Parallel multilevel constructions for constant dimension codes 1

Shuangqing Liu, Yanxun Chang, Tao Feng

Department of Mathematics, Beijing Jiaotong University, Beijing 100044, P. R. China 16118420@bjtu.edu.cn, yxchang@bjtu.edu.cn, tfeng@bjtu.edu.cn

Abstract: Constant dimension codes (CDCs), as special subspace codes, have received a lot of attention due to their application in random network coding. This paper introduces a family of new codes, called rank metric codes with given ranks (GRMCs), to generalize the parallel construction in [Xu and Chen, IEEE Trans. Inf. Theory, 64 (2018), 6315–6319] and the classic multilevel construction. A Singleton-like upper bound and a lower bound for GRMCs derived from Gabidulin codes are given. Via GRMCs, two effective constructions for CDCs are presented by combining the parallel construction and the multilevel construction. Many CDCs with larger size than the previously best known codes are given. The ratio between the new lower bound and the known upper bound for $(4\delta, 2\delta, 2\delta)_q$ -CDCs is calculated. It is greater than 0.99926 for any prime power q and any $\delta \geq 3$.

Keywords: constant dimension code, rank-metric code, multilevel construction, parallel construction.

1 Introduction

Subspace codes, constant dimension codes in particular, have drawn significant attention due to the work by Kötter and Kschischang [20], where they presented an application of such codes for error correction in random network coding.

Let \mathbb{F}_q be the finite field of order q, and \mathbb{F}_q^n be the set of all vectors of length n over \mathbb{F}_q . \mathbb{F}_q^n is an n-dimensional vector space over \mathbb{F}_q . Given a nonnegative integer $k \leq n$, the set of all k-dimensional subspaces of \mathbb{F}_q^n is called the *Grassmannian* $\mathcal{G}_q(n,k)$. The cardinality of $\mathcal{G}_q(n,k)$ is given by the q-ary Gaussian coefficient

$$|\mathcal{G}_q(n,k)| = \begin{bmatrix} n \\ k \end{bmatrix}_q \triangleq \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}$$

For any two subspaces $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(n,k)$, their subspace distance is defined by

$$d_S(\mathcal{U}, \mathcal{V}) \triangleq \dim \mathcal{U} + \dim \mathcal{V} - 2\dim(\mathcal{U} \cap \mathcal{V}) = 2(k - \dim(\mathcal{U} \cap \mathcal{V})).$$
(1.1)

A subset \mathcal{C} of the Grassmannian $\mathcal{G}_q(n,k)$ is called an $(n,d,k)_q$ constant-dimension code (CDC), if $d_S(\mathcal{U},\mathcal{V}) \geq d$ for all $\mathcal{U},\mathcal{V} \in \mathcal{C}$ and $\mathcal{U} \neq \mathcal{V}$. Elements in \mathcal{C} are called codewords. An $(n,d,k)_q$ -CDC with M codewords is written as an $(n,M,d,k)_q$ -CDC. Given n,d,k and q, denote by $A_q(n,d,k)$ the maximum number of codewords among all $(n,d,k)_q$ -CDCs. An $(n,d,k)_q$ -CDC with $A_q(n,d,k)$ codewords is said to be optimal.

Without loss of generality, assume that $n \ge 2k$. This assumption can be made as a consequence of the fact $A_q(n, d, k) = A_q(n, d, n - k)$, which can be obtained by taking

¹Supported by NSFC under Grant 11971053 (Y. Chang), and NSFC under Grant 11871095 (T. Feng).

orthogonal complements of subspaces (cf. [30]). Furthermore, by (1.1), when 2k < d, any nonempty $(n, d, k)_q$ -CDC consists of exactly one codeword. Therefore, we always assume that $n \ge 2k \ge d$.

A k-dimensional subspace \mathcal{U} of \mathbb{F}_q^n can be represented by a $k \times n$ generator matrix Uwhose rows form a basis of \mathcal{U} . Let $\mathbb{F}_q^{k \times n}$ denote the set of all $k \times n$ matrices over \mathbb{F}_q . For $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(n, k)$, the subspace distance on $\mathcal{G}_q(n, k)$ is also given by

$$d_{S}(\mathcal{U}, \mathcal{V}) = 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{U} \\ \mathbf{V} \end{pmatrix} - 2k, \qquad (1.2)$$

where $U, V \in \mathbb{F}_q^{k \times n}$ are matrices such that $\mathcal{U} = \operatorname{rowspace}(U)$ and $\mathcal{V} = \operatorname{rowspace}(V)$. The two matrices are usually not unique.

By (1.2), in a CDC, the minimum subspace distance d is even. If d = 0 or 2, then all k-dimensional subspaces of \mathbb{F}_q^n constitute an optimal $(n, d, k)_q$ -CDC. It follows that d = 4 is the minimum nontrivial value.

1.1 Lifted maximum rank distance codes

To obtain optimal CDCs, Silva, Kschischang and Kötter [25] pointed out that lifted maximum rank distance (MRD) codes can result in asymptotically optimal CDCs, and can be decoded efficiently in the context of random linear network coding.

For a matrix $A \in \mathbb{F}_q^{m \times n}$, the rank of A is denoted by rank(A). The set $\mathbb{F}_q^{m \times n}$ is an \mathbb{F}_q -vector space. The rank distance on $\mathbb{F}_q^{m \times n}$ is defined by

$$d_R(\boldsymbol{A}, \boldsymbol{B}) = \operatorname{rank}(\boldsymbol{A} - \boldsymbol{B}), \text{ for } \boldsymbol{A}, \boldsymbol{B} \in \mathbb{F}_q^{m \times n}.$$

An $[m \times n, k, \delta]_q$ rank-metric code \mathcal{D} is a k-dimensional \mathbb{F}_q -linear subspace of $\mathbb{F}_q^{m \times n}$ with minimum rank distance

$$\delta = \min_{\boldsymbol{A}, \boldsymbol{B} \in \mathcal{D}, \boldsymbol{A} \neq \boldsymbol{B}} \{ d_R(\boldsymbol{A}, \boldsymbol{B}) \}.$$

Clearly

$$\delta = \min_{\boldsymbol{A} \in \mathcal{D}, \boldsymbol{A} \neq \boldsymbol{0}} \{ \operatorname{rank}(\boldsymbol{A}) \}.$$

Elements in \mathcal{D} are called *codewords*. The Singleton-like upper bound for rank-metric codes implies that

$$k \le \max\{m, n\}(\min\{m, n\} - \delta + 1)$$

holds for any $[m \times n, k, \delta]_q$ code. When the equality holds, \mathcal{D} is called a *linear maximum* rank distance code, denoted by an MRD $[m \times n, \delta]_q$ code. Linear MRD codes exists for all feasible parameters (cf. [4, 10, 21]).

Write I_k as the $k \times k$ identity matrix.

Proposition 1.1 (Lifted MRD codes, [25]) Let $n \ge 2k$. The lifted MRD code

$$\mathcal{C} = \{ \operatorname{rowspace}(\boldsymbol{I}_k \mid \boldsymbol{A}) : \boldsymbol{A} \in \mathcal{D} \}$$

is an $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ -CDC, where \mathcal{D} is an $MRD[k \times (n-k), \delta]_q$ code.

We outline the proof of Proposition 1.1 for later use. It suffices to check the subspace distance of \mathcal{C} . For any $\mathcal{U}, \mathcal{V} \in \mathcal{C}$ and $\mathcal{U} \neq \mathcal{V}$, where $\mathcal{U} = \text{rowspace}(\mathbf{I}_k \mid \mathbf{A})$ and $\mathcal{V} = \text{rowspace}(\mathbf{I}_k \mid \mathbf{B})$, we have

$$d_{S}(\mathcal{U}, \mathcal{V}) = 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{I}_{k} & \mathbf{A} \\ \mathbf{I}_{k} & \mathbf{B} \end{pmatrix} - 2k = 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{I}_{k} & \mathbf{A} \\ \mathbf{O} & \mathbf{B} - \mathbf{A} \end{pmatrix} - 2k$$
$$= 2 \cdot \operatorname{rank}(\mathbf{B} - \mathbf{A}) \ge 2\delta.$$

Given n, δ, k and q, denote by $\bar{A}_q(n, 2\delta, k)$ the maximum number of codewords among all $(n, 2\delta, k)_q$ -CDC containing a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ -CDC as a subset. Many constructions for CDCs with large number of codewords known in the literature (cf. [6,11,22,24–26,28]) produce codes containing a lifted MRD code. This motivates the study, initialized by Etzion and Silberstein [7], on determining the lower and upper bounds on $\bar{A}_q(n, d, k)$.

Etzion and Silberstein [6] presented a simple but effective construction, named the multilevel construction, which generalizes the lifted MRD codes. Trautmann and Rosenthal in [28] improved the multilevel construction by pending dots. Using the idea of pending dots and graph matchings, Etzion, Silberstein in [7] and Silberstein, Trautmann in [24] constructed large subspace codes in $\mathcal{G}_q(n, k)$ of minimum subspace distance d = 4or 2k - 2. Xu and Chen [31] presented a new construction which can be also seen as a generalization of the lifted MRD codes. Heinlein [14] summarized the upper bounds of CDCs which contain lifted MRD codes as follows.

Theorem 1.2 [14, Theorem 1] For $n \ge 2k$, let C be an $(n, 2\delta, k)_q$ -CDC which contains a lifted MRD code.

- $\begin{array}{ll} (1) \ \ If \ k < 2\delta \ \ and \ n \ge 3\delta, \ then \ \ \bar{A}_q(n, 2\delta, k) \le q^{(n-k)(k-\delta+1)} + A_q(n-k, 2(2\delta-k), \delta). \\ If \ additionally \ k = \delta, \ n \equiv r \ (\mathrm{mod} \ k), \ 0 \le r < k, \ and \ \begin{bmatrix} r \\ 1 \end{bmatrix}_q < k, \ or \ (n, 2\delta, k) \in \{(6+3l, 4+2l, 3+l), (6l, 4l, 3l) \ | \ l \ge 1\}, \ then \ the \ bound \ can \ be \ achieved. \end{array}$
- (2) If $k < 2\delta$ and $n < 3\delta$, then $\bar{A}_q(n, 2\delta, k) = q^{(n-k)(k-\delta+1)} + 1$.
- (3) If $2\delta \leq k < 3\delta$, then

$$\begin{split} \bar{A}_q(n, 2\delta, k) &\leq q^{(n-k)(k-\delta+1)} + A_q(n-k, 6\delta - 2k, 2\delta) \\ &+ q^{(k-2\delta+1)(n-k-\delta)} \frac{\binom{n-k}{\delta}_q \binom{k}{2\delta-1}_q}{\binom{k-\delta}{\delta-1}_q} \end{split}$$

For more information on constructions and bounds for subspace codes, the interested reader is referred to [1,3,7–9,12,13,16,19,20,23,27,29,30].

1.2 Our contribution

This paper is devoted to constructing large constant dimension codes which contain a lifted MRD code as a subset.

Section 2 generalizes a construction for CDCs in [31] by introducing a family of new codes, called rank metric codes with given ranks (GRMCs). This generalized construction is called a parallel construction (see Construction 2.5). We shall establish a Singleton-like upper bound (see Proposition 2.3) and a lower bound (see Proposition 2.4) for GRMCs by using Gabidulin codes. Very recently, Heinlein [15] also introduced a similar concept to GRMCs. He presented several lower bounds for GRMCs, but most focus on special parameters. Here our construction is for general parameters. Applying Construction 2.5 together with Proposition 2.4, we give a lower bound on $\bar{A}_q(n, 2\delta, k)$ for any $n \geq 2k > 2\delta > 0$.

Section 3 presents two effective constructions for CDCs (see Constructions 3.10 and 3.17) by combining the parallel construction and the classic multilevel construction. Constructions 3.10 shows that if a multilevel construction satisfies the weight of the first n-k positions of every identifying vector is no less than δ , then the multilevel construction can be combined with a parallel construction. Constructions 3.17 shows that if identifying vectors in a multilevel construction dissatisfy the condition in Constructions 3.10, the multilevel construction is still possible to be combined with a parallel construction. In both construction, GRMCs play an important role.

In principle, people can always pick up suitable identifying vectors and then use the classic multilevel construction to construct optimal CDC. However, how to choose identifying vectors effectively is still an open and different problem. The combination of the parallel construction and the multilevel construction helps to weaken the requirement for identifying vectors and provides good constant dimension codes with large size. Applying Constructions 3.10 and 3.17, we establish new lower bounds for CDCs (see Theorems 3.12, 3.14, 3.16 and Corollary 3.18). Many CDCs with larger size than the previously best known codes in [16] are given (see Appendix B). We also calculate the ratio between our lower bound and the known upper bound for $(4\delta, 2\delta, 2\delta)_q$ -CDCs. It is greater than 0.99926 for any prime power q and any $\delta \geq 3$ (see Remark 3.20).

2 Parallel construction

In [31], Xu and Chen presented an interesting construction to establish new lower bounds for $A_q(2k, 2\delta, k)$, where $k \ge 2\delta$.

Theorem 2.1 [31, Theorem 3 and Corollary 4] For any positive integers k and δ such that $k \geq 2\delta$,

$$A_q(2k, 2\delta, k) \ge q^{k(k-\delta+1)} + \sum_{i=\delta}^{k-\delta} A_i,$$

where A_i denotes the number of codewords with rank *i* in an $MRD[k \times k, \delta]_q$ code.

Theorem 2.1 relies on the use of the rank distribution of an MRD code.

Theorem 2.2 (Rank distribution [4, 10]) Let $m \ge n$. Let \mathcal{D} be an $MRD[m \times n, \delta]_q$ code, and $A_i = |\{M \in \mathcal{D} : \operatorname{rank}(M) = i\}|$ for $0 \le i \le n$. Its rank distribution is given by $A_0 = 1, A_i = 0$ for $1 \le i \le \delta - 1$, and

$$A_{\delta+i} = \begin{bmatrix} n \\ \delta+i \end{bmatrix}_q \sum_{j=0}^i (-1)^{j-i} \begin{bmatrix} \delta+i \\ i-j \end{bmatrix}_q q^{\binom{i-j}{2}} (q^{m(j+1)} - 1)$$

for $0 \leq i \leq n - \delta$.

In this section, we shall generalize Theorem 2.1 by introducing the concept of rank metric codes with given ranks (GRMCs), which can be seen as a generalization of constant-rank codes. Constant-rank codes have been discussed systematically in [11].

We remark that very recently, based on Theorem 2.1, Chen et.al [3] and Heinlein [15] generalized linkage constructions to establish some lower bounds of CDCs independently.

2.1 Rank metric codes with given ranks

Let $K \subseteq \{0, 1, ..., n\}$ and δ be a positive integer. We say $\mathcal{D} \subseteq \mathbb{F}_q^{m \times n}$ is an $(m \times n, \delta, K)_q$ rank metric code with given ranks (GRMC) if it satisfies

- (1) $\operatorname{rank}(\boldsymbol{D}) \in K$ for any $\boldsymbol{D} \in \mathcal{D}$;
- (2) $d_R(\boldsymbol{D}_1, \boldsymbol{D}_2) = \operatorname{rank}(\boldsymbol{D}_1 \boldsymbol{D}_2) \ge \delta$ for any $\boldsymbol{D}_1, \boldsymbol{D}_2 \in \mathcal{D}$ and $\boldsymbol{D}_1 \neq \boldsymbol{D}_2$.

If $|\mathcal{D}| = M$, then it is often written as an $(m \times n, M, \delta, K)_q$ -GRMC. Given m, n, K and δ , denote by $A_q^R(m \times n, \delta, K)$ the maximum number of codewords among all $(m \times n, \delta, K)_q$ -GRMCs.

When $K = \{0, 1, ..., n\}$, a GRMC is just a usual rank-metric code (not necessarily linear). When $K = \{t\}$ for $0 \le t \le n$, a GRMC is often called a *constant-rank code* (cf. [11]). Consequently, when $m \ge n$,

$$\max_{t \in K} A_q^R(m \times n, \delta, t) \le A_q^R(m \times n, \delta, K) \le q^{m(n-\delta+1)}.$$

Usually, K is selected as a set of consecutive integers. Let $[t_1, t_2]$ denote the set of integers k such that $t_1 \le k \le t_2$.

Proposition 2.3 (Singleton-like upper bound) For all $0 \le i, j \le \min\{\delta - 1, t_1\}$,

$$A_{q}^{R}(m \times n, \delta, [t_{1}, t_{2}]) \leq A_{q}^{R}((m - i) \times (n - j), \delta - l, [t_{1} - l, t]),$$

where $l = \max\{i, j\}$ and $t = \min\{m - i, n - j, t_2\}$.

Proof Let \mathcal{D} be any $(m \times n, M, \delta, [t_1, t_2])_q$ -GRMC. For any $0 \le i, j \le \min\{\delta - 1, t_1\}$, the \mathcal{D}_{ij} is obtained by removing the same *i* rows and *j* columns from every codeword in \mathcal{D} . Then \mathcal{D}_{ij} is an $((m-i) \times (n-j), M, \delta - l, [t_1 - l, t])_q$ -GRMC. \Box

When $t_1 = t_2$, Proposition 2.3 provides an upper bound for constant-rank codes (see also [11, Proposition 7]).

We remark that very recently, Heinlein [15] also introduced the concept of GRMCs for the case $K = [0, t_2]$. He established a similar upper bound for GRMCs to that in Proposition 2.3. He also presented several lower bounds for GRMCs, but most focus on special parameters. Here we shall give a general construction on GRMCs.

To construct GRMCs, we need a special class of MRD codes, named Gabidulin codes. Let $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_{m-1})$ be an ordered basis of \mathbb{F}_{q^m} over \mathbb{F}_q . There is a natural bijective map Ψ_m from $\mathbb{F}_{q^m}^n$ to $\mathbb{F}_q^{m \times n}$ as follows:

$$\Psi_m : \mathbb{F}_{q^m}^n \longrightarrow \mathbb{F}_q^{m \times n}$$
$$\mathbf{a} = (a_0, a_1, \dots, a_{n-1}) \longmapsto \mathbf{A}$$

where $\mathbf{A} = \Psi_m(\mathbf{a}) \in \mathbb{F}_q^{m \times n}$ is defined such that

$$a_j = \sum_{i=0}^{m-1} A_{i,j} \beta_i$$

for any $j \in [n]$. The map Ψ_m will be used to facilitate switching between a vector in \mathbb{F}_{q^m} and its matrix representation over \mathbb{F}_q . In the sequel, we use both representations, depending on what is more convenient in the context.

For any positive integer *i* and any $a \in \mathbb{F}_{q^m}$, set $a^{[i]} \triangleq a^{q^i}$. Let $m \ge n$ and δ be a positive integer. A *Gabidulin code* $\mathcal{G}[m \times n, \delta]_q$ is an $\mathrm{MRD}[m \times n, \delta]_q$ code whose generator matrix \boldsymbol{G} in vector representation is

$$\boldsymbol{G} = \begin{pmatrix} g_0 & g_1 & \cdots & g_{n-1} \\ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-1}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-1}^{[n-\delta]} \end{pmatrix},$$

where $g_0, g_1, \ldots, g_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over \mathbb{F}_q (see [10]). Then $\mathcal{G}[m \times n, \delta]_q$ can be written as $\{ uG : u \in \mathbb{F}_{q^m}^{n-\delta+1} \}$.

Proposition 2.4 (Lower bound) Let $m \ge n$ and $1 \le \delta \le n$. Let t_1 be a nonnegative integer and t_2 be a positive integer such that $t_1 \le t_2 \le n$. Then

$$A_{q}^{R}(m \times n, \delta, [t_{1}, t_{2}]) \geq \begin{cases} \sum_{i=t_{1}}^{t_{2}} A_{i}(\delta), & t_{2} \geq \delta; \\ \\ \max_{\{1,t_{1}\} \leq a < \delta} \{ \lceil \frac{\sum_{i=\max\{1,t_{1}\}}^{t_{2}} A_{i}(a)}{q^{m(\delta-a)} - 1} \rceil \}, & t_{2} < \delta; \end{cases}$$

where $A_i(x)$ denotes the number of codewords with rank i in an $MRD[m \times n, x]_q$ code.

Proof When $t_2 \geq \delta$, all codewords with ranks from $[t_1, t_2]$ in an MRD $[m \times n, \delta]_q$ code form an $(m \times n, \delta, [t_1, t_2])_q$ -GRMC with $\sum_{i=t_1}^{t_2} A_i(\delta)$ codewords. Thus $A_q^R(m \times n, \delta, [t_1, t_2]) \geq \sum_{i=t_1}^{t_2} A_i(\delta)$ for any $t_2 \geq \delta$.

When $t_2 < \delta$, take any integer *a* such that $\max\{1, t_1\} \leq a < \delta$. Let \mathcal{D}_1 be a $\mathcal{G}[m \times n, \delta]_q$ code whose generator matrix \mathbf{G}_1 in vector representation is

$$m{G}_1 = \left(egin{array}{ccccc} g_0 & g_1 & \cdots & g_{n-1} \ g_0^{[1]} & g_1^{[1]} & \cdots & g_{n-1}^{[1]} \ dots & dots & \ddots & dots \ g_{0}^{[n-\delta]} & g_1^{[n-\delta]} & \cdots & g_{n-1}^{[n-\delta]} \end{array}
ight),$$

where $g_0, g_1, \ldots, g_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over \mathbb{F}_q . Let \mathcal{D}_2 be a $\mathcal{G}[m \times n, n - \delta + a + 1]_q$ code whose generator matrix \mathbf{G}_2 in vector representation is

$$\boldsymbol{G}_{2} = \begin{pmatrix} g_{0}^{[n-\delta+1]} & g_{1}^{[n-\delta+1]} & \cdots & g_{n-1}^{[n-\delta+1]} \\ g_{0}^{[n-\delta+2]} & g_{1}^{[n-\delta+2]} & \cdots & g_{n-1}^{[n-\delta+2]} \\ \vdots & \vdots & \ddots & \vdots \\ g_{0}^{[n-a]} & g_{1}^{[n-a]} & \cdots & g_{n-1}^{[n-a]} \end{pmatrix}.$$

Note that $g_0^{[n-\delta+1]}, g_1^{[n-\delta+1]}, \cdots, g_{n-1}^{[n-\delta+1]}$ are also linearly independent over \mathbb{F}_q . Then

$$\bigcup_{\boldsymbol{D}_2\in\mathcal{D}_2}\bigcup_{\boldsymbol{D}_1\in\mathcal{D}_1}(\boldsymbol{D}_2+\boldsymbol{D}_1)$$

is a $\mathcal{G}[m \times n, a]_q$ code whose generator matrix is $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$. For any $D_2 \in \mathcal{D}_2 \setminus \{O_{m \times n}\}$, where $O_{m \times n}$ is the $m \times n$ zero matrix, set

$$\mathcal{S}_{D_2} = \{ \boldsymbol{D} \in \bigcup_{\boldsymbol{D}_1 \in \mathcal{D}_1} (\boldsymbol{D}_2 + \boldsymbol{D}_1) \mid \operatorname{rank}(\boldsymbol{D}) \in [t_1, t_2] \}.$$

For any $D_1, D'_1 \in \mathcal{D}_1$ and $D_1 \neq D'_1$, we have rank $((D_2 + D_1) - (D_2 + D'_1)) \geq \delta$. It follows that S_{D_2} forms an $(m \times n, M_{D_2}, \delta, [t_1, t_2])_q$ -GRMC, where by the pigeonhole principle,

$$M_{D_2} \ge \lceil \frac{\sum_{i=\max\{1,t_1\}}^{t_2} A_i(a)}{|\mathcal{D}_2| - 1} \rceil = \lceil \frac{\sum_{i=\max\{1,t_1\}}^{t_2} A_i(a)}{q^{m(\delta - a)} - 1} \rceil.$$

Note that due to $\delta > t_2$, if $t_1 > 0$, then there is no codeword in \mathcal{D}_1 with rank $i \in [t_1, t_2]$. Therefore, $A_q^R(m \times n, \delta, [t_1, t_2]) \ge \max_{\max\{1, t_1\} \le a < \delta} \{ \lceil \frac{\sum_{i=\max\{1, t_1\}}^{t_2} A_i(a)}{q^{m(\delta-a)} - 1} \rceil \}.$

2.2 Generalization of Theorem 2.1

Construction 2.5 (Parallel construction) Let $n \ge 2k \ge 2\delta$. If there exists a $(k \times (n-k), M, \delta, [0, k-\delta])_q$ -GRMC, then there exists an $(n, q^{(n-k)(k-\delta+1)} + M, 2\delta, k)_q$ -CDC, which contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ -CDC as a subset.

Proof Let \mathcal{D}_1 be an MRD $[k \times (n-k), \delta]_q$ code and \mathcal{D}_2 be a $(k \times (n-k), M, \delta, [0, k-\delta])_q$ -GRMC. Let $\mathcal{C}_1 = \{\text{rowspace}(\boldsymbol{I}_k \mid \boldsymbol{A}) : \boldsymbol{A} \in \mathcal{D}_1\}$ and $\mathcal{C}_2 = \{\text{rowspace}(\boldsymbol{B} \mid \boldsymbol{I}_k) : \boldsymbol{B} \in \mathcal{D}_2\}$. Then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ forms an $(n, q^{(n-k)(k-\delta+1)} + M, 2\delta, k)_q$ -CDC.

It suffices to check the subspace distance of C. For any $\mathcal{U} = \operatorname{rowspace}(\mathbf{I}_k \mid \mathbf{A}) \in C_1$ and $\mathcal{V} = \operatorname{rowspace}(\mathbf{B} \mid \mathbf{I}_k) \in C_2$, where $\mathbf{A} = (\underbrace{\mathbf{A}_1}_{n-2k} \mid \underbrace{\mathbf{A}_2}_{k})$ and $\mathbf{B} = (\underbrace{\mathbf{B}_1}_{k} \mid \underbrace{\mathbf{B}_2}_{n-2k})$, we have

$$d_{S}(\mathcal{U}, \mathcal{V}) = 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{I}_{k} & \mathbf{A}_{1} & \mathbf{A}_{2} \\ \mathbf{B}_{1} & \mathbf{B}_{2} & \mathbf{I}_{k} \end{pmatrix} - 2k$$

$$= 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{I}_{k} & \mathbf{A}_{1} & \mathbf{A}_{2} \\ \mathbf{O} & \mathbf{B}_{2} - \mathbf{B}_{1}\mathbf{A}_{1} & \mathbf{I}_{k} - \mathbf{B}_{1}\mathbf{A}_{2} \end{pmatrix} - 2k$$

$$= 2 \cdot \operatorname{rank}(\mathbf{B}_{2} - \mathbf{B}_{1}\mathbf{A}_{1} \mid \mathbf{I}_{k} - \mathbf{B}_{1}\mathbf{A}_{2})$$

$$\geq 2 \cdot \operatorname{rank}(\mathbf{I}_{k} - \mathbf{B}_{1}\mathbf{A}_{2})$$

$$\geq 2 \cdot \operatorname{rank} \cdot (\mathbf{I}_{k} - \mathbf{B}_{1}\mathbf{A}_{2} + \mathbf{B}_{1}\mathbf{A}_{2}) - 2 \cdot \operatorname{rank}(\mathbf{B}_{1}\mathbf{A}_{2})$$

$$= 2k - 2 \cdot \operatorname{rank}(\mathbf{B}_{1}\mathbf{A}_{2})$$

$$\geq 2k - 2 \cdot \operatorname{rank}(\mathbf{B}_{1}) \geq 2k - 2 \cdot \operatorname{rank}(\mathbf{B}) \geq 2\delta.$$

Given $i \in \{1, 2\}$, for any $\mathcal{U}, \mathcal{V} \in \mathcal{C}_i$ and $\mathcal{U} \neq \mathcal{V}$, we have $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$ by the proof of Proposition 1.1.

Construction 2.5 comes from two parallel versions of lifted MRD codes, so it is named as a parallel construction. As a straightforward corollary of Construction 2.5, we have the following result.

Theorem 2.6 Let $n \ge 2k \ge 2\delta$. Then

$$\bar{A}_q(n,2\delta,k) \ge q^{(n-k)(k-\delta+1)} + A_q^R(k \times (n-k),\delta,[0,k-\delta]).$$

Theorem 2.7 Let $n \ge 2k > 2\delta > 0$. Then

$$\bar{A}_q(n, 2\delta, k) \ge q^{(n-k)(k-\delta+1)} + \begin{cases} \sum_{i=\delta}^{k-\delta} A_i(\delta) + 1, & k \ge 2\delta; \\ \max_{1 \le a < \delta} \{ \lceil \frac{\sum_{i=1}^{k-\delta} A_i(a)}{q^{m(\delta-a)} - 1} \rceil \}, & k < 2\delta, \end{cases}$$

where $A_i(x)$ denotes the number of codewords with rank *i* in an $MRD[m \times n, x]_q$ code.

Proof Apply Proposition 2.4 with $t_1 = 0$ and $t_2 = k - \delta$. Then apply Theorem 2.6. \Box

We remark that if $k \ge 2\delta$, then Theorem 2.1 is a corollary of Theorem 2.7 by taking n = 2k.

3 Parallel multilevel construction

This section is devoted to presenting two effective constructions for constant dimension codes (Constructions 3.10 and 3.17) by combining the parallel construction shown in Section 2 and the multilevel construction introduced in [6]. First we give a revisit of the multilevel construction in [6].

3.1 Preliminaries

Etzion and Silberstein [6] presented the multilevel construction for constant dimension codes by establishing relations between subspace distances and Hamming distances or rank distances (see Lemmas 3.7 and 3.8, respectively, below). The multilevel construction is a generalization of Proposition 1.1 by introducing Ferrers diagram rank-metric codes.

3.1.1 Ferrers diagram rank-metric codes

Given positive integers m and n, an $m \times n$ Ferrers diagram \mathcal{F} is an $m \times n$ array of dots and empty cells such that all dots are shifted to the right of the diagram, the number of dots in each row is less than or equal to the number of dots in the previous row, and the first row has n dots and the rightmost column has m dots.

We always denote by γ_i , $0 \leq i \leq n-1$, the number of dots in the *i*-th column of \mathcal{F} . Given positive integers m and n, and $1 \leq \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_{n-1} = m$, there exists a unique Ferrers diagram \mathcal{F} of size $m \times n$ such that the *i*-th column of \mathcal{F} has cardinality γ_i for any $0 \leq i \leq n-1$. In this case we write $\mathcal{F} = [\gamma_0, \gamma_1, \ldots, \gamma_{n-1}]$.

Example 3.1 Let $\mathcal{F} = [2, 3, 4, 5]$, where



be a 5×4 Ferrers diagram.

For a given $m \times n$ Ferrers diagram \mathcal{F} , an $[\mathcal{F}, k, \delta]_q$ Ferrers diagram rank-metric code (FDRMC), briefly an $[\mathcal{F}, k, \delta]_q$ code, is an $[m \times n, k, \delta]_q$ rank-metric codes in which for each $m \times n$ matrix, all entries not in \mathcal{F} are zero. If \mathcal{F} is a full $m \times n$ diagram with mn dots, then its corresponding FDRM code is just a classical rank-metric code. For a Ferrers diagram \mathcal{F} of size $m \times n$, one can transpose it to obtain a Ferrers diagram \mathcal{F}^t of size $n \times m$. If there exists an $[\mathcal{F}, k, \delta]_q$ code, then so does an $[\mathcal{F}^t, k, \delta]_q$ code. Etzion and Silberstein [6] established a Singleton-like upper bound on FDRMCs.

Lemma 3.2 [6, Theorem 1] Let δ be a positive integer. Let v_i , $0 \leq i \leq \delta - 1$, be the number of dots in a Ferrers diagram \mathcal{F} which are not contained in the first i rows and the rightmost $\delta - 1 - i$ columns. Then for any $[\mathcal{F}, k, \delta]_q$ code, $k \leq \min_{i \in \{0, 1, \dots, \delta - 1\}} v_i$.

An FDRMC attaining the upper bound in Lemma 3.2 is called *optimal*. Constructions for optimal FDRMCs can be found in [2,5,6,13,17,18,24,32]. We here only quote several known constructions for late use.

Theorem 3.3 [5, Theorem 3] Assume \mathcal{F} is an $m \times n$ $(m \ge n)$ Ferrers diagram and each of the rightmost $\delta - 1$ columns of \mathcal{F} has at least n dots. Then there exists an optimal $[\mathcal{F}, \sum_{i=0}^{n-\delta} \gamma_i, \delta]_q$ code for any prime power q.

Theorem 3.4 [5, Theorem 9] Let \mathcal{F}_i for i = 1, 2 be an $m_i \times n_i$ Ferrers diagram, and \mathcal{C}_i be an $[\mathcal{F}_i, k, \delta_i]_q$ code. Let \mathcal{F}_3 be an $m_3 \times n_3$ full Ferrers diagram with m_3n_3 dots, where $m_3 \ge m_1$ and $n_3 \ge n_2$. Let

$$\mathcal{F} = \left(\begin{array}{cc} \mathcal{F}_1 & \mathcal{F}_3 \\ & \mathcal{F}_2 \end{array}\right)$$

be an $m \times n$ Ferrers diagram \mathcal{F} , where $m = m_2 + m_3$ and $n = n_1 + n_3$. Then there exists an $[\mathcal{F}, k, \delta_1 + \delta_2]_q$ code \mathcal{D} such that for any $\mathbf{D} \in \mathcal{D}$, $\mathbf{D}|_{\mathcal{F}_3} = \mathbf{O}$, where $\mathbf{D}|_{\mathcal{F}_3}$ denotes the restriction of \mathbf{D} in \mathcal{F}_3 and \mathbf{O} is an $m_3 \times n_3$ zero matrix.

The following theorem is a variation of Theorem 10 in [5].

Theorem 3.5 Let $\mathcal{F}_{12} = [\gamma_0, \gamma_1, \ldots, \gamma_{n_1-1}, \gamma_{n_1}, \ldots, \gamma_{n_1+n_2-1}]$ be a $\gamma_{n_1+n_2-1} \times (n_1+n_2)$ Ferrers diagram, which induces a $\gamma_{n_1-1} \times n_1$ Ferrers diagram $\mathcal{F}_1 = [\gamma_0, \gamma_1, \ldots, \gamma_{n_1-1}]$ and a $\gamma_{n_1+n_2-1} \times n_2$ Ferrers diagram $\mathcal{F}_2 = [\gamma_{n_1}, \gamma_{n_1+1}, \ldots, \gamma_{n_1+n_2-1}]$. Let \mathcal{F}_3 be a $\gamma_{n_1-1} \times \gamma_{n_1+n_2-1}$ full Ferrers diagram with $\gamma_{n_1-1}\gamma_{n_1+n_2-1}$ dots. If there exists an $[\mathcal{F}_{12}, k, \delta]_q$ code, then there exists an $[\mathcal{F}, k, \delta]_q$ code \mathcal{D} , where

$$\mathcal{F} = \left(\begin{array}{cc} \mathcal{F}_1 & \mathcal{F}_3 \\ & \mathcal{F}_2^t \end{array}\right),$$

satisfying that for any codeword $D \in D$, $D|_{\mathcal{F}_3} = O$, where $D|_{\mathcal{F}_3}$ denotes the restriction of D in \mathcal{F}_3 and O is a $\gamma_{n_1-1} \times \gamma_{n_1+n_2-1}$ zero matrix.

Proof Let \mathcal{D}_{12} be the given $[\mathcal{F}_{12}, k, \delta]_q$ code. Set

$$\mathcal{D} = \left\{ \left(egin{array}{cc} oldsymbol{D}|_{\mathcal{F}_1} & oldsymbol{O} \ oldsymbol{O} & oldsymbol{D}|_{\mathcal{F}_2^t} \end{array}
ight) : oldsymbol{D} \in \mathcal{D}_{12}
ight\},$$

where $D|_{\mathcal{F}_1}$ denotes the restriction of D in \mathcal{F}_1 and $D|_{\mathcal{F}_2^t}$ denotes the restriction of D in \mathcal{F}_2^t . Then \mathcal{D} is an $[\mathcal{F}, k, \delta]_q$ code. One can easily verify the linearity and the dimension of \mathcal{D} . For any nonzero $D \in \mathcal{D}_{12}$, we have

$$\begin{aligned} \operatorname{rank} \begin{pmatrix} \boldsymbol{D}|_{\mathcal{F}_1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D}|_{\mathcal{F}_2^t} \end{pmatrix} &= \operatorname{rank}(\boldsymbol{D}|_{\mathcal{F}_1}) + \operatorname{rank}(\boldsymbol{D}|_{\mathcal{F}_2}) \\ &\geq \operatorname{rank}(\boldsymbol{D}|_{\mathcal{F}_1} \mid \boldsymbol{D}|_{\mathcal{F}_2}) = \operatorname{rank}(\boldsymbol{D}) \geq \delta, \end{aligned}$$

which proves the minimum rank distance of \mathcal{D} .

3.1.2 Multilevel construction

A matrix is said to be *in row echelon form* if each nonzero row has more leading zeros than the previous row. A matrix is *in reduced row echelon form* if (1) the leading coefficient of a row is always to the right of the leading coefficient of the previous row; (2) all leading coefficients are ones; (3) every leading coefficient is the only nonzero entry in its column.

A k-dimensional subspace \mathcal{U} of \mathbb{F}_q^n can be represented by a $k \times n$ generator matrix whose rows form a basis of \mathcal{U} . We usually represent a codeword of a constant dimension code by such a matrix. There is exactly one such matrix in reduced row echelon form and it will be denoted by $E(\mathcal{U})$ [6].

The *identifying vector* $\mathbf{v}(\mathcal{U})$ of a subspace $\mathcal{U} \in \mathcal{G}_q(n, k)$ is the binary vector of length n and weight k such that the 1's of $\mathbf{v}(\mathcal{U})$ are in the positions (columns) where $E(\mathcal{U})$ has its leading ones (of the rows). The *echelon Ferrers form* of a vector of length n and weight k, $EF(\mathbf{v})$, is the matrix in reduced row echelon form with leading entries (of rows) in the columns indexed by the nonzero entries of \mathbf{v} and \bullet (called a *dot*) in all entries which do not have terminals zeros or ones.

Example 3.6 Consider the subspace $\mathcal{U} \in \mathcal{G}_2(7,3)$ with reduced row echelon form

Its identifying vector $\mathbf{v}(\mathcal{U})$ is 1011000. For the identifying vector $\mathbf{v}(\mathcal{U}) = 1011000$, its echelon Ferrers form $EF(\mathbf{v}(\mathcal{U}))$ is the following 3×7 matrix:

$$EF(\boldsymbol{v}(\mathcal{U})) = \begin{pmatrix} 1 & \bullet & 0 & 0 & \bullet & \bullet \\ 0 & 0 & 1 & 0 & \bullet & \bullet \\ 0 & 0 & 0 & 1 & \bullet & \bullet \end{pmatrix}.$$

To present the multilevel construction, the following two lemmas are crucial.

Lemma 3.7 [6, Lemma 2] Let $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(n, k), \mathcal{U} = rowspace(U)$ and $\mathcal{V} = rowspace(V)$, where $U, V \in \mathbb{F}_q^{k \times n}$ are in reduced row echelon forms. Then

$$d_S(\mathcal{U}, \mathcal{V}) \geq d_H(\boldsymbol{v}(\boldsymbol{U}), \boldsymbol{v}(\boldsymbol{V}))$$

Lemma 3.8 ([6], [28, Proposition 1.2]) Let $\mathcal{U}, \mathcal{V} \in \mathcal{G}_q(n, k), \mathcal{U} = rowspace(U)$ and $\mathcal{V} = rowspace(V)$, where $U, V \in \mathbb{F}_q^{k \times n}$ are in reduced row echelon forms. If v(U) = v(V), then

$$d_S(\mathcal{U}, \mathcal{V}) = 2d_R(\boldsymbol{C}_{\boldsymbol{U}}, \boldsymbol{C}_{\boldsymbol{V}}),$$

where C_U and C_V denote the submatrices of U and V, respectively, without the columns of their pivots.

Construction 3.9 (Multilevel construction [6]) Let C be a binary Hamming code of length n, weight k and minimum Hamming distance 2δ . For each codeword $C \in C$, its echelon Ferrers form is EF(C). All dots in EF(C) produce a Ferrers diagram \mathcal{F}_C . If there exists an $[\mathcal{F}_C, k_C, \delta]_q$ code \mathcal{D}_C for each $C \in C$, then by Lemmas 3.7 and 3.8, the row spaces of the matrices in $\bigcup_{C \in C} \mathcal{D}_C$ form a $(n, 2\delta, k)_q$ -CDC.

3.2 Combination of parallel construction and multilevel construction

Although the multilevel construction is effective to construct CDCs, it is not known so far how to pick up identifying vectors such that the resulting CDCs are optimal. Also, only a few infinite families on optimal FDRMCs are known in the literature. The following constructions help to reduce the choice of identifying vectors.

3.2.1 The first construction

Construction 3.10 Let $n \ge 2k$. Suppose that there exists an $(n, M_1, 2\delta, k)_q$ -CDC which is constructed via the multilevel construction satisfying that for any of its identifying vectors $\mathbf{v} = (\underbrace{\mathbf{v}^{(1)}}_{n-k} | \underbrace{\mathbf{v}^{(2)}}_{k})$, it holds that $wt(\mathbf{v}^{(1)}) \ge s \ge \delta$, where $wt(\mathbf{v}^{(1)})$ is the weight

of $\boldsymbol{v}^{(1)}$. If there exists a $(k \times (n-k), M_2, \delta, [0, s-\delta])_q$ -GRMC, then there exists an $(n, M_1 + M_2, 2\delta, k)_q$ -CDC.

Proof Let C_1 be the given $(n, M_1, 2\delta, k)_q$ -CDC and \mathcal{D} be a $(k \times (n-k), M_2, \delta, [0, s-\delta])_q$ -GRMC. Set $C_2 = \{\text{rowspace}(\boldsymbol{B} \mid \boldsymbol{I}_k) : \boldsymbol{B} \in \mathcal{D}\}$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then \mathcal{C} is an $(n, M_1 + M_2, 2\delta, k)_q$ -CDC.

 $M_1 + M_2, 20, \kappa)_{q} \text{-CDC}.$ It suffices to examine the subspace distance of \mathcal{C} . For any $\mathcal{U} \in \mathcal{C}_1$, let $(\underbrace{\mathbf{A}}_{n-k} | \underbrace{\mathbf{D}}_{k})$ be

the reduced row echelon form of \mathcal{U} . We have rank $(\mathbf{A}) \geq s$ by the fact that any identifying vector \mathbf{v} satisfies wt $(\mathbf{v}^{(1)}) \geq s$. For any $\mathcal{V} \in \mathcal{C}_2$,

$$d_{S}(\mathcal{U}, \mathcal{V}) = 2 \cdot \operatorname{rank} \begin{pmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \boldsymbol{B} & \boldsymbol{I}_{k} \end{pmatrix} - 2k$$
$$= 2 \cdot \operatorname{rank} \begin{pmatrix} \boldsymbol{A} - \boldsymbol{D} \boldsymbol{B} & \boldsymbol{O} \\ \boldsymbol{B} & \boldsymbol{I}_{k} \end{pmatrix} - 2k$$
$$= 2 \cdot \operatorname{rank}(\boldsymbol{A} - \boldsymbol{D} \boldsymbol{B})$$
$$\geq 2s - 2 \cdot \operatorname{rank}(\boldsymbol{D} \boldsymbol{B})$$
$$\geq 2s - 2 \cdot \operatorname{rank}(\boldsymbol{B}) \geq 2s - 2(s - \delta) = 2\delta$$

For any $\mathcal{U}, \mathcal{V} \in \mathcal{C}_2$ and $\mathcal{U} \neq \mathcal{V}$, we know $d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta$ by Lemma 3.8.

Construction 3.10 shows a criteria to combine the parallel construction and the multilevel construction to produce CDCs with large size. When $n \ge 2k + \delta$ and $k \ge 2\delta$, applying Construction 3.10, in what follows we shall present better lower bounds on $\bar{A}_q(n, 2\delta, k)$ than that in Theorem 2.6.

Lemma 3.11 Let $n \ge 2k + \delta$ and $k \ge 2\delta$. Let q be any prime power and

$$M_1 = q^{(n-k)(k-\delta+1)} \frac{1 - q^{-\lfloor \frac{k}{\delta} \rfloor \delta^2}}{1 - q^{-\delta^2}} + q^{(n-k-\delta)(k-\delta+1)}.$$

Then there exists an $(n, M_1, 2\delta, k)_q$ -CDC constructed via the multilevel construction satisfying that for any of its identifying vectors $\mathbf{v} = (\underbrace{\mathbf{v}^{(1)}}_{n-k} | \underbrace{\mathbf{v}^{(2)}}_{k})$, it holds that $wt(\mathbf{v}^{(1)}) \ge k$, and this CDC contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ -CDC as a subset.

Proof We construct the set of identifying vectors of length n as follows:

$$\mathcal{A} = \{(\underbrace{1\ldots 1}_{k}\underbrace{0\ldots 0}_{n-k})\} \cup \{(\boldsymbol{u} \mid \underbrace{1\ldots 1}_{\delta}\underbrace{0\ldots 0}_{n-k-\delta}): \boldsymbol{u} \in \mathcal{B}\},$$

where

$$\mathcal{B} = \{(\underbrace{1\dots 1}_{k-\delta} \underbrace{0\dots 0}_{\delta}), (\underbrace{1\dots 1}_{k-2\delta} \underbrace{0\dots 0}_{\delta} \underbrace{1\dots 1}_{\delta}), \dots, (\underbrace{1\dots 1}_{k-\lfloor \frac{k}{\delta} \rfloor \delta} \underbrace{0\dots 0}_{\delta} \underbrace{1\dots 1}_{\lfloor \frac{k}{\delta} \rfloor -1)\delta})\}.$$

The size of \mathcal{B} is $\lfloor \frac{k}{\delta} \rfloor$, and each vector in \mathcal{B} has size k and weight $k - \delta$ (note that the δ zeros are always shifted to the left by δ positions). It is readily checked that for any $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{A}, \, \boldsymbol{v} \neq \boldsymbol{v}'$, we have $d_H(\boldsymbol{v}, \boldsymbol{v}') \geq 2\delta$.

For any identifying vector $\mathbf{v}_i = (\underbrace{\mathbf{v}_i^{(1)}}_{n-k} | \underbrace{\mathbf{v}_i^{(2)}}_{k}) = (\underbrace{1 \dots 1}_{k-i\delta} \underbrace{0 \dots 0}_{\delta} \underbrace{1 \dots 1}_{i\delta} \underbrace{0 \dots 0}_{n-k-\delta}) \in \mathcal{A}, \ 0 \leq \mathbf{v}_i = (\underbrace{\mathbf{v}_i^{(1)}}_{n-k} | \underbrace{\mathbf{v}_i^{(2)}}_{k}) = (\underbrace{1 \dots 1}_{k-i\delta} \underbrace{0 \dots 0}_{\delta} \underbrace{1 \dots 1}_{i\delta} \underbrace{0 \dots 0}_{n-k-\delta}) \in \mathcal{A}, \ 0 \leq \mathbf{v}_i = (\underbrace{\mathbf{v}_i^{(1)}}_{n-k} | \underbrace{\mathbf{v}_i^{(2)}}_{k}) = (\underbrace{1 \dots 1}_{k-i\delta} \underbrace{0 \dots 0}_{\delta} \underbrace{1 \dots 1}_{i\delta} \underbrace{0 \dots 0}_{n-k-\delta}) \in \mathcal{A}, \ 0 \leq \mathbf{v}_i = (\underbrace{1 \dots 1}_{k-i\delta} \underbrace{0 \dots 0}_{k-i\delta} \underbrace{1 \dots 1}_{i\delta} \underbrace{0 \dots 0}_{n-k-\delta}) \in \mathcal{A}, \ 0 \leq \mathbf{v}_i = (\underbrace{1 \dots 1}_{k-i\delta} \underbrace{0 \dots 0}_{k-i\delta} \underbrace{1 \dots 1}_{i\delta} \underbrace{0 \dots 0}_{n-k-\delta})$

 $i \leq \lfloor \frac{k}{\delta} \rfloor$, we have wt $(v_i^{(1)}) \geq k$ because of $n - k \geq k + \delta$. The echelon Ferrers form of v_i is

$$EF(\boldsymbol{v}_i) = \begin{pmatrix} \boldsymbol{I}_{k-i\delta} & \mathcal{F}_1 & \boldsymbol{O}_1 & \mathcal{F}_2 \\ \boldsymbol{O}_2 & \boldsymbol{O}_3 & \boldsymbol{I}_{i\delta} & \mathcal{F}_3 \end{pmatrix},$$

where \mathcal{F}_1 is a $(k - i\delta) \times \delta$ full Ferrers diagram, \mathcal{F}_2 is a $(k - i\delta) \times (n - k - \delta)$ full Ferrers diagram, \mathcal{F}_3 is an $i\delta \times (n - k - \delta)$ full Ferrers diagram, O_1 is a $(k - i\delta) \times i\delta$ zero matrix, O_2 is an $i\delta \times (k - i\delta)$ zero matrix and O_3 is an $i\delta \times \delta$ zero matrix. For $0 \le i \le \lfloor \frac{k}{\delta} \rfloor$, let

$$\mathcal{H}_i = [\underbrace{k - i\delta, \dots, k - i\delta}_{\delta}, \underbrace{k, \dots, k}_{n-k-\delta}]$$

be a $k \times (n-k)$ Ferrers diagram, and

$$\mathcal{C}_{i} = \left\{ \text{rowspace} \left(\begin{array}{ccc} \boldsymbol{I}_{k-i\delta} & \boldsymbol{D}_{1} & \boldsymbol{O}_{1} & \boldsymbol{D}_{2} \\ \boldsymbol{O}_{2} & \boldsymbol{O}_{3} & \boldsymbol{I}_{i\delta} & \boldsymbol{D}_{3} \end{array} \right) : \left(\begin{array}{ccc} \boldsymbol{D}_{1} & \boldsymbol{D}_{2} \\ \boldsymbol{O} & \boldsymbol{D}_{3} \end{array} \right) \in \mathcal{C}_{\mathcal{H}_{i}} \right\},$$

where $C_{\mathcal{H}_i}$ is an $[\mathcal{H}_i, (n-k)(k-\delta+1) - i\delta^2, \delta]_q$ code when $0 \le i \le \lfloor \frac{k}{\delta} \rfloor - 1$, and $C_{\mathcal{H}_i}$ is an $[\mathcal{H}_i, (n-k-\delta)(k-\delta+1), \delta]_q$ code when $i = \lfloor \frac{k}{\delta} \rfloor$.

Here we need to examine the existence of $\mathcal{C}_{\mathcal{H}_i}$. Consider the transpose of \mathcal{H}_i :

$$\mathcal{H}_{i}^{t} = [\underbrace{n-k-\delta, \dots, n-k-\delta}_{i\delta}, \underbrace{n-k, \dots, n-k}_{k-i\delta}].$$

For the $(n-k) \times k$ Ferrers diagram \mathcal{H}_i^t , every column of \mathcal{H}_i^t contains at least $n-k-\delta$ dots. Since $n - k - \delta \ge k$, by Theorem 3.3, there exists an optimal $[\mathcal{H}_i^t, \delta]_q$ code. Now it remains to analyse its dimension. If $k - i\delta \geq \delta$, i.e., $i \leq \lfloor \frac{k}{\delta} \rfloor - 1$, then by Theorem 3.3, the optimal dimension is equal to the number of dots in \mathcal{H}_i^t which are not contained in the rightmost $\delta - 1$ columns: $i\delta(n-k-\delta) + (k-i\delta-\delta+1)(n-k) = (n-k)(k-\delta+1) - i\delta^2$. Thus there exists an optimal $[\mathcal{H}_i^t, (n-k)(k-\delta+1) - i\delta^2, \delta]_q$ code. If $0 \leq k - i\delta < \delta$, i.e, $i = \lfloor \frac{k}{\delta} \rfloor$, then the optimal dimension is $(n - k - \delta)(k - \delta + 1)$. Thus there exists an optimal $[\mathcal{H}_i^t, (n-k-\delta)(k-\delta+1), \delta]_q$ code.

Let $\mathcal{C} = \bigcup_{i=0}^{\lfloor \frac{k}{\delta} \rfloor} \mathcal{C}_i$. Then

$$\begin{aligned} |\mathcal{C}| &= \sum_{i=0}^{\lfloor \frac{k}{\delta} \rfloor - 1} q^{(n-k)(k-\delta+1) - i\delta^2} + q^{(n-k-\delta)(k-\delta+1)} \\ &= q^{(n-k)(k-\delta+1)} \frac{1 - q^{-\lfloor \frac{k}{\delta} \rfloor \delta^2}}{1 - q^{-\delta^2}} + q^{(n-k-\delta)(k-\delta+1)} \end{aligned}$$

For any $\mathcal{U} \in \mathcal{C}_i$ and $\mathcal{V} \in \mathcal{C}_j$, $0 \leq i, j \leq \lfloor \frac{k}{\delta} \rfloor$, $i \neq j$, by Lemma 3.7, we have $d_s(\mathcal{U}, \mathcal{V}) \geq d_s(\mathcal{U}, \mathcal{V})$ $d_H(\boldsymbol{v}_i, \boldsymbol{v}_j) \geq 2\delta$. For any $\mathcal{U}, \mathcal{V} \in \mathcal{C}_i, 0 \leq i \leq \lfloor \frac{k}{\delta} \rfloor$, by Lemma 3.8, we have $d_s(\mathcal{U}, \mathcal{V}) \geq 2\delta$. Thus C is an $(n, M_1, 2\delta, k)_q$ -CDC. This CDC contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)})$ $(2\delta, k)_q$ -CDC as a subset which comes from the identifying vector $(\underbrace{1 \dots 1}_k \underbrace{0 \dots 0}_{n-k})$.

Applying Lemma 3.11 and Construction 3.10 with s = k, we obtain the following result.

Theorem 3.12 Let $n \ge 2k + \delta$ and $k \ge 2\delta$. Then

$$\bar{A}_{q}(n, 2\delta, k) \ge q^{(n-k)(k-\delta+1)} \frac{1-q^{-\lfloor \frac{k}{\delta} \rfloor \delta^{2}}}{1-q^{-\delta^{2}}} + q^{(n-k-\delta)(k-\delta+1)} + A_{q}^{R}(k \times (n-k), \delta, [0, k-\delta]).$$

Theorem 3.12 together with the use of Proposition 2.4 provides many new constant dimension codes with larger size than the previously best known codes in [16]. We list some of them in Table 1 in Appendix B.

When $\delta = 2$, Theorem 3.12 can be improved by using the following CDCs constructed in [24].

Lemma 3.13 [24, Construction B, Corollary 27] Let $n \ge 2k+2$ and $k \ge 4$. Let q be any prime power and

$$M_1 = \sum_{j=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} \left(q^{(k-1)(n-jk)} + \frac{(q^{2(k-2)} - 1)(q^{2(n-jk-1)} - 1)}{(q^4 - 1)^2} q^{(k-3)(n-jk-2) + 4} \right).$$

Then there exists an $(n, M_1, 4, k)_a$ -CDC constructed via the multilevel construction satis-

fying that for any of its identifying vectors $\boldsymbol{v} = (\underbrace{\boldsymbol{v}^{(1)}}_{n-k} | \underbrace{\boldsymbol{v}^{(2)}}_{k})$, it holds that $wt(\boldsymbol{v}^{(1)}) \ge k-2$, and this CDC contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 4, k)_q$ -CDC as a subset.

Proof (sketch only) We here employ the same set of identifying vectors as those in Construction B of [24] (note that the identifying vectors are exhibited in [24] recursively, and here we list all of them explicitly):

$$\mathcal{A} = \{ (\underbrace{0 \dots 0}_{ik} \underbrace{1 \dots 1}_{k} \underbrace{0 \dots 0}_{n-(i+1)k}) : 0 \le i \le \lfloor \frac{n-2}{k} \rfloor - 2 \} \cup \{ (\underbrace{0 \dots 0}_{ik} | \underbrace{\boldsymbol{w}}_{k} | \underbrace{\boldsymbol{z}}_{n-(i+1)k}) : \boldsymbol{w} \in \mathcal{B}, \ \boldsymbol{z} \in \mathcal{D}_{i+1}, \ 0 \le i \le \lfloor \frac{n-2}{k} \rfloor - 2 \},$$

where

$$\mathcal{B} = \{(\underbrace{1\dots1}_{k-2}00), (\underbrace{1\dots1}_{k-4}0011), \dots, u\}, \quad u = \begin{cases} (00\underbrace{11\dots1}_{k-2}), & \text{if } k \text{ is even}; \\ (100\underbrace{1\dots1}_{k-3}), & \text{if } k \text{ is odd}, \end{cases}$$

and

$$\mathcal{D}_{i+1} = \{ (11 \underbrace{00...00}_{n-(i+1)k-2}), (0011 \underbrace{00...00}_{n-(i+1)k-4}), \dots, u' \}, \\ u' = \begin{cases} (\underbrace{00...00}_{n-(i+1)k-2} \\ (\underbrace{00...00}_{n-(i+1)k-3} \\ 110), & \text{if } n-(i+1)k \text{ is odd.} \end{cases}$$

The size of \mathcal{B} is $\lfloor \frac{k}{2} \rfloor$, and each vector in \mathcal{B} has size k and weight k-2 (note that the two zeros are always shifted to the left by two positions). The size of \mathcal{D}_{i+1} is $\lfloor \frac{n-(i+1)k}{2} \rfloor$, and each vector in \mathcal{D}_{i+1} has size n-(i+1)k and weight 2 (note that the two ones are always shifted to the left by two positions).

Since $i \leq \lfloor \frac{n-2}{k} \rfloor - 2$, we have $n - k \geq (i+1)k$. Then for any identifying vector $\boldsymbol{v} = (\underbrace{\boldsymbol{v}^{(1)}}_{n-k} | \underbrace{\boldsymbol{v}^{(2)}}_{k})$ shown above, one can check that $\operatorname{wt}(\boldsymbol{v}^{(1)}) \geq k - 2$. By Construction B

and Corollary 27 in [24], these identifying vectors generate an $(n, M_1, 4, k)_q$ -CDC. This CDC contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 4, k)_q$ -CDC as a subset which comes from the identifying vector $(\underbrace{1 \dots 1}_{k} \underbrace{0 \dots 0}_{n-k})$.

Theorem 3.14 Let $n \ge 2k + 2$ and $k \ge 4$. Then

$$\bar{A}_q(n,4,k) \ge \sum_{j=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} \left(q^{(k-1)(n-jk)} + \frac{(q^{2(k-2)} - 1)(q^{2(n-jk-1)} - 1)}{(q^4 - 1)^2} q^{(k-3)(n-jk-2)+4} \right) + A_q^R(k \times (n-k), 2, [0, k-4]).$$

Proof Applying Lemma 3.13 and Construction 3.10 with s = k - 2.

With the aid of a computer, we compared the values of $\bar{A}_q(n, 4, k)$ from Theorems 3.12 and 3.14. It seems that when k > 4 and n is large enough, Theorem 3.14 always produces better lower bound on $\bar{A}_q(n, 4, k)$ than that in Theorem 3.12 (see Table 2 in Appendix B for example).

More specially, for $\delta = 4$ and k = 5, Silberstein and Trautmann in [24] presented an $(n, M_1, 4, 5)_q$ -CDC with larger size than that in Lemma 3.13 by choosing identifying vectors more carefully.

Lemma 3.15 [24, Construction C-5, Theorem 35] Let $n \ge 12$ and q be any prime power. If x is even, then let

$$\begin{split} M(x) &:= q^{4(x-5)} + (q^{2x-10} + q^{2x-14})(q^{2x-14} + \frac{x-8}{2}q^{x-9}) + (q^{2x-11} + q^{2x-13}) \\ &\times (\frac{x-8}{2}q^{x-10} + q^{2x-15}) + (q^{2x-12} + q^{2x-13})(2q^{2x-16} + \frac{x-10}{2}q^{x-11}) \\ &+ (q^{2x-12} + q^{2x-14}) \times \left[\sum_{i=3}^{\min\{\lceil \frac{q}{2} \rceil + 2, \lfloor \frac{x-5}{2} \rfloor\}} (iq^{2x-2i-12} + (\frac{x-6}{2} - i)q^{x-2i-7}) \right. \\ &+ \left. \sum_{i=2}^{\min\{\lfloor \frac{q}{2} \rfloor + 2, \lceil \frac{x-7}{2} \rceil\}} (iq^{2x-2i-13} + (\frac{x-6}{2} - i)q^{x-2i-8}) \right]. \end{split}$$

If x is odd, then let

$$\begin{split} M(x) &:= q^{4(x-5)} + (q^{2x-10} + q^{2x-14})(q^{2x-14} + \frac{x-9}{2}q^{x-8} + q^{\frac{x-9}{2}}) + (q^{2x-11} + q^{2x-13}) \\ &\times (\frac{x-9}{2}q^{x-9} + q^{2x-15} + q^{x-8}) + (q^{2x-12} + q^{2x-13})(q^{2x-16} + \frac{x-11}{2}q^{x-10} + q^{\frac{x-11}{2}}) \\ &+ (q^{2x-12} + q^{2x-14}) \times \left[\sum_{i=3}^{\min\{\lceil \frac{q}{2} \rceil + 2, \lfloor \frac{x-5}{2} \rfloor\}} (iq^{2x-2i-12} + (\frac{x-7}{2} - i)q^{x-2i-6} + q^{\frac{x-7}{2}-i}) \right. \\ &+ \sum_{i=2}^{\min\{\lfloor \frac{q}{2} \rfloor + 2, \lceil \frac{x-7}{2} \rceil\}} (iq^{2x-2i-13} + (\frac{x-7}{2} - i)q^{x-2i-7} + q^{x-i-7}) \right]. \end{split}$$

Let

$$M_1 = \sum_{j=0}^{\lfloor \frac{n-12}{5} \rfloor} M(n-5j).$$

Then there exists an $(n, M_1, 4, 5)_q$ -CDC constructed via the multilevel construction satisfying that for any of its identifying vectors $\mathbf{v} = (\underbrace{\mathbf{v}^{(1)}}_{n-5} | \underbrace{\mathbf{v}^{(2)}}_{5})$, it holds that $wt(\mathbf{v}^{(1)}) \ge 3$, and this CDC contains a lifted MRD code $(n, q^{4(n-5)}, 4, 5)_q$ -CDC as a subset.

Proof (sketch only) We here employ the same set of identifying vectors as those in Construction C-5 of [24] (note that the identifying vectors are exhibited in [24] recursively,

and here we list all of them explicitly):

$$\begin{split} &\{(11111 \mid \underbrace{0 \dots 0}_{n-5})\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 11100 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 10011 \mid u): \ u \in P_{j, \lceil \frac{n-5j-5}{2} \rceil+1}, \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 11010 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 01101 \mid u): \ u \in P_{j, 2}, \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 01110 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 10101 \mid u): \ u \in P_{j, \lceil \frac{n-5j-5}{2} \rceil+2}, \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 00111 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 11001 \mid u): \ u \in P_{j, 3}, \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 10110 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 01011 \mid u): \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\} \cup \\ &\{(\underbrace{0 \dots 0}_{5j} \mid 10110 \mid u), (\underbrace{0 \dots 0}_{5j} \mid 01011 \mid u): \ 0 \le j \le \lfloor \frac{n-12}{5} \rfloor\}, \\ &u \in \begin{pmatrix} \min\{\lceil \frac{q}{2} \rceil+2, \lfloor \frac{n-5j-5}{2} \rfloor\} \\ \bigcup \\ & \underset{i=3}{} P_{j, \lceil \frac{n-5j-5}{2} \rceil+i} \end{pmatrix} \cup \begin{pmatrix} \min\{\lfloor \frac{q}{2} \rfloor+3, \lceil \frac{n-5j-5}{2} \rceil\} \\ & \underset{i=4}{} P_{j,i} \end{pmatrix}\}, \end{split}$$

where each vector in $P_{j,l}$ has size n - 5j - 5 and weight 2; if n - 5j - 5 is even, then the positions of ones in vectors from $P_{j,2}, \ldots, P_{j,n-5j-5}$ correspond to a one-factorization of the complete graph K_{n-5j-5} ; if n - 5j - 5 is odd, then the positions of ones in vectors from $P_{j,1}, P_{j,2}, \ldots, P_{j,n-5j-5}$ correspond to a near one-factorization of the complete graph K_{n-5j-5} (see [24, Construction C-5] for more details).

Since $j \leq \lfloor \frac{n-12}{5} \rfloor$, we have 5j + 5 < n - 5. Then for any identifying vector $\boldsymbol{v} = (\underbrace{\boldsymbol{v}^{(1)}}_{n-5} | \underbrace{\boldsymbol{v}^{(2)}}_{5})$ shown above, one can check that $\operatorname{wt}(\boldsymbol{v}^{(1)}) \geq 3$. By Construction C-5 and

Theorem 35 in [24], these identifying vectors generate an $(n, M_1, 4, 5)_q$ -CDC. This CDC contains a lifted MRD code $(n, q^{4(n-5)}, 4, 5)_q$ -CDC as a subset which comes from the identifying vector (11111 | $\underbrace{0 \dots 0}_{n-5}$).

Theorem 3.16 Let M_1 be as in Lemma 3.15 and $n \ge 12$. Then

$$\bar{A}_q(n,4,5) \ge M_1 + q^4 + q^3 + q^2 + q + 1.$$

Proof Applying Lemma 3.15 and Construction 3.10 with s = 3, we have

$$\bar{A}_q(n,4,5) \ge M_1 + A_q^R(5 \times (n-5), 2, \{0,1\}).$$

By Proposition 2.4 and Theorem 2.2, one can check that

$$A_q^R(5 \times (n-5), 2, \{0, 1\}) \ge q^4 + q^3 + q^2 + q + 1.$$

This completes the proof.

Theorem 3.16 provides many new $(n, 4, 5)_q$ -CDCs with larger size than the previously best known codes in [16]. We list some of them in Table 2 in Appendix B. Compared with Theorem 3.16, Theorems 3.12 and 3.14 produce worse lower bounds of $\bar{A}_q(n, 4, 5)$ for those n in Table 2.

3.2.2 The second construction

In Construction 3.10, we start from a multilevel construction, and then choose an appropriate parallel construction based on the identifying vectors in the multilevel construction. Actually, we can also start from a parallel construction, and then choose appropriate identifying vectors to use the multilevel construction.

Construction 3.17 Let $n \ge 2k$ and $k \ge 2\delta$. If there exists a $(k \times (n-k), M, \delta, [0, k-\delta])_q$ -GRMC, then there exists an $(n, q^{(n-k)(k-\delta+1)} + M + q^{\max\{l_1, l_2\}} + q^{(n-k-\delta)(k-2\delta+1)}, 2\delta, k)_q$ -CDC, where

$$l_1 = \begin{cases} (k-\delta)\delta + n - k - \delta, & \text{if } n \ge k + 3\delta; \\ \delta(n-4\delta+2), & \text{if } n < k + 3\delta, \end{cases}$$

and

$$l_2 = \max_{1 \le j \le \delta - 1} \{ \min\{(\delta - j + 1)(k - \delta), (j + 1)(n - k - \delta) \} \}.$$

This CDC contains a lifted MRD code $(n, q^{(n-k)(k-\delta+1)}, 2\delta, k)_q$ -CDC as a subset.

Proof Let \mathcal{D}_1 be an MRD $[k \times (n-k), \delta]_q$ code and \mathcal{D}_2 be a $(k \times (n-k), M, \delta, [0, k-\delta])_q$ -GRMC. Set

 $C_1 = \{ \text{rowspace}(I_k \mid A) : A \in D_1 \}$

and

$$\mathcal{C}_2 = \{ \operatorname{rowspace}(\boldsymbol{B} \mid \boldsymbol{I}_k) : \boldsymbol{B} \in \mathcal{D}_2 \}.$$

Since $n \ge 2k$, $|\mathcal{C}_1| = q^{(n-k)(k-\delta+1)}$. Note that the identifying vector of each codeword in \mathcal{C}_1 is $\boldsymbol{n} = (\underbrace{1 \dots 1}_{k} \underbrace{0 \dots 0}_{n-k})$. Now we take two new identifying vectors \boldsymbol{n}_1 and \boldsymbol{n}_2 .

Step 1. Take

$$\boldsymbol{n}_1 = (\underbrace{1\ldots 1}_{k-\delta} \underbrace{0\ldots 0}_{\delta} \underbrace{1\ldots 1}_{\delta} \underbrace{0\ldots 0}_{n-k-\delta}).$$

Note that $d_H(\boldsymbol{n}, \boldsymbol{n}_1) = 2\delta$. Then

$$EF(\boldsymbol{n}_1) = \begin{pmatrix} \boldsymbol{I}_{k-\delta} & \mathcal{F}_1 & \boldsymbol{O}_1 & \mathcal{F}_3 \\ \boldsymbol{O}_2 & \boldsymbol{O}_3 & \boldsymbol{I}_\delta & \mathcal{F}_2 \end{pmatrix},$$

where \mathcal{F}_1 is a $(k-\delta) \times \delta$ full Ferrers diagram, \mathcal{F}_2 is a $\delta \times (n-k-\delta)$ full Ferrers diagram, \mathcal{F}_3 is a $(k-\delta) \times (n-k-\delta)$ full Ferrers diagram, O_1 is a $(k-\delta) \times \delta$ zero matrix, O_2 is a $\delta \times (k-\delta)$ zero matrix and O_3 is a $\delta \times \delta$ zero matrix. Let

$$\mathcal{F} = \left(\begin{array}{cc} \mathcal{F}_1 & \mathcal{F}_3 \\ & \mathcal{F}_2 \end{array}\right)$$

be a $k \times (n-k)$ Ferrers diagram and

$$\mathcal{C}_3 = \left\{ \text{rowspace} \left(\begin{array}{ccc} \boldsymbol{I}_{k-\delta} & \boldsymbol{D}_1 & \boldsymbol{O}_1 & \boldsymbol{D}_3 \\ \boldsymbol{O}_2 & \boldsymbol{O}_3 & \boldsymbol{I}_\delta & \boldsymbol{D}_2 \end{array} \right) : \left(\begin{array}{ccc} \boldsymbol{D}_1 & \boldsymbol{D}_3 \\ \boldsymbol{O}_3 & \boldsymbol{D}_2 \end{array} \right) \in \mathcal{D}_{\mathcal{F}} \right\},$$

where $\mathcal{D}_{\mathcal{F}}$ is a Ferrers diagram rank-metric code in \mathcal{F} with minimum rank distance δ , which will be constructed as follows in two different ways.

(1) We claim that there exists an $[\mathcal{F}, (k-\delta)\delta + n - k - \delta, \delta]_q$ code $\mathcal{D}'_{\mathcal{F}}$ when $n \ge k+3\delta$, and there exists an $[\mathcal{F}, \delta(n-4\delta+2), \delta]_q$ code $\mathcal{D}'_{\mathcal{F}}$ when $2k \le n < k+3\delta$.

To examine the existence of such FDRMC codes, take an $(n - k - \delta) \times 2\delta$ Ferrers diagram

$$\mathcal{F}_{12} = (\mathcal{F}_1 \mid \mathcal{F}_2^t) = [\underbrace{k - \delta, \dots, k - \delta}_{\delta}, \underbrace{n - k - \delta, \dots, n - k - \delta}_{\delta}].$$

Note that

$$\mathcal{F}_{12}^t = [\underbrace{\delta, \dots, \delta}_{n-2k}, \underbrace{2\delta, \dots, 2\delta}_{k-\delta}].$$

By Theorem 3.3, when $n - k - \delta \geq 2\delta$, i.e., $n \geq k + 3\delta$, there exists an optimal $[\mathcal{F}_{12}, (k - \delta)\delta + n - k - \delta, \delta]_q$ code, and when $n < k + 3\delta$, since $k - \delta \geq \delta$, there exists an optimal $[\mathcal{F}_{12}, \delta(n - 4\delta + 2), \delta]_q$ code, which yields an $[\mathcal{F}_{12}, \delta(n - 4\delta + 2), \delta]_q$ code. Now applying Theorem 3.5, we obtain an $[\mathcal{F}, (k - \delta)\delta + n - k - \delta, \delta]_q$ code $\mathcal{D}'_{\mathcal{F}}$ when $n \geq k + 3\delta$, and an $[\mathcal{F}, \delta(n - 4\delta + 2), \delta]_q$ code $\mathcal{D}'_{\mathcal{F}}$ when $2k \leq n < k + 3\delta$. Note that each codeword in $\mathcal{D}'_{\mathcal{F}}$ is of the form $\begin{pmatrix} * & \mathbf{O}_4 \\ \mathbf{O}_3 & * \end{pmatrix}$, where \mathbf{O}_4 is a $(k - \delta) \times (n - k - \delta)$ zero matrix. (2) We claim that there exists an

$$[\mathcal{F}, \max_{1 \le j \le \delta - 1} \{ \min\{(\delta - j + 1)(k - \delta), (j + 1)(n - k - \delta) \} \}, \delta]_q$$

code $\mathcal{D}''_{\mathcal{F}}$. Its existence comes from Theorem 3.4 by using an optimal $[\mathcal{F}_1, (\delta - j + 1)(k - \delta), j]_q$ code, i.e., an $\mathrm{MRD}[(k-\delta) \times \delta, j]_q$ code, and an optimal $[\mathcal{F}_2, (j+1)(n-k-\delta), \delta - j]_q$ code, i.e., an $\mathrm{MRD}[\delta \times (n-k-\delta), \delta - j]_q$ code (note that $\delta \leq n-k-\delta$ since $n \geq 2k$ and $k \geq 2\delta$). Each codeword in $\mathcal{D}''_{\mathcal{F}}$ is of the form $\begin{pmatrix} * & O_4 \\ O_3 & * \end{pmatrix}$.

Let l_1 and l_2 be as in the assumption. If $l_1 \ge l_2$, then take $\mathcal{D}_{\mathcal{F}} = \mathcal{D}'_{\mathcal{F}}$. Otherwise, take $\mathcal{D}_{\mathcal{F}} = \mathcal{D}''_{\mathcal{F}}$. Thus $|\mathcal{C}_3| = |\mathcal{D}_{\mathcal{F}}| = q^{\max\{l_1, l_2\}}$. Note that $\mathbf{D}_3 = \mathbf{O}_4$ in both cases. Step 2. Take

$$m{n}_2 = (\underbrace{1\ldots 1}_{k-2\delta} \underbrace{0\ldots 0}_{\delta} \underbrace{1\ldots 1}_{\delta} \underbrace{1\ldots 1}_{\delta} \underbrace{0\ldots 0}_{n-k-\delta}).$$

Note that $d_H(\boldsymbol{n}, \boldsymbol{n}_2) = d_H(\boldsymbol{n}_1, \boldsymbol{n}_2) = 2\delta$. Then

$$EF(\boldsymbol{n}_2) = \begin{pmatrix} \boldsymbol{I}_{k-2\delta} & \mathcal{F}_4 & \boldsymbol{O}_5 & \boldsymbol{O}_5 & \mathcal{F}_5 \\ \boldsymbol{O}_6 & \boldsymbol{O}_7 & \boldsymbol{I}_\delta & \boldsymbol{O}_7 & \mathcal{F}_6 \\ \boldsymbol{O}_6 & \boldsymbol{O}_7 & \boldsymbol{O}_7 & \boldsymbol{I}_\delta & \mathcal{F}_7 \end{pmatrix},$$

where \mathcal{F}_4 is a $(k-2\delta) \times \delta$ full Ferrers diagram, \mathcal{F}_5 is a $(k-2\delta) \times (n-k-\delta)$ full Ferrers diagram, \mathcal{F}_6 and \mathcal{F}_7 are $\delta \times (n-k-\delta)$ full Ferrers diagrams, \mathcal{O}_5 is a $(k-2\delta) \times \delta$ zero matrix, \mathcal{O}_6 is a $\delta \times (k-2\delta)$ zero matrix and \mathcal{O}_7 is a $\delta \times \delta$ zero matrix. Let

$$\mathcal{C}_4 = \left\{ \text{rowspace} \left(\begin{array}{cccc} \boldsymbol{I}_{k-2\delta} & \boldsymbol{O}_5 & \boldsymbol{O}_5 & \boldsymbol{O}_5 & \boldsymbol{D}_4 \\ \boldsymbol{O}_6 & \boldsymbol{O}_7 & \boldsymbol{I}_\delta & \boldsymbol{O}_7 & \boldsymbol{O}_8 \\ \boldsymbol{O}_6 & \boldsymbol{O}_7 & \boldsymbol{O}_7 & \boldsymbol{I}_\delta & \boldsymbol{D}_5 \end{array} \right) : \left(\begin{array}{c} \boldsymbol{D}_4 \\ \boldsymbol{D}_5 \end{array} \right) \in \mathcal{D}_3 \right\},$$

where \mathcal{D}_3 is an MRD $[(k - \delta) \times (n - k - \delta), \delta]_q$ code. Since $n - k \ge k$, $|\mathcal{C}_4| = |\mathcal{D}_3| = q^{(n-k-\delta)(k-2\delta+1)}$.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Then \mathcal{C} is an $(n, q^{(n-k)(k-\delta+1)} + M + q^{\max\{l_1, l_2\}} + q^{(n-k-\delta)(k-2\delta+1)}, 2\delta, k)_q$ -CDC.

It suffices to examine the subspace distance of C. Let $\mathcal{U}_i \in C_i$ for $1 \leq i \leq 4$. Since $d_H(\boldsymbol{n}, \boldsymbol{n}_1) = d_H(\boldsymbol{n}, \boldsymbol{n}_2) = d_H(\boldsymbol{n}_1, \boldsymbol{n}_2) = 2\delta$, by Lemma 3.7, $d_S(\mathcal{U}_1, \mathcal{U}_3)$, $d_S(\mathcal{U}_1, \mathcal{U}_4)$ and

 $d_S(\mathcal{U}_3, \mathcal{U}_4)$ are all no less than 2δ . Applying Construction 3.10 with s = k, we have $d_S(\mathcal{U}_1, \mathcal{U}_2) \geq 2\delta$. Since $k \geq 2\delta$,

$$d_{S}(\mathcal{U}_{2},\mathcal{U}_{3}) = 2 \cdot \operatorname{rank} \begin{pmatrix} \mathbf{I}_{k-\delta} & \mathbf{D}_{1} & \mathbf{O}_{9} & \mathbf{O}_{10} \\ \mathbf{O}_{2} & \mathbf{O}_{3} & \mathbf{D}_{6} & \mathbf{D}_{7} \\ \mathbf{B} & \mathbf{I}_{k} \end{pmatrix} - 2k$$
$$\geq 2 \left(\operatorname{rank} \left(\mathbf{I}_{k-\delta} & \mathbf{D}_{1} & \mathbf{O}_{9} \right) + \operatorname{rank}(\mathbf{I}_{k}) \right) - 2k = 2(k-\delta) \geq 2\delta,$$

where $(\underbrace{O_9}_{n-2k} | \underbrace{O_{10}}_k) = (O_1 | O_4)$ and $(\underbrace{D_6}_{n-2k} | \underbrace{D_7}_k) = (I_\delta | D_2)$. Similarly we have

$$d_{S}(\mathcal{U}_{2},\mathcal{U}_{4}) = 2 \cdot \operatorname{rank} \begin{pmatrix} I_{k-2\delta} & O_{5} & O_{5} & D_{8} & D_{9} \\ O_{6} & O_{7} & \boxed{I_{\delta}} & O_{11} & O_{12} \\ O_{6} & O_{7} & O_{7} & D_{10} & D_{11} \\ \hline B & \hline I_{k} \end{pmatrix} - 2k \geq 2\delta,$$

where $(\underbrace{D_8}_{n-2k} | \underbrace{D_9}_k) = (O_5 | D_4), (\underbrace{O_{11}}_{n-2k} | \underbrace{O_{12}}_k) = (O_7 | O_8) \text{ and } (\underbrace{D_{10}}_{n-2k} | \underbrace{D_{11}}_k) = (I_\delta | D_5).$ Finally, by Lemma 3.8, for any $\mathcal{U}, \mathcal{V} \in \mathcal{C}_i$ and $\mathcal{U} \neq \mathcal{V}, i \in \{1, 2, 3, 4\}, d_S(\mathcal{U}, \mathcal{V}) \geq 2\delta.$ \Box

In the proof of Construction 3.17, three identifying vectors are used: $(\underbrace{1...10...0}_{k})$,

 $\underbrace{(\underbrace{1\dots1}_{k-\delta}\underbrace{0\dots0}_{\delta}\underbrace{1\dots1}_{k-k-\delta}\underbrace{0\dots0}_{n-k-\delta})_{n-k-\delta} \text{ and } \underbrace{(\underbrace{1\dots1}_{k-2\delta}\underbrace{0\dots0}_{\delta}\underbrace{1\dots1}_{\delta}\underbrace{1\dots1}_{\delta}\underbrace{0\dots0}_{n-k-\delta})_{n-k-\delta}}_{\delta}. \text{ Since } n \ge 2k, \text{ for each each identifying vectors } \boldsymbol{v} = \underbrace{(\underbrace{\boldsymbol{v}}_{n-k}^{(1)} \mid \underbrace{\boldsymbol{v}}_{k}^{(2)})_{k}}_{k}, \text{ it holds that } wt(\boldsymbol{v}^{(1)}) \ge k-\delta. \text{ Therefore, if we}$

apply Construction 3.10 via the three identifying vectors to produce CDCs, then we have to construct a $(k \times (n - k), M_2, \delta, [0, k - 2\delta])_q$ -GRMC. However, in Construction 3.17, we can use a $(k \times (n - k), M, \delta, [0, k - \delta])_q$ -GRMC. Generally $M \ge M_2$. From this point of view, Construction 3.17 is better than Construction 3.10. But in Construction 3.10, one can choose other identifying vectors flexibly to change the details of the multilevel construction. From this point of view, Construction 3.10 is better. Anyway, compared with Construction 3.10, Construction 3.17 is easier to be used since its statement does not rely on the choice of identifying vectors.

Construction 3.17 together with the use of Proposition 2.4 provides many new constant dimension codes with larger size than the previously best known codes in [16]. We list some of them in Table 3 in Appendix B. Especially, when $n = 2k = 4\delta$, we obtain new lower bound on $\bar{A}_q(n, 2\delta, k)$.

 $\begin{array}{l} \textbf{Corollary 3.18} \ \ \bar{A}_q(4\delta, 2\delta, 2\delta) \geq q^{2\delta(\delta+1)} + (q^{2\delta}-1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + q^l + q^{\delta} + 1, \ where \\ \\ l = \begin{cases} 2, \quad \text{if } \delta = 1; \\ (\lfloor \frac{\delta}{2} \rfloor + 1)\delta, & \text{if } \delta \geq 2. \end{cases} \end{array}$

Proof By Theorems 2.4 and 2.2, $A_q^R(2\delta \times 2\delta, \delta, [0, \delta]) \ge (q^{2\delta} - 1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1$. Apply Construction 3.17 with $n = 4\delta$ and $k = 2\delta$. Then $l_1 = 2\delta$, $l_2 = (\lfloor \frac{\delta}{2} \rfloor + 1)\delta$, and so $\max\{l_1, l_2\} = l$.

Combining Corollary 3.18 and Theorem 1.2(3) with $n = 4\delta$ and $k = 2\delta$, we have the following corollary.

Corollary 3.19 Let $\delta \geq 2$. Then

$$q^{2\delta(\delta+1)} + (q^{2\delta} - 1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + q^{(\lfloor \frac{\delta}{2} \rfloor + 1)\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + (q^{2\delta} + q^{\delta}) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1 \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta(\delta+1)} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{2\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{\delta} + q^{\delta} + 1 \le \bar{A}_q(4\delta, \ 2\delta) \le q^{\delta} + 1 \le \bar$$

Remark 3.20 We can calculate the ratio between the lower bound and the upper bound of the CDCs in Corollary 3.19:

$$\frac{\text{the lower bound of } \mathcal{C}}{\text{the upper bound of } \mathcal{C}} = \frac{q^{2\delta(\delta+1)} + (q^{2\delta} - 1) \left\lfloor \frac{2\delta}{\delta} \right\rfloor_q + q^{(\lfloor\frac{\delta}{2}\rfloor+1)\delta} + q^{\delta} + 1}{q^{2\delta(\delta+1)} + (q^{2\delta} + q^{\delta}) \left\lfloor \frac{2\delta}{\delta} \right\rfloor_q + 1}$$
$$= 1 - \frac{(q^{\delta} + 1) \left\lfloor \frac{2\delta}{\delta} \right\rfloor_q - q^{(\lfloor\frac{\delta}{2}\rfloor+1)\delta} - q^{\delta}}{q^{2\delta(\delta+1)} + (q^{2\delta} + q^{\delta}) \left\lfloor \frac{2\delta}{\delta} \right\rfloor_q + 1} \ge \frac{4642}{4797} > 0.967688.$$

Furthermore, for $\delta \geq 3$,

$$\frac{\text{the lower bound of } \mathcal{C}}{\text{the upper bound of } \mathcal{C}} \ge \frac{16865174}{16877657} > 0.99926.$$

To ensure smooth reading of the paper, we move the proof to Appendix A.

There is no systematic approach so far to give a $(4\delta, 2\delta, 2\delta)_q$ -CDC attaining the lower bound in Corollary 3.19 for general δ . In principle, people can always pick up suitable identifying vectors and then use the multilevel construction to construct an optimal $(4\delta, 2\delta, 2\delta)_q$ -CDC. However, how to choose identifying vectors effectively is still an open and different problem. The combination of the parallel construction and the multilevel construction helps to weaken the requirement for identifying vectors and provides good constant dimension codes with large size.

4 Concluding remarks

Constructions 3.10 and 3.17 for CDCs are established in this paper by combining the parallel construction and the multilevel construction. How to choose identifying vectors compatible with a given parallel construction as many as possible is still an open problem. This paper initials the study.

GRMCs play a fundamental role in our constructions. It is meaningful to investigate various constructions for GRMCs as an independent research topic.

A Appendix

Proof of Remark 3.20 Write

$$f(\delta) = \frac{(q^{\delta}+1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q - q^{(\lfloor \frac{\delta}{2} \rfloor+1)\delta} - q^{\delta}}{q^{2\delta(\delta+1)} + (q^{2\delta}+q^{\delta}) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1}.$$

We claim that given any prime power q, $f(\delta)$ is a non-increasing function on δ for any $\delta \geq 2$. We have

$$\begin{split} f(\delta) &- f(\delta+1) \\ &= \frac{\left(\left(q^{\delta}+1\right) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q - q^{\left(\lfloor\frac{\delta}{2}\rfloor+1\right)\delta} - q^{\delta} \right) \left(q^{2(\delta+1)(\delta+2)} + \left(q^{2\delta+2} + q^{\delta+1}\right) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q + 1 \right)}{\left(q^{2(\delta+1)(\delta+2)} + \left(q^{2\delta+2} + q^{\delta+1}\right) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q + 1 \right) \left(q^{2\delta(\delta+1)} + \left(q^{2\delta} + q^{\delta}\right) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1 \right)} \\ &- \frac{\left(\left(q^{\delta+1}+1\right) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q - q^{\left(\lfloor\frac{\delta+1}{2}\rfloor+1\right)(\delta+1)} - q^{\delta+1} \right) \left(q^{2\delta(\delta+1)} + \left(q^{2\delta} + q^{\delta}\right) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1 \right)}{\left(q^{2(\delta+1)(\delta+2)} + \left(q^{2\delta+2} + q^{\delta+1}\right) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q + 1 \right) \left(q^{2\delta(\delta+1)} + \left(q^{2\delta} + q^{\delta}\right) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1 \right)}. \end{split}$$

Let

$$\begin{split} g(\delta) &= \left((q^{\delta}+1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q - q^{(\lfloor \frac{\delta}{2} \rfloor+1)\delta} - q^{\delta} \right) \left(q^{2(\delta+1)(\delta+2)} + (q^{2\delta+2}+q^{\delta+1}) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q + 1 \right) \\ &- \left((q^{\delta+1}+1) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q - q^{(\lfloor \frac{\delta+1}{2} \rfloor+1)(\delta+1)} - q^{\delta+1} \right) \left(q^{2\delta(\delta+1)} + (q^{2\delta}+q^{\delta}) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q + 1 \right) \end{split}$$

It suffices to verify $g(\delta) \ge 0$ for any $\delta \ge 2$.

Since
$$q^{\delta+j} - 1 \ge (q^j - 1)q^{\delta}$$
, when $\delta \ge 3$, we have $\begin{bmatrix} 2\delta\\ \delta \end{bmatrix}_q = \frac{(q^{2\delta} - 1)\cdots(q^{\delta+1} - 1)}{(q^{\delta} - 1)\cdots(q^{-1})} \ge q^{\delta^2} \ge q^{(\lfloor\frac{\delta}{2}\rfloor+1)\delta} + q^{\delta}$. When $\delta = 2$, $\begin{bmatrix} 4\\ 2 \end{bmatrix}_q = q^4 + q^3 + 2q^2 + q + 1 > q^{(\lfloor\frac{2}{2}\rfloor+1)2} + q^2$. So $\begin{bmatrix} 2\delta\\ \delta \end{bmatrix}_q \ge q^{(\lfloor\frac{\delta}{2}\rfloor+1)\delta} + q^{\delta}$ for any $\delta \ge 2$. It follows that

$$\begin{split} g(\delta) &\geq q^{\delta} \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_{q} \left(q^{2(\delta+1)(\delta+2)} + (q^{2\delta+2} + q^{\delta+1}) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_{q} \right) - \left((q^{\delta+1}+1) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_{q} \right) \times \\ \left(q^{2\delta(\delta+1)} + (q^{2\delta} + q^{\delta}) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_{q} + 1 \right). \end{split}$$

Write

$$g_1(\delta) = q^{\delta} \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q (q^{2\delta+2} + q^{\delta+1}) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q - (q^{\delta+1} + 1) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q (q^{2\delta} + q^{\delta}) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q$$

and

$$g_2(\delta) = q^{\delta} \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q q^{2(\delta+1)(\delta+2)} - (q^{\delta+1}+1) \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q \left(q^{2\delta(\delta+1)}+1\right).$$

Then $g(\delta) \ge g_1(\delta) + g_2(\delta)$. Clearly

$$g_1(\delta) = \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_q \begin{bmatrix} 2\delta+2 \\ \delta+1 \end{bmatrix}_q \left(q^{3\delta+2} - q^{3\delta+1} - q^{2\delta} - q^{\delta} \right) \ge 0.$$

It remains to examine $g_2(\delta) \ge 0$. Since $\begin{bmatrix} 2\delta+2\\\delta+1 \end{bmatrix}_q = \frac{(q^{2\delta+2}-1)(q^{2\delta+1}-1)}{(q^{\delta+1}-1)^2} \begin{bmatrix} 2\delta\\\delta \end{bmatrix}_q$ and

$$q^{\delta}(q^{\delta+1}-1) \leq q^{2\delta+1} - 1 \leq (q^{\delta}+1)(q^{\delta+1}-1),$$

we have $q^{\delta}(q^{\delta+1}+1) \begin{bmatrix} 2\delta\\\delta \end{bmatrix}_q \leq \begin{bmatrix} 2\delta+2\\\delta+1 \end{bmatrix}_q \leq (q^{\delta}+1)(q^{\delta+1}+1) \begin{bmatrix} 2\delta\\\delta \end{bmatrix}_q$. Thus

$$g_{2}(\delta) \geq q^{\delta} \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_{q} q^{2(\delta+1)(\delta+2)} - (q^{\delta+1}+1)(q^{\delta}+1)(q^{\delta+1}+1) \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_{q} (q^{2\delta(\delta+1)}+1)$$
$$= \begin{bmatrix} 2\delta \\ \delta \end{bmatrix}_{q} (q^{2\delta^{2}+7\delta+4} - q^{2\delta^{2}+5\delta+2} - q^{2\delta^{2}+4\delta+2} - 2q^{2\delta^{2}+4\delta+1} - 2q^{2\delta^{2}+3\delta+1} - q^{2\delta^{2}+3\delta}$$
$$- q^{2\delta^{2}+2\delta} - q^{3\delta+2} - q^{2\delta+2} - 2q^{2\delta+1} - 2q^{\delta+1} - q^{\delta} - 1) \geq 0.$$

So given any prime power $q, f(\delta)$ is a non-increasing function on δ for any $\delta \ge 2$. Since

$$\begin{split} f(2) &= \frac{\left(q^2+1\right) \left[\frac{4}{2}\right]_q - q^4 - q^2}{q^{12} + \left(q^4+q^2\right) \left[\frac{4}{2}\right]_q + 1} = \frac{\left(q^2+1\right) \left(q^4+q^3+q^2+q+1\right)}{q^{12} + \left(q^4+q^2\right) \left(q^2+1\right) \left(q^2+q+1\right) + 1} \\ &= \frac{q^6+q^5+2q^4+2q^3+2q^2+q+1}{q^{12}+q^8+q^7+3q^6+2q^5+3q^4+q^3+q^2+1} \leq \frac{155}{4797}, \end{split}$$

we have for any $\delta \geq 2$,

the lower bound of
$$\mathcal{C}$$

the upper bound of \mathcal{C} = 1 - $f(\delta) \ge 1 - f(2) \ge 1 - \frac{155}{4797} > 0.967688.$

Similarly since

$$f(3) = \frac{(q^3 + 1) \begin{bmatrix} 6\\3 \end{bmatrix}_q - q^6 - q^3}{q^{24} + (q^6 + q^3) \begin{bmatrix} 6\\3 \end{bmatrix}_q + 1} \le \frac{12483}{16877657},$$

we have for any $\delta \geq 3$,

$$\frac{\text{the lower bound of } \mathcal{C}}{\text{the upper bound of } \mathcal{C}} = 1 - f(\delta) \ge 1 - f(3) \ge 1 - \frac{12483}{16877657} > 0.99926.$$

_			
		L	
		L	
ᄂ			

B Appendix

Table 1:	Constant	dimension	codes	from	Theorem	3.12	and	[16]
Lower bounds for $\bar{A}_q(n, 2\delta, k)$								

\overline{A} (m ΩS la)	Theorem 2.19	[16]		
$A_q(n, 20, \kappa)$	1 HeOrein 5.12	150102261281024288		
$A_3(15, 0, 0)$	150102545990840750	150102201281924288		
$A_4(15, 6, 6)$	4722384778841908199452	4722384497336874172416		
$A_5(15, 6, 6)$	14551922738557090682988320	14551922678951263378341888		
$A_7(15, 6, 6)$	2651730911763599010817616	2651730911572017468075166		
	918746	138368		
$A_8(15, 6, 6)$	3245185560810007534450532	3245185560762783660124143		
_	05203320	69988608		
$A_9(15, 6, 6)$	2252839960316867802912978	2252839960308891273979904		
	0303636252	2841640960		
$\bar{A}_3(16,6,6)$	12158306011246213950	12158283163835867136		
$\bar{A}_4(16,6,6)$	1208930503358636748324892	1208930431318239788138496		
$\bar{A}_5(16,6,6)$	909495171159508342705798	909495167434454027503612		
	8320	7232		
$\bar{A}_7(16,6,6)$	6366805919144396570740848	6366805918684414419355934		
	233189146	306336768		
$\bar{A}_8(16,6,6)$	1329228005707779000468281	1329228005688436187186849		
	480881260920	259473338368		
$\bar{A}_9(16, 6, 6)$	1478088297963896954275969	1478088297958663518971939		
	56677838811932	31748755374080		
$\bar{A}_3(17,6,6)$	984822786754900790910	984820936270705197056		
$\bar{A}_4(17,6,6)$	309486208859711440279978012	309486190417469385763454976		
$\bar{A}_5(17,6,6)$	56843448197469116506558079	56843447964653369704091609		
	88320	98912		
$\bar{A}_7(17, 6, 6)$	15286701011865696133769150	15286701010761278864076273		
• • • • • • •	164214433946	642983391232		
$\bar{A}_{8}(17, 6, 6)$	54445179113790627852329396	54445179112998346227173345		
0(') ') ')	89634331511160	66802793955328		
$\bar{A}_{0}(17, 6, 6)$	96977373229411279169036992	96977373229067909497849520		
9(, -, -, -)	0953064514284572	2646735567454208		
$\bar{A}_{2}(17, 6, 7)$	717934513945606807214448	717934462541344066764800		
$\bar{A}_4(17, 6, 7)$	1267655437138714999914722	1267655435949954604087111		
114(11,0,1)	735680	581696		
$\bar{A}_{\rm F}(17, 6, 7)$	8881788744768854329316669	8881788744477090184835827		
113(11,0,1)	4301204500	3727594496		
$\bar{A}_{7}(17, 6, 7)$	1798465087215432610771286	1798465087215053471877135		
117(11,0,1)	951668456912186048	499494277221711872		
$\bar{A}_{0}(17, 6, 7)$	1/272/770333982/50/317286	1/272/77033307838/7337612		
118(11,0,1)	302786023730070637248	952679951618625503232		
$\bar{A}_{0}(17, 6, 7)$	5153775990699033098375560	5153775920622008/19230032		
A9(17, 0, 7)	7/561316380979313895860	5807388/692/98/76976736		
$\bar{A}_{a}(18.6.7)$	17//5808613305056060/000029	17//5807/3075/6630638906076		
	111111000001000000000000000000000000000	111110001100110000000000000000000000000		

$\bar{A}_q(n, 2\delta, k)$	Theorem 3.12	[16]		
$\bar{A}_4(18,6,7)$	129807916760321574216291241	129807916641275351458520225		
	3988416	9656704		
$\bar{A}_5(18,6,7)$	277555898273931997862551572	277555898264909066117850548		
	409926204500	774715260928		
$\bar{A}_7(18,6,7)$	302268027208297537838299541	302268027208234077195734138		
	69875398154049929084	40701999450419101696		
$\bar{A}_8(18,6,7)$	467680527430393663434529587	467680527430380371095589012		
	83305087356415231304192	33416654639120489906176		
$\bar{A}_9(18,6,7)$	304325273002563570083080541	304325273002562128138529778		
	77589445454500732900196836	45672945438783195662254080		
$\bar{A}_3(19,6,7)$	42393314923753431509633199	42393312078603826847037784		
	312	064		
$\bar{A}_4(19,6,7)$	132923306762526363919145683	132923306640665959893524711		
	3419327040	3888464896		
$\bar{A}_5(19,6,7)$	867362182106035125792002834	867362182077840679822534739		
	689855238704500	559109441355776		
$\bar{A}_7(19,6,7)$	508021873328985670761663974	508021873328878943122344222		
	813537143982278408961152	542078792818337377157120		
$\bar{A}_8(19,6,7)$	153249555228391395614936944	153249555228387040000602607		
	5210953425471736128141435392	5616596939214700213245575168		
$\bar{A}_9(19,6,7)$	17970103045528376249648755925	17970103045528289061322277127		
	65293397244462194049577733380	01292762916333299089237082112		

Table 1 (Cont.): Constant dimension codes from Theorem 3.12 and [16] Lower bounds for $\bar{A}_q(n, 2\delta, k)$

$\bar{A}_q(n, 2\delta, k)$	Theorem 3.16	Theorem 3.14	Theorem 3.12	[16]
$\bar{A}_3(13,4,5)$	187977330080	187644030023	187623140212	187646890063
	0662	1043	3284	3708
$\bar{A}_4(13,4,5)$	185244551321	185193502027	185191080330	185193668253
	42240085	94936427	51471424	31922597
$\bar{A}_5(13,4,5)$	233220333417	233204343725	233203799712	233204366687
	60498047656	35344411636	28104104500	01425801556
$\bar{A}_7(13,4,5)$	110489772145	110488804026	110488785701	110488804410
	826259551989	935735689413	592348093173	257915997937
	4274	6924	0404	0508
$\bar{A}_8(13,4,5)$	792478237928	792475147017	792475101353	792475147744
	289801654459	480054665841	872707790607	175434285903
	47977	57579	24224	14633
$\bar{A}_{9}(13,4,5)$	343421389872	343420732650	343420724881	343420732748
	099066234534	983522582815	350622237717	274503618449
	2116660	7580092	6247844	0814932
$\bar{A}_3(14,4,5)$	152290128114	151991664671	151971000771	151993980961
	549994	267113	313764	035277
$\bar{A}_4(14,4,5)$	474234835438	474095365198	474088568287	474095790728
	9224194389	0418663364	3806466624	5780822949
$\bar{A}_5(14,4,5)$	145763149391	145752714828	145752356230	145752729179
	911682128914	382638784555	952762603545	384155801334
	06	29	00	31
$\bar{A}_{7}(14,4,5)$	265285995217	265283618468	265283573374	265283619389
	197106992212	672861039951	849294506738	029256718151
	2343434	4145353	6763204	6018577
$\bar{A}_8(14,4,5)$	324599107938	324597820218	324597801169	324597820516
	548775888100	359848132646	235252976985	014257914299
	448539209	389432068	412575744	810657417
$\bar{A}_{9}(14,4,\overline{5})$	$225318779\overline{695}$	225318342692	225318337521	225318342756
	316687971945	310290649450	979435078382	142901825894
	70848710934	77022477769	85688931844	86571731339

Table 2: Constant dimension codes from Theorem 3.12, 3.14, 3.16 and [16] Lower bounds for $\bar{A}_q(n, 2\delta, k)$

$\bar{A}_q(n, 2\delta, k)$	Construction 3.17	[16]		
$\bar{A}_2(12,6,6)$	16865174	16865101		
$\bar{A}_3(12,6,6)$	282454201878	282454201122		
$\bar{A}_4(12,6,6)$	281476519731292	281476519727132		
$\bar{A}_5(12,6,6)$	59604684750285320	59604684750269570		
$\bar{A}_7(12,6,6)$	191581237048517757994	191581237048517640002		
$\bar{A}_8(12,6,6)$	4722366523787007642488	4722366523787007379832		
$\bar{A}_9(12,6,6)$	79766443311676870585932	79766443311676870053762		
$\bar{A}_2(14,6,7)$	34532242376	34532238023		
$\bar{A}_3(14,6,7)$	50035894106925204	$500358941\overline{06387202}$		
$\bar{A}_4(14,6,7)$	1180598085852258350656	1180598085852241507904		
$\bar{A}_5(14,6,7)$	2910384996920980879329500	2910384996920980634798250		
$\bar{A}_7(14,6,7)$	378818703472375564731912769036	378818703472375564718065717034		
$\bar{A}_8(14,6,7)$	40564819558769908757756030657	40564819558769908757687294403		
	024	072		
$\bar{A}_9(14,6,7)$	25031555123615248786076588088	25031555123615248786073763362		
	37716	54514		
$A_2(16, 8, 8)$	1099562832574	1099562828461		
$A_3(16, 8, 8)$	12157665957048196644	12157665957047665122		
$\bar{A}_4(16,8,8)$	1208925820022362634893084	1208925820022362618115612		
$A_5(16, 8, 8)$	9094947017807612368246590820	9094947017807612368002449570		
$A_7(16, 8, 8)$	63668057609092569002476703621	63668057609092569002476565208		
	23204	33602		
$A_8(16, 8, 8)$	13292279957849213674394204065	13292279957849213674394203378		
	92780664	73299832		
$A_9(16, 8, 8)$	14780882941434601431198749325	14780882941434601431198749296		
	1158647364	8729104322		
$A_2(16, 6, 8)$	282927684131264	282927683836351		
$A_3(16, 6, 8)$	79773403858211769073398	79773403858211367304002		
$A_4(16, 6, 8)$	79228596795209597355803963392	79228596795209597286010744832		
$A_5(16, 6, 8)$	3552716061446350478567982091	3552716061446350478564136876		
\overline{I} (10.0.0)	625000	781250		
$A_7(10, 6, 8)$	3670336930316550640268162626	3670336930316550640268162462		
<u> </u>	0010440740004	71/1000000000		
	0312448748394	7151289328002		
$A_8(16, 6, 8)$	0312448748394 2230074539175728767236156261	7151289328002 2230074539175728767236156259		
$A_8(16, 6, 8)$	0312448748394 2230074539175728767236156261 8047925701050368	7151289328002 2230074539175728767236156259 9998342819479552		
$\frac{A_8(16,6,8)}{\bar{A}_9(16,6,8)}$	$\begin{array}{r} 0312448748394 \\ 2230074539175728767236156261 \\ 8047925701050368 \\ 6362685459865446204861526038 \\ 705050041220515522 \end{array}$	7151289328002 2230074539175728767236156259 9998342819479552 63626854598654462048615260385 54750414000421762		
$\overline{A}_8(16, 6, 8)$ $\overline{A}_9(16, 6, 8)$	0312448748394 2230074539175728767236156261 8047925701050368 6362685459865446204861526038 705059941329515532	7151289328002 2230074539175728767236156259 9998342819479552 63626854598654462048615260385 54759414900421762		
$\overline{A}_{8}(16, 6, 8)$ $\overline{A}_{9}(16, 6, 8)$ $\overline{A}_{2}(18, 8, 9)$ $\overline{A}_{4}(18, 8, 9)$	0312448748394 2230074539175728767236156261 8047925701050368 6362685459865446204861526038 705059941329515532 18015215398134856 E8140720280412667241620716	7151289328002 2230074539175728767236156259 9998342819479552 63626854598654462048615260385 54759414900421762 18015215398068295 58140720280417667108523046		
$\overline{A}_{8}(16, 6, 8)$ $\overline{A}_{9}(16, 6, 8)$ $\overline{A}_{2}(18, 8, 9)$ $\overline{A}_{3}(18, 8, 9)$ $\overline{A}_{4}(18, 8, 9)$	0312448748394 2230074539175728767236156261 8047925701050368 6362685459865446204861526038 705059941329515532 18015215398134856 58149739380417667241629716 22451855276784208642221712	7151289328002 2230074539175728767236156259 9998342819479552 63626854598654462048615260385 54759414900421762 18015215398068295 58149739380417667198523946 29451855276784208642921288		
$\overline{A}_{8}(16, 6, 8)$ $\overline{A}_{9}(16, 6, 8)$ $\overline{A}_{2}(18, 8, 9)$ $\overline{A}_{3}(18, 8, 9)$ $\overline{A}_{4}(18, 8, 9)$	$\begin{array}{r} 0312448748394 \\ 2230074539175728767236156261 \\ 8047925701050368 \\ 6362685459865446204861526038 \\ 705059941329515532 \\ 18015215398134856 \\ 58149739380417667241629716 \\ 32451855376784298642321718 \\ 2266044 \end{array}$	$\begin{array}{r} 7151289328002\\ 2230074539175728767236156259\\ 9998342819479552\\ 63626854598654462048615260385\\ 54759414900421762\\ 18015215398068295\\ 58149739380417667198523946\\ 32451855376784298642321288\\ 6951072\end{array}$		
$\overline{A}_{8}(16, 6, 8)$ $\overline{A}_{9}(16, 6, 8)$ $\overline{A}_{2}(18, 8, 9)$ $\overline{A}_{3}(18, 8, 9)$ $\overline{A}_{4}(18, 8, 9)$ $\overline{A}_{5}(18, 8, 9)$	$\begin{array}{r} 0312448748394\\ 2230074539175728767236156261\\ 8047925701050368\\ 6362685459865446204861526038\\ 705059941329515532\\ 18015215398134856\\ 58149739380417667241629716\\ 32451855376784298642321718\\ 2266944\\ 55511151221725878257060116\\ \end{array}$	7151289328002 2230074539175728767236156259 9998342819479552 63626854598654462048615260385 54759414900421762 18015215398068295 58149739380417667198523946 32451855376784298642321288 6251072 55511151221725878257060116		
$\overline{A}_{8}(16, 6, 8)$ $\overline{A}_{9}(16, 6, 8)$ $\overline{A}_{2}(18, 8, 9)$ $\overline{A}_{3}(18, 8, 9)$ $\overline{A}_{4}(18, 8, 9)$ $\overline{A}_{5}(18, 8, 9)$	$\begin{array}{r} 0312448748394\\ 2230074539175728767236156261\\ 8047925701050368\\ 6362685459865446204861526038\\ 705059941329515532\\ 18015215398134856\\ 58149739380417667241629716\\ 32451855376784298642321718\\ 2266944\\ 55511151231735878357960116\\ 981761704500\\ \end{array}$	$\begin{array}{c} 7151289328002\\ 2230074539175728767236156259\\ 9998342819479552\\ 63626854598654462048615260385\\ 54759414900421762\\ 18015215398068295\\ 58149739380417667198523946\\ 32451855376784298642321288\\ 6251072\\ 55511151231735878357960116\\ 829164048250\\ \end{array}$		

Table 3: Constant dimension codes from Construction 3.17 and [16] Lower bounds for $\bar{A}_q(n,2\delta,k)$

$\bar{A}_a(n, 2\delta, k)$	Construction 3.17	[16]
$\bar{A}_7(18, 8, 9)$	43181145673965918176230160	43181145673965918176230160
	95285332497749370596	95285299264536325746
$\bar{A}_8(18,8,9)$	58460065493236358379340343	58460065493236358379340343
	02923933871658428965376	02923933590182378512896
$\bar{A}_9(18,8,9)$	33813919135227284246202802	33813919135227284246202802
	47018514715266280331502884	47018514713413256655866642
$\bar{A}_2(18,6,9)$	9271545156585415680	9271545156551861247
$\bar{A}_3(18,6,9)$	11446612801881132293137038	11446612801881132287488447
	59802	86840
$\bar{A}_4(18,6,9)$	85071058146182803276503914	85071058146182803276503351
	069802090496	119848669184
$\bar{A}_5(18,6,9)$	10842028996571097790669084	108420289965710977906690845
	5136306309921875000	017097020371093750
$\bar{A}_7(18,6,9)$	174251503388975551318884922	174251503388975551318884922
	599369849935281754612330830	599369466772818993479502028
$\bar{A}_8(18,6,9)$	78463772372191979113838163463	78463772372191979113838163463
	5235743275201736965558894592	5235733830468771226268467200
$\bar{A}_9(18,6,9)$	1310020512493866339206870302329	1310020512493866339206870302329
	188713507904303585133560754636	188713348371417431388541027914

Table 3 (Cont.): Constant dimension codes from Construction 3.17 and [16] Lower bounds for $\bar{A}_q(n, 2\delta, k)$

References

- R. Ahlswede and H. Aydinian, On error control codes for random network coding, in Proc. Workshop Netw. Coding Theory Appl., 2009, 68–73.
- [2] J. Antrobus and H. Gluesing-Luerssen, Maximal Ferrers diagram codes: constructions and genericity considerations, *IEEE Trans. Inf. Theory*, 65 (2019), 6204–6223.
- [3] H. Chen, X. He, J. Weng, and L. Xu, New constructions of subspace codes using subsets of MRD codes in several blocks, arXiv:1908.03804v1.
- [4] P. Delsarte, Bilinear forms over a finite field, with applications to coding theory, J. Combin. Theory A, 25 (1978), 226–241.
- [5] T. Etzion, E. Gorla, A. Ravagnani and A. Wachter-Zeh, Optimal Ferrers diagram rank-metric codes, *IEEE Trans. Inf. Theory*, 62 (2016), 1616–1630.
- [6] T. Etzion and N. Silberstein, Error-correcting codes in projective spaces via rankmetric codes and Ferrers diagrams, *IEEE Trans. Inf. Theory*, 55 (2009), 2909–2919.
- [7] T. Etzion and N. Silberstein, Codes and designs related to lifted MRD codes, *IEEE Trans. Inf. Theory*, 59 (2013), 1004–1017.
- [8] T. Etzion and A. Vardy, Error-correcting codes in projective spaces, *IEEE Trans.* Inf. Theory, 57 (2011), 1165–1173.

- [9] P. Frankl and R.M. Wilson, The Erdös-Ko-Rado theorem for vector spaces, J. Combin. Theory A, 43 (1986), 228–236.
- [10] É.M. Gabidulin, Theory of codes with maximum rank distance, Problems Inf. Transmiss., 21 (1985), 3–16.
- [11] M. Gadouleau and Z. Yan, Constant-rank codes and their connection to constantdimension codes, *IEEE Trans. Inform. Theory*, 56 (2010), 3207–3216.
- [12] H. Gluesing-Luerssen and C. Troha, Construction of subspace codes through linkage, Adv. in Math. of Comm., 10 (2016), 525–540.
- [13] E. Gorla and A. Ravagnani, Subspace codes from Ferrers diagrams, J. Algebra and its Appl., 16 (2017), 1750131.
- [14] D. Heinlein, New LMRD code bounds for constant dimension codes and improved constructions, *IEEE Trans. Inform. Theory*, 65 (2019), 4822–4830.
- [15] D. Heinlein, Generalized linkage construction for constant-dimension codes, arXiv:1910.11195v2.
- [16] D. Heinlein, M. Kiermaier, S. Kurz, and A. Wassermann, Tables of subspace codes, http://subspacecodes.uni-bayreuth.de.
- [17] S. Liu, Y. Chang and T. Feng, Constructions for optimal Ferrers diagram rankmetric codes, *IEEE Trans. Inf. Theory*, 65 (2019), 4115–4130.
- [18] S. Liu, Y. Chang and T. Feng, Several classes of optimal Ferrers diagram rank-metric codes, *Linear Algebra and its Appl.*, 581 (2019), 128–144.
- [19] A. Kohnert and S. Kurz, Construction of large constant dimension codes with a prescribed minimum distance, *Lecture Notes Comp. Sci.*, 5393 (2008), 31–42.
- [20] R. Kötter and F.R. Kschischang, Coding for errors and erasures in random network coding, *IEEE Trans. Inf. Theory*, 54 (2008), 3579–3591.
- [21] R.M. Roth, Maximum-rank array codes and their application to crisscross error correction, *IEEE Trans. Inf. Theory*, 37 (1991), 328–336.
- [22] N. Silberstein and T. Etzion, Large constant dimension codes and lexicodes, Adv. in Math. of Comm., 5 (2011), 177–189.
- [23] N. Silberstein and T. Etzion, Enumerative coding for Grassmannian space, IEEE Trans. Inf. Theory, 57 (2011), 365–374.
- [24] N. Silberstein and A.-L. Trautmann, Subspace codes based on graph matchings, Ferrers diagrams, and pending blocks, *IEEE Trans. Inf. Theory*, 61 (2015), 3937– 3953.
- [25] D. Silva, F.R. Kschischang, and R. Kötter, A rank-metric approach to error control in random network coding, *IEEE Trans. Inf. Theory*, 54 (2008), 3951–3967.
- [26] V. Skachek, Recursive code construction for random networks, *IEEE Trans. Inf. Theory*, 56 (2010), 1378–1382.

- [27] A.-L. Trautmann, F. Manganiello, M. Braun, and J. Rosenthal, Cyclic orbit codes, *IEEE Trans. Inf. Theory*, 59 (2013), 7386–7404.
- [28] A.-L. Trautmann and J. Rosenthal, New improvements on the Echelon-Ferrers construction, in *Proc.* 19th Int. Symp. Math. Theory Netw. Syst., Jul. (2010), 405–408.
- [29] H. Wang, C. Xing, and R. Safavi-Naini, Linear authentication codes: bounds and constructions, *IEEE Trans. Inf. Theory*, 49 (2003), 866–872.
- [30] S.-T. Xia and F.-W. Fu, Johnson type bounds on constant dimension codes, Des. Codes Cryptogr., 50 (2009), 163–172.
- [31] L. Xu and H. Chen, New constant-dimension subspace codes from maximum rank distance codes, *IEEE Trans. Inf. Theory*, 64 (2018), 6315–6319.
- [32] T. Zhang and G. Ge, Constructions of optimal Ferrers diagram rank metric codes, Des. Codes Cryptogr., 87 (2019), 107–121.