

# New Construction of Optimal Type-II Binary Z-Complementary Pairs

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## Abstract

A pair of sequences is called a Z-complementary pair (ZCP) if it has zero aperiodic autocorrelation sums at each of the non-zero time-shifts within a certain region, called the zero correlation zone (ZCZ). ZCPs are categorised into two types: Type-I ZCPs and Type-II ZCPs. Type-I ZCPs have the ZCZ around the in-phase position and Type-II ZCPs have the ZCZ around the end-shift position. Till now only a few constructions of Type-II ZCPs are reported in the literature, and all have lengths of the form  $2^m \pm 1$  or  $N + 1$  where  $N = 2^a 10^b 26^c$  and  $a, b, c$  are non-negative integers. In this paper, we propose a recursive construction of ZCPs based on concatenation of sequences. Inspired by Turyn's construction of Golay complementary pairs, we also propose a construction of Type-II ZCPs from known ones. The proposed constructions can generate optimal Type-II ZCPs with new flexible parameters and Z-optimal Type-II ZCPs with any odd length. In addition, we give upper bounds for the PMEPR of the proposed ZCPs. It turns out that our constructions lead to ZCPs with low PMEPR.

## Index Terms

Aperiodic correlation, Golay sequence, complementary pair, peak-to-mean envelope power ratio (PMEPR), Z-complementary pair.

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## I. INTRODUCTION

A PAIR of sequences is called a Golay complementary pair (GCP), if their aperiodic autocorrelation sums (AACSSs) are zero everywhere, except at the zero shift [1], [2]. GCPs were first introduced by Golay in 1961 in the context of an optical problem in multislit spectrometry. Since then, GCPs have found extensive engineering applications for its ideal correlation properties. For example, GCPs are useful in inter-symbol interference channel estimation [3], [4], radar waveform designs [5], [6], [7], asynchronous multi-carrier code-division multiple access (MC-CDMA) communications [8], [9], and peak-to-mean envelope power ratio (PMEPR) control in multi-carrier communications [10], [11].

The main drawback of the GCPs is their limited availability for various lengths. It was conjectured that binary GCPs are available only for lengths of the form  $2^a 10^b 26^c$  where  $a, b, c$  are non-negative integers [12]. By computer search, the conjecture has been verified for binary GCPs of length up to 100 [12]. In search of binary sequence pairs of other lengths, Fan, Yuan and Tu [13] proposed the concept of Z-complementary pair (ZCP) in 2007, which is a pair of sequences whose aperiodic autocorrelation sums are zero not everywhere but within a certain region called *zero correlation zone (ZCZ)*. Based on their lengths, binary ZCPs are categorised into two types: odd-length (OB-ZCPs) and even-length ZCPs (EB-ZCPs). It was further conjectured by Fan, Yuan and Tu [13] that “*For OB-ZCPs, the maximum zero correlation zone is given by  $Z_{\max} = (N + 1)/2$ , and for EB-ZCPs, given that the lengths  $N \neq 2^a 10^b 26^c$ , the ZCZ is upper bounded by  $N - 2$ .*” In 2011 Li *et al.* [14] proved the conjecture for OB-ZCPs. However, a systematic construction of OB-ZCPs was still unknown.

In 2014, Liu, Udaya and Guan [15] made remarkable progress towards this open problem and proposed a systematic construction of OB-ZCPs. The generated optimal sequence pairs achieve the maximum ZCZ of width  $(N + 1)/2$  as well as the minimum AACSSs magnitude of 2 at each time-shift outside the ZCZ [15]. The construction of optimal OB-ZCPs in [15] was given by applying the insertion method on binary Golay-Davis-Jedwab (GDJ) sequences. In 2014, Liu, Udaya and Guan [16] also confirmed the conjecture for EB-ZCPs that  $Z_{\max} \leq N - 2$ . Recently, a lot of work has been done for constructing EB-ZCPs (for example, see [17], [18], [19]). Besides the lengths, in [15] Liu, Udaya and Guan further categorised ZCPs based on their correlation properties: Type-I ZCPs are sequence pairs having zero AACSSs at each time-shift within the ZCZ around the in-phase position, while Type-II ZCPs are those having zero AACSSs

at each time-shift within the ZCZ around the end-shift position. Type-I ZCPs can be effectively used in quasi-synchronous CDMA (QS-CDMA) systems [20], [21], [22], [23], [24], [25], [26], which are tolerant of small-signal arrival delays. On the other hand, Type-II ZCPs are useful in wide-band wireless communication systems where the minimum interfering-signal delay can assume a large value. This is because the ZCZ of Type-II ZCP is designed for large time-shifts, and thus the asynchronous interfering signals arriving at the receiver after large delays can be rejected. A typical example of such a channel with large delays is the sparsely populated rural and mountainous areas [25]. In some other important applications, like designing preamble sequences in OFDMA systems [27], where PMEPR plays a very important role, Type-II ZCPs may be advantageous over Type-I ZCPs because of its huge availability with flexible lengths. Moreover, Type-I and Type-II ZCPs can also be used to construct complementary sets [28] and Z-complementary sets [29].

Till now there are only a few constructions of Type-I and Type-II constructions reported in the literature [15], [16], [17], [18], [19], [30], [31]. Note that most of the constructions are based on GCPs and thereby have lengths of the form of  $2^a + 2^v$ . Recently, based on generalized Boolean functions, Chen [17] gave a direct construction of those Type-I ZCPs having lengths of the form  $2^{m-1} + 2^v$  and a ZCZ of width  $2^{\pi(v+1)-1} + 2^v$ , where  $\pi$  is a permutation over  $\{1, 2, \dots, m-1\}$ . Adhikary *et al.* [30], [31] made further progress towards this problem and proposed a systematic construction of Type-I and Type-II ZCPs of lengths of the form  $2^a 10^b 26^c + 1$ , by applying the insertion method on binary GCPs. Very recently, Shen *et al.* [32] constructed Type-II ZCP of length  $2^m + 3$ , by inserting 3 elements into GDJ sequences. An overview of known Type-II binary ZCPs is given in Table I, together with their corresponding ZCZ width. For the definitions of Z-optimality and optimality of binary ZCPs, see Definitions 6 and 7, respectively.

To the best of our knowledge, the maximum ZCZ width for binary Type-II ZCPs of lengths of the form  $10^b + 1$ ,  $26^c + 1$  and  $10^b 26^c + 1$  are  $4 \times 10^{b-1} + 1$ ,  $12 \times 26^{c-1} + 1$  and  $12 \times 10^b 26^{c-1} + 1$ , respectively. Furthermore, there is no construction of Type-II EB-ZCPs in the literature.

Motivated by the constructions reported in [15], [30], in search of ZCPs with larger ZCZ widths, we propose an iterative construction of Type-II binary ZCPs of both even and odd lengths. Our proposed construction can generate Z-optimal Type-II OB-ZCPs having lengths of any odd length, and also optimal Type-II EB-ZCPs for certain cases. In fact, our proposed construction can generate Type-II ZCPs with more flexible lengths which were unknown before. As a comparison with previous results, our results are given in Table I. We further list down the

TABLE I: Existing and proposed binary Type-II ZCPs.

Ref.	Length	ZCZ width	Magnitude outside the ZCZ ( $v$ )	Remarks on Z-optimality	Remarks on optimality
[15]	$2^m + 1$	$2^{m-1} + 1$	2	Z-optimal	Optimal
[15]	$2^m - 1$	$2^{m-1}$	2	Z-optimal	Optimal
[30]	$2^a 10^b 26^c + 1$ , $a \geq 1$	$2^{a-1} 10^b 26^c + 1$	2	Z-optimal	Optimal
[30]	$10^b + 1$ , $b \geq 1$	$4 \times 10^{b-1} + 1$	2	Not Z-optimal	Not optimal
[30]	$26^c + 1$ , $c \geq 1$	$12 \times 26^{c-1} + 1$	2	Not Z-optimal	Not optimal
[30]	$10^b 26^c + 1$ , $b, c \geq 1$	$12 \times 10^b 26^{c-1} + 1$	2	Not Z-optimal	Not optimal
[32]	$2^m + 3$	$2^{m-1} + 2$	$v \in \{2, 6\}$	Z-optimal	Not optimal
Theorem 1	$2N + 1$ , $N \in \mathbb{Z}^+$	$N + 1$	$2 \leq v \leq 2(2N - 1)$	Z-optimal	Optimal when $ \rho_a(\tau) + \rho_b(\tau)  = 1$
Remark 3	$2N - 1$ , $N \in \mathbb{Z}^+$	$N$	$2 \leq v \leq 2(2N - 3)$	Z-optimal	Not optimal
Theorem 2	$2^k N + 2^{k-1}$ , $N \in \mathbb{Z}^+$ , $k \geq 2$	$2^k N + 2^{k-1} - N$	$4 \leq v \leq 2^k(2N - 1)$	Z-optimal when $N = 1$	Not optimal
Theorem 3	$3N$ , $N = 2^a 10^b 26^c$	$3N - 1$	$2N$	Z-optimal	Optimal when $N = 1, 2$
Theorem 3	$14N$ , $N = 2^a 10^b 26^c$	$14N - 1$	$4N$	Z-optimal	Optimal when $N = 1$
Theorem 5	$2N - 1$ , $N = 2^a 10^b 26^c$	$N$	2	Z-optimal	Optimal
Theorem 6	$2N + 1$ , $N = 2^a 10^b 26^c$	$N + 1$	2	Z-optimal	Optimal

“best possible” ZCPs up to length 30 in Table II. The term “best possible”, means that the ZCPs have the closest possible autocorrelation properties to those of the optimal Type-II ZCPs. Note that the sequence pairs, whose lengths are given in bold letters in Table II, were not reported before.

The rest of this paper is organized as follows. In Section II, we introduce some basic definitions and preliminary results about ZCPs, and the peak-to-mean envelope power ratio (PMEPR) control problem in code-keying MC communications. In Section III, we propose a generic construction of Type-II ZCPs, which allows generating both OB-ZCPs and EB-ZCPs. In addition, we propose a construction of optimal Type-II OB-ZCPs in Section IV. In Section V, we analyse the PMEPR of the proposed ZCPs. Finally, Section VI concludes the paper by some future work.

TABLE II: “Best possible” Type-II sequence pairs of length up to 30

$N$	Type	$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$	$ \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) _{\tau=0}^{N-1}$
2	GCP	$\begin{pmatrix} ++ \\ +- \end{pmatrix}$	(4,0)
3	Optimal OB-ZCP	$\begin{pmatrix} +++ \\ ++- \end{pmatrix}$	(6,2,0)
4	GCP	$\begin{pmatrix} +++- \\ ++-+ \end{pmatrix}$	(8, $\mathbf{0}_3$ )
5	Optimal OB-ZCP	$\begin{pmatrix} ---++ \\ --+-- \end{pmatrix}$	(10, $\mathbf{2}_2, \mathbf{0}_2$ )
6	Optimal EB-ZCP	$\begin{pmatrix} ++++-- \\ +++-++ \end{pmatrix}$	(12, 4, $\mathbf{0}_4$ )
7	Optimal OB-ZCP	$\begin{pmatrix} ---+--+ \\ --+--+ \end{pmatrix}$	(14, $\mathbf{2}_3, \mathbf{0}_3$ )
8	GCP	$\begin{pmatrix} +++-++-+ \\ +++-+-+ \end{pmatrix}$	(16, $\mathbf{0}_7$ )
9	Optimal OB-ZCP	$\begin{pmatrix} +-++++-+ \\ +-++++-+ \end{pmatrix}$	(18, $\mathbf{2}_4, \mathbf{0}_4$ )
10	GCP	$\begin{pmatrix} +-+--+--+ \\ +-+--+--+ \end{pmatrix}$	(20, $\mathbf{0}_9$ )
12	Z-optimal EB-ZCP	$\begin{pmatrix} ++++--+++- \\ ++++--+++- \end{pmatrix}$	(24, 8, $\mathbf{0}_{10}$ )
14	EB-ZCP	$\begin{pmatrix} ---+--+--+ \\ ---+--+--+ \end{pmatrix}$	(28, $\mathbf{4}_3, \mathbf{0}_{10}$ )
15	Optimal OB-ZCP	$\begin{pmatrix} -+--+--+--+ \\ -+--+--+--+ \end{pmatrix}$	(30, $\mathbf{2}_7, \mathbf{0}_7$ )
16	GCP	$\begin{pmatrix} +++-++-++- \\ +++-++-++- \end{pmatrix}$	(32, $\mathbf{0}_{15}$ )
17	Optimal OB-ZCP	$\begin{pmatrix} -++++-+--+ \\ -++++-+--+ \end{pmatrix}$	(34, $\mathbf{2}_8, \mathbf{0}_8$ )
18	EB-ZCP	$\begin{pmatrix} +-++++-+--+ \\ +-++++-+--+ \end{pmatrix}$	(36, $\mathbf{4}_4, \mathbf{0}_{13}$ )
19	Optimal OB-ZCP	$\begin{pmatrix} +-++++-+--+ \\ +-++++-+--+ \end{pmatrix}$	(22, $\mathbf{2}_9, \mathbf{0}_9$ )
20	GCP	$\begin{pmatrix} +++-+--+--+ \\ +++-+--+--+ \end{pmatrix}$	(20, $\mathbf{0}_9$ )
21	Optimal OB-ZCP	$\begin{pmatrix} -+--+--+--+ \\ -+--+--+--+ \end{pmatrix}$	(42, $\mathbf{2}_{10}, \mathbf{0}_{10}$ )
24	Z-optimal EB-ZCP	$\begin{pmatrix} ++++--+++- \\ ++++--+++- \end{pmatrix}$	(48, 16, $\mathbf{0}_{22}$ )
26	GCP	$\begin{pmatrix} +++-++-++- \\ +++-++-++- \end{pmatrix}$	(52, $\mathbf{0}_{25}$ )
28	Z-optimal EB-ZCP	$\begin{pmatrix} ---+--+--+ \\ ---+--+--+ \end{pmatrix}$	(56, $\mathbf{8}_3, \mathbf{0}_{24}$ )
30	Z-optimal EB-ZCP	$\begin{pmatrix} +-+--+--+ \\ +-+--+--+ \end{pmatrix}$	(60, 20, $\mathbf{0}_{28}$ )

## II. PRELIMINARIES

In this section, we recall some definitions and bounds of binary ZCPs. Before that, we fix some notations which will be used throughout the paper.

- $+$  and  $-$  denote 1 and  $-1$ , respectively.
- $\mathbf{0}_L$  and  $\mathbf{1}_L$  denote length- $L$  vectors whose elements are all 0 and 1, respectively.
- $\overleftarrow{\mathbf{c}} = (c_{N_1-1}, c_{N_1-2}, \dots, c_0)$  denotes the reverse of sequence  $\mathbf{c} = (c_0, \dots, c_{N_1-2}, c_{N_1-1})$ .
- $\mathbf{c} \parallel \mathbf{d}$  denote the horizontal concatenation of sequences  $\mathbf{c}$  and  $\mathbf{d}$ .
- $\mathbf{c} \otimes \mathbf{d}$  denotes the Kroneker product of the sequences  $\mathbf{c}$  and  $\mathbf{d}$  of lengths  $N_1$  and  $N_2$ , respectively, i.e.,

$$\mathbf{c} \otimes \mathbf{d} = (c_0 d_0, c_0 d_1, \dots, c_0 d_{N_2-1}, \dots, c_{N_1-1} d_0, c_{N_1-1} d_1, \dots, c_{N_1-1} d_{N_2-1}).$$

In the following, we first give the definition of aperiodic correlation, and then define the deletion function.

*Definition 1:* For two length- $N$  binary sequences  $\mathbf{c}$  and  $\mathbf{d}$ , their *aperiodic cross-correlation function* is defined as

$$\rho_{\mathbf{c}, \mathbf{d}}(\tau) = \begin{cases} \sum_{k=0}^{N-1-\tau} c_k d_{k+\tau}, & 0 \leq \tau \leq N-1, \\ \sum_{k=0}^{N-1-\tau} c_{k+\tau} d_k, & -(N-1) \leq \tau \leq -1, \\ 0, & |\tau| \geq N. \end{cases} \quad (1)$$

When  $\mathbf{c} = \mathbf{d}$ , the function  $\rho_{\mathbf{c}, \mathbf{d}}(\tau)$  is called the *aperiodic autocorrelation function* (AACF) of  $\mathbf{c}$ , denoted by  $\rho_{\mathbf{c}}(\tau)$  for simplicity.

*Definition 2:* (Deletion Function) For a sequence  $\mathbf{c} = (c_0, c_1, \dots, c_{N-1})$  and an integer  $r \in \{0, 1, \dots, N-1\}$ , define  $\mathcal{V}(\mathbf{c}, r)$  as a deletion function of  $\mathbf{c}$  as

$$\mathcal{V}(\mathbf{c}, r) = \begin{cases} (c_1, c_2, \dots, c_{N-1}), & \text{if } r = 0; \\ (c_0, c_1, \dots, c_{N-2}), & \text{if } r = N-1; \\ (c_0, c_1, \dots, c_{r-1}, c_{r+1}, \dots, c_{N-1}), & \text{otherwise,} \end{cases}$$

where  $r$  denotes the deletion position.

In what follows, we give a series of definitions on a pair of sequences with desirable aperiodic autocorrelation sums, and certain bounds on these sequence pairs.

*Definition 3:* A pair of sequences  $\mathbf{c}$  and  $\mathbf{d}$  of length  $N$  is called a *Golay complementary pair* (GCP) if and only if

$$\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau) = 0,$$

for all  $1 \leq \tau \leq N - 1$ .

*Definition 4:* (Type-I binary ZCPs) A pair of binary sequences  $\mathbf{c}$  and  $\mathbf{d}$  of length  $N$  is called a Type-I Z-complementary pair (ZCP) with ZCZ of width  $Z$ , if and only if

$$\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau) = 0,$$

for all  $1 \leq \tau \leq Z - 1$ , and  $\rho_{\mathbf{c}}(Z) + \rho_{\mathbf{d}}(Z) \neq 0$ .

*Definition 5:* (Type-II binary ZCPs) A pair of binary sequences  $\mathbf{c}$  and  $\mathbf{d}$  of length  $N$  is called a Type-II ZCP with ZCZ of width  $Z$ , if and only if

$$\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau) = 0, \text{ for all } N - Z + 1 \leq \tau \leq N - 1, \quad (2)$$

and  $\rho_{\mathbf{c}}(N - Z) + \rho_{\mathbf{d}}(N - Z) \neq 0$ .

Clearly, when  $Z = N$ , both Type-I and Type-II ZCPs become GCPs.

In the following lemma we recall the upper bounds of the ZCZ width for various types of binary ZCPs.

*Lemma 1:* Let  $(\mathbf{c}, \mathbf{d})$  be a binary ZCP of length  $N$  with ZCZ of width  $Z$ . Then

- 1)  $Z \leq (N + 1)/2$  if  $N$  is odd [15];
- 2)  $Z \leq N - 2$  if  $N$  is even and  $(\mathbf{c}, \mathbf{d})$  is Type-I ZCP [16]; and
- 3)  $Z \leq N - 1$  if  $N$  is even and  $(\mathbf{c}, \mathbf{d})$  is Type-II ZCP.

Note that bound 3) in Lemma 1 was obtained by exhaustive computer search. Based on the bounds above, we have the following definition on the Z-optimality of ZCPs.

*Definition 6:* [15], [16] A binary ZCP is said to be *Z-optimal* if the upper bound of the ZCZ width in Lemma 1 is achieved with equality.

The following lemma gives the lower bounds of the aperiodic autocorrelation sum magnitude outside the ZCZ of a Z-optimal (Type-I and Type-II) binary ZCP.

*Lemma 2:* Let  $(\mathbf{c}, \mathbf{d})$  be a binary ZCP of length  $N$  with ZCZ of width  $Z$ . Then we have the following bounds for odd-length ZCPs and even-length ZCPs.

- 1) [15] If  $(\mathbf{c}, \mathbf{d})$  is a Z-optimal Type-I OB-ZCP, then

$$|\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau)| \geq 2, \quad \text{for any } (N + 1)/2 \leq \tau \leq N - 1.$$

- 2) [16] If  $(\mathbf{c}, \mathbf{d})$  is a Z-optimal Type-II OB-ZCP, then

$$|\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau)| \geq 2, \quad \text{for any } 1 \leq \tau \leq (N - 1)/2.$$

3) [16] If  $(\mathbf{c}, \mathbf{d})$  is a Type-I EB-ZCP and  $Z \geq N/2$ , then

$$|\rho_{\mathbf{c}}(Z) + \rho_{\mathbf{d}}(Z)| \geq 4.$$

4) If  $(\mathbf{c}, \mathbf{d})$  is a Z-optimal Type-II EB-ZCP, then

$$|\rho_{\mathbf{c}}(1) + \rho_{\mathbf{d}}(1)| \geq 4.$$

The lower bounds above define the optimality of ZCPs, which achieve the smallest possible sum magnitude outside the ZCZ of Z-optimal binary ZCPs.

*Definition 7:* A Z-optimal Type-I OB-ZCP is called *optimal* if the lower bound 1) in Lemma 2 is met with equality. And a binary Z-optimal Type-II ZCP is called *optimal* if the lower bound 2) or 4) in Lemma 2 is met with equality.

*Remark 1:* The bounds in Lemma 1 may not be tight for all sequence length  $N$ . For example, as pointed out by one of the anonymous reviewers, there is no Type-I EB-ZCPs with length larger than 14 reported in the literature which can satisfy the upper bound in Lemma 1. Therefore it would be possible to derive tighter bounds on ZCZ widths of ZCPs for certain sequence lengths. In such cases, the bounds in Lemma 2 could be improved as well. Accordingly, the optimality in Definitions 6 and 7 should be changed with respect to the new bounds.

### III. A GENERAL CONSTRUCTION OF BINARY ZCPs

Infinite families of nontrivial (Type-I and Type-II) OB-ZCPs and Type-I EB-ZCPs were obtained in [15] and [16], [17], [18], [19], respectively. However, there was no infinite family of Type-II EB-ZCPs in the literature. In this section, we present a construction of ZCPs which can generate infinite families of Z-optimal Type-II OB-ZCPs and EB-ZCPs. We first present an example to show that an optimal Type-II EB-ZCP does exist.

*Example 1:* Suppose that

$$\begin{aligned} \mathbf{c} &= (+ - + + + + - + + + - - +), \\ \mathbf{d} &= (+ - + + + + - - - - + + -). \end{aligned} \tag{3}$$

Then  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-II EB-ZCP since it is easily verified that

$$|\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau)|_{\tau=0}^{13} = (28, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). \tag{4}$$

Now, we present a systematic construction of Type-II ZCPs in the following.

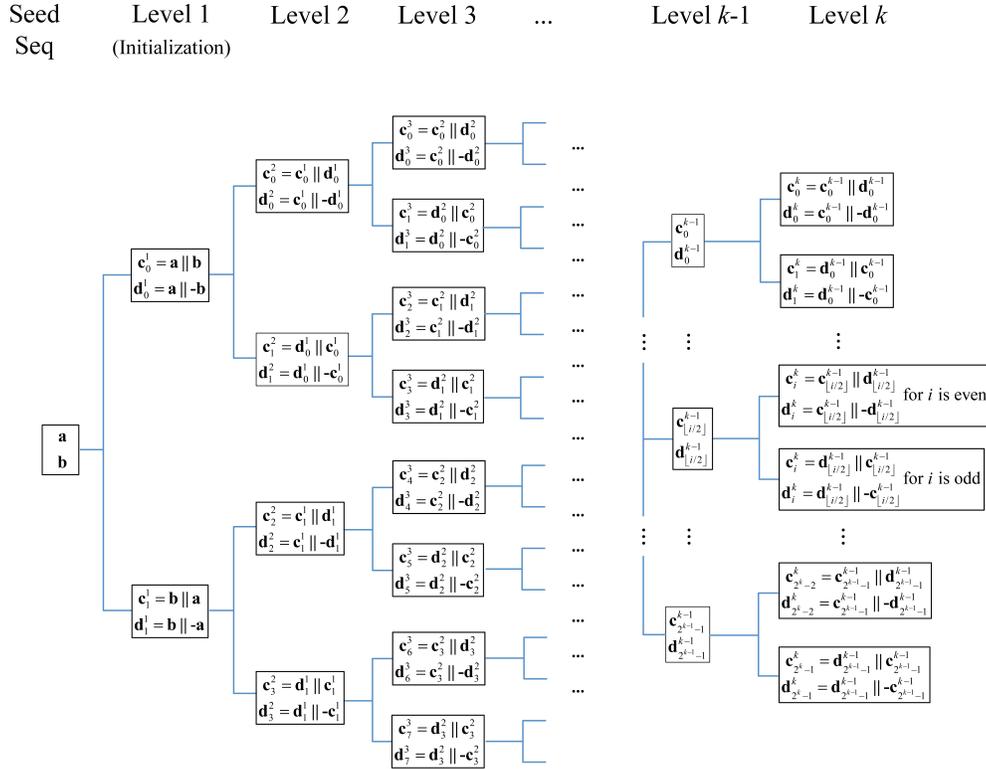


Fig. 1: Tree representation of recursive Type-II ZCP construction.

*Construction 1:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be two binary sequences of length  $N$  and  $N + 1$ , respectively. Taking  $\mathbf{a}$  and  $\mathbf{b}$  as seed sequences, and we then initialize two sequence pairs  $(\mathbf{c}_0^1, \mathbf{d}_0^1)$  and  $(\mathbf{c}_1^1, \mathbf{d}_1^1)$  as follows:

$$\begin{aligned} \mathbf{c}_0^1 &= \mathbf{a} \parallel \mathbf{b}, \mathbf{d}_0^1 = \mathbf{a} \parallel -\mathbf{b}; \\ \mathbf{c}_1^1 &= \mathbf{b} \parallel \mathbf{a}, \mathbf{d}_1^1 = \mathbf{b} \parallel -\mathbf{a}. \end{aligned} \quad (5)$$

At the  $k$ -th iteration, we have  $2^k$  pairs  $(\mathbf{c}_i^k, \mathbf{d}_i^k)$  for  $0 \leq i \leq 2^k - 1$ , given by

$$\mathbf{c}_i^k = \begin{cases} \mathbf{c}_{\lfloor \frac{i}{2} \rfloor}^{k-1} \parallel \mathbf{d}_{\lfloor \frac{i}{2} \rfloor}^{k-1} & \text{for } i \text{ even,} \\ \mathbf{d}_{\lfloor \frac{i}{2} \rfloor}^{k-1} \parallel \mathbf{c}_{\lfloor \frac{i}{2} \rfloor}^{k-1} & \text{for } i \text{ odd.} \end{cases} \quad (6)$$

and

$$\mathbf{d}_i^k = \begin{cases} \mathbf{c}_{\lfloor \frac{i}{2} \rfloor}^{k-1} \parallel -\mathbf{d}_{\lfloor \frac{i}{2} \rfloor}^{k-1} & \text{for } i \text{ even,} \\ \mathbf{d}_{\lfloor \frac{i}{2} \rfloor}^{k-1} \parallel -\mathbf{c}_{\lfloor \frac{i}{2} \rfloor}^{k-1} & \text{for } i \text{ odd.} \end{cases} \quad (7)$$

*Remark 2:* Figure 1 illustrates how Construction 1 generates sequence pairs recursively.

Based on the construction above, we have the following theorems to obtain Z-optimal (optimal) OB-ZCPs and EB-ZCPs.

*Theorem 1:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be binary sequences of lengths  $N$  and  $N + 1$ , respectively. By Construction 1, the sequence pairs at the initial step,  $(\mathbf{c}_0^1, \mathbf{d}_0^1)$  and  $(\mathbf{c}_1^1, \mathbf{d}_1^1)$  are Z-optimal Type-II OB-ZCPs of length  $L = 2N + 1$  with ZCZ of width  $Z = N + 1$ . In addition, for  $r = 0, 1$ , we have

$$\rho_{\mathbf{c}_r^1}(\tau) + \rho_{\mathbf{d}_r^1}(\tau) = \begin{cases} 2(\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)), & 1 \leq \tau \leq N, \\ 0, & N + 1 \leq \tau \leq 2N. \end{cases} \quad (8)$$

*Proof:* When  $r = 0$ , it is easy to see that

$$\rho_{\mathbf{c}_0^1}(\tau) = \begin{cases} \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) + \sum_{k=0}^{\tau-1} a_{N-1-k} b_{\tau-1-k}, & 1 \leq \tau \leq N, \\ \sum_{k=0}^{2N-\tau} a_k b_{k+\tau-N}, & N + 1 \leq \tau \leq 2N, \end{cases}$$

and

$$\rho_{\mathbf{d}_0^1}(\tau) = \begin{cases} \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) - \sum_{k=0}^{\tau-1} a_{N-1-k} b_{\tau-1-k}, & 1 \leq \tau \leq N, \\ - \sum_{k=0}^{2N-\tau} a_k b_{k+\tau-N}, & N + 1 \leq \tau \leq 2N. \end{cases}$$

Hence, we have

$$\rho_{\mathbf{c}_0^1}(\tau) + \rho_{\mathbf{d}_0^1}(\tau) = \begin{cases} 2(\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)), & 1 \leq \tau \leq N, \\ 0, & N + 1 \leq \tau \leq 2N, \end{cases}$$

i.e., the width of ZCZ is  $N + 1$ . Similarly, we can prove that (8) holds for  $r = 1$ . This completes the proof. ■

*Remark 3:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be binary sequences of lengths  $N - 1$  and  $N$ , respectively. Then Theorem 1 will produce Z-optimal Type-II OB-ZCPs of length  $2N - 1$ .

*Theorem 2:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be binary sequences of lengths  $N$  and  $N + 1$ , respectively. By Construction 1, at the  $k$ -th step of the iteration, sequence pairs  $(\mathbf{c}_r^k, \mathbf{d}_r^k)$  are Type-II EB-ZCPs of length  $L = 2^k N + 2^{k-1}$  having the ZCZ of width  $Z = 2^k N + 2^{k-1} - N$  when  $k \geq 2$  and  $r = 0, 1, \dots, 2^k - 1$ . In addition, we have

$$\rho_{\mathbf{c}_r^k}(\tau) + \rho_{\mathbf{d}_r^k}(\tau) = \begin{cases} 2^k(\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)), & 1 \leq \tau \leq N, \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

*Proof:* When  $r = 0, k \geq 2$ , it is easy to see that

$$\rho_{\mathbf{c}_r^k}(\tau) = \begin{cases} \rho_{\mathbf{c}_r^{k-1}}(\tau) + \rho_{\mathbf{d}_r^{k-1}}(\tau) + \rho_{\mathbf{d}_r^{k-1}, \mathbf{c}_r^{k-1}}(2^{k-1}N + 2^{k-2} - \tau), \\ \text{for } 1 \leq \tau \leq 2^{k-1}N + 2^{k-2} - 1; \\ \rho_{\mathbf{c}_r^{k-1}, \mathbf{d}_r^{k-1}}(\tau - (2^{k-1}N + 2^{k-2})), \\ \text{for } 2^{k-1}N + 2^{k-2} \leq \tau \leq 2^k N + 2^{k-1} - 1; \end{cases}$$

and

$$\rho_{\mathbf{d}_r^k}(\tau) = \begin{cases} \rho_{\mathbf{c}_r^{k-1}}(\tau) + \rho_{\mathbf{d}_r^{k-1}}(\tau) - \rho_{\mathbf{d}_r^{k-1}, \mathbf{c}_r^{k-1}}(2^{k-1}N + 2^{k-2} - \tau), \\ \text{for } 1 \leq \tau \leq 2^{k-1}N + 2^{k-2} - 1; \\ -\rho_{\mathbf{c}_r^{k-1}, \mathbf{d}_r^{k-1}}(\tau - (2^{k-1}N + 2^{k-2})), \\ \text{for } 2^{k-1}N + 2^{k-2} \leq \tau \leq 2^k N + 2^{k-1} - 1. \end{cases}$$

Hence, we have

$$\rho_{\mathbf{c}_r^k}(\tau) + \rho_{\mathbf{d}_r^k}(\tau) = \begin{cases} 2(\rho_{\mathbf{c}_r^{k-1}}(\tau) + \rho_{\mathbf{d}_r^{k-1}}(\tau)), & \text{for } 1 \leq \tau \leq 2^{k-1}N + 2^{k-2} - 1; \\ 0, & \text{for } 2^{k-1}N + 2^{k-2} \leq \tau \leq 2^k N + 2^{k-1} - 1. \end{cases} \quad (10)$$

From (8) and (10), it follows that

$$\rho_{\mathbf{c}_r^k}(\tau) + \rho_{\mathbf{d}_r^k}(\tau) = \begin{cases} 2^k(\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)), & 1 \leq \tau \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof.  $\blacksquare$

*Remark 4:* Note that the ZCZ width is independent of different selections of the seed sequences  $\mathbf{a}$  and  $\mathbf{b}$ .

In the following, we show some illustrative examples.

*Example 2:* Let  $\mathbf{a} = (++)$  and  $\mathbf{b} = (+++)$ , then according to Construction 1 and Figure 1, we have at the initial step,

$$\begin{aligned} \mathbf{c}_0^1 &= (++++), \\ \mathbf{d}_0^1 &= (++---). \end{aligned}$$

At the second iteration,

$$\begin{aligned} \mathbf{c}_0^2 &= (+++++---), \\ \mathbf{d}_0^2 &= (++++--++++). \end{aligned}$$

And at the third iteration,

$$\begin{aligned}\mathbf{c}_0^3 &= (+ + + + + + + - - - + + + + + - - + + +), \\ \mathbf{d}_0^3 &= (+ + + + + + + - - - - - - - - - + + - - -).\end{aligned}$$

Clearly, we have

$$\begin{aligned}\left| \rho_{\mathbf{c}_0^1}(\tau) + \rho_{\mathbf{d}_0^1}(\tau) \right|_{\tau=0}^4 &= (10, 6, 2, \mathbf{0}_2), \\ \left| \rho_{\mathbf{c}_0^2}(\tau) + \rho_{\mathbf{d}_0^2}(\tau) \right|_{\tau=0}^9 &= (20, 12, 4, \mathbf{0}_7), \\ \left| \rho_{\mathbf{c}_0^3}(\tau) + \rho_{\mathbf{d}_0^3}(\tau) \right|_{\tau=0}^{19} &= (40, 24, 8, \mathbf{0}_{17}).\end{aligned}$$

Hence,  $(\mathbf{c}_0^1, \mathbf{d}_0^1)$  is a Z-optimal Type-II OB-ZCP of length 5 having a ZCZ of width 3.  $(\mathbf{c}_0^2, \mathbf{d}_0^2)$  is a Z-optimal Type-II EB-ZCP of length 10 having a ZCZ of width 8 and  $(\mathbf{c}_0^3, \mathbf{d}_0^3)$  is a Type-II EB-ZCP of length 20 having a ZCZ of width 18. It is worth noting that  $(\mathbf{c}_0^2, \mathbf{d}_0^2)$  and  $(\mathbf{c}_0^3, \mathbf{d}_0^3)$  have large ZCZ ratios of 0.8 and 0.9, respectively.

*Remark 5:* Note that Z-optimal Type-II EB-ZCPs of length  $L = 2^k + 2^{k-1}$  with ZCZ width  $Z = 2^k + 2^{k-1} - 1$  can be generated by Theorem 2 when the length of the seed sequence  $\mathbf{a}$  is 1 and  $k \geq 2$ .

*Example 3:* Let  $\mathbf{a} = (+)$  and  $\mathbf{b} = (++)$ , then according to Construction 1 and Figure 1, we have at the initial step,

$$\begin{aligned}\mathbf{c}_0^1 &= (+ + +), \\ \mathbf{d}_0^1 &= (+ - -).\end{aligned}$$

At the second iteration,

$$\begin{aligned}\mathbf{c}_0^2 &= (+ + + + - -), \\ \mathbf{d}_0^2 &= (+ + + - + +).\end{aligned}$$

At the third iteration,

$$\begin{aligned}\mathbf{c}_0^3 &= (+ + + + - - + + + - + +), \\ \mathbf{d}_0^3 &= (+ + + + - - - - - + - -).\end{aligned}$$

Clearly, we have

$$\begin{aligned}\left| \rho_{\mathbf{c}_0^1}(\tau) + \rho_{\mathbf{d}_0^1}(\tau) \right|_{\tau=0}^2 &= (6, 2, 0), \\ \left| \rho_{\mathbf{c}_0^2}(\tau) + \rho_{\mathbf{d}_0^2}(\tau) \right|_{\tau=0}^5 &= (12, 4, \mathbf{0}_4), \\ \left| \rho_{\mathbf{c}_0^3}(\tau) + \rho_{\mathbf{d}_0^3}(\tau) \right|_{\tau=0}^{11} &= (24, 8, \mathbf{0}_{10}).\end{aligned}$$

Hence,  $(\mathbf{c}_0^1, \mathbf{d}_0^1)$  is an optimal Type-II OB-ZCP of length 3 having a ZCZ of width 2.  $(\mathbf{c}_0^2, \mathbf{d}_0^2)$  is an optimal Type-II EB-ZCP of length 6 having a ZCZ of width 5 and  $(\mathbf{c}_0^3, \mathbf{d}_0^3)$  is a Z-optimal Type-II EB-ZCP of length 12 having a ZCZ of width 11.

Example 3 gives us a construction of Z-optimal Type-II EB-ZCPs. However, the length of Z-optimal Type-II EB-ZCPs which are constructed through the above method is  $2^k + 2^{k-1}$ , and this makes the length very limited. By the following construction inspired by the well-known Turyn's construction, we can obtain Type-II ZCPs of length  $(2^k + 2^{k-1})10^b 26^c$ .

*Theorem 3:* Let  $(\mathbf{c}, \mathbf{d}), (\mathbf{e}, \mathbf{f})$  be Type-II ZCPs of length  $N_1, N_2$  with ZCZ of width  $Z_1, Z_2$ , respectively. Then  $(\mathbf{u}, \mathbf{v})$  is a Type-II ZCP of length  $N = N_1 N_2$  with ZCZ of width  $Z = N_1 Z_2 - N_1 + Z_1$ . Here  $(\mathbf{u}, \mathbf{v})$  is given by the following formula:

$$(\mathbf{u}, \mathbf{v}) = \left( \mathbf{e} \otimes \frac{\mathbf{c} + \mathbf{d}}{2} + \overleftarrow{\mathbf{f}} \otimes \frac{\mathbf{d} - \mathbf{c}}{2}, \mathbf{f} \otimes \frac{\mathbf{c} + \mathbf{d}}{2} - \overleftarrow{\mathbf{e}} \otimes \frac{\mathbf{d} - \mathbf{c}}{2} \right), \quad (11)$$

where  $\otimes$  denotes the Kronecker product. In particular, if  $(\mathbf{e}, \mathbf{f})$  is a GCP then  $(\mathbf{u}, \mathbf{v})$  is a Type-II ZCP of length  $N_1 N_2$  with ZCZ of width  $Z = N_1 N_2 - N_1 + Z_1$ .

*Proof:* See Appendix. ■

*Example 4:* Let  $\mathbf{c} = (+++)$  and  $\mathbf{d} = (+--)$ , then  $(\mathbf{c}, \mathbf{d})$  is a Type-II ZCP of length 3 with ZCZ of width 2. Let  $(\mathbf{e}, \mathbf{f})$  be a length-10 GCP as follows.

$$\mathbf{e} = (+ + - + - + - - ++),$$

$$\mathbf{f} = (+ + - + + + + - -).$$

By Theorem 3, we obtain a length-30 Z-optimal Type-II ZCP  $(\mathbf{u}, \mathbf{v})$ , as shown in (12),

$$\begin{aligned} \mathbf{u} &= (+ + + + + + - - - + + + - + + + + - + + - + + + - - + - -), \\ \mathbf{v} &= (+ + - + + - - - + + + - - - - + + - - - - - - + + + + + +), \end{aligned} \quad (12)$$

since it is computed that

$$|\rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau)|_{\tau=0}^{29} = (60, 20, \mathbf{0}_{28}).$$

#### IV. NEW FAMILIES OF OPTIMAL TYPE-II OB-ZCPs

In this section, we construct new families of optimal Type-II OB-ZCPs. According to Theorem 1, we have constructed Z-optimal Type-II OB-ZCP of any length. In Theorem 1, (8) shows that the key of construction of optimal Type-II OB-ZCPs is to find a sequence pair  $(\mathbf{a}, \mathbf{b})$  of length

$N$  and  $N + 1$  with low AACSSs. The following theorem gives a lower bound for the AACSSs of the above sequence pair  $(\mathbf{a}, \mathbf{b})$ .

*Theorem 4:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be two binary sequences of lengths  $N$  and  $N + 1$ , respectively. Then

$$|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| \geq 1, \text{ for all } 1 \leq \tau \leq N. \quad (13)$$

*Proof:* Clearly, we have

$$\rho_{\mathbf{a}}(\tau) \equiv N - \tau \pmod{2}, \text{ for all } 1 \leq \tau \leq N,$$

and

$$\rho_{\mathbf{b}}(\tau) \equiv N + 1 - \tau \pmod{2}, \text{ for all } 1 \leq \tau \leq N.$$

Therefore,

$$|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| \equiv 1 \pmod{2}, \text{ for all } 1 \leq \tau \leq N.$$

Hence,

$$|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| \geq 1, \text{ for all } 1 \leq \tau \leq N.$$

■

By exhaustive computer search, seed sequence pairs up to length  $N = 24$  which can achieve the bound derived in Theorem 4, are listed Table III.

*Example 5:* Let  $\mathbf{a} = (+ + + + + -)$  and  $\mathbf{b} = (+ + - - + - +)$ . Then the sequence pair  $(\mathbf{a}, \mathbf{b})$  meets the bound in Theorem 4, i.e.,

$$|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)|_{\tau=0}^6 = (13, \mathbf{1}_6).$$

Also, let

$$\mathbf{c} = \mathbf{a} \parallel \mathbf{b},$$

$$\mathbf{d} = \mathbf{a} \parallel -\mathbf{b}.$$

According to Theorem 1,  $(\mathbf{c}, \mathbf{d})$  is a length-13 optimal Type-II OB-ZCP having ZCZ width 7, because

$$|\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau)|_{\tau=0}^{12} = (26, \mathbf{2}_6, \mathbf{0}_6).$$

It is important to note that optimal Type-II OB-ZCP of length 13 has not been previously reported in the literature.

In what follows, we obtain optimal Type-II OB-ZCPs of lengths  $2N \pm 1$ , when  $N = 2^a 10^b 26^c$ , and  $a, b, c$  are non-negative integers.

TABLE III: Some seed sequence pair of Lengths up to 24.

$N$	$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$	$ \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) _{\tau=1}^{N-1}$
5	$\begin{pmatrix} + + + + - \\ + + - - +- \end{pmatrix}$	$\mathbf{1}_4$
6	$\begin{pmatrix} + + + + +- \\ + + - - +- + \end{pmatrix}$	$\mathbf{1}_5$
11	$\begin{pmatrix} + + + + + + - + - - + \\ + - + - - - + + + - - + \end{pmatrix}$	$\mathbf{1}_{10}$
12	$\begin{pmatrix} + + + + + + - + + - - + \\ + + - - - - + + - + - + - \end{pmatrix}$	$\mathbf{1}_{11}$
13	$\begin{pmatrix} + - - - - + + + - + + - + \\ - - - - - + + - - + - + - \end{pmatrix}$	$\mathbf{1}_{12}$
14	$\begin{pmatrix} - - + + + + - - - + - - + - \\ + + + + - + + + - + - + - - + \end{pmatrix}$	$\mathbf{1}_{13}$
17	$\begin{pmatrix} - + - + - - - + + + + + - - + \\ + + + - - + + - + + - + + + - + - \end{pmatrix}$	$\mathbf{1}_{16}$
18	$\begin{pmatrix} - - + + - - + - + - + + + + - - - \\ + + + - + + + + - + - + + - + + - - + \end{pmatrix}$	$\mathbf{1}_{17}$
21	$\begin{pmatrix} - - + - - + - + + - - - + - + + + - - - + \\ - + + - - - - - - - - + + + - + - + - - + \end{pmatrix}$	$\mathbf{1}_{20}$
22	$\begin{pmatrix} + + + - + - - - - + - - - + + - + - - + + - \\ + + + - + - - - - + - + - - - - + + - - + - \end{pmatrix}$	$\mathbf{1}_{21}$
23	$\begin{pmatrix} + - - - + - - - + - - - + + + + - + - + + - + \\ + + - - - - + - + + - + + - - - - + - - - + - \end{pmatrix}$	$\mathbf{1}_{22}$
24	$\begin{pmatrix} - + - - - - - + - + - - + + + - + - - + - - - + \\ - + - - - + - - - - + + - + + - - - + + + + - + \end{pmatrix}$	$\mathbf{1}_{23}$

*Theorem 5:* Let  $(\mathbf{x}, \mathbf{y})$  be a binary GCP of length  $N = 2^a 10^b 26^c$  and  $(\mathbf{a}, \mathbf{b}) = (\mathcal{V}(\mathbf{x}, 0), \mathbf{y})$ . Suppose that

$$\mathbf{c} = \mathbf{a} \parallel \mathbf{b},$$

$$\mathbf{d} = \mathbf{a} \parallel -\mathbf{b}.$$

Then  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-II OB-ZCP of length  $2N - 1$  having ZCZ of width  $N$ .

*Proof:* By the definition of AACF, we have

$$\begin{aligned}
\rho_{\mathbf{a}}(\tau) &= \sum_{i=0}^{N-2-\tau} a_i a_{i+\tau} \\
&= \sum_{i=1}^{N-1-\tau} x_i x_{i+\tau} \\
&= \left( \sum_{i=0}^{N-1-\tau} x_i x_{i+\tau} \right) - x_0 x_\tau \\
&= \rho_{\mathbf{x}}(\tau) - x_0 x_\tau,
\end{aligned}$$

for all  $\tau = 0, 1, \dots, N-1$ . As the sequence pair  $(\mathbf{x}, \mathbf{y})$  is a GCP, then

$$\begin{aligned}
\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) &= \rho_{\mathbf{x}}(\tau) + \rho_{\mathbf{y}}(\tau) - x_0 x_\tau \\
&= \begin{cases} 2N-1, & \tau = 0, \\ -x_0 x_\tau, & \tau = 1, 2, \dots, N-1. \end{cases}
\end{aligned}$$

Therefore, we have  $|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| = 1$  for all  $\tau = 1, 2, \dots, N-1$ . According to Theorem 1,  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-II OB-ZCP of length  $2N-1$ .  $\blacksquare$

*Theorem 6:* Let  $(\mathbf{x}, \mathbf{y})$  be a binary GCP of length  $N = 2^a 10^b 26^c$  and

$$\mathbf{c} = \mathbf{a} \parallel \mathbf{b},$$

$$\mathbf{d} = \mathbf{a} \parallel -\mathbf{b}.$$

Then  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-II OB-ZCP of length  $2N+1$  where  $(\mathbf{a}, \mathbf{b})$  is given by

$$\mathbf{a} = \mathbf{x},$$

$$\mathbf{b} = \lambda \parallel \mathbf{y}, \quad \text{where } \lambda \in \{+1, -1\}.$$

*Proof:* By the definition of AACF, we have

$$\begin{aligned}
\rho_{\mathbf{b}}(\tau) &= \sum_{i=0}^{N-\tau} b_i b_{i+\tau} \\
&= \left( \sum_{i=1}^{N-\tau} b_i b_{i+\tau} \right) + b_0 b_\tau \\
&= \left( \sum_{i=0}^{N-1-\tau} y_i y_{i+\tau} \right) + \lambda y_{\tau-1} \\
&= \rho_{\mathbf{y}}(\tau) + \lambda y_{\tau-1},
\end{aligned}$$

for all  $\tau = 1, 2, \dots, N$ . As the sequence pair  $(\mathbf{x}, \mathbf{y})$  is a GCP, then

$$\begin{aligned} \rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) &= \rho_{\mathbf{x}}(\tau) + \rho_{\mathbf{y}}(\tau) + \lambda y_{\tau-1} \\ &= \begin{cases} 2N + 1, & \tau = 0, \\ \lambda y_{\tau-1}, & \tau = 1, 2, \dots, N. \end{cases} \end{aligned}$$

Therefore,  $|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| = 1$  for all  $\tau = 1, 2, \dots, N$ . According to Theorem 1,  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-II OB-ZCP of length  $2N + 1$  having a ZCZ of width  $N + 1$ . ■

*Remark 6:* We can also obtain Type-I OB-ZCPs via Construction 1 as follows. Let  $(\mathbf{x}, \mathbf{y})$  be a binary GCP of length  $N = 2^a 10^b 26^c$  and

$$\begin{aligned} \mathbf{a} &= \mathbf{x}, \\ \mathbf{b} &= \mathbf{y} \parallel \lambda, \hat{\mathbf{b}} = \mathbf{y} \parallel -\lambda, \quad \text{where } \lambda \in \{+1, -1\}. \end{aligned} \tag{14}$$

Also let

$$\begin{aligned} \mathbf{c} &= \mathbf{a} \parallel \mathbf{b}, \\ \mathbf{d} &= \mathbf{a} \parallel -\hat{\mathbf{b}}. \end{aligned} \tag{15}$$

Then  $(\mathbf{c}, \mathbf{d})$  is an optimal Type-I OB-ZCP of length  $2N + 1$ .

Although the length of above optimal Type-I OB-ZCP has been reported in [30], [31], our construction is a new method. Besides, it is easy to see that  $|\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| = |\rho_{\mathbf{a}}(\tau) + \rho_{\hat{\mathbf{b}}}(\tau)| = 1$  for all  $\tau = 1, 2, \dots, N$ .

*Remark 7:* Compared with the results in [15], [30], Theorem 5 can construct optimal Type-II OB-ZCPs with more flexible parameters. The lengths of the optimal Type-II OB-ZCPs constructed in Theorem 5 are  $2N - 1$ , where  $N = 2^a 10^b 26^c$ . With Theorem 3, these optimal Type-II OB-ZCPs and EB-ZCPs can be used to generate EB-ZCPs with ZCZ of large width. For example, we can obtain Z-optimal Type-II EB-ZCPs of length  $3N$  or  $14N$  using Theorem 3. Theorem 2 can generate Type-II EB-ZCPs of flexible lengths having large ZCZ widths. The result of Theorem 6 is similar to the Type-II OB-ZCPs reported in [30]. In Table II we give a complete list of “best possible” Type-II binary ZCPs up to length 30 till now, which can be constructed by a systematic construction. Note that the lengths in the bold letter in Table II can only be generated by our systematic construction.

*Remark 8:* We note that the seed sequence pairs with low AACSSs, such as the pair used in Example 5, widely exist. We have verified the existence of all binary seed sequence pairs with low AACSSs of lengths up to 24 by computer search. Some examples of seed sequence pairs

with low AACSS up to 24, obtained by computer search, are shown in Table III. Based on Table III and Theorems 5 and 6, we are able to construct optimal Type-II OB-ZCP of length 3 to 53.

## V. PMEPR OF THE PROPOSED TYPE-II ZCPs

Sequences with low PMEPR are desirable in multi-carrier communications such as orthogonal frequency division multiplexing (OFDM) systems. In this section, we shall discuss the PMEPR of the proposed Type-II ZCPs. Before doing this, we give a short introduction to the definition of PMEPR of sequences.

We first define the OFDM signal of a sequence  $\mathbf{c} = (c_0, c_1, \dots, c_{L-1})$  to be the real part of

$$S_{\mathbf{c}}(t) = \sum_{k=0}^{L-1} c_k e^{2\pi j(f_c + k\Delta f)t}, 0 \leq t \leq \frac{1}{\Delta f},$$

where  $j = \sqrt{-1}$ ,  $f_c$  denotes the carrier frequency and  $\Delta f$  is the subcarrier spacing. Then, the PMEPR of  $\mathbf{c}$  (or its OFDM signal) is defined by

$$\text{PMEPR}(\mathbf{c}) = \frac{1}{L} \sup_{0 \leq t < \frac{1}{\Delta f}} |S_{\mathbf{c}}(t)|^2. \quad (16)$$

For a pair of sequence pair  $(\mathbf{c}, \mathbf{d})$ , its PMEPR is defined as

$$\text{PMEPR}(\mathbf{c}, \mathbf{d}) = \max\{\text{PMEPR}(\mathbf{c}), \text{PMEPR}(\mathbf{d})\}. \quad (17)$$

It turns out in [15] that

$$\text{PMEPR}(\mathbf{c}, \mathbf{d}) \leq 2 + \frac{2}{L} \sum_{\tau=1}^{L-1} |\rho_{\mathbf{c}}(\tau) + \rho_{\mathbf{d}}(\tau)| \quad (18)$$

which reveals a relationship between the PMEPR and autocorrelation of a sequence pair. Clearly, for a GCP  $(\mathbf{c}, \mathbf{d})$ , one immediately has  $\text{PMEPR}(\mathbf{c}, \mathbf{d}) \leq 2$ . Based on (18), upper bounds on PMEPR of some known Type-I ZCPs were given (see [15] and [33] for example). In the sequel, we shall discuss upper bounds for the PMEPR of the proposed Type-II ZCPs based on (18).

The following result follows directly from (18), (8) and (9).

*Theorem 7:* Let  $\mathbf{a}$  and  $\mathbf{b}$  be binary sequences of lengths  $N$  and  $N + 1$ , respectively. Then the PMEPR of Type-II ZCP generated by Construction 1 is upper bounded by

$$2 + \frac{4}{2N + 1} \sum_{\tau=1}^N |\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)|. \quad (19)$$

Theorems 7 tells us the upper bound of the PMEPR of the sequences generated by Construction 1 is determined by the aperiodic autocorrelation sums of the seed sequences  $\mathbf{a}$  and  $\mathbf{b}$ . According to Theorem 4, one has

$$\sum_{\tau=1}^N |\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau)| \geq N. \quad (20)$$

We thus have following corollary.

*Corollary 1:* Let  $\mathbf{a}, \mathbf{b}$  be the seed sequences meeting the bound in (20). Then each Type-II ZCP generated by Construction 1 has PMEPR upper bounded by  $2 + \frac{4N}{2N+1} \approx 4$ .

Corollary 1 means that each optimal Type-II OB-ZCP constructed in Section IV of this paper has PMEPR upper bounded by 4.

*Example 6:* Let  $(\mathbf{c}, \mathbf{d})$  be the optimal type-II OB-ZCP of length 13 in Example 5. Note that  $(\mathbf{c}, \mathbf{d})$  is constructed from the seed sequence  $\mathbf{a} = (+ + + + + -)$  and  $\mathbf{b} = (+ + - - + - +)$  meeting the bound in 20. It is easy to check that  $\text{PMEPR}(\mathbf{c}) = 2.4276$  and  $\text{PMEPR}(\mathbf{d}) = 2.4276$  which verifies the result in Corollary 1.

*Theorem 8:* Let  $(\mathbf{c}, \mathbf{d})$  be a Type-II ZCP of length  $N_1$  with ZCZ of width  $Z_1$ ,  $(\mathbf{e}, \mathbf{f})$  a GCP of length  $N_2$ . Let  $(\mathbf{u}, \mathbf{v})$  be the Type-II ZCP generated from  $(\mathbf{c}, \mathbf{d})$  and  $(\mathbf{e}, \mathbf{f})$  in Theorem 3. Then

$$\text{PMEPR}(\mathbf{u}, \mathbf{v}) \leq \text{UB}(\mathbf{c}, \mathbf{d}) \quad (21)$$

where

$$\text{UB}(\mathbf{c}, \mathbf{d}) = 2 + \frac{2}{N_1} \sum_{h=1}^{N_1-Z_1} |\rho_{\mathbf{c}}(h) + \rho_{\mathbf{d}}(h)| \quad (22)$$

is an upper bound of the PMEPR of  $(\mathbf{c}, \mathbf{d})$ .

*Proof:* According to Theorem 3,  $(\mathbf{u}, \mathbf{v})$  is a Type-II ZCP of length  $N = N_1 N_2$  and ZCZ width  $Z = N_1(Z_2 - 1) + Z_1$ . We then have

$$\sum_{\tau=1}^{N-1} |\rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau)| = \sum_{\tau=1}^{N-Z} |\rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau)| = N_2 \sum_{h=1}^{N_1-Z_1} |\rho_{\mathbf{c}}(h) + \rho_{\mathbf{d}}(h)| \quad (23)$$

where the second identity follows from (24) and the assumption that  $(\mathbf{e}, \mathbf{f})$  is a GCP. This together with (18) further leads to

$$\text{PMEPR}(\mathbf{u}, \mathbf{v}) \leq \text{UB}(\mathbf{c}, \mathbf{d}).$$

Note that  $\text{UB}(\mathbf{c}, \mathbf{d})$  in (22) is an upper bound of the PMPER of  $(\mathbf{c}, \mathbf{d})$  due to (18). This completes the proof. ■

The following result follows directly from Theorem 8 and Corollary 1.

*Corollary 2:* Let  $(\mathbf{e}, \mathbf{f})$  be a GCP and  $(\mathbf{u}, \mathbf{v})$  be the Type-II ZCP in Theorem 3. Then we have

- $\text{PMEPR}(\mathbf{u}, \mathbf{v}) < 4$  when  $(\mathbf{c}, \mathbf{d})$  is the Type-II ZCP generated by Construction 1 from the seed sequences  $\mathbf{a}, \mathbf{b}$  meeting the bound in (20);
- $\text{PMEPR}(\mathbf{u}, \mathbf{v}) \leq 2 + \frac{4}{3} \approx 3.33$  when  $\mathbf{c} = (+ + +)$ ,  $\mathbf{d} = (+ - -)$ ; and
- $\text{PMEPR}(\mathbf{u}, \mathbf{v}) \leq 2 + \frac{8}{14} \approx 2.57$  when  $\mathbf{c} = (+ - + + + + - + + + - - +)$ ,  $\mathbf{d} = (+ - + + + + - - - - + + -)$ .

*Example 7:* In Table IV, we list the PMEMR of some ZCPs generated by the construction in Theorem 3. Herein,  $(\mathbf{u}_{3,N_2}, \mathbf{v}_{3,N_2})$  (resp.,  $(\mathbf{u}_{14,N_2}, \mathbf{v}_{14,N_2})$ ) denotes the Type-II ZCP generated from GCP of length  $N_2$  and ZCP  $(\mathbf{c}, \mathbf{d})$  given by  $\mathbf{c} = (+ + +)$ ,  $\mathbf{d} = (+ - -)$  (resp.,  $\mathbf{c} = (+ - + + + + - + + + - - +)$ ,  $\mathbf{d} = (+ - + + + + - - - - + + -)$ ). It can be seen from the table that the PMEPR of these sequences are very close to the bounds in Corollary 2.

TABLE IV: PMEPR of some Type-II ZCPs in Theorem 3.

Length of GCP $N_2$	$\text{PMEPR}(\mathbf{u}_{3,N_2})$	$\text{PMEPR}(\mathbf{v}_{3,N_2})$	$\text{PMEPR}(\mathbf{u}_{14,N_2})$	$\text{PMEPR}(\mathbf{v}_{14,N_2})$
1	3.0000	1.6667	2.5714	2.4119
2	2.8452	2.6667	2.1373	2.5137
4	3.0000	1.7387	2.5714	2.5102
8	3.2545	3.0740	2.5243	2.5456
10	3.3333	3.0200	2.4851	2.5570
16	3.0312	3.2919	2.5714	2.5102
20	3.1834	3.3159	2.5669	2.5549
26	3.2902	3.2291	2.5542	2.5545
32	3.3290	3.1483	2.5714	2.5373
40	3.3333	3.0446	2.5624	2.5570
52	3.3064	3.3189	2.5682	2.5549
64	3.2902	3.3201	2.5714	2.5689
80	3.2037	3.2667	2.5669	2.5695

*Remark 9:* ZCPs with good PMEPR properties can be regarded as potential alternatives of GCPs in practical applications (see [27] for an application scenario) since they can exist for more lengths. Note that compared to the systematic constructions of Type-I ZCPs with low PMEPR, available in the literature, Type-II ZCPs are available with more flexible lengths. For example, for sequence lengths  $N \in \{N_1 \times N_2 : N_1 = 5, 11, 13, 14, N_2 = 2^a 10^b 26^c, a, b, c > 0\}$ , no Type-I ZCPs were reported in the literature. According to Theorem 3 and Corollary 2, Type-II

ZCPs with low PMEPR exist for such lengths. Therefore, Type-I ZCPs and Type-II ZCPs are two different ways of providing potential sequences with flexible lengths and low PMEPR for practical applications.

## VI. CONCLUDING REMARKS

In this paper, some properties and construction of optimal binary ZCPs are studied. Our motivation is the fact that all currently known binary GCPs have even-lengths of the form  $2^a 10^b 26^c$  only. We target at finding optimal binary sequence pairs of any length, which have the closest correlation property to that of GCPs. More precisely, we proposed a new method which horizontally concatenates sequences  $a$  and  $b$  of different lengths to construct the optimal binary ZCPs. Note that our construction is generic because in our construction  $N$  can be any number. Based on the new method, we constructed optimal and Z-optimal OB-ZCPs with more flexible parameters. The main contributions of this paper are listed in the following:

- 1) For even length of Type-II binary ZCPs, we proved that the width of its ZCZ can achieve  $N - 1$ , in which case the ZCP is called a Z-optimal Type-II EB-ZCP. We constructed the optimal Type-II EB-ZCPs of lengths  $6 \times 2^a 10^b 26^c$  and  $14 \times 2^a 10^b 26^c$  through Example 1, Theorem 2 and Theorem 3, where  $a, b, c$  are non-negative integers.
- 2) We proposed a new recursive construction of Type-II EB-ZCPs. By the construction, we can also generate ZCPs with large ZCZ ratio and flexible parameters.
- 3) By horizontally concatenating of sequence pair of different lengths, we constructed optimal Type-II OB-ZCPs of length  $2N \pm 1$ , where  $N$  is the Golay number, i.e.  $N = 2^a 10^b 26^c$ .
- 4) By horizontally concatenating of sequence pair of different lengths, we constructed Z-optimal Type-II OB-ZCPs of length  $2N \pm 1$ , where  $N$  can be any number.
- 5) We gave upper bounds for the PMEPR of ZCPs from the proposed constructions. It turns out that our constructions can generate Type-II ZCPs with low PMEPR.
- 6) Our construction can also be extended to obtain optimal Type-I OB-ZCPs. Although the length of generated Type-I OB-ZCP has been reported before, our construction is a new method. One of our near future work is to explore how to construct optimal Type-I ZCPs with new lengths from Type-II ones.

We conclude the present paper by proposing the following open questions:

- 1) Are there any systematic constructions of optimal Type-II OB-ZCPs in lengths other than the ones discussed in this paper?

2) Are there more optimal Type-II EB-ZCPs, except for the lengths 6 and 14?

### APPENDIX

#### PROOF OF THEOREM 3

We need the following lemma to prove the theorem.

*Lemma 3:* Let  $(\mathbf{c}, \mathbf{d})$  be a Type-II ZCP of length  $N$  with ZCZ of width  $Z$ , then so is  $(\mathbf{c}, \overleftarrow{\mathbf{d}})$ .

*Proof:* By the definition of AACF, we have

$$\begin{aligned} \rho_{\overleftarrow{\mathbf{d}}}(\tau) &= \sum_{i=0}^{N-1-\tau} \overleftarrow{d}_i \overleftarrow{d}_{i+\tau} \\ &= \sum_{i=0}^{N-1-\tau} d_{N-1-i} d_{N-1-(i+\tau)} \\ &= \sum_{t=0}^{N-1-\tau} d_{t+\tau} d_t \\ &= \rho_{\mathbf{d}}(\tau). \end{aligned}$$

Therefore, we have

$$\rho_{\mathbf{b}}(\tau) + \rho_{\overleftarrow{\mathbf{d}}}(\tau) = \rho_{\mathbf{b}}(\tau) + \rho_{\mathbf{d}}(\tau) = 0,$$

for all  $N - Z + 1 \leq \tau \leq N - 1$ , and  $\rho_{\mathbf{b}}(N - Z) + \rho_{\overleftarrow{\mathbf{d}}}(N - Z) \neq 0$ . ■

*Proof of Theorem 3:*

By the Euclidean division theorem, we have  $\tau = kN_1 + h$  where  $0 \leq k \leq N_2 - 1, 0 \leq h \leq N_1 - 1$ . By the definition of AACF, we have

$$\begin{aligned} \rho_{\mathbf{u}}(\tau) = & \sum_{m=0}^{N_2-1-k} \left[ \left( \frac{e_m + \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k} + \overleftarrow{f}_{m+k}}{2} \right) \rho_{\mathbf{c}}(h) + \left( \frac{e_m - \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k} - \overleftarrow{f}_{m+k}}{2} \right) \rho_{\mathbf{d}}(h) \right. \\ & + \left( \frac{e_m + \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k} - \overleftarrow{f}_{m+k}}{2} \right) \rho_{\mathbf{c},\mathbf{d}}(h) + \left( \frac{e_m - \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k} + \overleftarrow{f}_{m+k}}{2} \right) \rho_{\mathbf{d},\mathbf{c}}(h) \\ & + \left( \frac{e_m + \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k+1} + \overleftarrow{f}_{m+k+1}}{2} \right) \rho_{\mathbf{c}}(N_1 - h) \\ & + \left( \frac{e_m - \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k+1} - \overleftarrow{f}_{m+k+1}}{2} \right) \rho_{\mathbf{d}}(N_1 - h) \\ & + \left( \frac{e_m + \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k+1} - \overleftarrow{f}_{m+k+1}}{2} \right) \rho_{\mathbf{d},\mathbf{c}}(N_1 - h) \\ & \left. + \left( \frac{e_m - \overleftarrow{f}_m}{2} \right) \left( \frac{e_{m+k+1} + \overleftarrow{f}_{m+k+1}}{2} \right) \rho_{\mathbf{c},\mathbf{d}}(N_1 - h) \right] \end{aligned}$$

Therefore, we have

$$\begin{aligned} \rho_{\mathbf{u}}(\tau) = & \frac{\rho_{\mathbf{c}}(h)}{4} \left( \rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\ & + \frac{\rho_{\mathbf{d}}(h)}{4} \left( \rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\ & + \frac{\rho_{\mathbf{c},\mathbf{d}}(h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\ & + \frac{\rho_{\mathbf{d},\mathbf{c}}(h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\ & + \frac{\rho_{\mathbf{c}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k+1) + \rho_{\overleftarrow{\mathbf{f}}}(k+1) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k+1) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k+1) \right) \\ & + \frac{\rho_{\mathbf{d}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k+1) + \rho_{\overleftarrow{\mathbf{f}}}(k+1) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k+1) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k+1) \right) \\ & + \frac{\rho_{\mathbf{c},\mathbf{d}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k+1) - \rho_{\overleftarrow{\mathbf{f}}}(k+1) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k+1) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k+1) \right) \\ & + \frac{\rho_{\mathbf{d},\mathbf{c}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k+1) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k+1) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k+1) \right), \end{aligned}$$

and

$$\begin{aligned}
\rho_{\mathbf{v}}(\tau) = & \frac{\rho_{\mathbf{c}}(h)}{4} \left( \rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\
& + \frac{\rho_{\mathbf{d}}(h)}{4} \left( \rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\
& - \frac{\rho_{\mathbf{c},\mathbf{d}}(h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\
& - \frac{\rho_{\mathbf{d},\mathbf{c}}(h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k) \right) \\
& + \frac{\rho_{\mathbf{c}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}}}(k + 1) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k + 1) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k + 1) \right) \\
& + \frac{\rho_{\mathbf{d}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}}}(k + 1) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k + 1) \right) \\
& - \frac{\rho_{\mathbf{c},\mathbf{d}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k + 1) - \rho_{\overleftarrow{\mathbf{f}}}(k + 1) - \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k + 1) \right) \\
& - \frac{\rho_{\mathbf{d},\mathbf{c}}(N_1 - h)}{4} \left( \rho_{\mathbf{e}}(k) - \rho_{\overleftarrow{\mathbf{f}}}(k + 1) + \rho_{\mathbf{e},\overleftarrow{\mathbf{f}}}(k + 1) - \rho_{\overleftarrow{\mathbf{f}},\mathbf{e}}(k + 1) \right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau) \\
= & \frac{\rho_{\mathbf{c}}(h)}{2} (\rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k)) + \frac{\rho_{\mathbf{d}}(h)}{2} (\rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k)) \\
& + \frac{\rho_{\mathbf{c}}(N_1 - h)}{2} (\rho_{\mathbf{e}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}}}(k + 1)) + \frac{\rho_{\mathbf{d}}(N_1 - h)}{2} (\rho_{\mathbf{e}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}}}(k + 1)) \\
= & \frac{1}{2} \left[ (\rho_{\mathbf{c}}(h) + \rho_{\mathbf{d}}(h)) (\rho_{\mathbf{e}}(k) + \rho_{\overleftarrow{\mathbf{f}}}(k)) + (\rho_{\mathbf{c}}(N_1 - h) + \right. \\
& \left. \rho_{\mathbf{d}}(N_1 - h)) (\rho_{\mathbf{e}}(k + 1) + \rho_{\overleftarrow{\mathbf{f}}}(k + 1)) \right]. \tag{24}
\end{aligned}$$

This together with the definition of Type-II ZCP and Lemma 3 means that

- $\rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau) \neq 0$  for  $\tau = (N_2 - Z_2)N_1 + N_1 - Z_1$  (i.e.,  $k = N_2 - Z_2$  and  $h = N_1 - Z_1$ );  
and
- $\rho_{\mathbf{u}}(\tau) + \rho_{\mathbf{v}}(\tau) = 0$  for all  $\tau > (N_2 - Z_2)N_1 + N_1 - Z_1$  (i.e.,  $k > N_2 - Z_2$  or  $(k = N_2 - Z_2$   
and  $h > N_1 - Z_1)$ ).

Therefore, the ZCZ width of  $(\mathbf{u}, \mathbf{v})$  is  $Z = N_1(Z_2 - 1) + Z_1$ . This completes the proof of the theorem.

## REFERENCES

- [1] M. J. E. Golay, "Static multislit spectrometry and its application to the panoramic display of infrared spectra," *J. Opt. Soc. Am.*, vol. 41, no. 7, pp. 468-472, Jul. 1951.

- [2] M. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. 7, no. 2, pp. 82-87, Apr. 1961.
- [3] P. Spasojevic and C. N. Georghiades, "Complementary sequences for ISI channel estimation," *IEEE Trans. Inf. Theory*, vol. 47, no. 3, pp. 1145-1152, Mar. 2001.
- [4] S. Wang and A. Abdi, "MIMO ISI channel estimation using uncorrelated Golay complementary sets of polyphase sequences," *IEEE Trans. Veh. Technol.*, vol. 56, no. 5, pp. 3024-3039, Sep. 2007.
- [5] S. Z. Budisin, "Efficient pulse compressor for Golay complementary sequences," *Electron. Lett.*, vol. 27, no. 3, pp. 219-220, Jan. 1991.
- [6] A. Pezeshki, A. R. Calderbank, W. Moran, and S. D. Howard, "Doppler resilient Golay complementary waveforms," *IEEE Trans. Inf. Theory*, vol. 54, no. 9, pp. 4254-4266, Sep. 2008.
- [7] P. Kumari, J. Choi, N. González-Prelcic, and R. W. Heath, "IEEE 802.11ad-based radar: An approach to joint vehicular communication-radar system," *IEEE Trans. Veh. Technol.*, vol. 67, no. 4, pp. 3012-3027, Apr. 2018.
- [8] H. H. Chen, J. F. Yeh, and N. Suehiro, "A multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications," *IEEE Commun. Mag.*, vol. 39, no. 10, pp. 126-135, Oct. 2001.
- [9] Z. Liu, Y. L. Guan, and H. H. Chen, "Fractional-delay-resilient receiver design for interference-free MC-CDMA communications based on complete complementary codes," *IEEE Trans. Wireless Commun.*, vol. 14, no. 3, pp. 1226-1236, Mar. 2015.
- [10] J. A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 45, no. 7, pp. 2397-2417, Nov. 1999.
- [11] Z. Wang, M. G. Parker, G. Gong, and G. Wu, "On the PMEPR of binary Golay sequences of length  $2^n$ ," *IEEE Trans. Inf. Theory*, vol. 60, no. 4, pp. 2391-2398, Apr. 2014.
- [12] P. Borwein and R. Ferguson, "A complete description of Golay pairs for lengths up to 100," *Math. comput.*, vol. 73, no. 246, pp. 967-985, 2004.
- [13] P. Fan, W. Yuan, and Y. Tu, "Z-complementary binary sequences," *IEEE Signal Process. Lett.*, vol. 14, no. 8, pp. 509-512, Aug. 2007.
- [14] X. Li, P. Fan, X. Tang, and Y. Tu, "Existence of binary Z-complementary pairs," *IEEE Signal Process. Lett.*, vol. 18, no. 1, pp. 63-66, Jan. 2011.
- [15] Z. Liu, P. Udaya, and Y. L. Guan, "Optimal odd-length binary Z-complementary pairs," *IEEE Trans. Inf. Theory*, vol. 60, no. 9, pp. 5768-5781, Sep. 2014.
- [16] Z. Liu, P. Udaya, and Y. L. Guan, "On even-period binary Z-complementary pairs with large ZCZs," *IEEE Signal Process. Lett.*, vol. 21, no. 3, pp. 284-287, Mar. 2014.
- [17] C. Chen, "A novel construction of Z-complementary pairs based on generalized Boolean functions," *IEEE Signal Process. Lett.*, vol. 24, no. 7, pp. 987-990, Jul. 2017.
- [18] A. R. Adhikary, S. Majhi, Z. Liu, and Y. L. Guan, "New sets of even-length binary Z-complementary pairs with asymptotic ZCZ ratio of  $3/4$ ," *IEEE Signal Process. Lett.*, vol. 25, no. 7, pp. 970-973, Jul. 2018.
- [19] C. Xie and Y. Sun, "Constructions of even-period binary Z-complementary pairs with large ZCZs," *IEEE Signal Process. Lett.*, vol. 25, no. 8, pp. 1141-1145, Aug. 2018.
- [20] N. Suehiro, "A signal design without co-channel interference for approximately synchronized CDMA systems," *IEEE J. Sel. Areas Commun.*, vol. 12, no. 5, pp. 837-841, Jun. 1994.
- [21] B. Long, P. Zhang, and J. Hu, "A generalized QS-CDMA system and the design of new spreading codes," *IEEE Trans. Veh. Technol.*, vol. 47, no. 4, pp. 1268-1275, Nov. 1998.
- [22] P. Fan, N. Suehiro, N. Kuroyanagi, and X. M. Deng, "Class of binary sequences with zero correlation zone," *Electron. Lett.*, vol. 35, no. 10, pp. 777-779, May 1999.

- [23] X. Tang and W. H. Mow, "Design of spreading codes for quasi-synchronous CDMA with intercell interference," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 1, pp. 84–93, Jan. 2006.
- [24] X. Tang, P. Fan, and J. Lindner, "Multiple binary ZCZ sequence sets with good cross-correlation property based on complementary sequence sets," *IEEE Trans. Inf. Theory*, vol. 56, no. 8, pp. 4038–4045, Aug. 2010.
- [25] W. C. Lee, *Mobile communications design fundamentals*. John Wiley & Sons, 2010.
- [26] J. S. Lee and L. E. Miller, *CDMA Systems Engineering Handbook*. Mobile Communications series, Artech House Publishers, 1998.
- [27] "IEEE p802.11ax<sup>TM</sup>/d2.0: Draft standard for information technologytelecommunications and information exchange between systems local and metropolitan area networks-specific requirements," (Amendment to IEEE Std 802.11-2016).
- [28] A. R. Adhikary and S. Majhi, "New constructions of complementary sets of sequences of lengths non-power-of-two," *IEEE Commun. Lett.*, vol. 23, no. 7, pp. 1119–1122, July 2019.
- [29] A. R. Adhikary and S. Majhi, "New construction of optimal aperiodic Z-complementary sequence sets of odd-lengths," *Electron. Lett.*, vol. 55, no. 19, pp. 1043–1045, 2019.
- [30] A. R. Adhikary, S. Majhi, Z. Liu, and Y. L. Guan, "New sets of optimal odd-length binary Z-complementary pairs," *IEEE Trans. Inf. Theory*, vol. 66, no. 1, pp. 669-678, Jan. 2020.
- [31] A. R. Adhikary, S. Majhi, Z. Liu, and Y. L. Guan, "New optimal binary Z-complementary pairs of odd lengths," *2017 Eighth Int. workshop on signal design and its App. in Commun. (IWSDA)*, Sapporo, 2017, pp. 14–18.
- [32] B. Shen, Y. Yang, Z. Zhou, P. Fan, and Y. Guan, "New optimal binary Z-complementary pairs of odd length  $2^m + 3$ ," *IEEE Signal Process. Lett.*, vol. 26, no. 12, pp. 1931–1934, Dec. 2019.
- [33] C. Chen and C. Pai, "Binary Z-complementary pairs with bounded peak-to-mean envelope power ratios," *IEEE Commun. Lett.*, vol. 23, no. 11, pp. 1899-1903, Nov. 2019.