

Secure communication over fully quantum Gel'fand-Pinsker wiretap channel

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Abstract

In this work we study the problem of secure communication over a fully quantum Gel'fand-Pinsker channel. The best known achievability rate for this channel model in the classical case was proven by Goldfeld, Cuff and Permuter in [2]. We generalize the result of [2]. One key feature of the results obtained in this work is that all the bounds obtained are in terms of error exponent. We obtain our achievability result via the technique of simultaneous pinching. This in turn allows us to show the existence of a simultaneous decoder. Further, to obtain our encoding technique and to prove the security feature of our coding scheme we prove a bivariate classical-quantum channel resolvability lemma and a conditional classical-quantum channel resolvability lemma. As a by product of the achievability result obtained in this work, we also obtain an achievable rate for a fully quantum Gel'fand-Pinsker channel in the absence of Eve. The form of this achievable rate matches with its classical counterpart. The Gel'fand-Pinsker channel model had earlier only been studied for the classical-quantum case and in the case where Alice (the sender) and Bob (the receiver) have shared entanglement between them.

I. INTRODUCTION

The concept of communication over the wiretap channel was pioneered in the classical case by Wyner [3]. In this model the wiretapper (Eve) is aware of the encoding strategy used by the transmitter (Alice) to transmit the messages reliably to the legitimate receiver (Bob). A wiretap channel is classically modeled as a conditional probability distribution $p_{YZ|X}$, where X is the channel input supplied by Alice and (Y, Z) are the channel outputs with Y received by Bob and Z received by Eve. The goal here is to maximize the rate of reliable message transmission from Alice to Bob over this channel, such that Eve gets to know as little information as possible about the transmitted message.

This problem of secure communication over noisy wiretap channel was extended to the quantum domain in [4], [5]. In the quantum case, the wiretap channel is modeled as a CPTP (completely positive and trace preserving) map $\mathcal{N}_{A \rightarrow BE}$, where A is the input register supplied by Alice and B and E represent Bob's and Eve's respective shares of the channel output. The quantum wiretap channel model has also been well studied in the one-shot scenario, see for example [6], [7], [8], [9].

Recently, there has been an interest in studying the classical wiretap channel with *states*. A classical wiretap channel with states is modeled as $p_{YZ|XS}$. Similar to the wiretap channel as discussed above, it produces two outputs (Y, Z) with Y received by Bob and Z received by Eve. However, unlike the normal wiretap channel, in this case the channel takes two inputs X (supplied by Alice) and a random parameter S . This random parameter S is used to represent the channel state and is not controlled by the transmitter. A key motivation for studying this channel model is that it captures the scenario of communication both in the presence of a jammer and an eavesdropper. Further, this channel

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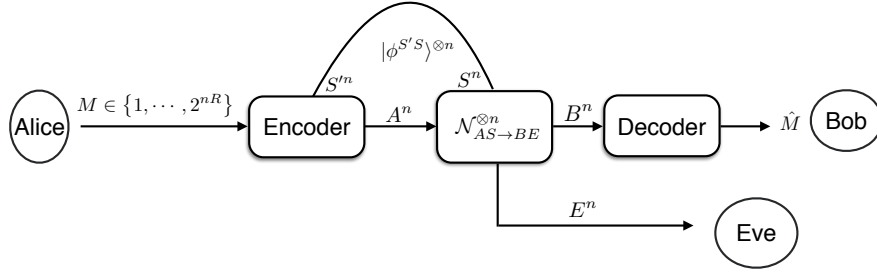


Fig. 1. A general block diagram for communication over n independent uses of the Gel'fand-Pinsker wiretap channel. In this model the encoder shares n copies of the entangled state $|\phi^{S'S}\rangle$ with the channel itself, where the register S'^n is held by the encoder and the register S^n is held by the channel.

also models the scenario of a memory with stuck-at faults (for more details, please see [10]). In [11], Chen and Vinck considered the problem of communication over this channel model in the presence of an eavesdropper. They combined the coding strategy for the normal wiretap channel along with the coding strategy for the Gel'fand-Pinsker (GP) channel and obtained a lower bound on the secrecy capacity. In [12], Chia and El-Gamal further advanced the theory of communication over this channel model by proposing a more sophisticated coding technique in the case when the full channel side information is causally available at both the encoder and the decoder. Even though their coding strategy is restricted to utilize the state information in a causal manner, the authors show that their technique allows to achieve a better transmission rate as compared to the one obtained in [11]. The GP channel model in the absence of Eve was first introduced by Gel'fand and Pinsker in their seminal work [13]. In this model there are two parties (Alice and Bob) and it takes two inputs X (supplied by Alice) and a random parameter S . However, unlike the GP-wiretap channel model this channel only produces one output Y received by Bob.

In [2] Goldfeld, Cuff and Permuter revisit this communication problem when the channel state side information is causally available at the encoder. The authors motivate this model by noting that having information about the extra randomness S (the channel state parameter) of the channel may help in secure transmission. They employ an encoding technique based on the *superposition* coding scheme [14] and obtain the best known lower bound on the secrecy capacity of the GP channel model and also recover the results of [11] and [12] as a special case. The converse result for this problem is not known except for some special cases (which do not seem to have natural interpretation in the quantum case). To obtain their results, the authors prove what they call a superposition covering lemma. Although the papers [2], [11], [12] call the approximation of the output distribution a covering lemma, this type of approximation was studied with the name of channel resolvability in the earlier papers [15], [16], [17], [7].

We study the problem of secure communication over the fully quantum Gel'fand-Pinsker wiretap channel and provide an exact quantum generalization of the results obtained in [2]. Fig 1 models this communication scheme. To derive the quantum generalization of the coding technique in [2] we prove a generalization of classical-quantum channel resolvability lemma [7]. This lemma is the quantum analogue of the [2, Lemma 7]. To prove the secrecy property of our coding technique we prove a conditional classical-quantum channel resolvability lemma. For the task of designing the decoding POVMs for our protocol we use the technique of *simultaneous pinching* (see [18] for details on the concept of pinching). Using this technique of simultaneous pinching we exhibit the existence of a *simultaneous decoder* in the single-shot case. One key feature of the single-shot bounds derived in this manuscript is that they are in terms of *error exponent*. The problem of reliable communication with no security constraint and in the presence of entanglement assistance was first studied in [10].

The result obtained in this manuscript allows us to recover the previous known results for classical message transmission over point-to-point quantum channels [19], [20] and the quantum wiretap channel (in the absence of the channel state) [4], [5]. Further, our result also implies an achievable rate for classical communication over fully quantum Gel'fand-Pinsker channel (in the absence of Eve). The form of our achievable rate for this problem is exactly similar to that obtained in [13]. We note here that the fully quantum Gel'fand-Pinsker channel has been studied in [10], [21] only in the case when Alice and Bob *share entanglement* and in [22] for classical-quantum channels. Our

work is the first work to study this model in the *absence* of entanglement assistance between Alice and Bob. We discuss these results in Corollary 1.

II. PRELIMINARIES

Consider a finite dimensional Hilbert space \mathcal{H} endowed with an inner product $\langle \cdot, \cdot \rangle$ (in this paper, we only consider finite dimensional Hilbert-spaces). The ℓ_1 norm of an operator X on \mathcal{H} is $\|X\|_1 := \text{Tr}\sqrt{X^\dagger X}$ and ℓ_2 norm is $\|X\|_2 := \sqrt{\text{Tr}XX^\dagger}$. A quantum state (or a density matrix or a state) is a positive semi-definite matrix on \mathcal{H} with trace equal to 1. It is called *pure* if and only if its rank is 1. A sub-normalized state is a positive semi-definite matrix on \mathcal{H} with trace less than or equal to 1. Let $|\psi\rangle$ be a unit vector on \mathcal{H} , that is $\langle\psi, \psi\rangle = 1$. With some abuse of notation, we use ψ to represent the state and also the density matrix $|\psi\rangle\langle\psi|$, associated with $|\psi\rangle$. Given a quantum state ρ on \mathcal{H} , *support* of ρ , called $\text{supp}(\rho)$ is the subspace of \mathcal{H} spanned by all eigen-vectors of ρ with non-zero eigenvalues.

A *quantum register* A is associated with some Hilbert space \mathcal{H}_A . Define $|A| := \dim(\mathcal{H}_A)$. Let $\mathcal{L}(A)$ represent the set of all linear operators on \mathcal{H}_A . Let $\mathcal{P}(A)$ represent the set of all positive semidefinite operators on \mathcal{H}_A . We denote by $\mathcal{D}(A)$, the set of quantum states on the Hilbert space \mathcal{H}_A . State ρ with subscript A indicates $\rho_A \in \mathcal{D}(A)$. If two registers A, B are associated with the same Hilbert space, we shall represent the relation by $A \equiv B$. Composition of two registers A and B , denoted AB , is associated with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. For two quantum states $\rho \in \mathcal{D}(A)$ and $\sigma \in \mathcal{D}(B)$, $\rho \otimes \sigma \in \mathcal{D}(AB)$ represents the tensor product (Kronecker product) of ρ and σ . The identity operator on \mathcal{H}_A (and associated register A) is denoted \mathbb{I}_A .

Let $\rho_{AB} \in \mathcal{D}(AB)$. We define

$$\rho_B := \text{Tr}_A \rho_{AB} := \sum_i (\langle i| \otimes \mathbb{I}_B) \rho_{AB} (|i\rangle \otimes \mathbb{I}_B),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for the Hilbert space \mathcal{H}_A . The state $\rho_B \in \mathcal{D}(B)$ is referred to as the marginal state of ρ_{AB} . Unless otherwise stated, a missing register from subscript in a state will represent partial trace over that register. Given a $\rho_A \in \mathcal{D}(A)$, a *purification* of ρ_A is a pure state $\rho_{AB} \in \mathcal{D}(AB)$ such that $\text{Tr}_B \rho_{AB} = \rho_A$. Purification of a quantum state is not unique. A quantum map $\mathcal{E} : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$ is a completely positive and trace preserving (CPTP) linear map (mapping states in $\mathcal{D}(A)$ to states in $\mathcal{D}(B)$). A *unitary* operator $U_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ is such that $U_A^\dagger U_A = U_A U_A^\dagger = \mathbb{I}_A$. An *isometry* $V : \mathcal{H}_A \rightarrow \mathcal{H}_B$ is such that $V^\dagger V = \mathbb{I}_A$ and $VV^\dagger = \mathbb{I}_B$.

Our model is given as the following pair. One is a CPTP map $\mathcal{N}_{AS \rightarrow BE}$ from the joint system (A, S) to the joint system (B, E) , where A is the input system, S is the channel internal system, B is the legitimate receiver (Bob)'s system, and E is the wiretapper (Eve)'s system. The other is an entangled state $|\phi^{S'S}\rangle$ across the channel internal system S and the system S' of side information available to the transmitter (Alice). Using the information in S' , Alice can choose the encoder dependently of the channel internal system S . That is, the pair of a CPTP map $\mathcal{N}_{AS \rightarrow BE}$ and an entangled state $|\phi^{S'S}\rangle$ gives our model.

Definition 1. We shall consider the following information theoretic quantities.

- 1) **Fidelity** ([23], see also [24]) For $\rho_A, \sigma_A \in \mathcal{D}(A)$,

$$F(\rho_A, \sigma_A) \stackrel{\text{def}}{=} \|\sqrt{\rho_A} \sqrt{\sigma_A}\|_1.$$

- 2) **Purified distance** ([25]) For $\rho_A, \sigma_A \in \mathcal{D}(A)$,

$$P(\rho_A, \sigma_A) = \sqrt{1 - F^2(\rho_A, \sigma_A)}.$$

This is different from the Hellinger distance which is defined as $\sqrt{1 - F(\rho_A, \sigma_A)}$.

- 3) **Sandwiched Rényi relative entropies** ([26], [27]) Let $\rho, \sigma \in \mathcal{D}(A)$ and let $\alpha > 0$ we define the following two kinds of Rényi relative entropies:

$$\underline{D}_{1+\alpha}(\rho \parallel \sigma) := \frac{1}{\alpha} \log \text{Tr}(\sigma^{-\frac{\alpha}{2(1+\alpha)}} \rho \sigma^{-\frac{\alpha}{2(1+\alpha)}})^{1+\alpha}.$$

4) **Rényi mutual information and Rényi conditional mutual information ([28])** *Let*

$$\begin{aligned}\rho_{UVB} &:= \sum_{u,v} p_{UV}(u,v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{B|u,v}; \\ \rho_{V-U-B} &:= \sum_u p_U(u) |u\rangle\langle u|_U \otimes \rho_{V|u} \otimes \rho_{B|u},\end{aligned}$$

where in the above $\rho_{V|u}$ and $\rho_{B|u}$ are appropriate marginals with respect to the state ρ_{UVB} . We define the Rényi mutual information

$$\underline{I}_{1+\alpha}(UV; B)_{\rho_{UVB}|\rho_{UV}} := \min_{\sigma_B} \underline{D}_{1+\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \sigma_B),$$

where σ_B is an arbitrary state on \mathcal{H}_S . Also, we define the Rényi conditional mutual information

$$\underline{I}_{1+\alpha}(V; B|U)_{\rho_{UVB}|\rho_{UV}} := \min_{\sigma_{UVB}} \underline{D}_{1+\alpha}(\rho_{UVB} \| \sigma_{UVB}), \quad (1)$$

where σ_{UVB} is given with an arbitrary state $\sigma_{B|u}$ as

$$\sigma_{UVB} = \sum_u p_U(u) |u\rangle\langle u|_U \otimes \rho_{V|u} \otimes \sigma_{B|u}.$$

We will use the following facts.

Fact 1 (Minimum achieving state, [29]). *For $1 + \alpha \geq \frac{1}{2}$, the minimum in (1) is uniquely attained when $\sigma_{B|u}$ satisfies*

$$\sigma_{B|u} = \frac{\text{Tr}_A[(\rho_{V|u} \otimes \rho_{B|u})^{-\frac{\alpha}{2(1+\alpha)}} \rho_{VB|u} (\rho_{V|u} \otimes \rho_{B|u})^{-\frac{\alpha}{2(1+\alpha)}}]}{\text{Tr}[(\rho_{V|u} \otimes \rho_{B|u})^{-\frac{\alpha}{2(1+\alpha)}} \rho_{VB|u} (\rho_{V|u} \otimes \rho_{B|u})^{-\frac{\alpha}{2(1+\alpha)}}]}. \quad (2)$$

Lemma 5 of [29] showed the above inequality without the classical system U . Since U is a classical system, we can apply Lemma 5 of [29] to the state $\rho_{VB|u}$ for each element u , which implies (2).

Fact 2 (Triangle inequality for purified distance, [30]). *For states $\rho_A, \sigma_A, \tau_A \in \mathcal{D}(A)$,*

$$P(\rho_A, \sigma_A) \leq P(\rho_A, \tau_A) + P(\tau_A, \sigma_A),$$

which implies that

$$P(\rho_A, \sigma_A)^2 \leq (P(\rho_A, \tau_A) + P(\tau_A, \sigma_A))^2 \leq 2(P(\rho_A, \tau_A)^2 + P(\tau_A, \sigma_A)^2).$$

Fact 3 (Monotonicity under quantum operations, [31],[32]). *For quantum states ρ, σ and quantum operation $\mathcal{E}(\cdot) : \mathcal{L}(A) \rightarrow \mathcal{L}(B)$, it holds that*

$$\underline{D}_{1+\alpha}(\mathcal{E}(\rho) \| \mathcal{E}(\sigma)) \leq \underline{D}_{1+\alpha}(\rho \| \sigma) \quad \text{and} \quad P(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq P(\rho, \sigma)$$

Fact 4 (Uhlmann's Theorem, [24]). *Let $\rho_A, \sigma_A \in \mathcal{D}(A)$. Let $\rho_{AB} \in \mathcal{D}(AB)$ be a purification of ρ_A and $|\sigma\rangle_{AC} \in \mathcal{D}(AC)$ be a purification of σ_A . There exists an isometry $V : C \rightarrow B$ such that,*

$$F(|\theta\rangle\langle\theta|_{AB}, |\rho\rangle\langle\rho|_{AB}) = F(\rho_A, \sigma_A),$$

where $|\theta\rangle_{AB} = (\mathbb{I}_A \otimes V)|\sigma\rangle_{AC}$.

Fact 5. *For quantum states $\rho_A, \sigma_A \in \mathcal{D}(A)$,*

$$F^2(\rho, \sigma) \geq 2^{-\underline{D}_{1+\alpha}(\rho \| \sigma)}.$$

The fact follows from [33, Lemma 5], see also [34, Corollary 4.3,] and from the monotonicity of sandwiched Rényi relative entropy.

Fact 6. Let ρ and σ be two quantum states. We have the following relation:

$$P(\rho, \sigma) \leq \sqrt{2\|\rho - \sigma\|_1}. \quad (3)$$

Fact 7 ([21]). Let ρ and σ be quantum states. Then, for every let $0 < \Lambda < \mathbb{I}$ be an operator,

$$|\sqrt{\text{Tr}[\Lambda\rho]} - \sqrt{\text{Tr}[\Lambda\sigma]}| \leq P(\rho, \sigma).$$

Fact 8 (Hayashi-Nagaoka inequality, [35]). Let $0 < S < \mathbb{I}, T$ be positive semi-definite operators. Then

$$\mathbb{I} - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \leq 2(\mathbb{I} - S) + 4T.$$

Fact 9 (Hayashi, [18]). Let ρ and σ be two quantum states. Further, let \mathcal{E} be the pinching operation with respect to the basis of σ . Then,

$$\rho \leq v\mathcal{E}(\rho),$$

where v represents the distinct number of eigenvalues of σ and is sometimes also called as the pinching constant.

Fact 10. (Jensen's inequality) Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be a concave function. Then, $\mathbb{E}_X[f(X)] \leq f(\mathbb{E}[X])$.

III. MAIN RESULT

Before giving our main result we first give the following definitions:

Definition 2. (Encoding, Decoding, Error, Secrecy) An $(n, M_n, \varepsilon_n, \delta_n)$ secrecy code for communication over a quantum Gel'fand-Pinsker wiretap channel $\mathcal{N}_{AS \rightarrow BE}^{\otimes n}$ with channel state side information available at the encoder (i.e, when the sender shares an entangled state $|\phi^{S^n}\rangle^{\otimes n}$ with the channel itself) consists of

- an encoding operation (for Alice) $\mathcal{E} : MS^n \rightarrow A^n$, where $S^n \equiv S^n$ and $|M| = M_n$, such that

$$P(\rho_{ME^n}, \text{Tr}_{E^n}[\rho_{ME^n}] \otimes \text{Tr}_M[\rho_{ME^n}]) \leq \delta_n,$$

where $\rho_{ME^n} := \frac{1}{M_n} \sum_{m \in [1:M_n]} |m\rangle\langle m|_M \otimes \mathcal{N}_{AS \rightarrow BE}^{\otimes n}(\mathcal{E}(m, S^n), S^n)$ and $P(\cdot, \cdot)$ is the purified distance.

- a decoding operation (for Bob) $\mathcal{D} : B^n \rightarrow \hat{M}_n$, with $\hat{M}_n \equiv M_n$. such that

$$\Pr \{M \neq \hat{M}\} \leq \varepsilon_n$$

Definition 3. A rate R is said to be achievable if there exists a sequence of $(n, M_n, \varepsilon_n, \delta_n)$ - codes such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n &\geq R; \\ \limsup_{n \rightarrow \infty} \varepsilon_n &\rightarrow 0; \\ \lim_{n \rightarrow \infty} \delta_n &\rightarrow 0. \end{aligned}$$

The supremum of all the achievable rates is called the secrecy capacity of the Gel'fand-Pinsker channel.

The following theorem is one of our main result. It can be considered as the quantum generalization of the achievability result in [2, Equations 22 and 24].

Theorem 1. Let $\mathcal{N}_{AS \rightarrow BE}$ be a quantum Gel'fand-Pinsker wiretap channel. Further, let $|\phi\rangle_{S'S}$ be the shared entanglement between the sender and the channel. We choose a joint distribution p_{UV} and conditional states $\{\rho_{AS|u,v}\}_{u,v}$ such that $\text{Tr}_{UV A} \rho_{UVAS} = \text{Tr}_{S' \phi_{S'S}}$, where $\rho_{UVAS} := \sum_{(u,v)} p_{UV}(u, v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{AS|u,v}$. Then, a rate R is achievable if

$$R \leq R_a(\rho_{UVAS}) := \min \{I[V; B | U] - I[V; E | U], I[UV; B] - I[UV; S], I[UV; B] - I[U; S] - I[V; E | U]\}, \quad (4)$$

where the information theoretic quantities above are calculated with respect to the state ρ_{UVAS} .

We denote the set of ρ_{UVAS} to satisfy the condition given in Theorem 1 by \mathcal{S}_1 . Then, the rate

$$\max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS})$$

is achievable. To simplify this rate, we introduce the set $\mathcal{S}_2 := \{\rho_{UVAS} \in \mathcal{S}_1 | I[U; B] - I(U; S) \geq 0\}$. Then, we have the following lemma.

Lemma 1.

$$\max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}) = \max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS}), \quad (5)$$

where

$$R_{alt}(\rho_{UVAS}) := \min \{I[V; B | U] - I[V; E | U], I[UV; B] - I[UV; S]\}. \quad (6)$$

Therefore, Theorem 1 guarantees that the rate $\max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS})$ is also achievable. The proof of this lemma follows exactly similar to that given in [2, Appendix A]. However, we repeat the same proof in the Appendix just for completeness.

To achieve the rate given in (4), we employ superposition coding, in which we randomly choose U and we make an encoder with respect to V conditioned on U . Here, we elaborate upon the roles of U and V . In Gel'fand-Pinsker wiretap channel, the register S some correlation with the systems E and B , which makes our analysis difficult. We convert these correlations to the correlation between the register S and the message. Therefore, we need three types of evaluations. The first one is the evaluation of the correlation between the register S and the message. The second one is the evaluation of the decoding error probability with ignoring the correlation between the register S and the receiver B . It can be evaluated as a correlation between U, V and S . The third one is the evaluation of the information leakage while ignoring the correlation between the register S and the eavesdropper E . It can be evaluated as the correlation between V and E conditioned with U .

To realize the third type of evaluation, we need a scramble variable related to V with the rate R_1 . This type of analysis requires the condition

$$R_1 > I[V; E | U]. \quad (7)$$

In contrast, to realize the second type of analysis, we need another scramble variable related to U with the rate r as well as the scramble variable related to V with the rate R_1 . This type of analysis requires the conditions

$$r > I[U; S]; \quad (8)$$

$$R_1 + r > I[UV; S]. \quad (9)$$

In addition, the first type of analysis requires the condition for the coding rate R ;

$$R + R_1 + r < I[UV; B]; \quad (10)$$

$$R + R_1 < I[V; B | U]. \quad (11)$$

As explained in the final part of our proof, combining the conditions (7) – (11), we can show that the rate given in (4) is achievable.

An important consequence of our achievability result is the following corollary:

Corollary 1. (a) (Communication over point-to-point channel, [19], [20]) Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel. Further, let $\chi(\mathcal{N}) := \max_{\rho} I[X; B]$, where the maximization is over the states of the following form: $\sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A \rightarrow B}(\rho_{A|x})$. Then every rate R satisfying the following constraint

$$R \leq \lim_{k \rightarrow \infty} \frac{1}{k} \chi(\mathcal{N}^{\otimes k})$$

is achievable.

(b) (Communication over point-to-point wiretap channel, [4], [5]) Let $\mathcal{N}_{A \rightarrow BE}$ be a quantum wiretap channel.

Further, let $P(\mathcal{N}) := \max_{\rho} (I[X; B] - I[X; E])$, where the maximization is over the states of the following form: $\sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A \rightarrow BE}(\rho_{A|x})$. Then every rate R satisfying the following constraint

$$R \leq \lim_{k \rightarrow \infty} \frac{1}{k} P(\mathcal{N}^{\otimes k})$$

is achievable for the wiretap channel $\mathcal{N}_{A \rightarrow BE}$.

- (c) (Entanglement unassisted communication over Gel'fand-Pinsker quantum channel) Let $\mathcal{N}_{AS \rightarrow B}$ be a Gel'fand-Pinsker quantum channel. Further, let $\chi(\mathcal{N})_{GP} := \max_{\rho} (I[U; B] - I[U; S])$, where the maximization is over the states of the following form: $\rho_{UAS} = \sum_u p_{UV}(u, v) |u\rangle\langle u|_U \otimes \rho_{AS|u}$ such that $\text{Tr}_{UA} \rho_{UAS} = \text{Tr}_{S'} \phi_{S'S}$. Then any rate R satisfying the following constraint

$$R \leq \lim_{k \rightarrow \infty} \frac{1}{k} \chi_{GP}(\mathcal{N}^{\otimes k})$$

is achievable for the channel $\mathcal{N}_{AS \rightarrow B}$.

- Proof.* (a) The proof follows by setting $U = \emptyset, V = X, S = \emptyset, E = \emptyset$ in (6) and then using the coding strategy in the proof of Theorem 1 for $\mathcal{N}_{A \rightarrow B}^{\otimes k}$.
 (b) The proof follows by setting $U = \emptyset, V = X, S = \emptyset$ in (6) and then using the coding strategy in the proof of Theorem 1 for $\mathcal{N}_{A \rightarrow BE}^{\otimes k}$.
 (c) The proof follows by setting $V = X, E = \emptyset$ in (6) and then using the coding strategy in the proof of Theorem 1 for $\mathcal{N}_{AS \rightarrow B}^{\otimes k}$.
 This completes the proof. \square

Before giving the proof of Theorem 1 we first study the single-shot version of the task mentioned in Fig 1. For the single-shot case we obtain an error exponent like bound on the decoding error probability and the secrecy criterion.

IV. CODE CONSTRUCTION IN SINGLE-SHOT FORM

In this section, we give the construction of our code in the single-shot form, and evaluate its performance. Let $\mathcal{N}_{AS \rightarrow BE}$ be a quantum Gel'fand-Pinsker wiretap channel. Further, let $|\phi\rangle_{S'S}$ be the shared entanglement between the sender and the channel.

Let ρ_{UVAS} be as defined in Theorem 1 and define the following states:

$$\rho_{UVBE} := \mathcal{N}_{AS \rightarrow BE}(\rho_{UVAS}); \quad (12)$$

$$\rho_B := \text{Tr}_{UVE} \mathcal{N}_{AS \rightarrow BE}(\rho_{UVAS}); \quad (13)$$

$$\rho_{B|u} := \sum_v p_{V|U}(v | u) \rho_{B|u,v}. \quad (14)$$

The codebook: We choose real numbers $R, R_1, r > 0$. Let $U(1), \dots, U(2^r)$ be drawn independently according to p_U . Further, for every $i \in [1 : 2^r]$ and for every message $m \in [1 : 2^R]$, generate $V(m, i, 1), \dots, V(m, i, 2^{R_1})$ independently, where for every $j \in [1 : 2^{R_1}]$, $V(m, i, j) \sim p_{V|U(i)}$. The distribution $p_{V|U(i)}$ is with respect to the conditional distribution of the joint distribution p_{UV} . In what follows we will use the notation $\mathcal{C}_U := \{U(1), U(2), \dots, U(2^r)\}$ and $\mathcal{C}_{m,i} := \{V(m, i, 1), \dots, V(m, i, 2^{R_1})\}$. Both \mathcal{C}_U and $\mathcal{C}_{m,i}$ for all $(m, i) \in [1 : 2^r] \times [1 : 2^R]$, are known to all the parties involved, i.e., Alice, Bob and Eve. We will use the notation $\mathcal{C}_m := \cup_i \mathcal{C}_{m,i}$.

Remark: In the above R stands for the final rate of communication. Our encoding scheme mentioned below is a multi-level coding scheme which has the dual feature of being a good wiretap channel code along with being a good code for the Gel'fand-Pinsker channel. Intuitively, the coding scheme should be such that it should be able to keep the messages secure from Eve. Further, since Bob does not have any information about S therefore the encoding scheme should be such that it should somehow help Bob in decoding. These two features about our encoding schemes are reflected by bounds on r and R_1 derived below.

Encoding: To define our encoder, we introduce a register C such that $|\rho_{CAS|u,v}\rangle$ is a purification of the state $\rho_{AS|u,v}$, which is given in Theorem 1. Thus, we have the following purification of the state $\frac{1}{2^{(R_1+r)}} \sum_{(i,j)} \rho_{S|U(i)V(m,i,j)}$:

$$|\tau_{CASIJ|U(1),\dots,U(2^r),V(m,1,1),\dots,V(m,2^r,2^{R_1})}\rangle := \frac{1}{\sqrt{2^{(R_1+r)}}} \sum_{(i,j)} |\rho_{CAS|U(i)V(m,i,j)}\rangle |i\rangle_I |j\rangle_J. \quad (15)$$

It follows from the Uhlmann's theorem (Fact 4) that for every $m \in [1 : 2^R]$, there exists a set of isometries $\{W_{\mathcal{C}_U, \mathcal{C}_m}^{S' \rightarrow ACIJ}\}$ such that

$$\begin{aligned} & \mathbb{P} \left(\tau_{CASIJ|U(1),\dots,U(2^r),V(m,1,1),\dots,V(m,2^r,2^{R_1})}, W_{\mathcal{C}_U, \mathcal{C}_m}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_m}^{\dagger S' \rightarrow ACIJ} \right) \\ &= \mathbb{P} \left(\frac{1}{2^{n(R_1+r)}} \sum_{(i,j)} \rho_{S|U(i)V(m,i,j), \rho_S} \right), \end{aligned} \quad (16)$$

where $\rho_{S|u,v} := \text{Tr}_A \rho_{AS|u,v}$. Using these notations, we define our encoder depending on the codewords in the codebook \mathcal{C}_U and $\{\mathcal{C}_m\}_{m \in [1:2^R]}$ as follows. When Alice intends to send the message m , she applies the isometry $W_{\mathcal{C}_U, \mathcal{C}_m}^{S' \rightarrow ACIJ}$ (obtained in the derivation of (16)) on her register S' and transmits the register A across the channel $\mathcal{N}_{AS \rightarrow B}$.

Pinching: Our decoder will be based on the method of pinching. Therefore, before designing our decoder we first discuss this method. Consider the following classical-quantum states

$$\rho_{UVB} := \sum_{u,v} p_{UV}(u,v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{B|u,v}; \quad (17)$$

$$\rho_{V-U-B} := \sum_u p_U(u) |u\rangle\langle u|_U \otimes \rho_{V|u} \otimes \rho_{B|u}, \quad (18)$$

where in the above $\rho_{V|u}$ and $\rho_{B|u}$ are appropriate marginals with respect to the state ρ_{UVB} .

In the subsequent discussions the main aim is to come up with completely positive and trace preserving operations such that at the end of these operations the states ρ_{UVB} , $\rho_{UV} \otimes \rho_B$ and ρ_{V-U-B} start commuting. Towards this we define the following operations: \mathcal{E}_1 be the pinching operation with respect to the spectral decomposition of the state ρ_B . Further, for every u , let $\mathcal{E}_{2|u}$ be the pinching operation with respect to the spectral decomposition of the operator $\mathcal{E}_1(\rho_{B|u})$. Then, \mathcal{E}_2 is defined as $\mathcal{E}_2(\rho) := \sum_u |u\rangle\langle u| \otimes \mathcal{E}_{2|u}(\langle u|\rho|u\rangle)$. It easy to observe that $\mathcal{E}_1(\rho_{V-U-B})$, $\mathcal{E}_2(\rho_{UVB})$ and the state $\rho_{UV} \otimes \rho_B$ commute with each other. In what follows further in this section we will use the notation v_1 and v_2 to represent the maximum number of components of the pinching map \mathcal{E}_1 and the maximum number of components of the pinching maps $\{\mathcal{E}_{2|u}\}_u$. Further, in the discussions below we will define pinching maps \mathcal{E}_3 , $\mathcal{E}_{4|u}$ and $\mathcal{E}_{5|u}$, where \mathcal{E}_3 is the pinching map with respect to the the spectral basis of ρ_S , $\mathcal{E}_{4|u}$ is the pinching map with respect to the spectral basis of the operator $\mathcal{E}_3(\rho_{S|u})$ and $\mathcal{E}_{5|u}$ is defined with respect to the state $\rho_{E|u}$. Then, \mathcal{E}_4 and \mathcal{E}_5 are defined from $\mathcal{E}_{4|u}$ and $\mathcal{E}_{5|u}$ in the same way as \mathcal{E}_2 . Further, let v_1, v_2, v_3, v_4 , and v_5 be defined as follows:

$$\begin{aligned} v_1 &:= \text{distinct components of the pinching map } \mathcal{E}_1; \\ v_2 &:= \text{maximum number of distinct components of the pinching maps } \{\mathcal{E}_{2|u}\}_u; \\ v_3 &:= \text{distinct components of the pinching map } \mathcal{E}_3; \\ v_4 &:= \text{maximum number of distinct components of the pinching maps } \{\mathcal{E}_{4|u}\}_u; \\ v_5 &:= \text{maximum number of distinct components of the pinching map } \{\mathcal{E}_{5|u}\}_u. \end{aligned} \quad (19)$$

Decoding: First, for two Hermitian matrices A and B , we define the projection $\{A \geq B\}$ as $\sum_{j: \lambda_j \geq 0} P_j$, where the spectral decomposition of $A - B$ is given as $\sum_j \lambda_j P_j$. In this notation, P_j is the projection to the eigenspace

corresponding to the eigenvalue λ_j . Then, we define the following projectors:

$$\Pi_{UVB}(1) := \{\mathcal{E}_2(\rho_{UVB}) \geq 2^{R+R_1+r} \rho_{UV} \otimes \rho_B\}, \quad (20)$$

$$\Pi_{UVB}(2) := \{\mathcal{E}_2(\rho_{UVB}) \geq 2^{R+R_1} \mathcal{E}_1(\rho_{V-U-B})\}. \quad (21)$$

Let $\Pi_{UVB} := \Pi_{UVB}(1)\Pi_{UVB}(2) = \Pi_{UVB}(2)\Pi_{UVB}(1)$. For every $(m, i, j) \in [1 : 2^R] \times [1 : 2^r] \times [1 : 2^{R_1}]$ define the following operator:

$$\gamma(m, i, j) := \text{Tr}_{UV} [\Pi_{UVB} (|U(i)\rangle\langle U(i)|_U \otimes |V(m, i, j)\rangle\langle V(m, i, j)|_V \otimes \mathbb{I}_B)]. \quad (22)$$

We now scale these operators to obtain a valid set of POVM operators as follows:

$$\beta(m, i, j) := \left(\sum_{(m', i', j')} \gamma(m', i', j') \right)^{-\frac{1}{2}} \gamma(m, i, j) \left(\sum_{(m', i', j')} \gamma(m', i', j') \right)^{-\frac{1}{2}}. \quad (23)$$

Bob uses the above set of decoding POVM operators to decode the transmitted message.

Average performance: Under the above random construction, we can evaluate the average performances. Let M be the message which was transmitted by Alice using the strategy above and let \hat{M} be the decoded message by Bob using the decoding POVMs defined in (23). Notice that by the symmetry of the encoding and decoding strategy, it is enough to bound $\Pr \{\hat{M} \neq 1 | M = 1\}$. The following lemma discusses the average performance of our protocol.

Lemma 2. *The average performances are evaluated with $\alpha \in (0, 1)$ as follows.*

$$\begin{aligned} \mathbb{E}_{\mathcal{C}} \Pr \{\hat{M} \neq 1 | M = 1\} &\leq 20 \left(v_2^\alpha 2^{\alpha(R+R_1+r-\underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B))} + v_2^\alpha 2^{\alpha(R+R_1-\underline{I}_{1-\alpha}[V;B|U])} \right) \\ &\quad + \frac{2}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_{UV} \otimes \rho_S)} \right), \end{aligned} \quad (24)$$

and

$$\begin{aligned} \mathbb{E}_{\mathcal{C}} \left[P(\rho_{ME}, \text{Tr}_E[\rho_{ME}] \otimes \text{Tr}_M[\rho_{ME}])^2 \right] &\leq \frac{8}{\alpha} \left(\left(\frac{v_1}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \left(\frac{v_2}{2^{R_1+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_{UV} \otimes \rho_S)} \right) \\ &\quad + \frac{8}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha R_1}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVE} \|\rho_{V-U-E})} \right), \end{aligned} \quad (25)$$

where v_1, v_2, v_3, v_4 , and v_5 are constants defined in (19).

This lemma will be proven in Section VI. For now we assume this lemma and prove the existence of a code which is robust to both decoding error and secrecy.

Existence of good code: Applying expurgation to this construction, we obtain the following theorem.

Theorem 2. *For $\alpha \in (0, 1)$ and for every $R, R_1, r > 0$, there exists a code such that*

$$\begin{aligned} \Pr \{M \neq \hat{M}\} &\leq 42 \left(v_2^\alpha 2^{\alpha(R+R_1+r-\underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B))} + v_2^\alpha 2^{\alpha(R+R_1-\underline{I}_{1-\alpha}[V;B|U])} \right) \\ &\quad + \frac{5}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_{UV} \otimes \rho_S)} \right), \end{aligned}$$

and

$$\begin{aligned} P(\rho_{ME}, \text{Tr}_E[\rho_{ME}] \otimes \text{Tr}_M[\rho_{ME}])^2 &\leq 20 \left(\frac{1}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_{UV} \otimes \rho_S)} \right) \right. \\ &\quad \left. + \frac{1}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha R_1}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVE} \|\rho_{V-U-E})} \right) \right), \end{aligned}$$

where v_1, v_2, v_3, v_4 , and v_5 are constants defined in (19) and the information theoretic quantities above are calculated with respect to the state $\rho_{UVAS} = \sum_{(u,v)} p_{UV}(u, v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{AS|u,v}$ such that $\text{Tr}_{UV} \rho_{UVAS} = \text{Tr}_S \phi_{S'S}$.

Proof. We now show the existence of a code which simultaneously satisfies both the reliability and the secrecy criterion as discussed in the Definition (2). Towards this let $\varepsilon(\mathcal{C})$ and $\delta(\mathcal{C})$ represent the decoding error and secrecy parameter of a random codebook \mathcal{C} . Define the following events:

$$\mathcal{B}_1 := \{\varepsilon(\mathcal{C}) \leq (1 + \beta)\mathbb{E}_{\mathcal{C}}[\varepsilon(\mathcal{C})]\}; \quad (26)$$

$$\mathcal{B}_2 := \{\delta(\mathcal{C}) \leq (1 + \beta)\mathbb{E}_{\mathcal{C}}[\delta(\mathcal{C})]\}, \quad (27)$$

where $\beta > 1$ is an arbitrary constant. From Markov's inequality and union bound it now easily follows that

$$\Pr\{\mathcal{B}_1, \mathcal{B}_2\} \geq \frac{\beta - 1}{\beta + 1}. \quad (28)$$

Thus, from (24), (25), (26), (27) and setting $\beta = 1.1$ in (28) we now conclude that there exists a codebook such that:

$$\begin{aligned} \Pr\{M \neq \hat{M}\} &\leq 42 \left(v_2^\alpha 2^{\alpha(R+R_1+r-\underline{D}_{1-\alpha}(\rho_{UVB}\|\rho_{UV}\otimes\rho_B))} + v_2^\alpha 2^{\alpha(R+R_1-\underline{I}_{1-\alpha}[V;B|U])} \right) \\ &\quad + \frac{5}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha\underline{D}_{1+\alpha}(\rho_{US}\|\rho_U\otimes\rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha\underline{D}_{1+\alpha}(\rho_{UVS}\|\rho_{UV}\otimes\rho_S)} \right), \end{aligned}$$

and

$$\begin{aligned} P(\rho_{ME}, \text{Tr}_E[\rho_{ME}] \otimes \text{Tr}_M[\rho_{ME}])^2 &\leq 20 \left(\frac{1}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha\underline{D}_{1+\alpha}(\rho_{US}\|\rho_U\otimes\rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha\underline{D}_{1+\alpha}(\rho_{UVS}\|\rho_{UV}\otimes\rho_S)} \right) \right. \\ &\quad \left. + \frac{1}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha R_1}} 2^{\alpha\underline{D}_{1+\alpha}(\rho_{UVE}\|\rho_{U-V-E})} \right) \right). \end{aligned}$$

This completes the proof. \square

V. ASYMPTOTIC ANALYSIS

A. Preparation

To analyze the asymptotic case, first we bound the number of distinct components of the pinching maps \mathcal{E}_1 and \mathcal{E}_2 in the asymptotic and i.i.d. case. That is, we consider the case when there are n independent copies of the states ρ_{UVB} and ρ_{V-U-B} . Let d_U and d_B be the dimensions of \mathcal{H}_U and \mathcal{H}_B . The lemma below gives an upper bound on the number of distinct components of the maps \mathcal{E}_1 and \mathcal{E}_2 discussed above.

Lemma 3. *Let \mathcal{E}_1 and \mathcal{E}_2 be the pinching maps as defined above. Further, let v_1, v_2 represent the number of distinct components of the map \mathcal{E}_1 and \mathcal{E}_2 respectively. Then,*

$$v_1 \leq (n+1)^{d_B-1}, \quad v_2 \leq (n+1)^{d_U(d_B+2)(d_B-1)/2}.$$

Proof. $\rho_B^{\otimes n}$ has $(n+1)^{d_B-1}$ eigenvalues at most. Hence, $v_1 \leq (n+1)^{d_B-1}$.

We now prove the upper bound on v_2 . Towards this let $\{|u_1\rangle, \dots, |u_{d_U}\rangle\}$ represent the basis of \mathcal{H}_U . We now focus on the the number of components of the pinching map $\mathcal{E}_{2|\vec{u}}$, where $\vec{u} := (\underbrace{u_1, \dots, u_1}_{n_1}, \dots, \underbrace{u_{d_U}, \dots, u_{d_U}}_{n_{d_U}})$. The state

$\rho_{B^n|\vec{u}}$ is written as $\rho_{B|u_1}^{\otimes n_1} \otimes \dots \otimes \rho_{B|u_{d_U}}^{\otimes n_{d_U}}$. Then, the space $\mathcal{H}_B^{\otimes n_j}$ is decomposed to

$$\mathcal{H}_B^{\otimes n_j} = \bigoplus_{\lambda \in Y_d^{n_j}} \mathcal{U}_\lambda(\text{SU}) \otimes \mathcal{U}_\lambda(S_{n_j}), \quad (29)$$

where $Y_d^{n_j}$ is the set of indexes of size n_j and depth not greater than d_B . We have $|Y_d^{n_j}| \leq n_j^{d_B-1}$ and Weyl's dimension formula shows that $\dim \mathcal{U}_\lambda(\text{SU}) \leq (n+1)^{d_B(d_B-1)/2}$.

We denote the pinching whose components are $\{\mathcal{U}_\lambda(\text{SU}) \otimes \mathcal{U}_\lambda(S_{n_j})\}_{\lambda \in Y_d^{n_j}}$ by \mathcal{E}_{n_j} . The states $\rho_{B|\vec{u}}$ and $\rho_B^{\otimes n}$ are invariant with respect to $\mathcal{E}_{n_1} \otimes \dots \otimes \mathcal{E}_{n_{d_U}}$. Therefore, $\mathcal{E}_1(\rho_{V-U-B}) = \mathcal{E}_1 \circ \mathcal{E}_{n_1} \otimes \dots \otimes \mathcal{E}_{n_{d_U}}(\rho_{V-U-B})$.

Now, we consider each component of $\mathcal{E}_{n_1} \otimes \dots \otimes \mathcal{E}_{n_{d_U}}$. That is, we consider the subspace $\mathcal{U}_{\lambda_1}(\text{SU}) \otimes \mathcal{U}_{\lambda_1}(S_{n_1}) \otimes \dots \otimes \mathcal{U}_{\lambda_{d_U}}(\text{SU}) \otimes \mathcal{U}_{\lambda_{d_U}}(S_{n_{d_U}}) = (\mathcal{U}_{\lambda_1}(\text{SU}) \otimes \dots \otimes \mathcal{U}_{\lambda_{d_U}}(\text{SU})) \otimes (\mathcal{U}_{\lambda_1}(S_{n_1}) \otimes \dots \otimes \mathcal{U}_{\lambda_{d_U}}(S_{n_{d_U}}))$. Both states are the

identity on $\mathcal{U}_{\lambda_1}(S_{n_1}) \otimes \cdots \otimes \mathcal{U}_{\lambda_{d_U}}(S_{n_{d_U}})$. Thus, on this subspace, the number of eigenvalues of $\mathcal{E}_1(\rho_{V-U-B})$ is the dimension of $\mathcal{U}_{\lambda_1}(\text{SU}) \otimes \cdots \otimes \mathcal{U}_{\lambda_{d_U}}(\text{SU})$ at most. The dimension is $(n+1)^{d_U d_B (d_B-1)/2}$ at most. Further, the number of components of $\mathcal{E}_{n_1} \otimes \cdots \otimes \mathcal{E}_{n_{d_B}}$ is $(n+1)^{d_B(d_U-1)}$ at most. Therefore, the number of eigenvalues of $\mathcal{E}_1(\rho_{V-U-B})$ is $(n+1)^{d_U d_B (d_B-1)/2} (n+1)^{d_U (d_B-1)} = (n+1)^{d_U (d_B+2)(d_B-1)/2}$ at most. \square

Further, we have the following additivity property.

Lemma 4.

$$\underline{I}_{1-\alpha}[V; B | U]_{\rho_{UVB}^{\otimes n} | \rho_{UV}^{\otimes n}} = n \underline{I}_{1-\alpha}[V; B | U]_{\rho_{UVB} | \rho_{UV}}. \quad (30)$$

Proof. Let σ_{UVB} be the state which attains the minimum in the definition of $\min_{\sigma_B} \underline{D}_{1+\alpha}(\rho_{UVB} \| \rho_{UV})$ (see (1)) It then follows from the uniqueness condition that (see (2)), $\sigma_{UVB}^{\otimes n}$ satisfies the condition (2) for the n -copy case. The statement of the lemma now follows from the uniqueness condition. Due to the uniqueness condition, we obtain (30). \square

B. Proof of Theorem 1

Now, we proceed to our proof for Theorem 1. From Theorem 2 it is easy to see that if the channel $\mathcal{N}_{AS \rightarrow BE}$ is used n times independently, then there exists a code such that

$$\begin{aligned} & \Pr \{M \neq \hat{M}\} \\ & \leq 42 \left(v_2^\alpha 2^{\alpha(n(R+R_1+r) - \underline{D}_{1-\alpha}(\rho_{UVB}^{\otimes n} \| \rho_{UV}^{\otimes n} \otimes \rho_B^{\otimes n}))} + v_2^\alpha 2^{\alpha(n(R+R_1) - \underline{I}_{1-\alpha}[V^n; B^n | U^n]_{\rho_{UVB}^{\otimes n} | \rho_{UV}^{\otimes n}})} \right) \\ & \quad + \frac{5}{\alpha} \left(\left(\frac{v_3}{2^{nr}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US}^{\otimes n} \| \rho_U^{\otimes n} \otimes \rho_S^{\otimes n})} + \left(\frac{v_4}{2^{n(R_1+r)}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS}^{\otimes n} \| \rho_{UV}^{\otimes n} \otimes \rho_S^{\otimes n})} \right), \end{aligned} \quad (31)$$

and

$$\begin{aligned} \mathbb{P}(\rho_{ME}^{\otimes n}, \text{Tr}_E[\rho_{ME}^{\otimes n}] \otimes \text{Tr}_M[\rho_{ME}^{\otimes n}])^2 & \leq 20 \left(\frac{1}{\alpha} \left(\left(\frac{v_3}{2^{nr}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US}^{\otimes n} \| \rho_U^{\otimes n} \otimes \rho_S^{\otimes n})} + \left(\frac{v_4}{2^{n(R_1+r)}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS}^{\otimes n} \| \rho_{UV}^{\otimes n} \otimes \rho_S^{\otimes n})} \right) \right. \\ & \quad \left. + \frac{1}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha n R_1}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVE}^{\otimes n} \| \rho_{V-U-E}^{\otimes n})} \right) \right). \end{aligned} \quad (32)$$

Hence, using Lemma 3 and Lemma 4, we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{-1}{n} \log \Pr \{M \neq \hat{M}\} \\ & \geq \min \left(\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B) - (R + R_1 + r), \right. \\ & \quad \alpha (\underline{I}_{1-\alpha}[V; B | U]_{\rho_{UVB} | \rho_{UV}} - (R + R_1)), \\ & \quad \alpha (r - \underline{D}_{1+\alpha}(\rho_{US} \| \rho_U \otimes \rho_S)), \\ & \quad \left. \alpha ((R_1 + r) - \underline{D}_{1+\alpha}(\rho_{UVS} \| \rho_{UV} \otimes \rho_S)) \right), \end{aligned} \quad (33)$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mathbb{P}(\rho_{ME}^{\otimes n}, \text{Tr}_E[\rho_{ME}^{\otimes n}] \otimes \text{Tr}_M[\rho_{ME}^{\otimes n}]) \\ & \geq \min \left(\alpha (r - \underline{D}_{1+\alpha}(\rho_{US} \| \rho_U \otimes \rho_S)), \right. \\ & \quad \alpha ((R_1 + r) - \underline{D}_{1+\alpha}(\rho_{UVS} \| \rho_{UV} \otimes \rho_S)), \\ & \quad \left. \alpha (R_1 - \underline{D}_{1+\alpha}(\rho_{UVE} \| \rho_{V-U-E})) \right). \end{aligned} \quad (34)$$

Thus, it now follows from (33), (34) that as $n \rightarrow \infty$ and $\alpha \rightarrow 0$, then there exists a code such that

$$\lim_{n \rightarrow \infty} \Pr \left\{ M \neq \hat{M} \right\} \rightarrow 0,$$

$$\lim_{n \rightarrow \infty} P \left(\rho_{ME}^{\otimes n}, \text{Tr}_E[\rho_{ME}^{\otimes n}] \otimes \text{Tr}_M[\rho_{ME}^{\otimes n}] \right) \rightarrow 0;$$

if,

$$R + R_1 + r < I[UV; B]; \quad (35)$$

$$R + R_1 < I[V; B | U]; \quad (36)$$

$$R_1 + r > I[UV; S]; \quad (37)$$

$$r > I[U; S]; \quad (38)$$

$$R_1 > I[V; E | U]. \quad (39)$$

Now, for arbitrary small real numbers $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, we set $R_1 := I[V; E | U] + \epsilon_1$ and $r := \max(I[U; S], I[UV; S] - I[V; E | U]) + \epsilon_2$, which implies that $R_1 + r = \max(I[U; S] + I[V; E | U], I[UV; S]) + \epsilon_1 + \epsilon_2$. Then, we set $R := \min(I[UV; B] - (R_1 + r), I[V; B | U] - R) - \epsilon_3$. With this choice, the aforementioned conditions are satisfied. In this case, the rate R can be written as

$$\begin{aligned} R &= \min(I[UV; B] - (R_1 + r), I[V; B | U] - R) - \epsilon_3 \\ &= \min \left(I[UV; B] - \max(I[U; S] + I[V; E | U], I[UV; S]) - \epsilon_1 - \epsilon_2, \right. \\ &\quad \left. I[V; B | U] - I[V; E | U] - \epsilon_1 \right) - \epsilon_3 \\ &= \min \left(I[UV; B] - I[V; E | U] - I[U; S] - \epsilon_1 - \epsilon_2, \right. \\ &\quad \left. I[UV; B] - I[UV; S] - \epsilon_1 - \epsilon_2, \right. \\ &\quad \left. I[V; B | U] - I[V; E | U] - \epsilon_1 \right) - \epsilon_3. \end{aligned} \quad (40)$$

Since $\epsilon_1, \epsilon_2, \epsilon_3$ are arbitrary small real numbers, the rate given in (4) is achievable. This completes the proof of Theorem 1.

VI. PROOF OF LEMMA 2

A. Error Analysis

First, we show (24) of Lemma 2. Our proof employs Lemmas 5 and 6, which are given in latter sections. Towards this let $\Theta_B(1)$ be the state received by the Bob when Alice transmits the message $m = 1$. Hence, it is given as $\text{Tr}_{CEIJ} \mathcal{N}_{AS \rightarrow BE} \left(W_{\mathcal{C}_U, \mathcal{C}_1}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_1}^{\dagger S' \rightarrow ACIJ} \right)$. Further, let $\hat{\Theta}_B(1)$ be defined as follows:

$$\hat{\Theta}_B(1) := \frac{1}{2^{(R_1+r)}} \sum_{(i,j) \in [1:2^r] \times [1:2^{R_1}]} \text{Tr}_{CE} \mathcal{N}_{AS \rightarrow BE} \left(\rho_{CAS|U(i)V(m,i,j)} \right). \quad (41)$$

We now bound $\Pr \left\{ \hat{M} \neq 1 | M = 1 \right\}$ average over the random choice of the codebook. In what follows we will use the notation \mathcal{C} to denote the random choice of sequences mentioned in the codebook above. The error is now

bounded as follows:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \Pr \left\{ \hat{M} \neq 1 | M = 1 \right\} \\
&= \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \Theta_B(1) \right] \right] \\
&\stackrel{a}{\leq} 2 \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \hat{\Theta}_B(1) \right] \right] \\
&\quad + 2 \mathbb{E}_{\mathcal{C}} \left| \sqrt{\left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \Theta_B(1) \right] \right]} - \sqrt{\left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \hat{\Theta}_B(1) \right] \right]} \right|^2 \\
&\stackrel{b}{\leq} 2 \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \hat{\Theta}_B(1) \right] \right] + 2 \mathbb{E}_{\mathcal{C}} \left[P(\Theta_B(1), \hat{\Theta}_B(1))^2 \right], \tag{42}
\end{aligned}$$

where a follows from the generic inequality $(x + y)^2 \leq 2(x^2 + y^2)$; b follows from the Fact 7 and the facts that $\sum_{(m' \neq 1), k, l} \beta(m', k, l) \preceq \mathbb{I}$.

Using $\tau_{CAS|\mathcal{C}} := \frac{1}{2^{(R_1+r)}} \sum_{(i,j) \in [1:2^r] \times [1:2^{R_1}]} \rho_{CAS|U(i)V(1,i,j)}$ and $\tau_{S|\mathcal{C}} := \frac{1}{2^{(R_1+r)}} \sum_{(i,j) \in [1:2^r] \times [1:2^{R_1}]} \text{Tr}_{CA} \rho_{CAS|U(i)V(1,i,j)}$, we will first bound the second term in (42) as follows:

$$\begin{aligned}
& 2 \mathbb{E}_{\mathcal{C}} \left[P(\hat{\Theta}_B(1), \Theta_B(1))^2 \right] \\
&= 2 \mathbb{E}_{\mathcal{C}} \left[P \left(\frac{1}{2^{(R_1+r)}} \sum_{(i,j) \in [1:2^r] \times [1:2^{R_1}]} \text{Tr}_{CE} \mathcal{N}_{AS \rightarrow BE} \left(\rho_{CAS|U(i)V(m,i,j)} \right), \right. \right. \\
&\quad \left. \left. \text{Tr}_{CEIJ} \mathcal{N}_{AS \rightarrow BE} \left(W_{\mathcal{C}_U, \mathcal{C}_1}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_1}^{\dagger S' \rightarrow ACIJ} \right) \right)^2 \right] \\
&\stackrel{a}{\leq} 2 \mathbb{E}_{\mathcal{C}} \left[P \left(\tau_{CAS|\mathcal{C}}, W_{\mathcal{C}_U, \mathcal{C}_1}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_1}^{\dagger S' \rightarrow ACIJ} \right)^2 \right] \\
&\stackrel{b}{=} 2 \mathbb{E}_{\mathcal{C}} \left[P \left(\tau_{S|\mathcal{C}}, \rho_S \right)^2 \right] \\
&\stackrel{c}{\leq} 2 \mathbb{E}_{\mathcal{C}} \left[1 - 2^{\alpha \underline{D}_{1+\alpha}(\tau_{S|\mathcal{C}} \| \rho_S)} \right] \\
&\stackrel{d}{\leq} 2 \ln 2 \cdot \mathbb{E}_{\mathcal{C}} \left[\underline{D}_{1+\alpha}(\tau_{S|\mathcal{C}} \| \rho_S) \right] \\
&\stackrel{e}{\leq} \frac{2}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^{\alpha} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UV} \| \rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^{\alpha} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \| \rho_{UV} \otimes \rho_S)} \right) \tag{43}
\end{aligned}$$

where a follows from Fact 3 with respect to the map $\text{Tr}_{CE} \mathcal{N}_{AS \rightarrow BE}$; b follows from (16); c follows from Fact 5; d follows from the following relation (44); and e follows from Lemma 6.

$$1 - 2^{-\frac{x}{\ln 2}} = 1 - e^{-x} \leq x. \tag{44}$$

We now bound the first term on the R.H.S of (42) by using several steps as follows.

$$\begin{aligned}
& \sum_{(m' \neq 1), k, l} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\beta(m', k, l) \hat{\Theta}_B(1) \right] \right] \\
&= \frac{1}{2^{(R_1+r)}} \sum_{i, j} \sum_{(m' \neq 1), k, l} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\beta(m', k, l) \rho_{B|U(i)V(1, i, j)} \right] \right] \\
&\stackrel{a}{=} \sum_{(m' \neq 1), k, l} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\beta(m', k, l) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\leq \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[(\mathbb{I} - \beta(1, 1, 1)) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\stackrel{b}{\leq} 2\mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[(\mathbb{I} - \gamma(1, 1, 1)) \rho_{B|U(1)V(1, 1, 1)} \right] \right] + 4 \sum_{(m', k, l) \neq (1, 1, 1)} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)} \right] \right], \quad (45)
\end{aligned}$$

where a follows from the symmetry of the code construction and b follows from the Hayashi-Nagaoka operator inequality (Fact 8). We now bound each of the terms on the right hand side of (45).

Consider $2\mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[(\mathbb{I} - \gamma(1, i, j)) \rho_{B|U(i)V(1, i, j)} \right] \right]$:

$$\begin{aligned}
& 2 \cdot \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[(\mathbb{I} - \gamma(1, i, j)) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\stackrel{a}{=} 2 \cdot \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[(\mathbb{I} - \text{Tr}_{UV} [\Pi_{UVB} (|U(1)\rangle\langle U(i)|_U \otimes |V(1, 1, 1)\rangle\langle V(1, 1, 1)|_V \otimes \mathbb{I})]) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\stackrel{b}{=} 2 \cdot \text{Tr} \left[(\mathbb{I} - \Pi_{UVB}) \rho_{UVB} \right] \\
&\stackrel{c}{\leq} 2 \cdot \text{Tr} \left[(\mathbb{I} - \Pi_{UVB}(1)) \rho_{UVB} \right] + 2\text{Tr} \left[(\mathbb{I} - \Pi_{UVB}(1)) \rho_{UVB} \right] \\
&\stackrel{d}{\leq} 2 \cdot v_2^{\alpha} 2^{\alpha(R+R_1+r)} 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B)} + 2 \cdot v_2^{\alpha} 2^{\alpha(R+R_1)} 2^{-\alpha \underline{L}_{1-\alpha}[V; B|U]_{\rho_{UVB} | \rho_{UV}}}, \quad (46)
\end{aligned}$$

where a follows from the definition of $\gamma(1, i, j)$ mentioned in (22); b follows from the linearity of the trace operation; c follows from the definition of Π_{UVB} and the fact $\mathbb{I} - \Pi_{UVB} \preceq \mathbb{I} - \Pi_{UVB}(1) + \mathbb{I} - \Pi_{UVB}(2)$ and d follows from (62) and (63) proven in Lemma 5.

We now bound the second term on the right hand side of (45) as follows:

$$\begin{aligned}
& 4 \cdot \sum_{(m', k, l) \neq (1, 1, 1)} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&= 4 \cdot \sum_{k \neq 1} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(1, k, 1) \rho_{B|U(1)V(1, 1, 1)} \right] \right] + 4 \cdot \sum_{(m', l) \neq (1, 1)} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(m', 1, l) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\quad + 4 \cdot \sum_{m' \neq 1, k \neq 1, l \neq 1} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)} \right] \right]. \quad (47)
\end{aligned}$$

We now bound each of the terms on the right hand side of (47). Consider $\sum_{k \neq 1} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(1, k, 1) \rho_{B|U(1)V(1, 1, 1)} \right] \right]$:

$$\begin{aligned}
& 4 \cdot \sum_{k \neq 1} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\gamma(1, k, 1) \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\stackrel{a}{=} 4 \cdot \sum_{k \neq 1} \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\text{Tr}_{UV} [\Pi_{UVB} (|U(k)\rangle\langle U(k)|_U \otimes |V(1, k, 1)\rangle\langle V(1, k, 1)|_V \otimes \mathbb{I})] \rho_{B|U(1)V(1, 1, 1)} \right] \right] \\
&\stackrel{b}{=} 4 \cdot 2^r \text{Tr} [\Pi_{UVB} \rho_{UV} \otimes \rho_B] \\
&\stackrel{c}{\leq} 4 \cdot 2^r \text{Tr} [\Pi_{UVB}(1) \rho_{UV} \otimes \rho_B] \\
&\stackrel{d}{\leq} 4 \cdot 2^r v_2^{\alpha} 2^{-(1-\alpha)(R+r+R_1)} 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B)} \\
&\leq 4 \cdot v_2^{\alpha} 2^{\alpha((R+r+R_1) - \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B))}, \quad (48)
\end{aligned}$$

where a follows from the definition of $\gamma(1, k, 1)$ mentioned in (22); b follows from the independence of the random variables involved, linearity of trace operation, from the definition of the states ρ_{UV} and ρ_B and by the symmetry of the code construction c follows because $\Pi_{UVB} \preceq \Pi_{UVB}(1)$ and d follows from (60) proven in Lemma 5.

We now bound the second term on the right hand side of (47) as follows:

$$\begin{aligned}
& 4 \cdot \sum_{(m', l) \neq (1, 1)} \mathbb{E}_{\mathcal{C}} [\text{Tr} [\gamma(m', 1, l) \rho_{B|U(1)V(1, 1, 1)}]] \\
& \stackrel{a}{=} 4 \cdot \sum_{(m', l) \neq (1, 1)} \mathbb{E}_{\mathcal{C}} [\text{Tr} [\text{Tr}_{UV} [\Pi_{UVB} (|U(1)\rangle\langle U(1)|_U \otimes |V(m', 1, l)\rangle\langle V(m', 1, l)|_V \otimes \mathbb{I})] \rho_{B|U(1)V(1, 1, 1)}]] \\
& \stackrel{b}{=} 4 \cdot 2^{R+R_1} \text{Tr} [\Pi_{UVB} \rho_{V-U-B}] \\
& \stackrel{c}{\leq} 4 \cdot 2^{R+R_1} \text{Tr} [\Pi_{UVB}(2) \rho_{V-U-B}] \\
& \stackrel{d}{\leq} 4 \cdot 2^{R+R_1} v_2^\alpha 2^{-(1-\alpha)(R+R_1)} 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]} \\
& = 4 \cdot v_2^\alpha 2^{\alpha((R+R_1)-\underline{I}_{1-\alpha}[V; B|U])}, \tag{49}
\end{aligned}$$

where a follows from the definition of $\gamma(m', i, l)$ mentioned in (22); b follows from the independence of the random variables involved, linearity of trace operation and from the definition of the state ρ_{V-U-B} and from the symmetry of the code construction; c follows because $\Pi_{UVB} \preceq \Pi_{UVB}(2)$ d follows from (61) proven in Lemma 5.

The third term on the right hand side of (47) is bounded as follows:

$$4 \cdot \sum_{m' \neq 1, k \neq 1, l \neq 1} \mathbb{E}_{\mathcal{C}} [\text{Tr} [\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)}]] \leq 4 \cdot v_2^\alpha 2^{\alpha((R+r+R_1)-\underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B))}. \tag{50}$$

The proof for (50) follows using exactly similar steps and techniques as that used in the proof of (48).

Combining with the above discussion, we now obtain an upper bound for the first term on the R.H.S of (42);

$$\begin{aligned}
& \stackrel{a}{\leq} 2 \mathbb{E}_{\mathcal{C}} \left[\text{Tr} \left[\left(\sum_{(m' \neq 1), k, l} \beta(m', k, l) \right) \hat{\Theta}_B(1) \right] \right], \\
& \stackrel{b}{\leq} 4 \mathbb{E}_{\mathcal{C}} [\text{Tr} [(\mathbb{I} - \gamma(1, 1, 1)) \rho_{B|U(1)V(1, 1, 1)}]] + 8 \sum_{(m', k, l) \neq (1, 1, 1)} \mathbb{E}_{\mathcal{C}} [\text{Tr} [\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)}]] \\
& \stackrel{c}{\leq} 4 \left(v_2^\alpha 2^{\alpha(R+R_1+r)} 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B)} + v_2^\alpha 2^{\alpha(R+R_1)} 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]_{\rho_{UVB} \|\rho_{UV}}} \right) \\
& \quad + 8 \sum_{(m', k, l) \neq (1, 1, 1)} \mathbb{E}_{\mathcal{C}} [\text{Tr} [\gamma(m', k, l) \rho_{B|U(1)V(1, 1, 1)}]] \\
& \stackrel{d}{\leq} 4 \left(v_2^\alpha 2^{\alpha(R+R_1+r)} 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B)} + v_2^\alpha 2^{\alpha(R+R_1)} 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]_{\rho_{UVB} \|\rho_{UV}}} \right) \\
& \quad + 8 \left(2 \cdot v_2^\alpha 2^{\alpha(R+R_1+r)} 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B)} + v_2^\alpha 2^{\alpha(R+R_1)} 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]_{\rho_{UVB} \|\rho_{UV}}} \right) \\
& \leq 20 \left(v_2^\alpha 2^{\alpha(R+R_1+r-\underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B))} + v_2^\alpha 2^{\alpha(R+R_1-\underline{I}_{1-\alpha}[V; B|U])} \right) \tag{51}
\end{aligned}$$

where a follows (42); b follows from (45); c follows from (46); d follows from (47), (48), (49) and (50).

Thus, combining (42), (43) and (51), we now have the following bound on error probability:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \Pr \left\{ \hat{M} \neq 1 \mid M = 1 \right\} \\
& \leq 20 \left(v_2^\alpha 2^{\alpha(R+R_1+r-\underline{D}_{1-\alpha}(\rho_{UVB} \|\rho_{UV} \otimes \rho_B))} + v_2^\alpha 2^{\alpha(R+R_1-\underline{I}_{1-\alpha}[V; B|U])} \right) \\
& \quad + \frac{2}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_{UV} \otimes \rho_S)} \right). \tag{52}
\end{aligned}$$

Therefore, we obtain (24) of Lemma 2. In this derivation, Lemma 5 is used for the evaluation for the first term of (42) and Lemma 6 is used for the evaluation for the second term of (42).

B. Secrecy analysis

Next, we show (25) of Lemma 2. Our proof employs Lemmas 6 and 7, which are given in latter sections. Let $\rho_{ME} := \frac{1}{2^R} \sum_{m \in [1:2^R]} |m\rangle\langle m|_M \otimes \rho_{E|m}$ be the joint state between the register M and E . Notice that if the message $m \in [1:2^R]$ is transmitted using the encoding strategy discussed above then the state in Eve's possession at the end of this transmission is the following:

$$\rho_{E|m} = \text{Tr}_B \left[\mathcal{N}_{AS \rightarrow BE} \left(W_{\mathcal{C}_U, \mathcal{C}_m}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_m}^{\dagger S' \rightarrow ACIJ} \right) \right]. \quad (53)$$

We now have the following set of inequalities:

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\text{P} (\rho_{ME}, \text{Tr}_E[\rho_{ME}] \otimes \text{Tr}_M[\rho_{ME}])^2 \right] \\ &= \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\frac{1}{2^R} \sum_{m \in [1:2^R]} |m\rangle\langle m| \otimes \rho_{E|m}, \frac{1}{2^R} \sum_{m \in [1:2^R]} |m\rangle\langle m| \otimes \sum_{m \in [1:2^R]} \rho_E \right)^2 \right] \\ &= \mathbb{E}_{\mathcal{C}} \left[\left(\frac{1}{2^R} \sum_{m \in [1:2^R]} \text{P} (|m\rangle\langle m| \otimes \rho_{E|m}, |m\rangle\langle m| \otimes \rho_E) \right)^2 \right] \\ &\stackrel{a}{\leq} \frac{1}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} (|m\rangle\langle m| \otimes \rho_{E|m}, |m\rangle\langle m| \otimes \rho_E)^2 \right] \\ &= \frac{1}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} (\rho_{E|m}, \rho_E)^2 \right] \\ &\stackrel{b}{\leq} \frac{2}{2^R} \sum_{m \in [1:2^R]} \left(\mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\rho_{E|m}, \frac{1}{2^r} \sum_i \rho_{E|U(i)} \right)^2 \right] + \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\text{Tr}_M[\rho_{ME}], \frac{1}{2^r} \sum_i \rho_{E|U(i)} \right)^2 \right] \right) \\ &\stackrel{c}{\leq} \frac{4}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\rho_{E|m}, \frac{1}{2^r} \sum_i \rho_{E|U(i)} \right)^2 \right] \\ &\stackrel{d}{\leq} \frac{8}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\rho_{E|m}, \frac{1}{2^{(R_1+r)}} \sum_{(i,j)} \rho_{E|U(i), V(m,i,j)} \right)^2 \right] \\ &\quad + \frac{8}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\frac{1}{2^r} \sum_i \rho_{E|U(i)}, \frac{1}{2^{(R_1+r)}} \sum_{(i,j)} \rho_{E|U(i), V(m,i,j)} \right)^2 \right] \\ &\stackrel{e}{\leq} \frac{8}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\rho_{E|m}, \frac{1}{2^{(R_1+r)}} \sum_{(i,j)} \rho_{E|U(i), V(m,i,j)} \right)^2 \right] \\ &\quad + \frac{8}{2^R} \sum_{m \in [1:2^R]} \frac{1}{2^r} \sum_i \mathbb{E}_{\mathcal{C}} \left[\text{P} \left(\rho_{E|U(i)}, \frac{1}{2^{R_1}} \sum_j \rho_{E|U(i), V(m,i,j)} \right)^2 \right] \end{aligned} \quad (54)$$

where a follows from the convexity of $x \mapsto x^2$; b follows from Fact 2 (the triangle inequality for the purified distance) and the relation that $\text{Tr}_M[\rho_{ME}] = \rho_E$; c follows from the inequality $\text{P}(\text{Tr}_M[\rho_{ME}], \frac{1}{2^r} \sum_i \rho_{E|U(i)}) \leq \frac{1}{2^R} \sum_{m \in [1:2^R]} \mathbb{E}_{\mathcal{C}} [\text{P}(\rho_{E|m}, \frac{1}{2^r} \sum_i \rho_{E|U(i)})]$, which can be shown by the relation $\frac{1}{2^R} \sum_{m \in [1:2^R]} \rho_{E|m} = \text{Tr}_M[\rho_{ME}]$; d follows from Fact 2; e follows from Fact 3 and convexity of square function;

We now bound each of the terms on the right hand side of (54). Towards this consider the first term:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[\mathbb{P} \left(\frac{1}{2^{(R_1+r)}} \sum_{(i,j)} \rho_{E|U(i),V(m,i,j)}, \rho_{E|m} \right)^2 \right] \\
& \stackrel{a}{=} \mathbb{E}_{\mathcal{C}} \left[\mathbb{P} \left(\frac{1}{2^{(R_1+r)}} \sum_{(i,j) \in [1:2^r] \times [1:2^{R_1}]} \text{Tr}_{CB} \mathcal{N}_{AS \rightarrow BE} \left(\rho_{CAS|U(i)V(m,i,j)} \right), \right. \right. \\
& \quad \left. \left. \text{Tr}_{CBIJ} \mathcal{N}_{AS \rightarrow BE} \left(W_{\mathcal{C}_U, \mathcal{C}_m}^{S' \rightarrow ACIJ} (\phi_{S'S}) W_{\mathcal{C}_U, \mathcal{C}_m}^{\dagger S' \rightarrow ACIJ} \right) \right)^2 \right] \\
& \stackrel{b}{\leq} \frac{1}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{US} \parallel \rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{UVS} \parallel \rho_{UV} \otimes \rho_S)} \right), \tag{55}
\end{aligned}$$

where a follows from (53); b can be shown by replacing B by E in the derivation of (43).

We now bound the second term in (54) as follows:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[\mathbb{P} \left(\rho_{E|U(i)}, \frac{1}{2^{R_1}} \sum_j \rho_{E|U(i),V(m,i,j)} \right)^2 \right] \\
& \stackrel{a}{\leq} \mathbb{E}_{\mathcal{C}} \left[1 - 2^{\left(\mathbb{D}_{1+\alpha} \left(\frac{1}{2^{R_1}} \sum_j \rho_{E|U(i),V(m,i,j)} \parallel \rho_{E|U(i)} \right) \right)} \right] \\
& \stackrel{c}{\leq} \ln 2 \cdot \mathbb{E}_{\mathcal{C}} \left[\mathbb{D}_{1+\alpha} \left(\frac{1}{2^{R_1}} \sum_j \rho_{E|U(i),V(m,i,j)} \parallel \rho_{E|U(i)} \right) \right] \\
& \stackrel{d}{\leq} \frac{1}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha R_1}} 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{UV E} \parallel \rho_{U-V-E})} \right), \tag{56}
\end{aligned}$$

where a follows from Fact 5; b follows from the Fact 10; c follows from (44); and d follows from Lemma 7.

Thus, from (54), (55) and (56) we have the following bound:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[\mathbb{P} (\rho_{ME}, \text{Tr}_E[\rho_{ME}] \otimes \text{Tr}_M[\rho_{ME}])^2 \right] \\
& \leq \frac{8}{\alpha} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{US} \parallel \rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R_1+r}} \right)^\alpha 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{UVS} \parallel \rho_{UV} \otimes \rho_S)} \right) \\
& \quad + \frac{8}{\alpha} \left(\frac{v_5^\alpha}{2^{\alpha R_1}} 2^{\alpha \mathbb{D}_{1+\alpha}(\rho_{UV E} \parallel \rho_{U-V-E})} \right), \tag{57}
\end{aligned}$$

which implies (25). In this derivation, Lemma 6 is used for the evaluation of the first term in (54) and Lemma 7 is used for the evaluation of the second term in (54).

VII. HYPOTHESIS TESTING

We consider hypothesis testing on three quantum systems U, V , and B . The null hypothesis is ρ_{UVB} , and the alternative hypothesis is composed of the product state $\rho_{UV} \otimes \rho_B$ and the state ρ_{V-U-B} . To give our test, we fix two real numbers M_1 and M_2 and define the following projectors:

$$\Pi_1 := \{\mathcal{E}_2(\rho_{UVB}) \geq M_1 \rho_{UV} \otimes \rho_B\}, \tag{58}$$

$$\Pi_2 := \{\mathcal{E}_2(\rho_{UVB}) \geq M_2 \mathcal{E}_1(\rho_{V-U-B})\}. \tag{59}$$

Since it follows from the property of the pinching operations defined above that Π_1 and Π_2 commute, the test $\Pi := \Pi_1 \Pi_2$ satisfies the properties $\Pi \leq \Pi_1, \Pi_2$ and $(\mathbb{I} - \Pi) \leq (\mathbb{I} - \Pi_1) + (\mathbb{I} - \Pi_2)$. The following lemma shows the performance of the test Π .

Lemma 5. For $\alpha \in (0, 1)$

$$\text{Tr} \Pi_1(\rho_{UV} \otimes \rho_B) \leq v_2^\alpha M_1^{-(1-\alpha)} 2^{-s \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B)}; \quad (60)$$

$$\text{Tr} \Pi_2 \rho_{U-V-B} \leq v_2^\alpha M_2^{-(1-\alpha)} 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]_{\rho_{UVB} | \rho_{UV}}}; \quad (61)$$

$$\text{Tr}(\mathbb{I} - \Pi_1) \rho_{UVB} \leq v_2^\alpha M_1^\alpha 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B)}; \quad (62)$$

$$\text{Tr}(\mathbb{I} - \Pi_2) \rho_{UVB} \leq v_2^\alpha M_2^\alpha 2^{-\alpha \underline{I}_{1-\alpha}[V; B|U]_{\rho_{UVB} | \rho_{UV}}}. \quad (63)$$

Proof. We will only give the proof for (60). The proof for the other inequalities follows using exactly similar techniques.

Notice the following set of inequalities.

$$\begin{aligned} & \text{Tr}[(\mathbb{I} - \Pi_1) \rho_{UVB}] \\ & \stackrel{a}{=} \text{Tr}[\mathcal{E}_2(\mathbb{I} - \Pi_1) \rho_{UVB}] \\ & \stackrel{b}{=} \text{Tr}[(\mathbb{I} - \Pi_1) \mathcal{E}_2(\rho_{UVB})] \\ & = \text{Tr}[(\mathbb{I} - \Pi_1) (\mathcal{E}_2(\rho_{UVB}))^{1-\alpha} (\mathcal{E}_2(\rho_{UVB}))^\alpha] \\ & \stackrel{c}{\leq} M_1^\alpha \text{Tr}[(\mathbb{I} - \Pi_1) \mathcal{E}_2(\rho_{UVB})^{1-\alpha} (\rho_{UV} \otimes \rho_S)^\alpha] \\ & \stackrel{d}{\leq} M_1^\alpha \text{Tr}[\mathcal{E}_2(\rho_{UVB})^{1-\alpha} (\rho_{UV} \otimes \rho_S)^\alpha] \\ & = M_1^\alpha \text{Tr} \left[\left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \mathcal{E}_2(\rho_{UVB}) (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right)^{1-\alpha} \right] \\ & = M_1^\alpha \text{Tr} \left[\left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \mathcal{E}_2(\rho_{UVB}) (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right) \cdot \left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \mathcal{E}_2(\rho_{UVB}) (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right)^{-\alpha} \right] \\ & = M_1^\alpha \text{Tr} \left[\left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \rho_{UVB} (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right) \cdot \left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \mathcal{E}_2(\rho_{UVB}) (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right)^{-\alpha} \right] \\ & \stackrel{e}{\leq} v_2^\alpha M_1^\alpha \text{Tr} \left[\left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \rho_{UVB} (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right) \cdot \left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} (\rho_{UVB}) (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right)^{-\alpha} \right] \\ & = v_2^\alpha M_1^\alpha \text{Tr} \left((\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \rho_{UVB} (\rho_{UV} \otimes \rho_B)^{\frac{\alpha}{2(1-\alpha)}} \right)^{1-\alpha} \\ & = v_2^\alpha M_1^\alpha 2^{-\alpha \underline{D}_{1-\alpha}(\rho_{UVB} \| \rho_{UV} \otimes \rho_B)}, \end{aligned} \quad (64)$$

where a and b both follow from the definition of Π_1 and \mathcal{E}_2 along with the fact that after applying the pinching operation \mathcal{E}_2 , $(\mathbb{I} - \Pi_1)$ and $\mathcal{E}_2(\rho_{UVB})$ commute; c follows from the definition of Π_1 ; d follows from the monotonicity of the trace operation and e follows because of the Fact 9 and the operator monotonicity of the function $x^{-\alpha}$. \square

Here, we discuss the relation between Lemma 5 and existing results for quantum hypothesis testing. For this aim, we consider two states ρ and σ on the single system because since existing studies mainly discuss such a case. Let \mathcal{E}_σ be the pinching with respect to

$$\Pi_\sigma := \{\mathcal{E}_\sigma(\rho) \geq M\sigma\}. \quad (65)$$

Applying the same method as the proof of Lemma 5, we have

$$\begin{aligned} \text{Tr} \Pi_\sigma \sigma & \leq v_\sigma^\alpha M^{-(1-\alpha)} 2^{-s \underline{D}_{1-\alpha}(\rho \| \sigma)}; \\ \text{Tr}(\mathbb{I} - \Pi_\sigma) \rho & \leq v_\sigma^\alpha M^\alpha 2^{-\alpha \underline{D}_{1-\alpha}(\rho \| \sigma)}, \end{aligned} \quad (66)$$

where v_σ is the number of distinct eigenvalues of σ . In contrast, the paper [36] showed

$$\begin{aligned} \text{Tr} \Pi \sigma & \leq M^{-(1-\alpha)} 2^{-s \underline{D}_{1-\alpha}(\rho \| \sigma)}; \\ \text{Tr}(\mathbb{I} - \Pi) \rho & \leq M^\alpha 2^{-\alpha \underline{D}_{1-\alpha}(\rho \| \sigma)}, \end{aligned} \quad (67)$$

with $\Pi := \{\rho \geq M\sigma\}$ and $-\alpha \underline{D}_{1-\alpha}(\rho \| \sigma) := \log \text{Tr} \rho^{1-\alpha} \sigma^\alpha$.

Since $D_{1-\alpha}(\rho\|\sigma) \geq \underline{D}_{1-\alpha}(\rho\|\sigma)$, the evaluation (67) is better than the evaluation (66). While the evaluation (67) is obtained from the optimal testing Π , whose optimality is shown in [37], [38][39, Eq. (14)], the evaluation (66) is obtained from the testing Π_σ based on the pinching \mathcal{E}_σ , which is not optimal in general. Hence, to address the merit of the evaluation (66), we compare it with existing evaluation for the error probability of the same testing Π_σ . Before the paper [36], the paper [40] showed the following evaluation;

$$\begin{aligned} \text{Tr} \Pi_\sigma \sigma &\leq v_\sigma^\alpha M^{-(1-\alpha)} 2^{-s \hat{D}_{1-\alpha}(\rho\|\sigma)}; \\ \text{Tr}(\mathbb{I} - \Pi_\sigma) \rho &\leq v_\sigma^\alpha M^\alpha 2^{-\alpha \hat{D}_{1-\alpha}(\rho\|\sigma)}, \end{aligned} \quad (68)$$

where $-\alpha \hat{D}_{1-\alpha}(\rho\|\sigma) := \log \text{Tr} \rho \sigma^{\alpha/2} \rho^{-\alpha} \sigma^{\alpha/2}$. One might consider that we can replace Lemma 5 by the evaluation similar to (68). However, the information processing inequality of $\hat{D}_{1-\alpha}(\rho\|\sigma)$ has not been shown. Since we employ the information processing inequality of $\underline{D}_{1-\alpha}(\rho\|\sigma)$ for $\alpha \in (0, 1/2]$ in the latter discussion, we cannot replace Lemma 5 by such an evaluation.

VIII. BIVARIATE CLASSICAL-QUANTUM CHANNEL RESOLVABILITY LEMMA WITH ERROR EXPONENT

Lemma 6. *Let $\rho_{UVS} := \sum_{(u,v) \in \mathcal{U} \times \mathcal{V}} p(u,v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{S|u,v}$ be a classical-quantum state. Let, $\{U(1), \dots, U(2^r)\}$, be a set of independent and identically distributed random variables where for every $i \in [1 : 2^r]$, $U_i \sim p_U$. Further for every $(i, j) \in [1 : 2^r] \times [1 : 2^R]$, let $\{V(1, 1), \dots, V(2^r, 2^R)\}$ be a collection of independent sequences and for every (i, j) , $V(i, j) \sim p_{V|U(i)}$. Let $\mathcal{C} := \{U(1) \dots U(2^r), V(1, 1) \dots V(2^r, 2^R)\}$ and $\tau_{S|\mathcal{C}} := \frac{1}{2^{(R+r)}} \sum_{(i,j)} \rho_{S|U(i), V(i,j)}$. Then for $\alpha \in [0, 1]$ there exists constants $v_3, v_4 > 0$ such that,*

$$\begin{aligned} \mathbb{E}_{\mathcal{C}} [\underline{D}_{1+\alpha}(\tau_{S|\mathcal{C}}\|\rho_S)] &\leq \frac{1}{\alpha} \log_2 \left(\mathbb{E}_{\mathcal{C}} \left[2^{\alpha \underline{D}_{1+\alpha}(\tau_{S|\mathcal{C}}\|\rho_S)} \right] \right) \\ &\leq \frac{1}{\alpha \ln 2} \left(\left(\frac{v_3}{2^r} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{VS}\|\rho_U \otimes \rho_S)} + \left(\frac{v_4}{2^{R+r}} \right)^\alpha 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS}\|\rho_U \otimes \rho_V \otimes \rho_S)} \right), \end{aligned}$$

where in the above the first inequality follows because of the concavity of the $\log(\cdot)$.

Proof. In the proof of this lemma we will need a pair of pinching maps similar to the pair \mathcal{E}_1 and \mathcal{E}_2 defined in earlier sections. Towards this consider the following states:

$$\begin{aligned} \rho_{UVS} &= \sum_{(u,v) \in \mathcal{U} \times \mathcal{V}} p(u,v) |u\rangle\langle u|_U \otimes |v\rangle\langle v|_V \otimes \rho_{S|u,v}; \\ \rho_{V-U-S} &= \sum_{u \in \mathcal{U}} p_U(u) |u\rangle\langle u|_U \otimes \rho_{V|u} \otimes \rho_{S|u}, \end{aligned}$$

where in the above $\rho_{V|u}$ and $\rho_{S|u}$ are appropriate marginals of the state ρ_{UVS} , and $p_U(u)$ is the marginal distribution with respect to U . Here, the pinching operations \mathcal{E}_3 and \mathcal{E}_4 are defined in Section IV. Notice that \mathcal{E}_3 and \mathcal{E}_4 are defined in a manner similar to \mathcal{E}_1 and \mathcal{E}_2 , by replacing the system B with the system S .

We now have the following set of inequalities:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[2^{\alpha \underline{D}_{1+\alpha}(\tau_{S|C} \|\rho_S)} \right] \\
&= \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \tau_{S|C} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{1+\alpha} \right] \\
&= \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{(R+r)}} \sum_{(i,j)} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{1+\alpha} \right] \\
&= \frac{1}{2^{(R+r)}} \sum_{(i,j)} \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{(R+r)}} \sum_{(i',j')} \rho_{S|U(i'),V(i',j')} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&= \frac{1}{2^{(R+r)}} \sum_{(i,j)} \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \right. \\
&\quad \cdot \left. \left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{R+r}} \left(\rho_{S|U(i),V(i,j)} + \sum_{j' \neq j} \rho_{S|U(i),V(i,j')} + \sum_{i' \neq i, j' \neq j} \rho_{S|U(i'),V(i',j')} \right) \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{a}{\leq} \frac{1}{2^{(R+r)}} \sum_{(i,j)} \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \right. \\
&\quad \cdot \left. \left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{R+r}} \left(\rho_{S|U(i),V(i,j)} + \sum_{j' \neq j} \mathbb{E}_{V|U} [\rho_{S|U(i),V(i,j')}] + \sum_{i' \neq i, j' \neq j} \mathbb{E}_{U,V} [\rho_{S|U(i'),V(i',j')}] \right) \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{b}{\leq} \frac{1}{2^{(R+r)}} \sum_{(i,j)} \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{R+r}} (\rho_{S|U(i),V(i,j)} + 2^R \rho_{S|U(i)} + 2^{R+r} \rho_S) \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{c}{\leq} \sum_{i,j} \frac{1}{2^{R+r}} \mathbb{E}_{U(i),V(i,j)} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \right. \\
&\quad \cdot \left. \left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2^{R+r}} (v_4 \mathcal{E}_{4|U(i)}(\rho_{S|U(i),V(i,j)}) + v_3 2^R \mathcal{E}_3(\rho_{S|U(i)}) + 2^{R+r} \rho_S) \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{d}{\leq} \sum_{i,j} \frac{1}{2^{R+r}} \mathbb{E}_{U(i),V(i,j)} \text{Tr} \left[\left(\rho_S^{-\frac{\alpha}{2(1+\alpha)}} \rho_{S|U(i),V(i,j)} \rho_S^{-\frac{\alpha}{2(1+\alpha)}} \right) \right. \\
&\quad \cdot \left. \left(\rho_S^{-\frac{\alpha^2}{2(1+\alpha)}} \frac{1}{2^{\alpha(R+r)}} \left(v_4^{\alpha} (\mathcal{E}_{4|U(i)}(\rho_{S|U(i),V(i,j)}))^{\alpha} + v_3^{\alpha} 2^{\alpha R} (\mathcal{E}_3(\rho_{S|U(i)}))^{\alpha} + 2^{\alpha(R+r)} \rho_S^{\alpha} \right) \rho_S^{-\frac{\alpha^2}{2(1+\alpha)}} \right) \right] \\
&\stackrel{e}{=} 1 + \mathbb{E}_{U',V'} \text{Tr} \left[\frac{v_4^{\alpha}}{2^{\alpha(r+R)}} \rho_{S|U',V'} (\mathcal{E}_{4|U'}(\rho_{S|U',V'}))^{\alpha} \rho_S^{-\alpha} \right] + \mathbb{E}_{U',V'} \text{Tr} \left[\frac{v_3^{\alpha}}{2^{\alpha r}} \rho_{S|U',V'} \mathcal{E}_3(\rho_{S|U'})^{\alpha} \rho_S^{-\alpha} \right] \\
&\stackrel{f}{=} 1 + \frac{v_4^{\alpha}}{2^{\alpha(r+R)}} \text{Tr} \left[\mathbb{E}_{U',V'} (\mathcal{E}_{4|U'}(\rho_{S|U',V'}))^{(1+\alpha)} \rho_S^{-\alpha} \right] + \frac{v_3^{\alpha}}{2^{\alpha r}} \text{Tr} \left[\mathbb{E}_{U'} (\mathcal{E}_3(\rho_{S|U'}))^{(1+\alpha)} \rho_S^{-\alpha} \right] \\
&\stackrel{g}{\leq} 1 + \frac{v_3^{\alpha}}{2^{\alpha r}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{US} \|\rho_U \otimes \rho_S)} + \frac{v_4^{\alpha}}{2^{\alpha(r+R)}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UVS} \|\rho_U \otimes \rho_S)},
\end{aligned}$$

where from step e , the variables U' and V' are subject to the joint distribution $p(u, v)$. Here, a follows from the fact that when $j \neq j'$ then the random variables $(V(i, j), V(i, j'))$ are independent of each other and from the operator Jensen's inequality; b follows from symmetry and from the definition of $\rho_{S|u}$ and ρ_S ; c follows from the Fact 9; d follows because the terms in the second terms inside the trace commute and from the fact that $(a+b)^x \leq a^x + b^x$ ($a, b > 0; x < 1$); e follows from the circular and linear property of the trace operation; f follows from the circularity of trace operation and g follows from the definition of $\underline{D}_{1+\alpha}(\cdot \|\cdot)$, the data-processing inequality

(Fact 3), and the fact that the states involved are classical-quantum states. The desired bound now follows from the fact that $\log_2(1+x) \leq \frac{x}{\ln 2}$. \square

IX. CONDITIONAL CLASSICAL-QUANTUM CHANNEL RESOLVABILITY LEMMA WITH ERROR EXPONENT

Lemma 7. *Let $\rho^{UVE} := \sum_{(u,v) \in \mathcal{U} \times \mathcal{V}} p_{UV}(u,v) (u,v) |u\rangle\langle u|^U \otimes |v\rangle\langle v|^V \otimes \rho_{E|u,v}$ be a classical-quantum state. Further, let $\mathcal{C} := \{U', V(1), \dots, V(2^R)\}$ be a collection of random variables where for every $i \in [1 : 2^R]$, $(U', V(i)) \sim p_{UV}$ and for $i \neq i'$, $(V(i), V(i')) \sim p_{V(i)|U'} \cdot p_{V(i')|U'}$. Consider the following state:*

$$\tau_{E|\mathcal{C}} := \frac{1}{2^R} \sum_i \rho_{E|U', V(i)}.$$

Then, for $\alpha \in [0, 1]$ there exists constant $v_5 > 0$ such that,

$$\begin{aligned} \mathbb{E}_{\mathcal{C}} [\underline{D}_{1+\alpha}(\tau_{E|\mathcal{C}} \| \rho_{E|U'})] &\leq \frac{1}{\alpha} \log_2 \left(\mathbb{E}_{\mathcal{C}} \left[2^{\alpha \underline{D}_{1+\alpha}(\tau_{E|\mathcal{C}} \| \rho_{E|U'})} \right] \right) \\ &\leq \frac{1}{\alpha \ln 2} \left(\frac{v_5^\alpha}{2^{\alpha R}} 2^{\alpha \underline{D}_{1+\alpha}(\rho_{UV E} \| \rho_{V-U-E})} \right), \end{aligned}$$

where in the above $\rho_{V-U-E} := \sum_u p_U(u) |u\rangle\langle u|^U \otimes \rho_{V|u} \otimes \rho_{E|u}$ and the first inequality is because of the concavity of the $\log(\cdot)$.

Proof. Let $\mathcal{E}_{5|u}$ be the pinching maps with respect to the spectral decomposition of $\rho_{E|u}$. Further, let v_5 represent the maximum number of distinct components of the pinching map $\{\mathcal{E}_{5|u}\}_u$.

We now have the following inequalities:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[2^{\alpha \underline{D}_{1+\alpha}(\tau_{E|C} \|\rho_{E|U'})} \right] \\
&= \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \tau_{E|C} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{1+\alpha} \right] \\
&= \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} \sum_i \rho_{E|U',V(i)} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{1+\alpha} \right] \\
&= \frac{1}{2R} \sum_i \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V(i)} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} \sum_{i'} \rho_{E|U',V(i')} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&= \frac{1}{2R} \sum_i \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V(i)} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} \left(\rho_{E|U',V(i)} + \sum_{i' \neq i} \rho_{E|U',V(i')} \right) \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{a}{\leq} \frac{1}{2R} \sum_i \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V(i)} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} (\rho_{E|U',V(i)} + 2^R \rho_{E|U'}) \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{b}{\leq} \frac{1}{2R} \sum_i \mathbb{E}_{\mathcal{C}} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V(i)} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} (v_5 \mathcal{E}_{5|U'}(\rho_{E|U',V(i)}) + 2^R \rho_{E|U'}) \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{c}{=} \mathbb{E}_{U',V'} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V'} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \frac{1}{2R} (v_5 \mathcal{E}_{5|U'}(\rho_{E|U',V'}) + 2^R \rho_{E|U'}) \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right)^{\alpha} \right] \\
&\stackrel{d}{\leq} \mathbb{E}_{U',V'} \text{Tr} \left[\left(\rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \rho_{E|U',V'} \rho_{E|U'}^{-\frac{\alpha}{2(1+\alpha)}} \right) \cdot \left(\rho_{E|U'}^{-\frac{\alpha^2}{2(1+\alpha)}} \frac{1}{2^{\alpha R}} (v_5^{\alpha} (\mathcal{E}_{5|U'}(\rho_{E|U',V'}))^{\alpha} + 2^{\alpha R} \rho_{E|U'}^{\alpha}) \rho_{E|U'}^{-\frac{\alpha^2}{2(1+\alpha)}} \right) \right] \\
&\stackrel{e}{=} 1 + \mathbb{E}_{U',V'} \text{Tr} \left[\frac{v_5^{\alpha}}{2^{\alpha R}} \rho_{E|U',V'} (\mathcal{E}_{5|U'}(\rho_{E|U',V'}))^{\alpha} \rho_{E|U'}^{-\alpha} \right] \\
&\stackrel{f}{=} 1 + \frac{v_5^{\alpha}}{2^{\alpha R}} \mathbb{E}_{U',V'} \text{Tr} \left[(\mathcal{E}_{5|U'}(\rho_{E|U',V'}))^{(1+\alpha)} \rho_{E|U'}^{-\alpha} \right] \\
&\stackrel{g}{\leq} 1 + \frac{v_5^{\alpha}}{2^{\alpha R}} 2^{\underline{D}_{1+\alpha}(\rho_{UV} \|\rho_{V-U-E})},
\end{aligned}$$

where in step c , U', V' are distributed the same as U, V . Here, a follows from the fact that when $i' \neq i$ then the random variables $(V(i), V(i')) \sim p_{V(i)|U} \cdot p_{V(i')|U}$ and from the operator Jensen's inequality and from the definition of $\rho_{E|U}$; b follows from Fact 9; c follows from the symmetry of \mathcal{C} ; d follows because the terms in the second terms inside the trace is completely classical and from the fact that $(a+b)^x \leq a^x + b^x$ ($a, b > 0; x < 1$); e follows from the circular and linear property of the trace operation; f follows from the circularity of trace operation and g follows from the definition of $\underline{D}_{1+\alpha}(\cdot \|\cdot)$, the data-processing inequality (Fact 3), and the fact that the states involved are classical-quantum states. The desired bound now follows from the fact that $\log_2(1+x) \leq \frac{x}{\ln 2}$. \square

X. DISCUSSION

We have derived an achievable rate for secure communication over fully quantum Gel'fand-Pinsker wiretap channel. This rate is a natural quantum extension of the achievable rate of secure communication over Gel'fand-Pinsker wiretap channel, as given in [2]. Here, we emphasize that even in the classical case, a matching converse is known only in a special case [2, Remark 7] and the question of finding a matching converse is still open in the general case.

Further, since our proof is based on an exponential upper bound of decoding error probability, our method has the potential to improve the evaluation of error probabilities for various kinds of coding problems. Our method has two key points. The first key point is the removal of the correlations between the system S and the systems B, E in the analysis. In our method, instead of these correlations, we evaluate the correlation between the system S and the message.

The second key point is composed of three types of evaluations. The first is the evaluation of the decoding error probability of super-position coding, in which the code is randomly generated by conditional distribution. This evaluation is based on a special type of hypothesis testing on three systems, which is discussed in Section VII. The second is the bivariate classical-quantum channel resolvability, which is given in Section VIII. We have evaluated the quality of approximation of the average output state when the superposition coding is applied. Similar to the evaluation in [7], our upper bound has the exponential form. The third is conditional classical-quantum channel resolvability, which is given in Section IX. We have derived the conditional evaluation of approximation of the average output state when the superposition coding based on p_{UV} is applied. Here, we take the condition for the choice of V . Similar to the above case, our upper bound also has the exponential form. Combining the first and second types of evaluations, we have analyzed the decoding error probability of our code for our main problem. Further, combining the second and third types of evaluations, we have evaluated the secrecy of our code.

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APPENDIX

Here we show Lemma 1 by following 7 steps.

Step 1: Analysis of easy case.

We first prove that $\max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}) \geq \max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS})$. For $\rho_{UVAS} \in \mathcal{S}_2$, the inequality $I[UV; B] - I[U; S] - I[V; E | U] \geq I[V; B | U] - I[V; E | U]$ follows from $I[UV; B] - I[U; S] - I[V; E | U] = I[V; B | U] - I[V; E | U] + I[U; B] - I[U; S]$ and the assumption $I[U; B] - I[U; S] \geq 0$. Hence, $R_a(\rho_{UVAS}) = R_{alt}(\rho_{UVAS})$. Thus, $\max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}) \geq \max_{\rho_{UVAS} \in \mathcal{S}_2} R_a(\rho_{UVAS}) = \max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS})$.

We now prove the opposite inequality

$$\max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS}) \geq \max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}). \quad (69)$$

Let ρ_{UVAS}^* be a state such that $R_a(\rho_{UVAS}^*) = \max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}) > 0$, (if $R_a \leq 0$ then the bound is trivial to prove). If the state ρ_{UVAS}^* satisfies the inequality $I[U; B] - I[U; S] > 0$, we have $\max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}) = R_a(\rho_{UVAS}^*) \leq R_{alt}(\rho_{UVAS}^*) \leq \max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS})$. Thus, the inequality of interest holds.

Step 2: Main part.

We show (69) when ρ_{UVAS}^* satisfies the opposite inequality, i.e.,

$$I[U; B] - I[U; S] \leq 0. \quad (70)$$

For this aim, we introduce the variable \tilde{V} as the output of the erasure channel of erasure probability $\varepsilon \in [0, 1]$ with the input variable V . Using the transition probability $p_{\tilde{V}|V}$ of the erasure channel with erasure probability ε , we define the variables $U' := (U, \tilde{V})$, $V' := V$, and the following classical-quantum state:

$$\rho_{UV\tilde{V}U'V'AS} := \sum_{(u,v)} p_{UV}(u, v) p_{\tilde{V}|V}(\tilde{v} | v) \delta_{\{U'=(U,\tilde{V}), V'=V\}} |u\rangle\langle u| \otimes |v\rangle\langle v| \otimes |u'\rangle\langle u'| \otimes |v'\rangle\langle v'| \otimes |\tilde{v}\rangle\langle \tilde{v}| \otimes \rho_{AS|u,v}^*.$$

In what follows all the calculations would be with respect to the state $\rho_{UV\tilde{V}U'V'AS}$ and the channel $\mathcal{N}_{AS \rightarrow BE}$. Hence, the state $\text{Tr}_{UV\tilde{V}}[\rho_{UV\tilde{V}U'V'AS}]$ is a valid state for R_{alt} . As shown later (Step 3), there exists an $\varepsilon \in (0, 1)$ such that

$$I[U'; B] - I[U'; S] = 0 \quad (71)$$

$$I[U; B] - I[U; S] + (1 - \varepsilon)(I[V; B | U] - I[V; S | U]) = 0. \quad (72)$$

Then, we choose such an $\varepsilon \in [0, 1]$. Also, as shown later (Steps 4 and 5), we have

$$I[U'V'; B] - I[U'V'; S] \geq R_a(\rho_{UVAS}^*) \quad (73)$$

$$I[V'; B | U'] - I[V'; E | U'] \geq R_a(\rho_{UVAS}^*). \quad (74)$$

Hence, considering the definition of $R_{alt}(\text{Tr}_{UV\tilde{V}}[\rho_{UV\tilde{V}U'V'AS}])$, we obtain (69) as

$$\begin{aligned} \max_{\rho_{UVAS} \in \mathcal{S}_2} R_{alt}(\rho_{UVAS}) &\geq R_{alt}(\text{Tr}_{UV\tilde{V}}[\rho_{UV\tilde{V}U'V'AS}]) \\ &\geq R_a(\rho_{UVAS}^*) = \max_{\rho_{UVAS} \in \mathcal{S}_1} R_a(\rho_{UVAS}). \end{aligned}$$

Step 3: Existence of ε satisfying (71) and (72).

Now, we show that there exists an $\varepsilon \in [0, 1]$ satisfying (71) and (72). We find

$$\begin{aligned} I[U'; B] - I[U'; S] &= I[U\tilde{V}; B] - I[U\tilde{V}; S] \\ &= I[U; B] - I[U; S] + I[\tilde{V}; B | U] - I[\tilde{V}; S | U] \\ &= I[U; B] - I[U; S] + (1 - \varepsilon)(I[V; B | U] - I[V; S | U]). \end{aligned} \quad (75)$$

Notice that in the above if $\varepsilon = 1$ then $I[U'; B] - I[U'; S] < 0$, this follows from the assumption that $I[U; B] - I[U; S] < 0$. On the other hand if $\varepsilon = 0$ then $I[U'; B] - I[U'; S] = I[UV; B] - I[UV; S]$. Our assumption that $R_a(\rho_{UVAS}^*) > 0$ guarantees that $I[UV; B] - I[UV; S] > 0$. Thus, $I[U'; B] - I[U'; S] > 0$. Hence, we can choose an $\varepsilon \in [0, 1]$ such that $I[U'; B] - I[U'; S] = 0$, i.e., (71), which implies (72).

Step 4: Proof of (73).

Since \tilde{V} is obtained by passing V through an erasure channel, we have $I[UV\tilde{V}; B] = I[UV; B]$ and $I[UV\tilde{V}; S] = I[UV; S]$. Hence, we can show (73) as follows:

$$\begin{aligned} I[U'V'; B] - I[U'V'; S] &= I[UV\tilde{V}; B] - I[UV\tilde{V}; S] \\ &= I[UV; B] - I[UV; S] \\ &\stackrel{a}{\geq} R_a(\rho_{UVAS}^*), \end{aligned} \quad (76)$$

where a follows from the definition of $R_a(\rho_{UVAS}^*)$.

Step 5: Proof of (74).

Since \tilde{V} is the output of the erasure channel of erasure probability ε with the input random variable V , we have $I[\tilde{V}; B | U] = (1 - \varepsilon)I[V; B | U]$ and $I[\tilde{V}; E | U] = (1 - \varepsilon)I[V; E | U]$. Hence, we can show (74) as follows:

$$\begin{aligned} I[V'; B | U'] - I[V'; E | U'] &= I[V'; B | U\tilde{V}] - I[V'; E | U\tilde{V}] \\ &\stackrel{a}{=} I[V; B | U] - I[V; E | U] - (I[\tilde{V}; B | U] - I[\tilde{V}; E | U]) \\ &= I[V; B | U] - I[V; E | U] - (1 - \varepsilon)(I[V; B | U] - I[V; E | U]) \\ &= \varepsilon(I[V; B | U] - I[V; E | U]), \end{aligned} \quad (77)$$

where a follows from the chain rule of mutual information. Using exactly similar steps we can prove that

$$I[V'; B | U'] - I[V'; S | U'] = \varepsilon(I[V; B | U] - I[V; S | U]). \quad (78)$$

As shown later (Steps 6 and 7), using (78), we have

$$\varepsilon (I[V; B | U] - I[V; E | U]) \geq R_a(\rho_{UVAS}^*). \quad (79)$$

Hence, we obtain (74).

Step 6: Proof of (79) in First case.

We show (79) when

$$I[V; S | U] \geq I[V; E | U]. \quad (80)$$

We have the following set of inequalities:

$$\begin{aligned} \varepsilon (I[V; B | U] - I[V; E | U]) &\stackrel{a}{\geq} \varepsilon (I[V; B | U] - I[V; S | U]) \\ &\stackrel{b}{=} I[V'; B | U'] - I[V'; S | U'] \\ &\stackrel{c}{=} I[U'V'; B] - I[U'V'; S] \\ &\stackrel{d}{\geq} R_a(\rho_{UVAS}^*), \end{aligned} \quad (81)$$

where a follows from (80); b follows from (78); c follows from (71) and d follows from (73). Hence, we have (79).

Step 7: Proof of (79) in Second case.

We show (79) when

$$I[V; S | U] < I[V; E | U]. \quad (82)$$

The assumption $I[U; B] - I[U; S] \leq 0$ (see (70)) implies that

$$I[V; B | U] - I[V; E | U] + I[U; B] - I[U; S] \leq I[V; B | U] - I[V; E | U]. \quad (83)$$

The assumption $I[V; S | U] < I[V; E | U]$ (see (82)) implies that

$$I[UV; B] - I[U; S] - I[V; E | U] < I[UV; B] - I[UV; S]. \quad (84)$$

Using the chain rule, we have

$$I[UV; B] - I[U; S] - I[V; E | U] = I[V; B | U] - I[V; E | U] + I[U; B] - I[U; S]. \quad (85)$$

Combining (83), (84), and (85), we have

$$I[UV; B] - I[U; S] - I[V; E | U] \leq \min\{I[V; B | U] - I[V; E | U], I[UV; B] - I[UV; S]\}. \quad (86)$$

Hence, the minimum $R_a(\rho_{UVAS}^*)$ is given as

$$R_a(\rho_{UVAS}^*) = I[UV; B] - I[U; S] - I[V; E | U]. \quad (87)$$

Now, we have the following set of inequalities:

$$\begin{aligned} \varepsilon (I[V; B | U] - I[V; E | U]) &= I[V; B | U] - I[V; E | U] - (1 - \varepsilon) (I[V; B | U] - I[V; E | U]) \\ &\stackrel{a}{>} I[V; B | U] - I[V; E | U] - (1 - \varepsilon) (I[V; B | U] - I[V; S | U]) \\ &\stackrel{b}{=} I[V; B | U] - I[V; E | U] + I(U; B) - I(U; S) \\ &\stackrel{c}{=} R_a(\rho_{UVAS}^*), \end{aligned} \quad (88)$$

where a follows from the assumption that $I[V; S | U] < I[V; E | U]$ (see (82)); b follows from (72) and c follows from (87). Hence, we obtain (79). This completes the proof.