} E_1^{\dagger}$  with

$$E_{0} \triangleq \begin{bmatrix} \sqrt{1-p_{\rm g}} & 0 & 0 & 0 \\ 0 & \sqrt{1-p_{\rm g}} & 0 & 0 \\ 0 & 0 & \sqrt{1-p_{\rm b}} & 0 \\ 0 & 0 & 0 & \sqrt{1-p_{\rm b}} \end{bmatrix}, \ E_{1} \triangleq \begin{bmatrix} 0 & \sqrt{p_{\rm g}} & 0 & 0 \\ \sqrt{p_{\rm g}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{p_{\rm b}} \\ 0 & 0 & \sqrt{p_{\rm b}} & 0 \end{bmatrix};$$

and where  $V_S$  is some unitary operator on  $\mathcal{H}_S$  to be specified later. The controlled bit-flip channel  $\Phi^{\rm CBF}$  applies a quantum bit-flip channel on system A with flipping probability  $p_{\rm g}$ when the system S is in the state of  $|0\rangle$ , and with flipping probability  $p_{\rm b}$  when the system S is in the state of  $|1\rangle$ . The action of a QGEC is the combined effect of a controlled bitflip channel and a unitary evolution on S; as depicted in the

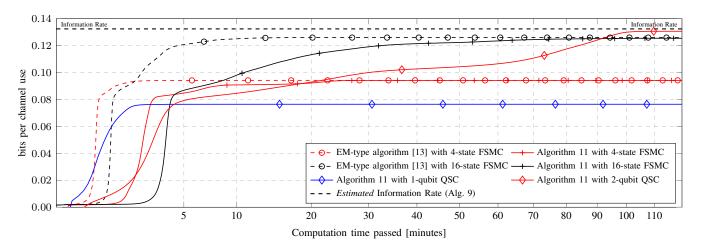


Fig. 11. Minimizing the difference function  $\Delta_W^{(n)}$  using different methods. The markets appear after every 400 updates.

following circuit diagram in Fig. 9, where  $U^{|s\rangle}$  is a Stinespring representation of  $\Phi^{\rm CBF}$ :

$$\begin{split} U^{|0\rangle} &\triangleq \begin{bmatrix} \sqrt{1-p_{\rm g}} & 0 & 0 & -\sqrt{p_{\rm g}} \\ 0 & \sqrt{1-p_{\rm g}} & \sqrt{p_{\rm g}} & 0 \\ 0 & -\sqrt{p_{\rm g}} & \sqrt{1-p_{\rm g}} & 0 \\ \sqrt{p_{\rm g}} & 0 & 0 & \sqrt{1-p_{\rm g}} \end{bmatrix}, \\ U^{|1\rangle} &\triangleq \begin{bmatrix} \sqrt{1-p_{\rm b}} & 0 & 0 & -\sqrt{p_{\rm b}} \\ 0 & \sqrt{1-p_{\rm b}} & \sqrt{p_{\rm b}} & 0 \\ 0 & -\sqrt{p_{\rm b}} & \sqrt{1-p_{\rm b}} & 0 \\ \sqrt{p_{\rm b}} & 0 & 0 & \sqrt{1-p_{\rm b}} \end{bmatrix}. \end{split}$$

In Fig. 10, we present some numerical information rate lower bounds estimated for a QGEC and a variant of a QGEC (as depicted in Fig. 9), equipped with "trivial" orthonormal ensemble and projective measurements. Namely, the original channel in Fig. 10(a) and 10(c) can be described by the CC-QSC

$$\mathcal{N}^{y|x}(\rho_{\mathsf{S}}) = \operatorname{tr}_{\mathsf{B}}\left( (V_{\mathsf{S}}^{\dagger} V_{\mathsf{S}} \otimes |y\rangle \langle y|) \cdot \Phi^{\operatorname{CBF}}(\rho_{\mathsf{S}} \otimes |x\rangle \langle x|) \right),$$
(95)

whereas that in Fig. 10(b) and 10(d) is described by

$$\mathcal{N}^{y|x}(\rho_{\mathsf{S}}) = \operatorname{tr}_{\mathsf{B}}\left( (\tilde{V}_{\mathsf{S}}^{\dagger} \tilde{V}_{\mathsf{S}} \otimes |y\rangle \langle y|) \cdot (\mathcal{I} \otimes \Phi^{\operatorname{CBF}})(\rho_{\mathsf{S}} \otimes |x\rangle \langle x|) \right) (96)$$

where  $\{|x\rangle\}_{x\in\mathcal{X}}$  and  $\{|y\rangle\}_{y\in\mathcal{Y}}$  are some orthonormal basis of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. In the latter case, the memory system S is extended as  $\mathcal{H}_{S} = \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_0}$ . More specifically, in (96),  $\rho_{\rm S}$  and  $\tilde{V}_{\rm S}$  are operators on  $\mathcal{H}_{\rm S}$ , and  $\Phi^{\rm CBF}$  acts on  $\mathfrak{D}(\mathcal{H}_{S_0} \otimes \mathcal{H}_A)$ , and  $\mathcal{I}$  is the identity map on  $S_1$ . For both scenarios, the input processes are binary symmetric i.i.d. processes, *i.e.*,  $Q^{(n)}(\boldsymbol{x}_1^n) \triangleq 2^{-n}$  for all  $\boldsymbol{x}_1^n \in \{0,1\}^n$ . The lower bounds in those figures were obtained by minimizing the difference function  $\Delta_W^{(n)}$  defined in (30) w.r.t. different classes of auxiliary channels (subject to certain time and threshold constraints). For the case where the auxiliary channels are CC-QSCs, Alg. 10 was applied. For FSMC auxiliary channels, we implemented the expectation-maximization type algorithm in [14] for comparison. As already emphasized beforehand, these lower bounds represent rates that are achievable with the help of a mismatched decoder [18]. Fig. 11 is an example illustrating the typical convergence time of different methods (including our own) for minimizing the difference function. In all of the above figures,  $n = 10^5$ , and we have used Alg. 9 to *estimate* the information rate. According to our experience, the error of the estimation in this case lies within the line-width in the figures.

#### VII. CONCLUSION

In this article, we have considered the scenario of transmitting classical information over a quantum channel with finite memory using separable-state ensembles and local measurements. We defined the notion of CC-QSCs as an equivalent way to describe such communication setups, and demonstrated how NFGs can be used to visualize such channels. We have shown that the information rate of a quantum-state channel is independent of the initial density operator under suitable conditions, and proposed algorithms for estimating and bounding such information rate. The computations in such algorithms can be carried out using the corresponding NFGs of the CC-QSC. We emphasize that our approach for optimizing the lower bound is data-driven, and does not require the knowledge of the true channel model.

#### ACKNOWLEDGMENT

It is a great pleasure to acknowledge discussions on topics related to this paper with Andi Loeliger.

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## APPENDIX

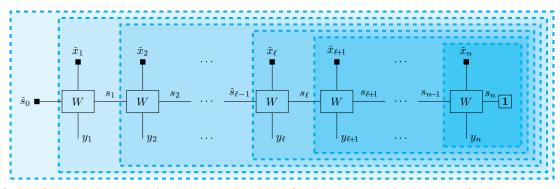


Fig. 12. Verification of (7). Note that every closing-the-box operation yields a function node representing the constant function 1.

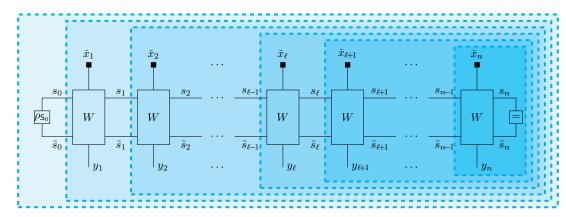


Fig. 13. Counterpart of Fig. 12 for QSCs. Note that every closing-the-box operation yields a function node representing a Kronecker-delta function node, i.e., a degree-two equality function node.

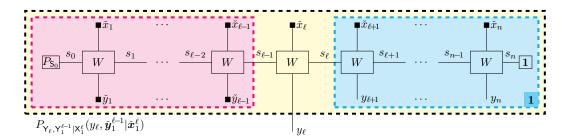


Fig. 14. Efficient simulation of the channel output at step  $\ell$  given channel input  $\check{x}_1^n$  and channel output  $\check{y}_1^{\ell-1}$  for an FSMC.

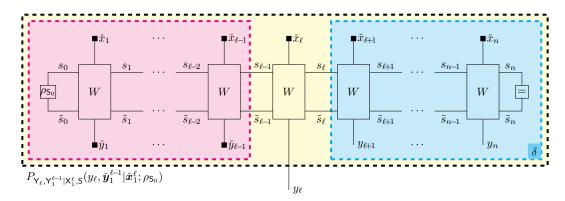


Fig. 15. Efficient simulation of the channel output at step  $\ell$  given channel input  $\check{x}_1^n$  and channel output  $\check{y}_1^{\ell-1}$  for a QSC.

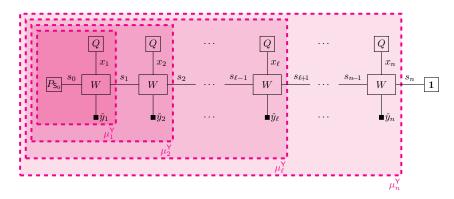


Fig. 16. The iterative computation of  $\mu_{\ell}^{Y}$  as in (20) can be understood as a sequence of CTB operations as shown above.

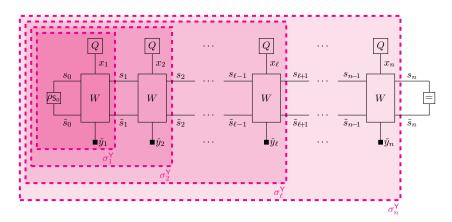


Fig. 17. The iterative computation of  $\sigma_{\ell}^{Y}$  as in (70) can be understood as a sequence of CTB operations as shown above.

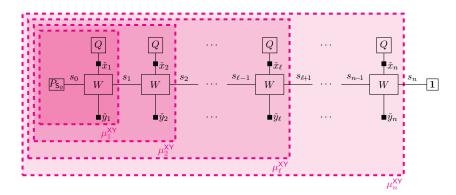


Fig. 18. The iterative computation of  $\mu_{\ell}^{XY}$  can be understood as a sequence of CTB operations as shown above.

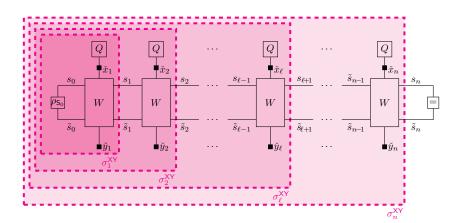


Fig. 19. The iterative computation of  $\sigma_{\ell}^{XY}$  as in (71) can be understood as a sequence of CTB operations as shown above.