Asymptotically Optimal Change Point Detection for Composite Hypothesis in State Space Models

Cheng-Der Fuh

Abstract—This paper investigates change point detection in state space models, in which the pre-change distribution f^{θ_0} is given, while the poster distribution f^{θ} after change is unknown. The problem is to raise an alarm as soon as possible after the distribution changes from f^{θ_0} to f^{θ} , under a restriction on the false alarms. We investigate theoretical properties of a weighted Shiryayev-Roberts-Pollak (SRP) change point detection rule in state space models. By making use of a Markov chain representation for the likelihood function, exponential embedding of the induced Markovian transition operator, nonlinear Markov renewal theory, and sequential hypothesis testing theory for Markov random walks, we show that the weighted SRP procedure is second-order asymptotically optimal. To this end, we derive an asymptotic approximation for the expected stopping time of such a stopping scheme when the change time $\omega = 1$. To illustrate our method we apply the results to two types of state space models: general state Markov chains and linear state space models.

Index Terms

Asymptotic optimality, change point detection, first passage time, iterated random functions system, nonlinear Markov renewal theory, sequential analysis, Shiryayev-Roberts-Pollak procedure.

I. INTRODUCTION

A prototypical problem of detecting abrupt changes can be found in instruction detection in distributed computer networks. Large scale attacks, denial of service attacks, occur at unknown points in time and need to be detected at the early stages by observing abrupt changes in the computer network traffic. Further applications are in, for example, biomedical signal processing, industrial quality control, segmentation of signals, financial engineering, edge detection in images, and the diagnosis of faults in the elements of computer communication networks. The reader is referred to Lai [12], [13] and Tartakovsky et al. [28] for a comprehensive summary in this area. A standard formulation of the change point detection problem is that there is a sequence of observations whose distribution changes at some unknown time ω , and the goal is to detect this change as soon as possible under false alarm constraints.

When the observations Y_n are independent with a common density function f^{θ_0} for $n < \omega$ and with another common density function f^{θ} for $n \ge \omega$, where ω is unknown and both θ_0 and θ are given, there are two standard formulations for the optimum tradeoff problem. The first is a minimax formulation proposed by Lorden [15], in which he shows that subject to the "average run length" (ARL) constraint, Page's CUSUM procedure asymptotically minimizes the "worst case" detection delay. The second is a Bayesian formulation, proposed by Shiryayev [25], [26], in which the change point has a geometric prior distribution on, and the goal is to minimize the expected delay subject to an upper bound on false alarm probability. He uses optimal stopping theory to show that the Bayes rule triggers an alarm as soon as the posterior probability that a change has occurred exceeds some fixed level. Roberts [24] considers the non-Bayesian setting, and studies by simulation the average run length of this rule, and finds it to be very good. Pollak [21] shows that the (modified) Shiryayev-Roberts rule is asymptotically minimax. When θ is unknown, Pollak and Siegmund [23] extends Shiryayev's work in a non-Bayesian setting, and calculates the expected value of a weighted likelihood ratio test as well as the average run lengths of a CUSUM rule. Then Pollak [22] provides average run lengths of the weighted Shiryayev-Roberts change point detection rule.

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Regarding change point detection rules in dynamic systems beyond independent assumption. In the case of using CUSUM type change point detection rules, Bansal and Papantoni-Kazakos [2] extends Lorden's asymptotic theory to the case where Y_i are stationary ergodic sequences, under the condition that $\{Y_j, j < \omega\}$ (before the change point) and $\{Y_j, j \ge \omega\}$ (after the change point) are independent, and proves the asymptotic optimality of the CUSUM algorithm. Further extensions to general stochastic sequences Y_n were obtained by Lai [12], [13], and Tartakovsky and Veeravalli [29]. When both θ_0 and θ are given, Fuh [6] proves that the CUSUM scheme is asymptotically optimal, in the sense of Lorden [15], in hidden Markov models. In the domain of Shiryayev-Roberts type change point detection rules, Yakir [30] generalizes the result to a finite state Markov chain, while Bojdecki [3] studies a different loss function and applies optimal stopping theory to find the Bayes rule. Tartakovsky [27] considers a sequential Bayesian changepoint detection problem for a general stochastic model. Fuh [8] investigates the Shiryayev-Roberts-Pollak (SRP) change point detection rule in hidden Markov models, in which he proves the asymptotic minimax property and derives an asymptotic approximation for the average run lengths when $\omega = 1$. Fuh and Tartakovsky [10] considers asymptotic Bayesian change point detection in hidden Markov models.

It is noted that many practical problems for change point detection are beyond independent assumption. Some useful class of such models are AR models, ARMA models, and

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linear state space models, cf. Tartakovsky et al. [28]. Along this line, in this paper, we study change point detection in state space models. A prototypical *state space model* can be formulated as follows: for n = 1, 2, ..., define

$$Y_n = G_\theta(X_n, \varepsilon_n), \text{ and } X_n = F_\theta(X_{n-1}, \eta_n),$$
 (1.1)

where Y_n is the observed value, X_n is a *d*-dimensional vector representing an unobservable state, and (ε_n, η_n) are independent random vectors representing random disturbances and having a common density function ϕ_{θ} . Furthermore, we assume $\{\varepsilon_n, n \ge 0\}$ and $\{\eta_n, n \ge 0\}$ are independent. Here the system dynamics are given by the second equation in (1.1). Note that the state vectors X_n are not directly observable and the observations are Y_n which are related to X_n and measurement error ε_n in the first equation of (1.1).

Specifically, a simple linear state space model given by MacGregor and Harris (1990) to study the problem of monitoring process means with the sample means Y_n :

$$Y_n = X_n + \varepsilon_n$$
, and $X_n - \mu = \alpha (X_{n-1} - \mu) + \eta_n$, (1.2)

where ε_n and η_n are independent normal random variables with zero means, and $var(\varepsilon_n) = \sigma_{\varepsilon}^2$ and $var(\eta_n) = \sigma_{\eta}^2$. Here $\theta = (\mu, \alpha, \sigma_{\varepsilon}^2, \sigma_{\eta}^2)$ with $|\alpha| < 1$, and the target value of the production process is $\mu = \mu^*$. If we are interested primarily in shifts in the overall mean and treat $\alpha, \sigma_{\varepsilon}^2$ and σ_{η}^2 as unknown nuisance parameters, then we have an incomplete base-line information and can apply the change point detection rule, described in Section II, to this case.

In this paper, we will study the change point problem that the pre-change is given while the after change is unknown. It is reasonable to assume that the pre-change distribution is known, because in most practical applications, a large amount of data generated by the pre-change distribution is available to the observer who may use this data to obtain an accurate approximation of the pre-change distribution. However, estimating or even modelling the post-change distribution is often impractical as we may not know a priori what kind of change will happen. We seek to design a change point detection algorithm that allows us to quickly detect the change, under false alarm constraints, and with suitable knowledge of the post-change distribution. To this end, the primary goal of this paper is to investigate theoretical properties of a weighted Shiryayev-Roberts-Pollak (SRP) change point detection rule in state space models.

There are two main contributions in this study. First, we consider a state space model (1.1) in which the underlying state space is neither finite nor compact, and includes (finite state) hidden Markov models, linear state space model, and AR/ARMA models as special cases. Second, the parameter of the distribution after change is assumed to be unknown for practical applications.

The remainder of the paper is organized as follows. Our main results are in Section II, in which we derive a secondorder asymptotic approximation for the expected stopping scheme when $\omega = 1$, not worst case, and prove the weighted SRP rule is second-order asymptotically optimal under a falsealarm constraint. In Section III we illustrate our method by considering two interesting examples: general state Markov chains and linear state space models. Section IV presents the pre-required methods used in the proofs of our results. We first give a Markov chain representation of the likelihood ratio, and then study exponential embedding for the induced Markovian transition kernel in state space models. Based on a nonlinear Markov renewal theory, we characterize the constant term of the second order approximation in Section V. The proofs are given in Sections VI, VII and Appendix, respectively.

II. ASYMPTOTIC OPTIMALITY OF THE WEIGHTED SRP DETECTION PROCEDURE

In this section, we define a state space model as a parameterized Markov random walk, in which the underlying environmental Markov chain can be viewed as a latent variable. To be more precise, for each $\theta \in \Theta \subset \mathbf{R}$, the unknown parameter, let $\mathbf{X} = \{X_n, n \geq 0\}$ be a Markov chain on a general state space \mathcal{X} , with transition probability kernel $P^{\theta}(x, \cdot) = P^{\theta}\{X_1 \in \cdot | X_0 = x\}$ and stationary probability $\pi_{\theta}(\cdot)$. Suppose that a random sequence $\{Y_n\}_{n=0}^{\infty}$, taking values in \mathbf{R}^d , is adjoined to the chain such that $\{(X_n, Y_n), n \geq 0\}$ is a Markov chain on $\mathcal{X} \times \mathbf{R}^d$ satisfying $P^{\theta}\{X_1 \in A | X_0 = x, Y_0 = y\} = P^{\theta}\{X_1 \in A | X_0 = x\}$ for $A \in \mathcal{B}(\mathcal{X})$, the σ -algebra of \mathcal{X} . And conditioning on the full \mathbf{X} sequence, we have

$$P^{\theta}\{Y_{n+1} \in B | X_0, X_1, \dots; Y_0, Y_1, \dots, Y_n\} \quad (2.1)$$

= $P^{\theta}\{Y_{n+1} \in B | X_{n+1}\} = P^{\theta}(X_{n+1} : B) \quad a.s.$

for each n and $B \in \mathcal{B}(\mathbb{R}^d)$, the Borel σ -algebra of \mathbb{R}^d . Furthermore, we assume the existence of a transition probability density $p_{\theta}(x, y)$ for the Markov chain $\{X_n, n \geq 0\}$ with respect to a σ -finite measure m on \mathcal{X} such that

$$P^{\theta} \{ X_{1} \in A, Y_{1} \in B | X_{0} = x \}$$

$$= \int_{x' \in A} \int_{y \in B} p_{\theta}(x, x') f(y; \theta | x') Q(dy) m(dx'),$$
(2.2)

for $B \in \mathcal{B}(\mathbf{R}^d)$. Here $f(Y_k; \theta | X_k)$ is the conditional probability density of Y_k given X_k , with respect to a σ -finite measure Q on \mathbf{R}^d . We also assume that the Markov chain $\{(X_n, Y_n), n \ge 0\}$ has a stationary probability with probability density function $\pi_{\theta}(x)f(\cdot; \theta | x)$ with respect to $m \times Q$. For convenience of notation, we will use $\pi(x)$ for $\pi_{\theta}(x), p(x, x')$ for $p_{\theta}(x, x')$, and $f(Y_k | X_k)$ for $f(Y_k; \theta | X_k)$, respectively, here and in the sequel. We give a formal definition as follows.

Definition 1: $\{Y_n, n \ge 0\}$ is called a state space model if there is an unobserved Markov chain $\{X_n, n \ge 0\}$ such that the process $\{(X_n, Y_n), n \ge 0\}$ satisfies (2.1).

Note that the general state space model defined in Definition 1 includes (1.1), ARMA models, (G)ARCH models and stochastic volatility models. cf. Fan and Yao [5] and Fuh [9].

To formulate the change point detection problem, let $Y_1, \ldots, Y_{\omega-1}$ be a sequence of random variables from the state space model $\{Y_n, n \ge 1\}$ with distribution P^{θ_0} , and let $Y_{\omega}, Y_{\omega+1}, \ldots$ be a sequence of random variables from the state space model $\{Y_n, n \ge 1\}$ with distribution P^{θ} at some unknown time ω . The parameter of pre-change $\theta_0 \in \Theta \subset \mathbf{R}$ is given; while the parameter of after change $\theta \in J = (a, b) \subset \Theta$ is unknown. Moreover, we assume $\theta_0 < a < b < \infty$. We shall

use P_{ω} to denote such a probability measure (with change time ω) and use P_{∞} to denote the case $\omega = \infty$ (no change point). Denote E_{ω} as the corresponding expectation under P_{ω} . The objectives are to raise an alarm as soon as possible after the change and to avoid false alarms. A sequential detection scheme N is a stopping time on the sequence of observations $\{Y_n, n \ge 1\}$. A false alarm is raised whenever the detection is declared before the change occurs. A good detection procedure should minimize the number of post change observations, provided that there is no false alarm, while the rate of false alarms should be low. Hence, the stopping time N should satisfy $\{N \ge \omega\}$ but, at the same time, keep $N - \omega$ small. Specifically we will find a stopping time N to minimize

$$\sup_{1 \le k < \infty} \sup_{\theta \in J} E_k^{\theta} (N - k | N \ge k)$$
(2.3)

subject to

$$E_{\infty}^{\theta_0} N \ge \gamma, \tag{2.4}$$

for some specified (large) constant γ . A detection scheme is called second-order asymptotically optimal, if it minimizes (2.3), within an O(1) order, among all stopping rules that satisfy (2.4), where O(1) converges to a constant as $\gamma \to \infty$.

When both $\theta_0 \in \Theta$ and $\theta \in J \subset \Theta$ are given, the Shiryayev-Roberts-Pollak change point detection scheme in state space models can be described as follows. Let Y_1, \ldots, Y_n be a sequence of random variables from the state space model $\{Y_n, n \geq 1\}$, denote

$$LR_{n}(\theta) := \frac{p_{n}(Y_{1}, \dots, Y_{n}; \theta)}{p_{n}(Y_{1}, \dots, Y_{n}; \theta_{0})}$$

$$:= \frac{\int_{x_{0} \in \mathcal{X}, \dots, x_{n} \in \mathcal{X}} \pi_{\theta}(x_{0}) \prod_{l=1}^{n} p_{\theta}(x_{l-1}, x_{l}) f(Y_{l}; \theta | x_{l})}{\int_{x_{0} \in \mathcal{X}, \dots, x_{n} \in \mathcal{X}} \pi_{\theta_{0}}(x_{0}) \prod_{l=1}^{n} p_{\theta_{0}}(x_{l-1}, x_{l}) f(Y_{l}; \theta_{0} | x_{l})}$$

$$\times \frac{m(dx_{n}) \cdots m(dx_{0})}{m(dx_{n}) \cdots m(dx_{0})}$$

$$(2.5)$$

as the likelihood ratio. For $0 \le k \le n$, denote the detection scheme as

$$LR_{n}^{k}(\theta) := \frac{p_{n}(Y_{k}, Y_{k+1}, \dots, Y_{n}; \theta)}{p_{n}(Y_{k}, Y_{k+1}, \dots, Y_{n}; \theta_{0})}$$
(2.6)
$$:= \frac{\int_{x_{k} \in \mathcal{X}, \dots, x_{n} \in \mathcal{X}} \prod_{l=k}^{n} p_{\theta}(x_{l-1}, x_{l}) f(Y_{l}; \theta | x_{l})}{\int_{x_{k} \in \mathcal{X}, \dots, x_{n} \in \mathcal{X}} \prod_{l=k}^{n} p_{\theta_{0}}(x_{l-1}, x_{l}) f(Y_{l}; \theta_{0} | x_{l})}$$
$$\times \frac{m(dx_{n}) \cdots m(dx_{k})}{m(dx_{n}) \cdots m(dx_{k})}$$

Given an approximate threshold B > 0 and setting $b = \log B$, define the Shiryayev-Roberts scheme as

$$N_{b}(\theta) := \inf\{n : \sum_{k=0}^{n} LR_{n}^{k}(\theta) \ge B\}$$

$$= \inf\{n : \log\sum_{k=0}^{n} LR_{n}^{k}(\theta) \ge b\}.$$

$$(2.7)$$

A simple modification of (2.7) was given in Pollak [21] by adding a randomization on the initial $LR_n^0(\theta)$. This is the celebrated Shiryayev-Roberts-Pollak (SRP) change point detection scheme. An extension to finite state hidden Markov models can be found in Fuh [8].

When $\theta_0 \in \Theta$ is given and $\theta \in J$ is unknown, we apply a similar idea as that in Pollak and Siegmund [23], and Pollak [22] for independent observations, extending (2.7) to have a weight function of $LR_n^k(\theta)$

$$LR_n^k(F) := \int_{\theta \in J} LR_n^k(\theta) dF(\theta), \qquad (2.8)$$

where F is a probability measure on J with $F(\{\theta_0\}) = 0$. Given an approximate threshold B > 0 and setting $b = \log B$, define

$$N_{b}(F) := \inf\{n : \sum_{k=0}^{n} LR_{n}^{k}(F) \ge B\}$$
(2.9)
= $\inf\{n : \log\sum_{k=0}^{n} LR_{n}^{k}(F) \ge b\}.$

Then (2.9) is the weighted SRP change point detection rule in state space models. A formal definition will be given in Section 5, in which we will show that the SRP scheme is an "equalizer rule" in the sense that $E_k(N_b(\theta) - k + 1|N_b(\theta) \ge k - 1) = E_1N_b(\theta)$, for all k > 1.

REMARK 1. Note that the weighted SRP change point detection rule (2.9) involves two mixture components. One is an integration over the unknown parameter θ with respect to a prior distribution. The other is an integration over unknown states in the state space models, which is related to the nonlinear filtering problem. In practice, it is usually difficult to carry out the computation of $LR_n^k(F)$ in (2.8). A natural substitution is to replace it by $LR_n^k(\hat{\theta}_{l,k})$ with $\hat{\theta}_{l,k}$ is an estimator of θ based on Y_k, \ldots, Y_{l-1} , then apply Markov chain Monte Carlo method, in particular particle filtering algorithm, to approximate the change point detection rule (2.9). Theoretical justification and empirical study of this change point detection rule are interesting tasks for further investigation.

To derive asymptotic approximation of the average run length, and to prove asymptotic optimality of the weighted SRP rule in state space models, the following condition C will be assumed throughout this paper. Before that, we need some definitions first.

A Markov chain $\{X_n, n \ge 0\}$ on a state space \mathcal{X} is called *V*-uniformly ergodic if there exists a measurable function $V : \mathcal{X} \to [1, \infty)$, with $\int V(x)m(dx) < \infty$, and

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$$\lim_{n \to \infty} \sup_{x \in \mathcal{X}} \left\{ \frac{\left| E[h(X_n) | X_0 = x] - \int h(x') m(dx') \right|}{V(x)} : |h| \le V \right\} = 0.$$

$$(2.10)$$

A Markov chain $\{X_n, n \ge 0\}$ is called Harris recurrent if there exist a recurrent set $\mathcal{R} \in \mathcal{B}(\mathcal{X})$, a probability measure φ on \mathcal{R} and an integer n_0 such that $P\{X_n \in \mathcal{R} \text{ for some } n \ge n_0 | X_0 = x\} = 1$, for all $x \in \mathcal{X}$, and there exists $\lambda > 0$ such that

$$P\{X_n \in A | X_0 = x\} \ge \lambda \varphi(A), \tag{2.11}$$

for all $x \in \mathcal{R}$ and $A \subset \mathcal{R}$. Under (2.11), Athreya and Ney [1], and Nummelin [19] show that X_n admits a regenerative scheme with i.i.d. interregeneration times for an augmented Markov chain, which is called the "split chain". It is known that under irreducibility and aperiodicity assumption, w-uniform ergodicity implies that $\{X_n, n \ge 0\}$ is Harris recurrent.

Denote $S_n := \log LR_n(\theta)$, where $LR_n(\theta)$ is defined in (2.5). Let ρ be the first time (> 0) to reach the atom of the split chain, and define $u(\alpha, \zeta) = E_{\nu}e^{\alpha S_{\rho}-\zeta\rho}$ for $\zeta \in \mathbf{R}$, where ν is an initial distribution on \mathcal{X} . Assume that

$$W := \{(\alpha, \zeta) : u(\alpha, \zeta) < \infty\}$$
 is an open subset on \mathbb{R}^2 (2.12)

Denote $\zeta_1 = \zeta_1(\theta) := \log LR_1(\theta)$. Ney and Nummelin [18] shows that $D = \{\alpha : u(\alpha, \zeta) < \infty \text{ for some } \zeta\}$ is an open set and that for $\alpha \in D$, the transition kernel $\hat{P}_{\alpha}(x, A) = E_x \{e^{\alpha \zeta_1} I_{\{X_1 \in A\}}\}$ has a maximal simple real eigenvalue $e^{\Psi(\alpha)}$, where $\Psi(\alpha)$ is the unique solution of the equation $u(\alpha, \Psi(\alpha)) = 1$, with corresponding eigenfunction $r^*(x; \alpha) := E_x \exp\{\alpha S_{\varrho} - \Psi(\alpha)\varrho\}$. For a measurable subset $A \in \mathcal{B}(\mathcal{X})$ and $x \in \mathcal{X}$, define

$$\mathcal{L}(A;\alpha) = E_{\nu} \left[\sum_{n=0}^{\varrho-1} e^{\alpha S_n - n\Psi(\alpha)} I_{\{X_n \in A\}} \right], \quad (2.13)$$
$$\mathcal{L}_x(A;\alpha) = E_x \left[\sum_{n=0}^{\varrho-1} e^{\alpha S_n - n\Psi(\alpha)} I_{\{X_n \in A\}} \right]. \quad (2.14)$$

For each $\theta \in J$, denote $K(P^{\theta}, P^{\theta_0})$ as the Kullback-Leibler information numbers which will be defined precisely in (5.8) of Section V.

The following assumptions will be used throughout this paper.

Condition C:

C1. For each $\theta \in \Theta$, the Markov chain $\{X_n, n \ge 0\}$ defined in (2.1) and (2.2) is aperiodic, irreducible, and V-uniformly ergodic for some V on \mathcal{X} , such that there exists $p \ge 1$,

$$\sup_{x \in \mathcal{X}} E_x^{\theta} \left\{ \frac{V(X_p)}{V(x)} \right\} < \infty.$$
(2.15)

C2. For each $\theta \in \Theta$, assume $0 < p_{\theta}(x, x') < \infty$ for all $x, x' \in \mathcal{X}$, and $0 < \sup_{x \in \mathcal{X}} f(y; \theta | x) < \infty$, for all $y \in \mathbf{R}^d$. Denote $h_{\theta}(Y_1) = \sup_{x_0 \in \mathcal{X}} \int p_{\theta}(x_0, x_1) f(Y_1; \theta | x_1) m(dx_1)$, and assume there exists $p \ge 1$ as in C1 such that

$$\sup_{x \in \mathcal{X}} E_x^{\theta} \left\{ \log \left(h_{\theta}(Y_1)^p \frac{V(X_p)}{V(x)} \right) \right\} < 0, \quad (2.16)$$

$$\sup_{x \in \mathcal{X}} E_x^{\theta} \left\{ h_{\theta}(Y_1) \frac{V(X_1)}{V(x)} \right\} < \infty.$$
(2.17)

C3. For each $\theta \in J$, assume $0 < K(P^{\theta}, P^{\theta_0}) < \infty$. For each $\theta \in \Theta$, assume

$$\sup_{x_0 \in \mathcal{X}} \left| \int_{x_1 \in \mathcal{X}} \int_{y \in \mathbf{R}^d} \pi_{\theta}(x_0) p_{\theta}(x_0, x_1) f(y; \theta | x_1) \right|$$
$$Q(dy) m(dx_1) | < \infty.$$

C4. Assume (2.12) hold. Let C be a measurable subset of \mathcal{X} such that

$$\mathcal{L}(C;\alpha) < \infty \text{ and } \mathcal{L}_x(C;\alpha) < \infty \text{ for all } x \in \mathcal{X}.$$
 (2.18)

Let $V : \mathcal{X} \to [1, \infty)$ be a measurable function such that for some $0 < \beta < 1$ and K > 0,

$$E_x[e^{\alpha\zeta_1 - \Psi(\alpha)}V(X_1)] \le (1 - \beta)V(x) \quad \forall \ x \notin Q(2.19)$$

$$\sup_{x \in C} E_x[e^{\alpha\zeta_1 - \Psi(\alpha)}V(X_1)] = K < \infty$$
(2.20)
and $\int V(x)\varphi(dx) < \infty$,

where φ is defined in (2.11).

REMARK 2: C1 is an ergodic condition for the underlying Markov chain. The weighted mean contraction property (2.16) and the finite weighted mean average property (2.17), appeared in C2, guarantee that the induced Markovian iterated random functions system satisfies uniformly ergodic condition with respect to a given norm. In Section III, we show that several interesting models satisfy these conditions. C3 is a constraint of the Kullback-Leibler information numbers and a standard moment condition. Note that positiveness of the Kullback-Leibler information numbers is not at all restrictive, since it holds whenever the probability density functions of P^{θ} and P^{θ_0} do not coincide almost surely. The finiteness condition is quite natural and holds in most cases. C4 ensures the finiteness of the eigenfunction $r(x; \alpha)$ and the eigenmeasure $\mathcal{L}(A; \alpha)$, cf. Theorem 4 of Chan and Lai [4]. These properties are useful for defining the exponential embedding in (4.18) and (4.20) below.

The next theorem establishes second order approximation of the weighted SRP rule.

Theorem 1: Let Y_1, \ldots, Y_n be a sequence of random variables from a state space model $\{Y_n, n \ge 1\}$ satisfying conditions C1-C4. Suppose $F'(\theta) = dF(\theta)/d\theta$ exists, positive and continuous in an open neighborhood of $\theta \in \Theta$. Assume that S_1 is nonarithmetic with respect to P_{∞}^{θ} and P_1^{θ} . Then for given $x_0 \in \mathcal{X}$, as $b \to \infty$

$$E_{1}^{\theta}(N_{b}(F)|X_{0} = x_{0})$$

$$\frac{1}{(b+1)} \left(\begin{array}{c} b \\ b \\ b \end{array} + C(\theta) \right) + c(1)$$
(2.21)

$$= \frac{1}{K(P^{\theta}, P^{\theta_0})} \left(b + \frac{1}{2} \log \frac{1}{K(P^{\theta}, P^{\theta_0})} + C(\theta) \right) + o(1),$$

where $C(\theta)$ will be defined precisely in (5.21) of Section V. The proof of Theorem 1 is given in Section VI.

The next theorem establishes asymptotic optimality of the weighted SRP rule.

Theorem 2: Let Y_1, \ldots, Y_n be a sequence of random variables from a state space model $\{Y_n, n \ge 1\}$ satisfying conditions C1-C4. Assume $\theta_0 \in \Theta$, and suppose that there exists $J \subset \Theta$ with F(J) > 0. Assume that for all $\theta \in J \subset \Theta$, S_1 is nonarithmetic with respect to P_{∞}^{θ} and P_1^{θ} . Then for any given change point detection rule $N \in \mathcal{C} := \{E_{\infty}^{\theta_0}N \ge 1/B\}$, we have

$$\inf_{N \in \mathcal{C}} \sup_{1 \le \omega < \infty} \sup_{\theta \in J} 2K(P^{\theta}, P^{\theta_0}) E^{\theta}_{\omega}(N - \omega | N \ge \omega)$$

$$\ge 2b + \log b + O_{\theta}(1),$$
(2.22)

where $\limsup_{b\to\infty} \sup_{\theta\in J} |O_{\theta}(1)| < \infty$, and equality is attained by the weighted SRP rule.

The proof of Theorem 2 is given in Section VII.

III. EXAMPLES AND APPLICATIONS

In this section, we demonstrate the application of our results to models of general state Markov models and linear state space models, which are commonly used in practice for change point detection, cf. Tartakovsky et al. [28].

Example 1. General state Markov models

When Y_n equals X_n in (2.1), one has a general state Markov chain. Under the uniform recurrent condition for the Markov chain, and using the characterization of the Kullback-Leibler distance $K(\theta, \theta_0)$ $\int_{x \in \mathcal{X}} \pi_{\theta}(x) \int_{x' \in \mathcal{X}} p_{\theta}(x, x') \log \frac{p_{\theta}(x, x')}{p_{\theta_0}(x, x')} dx' dx, \text{ Lai [13] investigates the optimality property of generalized CUSUM rule$ under error probability constraint. In this paper, we prove that the SRP rule is second-order asymptotic optimal, and present an asymptotic expansion of the average run length under conditions C1-C4. Note that the V-uniformly ergodic condition appeared in C1 is weaker than the uniform recurrent condition, and covers several interesting examples. For instance an AR(1) model with normal innovation is Vuniformly ergodic with V(x) = |x| + 1 [cf. pages 380 and 383 of Meyn and Tweedie [17]]; while it does not satisfy the assumption of the transition density function $p_{\theta_1}(\cdot, \cdot)$ is uniform recurrent, in the sense that there exist $c_2 > c_1 >$ 0, $m \geq 1$ and a probability measure μ^* on \mathcal{X} such that $c_1\mu^*(A) \leq P\{X_m \in A | X_0 = x\} \leq c_2\mu^*(A)$ for all measurable subsets A and all $x \in \mathcal{X}$.

To discuss condition C, appeared in Section II, for this model. Suppose that $\{X_n, n \ge 0\}$ is a Markov chain with transition density function $p_{\theta_0}(\cdot, \cdot)$ for $n < \omega$ and $p_{\theta}(\cdot, \cdot)$ for $n \ge \omega$, with respect to some σ -finite measure m on the state space \mathcal{X} . Condition C1 requires that $\{X_n, n \ge 0\}$ is V-uniformly ergodic. By choosing p = 1, (2.15) reduces to $\sup_x E_x^{\theta} V(X_1)/V(x) < \infty$. Condition C3 reduces to that for each $\theta \in \Theta$, $0 < p_{\theta}(x, y) < \infty$, for all $x, y \in \mathcal{X}$, which is also required in C2. Note that $h(Y_1)$, used in (2.16) and (2.17), reduces to $\sup_{x_0} \int p_{\theta}(x_0, x_1)\pi_{\theta}(x_1)m(dx_1)$. Condition C4 reduces to a condition involves X_0 and X_1 only; see conditions (W1) and (W2) in Chan and Lai [4].

One can show that many practical used models satisfy condition C. For instance, we consider an AR(1) model $X_n = \alpha X_{n-1} + \varepsilon_n$, where $|\alpha| < 1$, and ε_n are independent and identically distributed standard normal random variables. Under the normal errors assumption, it is straightforward to check that C1 and C3 hold. To check condition C2, we only show that (2.16) holds since the verification of (2.17) is the same. Note that X_1 has stationary distribution $N(0, a^2)$ with $a = 1/(1 - \alpha^2)$. Observe that $Y_1 = X_1$ and $h_{\theta}(Y_1)$ reduces to

$$\sup_{x \in \mathbf{R}} \int_{-\infty}^{\infty} \frac{\exp\{-(y - \alpha x)^2/2\}}{\sqrt{2\pi}} \frac{\exp\{-y^2/2a^2\}}{\sqrt{2\pi}} dy$$

=
$$\sup_{x \in \mathbf{R}} \frac{1}{\sqrt{2\pi(1 + a^2)}} \exp\left\{-\frac{\alpha^2 x^2}{2(1 + a^2)}\right\} \int_{-\infty}^{\infty} \frac{\sqrt{1 + a^2}}{\sqrt{2\pi a}} \times \exp\left\{-\frac{1 + a^2}{2a^2} \left(y - \frac{a^2 \alpha x}{1 + a^2}\right)^2\right\} dy$$

$$= \frac{1}{\sqrt{2\pi(1+a^2)}} \sup_{x \in \mathbf{R}} \exp\left\{-\frac{\alpha^2 x^2}{2(1+a^2)}\right\} = \frac{1}{\sqrt{2\pi(1+a^2)}}$$

Consider p = 1, a simple calculation leads that

$$\sup_{x_{0}\in\mathbf{R}} E_{x_{0}}^{\theta} \left\{ \log \left(h_{\theta}(X_{1}) \frac{V(X_{1})}{V(x_{0})} \right) \right\}$$
(3.1)
$$< \log \sup_{x_{0}\in\mathbf{R}} E_{x_{0}}^{\theta} \left\{ \frac{|\alpha x_{0} + \varepsilon_{1}| + 1}{\sqrt{2\pi(1 + a^{2})}(|x_{0}| + 1)} \right\}$$
(3.1)
$$\leq \log \sup_{x_{0}\in\mathbf{R}} \left\{ \frac{|\alpha x_{0}| + E_{x_{0}}^{\theta}|\varepsilon_{1}| + 1}{\sqrt{2\pi(1 + a^{2})}(|x_{0}| + 1)} \right\}$$
(3.1)
$$= \log \sup_{x_{0}\in\mathbf{R}} \left\{ \frac{|\alpha x_{0}| + \frac{2}{\sqrt{2\pi}} + 1}{\sqrt{2\pi(1 + a^{2})}(|x_{0}| + 1)} \right\} < 0.$$

This implies (2.16) hold. The verification of C4 is similar to Example 2 in Chan and Lai [4].

Next we consider the following example which involves change in the mean value θ of a stable autoregressive sequence:

$$X_n = \sum_{k=1}^p a_k X_{n-k} + v_k + (1 - \sum_{k=1}^p a_k)\theta,$$
 (3.2)

where a_1, \ldots, a_p are autoregressive coefficients and v_k is a Gaussian sequence with zero mean and variance σ^2 . By Theorem 16.5.1 of Meyn and Tweedie [17], X_n defined in (3.2) is a V-uniformly ergodic Markov chain with $V(x) = x^2 + 1$. It is easy to see condition C1 holds. Since the verification of C2 can be done as that in (3.1), we will not repeat it here. Note that the assumption of normal distributed innovation (with mean zero and finite variance σ^2) implies that the moment condition C3 holds. The verification of condition C4 is similar to Example 2 in Chan and Lai [4]. Note that this example can be generalized to the case of random coefficient autoregression appeared on page 404 of Meyn and Tweedie [17].

Example 2. Linear state space models

Consider the stochastic system

$$X_{n+1} = FX_n + Gu_n + \delta_n, \qquad (3.3)$$

$$\|F\| = \sup \|Fx\| < 1,$$

$$Y_n = HX_n + Ju_n + \varepsilon_n, \qquad (3.4)$$

in which the unobservable state vector X_n , the input vector u_n , and the measurement vector Y_n have dimensions p, q, and r, respectively, and δ_n , ε_n are independent Gaussian vectors with zero means and $cov(\delta_n) = \Sigma_1$, $cov(\varepsilon_n) = \Sigma_2$. We assume G, J, Σ_1 and Σ_2 are given, and the unknown parameter is $(F, H)^t$, where t denotes transpose. The problem of additive change point detection can be found in Tartakovsky et al. [28] and Lai [13]. Here we consider the problem of nonadditive change. Suppose at an unknown time ω the system undergoes some change in the sense that the parameter is changed from θ_0 to θ , where θ_0 is given while $\theta \in J \subset \Theta$ is unknown. Here we consider θ is one dimensional unknown parameter, which can be one of the component in $(F, H)^t$, the other parts are treated as nuisance parameters.

Let \hat{H}_n and \hat{F}_n be the estimators of H and F, respectively. The Kalman filter provides a recursive algorithm to compute the conditional expectation $\hat{X}_{n|n-1}$ of the state X_n given the past observations $Y_{n-1}, u_{n-1}, Y_{n-2}, u_{n-2}, \ldots$ The innovations $e_n = Y_n - \hat{H}_n \hat{X}_{n|n-1} - \hat{J}_n u_n$ are independent zero-mean Gaussian vectors with $cov(e_n) = V_n$ given recursively by

$$V_n = \hat{H}_n P_{n|n-1} \hat{H}_n^t + \Sigma_2,$$
(3.5)

where

$$= \hat{F}_n(P_{n|n-1} - P_{n|n-1}\hat{H}_n^t V_n^{-1} \hat{H}_n P_{n|n-1})\hat{F}_n^t + \Sigma_1.$$
(3.6)

When the parameter $\theta = \theta_0$, the innovations e_n^0 are independent Gaussian vectors with covariance matrices V_n^0 , and means $\mu_n^0 = E(e_n^0)$ for $n \le \omega$, while when the parameter is changed to $\theta \in J$, the innovations e_n^{θ} are independent Gaussian vectors with covariance matrices V_n^{θ} , and means $\mu_n^{\theta} = E(e_n^{\theta})$ for $n \ge \omega$. Consider the weighted likelihood

$$LR_n^k(F) = \int_{\theta \in J} \prod_{l=k}^n \frac{f(e_l^\theta/\sqrt{V_l^\theta})}{f(e_l^0/\sqrt{V_l^0})} dF(\theta), \qquad (3.7)$$

where $f(s) = e^{-||s||^2/2}/(2\pi)^{d/2}$ denotes the *d*-dimensional standard normal density, d = p + r, and assume that the matrix whose inverse appears in (3.7) is nonsingular.

To illustrate the computation of (3.7), we consider a simple case that there is only a one-dimensional unknown parameter $H = \theta \in J = (0,1)$ and $F(\theta)$ is uniform distributed on (0,1). Let $\theta + a_n = \mu_n$ for given a_n , and denote $\sigma_{n,\theta}^2$ as V_n^{θ} . When $\theta = \theta_0$, simply denote $a_l = 0$ and $\sigma_{n,0}^2$ as σ_{n,θ_0}^2 . That is $e_l^{\theta} \sim N(\theta + a_l, \sigma_{l,\theta}^2)$, and $e_l^0 \sim N(0, \sigma_{l,0}^2)$. Then a simple calculation leads that

$$(3.7) = \int_{0}^{1} \prod_{l=k}^{n} \left[\exp\left\{ -\frac{(e_{l}^{\theta} - (\theta + a_{l}))^{2}}{2\sigma_{l,\theta}^{2}} + \frac{(e_{l}^{0} - \theta_{0})^{2}}{2\sigma_{l,0}^{2}} \right\} \right] d\theta$$
$$= \exp\left\{ \sum_{l=k}^{n} \left[\frac{(e_{l}^{0} - \theta_{0})^{2}}{2\sigma_{l,0}^{2}} - \frac{(e_{l}^{\theta} - a_{l})^{2}}{2\sigma_{l,\theta}^{2}} \right] + \frac{\left(\sum_{l=k}^{n} (e_{l}^{\theta} - a_{l})\right)^{2}}{2\alpha} \right\} \cdot \frac{\sqrt{2\pi}}{\sqrt{\alpha}} \left(\Phi(b) - \Phi(a) \right),$$

where $\alpha = \sum_{l=k}^{n} 1/\sigma_{l,\theta}^2$, $a = -\sum_{l=k}^{n} \frac{e_l^{\sigma} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - a_l}{\sqrt{\alpha}}$, $b = \sqrt{\alpha} - \sum_{l=k}^{n} \frac{e_l^{\theta} - \alpha}{\sqrt{\alpha}}$, $b = \sqrt{$

 $\sum_{\substack{l=k\\ e_l^{\prime\prime} = a_l}} \frac{e_l^{\prime\prime} - a_l}{\sqrt{\alpha}}, \text{ and } \Phi(\cdot) \text{ is the cumulative distribution function}$

of standard normal random variable.

Without assuming prior knowledge for the parameter after change θ and the change time ω , the weighted SRP change point detection rule, defined in Section II, has the form

$$N_b = \inf \left\{ n : \sum_{k=0}^n LR_n^k(F) \ge B \right\}$$
(3.8)
$$= \inf \left\{ n : \log \sum_{k=0}^n LR_n^k(F) \ge b \right\},$$

where B > 0 is a given threshold and $b = \log B$.

To check the regularity condition C hold, we assume that there is no input vector u_n for simplicity. We first consider condition C1. Note that Y_n are independent for given X_n , therefore the weight function V depends on X_0 only and one can choose $V(x) = e^{\gamma ||x||}$ for some γ to be specified later. Let $C = \{\mu : ||\mu|| \le N\}$, and denote λ as the Lebesgue measure on \mathbb{R}^p . Recall that δ_1 has normal density function ϕ with zero mean vector and variance-covariance matrix $cov(\delta_1) = \Sigma_1$, which is positive and continuous, and this implies $\eta := \inf\{\phi(\delta - Fx) : x \in C \text{ and } Fx + \delta \in C\} > 0$. Since $P\{Fx_1 + \delta_1 \in d\delta\} \ge \phi(\delta - Fx)d\delta$, we have for all $x \in \mathbb{R}^p$, $P_x\{X_1 \in A\} \ge \delta I_{\{x \in C\}}\lambda(A \cap C)$, and therefore the minorization condition holds with $h(x) = \delta\lambda(C) \times I_{\{x \in C\}}$. Under the normal error assumptions, it is easy to see that (2.15) and C3 hold.

To check condition C4 hold. Let

$$\begin{aligned} \zeta_1 &:= \zeta_1(\theta) \\ \vdots \\ \vdots \\ &:= \log \frac{\int_{x_0, x_1 \in \mathcal{X}} \pi_\theta(x_0) p_\theta(x_0, x_1) f(Y_1; \theta | x_1) m(dx_1) m(dx_0)}{\int_{x_0, x_1 \in \mathcal{X}} \pi_{\theta_0}(x_0) p_{\theta_0}(x_0, x_1) f(Y_1; \theta_0 | x_1) m(dx_1) m(dx_0)}, \end{aligned}$$
(3.9)

where $\pi_{\theta}(x_0)$ is the *p*-variate normal density function with zero mean vector and variance-covariance matrix $\Sigma_1/(1 - ||F||)$, $p_{\theta}(x_0, x_1)$ is the *p*-variate normal density function with mean vector Fx_0 and variance-covariance matrix Σ_1 , and $f(Y_1; \theta | x_1)$ is the *p*-variate normal density function with mean vector Hx_1 and variance-covariance matrix Σ_2 . Denote the conditional distribution of ζ_1 given (X_1, Y_1) has the form $F_{(X_1, Y_1)}$. Since $\zeta_n = g(Y_n)$ for some *g* by (3.9), $F_{(X_1, Y_1)}$ degenerates to F_{Y_1} . By (3.3), (3.4) and (3.9), it is easy to see that for any given $\alpha \in \mathbf{R}$, there exists a positive constant ρ_{α} such that

$$\int e^{\alpha g(s_1)} dF_{s_1}(s_1) \le \exp\{\rho_\alpha \|s_1\|\} \text{ for all } s_1 \in \mathbf{R}^r.$$
(3.10)

This implies that

$$E_{x}[e^{\theta\zeta_{1}}V(X_{1})]$$

$$\leq E \exp\{\rho_{\theta}(\|Hx + \varepsilon_{1}\|) + \gamma\|Fx + \delta_{1}\|\}$$

$$\leq \Lambda(\rho_{\theta} + \gamma) \exp\{(\rho_{\theta}(1 + \|H\|) + \gamma\|F\|)\|x\|\}.$$
(3.11)

Since ||F|| < 1, we can choose γ large enough so that $2\rho_{\theta} + \gamma ||H|| < \gamma$, and then (2.19) is satisfied if N is chosen large enough. Since C is compact and $\lambda(\cdot \cap C)$ has support C, (2.20) also holds for sufficiently large L.

Finally we need to verify C2 hold. For simplicity, let p = r = d in (3.3) and (3.4). After normalization, we may assume the variance parts in Σ_1 and Σ_2 are both equal to 1. That is, define

$$\Sigma_{1} = \begin{pmatrix} 1 & \rho_{1}^{1} & \cdots & \rho_{d}^{1} \\ \rho_{1}^{1} & 1 & \cdots & \rho_{d-1}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d}^{1} & \rho_{d-1}^{1} & \cdots & 1 \end{pmatrix}$$
$$\Sigma_{2} = \begin{pmatrix} 1 & \rho_{1}^{2} & \cdots & \rho_{d}^{2} \\ \rho_{1}^{2} & 1 & \cdots & \rho_{d-1}^{2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{d}^{2} & \rho_{d-1}^{2} & \cdots & 1 \end{pmatrix}$$

Let

$$x_{0} = x = \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{d} \end{pmatrix}, x_{1} = x' = \begin{pmatrix} x'_{1} \\ x'_{2} \\ \vdots \\ x'_{d} \end{pmatrix}, y = \begin{pmatrix} y_{1} \\ y_{2} \\ \vdots \\ y_{d} \end{pmatrix}$$
$$z = Hx' = \begin{pmatrix} h_{11} & \cdots & \cdots & h_{1d} \\ \vdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{d1} & \cdots & \cdots & h_{dd} \end{pmatrix} \begin{pmatrix} x'_{1} \\ x'_{2} \\ \vdots \\ x'_{d} \end{pmatrix},$$
$$\mu = FX = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1d} \\ \vdots & \cdots & \cdots & h_{dd} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{d1} & \cdots & \cdots & \alpha_{dd} \end{pmatrix} \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x'_{d} \end{pmatrix},$$

$$\mu^* = (\Sigma_1^{-1} + H^t \Sigma_2^{-1} H)^{-1} (\Sigma_1^{-1} \mu + H^t \Sigma_2^{-1} y)$$

and

$$\Sigma^{*-1} = \Sigma_1^{-1} + H^t \Sigma_2^{-1} H.$$

Denote $|\Sigma|$ as the determinant of the matrix Σ . Then a simple calculation leads that

$$\begin{split} & \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\exp\{-\frac{1}{2}(x'-\mu)^{t}\Sigma_{1}^{-1}(x'-\mu)\}}{(2\pi)^{d/2}|\Sigma_{1}|^{1/2}} \\ & \times \frac{\exp\{-\frac{1}{2}(y-z)^{t}\Sigma_{2}^{-1}(y-z)\}}{(2\pi)^{d/2}|\Sigma_{2}|^{1/2}} dx'_{1} \cdots dx'_{d} \\ = & \frac{|\Sigma^{*}|^{1/2}}{(2\pi)^{d/2}|\Sigma_{1}|^{1/2}|\Sigma_{2}|^{1/2}} \exp\left\{\mu^{t}\Sigma_{1}^{-1}\mu + s^{t}\Sigma_{2}^{-1}y\right. \\ & \left. - \left[(\Sigma_{1}^{-1} + H^{t}\Sigma_{2}^{-1}H)^{-1}(\Sigma_{1}^{-1}\mu + H^{t}\Sigma_{2}^{-1}y) \right]^{t} \\ & \times \Sigma^{*-1} \left[(\Sigma_{1}^{-1} + H^{t}\Sigma_{2}^{-1}H)^{-1}(\Sigma_{1}^{-1}\mu + H^{t}\Sigma_{2}^{-1}y) \right]^{t} \end{split}$$

Note that

$$\begin{split} \Sigma^{*-1} &= \Sigma_1^{-1} + H^t \Sigma_2^{-1} H \Longrightarrow \Sigma^* = (\Sigma_1^{-1} + H^t \Sigma_2^{-1} H)^- \\ \Longrightarrow |\Sigma^*| &= |(\Sigma_1^{-1} + H^t \Sigma_2^{-1} H)^{-1}|. \end{split}$$

Therefore

$$h(y) \qquad (3.12)$$

$$= \sup_{x \in \mathbf{R}^{d}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\exp\{-\frac{1}{2}(x'-\mu)^{t}\Sigma_{1}^{-1}(x'-\mu)\}}{(2\pi)^{d/2}|\Sigma_{1}|^{1/2}} \times \frac{\exp\{-\frac{1}{2}(y-z)^{t}\Sigma_{2}^{-1}(y-z)\}}{(2\pi)^{d/2}|\Sigma_{2}|^{1/2}} dx'_{1} \cdots dx'_{d}$$

$$= \frac{1}{(2\pi)^{d/2}|\Sigma_{1}|^{1/2}|\Sigma_{2}|^{1/2}} \cdot |(\Sigma_{1}^{-1} + H^{t}\Sigma_{2}^{-1}H)|^{-1/2}.$$

Assume $\frac{|(\Sigma_1^{-1} + H^t \Sigma_2^{-1} H)|^{-1/2}}{(2\pi)^{d/2} |\Sigma_1|^{1/2} |\Sigma_2|^{1/2}} = a < 1$. A simple calculation leads that

$$\sup_{x_0 \in \mathbf{R}^d} E_{x_0}^{\alpha} \left\{ \log \left(h(Y_1)^p \frac{w(X_p)}{w(x_0)} \right) \right\}$$
(3.13)

$$= \sup_{x_0 \in \mathbf{R}^d} E_{x_0}^{\alpha} \log \left\{ \frac{a^p \exp\{\gamma(\alpha^p \| x_0 \| + \sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k})\}}{\exp\{\gamma \| x_0 \|\}} \right\}$$
$$= \sup_{x_0 \in \mathbf{R}^d} E_{x_0}^{\alpha} \left\{ \gamma \alpha^p \| x_0 \| + \sum_{k=0}^{p-1} \alpha^k \varepsilon_{p-k} - \gamma \| x_0 \| + p \log a \right\}$$
$$= p \log a < 0.$$

This implies (2.16) hold. By using the same argument, we have (2.17) hold.

To illustrate (3.12) and (3.13), we consider a simple case of d = 2. Denote $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x' = \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix}$, $\mu = \begin{pmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\Sigma_1 = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}$ and $\Sigma_2 = \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix}$. Simple calculation leads that

$$\Sigma^{*-1} = \frac{2 - \rho_1^2 - \rho_2^2}{(1 - \rho_1^2)(1 - \rho_2^2)} \times \begin{pmatrix} 1 & \frac{-\rho_1(1 - \rho_2^2) - \rho_2(1 - \rho_1^2)}{2 - \rho_1^2 - \rho_2^2} \\ \frac{-\rho_1(1 - \rho_2^2) - \rho_2(1 - \rho_1^2)}{2 - \rho_1^2 - \rho_2^2} & 1 \end{pmatrix},$$

and

$$\mu^* = \begin{pmatrix} \frac{(\rho_1 - \rho_2)(\mu_2 - y_2) + (\rho_1^2 + \rho_1 \rho_2 - 2)s_1 + (\rho_2^2 + \rho_1 \rho_2 - 2)\mu_1}{(\rho_1 + \rho_2)^2 - 4} \\ \frac{(\rho_1 - \rho_2)(\mu_1 - y_1) + (\rho_1^2 + \rho_1 \rho_2 - 2)s_2 + (\rho_2^2 + \rho_1 \rho_2 - 2)\mu_2}{(\rho_1 + \rho_2)^2 - 4} \end{pmatrix}.$$

Then $h(y) = \frac{1}{2\pi\sqrt{4-(\rho_1+\rho_2)^2}}$, and the condition reduces to $|\rho_1 + \rho_2| < \frac{\sqrt{16\pi^2-1}}{2\pi} \approx 1.994.$

IV. LIKELIHOOD REPRESENTATION AND EXPONENTIAL EMBEDDING

In this section, we investigate the weighted Shiryayev-Roberts change point detection rule (2.8)-(2.9). Due to the change point detection rule involves $LR_n^k(\theta)$ defined in (2.6), we study the likelihood ratio LR_n appeared in (2.5) first. A major difficulty for analysing the likelihood ratio (2.5) is its integral form. To overcome this obstacle, we represent (2.5) as the ratio of L_1 -norms of a Markovian iterated random functions system. Specifically, let

$$\mathbf{H} = \{h|h: \mathcal{X} \to \mathbf{R}^+ \text{ is } m\text{-measurable}, \qquad (4.1)$$
$$\int h(x)m(dx) < \infty \text{ and } \sup_{x \in \mathcal{X}} h(x) < \infty\},$$

and define the variation distance between any two elements h_1, h_2 in **H** by

$$d(h_1, h_2) = \sup_{x \in \mathcal{X}} |h_1(x) - h_2(x)|.$$
(4.2)

For j = 1, ..., n, define the random functions $\mathbf{P}_{\theta}(Y_j)$ on f

$$\begin{aligned} & \mathbf{P}_{\theta}(Y_{0})h(x) & (4.3) \\ & = \int_{x' \in \mathcal{X}} f(Y_{0}; \theta | x')h(x')m(dx') \text{ a constant}, \\ & \mathbf{P}_{\theta}(Y_{j})h(x) & (4.4) \\ & = \int_{x' \in \mathcal{X}} p_{\theta}(x, x')f(Y_{j}; \theta | x')h(x')m(dx'), \end{aligned}$$

and denote the composition of two random functions as

$$\mathbf{P}_{\theta}(Y_{j+1}) \circ \mathbf{P}_{\theta}(Y_{j})h(x) \tag{4.5}$$

$$= \int_{x_{j} \in \mathcal{X}} p_{\theta}(x, x_{j})f(Y_{j}; \theta | x_{j}) \left(\int_{x_{j+1} \in \mathcal{X}} p_{\theta}(x_{j}, x_{j+1}) \times f(Y_{j+1}; \theta | x_{j+1})h(y)m(dx_{j+1}) \right) m(dx_{j}).$$

Furthermore, let

 $\mathcal{X} \times \mathbf{H}$ as

$$\mathbf{M} = \{ M : \mathbf{H} \to \mathbf{H} | M \text{ is a linear}$$
(4.6)
and bounded operator $P_{\theta^*}^{\pi} - a.s. \},$

be equipped with the operator norm $\|\cdot\|$ with respect to the sup-norm, i.e.

$$||M|| = \sup_{h \in \mathbf{H}: ||h||_{\infty} = 1} ||M(h)||_{\infty}.$$
 (4.7)

We define the iterated random functional system as

$$M_{\theta,0}(h) = \mathbf{P}_{\theta}(Y_0)h \tag{4.8}$$

$$M_{\theta,n}(h) = F(Y_n, M_{\theta,n-1})(h)$$
(4.9)
:= $\frac{M_{\theta,n-1}(P_{\theta}(Y_n)h)}{\int M_{\theta,n-1}(P_{\theta}(Y_n)1)(x)m(dx)},$

for $n \ge 1$. Note that

$$M_{\theta,n}(h)(x)$$

$$= \int_{x_n \in \mathcal{X}} h(x_n) p_{\theta} \left(X_0 = x, X_n = x_n | Y_0, \cdots, Y_n \right) m(dx_n)$$
(4.10)

For $h \in \mathbf{M}$, let $||h|| := \int_{x \in \mathcal{X}} h(x)m(dx)$ be the L^1 -norm on \mathbf{M} with respect to m. Then, the likelihood ratio (2.5) can be represented as

$$LR_n(\theta) = \frac{||\mathbf{P}_{\theta}(Y_n) \circ \cdots \circ \mathbf{P}_{\theta}(Y_1)\pi_{\theta}||}{||\mathbf{P}_{\theta_0}(Y_n) \circ \cdots \circ \mathbf{P}_{\theta_0}(Y_1)\pi_{\theta_0}||}.$$
 (4.11)

Let $\{(X_n, Y_n), n \ge 0\} := \{(X_n^{\theta}, Y_n^{\theta}), n \ge 0\}$ be the Markov chain defined in (1.1) and (2.2). Abuse the notation a little bit, we denote $\theta = (\theta_0, \theta)$ because θ_0 is given. For each n, let

$$M_{n}(\theta) = \mathbf{P}_{\theta}(Y_{n}) \circ \cdots \circ \mathbf{P}_{\theta}(Y_{1}) = (M_{n}(\theta_{0}), M_{n}(\theta))$$
(4.12)
$$= (\mathbf{P}_{\theta_{0}}(Y_{n}) \circ \cdots \circ \mathbf{P}_{\theta_{0}}(Y_{1}), \mathbf{P}_{\theta}(Y_{n}) \circ \cdots \circ \mathbf{P}_{\theta}(Y_{1}))$$

be the Markovian iterated random functions system on **M** induced from (4.4). Then $\{W_n^{\theta}, n \ge 0\} := \{(X_n^{\theta}, M_n(\theta)), n \ge 0\}$ is a Markov chain on the state space $\mathcal{X} \times \mathbf{M}$, with transition probability kernel

$$\mathbb{P}^{\theta}((x_0,h), A \times \Gamma) := \int_{x_1 \in A} \int_{y \in B} I_{\Gamma}(\mathbf{P}_{\theta}(y)h) \qquad (4.13)$$
$$\times p_{\theta}(x_0, x_1) f(y; \theta | x_1) Q(dy) m(dx_1)$$

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for all $x_0 \in \mathcal{X}$, $h \in \mathbf{M}$, $A \in \mathcal{B}(\mathcal{X})$ and $\Gamma \in \mathcal{B}(\mathbf{M})$, where I_{Γ} denotes the indicator function on the set Γ . For $(x,h) \in \mathcal{X} \times \mathbf{M}$, let $\mathbb{P}_{(x,h)}$ be the probability measure on the underlying measurable space under which $X_0 = x, M_0 = h$. The associated expectation is denoted $\mathbb{E}_{(x,h)}$, as usual. For an arbitrary distribution ν on $\mathcal{X} \times \mathbf{M}$, we put $\mathbb{P}_{\nu}(\cdot) :=$ $\int \mathbb{P}_{(x,h)}(\cdot) \nu(dx \times dh)$ with associated expectation \mathbb{E}_{ν} . We use \mathbb{P} and \mathbb{E} for probabilities and expectations, respectively, that do not depend on the initial distribution. Since the Markov chain $\{X_n, n \ge 0\}$ has transition probability density and the iterated random function $M_1(\theta)$, defined in (4.12), is driven by $\{(X_n, Y_n), n \geq 0\}$, the induced transition probability $\mathbb{P}(\cdot, \cdot)$ has a density with respect to $m \times Q$. Denote it as \mathbb{P} for simplicity. According to Theorem 1(iii) in Fuh [9], the stationary distribution of $\{W_n^{\theta}, n \geq 0\}$ exists, and denote it by Π_{θ} .

Now the log-likelihood ratio can be written as an additive functional of the Markov chain $\{W_n^{\theta}, n \ge 0\}$. That is

$$\log LR_n(\theta) = \sum_{k=1}^{n} g(W_{k-1}^{\theta}, W_k^{\theta}),$$
 (4.14)

where

$$g(W_{k-1}^{\theta}, W_{k}^{\theta}) := \log \frac{||\mathbf{P}_{\theta}(Y_{k}) \circ \cdots \circ \mathbf{P}_{\theta}(Y_{1})\pi_{\theta}||}{||\mathbf{P}_{\theta_{0}}(Y_{k}) \circ \cdots \circ \mathbf{P}_{\theta_{0}}(Y_{1})\pi_{\theta_{0}}||} (4.15)$$
$$-\log \frac{||\mathbf{P}_{\theta}(Y_{k-1}) \circ \cdots \circ \mathbf{P}_{\theta}(Y_{1})\pi_{\theta}||}{||\mathbf{P}_{\theta_{0}}(Y_{k-1}) \circ \cdots \circ \mathbf{P}_{\theta_{0}}(Y_{1})\pi_{\theta_{0}}||}.$$

To analyze the weighted SRP change point detection rule in state space models, we need first to construct an exponential embedding of the transition probability operator for the induced Markov chain $\{W_n, n \ge 0\}$ with state space $\mathcal{W} :=$ $\mathcal{X} \times \mathbf{M}$, and then to represent the weighted likelihood ratio as an additive functional of the Markov chain $\{W_n, n \ge 0\}$. To this end, we show that the induced Markov chain $\{W_n, n \ge 0\}$ satisfies some required recurrent and ergodic conditions.

For any given two transition probability kernels $Q(w, A), K(w, A), w \in \mathcal{W}, A \in \mathcal{B}(\mathcal{W})$, the σ -algebra of \mathcal{W} , and for all measurable functions $h(w), w \in \mathcal{W}$, define Qh and QK by $Qh(w) = \int Q(w, dw')h(w')$ and $QK(w, A) = \int K(w, dw')Q(w', A)$, respectively. Let \mathcal{N} be the Banach space of measurable functions $h : \mathcal{W} \to \mathbb{C}$ (:= the set of complex numbers) with norm $||h|| < \infty$. We also introduce the Banach space \mathcal{B} of transition probability kernels Q such that the operator norm $||Q|| = \sup\{||Qg||; ||g|| \leq 1\}$ is finite.

Denote by $P^n(y, A) = P\{W_n \in A | W_0 = y\}$, the transition probabilities over *n* steps. The kernel P^n is a *n*-fold power) of *P*. Define also the Césaro averages $P^{(n)} = \sum_{j=0}^{n} P^j/n$,) where $P^0 = P^{(0)} = I$ and *I* is the identity operator on \mathcal{B} .

Definition 2: A Markov chain $\{X_n, n \ge 0\}$ is said to be uniformly ergodic with respect to a given norm $|| \cdot ||$, if there exists a stochastic kernel Π such that $P^{(n)} \to \Pi$ as $n \to \infty$ in the induced operator norm in \mathcal{B} .

Definition 3: Let $\omega : \mathcal{X} \to [1,\infty)$ be the weight function defined in C1, and **M** be defined in (4.1). For any measurable function $g : \mathcal{X} \times \mathbf{M} \to [1,\infty)$, define $||g||_V := \sup_{(x,h) \in \mathcal{X} \times \mathbf{M}} \frac{|g(x,h)|}{V(x)}$, and $||g||_h :=$
$$\begin{split} \sup_{x\in\mathcal{X},h_1,h_2:0< d(h_1,h_2)\leq 1} \frac{|g(x,h_1)-g(x,h_2)|}{(V(x)d(h_1,h_2))^{\delta}}, \text{ for } 0 < \delta < 1.\\ \text{We define } \mathcal{H} \text{ as the set of } g \text{ on } \mathcal{X}\times\mathbf{M} \text{ for which } \|g\|_{Vh} := \|g\|_V + \|g\|_h \text{ is finite, where } Vh \text{ represents a combination } \\ \text{of the weighted variation norm and the bounded weighted } \\ \text{Hölder's norm.} \end{split}$$

Theorem 3: Let $\{(X_n^{\theta}, Y_n^{\theta}), n \ge 0\}$ be the state space model given in (2.1), satisfying C1-C3, where $\theta = (\theta_0, \theta) \in \Theta \times J$ is the unknown parameter. Then the induced Markov chain $\{W_n^{\theta}, n \ge 0\}$ is an aperiodic, irreducible and Harris recurrent Markov chain. Moreover, it is uniformly ergodic with respect to the norm defined in Definition 3. Furthermore there exist a, C > 0, such that $\mathbb{E}_w(\exp\{ag(W_0, W_1)\}) \le C < \infty$ for all $w \in \mathcal{W}$.

Since the proof is the same as those in Lemmas 3 and 4 of Fuh [9], it is omitted.

Next we define Laplace transform of the transition operator and introduce the twisting probability measure for $\{W_n, n \ge 0\}$. Denote w := (x, h) and $\tilde{w} := (x_0, \pi)$, where x_0 is the initial state of X_0 taken from $\pi(X_0)$. Recall $g(W_0, W_1)$ defined in (4.15). For given $w \in \mathcal{W}$, $A \times \Gamma \in \mathcal{B}(\mathcal{W})$, and $\alpha \in \mathbf{R}$, define the linear operator $\hat{\mathbf{P}}_{\alpha}$ by

$$\hat{\mathbf{P}}_{\alpha}(w, A \times \Gamma) = \mathbb{E}_{w} \bigg\{ e^{\alpha g(W_{0}, W_{1})} I_{\{W_{1} \in A \times \Gamma\}} \bigg\}.$$
 (4.16)

Under conditions C1-C3, Theorem 3 leads that $\{W_n, n \ge 0\}$ is an aperiodic, irreducible and Harris recurrent Markov chain, and conditions in Theorem 4.1 of Ney and Nummelin [18] hold. Therefore, $\hat{\mathbf{P}}_{\alpha}$ has a maximal simple real eigenvalue $\lambda(\alpha)$ with associated right eigenfunction $r(\cdot; \alpha)$ such that $\Lambda(\alpha) = \log \lambda(\alpha)$ is analytic and strictly convex on $\mathcal{D} = \{\alpha : \Lambda(\alpha) < \infty\}$.

Let τ be the first time (> 0) to reach the atom of the split chain for $\{W_n, n \ge 0\}$. For each $w \in \mathcal{W}$ and $A \times \Gamma \in \mathcal{B}(\mathcal{W})$, define the left eigenmeasures

$$\ell(A \times \Gamma; \alpha) := \mathbb{E}_{\nu} \left\{ \sum_{n=0}^{\tau-1} e^{\alpha S_n - n\Lambda(\alpha)} I_{\{W_n \in A \times \Gamma\}} \right\} (4.17)$$
$$\ell_w(A \times \Gamma; \alpha) := \mathbb{E}_w \left\{ \sum_{n=0}^{\tau-1} e^{\alpha S_n - n\Lambda(\alpha)} I_{\{W_n \in A \times \Gamma\}} \right\}.$$

Recall $S_n = \sum_{k=1}^n g(W_{k-1}, W_k)$, and $g(W_{k-1}, W_k)$ defined in (4.15) is an additive functional of the Markov chain $\{(W_{n-1}, W_n), n \geq 1\}$. Since $r(w; \alpha)^{-1} \pi_\alpha(dw) = L_\alpha \ell(dw; \alpha)$ for some constant L_α [cf. Ney and Nummelin [18], page 581], the finiteness of $\ell(A \times \Gamma; \alpha)$ implies that $r(w; \alpha) > 0$ uniformly for $w \in \mathcal{W}$. On the other hand, Theorem 4 of Chan and Lai [4] establishes the finiteness of $\ell(A \times \Gamma; \alpha)$ and $\ell_w(A \times \Gamma; \alpha)$.

Denote $\theta := (\theta_0, \theta) \in \Theta \times J$ as the parameter. Assume $\theta = \Lambda'(\alpha)$ is a one to one function, so one can indifferently consider θ to be a function of α or α a function of θ . Here ' denotes derivative. For simplicity, we replace α by θ in (4.16), and let $\mathcal{D} = J$ here and in the sequel. Then under conditions C1-C4, by using Theorem 1 of Ney and Nummelin [18] and Theorem 4 of Chan and Lai [4], we have $r(\cdot; \theta)$ is uniformly positive, bounded and analytic on J for each $w \in \mathcal{W}$. For

 $\theta \in J$, define the twisting transformation for the transition probability of $\{W_n, n \ge 0\}$ as

$$\mathbb{P}^{\theta}(w, dw') = \frac{r(w'; \theta)}{r(w; \theta)} e^{-\Lambda(\theta) + \theta g(W_0, W_1)} \mathbb{P}(w, dw').$$
(4.18)

For given $\theta \in J \subset \Theta \subset \mathbf{R}$, let $\{W_n^{\theta}, n \geq 0\}$ be the Markov chain with transition kernel \mathbb{P}^{θ} and invariant probability Π^{θ} . If the function $\Lambda(\theta)$ is normalized so that $\Lambda(0) = \Lambda'(0) = 0$, then $\mathbb{P} = \mathbb{P}^0$ is the transition probability of the Markov chain $\{W_n, n \geq 0\}$, with invariant probability $\Pi = \Pi^0$.

By making use of (4.18) and repeat the same idea as (4.14), we have representations for

$$LR_n^k(\theta) = \exp\bigg(\sum_{l=k+1}^n g(W_{l-1}^\theta, W_l^\theta)\bigg),\tag{4.19}$$

and

$$LR_{n}^{k}(F) = \int_{\theta \in J} \frac{r(W_{n};\theta)}{r(W_{k};\theta)} \exp\left\{-(n-k)\Lambda(\theta) + \theta \sum_{l=k+1}^{n} g(W_{l-1},W_{l})\right\} dF(\theta).$$
(4.20)

V. SECOND ORDER APPROXIMATION OF THE WEIGHTED SRP DETECTION RULE

By using the same idea as that in Pollak [21] and Fuh [8], we introduce a randomization on the initial $LR_n^0(\theta)$ for the Shiryayev-Roberts scheme, and call it the Shiryayev-Roberts-Pollak (SRP) change point detection rule in state space models. Before that, we need the following notations first.

Given $0 \leq k \leq n$, denote $\beta(W_{k-1}^{\theta}, W_k^{\theta}) = \exp\{g(W_{k-1}^{\theta}, W_k^{\theta})\}$. For 0 and <math>q = 1 - p, let

$$R_{n,p} := \sum_{k=1}^{n} \frac{1}{q} \frac{p_n(Y_k, Y_{k+1}, \dots, Y_n; \theta)}{p_n(Y_k, Y_{k+1}, \dots, Y_n; \theta_0)}$$
(5.1)
$$= \sum_{k=1}^{n} \frac{1}{q} \beta(W_{n-1}^{\theta}, W_n^{\theta}) \cdots \beta(W_{k-1}^{\theta}, W_k^{\theta}).$$

Note that $R_{n+1,p} = \beta(W_n^{\theta}, W_{n+1}^{\theta})\frac{1}{q}(1+R_{n,p})$. Define

$$\begin{split} N_{q,b} &= \inf\{n: R_{n,p} \geq B\} = \inf\{n: R_{n,p} \geq B(W_n)\}\\ H_n(y,w) &= \mathbb{P}_{\infty}\{R_{n,p} \leq y | N_{q,b} > n, W_n = w\},\\ \rho(t,y,w) &= \mathbb{P}_{\infty}\{R_{n+1,p} \leq y | R_{n,p} = t, N_{q,b} > n+1,\\ W_{n+1} = w\},\\ \zeta(t,w,w') &= \mathbb{P}_{\infty}\{N_{q,b} > n+1, W_{n+1} \in dw' | R_{n,p} = t,\\ N_{q,b} > n, W_n = w\}. \end{split}$$

For a given set of non-negative boundary points $B = \{B(w) : w \in W\}$ (infinity is not excluded), consider the set $S_B = \{(r, w) : w \in W, 0 < r < B(w)\}$. Let \mathcal{F}_B be the set of distribution functions with support in S_B . For given $H(\cdot, \cdot) \in \mathcal{F}_B$, let T_B be the transformation on \mathcal{F}_B defined by

$$T_B H(r, w) = \frac{1}{\mathbb{Q}(H)} \int_{w' \in \mathcal{W}} \int_0^{B(w')} \rho(t, r, w) \zeta(t, w', w)$$
$$dH(t, w') \mathbb{P}(w', dw), \tag{5.2}$$

where

$$\mathbb{Q}(H) = \int_{w,w'\in\mathcal{W}} \int_0^{B(w')} \zeta(t,w',w) dH(t,w') \mathbb{P}(w',dw).$$
(5.3)

The following proposition characterizes the behavior of T_B . *Proposition 1:* For each given B, we have $T_BH_n = H_{n+1}$. Therefore there associates a set of invariant measures Φ_B such that $T_B\phi = \phi$ for all $\phi \in \Phi_B$.

The proof of Proposition 1 is given in the Appendix.

By Proposition 1, we have that for each B there is an associated set of invariant measures Φ_B , i.e., $T_B\phi = \phi$ for all $\phi \in \Phi_B$. Define $\tilde{\phi}$ as

$$d\tilde{\phi}(y,w) = \frac{\int_{w'\in\mathcal{W}} (1+py)d\phi(y,w)\mathbb{P}(w,dw')}{\int_{w,w'\in\mathcal{W}} \int_0^{B(w')} (1+pt)d\phi(t,w)\mathbb{P}(w,dw')}$$

It is easy to see that if the distribution of $R_{0,p}$ is ϕ , then the distribution of $R_{0,p}$ conditional on $\{\omega > 0\}$ is ϕ . Note that ϕ depends on p. Let $0 < c < \infty$ and $0 be such that <math>N_{q,b}$ is the Bayes rule for B(0, p, c). By using the same argument as that in Theorem 4 of Fuh [8], we can choose a subsequence $\{T_B^i, p_i, c_i, \phi_i\}$ such that as $i \to \infty, p_i \to 0, c_i \to c^*$ and ϕ_i converges in distribution to a limit ψ .

Given the value of the initial state $W_0 = \tilde{w}$, the initial $R_0^*(\theta)$ is simulated from the distribution ψ , conditioned on the event $\{W_0 = \tilde{w}\}$. Define recursively

$$R_{n+1}^{*}(\theta) = \beta(W_{n}^{\theta}, W_{n+1}^{\theta})(1 + R_{n}^{*}(\theta)).$$
(5.4)

Let

$$R_n^*(F) = \int_{\theta \in J} R_n^*(\theta) dF(\theta).$$
(5.5)

Denote $b = \log B$, and define the weighted Shiryayev-Roberts-Pollak (SRP) rule as

$$N_b^{\psi} := \inf\{n : R_n^*(F) \ge B\} = \inf\{n : \log R_n^*(F) \ge b\}.$$
(5.6)

Note that each one of these detection policies (5.2) and (5.6) is an "equalizer rule" in the sense that

$$\mathbb{E}_{k}(N_{b}^{\psi} - k + 1 | N_{b}^{\psi} \ge k - 1) = \mathbb{E}_{1}N_{b}^{\psi},$$
(5.7)

for all k > 1. The same is true for the case where ψ has atoms on the boundary, since the randomization law is time independent. Note that the threshold of the Bayes rule (5.2) depends on the current state of the Markov chain, while the threshold of the SRP rule (5.6) is a constant. By using an argument similar to Lemma 7 of Fuh [8], we have that the difference between these two rules is o(1) as $p \to 0$ and $b \to \infty$.

Next, we will study asymptotic approximations for the average run length in the weighted SRP detection rule when w is finite. Since N_b^{ψ} is an equalizer rule, we only consider the approximation of $\mathbb{E}_1 N_b^{\psi}$. Given $\theta = \theta^0$ or $\theta \in J$, let π_{θ} denote the stationary distribution of $\{X_n, n \ge 0\}$ under P^{θ} . For given P^{θ_0} and P^{θ} and denote \mathbb{P}^{θ_0} and \mathbb{P}^{θ} as the induced probabilities, define the Kullback-Leibler information number

$$K(P^{\theta}, P^{\theta_0}) = K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}) = \mathbb{E}^{\theta} \left(\log \frac{\|\mathbf{P}_{\theta}(Y_1)\pi_{\theta}\|}{\|\mathbf{P}_{\theta_0}(Y_1)\pi_{\theta_0}\|} \right).$$
(5.8)

By assumption C3, we have $0 < K(P^{\theta}, P^{\theta_0}) < \infty$.

To derive a second-order approximation for the average run lengths of the weighted SRP rule, we will apply relevant results from nonlinear Markov renewal theory developed in Section 3 of Fuh [8]. To this end, we rewrite the stopping time $N_b := N_b^{\psi}$ (we delete ψ for simplicity) in the form of a Markov random walk crossing a constant threshold plus a nonlinear term that is slowly changing. Note that the stopping time N_b can be written in the following form

$$N_b = \inf\{n \ge 1 : \mathbb{S}_n + \eta_n \ge b\}, \quad b = \log B, \tag{5.9}$$

where for $n \ge 1$,

$$S_n = S_n(\theta)$$

$$= (\theta - \theta_0) \sum_{k=1}^n g(W_{k-1}, W_k) - n(\Lambda(\theta) - \Lambda(\theta_0)),$$
(5.10)

is a Markov random walk with mean $\mathbb{E}^{\theta}\mathbb{S}_1 = K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})$, and

$$\eta_{n} = \eta(\theta)$$

$$= \log \int_{\Theta} \frac{r(W_{n}; \alpha)}{r(W_{0}; \alpha)} \exp \left\{ (\alpha - \theta) \sum_{k=1}^{n} g(W_{k-1}, W_{k}) -n(\Lambda(\alpha) - \Lambda(\theta)) \right\} \left\{ 1 + \sum_{k=1}^{n} \exp(-\mathbb{S}_{k}(\alpha)) \right\} dF(\alpha).$$
(5.11)

Suppose there exists θ such that $\Lambda'(\theta) = \mathbb{E}_{\Pi}g(W_0, W_1)$, and denote $\hat{\theta}_n = \theta(\sum_{k=1}^n g(W_{k-1}, W_k)/n)$. Then η_n can be further decomposed as $l_n + V_n$, where

$$l_n = -\frac{1}{2}\log n, (5.12)$$

$$T_n = (\hat{\theta}_n - \theta) \sum_{k=1} g(W_{k-1}, W_k) - n(\Lambda(\hat{\theta}_n) - \Lambda(\theta)) \quad (5.13)$$

+
$$\log n^{1/2} \int_{\Theta} \frac{r(W_n; \alpha)}{r(W_0; \alpha)} \exp\left\{ (\alpha - \hat{\theta}_n) \sum_{k=1}^n g(W_{k-1}, W_k) - n(\Lambda(\alpha) - \Lambda(\hat{\theta}_n)) \right\} \left\{ 1 + \sum_{k=1}^n \exp(-\mathbb{S}_k(\alpha)) \right\} dF(\alpha)$$

:= $nK(\mathbb{P}^{\theta}, \mathbb{P}^{\hat{\theta}_n}) + \log u_n(\sum_{k=1}^n g(W_{k-1}, W_k)/n).$

For b > 0, define

V

$$N_b^* = \inf\{n \ge 1 : \mathbb{S}_n \ge b\},$$
 (5.14)

and let $R_b = \mathbb{S}_{N_b^*} - b$ (on $\{N_b^* < \infty\}$) denote the overshoot of the statistic \mathbb{S}_n crossing the threshold b at time $n = N_b^*$. When b = 0, we denote N_b^* in (5.14) as N_+^* . For given $\tilde{w} :=$ $(x_0, \pi) \in \mathcal{W}$, with x_0 is the initial state of X_0 taken from $\pi(x_0)$, let

$$G(u) = \lim_{b \to \infty} \mathbb{P}^{\theta} \{ R_b \le u | W_0 = \tilde{y} \}$$
(5.15)

be the limiting distribution of the overshoot. It is known [cf. Theorem 1 of Fuh [7]] that

$$\lim_{b \to \infty} \mathbb{E}^{\theta}(R_b | W_0 = \tilde{w}) = \int_0^\infty u dG(u) = \frac{\mathbb{E}_{m_+}^{\theta} S_{N_+}^{2*}}{2\mathbb{E}_{m_+}^{\theta} S_{N_+}^{**}}, (5.16)$$

where $m_+ := m_+^{\theta}$ is defined in the same way as π_+^{θ} defined in Section 3 of Fuh [8].

Note that by (5.9), we have

$$\mathbb{S}_{N_b} = b - \eta_{N_b} + O_b \quad \text{ on } \{N_b < \infty\}, \tag{5.17}$$

where $O_b = S_{N_b} + \eta_{N_b} - b$ is the overshoot of $S_n + \eta_n$ crossing the boundary *b* at time N_b . Taking the expectations on both sides of (5.17), and applying Wald's identity for Markov random walks [cf. Corollary 1 of Fuh and Zhang [11]], we obtain

$$K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_{0}})\mathbb{E}^{\theta}(N_{b}|W_{0} = \tilde{w}) + \int_{\mathcal{W}} \Delta_{\theta}(w)m_{+}^{\theta}(dw) - \Delta_{\theta}(\tilde{w}) = \mathbb{E}^{\theta}(\mathbb{S}_{N_{b}}|W_{0} = \tilde{w}) = b - \mathbb{E}^{\theta}(\eta_{N_{b}}|W_{0} = \tilde{w}) + \mathbb{E}^{\theta}(O_{b}|W_{0} = \tilde{w}),$$
(5.18)

where $\Delta_{\theta} : \mathcal{W} \to \mathbf{R}^d$ solves the Poisson equation

$$\mathbb{E}_{w}^{\theta} \Delta_{\theta}(W_{1}) - \Delta_{\theta}(w) = \mathbb{E}_{w}^{\theta} \mathbb{S}_{1} - \mathbb{E}_{m_{+}}^{\theta} \mathbb{S}_{1}$$
(5.19)

for almost all $w \in \mathcal{W}$ with $\mathbb{E}_{m_{+}}^{\theta} \Delta_{\theta}(W_{1}) = 0$.

The crucial observations are that the sequence $\{V_n, n \ge 1\}$ is slowly changing, and that V_n converges in \mathbb{P}^{θ} -distribution, as $n \to \infty$, to the random variable

$$\tilde{V}$$

$$= \frac{1}{2}\chi_1^2 + \frac{1}{2}\log\frac{2\pi F'(\theta)}{\Lambda''(\theta)} + \log\left\{1 + \sum_{k=1}^{\infty}\exp(-\mathbb{S}_k(\theta))\right\}$$

$$+ \log\left\{\frac{\mathbb{E}_{m_+}^{\theta}r(W_{N_+^*};\theta)}{r(W_0;\theta)}\right\},$$
(5.20)

where χ_1^2 denotes a random variable having the chi-squared distribution with one degree of freedom.

Denote $\gamma_{\theta} = \log \{1 + \sum_{k=1}^{\infty} \exp(-\mathbb{S}_k(\theta))\}\)$, we will show in Section 6 that $\mathbb{E}_{m_+}^{\theta} \gamma_{\theta} < \infty$. Here the expectation $\mathbb{E}_{m_+}^{\theta}$ is taken under $\omega = 1$ and the initial distribution of Y_0 is m_+ , we omit 1 for simplicity. An important consequence of the slowly changing property is that, under mild conditions, the limiting distribution of the overshoot of a Markov random walk over a fixed threshold does not change by the addition of a slowly changing nonlinear term [cf. Theorem 1 in Section 3 of Fuh [8]]. More importantly, nonlinear Markov renewal theory allows us to obtain an asymptotically accurate approximation for $\mathbb{E}N_b$, that takes the overshoot into account. Now we can characterize the constant $C(\theta)$ appeared in Theorem 1,

$$C(\theta) = \frac{\mathbb{E}_{m_{+}}^{\theta} S_{N_{+}}^{2}}{2\mathbb{E}_{m_{+}}^{\theta} S_{N_{+}}^{*}} - \mathbb{E}_{m_{+}}^{\theta} \gamma_{\theta} - \frac{1}{2} \log \frac{2\pi F'(\theta)}{\Lambda''(\theta)} - \frac{1}{2} (5.21)$$
$$(\int_{\mathcal{W}} \Delta(w) m_{+}^{\theta}(dw) - \Delta(\tilde{w})) - \log \left\{ \frac{\mathbb{E}_{m_{+}}^{\theta} r(Y_{N_{+}}^{*};\theta)}{r(Y_{0};\theta)} \right\}.$$

When $\theta = \theta_1$ is known, we have the following approximation of the average run length. Since the proof is similar to that of Theorem 6 in Fuh [8], we will not repeat it here.

Proposition 2: Let Y_1, \ldots, Y_n be a sequence of random variables from a state space model $\{Y_n, n \ge 1\}$ satisfying

conditions C1-C4. Assume that S_1 is nonarithmetic with respect to P_{∞} and P_1 . Then for $\tilde{w} \in \mathcal{W}$, as $b \to \infty$

$$\mathbb{E}_{1}(N_{b}|W_{0} = \tilde{w})$$

$$= \frac{1}{K(\mathbb{P}^{\theta_{1}}, \mathbb{P}^{\theta_{0}})} \left(b - \mathbb{E}_{m_{+}}\gamma + \frac{\mathbb{E}_{m_{+}}S_{N_{+}^{*}}^{2}}{2\mathbb{E}_{m_{+}}S_{N_{+}^{*}}} - \int_{\mathcal{W}}\Delta(w)m_{+}(dw) + \Delta(\tilde{w})\right) + o(1).$$
(5.22)

VI. PROOF OF THEOREM 1

To prove Theorem 1, without loss of generality, we assume that $J = \Theta = [\theta_0, \theta_1] \subset \mathbf{R}$ and $\theta \geq 0$. Note that the proof of (2.21) rests on the nonlinear Markov renewal theory from Theorem 3 and Corollary 1 in Fuh [8]. Indeed, by (5.9), the stopping time N_h^{ψ} is based on the thresholding of the sum of the Markov random walk \mathbb{S}_n and the nonlinear term η_n . From (5.12) and (5.13), we have $\eta_n = l_n + V_n$, with $l_n = -(1/2) \log n$. It is easy to see that $\lim_{n \to \infty} \max_{0 \le j \le \sqrt{n}} |-(1/2)\log(n+j) + (1/2)\log n| = 0.$ In order to apply Theorem 3 and Corollary 1 in Fuh [8], we need to check the validity of the conditions which are stated in the following lemmas, respectively. Relation (2.21) will then follow by specialization. Note that although the nonlinear Markov renewal theory developed in Fuh [8] is under the condition of w-uniformly ergodic, it can be generalized to the norm in Definition 3. A heuristic explanation of this result can be described as follows: we first investigate the difference between a stopping time crossing nonlinear boundaries and a stopping time crossing linear boundaries with varying drift, then derive nonlinear Markov renewal theory directly from parallel results in the linear case with varying drift via the uniform integrabilities and the weak convergence of the overshoot. Because the uniform Markov renewal theory developed in Fuh [7] is under a general norm, therefore the extension of the proofs in Fuh [8] is straightforward. The details are omitted.

In the proof of the following lemmas, we will assume the conditions of Theorem 1 hold. We first consider the case that F is concentrated on $[\theta_0, \theta_1]$, where $0 < \theta_0 < \theta < \theta_1 < \infty$ are such that $\alpha \Lambda'(\alpha) - \Lambda(\alpha) > 0$ for $\theta_0 \le \alpha \le \theta_1$ and F has a derivative F' which is positive and continuous on $[\theta_0, \theta_1]$. The probability \mathbb{P}_1 and expectation \mathbb{E}_1 in this section are taken under $Y_0 = \tilde{y}$, and we omit it for simplicity.

Lemma 1: Under assumptions of Theorem 1, $\mathbb{S}_1 = \mathbb{S}_1(\theta) = (\theta - \theta_0)g(Y_0, Y_1) - (\Lambda(\theta) - \Lambda(\theta_0))$ has a nonarithmetic distribution.

PROOF. Suppose the $\mathbb{S}_1(\theta)$ has an arithmetic distribution for some $\theta \neq \theta_0$, say $\theta = \theta^*$, and let d_1 be the span of $\mathbb{S}_1(\theta^*)$. Then $g(Y_0, Y_1)$ must take values of the form $(\frac{1}{\theta^* - \theta_0})\{kd_1 + [\Lambda(\theta^*) - \Lambda(\theta_0)]\} = dk + \gamma$, say, where $k = 0, \pm 1, \pm 2, \ldots$. Moreover, since $g(Y_0, Y_1)$ is assumed to have a nondegenerate distribution, there are $k_1 \neq k_2$ for which $\mathbb{P}_y\{dk_1 + \gamma\} > 0 < \mathbb{P}_y\{dk_2 + \gamma\}$ for all $y \in \mathcal{Y}$. Now suppose that $\mathbb{S}_1(\theta)$ has an arithmetic distribution for some θ with $\theta_0 \neq \theta \neq \theta^*$ and let $d(\theta) > 0$ denote the span of $\mathbb{S}_1(\theta)$. Then there are j_1 and j_2 for which $j_1 \neq j_2$ and $j_i d(\theta) = (\theta - \theta_0)(dk_i + \gamma) - [\Lambda(\theta) - \Lambda(\theta_0)], i = 1, 2$. Therefore

$$\frac{\Lambda(\theta) - \Lambda(\theta_0)}{\theta - \theta_0} = \gamma + d(\frac{k_1 j_2 - k_2 j_1}{j_2 - j_1}). \tag{6.1}$$

Thus, the set of θ for which $\theta_0 \neq \theta \neq \theta^*$ and $\mathbb{S}_1(\theta)$ has an arithmetic distribution is contained in the set of θ for which (6.1) holds for some $j_1 \neq j_2$; the latter set is countable, since Λ is convex. \Box

Lemma 2: Under assumptions of Theorem 1, we have

$$\sum_{n=1}^{\infty} \mathbb{P}_1\{V_n \le -\varepsilon n\} < \infty \text{ for some } 0 < \varepsilon < K(\mathbb{P}^{\theta_1}, \mathbb{P}^{\theta_0}).$$
(6.2)

Condition (6.2) holds trivially because $r(y; \theta)$ is uniformly positive and hence $V_n \ge 0$.

Lemma 3: Under assumptions of Theorem 1, then

 $\max_{0 \le l \le n} |V_{n+l}|, \ n \ge 1, \ \text{are } \mathbb{P}_1 - uniformly \ integrable. (6.3)$

PROOF. To show (6.3) holds, we first prove

$$\max_{0 \le l \le n} (\hat{\theta}_{n+l} - \theta) \sum_{k=1}^{n+l} g(Y_{k-1}, Y_k) - (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (6.4)^{\text{xists } 2} d\theta_{n+l} + (n+l) (\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta)) (\theta_{n+l}) + (n+l) (\Lambda(\hat{\theta}_{n+l}) - (n+l) (\Lambda(\theta_{n+l}) - \Lambda(\theta))) (\theta_{n+l}) + (n+l) (\Lambda(\theta_{n+l}) - (n+l) (\Lambda$$

are \mathbb{P}_1 -uniformly integrable, where θ_{n+l} is the maximum likelihood estimator of θ . Note that on the event A_n of $|(1/n)\sum_{k=1}^n g(Y_{k-1},Y_k) - \Lambda'(\theta)| < \varepsilon$ for some $\varepsilon > 0$, we have for all $n \ge 1$,

$$(\hat{\theta}_n - \theta) \sum_{k=1}^n g(Y_{k-1}, Y_k) - n(\Lambda(\hat{\theta}_n) - \Lambda(\theta))$$

$$\leq Bn\left(\frac{1}{n} \sum_{k=1}^n g(Y_{k-1}, Y_k) - \Lambda'(\theta)\right)^2$$

on A_n , for some constant B. Therefore,

$$\mathbb{P}_{1}\left\{\max_{0\leq l\leq n}\left\{\left(\hat{\theta}_{n+l}-\theta\right)\sum_{k=1}^{n+l}g(Y_{k-1},Y_{k})-(n+l)(\Lambda(\hat{\theta}_{n+l})-\Lambda(\theta))\right\}>a\right\}$$

$$(6.5)$$

$$\leq \mathbb{P}_1\bigg\{\max_{0\leq l\leq 2n}l\big|\frac{1}{l}\sum_{k=1}^{l}g(Y_{k-1},Y_k)-\Lambda'(\theta)\big|>\sqrt{Bna}\bigg\}.$$

Since conditions of Theorem 2 imply that conditions of Theorem 2 in Fuh and Zhang [11] hold, we have that for all $\varepsilon > 0$ and $r \ge 0$

$$\sum_{n=1}^{\infty} n^{r-1} \mathbb{P}_1 \left\{ \max_{1 \le l \le n} (\mathbb{S}_l - \Lambda(\theta)l) \ge \varepsilon n \right\} < \infty.$$
 (6.6)

Hence $(6.6) \leq Ca^{-r}$, for some C > 0 and r > 1. This imply (6.4) hold.

Denote

$$H_n$$

$$= n^{1/2} \int_{\Theta} \frac{r(Y_n; \alpha)}{r(Y_0; \alpha)} \exp\left\{ (\alpha - \hat{\theta}_n) \sum_{k=1}^n g(Y_{k-1}, Y_k) -n(\Lambda(\alpha) - \Lambda(\hat{\theta}_n)) \right\} \times \left\{ 1 + \sum_{k=1}^{n-1} \exp(-\mathbb{S}_k(\alpha)) \right\} dF(\alpha).$$
(6.7)

To complete the proof, we need to show that $\max_{0 \le l \le n} H_{n+l}$ are \mathbb{P}_1 -uniformly integrable. First, we note that $(\hat{\theta}_{n+l} - \theta) \sum_{k=1}^{n+l} g(Y_{k-1}, Y_k) - (n+l)(\Lambda(\hat{\theta}_{n+l}) - \Lambda(\theta))$ are uniformly bounded on A_n and $0 < r(y; \theta) < \infty$ uniformly for $y \in \mathcal{Y}$ by Theorem 4 of Chan and Lai [4].

To analyze the term appeared in (6.8), denote $W_{\alpha}^{n} = 1 + \sum_{k=1}^{n-1} \exp(-\mathbb{S}_{k}(\alpha))$, for $\theta_{0} \leq \alpha \leq \theta_{1}$. Note that W_{α}^{n} converges \mathbb{P}_{1}^{θ} -a.s. as $n \to \infty$ to a random variable $W_{\alpha}^{\theta} := 1 + \sum_{k=1}^{\infty} \exp(-\mathbb{S}_{k}(\alpha))$. Since

$$\sum_{n=m}^{\infty} (W_{\alpha}^{n+1} - W_{\alpha}^{n})$$

=
$$\sum_{n=m}^{\infty} \exp\left\{-\left[\alpha \sum_{k=1}^{n} g(X_{k-1}, X_{k}) - n\Lambda(\alpha)\right]\right\} \rightarrow_{m \to \infty} 0$$

 $\mathbb{P}_1^{\theta} - a.s.$, uniformly in $\alpha \in [\theta_0, \theta_1]$, it follows that W_{α}^{θ} is $\mathbb{P}_1^{\theta} - a.s.$ continuous in $\alpha \in [\theta_0, \theta_1]$.

Next we will show, which is more than it suffices, that there exists a constant a > 0 such that

$$\mathbb{E}_{1}^{\theta}\left(\int_{\theta_{0}}^{\theta_{1}}\left\{1+\sum_{k=1}^{\infty}\exp(-\mathbb{S}_{k}(\alpha))\right\}dF(\alpha)\right)^{a}<\infty.$$
 (6.8)

For given $\varepsilon > 0$, let $\Gamma = \min\{n || \sum_{k=1}^{m} g(Y_{k-1}, Y_k)/m - \Lambda'(\theta)| \le \varepsilon$ for all $m \ge n\}$. Suppose that ε is chosen small enough so that there exists $\beta > 0$ such that $\mathbb{S}_n(\alpha) \ge \beta n$ if $n \ge \Gamma$ for all $\theta_0 \le \alpha \le \theta_1$. There exists a constant $\eta > 0$ such that $|\Lambda(\theta - \alpha) + \Lambda(\alpha) - \Lambda(\theta)| < \eta$ for all $\theta_0 \le \alpha \le \theta_1$. By using the large deviation result for Markov random walks (cf. Ney and Nummelin [18]) we can choose a constant $\delta > 0$ such that $\mathbb{P}_1^{\theta}(\Gamma = \lambda) \le \exp\{-\delta\lambda\}$. Furthermore we choose 1 > a > 0 such that $a\eta - \delta(1 - a) < 0$. Now

$$\int_{\theta_{0}}^{\theta_{1}} W_{\alpha}^{\theta} dF(\alpha)$$

$$= \int_{\theta_{0}}^{\theta_{1}} \left(1 + \sum_{k=1}^{\Gamma-1} \exp(-\mathbb{S}_{k}(\alpha)) + \sum_{k=\Gamma}^{\infty} \exp(-\mathbb{S}_{k}(\alpha)) \right) dF(\alpha)$$

$$\leq \int_{\theta_{0}}^{\theta_{1}} \left(1 + \sum_{k=1}^{\Gamma-1} \exp(-\mathbb{S}_{k}(\alpha)) + \frac{1}{1 - e^{-\beta}} \right) dF(\alpha).$$
(6.9)

To evaluate the second term in the integrand of (6.9), we have

$$\begin{split} & \mathbb{E}_{1}^{\theta} \left(\int_{\theta_{0}}^{\theta_{1}} \sum_{k=1}^{b-1} \exp(-\mathbb{S}_{k}(\alpha)) dF(\alpha) \middle| \Gamma = b \right) \\ & \leq \quad \frac{\mathbb{E}_{1}^{\theta} \int_{\theta_{0}}^{\theta_{1}} \sum_{k=1}^{b-1} \exp(-\mathbb{S}_{k}(\alpha)) dF(\alpha)}{\mathbb{P}_{1}^{\theta}(\Gamma = b)} \\ & = \quad \frac{1}{\mathbb{P}_{1}^{\theta}(\Gamma = b)} \int_{\theta_{0}}^{\theta_{1}} \sum_{k=1}^{b-1} \{ e^{[\Lambda(\theta - \alpha) + \Lambda(\alpha) - \Lambda(\theta)]k} + O(\rho^{k}) \} dF(\alpha) \\ & \leq \quad \frac{1}{\mathbb{P}_{1}^{\theta}(\Gamma = b)} \left(\frac{1}{\eta} e^{\eta b} + \frac{\rho^{2}}{1 - \rho} \right), \end{split}$$

where $0 < \rho < 1$. By Jensen's inequality,

$$\mathbb{E}_{1}^{\theta} \left(\int_{\theta_{0}}^{\theta_{1}} W_{\alpha}^{\theta} dF(\alpha) \right)^{a}$$

$$= \mathbb{E}_{1}^{\theta} \left(\mathbb{E}_{1}^{\theta} \left[\left(\int_{\theta_{0}}^{\theta_{1}} W_{\alpha}^{\theta} dF(\alpha) \right)^{a} \middle| \Gamma \right] \right)$$

$$\leq \sum_{b=1}^{\infty} \left(\frac{1}{\mathbb{P}_{1}^{\theta}(\Gamma = b)} \left(\frac{1}{\eta} e^{\eta b} + \frac{\rho^{2}}{1 - \rho} \right) + \frac{2 - e^{-\beta}}{1 - e^{-\beta}} \right)^{a}$$

$$\mathbb{P}_{1}^{\theta}(\Gamma = b).$$
(6.10)

The inequality (6.8) now follows because there exist constants C_1, \dots, C_a such that

$$\begin{split} &\sum_{b=1}^{\infty} \left(\frac{1}{\mathbb{P}_{1}^{\theta}(\Gamma=b)} \left(\frac{e^{\eta b}}{\eta} + \frac{\rho^{2}}{1-\rho} \right) \right)^{a} \mathbb{P}_{1}^{\theta}(\Gamma=b) \\ &= \sum_{b=1}^{\infty} \left(\frac{e^{\eta b}}{\eta} + \frac{\rho^{2}}{1-\rho} \right)^{a} [\mathbb{P}_{1}^{\theta}(\Gamma=b)]^{1-a} \\ &\leq \frac{1}{\eta^{a}} \sum_{b=1}^{\infty} e^{b(a\eta-\delta(1-a))} + C_{1} \frac{1}{\eta^{a}} \sum_{b=1}^{\infty} e^{b((a-1)\eta-\delta(1-a))} + \cdots \\ &+ C_{a} \frac{1}{\eta^{a}} \sum_{b=1}^{\infty} e^{b(\eta-\delta(1-a))} < \infty. \quad \Box \end{split}$$

Lemma 4: Let V_n be defined in (5.13) and \tilde{V} be defined in (5.20). Then under assumptions of Theorem 1, we have

$$V_n \longrightarrow_{n \to \infty} V \quad in \mathbb{P}_1\text{-distribution} \quad (6.11)$$

and $\mathbb{E}_1 V_n \longrightarrow_{n \to \infty} \mathbb{E}_1 \tilde{V}.$

PROOF. Let $A_n = \{ | \sum_{k=1}^n g(Y_{k-1}, Y_k)/n - \Lambda'(\theta) | < \varepsilon \}$. Then by using a result of large deviations in Markov random walks [cf. Ney and Nummelin [18]], there exists a $\delta > 0$ such that $\mathbb{P}_1\{A_n^c\} \le \delta$. Let θ be defined such that $\Lambda'(\theta) = \mathbb{E}_{\Pi}g(Y_0, Y_1)$. Under the event A_n , the maximum likelihood estimate $\hat{\theta}_n = \theta(\sum_{k=1}^n g(Y_{k-1}, Y_k)/n)$ is well defined. Recall $\eta_n = l_n + V_n$, where η_n is defined in (5.11), $l_n = (-1/2) \log n$, and

$$V_{n} \qquad (6.12)$$

$$= (\hat{\theta}_{n} - \theta) \sum_{k=1}^{n} g(Y_{k-1}, Y_{k}) - n(\Lambda(\hat{\theta}_{n}) - \Lambda(\theta))$$

$$+ \log n^{1/2} \int_{\Theta} \frac{r(Y_{n}; \alpha)}{r(Y_{0}; \alpha)}$$

$$\times \exp\left\{ (\alpha - \hat{\theta}_{n}) \sum_{k=1}^{n} g(Y_{k-1}, Y_{k}) - n(\Lambda(\alpha) - \Lambda(\hat{\theta}_{n})) \right\}$$

$$\left\{ 1 + \sum_{k=1}^{n} \exp(-\mathbb{S}_{k}(\alpha)) \right\} dF(\alpha)$$

$$:= nK(\mathbb{P}^{\theta}, \mathbb{P}^{\hat{\theta}_{n}}) + \log u_{n}(\sum_{k=1}^{n} g(Y_{k-1}, Y_{k})/n).$$

We first analyze the second term in (6.12) and show that for any $\theta \in \Theta$

$$\log u_n \left(\sum_{k=1}^n g(Y_{k-1}, Y_k)/n\right)$$

$$\to \frac{1}{2} \log \frac{2\pi F'(\theta)}{\Lambda''(\theta)} + \log \left\{ 1 + \sum_{k=1}^\infty \exp(-\mathbb{S}_k(\theta)) \right\}$$

$$+ \log \left\{ \frac{\mathbb{E}_{m_+}^\theta r(Y_{N_+^*}; \theta)}{r(Y_0; \theta)} \right\},$$
(6.13)

 \mathbb{P}_1^{θ} -a.s. as $n \to \infty$.

To complete the proof of (6.13). By (6.8), and $0 < r(y; \alpha) < \infty$ uniformly for $y \in \mathcal{Y}$ for all $\alpha \in \Theta$ via Theorem 4 of Chan and Lai [4], we need only to show that

$$\log n^{1/2} \int_{\Theta} \exp\left\{ (\alpha - \hat{\theta}_n) \sum_{k=1}^n g(Y_{k-1}, Y_k) - n(\Lambda(\alpha)) - \Lambda(\hat{\theta}_n) \right\} dF(\alpha)$$

$$\frac{1}{2} \log \frac{2\pi F'(\theta)}{\Lambda''(\theta)} \quad \mathbb{P}_1^{\theta} - a.s. \text{ as } n \to \infty,$$
(6.14)

For given $\alpha \in \Theta$ and $y \in \mathbf{R}$, let

$$H(\alpha, y) = (\Lambda(\alpha) - \Lambda(\hat{\theta}_n)) - (\alpha - \hat{\theta}_n) \frac{1}{n} \sum_{k=1}^n g(Y_{k-1}, Y_k).$$

Then

$$(6.14) = \int_{\Theta} \exp[-nH(\alpha, y)] dF(\alpha), \qquad y \in \mathbf{R}.$$
 (6.15)

Observe that $H(\alpha, y)$ is convex in Θ for fixed $y \in \mathbf{R}$, since Λ is convex. Moreover, for fixed y, $H(\alpha, y) = \frac{1}{2}\Lambda''(\alpha^*)(\alpha - \hat{\theta}_n)^2$, where $\alpha^* = \alpha^*(\alpha, y)$ is an intermediate point between α and $\hat{\theta}_n$. Let K be any compact subinterval of \mathbf{R} . Then there are a $\sigma > 0$ and a compact $J \subset \Theta$ for which $[\hat{\theta}_n - \delta, \hat{\theta}_n + \delta] \subset J$ for all $y \in K$ and, since Λ'' is positive and continuous, there is an $\varepsilon > 0$ for which $\Lambda''(\alpha^*) \ge \varepsilon$ for $|\alpha - \hat{\theta}_n| \le \delta$ and $y \in K$. In particular, $H(\alpha, y) \ge \frac{1}{2}\varepsilon(\alpha - \hat{\theta}_n)^2$ for $|\alpha - \hat{\theta}_n| \le \delta$ and $y \in K$. Since H is convex in α for fixed y, it follows that $H(\alpha, y) \ge \frac{1}{2}\varepsilon\delta^2$ for $|\alpha - \hat{\theta}_n| \ge \delta$ and $y \in K$ and, consequently, that

)
$$\int_{|\alpha - \hat{\theta}_n| \ge \delta} e^{-nH} dF(\alpha) \le e^{-\varepsilon \delta^2 n/2}, \ y \in K, \ n \ge 1.$$
(6.16)

Next, consider the change of variables $\hat{\theta}_n = \hat{\theta}_n + n^{-1/2} \alpha$ shows that

$$\sqrt{n} \int_{|\alpha - \hat{\theta}_n| < \delta} \exp\{-nH\} dF(\alpha)$$

$$= \int_{-\delta\sqrt{n}}^{\delta\sqrt{n}} \exp[-\frac{1}{2}\Lambda''(\alpha_n^*)\alpha^2] F'(\hat{\theta}_n + \frac{\alpha}{\sqrt{n}}) d\alpha,$$
(6.17)

where $\alpha_n^* = \alpha^*(\hat{\theta}_n + n^{-1/2}\alpha, y_n), n \ge 1$. As $n \to \infty$, the integrand on the right side of (6.17) converges to $\exp[-\frac{1}{2}\Lambda''(\theta)\alpha^2]F'(\theta)$; and the integrand is dominated by

 $C \exp(-\frac{1}{2}\varepsilon\alpha^2)$ for some C. So, the right hand side of (6.17) converges to

$$\int_{-\infty}^{\infty} \exp[-\frac{1}{2}\Lambda''(\theta)\alpha^2]F'(\theta)d\alpha = \sqrt{\frac{2\pi}{\Lambda''(\theta)}}F'(\theta) \quad (6.18)$$

by the dominated convergence theorem.

Finally, using Theorem 17.2.2 of Meyn and Tweedie [17], we have as $n \to \infty$,

$$(\hat{\theta}_n - \theta) \sum_{k=1}^n g(Y_{k-1}, Y_k) - n(\Lambda(\hat{\theta}_n) - \Lambda(\theta)) \longrightarrow \chi_1^2, (6.19)$$

where χ_1^2 is a random variable with chi-squared distribution with one degree of freedom.

Combining (6.13) and (6.19), we have the proof. \Box Lemma 5: Under assumptions of Theorem 1, we have for some $0 < \varepsilon < 1$,

$$\lim_{b \to \infty} b \mathbb{P}_1 \left\{ N_b^{\psi} \le \frac{\varepsilon b}{K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})} \right\} = 0.$$
 (6.20)

PROOF. By using $\mathbb{E}_1 g(Y_0, Y_1) > 0$, and $0 < K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}) < \infty$, we will prove that

$$\mathbb{P}_1\left\{N_b^{\psi} < \frac{(1-\varepsilon)b}{K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})}\right\} \le e^{-y_{\varepsilon}b} + \alpha_1(\varepsilon, b), \qquad (6.21)$$

where $y_{\varepsilon} > 0$ for all $\varepsilon > 0$, and

$$\alpha_1(\varepsilon, b) = \mathbb{P}_1 \left\{ \max_{1 \le n < K_{\varepsilon, b}} \mathbb{S}_n \ge (1 + \varepsilon)(1 - \varepsilon)b \right\} (6.22)$$

$$K_{\varepsilon, b} = \frac{(1 - \varepsilon)b}{K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})}.$$

If (6.21) is correct, then the first term on the right hand side of (6.21) is o(1/b) as $b \to \infty$. All it remains to do is to show that $\alpha_1(\varepsilon, b)$ in (6.22) is o(1/b).

Note that Theorem 1 implies conditions of Theorem 2 in Fuh and Zhang [11] hold. Hence for all $\varepsilon > 0$ and $r \ge 0$

$$\sum_{n=1}^{\infty} n^{r-1} \mathbb{P}_1 \left\{ \max_{1 \le k \le n} (\mathbb{S}_k - K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})k) \ge \varepsilon n \right\} < \infty, (6.23)$$

whenever $\mathbb{E}_1|\mathbb{S}_1|^2 < \infty$ and $\mathbb{E}_1[(\mathbb{S}_1 - K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}))^+]^{r+1} < \infty$. Recall that under conditions of Theorem 1, $\mathbb{E}_1|\mathbb{S}_1|^2 < \infty$, and hence, the sum on the left hand side of the inequality (6.23) is finite for r = 1 and all $\varepsilon > 0$, which implies that the summand should be o(1/n). Since $\alpha_1(\varepsilon, b) \leq \mathbb{P}_1\left\{\max_{n < K_{\varepsilon,b}}(\mathbb{S}_n - K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})n) \geq \varepsilon(1 - \varepsilon)b\right\}$, it follows that $\alpha_1(\varepsilon, b) = o(1/b)$.

Next, we need to prove (6.21). We only consider the case that $\theta_0 < \theta$, as the other case can be done by using a similar way. Denote $\mathbb{S}_n^k = \log LR_n^k$, and let $N = N_b^{\psi}$ for simplicity. Let $I\{\cdot\}$ be the indicator function. Recall from (4.18), we have

$$= \frac{\mathbb{P}^{\theta_0}(y, dz)}{r(z; \theta_0)} \frac{r(y; \theta)}{r(z; \theta)} e^{-(\Lambda(\theta_0) - \Lambda(\theta)) + (\theta_0 - \theta)g(Y_0, Y_1)} \mathbb{P}^{\theta}(y, dz)$$

By Proposition 1, for all $\theta \in \Theta$ $0 < r(z; \theta) < \infty$ uniformly for $z \in \mathcal{Y}$. For any C > 0, by using a change of measure argument, we have

$$\mathbb{P}_{\infty}\left\{N < (1-\varepsilon)bK(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_{0}})^{-1}\right\}$$

$$= \mathbb{E}_{1}\left\{I\{N < K_{\varepsilon,b}\}\frac{r(Y_{N};\theta_{0})}{r(Y_{k};\theta_{0})}\frac{r(Y_{k};\theta)}{r(Y_{N};\theta)}e^{-(\Lambda(\theta_{0})-\Lambda(\theta))+(\theta_{0}-\theta)\mathbb{S}_{N}^{k}}\right\}$$

$$\geq K\mathbb{E}_{1}\left\{I\{N < K_{\varepsilon,b}, \mathbb{S}_{N}^{k} < C\}\exp(-k\mathbb{S}_{N}^{k})\right\}$$

$$\geq e^{-kC}\mathbb{P}_{1}\left\{N < K_{\varepsilon,b}, \max_{n < K_{\varepsilon,b}}\mathbb{S}_{n}^{k} < C\right\}$$

$$\geq e^{-kC}\left[\mathbb{P}_{1}\left\{N < K_{\varepsilon,b}\right\} - \mathbb{P}_{1}\left\{\max_{n < K_{\varepsilon,b}}\mathbb{S}_{n}^{k} \ge C\right\}\right],$$

where K > 0 is a constant such that $\left|\frac{r(Y_N;\theta_0)}{r(Y_k;\theta_0)}\frac{r(Y_k;\theta)}{r(Y_N;\theta)}\right| > K$, and $k = \theta - \theta_0 > 0$. Choosing $kC \le (1 + \varepsilon)(1 - \varepsilon)b$, then, we have

$$\mathbb{P}_{1}\left\{N < \frac{(1-\varepsilon)b}{K(\mathbb{P}^{\theta},\mathbb{P}^{\theta_{0}})}\right\}$$

$$\leq e^{kC}\mathbb{P}_{\infty}\left\{N < (1-\varepsilon)bK(\mathbb{P}^{\theta},\mathbb{P}^{\theta_{0}})^{-1}\right\} + \alpha_{1}(\varepsilon,b).$$
(6.24)

Recall that $R_n^*(F)$ is defined in (5.5). Note that under the condition of $0 < K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}) < \infty$, we have $\mathbb{P}_{\infty}\left\{N < K_{\varepsilon,b}\right\} = \sum_{i=1}^{K_{\varepsilon,b}} \mathbb{P}_{\infty}\left\{R_i^*(F) > B\right\} \leq \sum_{i=1}^{K_{\varepsilon,b}} \frac{i}{B} \leq \frac{(\log B)^2}{(K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}))^2 B}$. By letting a suitable kC, we have the first term on the right hand side of (6.24) $\leq e^{-y_{\varepsilon}b}$, for some $y_{\varepsilon} > 0$, and get the proof of (6.21). \Box

Next we consider the case that F is a measure on the real line. Assume there exist constants $0 < c < K(\mathbb{P}^{\theta_1}, \mathbb{P}^{\theta_0})/2$, w > 0, and $0 < \theta_0 < \theta < \theta_1 < \infty$ such that $\alpha \Lambda'(\theta) - \Lambda(\alpha) > 0$ for $\alpha \in [\theta_0, \theta_1]$, $\max\{\alpha \Lambda'(\theta - w) - \Lambda(\alpha), \alpha \Lambda'(\theta + w) - \Lambda(\alpha)\} < c$ for $\alpha \in [\theta_0, \theta_1]$, and $F(\alpha)$ has a derivative $F'(\alpha)$, which is positive and continuous for $\theta_0 \leq \alpha \leq \theta_1$. Since $\mathbb{P}_1^{\theta}\{N \geq (2 \log B)/K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})\}$ is arbitrarily small when B is large enough, and since for all C > 0, $\mathbb{E}_1^{\theta}(N|N > C) \leq C + (2 \log B)/K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})$ for large enough B, it suffices to show that

$$(\log B)\mathbb{P}_{1}^{\theta}\left\{\max_{n=1,\ldots,(2\log B)/K(\mathbb{P}^{\theta_{1}},\mathbb{P}^{\theta_{0}})}\int_{\Theta\setminus[\theta_{0},\theta_{1}]}(6.25)\right\}$$
$$\times\sum_{k=1}^{n}\exp\left(\alpha\sum_{i=k}^{n}g(Y_{i},Y_{i+1})-(n-k+1)\Lambda(\alpha)\right)dF(\alpha)$$
$$\geq\frac{4B}{\log B}\right\}\longrightarrow_{B\to\infty}0.$$

Since the proof of (6.25) follows directly as that in (39) of Pollak [22], we will not repeat it here.

Thus, by Lemmas 1-5, all conditions of Theorems 3 in Fuh [8] are satisfied, and so the proof of Theorem 1 is complete.

VII. PROOF OF THEOREM 2

To prove Theorem 2, we need the following lemmas first. Note that the probability and expected value are taken under \mathbb{P}_{ω} and \mathbb{E}_{ω} for $1 \leq \omega < \infty$, we delete ω for simplicity. Lemma 6: Under assumptions of Theorem 2. Let $0 < a \le b < \infty$ satisfy $\Lambda'(a) > \Lambda(b)/b$,

 $[a,b] \subset J$. For any c > 1 and probability measure G on [a,b], define

$$N(c; a, b, G)$$

$$= \inf \left\{ n | \int_{a}^{b} \frac{r(Y_{n}, \alpha)}{r(Y_{0}, \alpha)} \exp\{\alpha \mathbb{S}_{n} - n\Lambda(\alpha)\} dG(\alpha) \ge c \right\}.$$
(7.1)

Then there exist constants $0 < A, B < \infty$ independent of c, G such that

$$\mathbb{E}^{\theta} N(c; a, b, G) \le A \log c + B$$

for all $\theta \in [a, b]$ and c > 1.

PROOF. Define $M(\gamma) = \inf\{n|\frac{r(Y_n,\gamma)}{r(Y_0,\gamma)}\exp\{\gamma \mathbb{S}_n - n\Lambda(\gamma)\} \geq c\}$. It follows from a simple modification of Lemma 2 of Fuh [7] that there exists $0 < D < \infty$ such that $\mathbb{E}^{\theta}\{\mathbb{S}_{M(\gamma)} - [M(\gamma)\Lambda(\gamma) + \log c - \log \frac{r(Y_n,\alpha)}{r(Y_0,\alpha)}]/\gamma\} \leq D$ uniformly in $\theta \in [a, b], \gamma \in [a, b], c > 1$. Therefore, by Wald's identity for Markov random walks (cf. Fuh and Zhang [11]) that for all θ , $\gamma \in [a, b]$, there exists a constant C

$$\mathbb{E}^{\theta} M(\gamma) \tag{7.2}$$

$$\leq [(\log c - \log \frac{r(Y_n, \alpha)}{r(Y_0, \alpha)})/\gamma + D]/[\Lambda'(\theta) - \Lambda(\gamma)/\gamma + C]$$

$$\leq [(\log c - \log \frac{r(Y_n, \alpha)}{r(Y_0, \alpha)})/a + D]/[\Lambda'(a) - \Lambda(b)/b + C].$$

From $0 < r(y,\alpha) < \infty$ for all y via Proposition 1, and $\int_a^b \frac{r(Y_n,\alpha)}{r(Y_0,\alpha)} \exp\{\alpha \mathbb{S}_n - n\Lambda(\alpha)\} dG(\alpha) \geq \min(\frac{r(Y_n,a)}{r(Y_0,a)} \exp\{a \mathbb{S}_n - n\Lambda(a)\}, \frac{r(Y_n,b)}{r(Y_0,b)} \exp\{b \mathbb{S}_n - n\Lambda(b)\})$ it follows that $N(c; a, b, G) \leq$

 $\max(M(a), M(b)) \le M(a) + M(b)$. This and (7.2) complete the proof of Lemma 6.

Lemma 7: For given $0 < a \le b < \infty$, $[a, b] \subset J$, $\Lambda'(a) > \Lambda(b)/b$, let G be a probability on [a, b], and denote $F = \gamma F_0 + (1 - \gamma)G$, where F_0 is the probability measure wholly concentrated at $\{0\}$ and $\gamma \in (0, 1)$. Consider the optimal stopping problem defined by a prior distribution F on θ when Y_0, Y_1, Y_2, \ldots are a sequence of random variables from a state space model satisfying C1-C4. Assume each observation costs c > 0 if $\theta \neq \theta_0$, zero if $\theta = \theta_0$, with loss = 1 for stopping if $\theta = \theta_0$. Then there exists a constant $0 < M < \infty$ independent of c, F such that a Bayes procedure (with probability one) continues sampling whenever the posterior risk of stopping is at least Mc.

PROOF. By making use of a similar procedure as that in pages 2317-2318 of Fuh [8], a Bayes rule exists.

Let $\infty > Q > A/e$ where A is defined in Lemma 6 and define T_{Qc} to be the first time $n \leq \infty$ that the posterior risk of stopping is at most Qc. It is sufficient to prove for some $Q < M < \infty$ that the (integrated) risk of T_{Qc} is less than γ if $\gamma \geq Mc$. Since the (integrated) risk of any generalized stopping time T is the expected posterior risk of stopping plus $c(1-\gamma) \int_a^b \mathbb{E}^\theta T dG(\theta)$, it is sufficient to prove for some $0 < M < \infty$ that $(1-\gamma) \int_a^b \mathbb{E}^\theta T_{Qc} dG(\theta) < \gamma/c - Q$ if $\gamma \geq Mc$.

Choose M > Q such that (1 - A/(Qe))M - (B + A/e) > Q where A, B are the constants defined by Lemma 6. It is

enough to look at c for which Qc < 1. Denote $p_n(\underline{Y}_n; \theta_0) := p_n(Y_1, \ldots, Y_n; \theta_0)$. Note that

$$T_{Qc} \qquad (7.3)$$

$$= \inf \left\{ n | Qc \ge \frac{\gamma p_n(\underline{Y}_n; \theta_0)}{\gamma p_n(\underline{Y}_n; \theta_0) + (1 - \gamma) \int_a^b p_n(\underline{Y}_n; \theta) dG(\theta)} \right\}$$

$$= \inf \left\{ n | \int_a^b \frac{r(W_0, \theta_0)}{r(W_n, \theta_0)} \frac{r(W_n, \alpha)}{r(W_0, \alpha)} \exp\{(\alpha - \theta_0) \mathbb{S}_n - n(\Lambda(\alpha) - \Lambda(\theta_0))\} dG(\alpha) \ge \frac{\gamma}{1 - \gamma} \frac{1 - Qc}{Qc} \right\}$$

$$\leq \inf \left\{ n | \int_a^b \frac{r(W_0, \theta_0)}{r(W_n, \theta_0)} \frac{r(W_n, \alpha)}{r(W_0, \alpha)} \exp\{(\alpha - \theta_0) \mathbb{S}_n - n(\Lambda(\alpha) - \Lambda(\theta_0))\} dG(\alpha) \ge \frac{\gamma}{(1 - \gamma)Qc} \right\}.$$

Note that $\sup_{0 < \alpha < 1} -\alpha(\log \alpha) = 1/e$, applying Lemma 6 to get that if $1 > \gamma \ge Mc$

$$(1-\gamma)\int_{a}^{b} \mathbb{E}^{\theta} T_{Qc} dG(\theta)$$

$$\leq (1-\gamma) \left[A \left(\log \frac{\gamma}{Qc} + \log \frac{1}{1-\gamma} \right) + B \right]$$

$$\leq \frac{\gamma}{c} \frac{A}{Q} \frac{Qc}{\gamma} \log \frac{\gamma}{Qc} + B + A(1-\gamma) \log \frac{1}{1-\gamma}$$

$$\leq \frac{\gamma}{c} \frac{A}{Qe} + B + \frac{A}{e}$$

$$\leq \frac{\gamma}{c} - \left(1 - \frac{A}{Qe} \right) M + B + \frac{A}{e} \leq \frac{\gamma}{c} - Q.$$

This completes the proof of Lemma 7. \Box

PROOF OF THEOREM 2. Without loss of generality, we assume $\theta_0 = 0$, 0 < a < b, (a, b) = J and $\Lambda'(a) > \Lambda(b)/b$. We first show that the right hand side of (2.22) is a lower bound of the left hand side of (2.22). Consider the Bayesian problem defined in Lemma 7 when $\gamma = \frac{1}{2}$ and $dG(\theta)/d\theta = K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0}) / \int_a^b K(\mathbb{P}^{\alpha}, \mathbb{P}^{\theta_0}) d\alpha$ on [a, b]. Let M be the constant derived in Lemma 7 and let T_{Mc} be T_{Qc} for Q=M where T_{Qc} is defined in (7.3). T_{Mc} is a mixture stopping rule defined by G and B = (1 - Mc)/(Mc). By virtue of Lemma 7 there exists a Bayes rule which continues sampling at least as long as T_{Mc} . Hence the Bayes risk is at least the sampling cost of T_{Mc} , whence for any stopping rule T

$$\mathbb{P}^{\theta_0}(T<\infty) + c \int_a^b \mathbb{E}^{\theta} T dG(\theta) \ge c \int_a^b \mathbb{E}^{\theta} T_{Mc} dG(\theta).$$

Thus if $\mathbb{P}^{\theta_0}(T < \infty) \leq 1/\varepsilon = Mc/(1 - Mc)$, then

$$\int_{a}^{b} \mathbb{E}^{\theta} T dG(\theta) \ge \int_{a}^{b} \mathbb{E}^{\theta} T_{Mc} dG(\theta) - M/(1 - Mc).$$
(7.4)

There exist a_1, b_1 such that $0 < a_1 < a < b < b_1 < \infty$ and $\Lambda'(a_1) > \Lambda(b_1)/b_1$. Define $\Lambda = \inf\{n | \int_{a_1}^{b_1} \exp\{\theta \mathbb{S}_n - n\Lambda(\theta)\}K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})d\theta / \int_a^b K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})d\theta \geq B\}$. By definition, $T_{Mc} \geq \Lambda$. Λ is a mixture stopping rule defined by $dF(\theta)/d\theta = K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})/\int_{a_1}^{b_1} K(\mathbb{P}^{\alpha}, \mathbb{P}^{\theta_0})d\alpha$ on $[a_1, b_1]$ and $B'=B\int_a^b K(\mathbb{P}^\theta,\mathbb{P}^{\theta_0})d\theta/\int_{a_1}^{b_1} K(\mathbb{P}^\theta,\mathbb{P}^{\theta_0})d\theta.$ Thus by Theorem 1

$$= \frac{\mathbb{E}^{\theta} T_{Mc} \geq \mathbb{E}^{\theta} \Lambda}{2K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_0})} [2 \log B' + \log \log B'] + O_{\theta}(1),$$

$$(7.5)$$

where $\limsup_{\varepsilon \to \infty} \sup_{a \le \theta \le b} |O_{\theta}(1)| \le \infty$. Combining (7.4) and (7.5), and replacing B^{T} by B yields

$$\int_{a}^{b} \mathbb{E}^{\theta} T dG(\theta)$$

$$\geq \int_{a}^{b} [2 \log B + \log \log B + O(1)] d\theta / 2(\int_{a}^{b} K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_{0}}) d\theta)$$

Hence by definition of G, we have

$$\int_{a}^{b} [2K(\mathbb{P}^{\theta}, \mathbb{P}^{\theta_{0}})\mathbb{E}^{\theta}T - (2\log B + \log\log B)]d\theta \ge O(1)$$

for all T satisfying $\mathbb{P}^{\theta_0} \{T < \infty\} \leq 1/B$.

To show that the equality is attained by the weighted SRP rule. By Theorem 1, we need only to show that for the weighted SRP detection rule (5.6) satisfies $\mathbb{P}^{\theta_0}\{N_b < \infty\} \leq 1/c$, for any c > 1.

Recall that $\mathbb{P}^{\theta}(y, dz)$ defined in (4.18), and denote $\mathbb{Q}(w, dw') := \int_{\theta \in J} \mathbb{P}^{\theta}(w, dw') dF(\theta)$. Then it is easy to see that $\mathbb{Q}(w, \cdot)$ is a transition kernel. By definition of N_b , we have

$$\mathbb{P}^{\theta_0}\{N_b < \infty\}$$

$$= \int_{\{N_b < \infty\}} \frac{1}{LR_n(F)} d\mathbb{Q} \le \frac{1}{c} \mathbb{Q}\{N_b < \infty\} \le \frac{1}{c}.$$
(7.6)

This establishes the desired property, and thus completing the proof of (2.22).

APPENDIX

We give a proof of Proposition 1 which also corrects notations error in Lemma 8 of Fuh [8], in the setting of hidden Markov models.

PROOF OF PROPOSITION 1. Note that

$$\begin{split} &H_{n+1}(y,w) \\ = & \mathbb{P}_{\infty}\{R_{n+1,p} \leq y | N_{q,b} > n+1, W_{n+1} = w\} \\ = & \int_{w' \in \mathcal{W}} \int_{0}^{B(w')} \mathbb{P}_{\infty}\{R_{n+1,p} \leq y, W_{n} \in dw', \\ & R_{n,p} \in dt | N_{q,b} > n+1, W_{n+1} = w\} \\ = & \int_{w' \in \mathcal{W}} \int_{0}^{B(w')} \mathbb{P}_{\infty}\{R_{n+1,p} \leq y | W_{n} = w', R_{n,p} = t, \\ & N_{q,b} > n+1, W_{n+1} = w\} \\ & \times \mathbb{P}_{\infty}\{R_{n,p} \in dt, W_{n} \in dw' | N_{q,b} > n+1, W_{n+1} = w\} \\ = & \int_{w' \in \mathcal{W}} \int_{0}^{B(w')} \rho(t, y, w) \mathbb{P}_{\infty}\{R_{n,p} \in dt, W_{n} \in dw' | \\ & N_{q,b} > n+1, W_{n+1} = w\}. \end{split}$$

Since

$$\begin{split} & \mathbb{P}_{\infty}\{R_{n,p} \in dt, W_{n} \in dw' | N_{q,b} > n + 1, W_{n+1} = w\} \\ &= \mathbb{P}_{\infty}\{R_{n,p} \in dt, W_{n} \in dw' | N_{q,b} > n, \\ & N_{q,b} > n + 1, W_{n+1} = w\} \\ &= \left(\mathbb{P}_{\infty}\{N_{q,b} > n + 1, W_{n+1} \in dy | R_{n,p} = t, N_{q,b} > n, \\ & W_{n} = w'\}\right) \Big/ \left(\int_{w,w' \in \mathcal{W}} \int_{0}^{B(w')} \mathbb{P}_{\infty}\{N_{q,b} > n + 1, \\ & W_{n+1} \in dw | R_{n,p} = t, N_{q,b} > n, W_{n} = w'\} \\ & \times \frac{\mathbb{P}_{\infty}\{R_{n,p} \in dt | N_{q,b} > n, W_{n} = w'\}}{\mathbb{P}_{\infty}\{R_{n,p} \in dt | N_{q,b} > n, W_{n} = w'\}} \\ & \times \frac{\mathbb{P}_{\infty}\{R_{n,p} \in dt | N_{q,b} > n, W_{n} = w'\}}{\mathbb{P}_{\infty}\{N_{q,b} > n | W_{n} = w'\}\mathbb{P}(w', dw)} \\ & = \frac{\zeta(t, w', w) dH_{n}(t, w')\mathbb{P}(w', dw)}{\int_{w,w' \in \mathcal{W}} \int_{0}^{B(w')} \zeta(t, w', w) dH_{n}(t, w')\mathbb{P}(w', dw)}. \end{split}$$

It follows that

$$H_{n+1}(y,w)$$

$$= \frac{\int_{w'\in\mathcal{W}} \int_0^{B(w')} \rho(t,y,w)\zeta(t,w',w)dH_n(t,w')\mathbb{P}(w',dw)}{Q(H_n)}$$

$$= T_B H_n(y,w).$$

The existence of the fixed point follows the same argument as that of Lemma 11 in Pollak [21]. \Box

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