# Embedding linear codes into self-orthogonal codes and their optimal minimum distances

Jon-Lark Kim, Young-Hun Kim, and Nari Lee

Abstract—We obtain a characterization on self-orthogonality for a given binary linear code in terms of the number of column vectors in its generator matrix, which extends the result of Bouyukliev et al. (2006). As an application, we give an algorithmic method to embed a given binary k-dimensional linear code C (k = 2, 3, 4) into a self-orthogonal code of the shortest length which has the same dimension k and minimum distance  $> d(\mathcal{C})$ . For k > 4, we suggest a recursive method to embed a k-dimensional linear code to a self-orthogonal code. We also give new explicit formulas for the minimum distances of optimal self-orthogonal codes for any length n with dimension 4 and any length  $n \not\equiv 6, 13, 14, 21, 22, 28, 29 \pmod{31}$  with dimension 5. We determine the exact optimal minimum distances of [n, 4]self-orthogonal codes which were left open by Li-Xu-Zhao (2008) when  $n \equiv 0, 3, 4, 5, 10, 11, 12 \pmod{15}$ . Then, using MAGMA, we observe that our embedding sends an optimal linear code to an optimal self-orthogonal code.

Index Terms—Binary linear code, self-orthogonal code, optimal code.

#### I. INTRODUCTION

SELF-orthogonal codes have been extensively studied for their interesting structures and applications. In particular, self-dual codes, a special class of self-orthogonal codes, have attracted much attention because of their connections to other fields of mathematics such as unimodular lattices, secret sharing schemes, and designs ([1], [2], [3], [4], [5], [6]).

Since the 1970s, a lot of researchers have studied self-orthogonal codes. For instance, constructions and classification of self-orthogonal codes were steadily studied ([7], [8], [9], [10], [11]). Self-orthogonal codes were also studied due to their connections to quantum codes ([12], [13], [14], [15], [16]) and their applications to the resistance to side-channel attacks ([17], [18]). However, several questions for self-orthogonal codes remain. To mention a few, the classification of self-orthogonal codes and the explicit formulas for the minimum distances of optimal self-orthogonal codes are partially computed.

From now on, we will only consider binary linear codes. Pless [7] classified certain self-orthogonal codes. Since then people have gotten more results for the classification of self-orthogonal codes. In [19, Section 3], Bouyukliev et al. introduced a noteworthy characterization for self-orthogonality of three-dimensional codes in terms of the number of column vectors in a generator matrix. As a consequence, they gave the complete classification of three-dimensional optimal self-orthogonal codes. In [10], Li, Xu, and Zhao characterized four-dimensional optimal self-orthogonal codes by systems of linear equations. They also obtained the complete classification of optimal [n,4] self-orthogonal codes for  $n\equiv 1,2,6,7,8,9,13,14 \pmod{15}$  and left the other cases open.

In this paper, we generalize the characterization in [19, Section 3] for arbitrary dimensions. In particular, we give an explicit characterization for [n,k] self-orthogonal codes for k=2,4 and reprove the characterization for [n,3] self-orthogonal codes, which was introduced in [19].

As a consequence of our characterizations, we construct an algorithm that embeds (or extends) an [n,k] linear code to a self-orthogonal code for k=2,3,4 in Section IV. Precisely, if we input a generator matrix G of a linear code C, then the algorithm will give a matrix  $\widetilde{G}$  by adding more columns to G which generates a self-orthogonal code  $\widetilde{C}$  and produces a minimum distance greater than or equal to the minimum distance of C. Moreover, we prove that  $\widetilde{C}$  is a shortest (length) self-orthogonal embedding. In [20], Kobayashi and Takada introduced a similar embedding with a critical error. We point out their error (see Remark IV.6). For k>4, we suggest a recursive method to embed a k-dimensional linear code to a self-orthogonal code.

It is a quite natural question whether an optimal linear code results in an optimal self-orthogonal code when it is embedded by our algorithm in Section IV. Therefore, we will also discuss the explicit formulas for the minimum distances of optimal linear codes and optimal SO codes in Section V.

There have been a number of studies on bounds about the minimum distances of linear codes. In particular, Griesmer [21] introduced a remarkable bound, called the Griesmer bound. Due to this bound, researchers obtained a lot of minimum distances of optimal linear codes. For the details, readers can refer to [22].

Manipulating the Griesmer bound, we calculate an explicit formula for an upper bound of the minimum distance d(n,k) of optimal linear codes of length n for k=1,2,3,4,5. More precisely, we obtain new explicit formulas for the minimum distances  $d_{\rm so}(n,k)$  of optimal self-orthogonal codes for any length n with k=4 and for any length  $n\not\equiv 6,13,14,21,22,28,29\pmod{31}$  with k=5. We determine the exact optimal minimum distances of [n,4] self-orthogonal

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codes which were left open by Li-Xu-Zhao in [10] when  $n \equiv 0, 3, 4, 5, 10, 11, 12 \pmod{15}$ . Using the code database in MAGMA [23] and our explicit formulas, we observe that our embedding algorithm sends an optimal linear code to an optimal self-orthogonal code.

Readers may refer to [22] to see the list of d(n,k) for  $n,k \leq 256$  and refer to [19] to see the known  $d_{\rm so}(n,k)$  for  $n \leq 40$  and  $k \leq 10$ . For  $40 < n \leq 100$ , the upper bound of  $d_{\rm so}(n,5)$  is obtained from the Griesmer bound [21]. In Tables I.1 and I.2, we list  $d_{\rm so}(n,k)$  for  $n \leq 100$  and k = 4,5. Here the superscript \* and  $\dagger$  respectively denote our exact value and the conjectured value from Conjecture V.19.

This paper is organized as follows. In Section II, we introduce basic facts. In Section III, we give characterizations for self-orthogonal codes. In Section IV, we introduce algorithms which give shortest self-orthogonal embeddings using our characterizations. In Section V, we provide explicit formulas for d(n,k) and  $d_{\rm so}(n,k)$  for k=1,2,3,4,5.

8 9 10 11	$d_{ m so}(n$ Known [ref] $4$ $4$ $4$	Improved -	$d_{\rm so}(n$ Known [ref]	Improved
9 10 11	4	-		
10 11				
11	4	-		,
	4	-	4	-
	4	-	4	-
12	4	-	4	-
13	4	-	4	-
14	6	-	4	-
15	8	-	6	-
16	8	-	8	-
17	8	-	8	-
18	8	-	8	-
19	8	-	8	-
20	8	-	8	-
21	10	-	8	-
22	10	-	8	-
23	12	-	10	-
24	12	-	12	-
25	12	-	12	-
26	12	-	12	-
27	12	-	12	-
28	14	-	12	-
29	14	-	12	-
30	16	-	14	
31	16	-	16	-
32	16	-	16	-
33	16	-	16 16	
35	16 16		16	
36	18	-	16	-
37	18	-		-
38	20	-	16 18	-
39	20	-	18	
40	20	-	20	-
41	< 20 [21]	20*	20	
42	$\leq 20 [21]$ $\leq 20 [10]$	20*	20	
43	$\frac{\leq 20 \ [10]}{22 \ [10]}$	-	< 20 [21]	20*
44	22 [10]	-	$\leq 20 [21]$ $\leq 22 [21]$	20 <sup>†</sup>
45	24 [23]		$\leq 22 [21]$ $\leq 22 [21]$	20†
46	24 [23]	-	$\leq 22 [21]$ $\leq 22 [21]$	20° 22*
47	24 [10]	-	$\frac{\leq 22 \ [21]}{24 \ [23]}$	-
48	24 [10]	-	24 [23]	
49	< 24 [23]	24*	24 [23]	-
50	_ [ ]	2/1*	24 [23]	
50	[10]	Table I.	1 24 [23]	1

 $d_{\mathrm{so}}(n,k)$  for  $n\leq 50$  and k=4,5

1	$d_{\rm so}(n$	, 4)	$d_{\rm so}(n,5)$				
n	Known [ref]	Improved	Known [ref]	Improved			
51	26 [10]	-	$\leq 24 \ [21]$	24*			
52	26 [10]	-	$\leq 26  [21]$	$24^{\dagger}$			
53	28 [10]	-	≤ 26 [21]	$24^{\dagger}$			
54	28 [10]	-	$\leq 26  [21]$	26*			
55	28 [23]	-	28 [23]	-			
56	$\leq 28 \ [21]$	28*	28 [23]	-			
57	$\leq 28 \ [10]$	28*	28 [23]	-			
58	30 [10]	-	$\leq 28 \ [21]$	28*			
59	30 [10]	-	$\leq 30 \ [21]$	$28^{\dagger}$			
60	32 [23]	-	$\leq 30 \ [21]$	$28^{\dagger}$			
61	32 [10]	-	$\leq 30 \ [21]$	30*			
62	32 [10]	-	32 [23]	-			
63	32 [23]	-	32 [23]	-			
64	$\leq 32 [21]$	32*	32 [23]	-			
65	$\leq 32 [10]$	32*	32 [23]	-			
66	34 [10]	-	32 [23]	-			
67	34 [10]	-	$\leq 32 [21]$	32*			
68	36 [10]	-	≤ 34 [21]	32 <sup>†</sup>			
69	36 [10]	-	$\leq 34 \ [21]$	34*			
70	36 [23]	-	$\leq 34 [21]$	$34^*$			
71	$\leq 36 [21]$	36*	36 [23]	-			
72	$\leq 36 [10]$	36*	36 [23]	-			
73	38 [10]	-	$\leq 36 [21]$	36* 36*			
74	38 [10]	-	$\leq 36 [21]$	36 <sup>†</sup>			
75	40 [23]	-	$\leq 38 [21]$				
76	40 [10]	-	$\leq 38 [21]$	36 <sup>†</sup>			
77	40 [10] 40 [23]	-	$\leq 38 [21]$ 40 [23]	38*			
78 79	< 40 [23] < 40 [21]	40*	40 [23]	-			
80	$\leq 40 [21]$ $\leq 40 [10]$	40*	40 [23]				
81	42 [10]	-	40 [23]	-			
82	42 [10]	_	< 40 [23]	40*			
83	44 [10]	-	< 42 [21]	40 <sup>†</sup>			
84	44 [10]	-	< 42 [21]	40 <sup>†</sup>			
85	44 [23]	-	< 42 [21]	42*			
86	< 44 [21]	44*	44 [23]	-			
87	< 44 [10]	44*	44 [23]	-			
88	46 [10]	-	44 [23]	-			
89	46 [10]	-	< 44 [21]	44*			
90	48 [23]	-	< 46 [21]	$44^{\dagger}$			
91	48 [10]	-	< 46 [21]	$44^{\dagger}$			
92	48 [10]	-	< 46 [21]	46*			
93	48 [23]	-	48 [23]	-			
94	≤ 48 [21]	48*	48 [23]	-			
95	$\leq 48 \ [10]$	48*	48 [23]	-			
96	50 [10]	-	48 [23]	-			
97	50 [10]	-	48 [23]	-			
98	52 [10]	-	$\leq 48 \ [21]$	48*			
99	52 [10]	-	$\leq 50 [21]$	$48^{\dagger}$			
100	$\leq 52 [21]$	<b>52</b> *	$\leq 50 [21]$	50*			
-	d (n h) E0	Table I.2	$\frac{1}{2}$	-			

 $d_{\mathrm{so}}(n,k)$  for  $50 < n \leq 100$  and k = 4,5

#### II. PRELIMINARIES

Let GF(q) be a finite field with q elements. We consider the case where q=2 only. A subspace  $\mathcal{C}$  of  $GF(q)^n$  is called a linear code of length n. For  $n,k\in\mathbb{Z}^+$ , a k-dimensional linear code  $\mathcal{C}\subset GF(q)^n$  is called an [n,k] code. The elements of  $\mathcal{C}$  are called codewords. A generator matrix for  $\mathcal{C}$  is a  $k\times n$  matrix G whose rows form a basis for  $\mathcal{C}$ .

For  $\mathbf{x} = (x_1, x_2, \dots, x_n), \mathbf{y} = (y_1, y_2, \dots, y_n) \in GF(q)^n$ , the *ordinary inner product*  $\mathbf{x} \cdot \mathbf{y}$  of  $\mathbf{x}$  and  $\mathbf{y}$  is  $\sum_{i=1}^n x_i y_i$ . For a linear code  $\mathcal{C}$ , the code

$$\mathcal{C}^{\perp} := \left\{ \mathbf{x} \in GF(q)^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in \mathcal{C} \right\}.$$

is called the *dual* of C. A linear code C satisfying  $C \subseteq C^{\perp}$  (resp.  $C = C^{\perp}$ ) is called *self-orthogonal* (abbr. SO) (resp. *self-dual*).

The (Hamming) weight  $\operatorname{wt}(\mathbf{x})$  of a vector  $\mathbf{x}$  in  $GF(q)^n$  is the total number of nonzero coordinates in  $\mathbf{x}$ . For  $\mathbf{x}, \mathbf{y} \in GF(q)^n$ , we define the (Hamming) distance  $d(\mathbf{x}, \mathbf{y})$  between  $\mathbf{x}$  and  $\mathbf{y}$  by the number of coordinates in which  $\mathbf{x}$  and  $\mathbf{y}$  differ. The minimum distance of a code  $\mathcal{C}$  is the smallest nonzero distance between any two distinct codewords. For  $n, k, d \in \mathbb{Z}^+$ , an [n, k, d] code  $\mathcal{C}$  is an [n, k] code whose minimum distance is d. We call an [n, k, d] code  $\mathcal{C}$  optimal if its minimum distance d is the highest among all [n, k] linear codes. We denote by d(n, k) the minimum distance of an optimal [n, k] so code. An [n, k, d] so code  $\mathcal{C}$  is called by optimal SO if its minimum distance d is the highest among all [n, k] so codes. We denote by  $d_{so}(n, k)$  the minimum distance of an optimal [n, k] so code.

For  $k, d \in \mathbb{Z}^+$ , let n(k, d) be the smallest value of n for which an [n, k, d] code exists. A notable lower bound on n(k, d) was obtained by Griesmer as follows.

**Theorem II.1.** ([21], [24]) Let C be an [n, k, d] code over GF(2) with  $k \ge 1$ . Then

$$n(k,d) \ge g(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil.$$

Let us collect some required notations. For any [n,k] code  $\mathcal C$  generated by G, we denote by  $r_i(G)$  the ith row of G from the top for  $1 \leq i \leq k$  and  $c_j(G)$  the jth column of G from the left for  $1 \leq j \leq n$ . If there is no danger of confusion to the matrix G, then we will write  $r_i$  (resp.  $c_j$ ) for  $r_i(G)$  (resp.  $c_j(G)$ ). For a positive integer s, we denote by  $sG = (G, G, \ldots, G)$  the juxtaposition of s copies of G.

For  $k \in \mathbb{Z}^+$ , we denote by  $\mathcal{S}_k$  the  $[2^k-1,k]$  simplex code and  $H_k$  the generator matrix of  $\mathcal{S}_k$  whose *i*th column is the k-dimensional binary representation of i for  $1 \le i \le 2^k-1$ . For example, the first column is written as  $[0,0,\ldots,0,1]^T$ . For  $i=1,2,\ldots,2^k-1$ , we let

 $h_i :=$  the *i*th column vector of  $H_k$  (from the left hand side).

Let  $m_1, m_2, \ldots, m_t$  be nonnegative integers and  $a, n \in \mathbb{Z}^+$ . If  $n \equiv m_1, m_2, \ldots$ , or  $m_t \pmod{a}$ , then we write

$$n \equiv_a m_1, m_2, \dots, m_t.$$

Otherwise, we denote

$$n \not\equiv_a m_1, m_2, \dots, m_t.$$

If  $n \equiv m_1 \pmod{a}$ ,  $n \equiv m_2 \pmod{a}$ ,..., and  $n \equiv m_t \pmod{a}$ , then we write

$$n \equiv_a m_1 \equiv_a m_2 \cdots \equiv_a m_t$$
.

For a statement P, we define

$$\delta(P) := \begin{cases} 1 & \text{if } P \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

For instance, for nonnegative integers  $m_1, m_2, \ldots, m_t$  and  $a, n \in \mathbb{Z}^+$ ,

$$\delta(n \equiv_a m_1, m_2, \dots, m_t) = \begin{cases} 1 & \text{if } n \equiv_a m_1, m_2, \dots, m_t, \\ 0 & \text{otherwise.} \end{cases}$$

#### III. CHARACTERIZATIONS FOR SELF-ORTHOGONALITY

In this section, we obtain characterizations for self-orthogonality by reading column vectors of a generator matrix.

For a  $k \times n$  matrix G and  $i = 1, 2, \dots, 2^k - 1$ , we define

$$\ell_{\mathsf{h}_i}(G) := \mathsf{the} \ \mathsf{number} \ \mathsf{of} \ \mathsf{h}_i \ \mathsf{among} \ \mathsf{the} \ \mathsf{columns} \ \mathsf{of} \ G.$$

If there is no danger of confusion to the matrix G, then we will write  $\ell_i$  for  $\ell_{h_i}(G)$ .

For a  $k \times n$  matrix G and  $0 < j \le k$ , we define a multiset

$$\mathbf{I}(j) := \left\{ c_i(G) \left| \begin{array}{l} \text{(i) } 0 < i \leq n, \\ \text{(ii) } c_i(G) = \mathsf{h}_t \text{ for } 1 \leq t \leq 2^k - 1 \\ \text{satisfying } \left\lfloor \frac{t}{2^{j-1}} \right\rfloor \equiv_2 1 \end{array} \right. \right\}.$$

**Example III.1.** Let  $C_{10,3}$  be a [10,3] code generated by

Then we have

$$\ell_1 = \ell_2 = \ell_4 = \ell_5 = \ell_7 = 1$$
,  $\ell_3 = 2$ , and  $\ell_6 = 3$ .

and

$$\mathbf{I}(1) = \{c_{1}, c_{3}, c_{4}, c_{6}, c_{10}\} 
= \{h_{1}, h_{3}, h_{3}, h_{5}, h_{7}\}, 
\mathbf{I}(2) = \{c_{2}, c_{3}, c_{4}, c_{7}, c_{8}, c_{9}, c_{10}\} 
= \{h_{2}, h_{3}, h_{3}, h_{6}, h_{6}, h_{6}, h_{7}\}, 
\mathbf{I}(3) = \{c_{5}, c_{6}, c_{7}, c_{8}, c_{9}, c_{10}\} 
= \{h_{4}, h_{5}, h_{6}, h_{6}, h_{6}, h_{7}\}.$$
(III.1)

In terms of I(j), we obtain the following characterization for self-orthogonality.

**Theorem III.2.** Let C be an [n, k] code generated by G. Then, C is SO if and only if for all  $0 < j \le j' \le k$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even.

*Proof.* Note that for  $i = 1, 2, ..., 2^k - 1$ ,

$$\mathsf{h}_i = \begin{bmatrix} \delta\left(\left\lfloor\frac{i}{2^{k-1}}\right\rfloor \equiv_2 1\right) \\ \delta\left(\left\lfloor\frac{i}{2^{k-2}}\right\rfloor \equiv_2 1\right) \\ \vdots \\ \delta\left(\left\lfloor\frac{i}{2^0}\right\rfloor \equiv_2 1\right) \end{bmatrix},$$

that is,  $\left\lfloor \frac{i}{2^{j-1}} \right\rfloor$  is odd for  $i=1,2,\ldots,2^k-1$  and  $j=1,2,\ldots,k$  if and only if the jth component of  $h_i$  from the bottom is 1. Therefore,  $|\mathbf{I}(j)\cap\mathbf{I}(j')|$  counts the number of columns of G whose jth and j'th components are both 1. Thus, in terms of row indices, we have

$$|\mathbf{I}(j) \cap \mathbf{I}(j')| = r_{k-j+1} \cdot r_{k-j'+1}.$$

for  $0 < j \le j' \le k$  and hence, our assertion follows.  $\square$ 

#### Example III.3.

(1) Let  $C_{10,3}$  be the [10,3] code generated by  $G_{10,3}$  appeared in Example III.1. From Equation (III.1), we see that

$$|\mathbf{I}(1) \cap \mathbf{I}(1)| = |\{c_1, c_3, c_4, c_6, c_{10}\}| = 5$$

is odd. Thus, by Theorem III.2,  $C_{10,3}$  is not SO.

(2) Let  $\widetilde{\mathcal{C}}_{10,3}$  be an [11,3] code generated by

It is easy to obtain I(j) as follows:

$$\mathbf{I}(1) = \{c_1, c_3, c_4, c_6, c_{10}, c_{11}\},\,$$

$$\mathbf{I}(2) = \{c_2, c_3, c_4, c_7, c_8, c_9, c_{10}, c_{11}\},\$$

$$\mathbf{I}(3) = \{c_5, c_6, c_7, c_8, c_9, c_{10}\}.$$

Therefore, we have

$$\mathbf{I}(1) \cap \mathbf{I}(2) = \{c_3, c_4, c_{10}, c_{11}\},\,$$

$$\mathbf{I}(1) \cap \mathbf{I}(3) = \{c_6, c_{10}\},\$$

$$\mathbf{I}(2) \cap \mathbf{I}(3) = \{c_7, c_8, c_9, c_{10}\}.$$

Since  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even for all  $0 < j \le j' \le 3$ , by Theorem III.2,  $\widetilde{\mathcal{C}}_{10.3}$  is SO.

#### A. Self-orthogonality for dimension 2 and 3

In this subsection, we characterize self-orthogonality for [n,2] codes in terms of  $\ell_i$  using Theorem III.2. For [n,3] codes, we introduce the result in [19] which gives a characterization for self-orthogonality and we reprove it using Theorem III.2.

For [n,2] codes, we obtain the following characterization for self-orthogonality.

**Lemma III.4.** Let C be an [n,2] code. The code C is SO if and only if C is generated by a matrix G satisfying

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 0. \tag{III.2}$$

*Proof.* Let G be a generator matrix of  $\mathcal{C}$ . By Theorem III.2, it suffices to show that Equation (III.2) holds if and only if for all  $0 < j \le j' \le 2$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even. By definition of  $\mathbf{I}(j)$ , we obtain that

$$|\mathbf{I}(1)| = \ell_1 + \ell_3, \quad |\mathbf{I}(2)| = \ell_2 + \ell_3, \quad \text{and} \quad |\mathbf{I}(1) \cap \mathbf{I}(2)| = \ell_3.$$

This shows that for all  $0 < j \le j' \le 2$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even if and only if

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 0.$$

For [n, 3] codes, a characterization for the self-orthogonality was introduced in [19]. We reprove this characterization using Theorem III.2.

**Lemma III.5.** [19, Lemma 3] Let C be an [n,3] code. The code C is SO if and only if C is generated by a matrix G satisfying

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_7. \tag{III.3}$$

*Proof.* Let G be a generator matrix of C. By Theorem III.2, it suffices to show that Equation (III.3) holds if and only if

for all  $0 < j \le j' \le 3$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even. By definition of  $\mathbf{I}(j)$ , we obtain that

$$|\mathbf{I}(1)| = \ell_1 + \ell_3 + \ell_5 + \ell_7, \quad |\mathbf{I}(1) \cap \mathbf{I}(2)| = \ell_3 + \ell_7,$$

$$|\mathbf{I}(2)| = \ell_2 + \ell_3 + \ell_6 + \ell_7, \quad |\mathbf{I}(1) \cap \mathbf{I}(3)| = \ell_5 + \ell_7,$$

$$|\mathbf{I}(3)| = \ell_4 + \ell_5 + \ell_6 + \ell_7, \quad |\mathbf{I}(2) \cap \mathbf{I}(3)| = \ell_6 + \ell_7.$$

This shows that for all  $0 < j \le j' \le 3$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even if and only if

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_7.$$

#### B. Self-orthogonal codes of dim 4

In this subsection, we provide a characterization for self-orthogonality for [n, 4] codes in terms of congruence equations on  $\ell_i$ 's.

Recall that for an [n, 4] code C generated by G,

$$\mathbf{I}(1) = \{c_i \mid c_i = \mathsf{h}_t \text{ for } t \in \{1, 3, 5, 7, 9, 11, 13, 15\}\},\$$

$$\mathbf{I}(2) = \{c_i \mid c_i = \mathsf{h}_t \text{ for } t \in \{2, 3, 6, 7, 10, 11, 14, 15\}\},\$$

$$I(3) = \{c_i \mid c_i = h_t \text{ for } t \in \{4, 5, 6, 7, 12, 13, 14, 15\}\},\$$

$$\mathbf{I}(4) = \{c_i \mid c_i = \mathsf{h}_t \text{ for } t \in \{8, 9, 10, 11, 12, 13, 14, 15\}\}.$$

Therefore, we have

$$|\mathbf{I}(1)| = \ell_1 + \ell_3 + \ell_5 + \ell_7 + \ell_9 + \ell_{11} + \ell_{13} + \ell_{15},$$

$$|\mathbf{I}(2)| = \ell_2 + \ell_3 + \ell_6 + \ell_7 + \ell_{10} + \ell_{11} + \ell_{14} + \ell_{15},$$

$$|\mathbf{I}(3)| = \ell_4 + \ell_5 + \ell_6 + \ell_7 + \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15},$$

$$|\mathbf{I}(4)| = \ell_8 + \ell_9 + \ell_{10} + \ell_{11} + \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15},$$

$$|\mathbf{I}(1) \cap \mathbf{I}(2)| = \ell_3 + \ell_7 + \ell_{11} + \ell_{15},$$
(III.4)

$$|\mathbf{I}(1) \cap \mathbf{I}(3)| = \ell_5 + \ell_7 + \ell_{13} + \ell_{15},$$
  
 $|\mathbf{I}(1) \cap \mathbf{I}(4)| = \ell_9 + \ell_{11} + \ell_{13} + \ell_{15},$ 

$$|\mathbf{I}(2) \cap \mathbf{I}(3)| = \ell_6 + \ell_7 + \ell_{14} + \ell_{15},$$

$$|\mathbf{I}(2) \cap \mathbf{I}(4)| = \ell_{10} + \ell_{11} + \ell_{14} + \ell_{15},$$

$$|\mathbf{I}(3) \cap \mathbf{I}(4)| = \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15}.$$

We define the sets  $\mathcal{P}_s^{(1)}$  and  $\mathcal{P}_s^{(2)}$  for  $s=1,2,\ldots,15$  as in Table III.1. Although we only use the union  $\mathcal{P}_s^{(1)} \cup \mathcal{P}_s^{(2)}$  of these sets in the following lemma, we define them separately for later use.

**Lemma III.6.** Let C be an [n,4] code. If C is SO, then any generator matrix G of C satisfies that

$$\ell_{i_1} + \ell_{i_2} \equiv_2 \ell_{j_1} + \ell_{j_2} \tag{III.5}$$

for 
$$s = 1, 2, ..., 15$$
 and all  $(i_1, i_2), (j_1, j_2) \in \mathcal{P}_s^{(1)} \cup \mathcal{P}_s^{(2)}$ .

*Proof.* Since C is SO, for all  $0 < j \le j' \le 4$ ,  $|\mathbf{I}(j) \cap \mathbf{I}(j')| \equiv_2 0$  by Theorem III.2. Thus, by Equation (III.4), we obtain the following:

- Since  $\ell_2 + \ell_3 + \ell_6 + \ell_7 \equiv_2 |\mathbf{I}(2)| + |\mathbf{I}(2) \cap \mathbf{I}(4)| \equiv_2 0$ , we have  $\ell_2 + \ell_3 \equiv_2 \ell_6 + \ell_7$ .
- Since  $\ell_4 + \ell_5 + \ell_6 + \ell_7 \equiv_2 |\mathbf{I}(3)| + |\mathbf{I}(3) \cap \mathbf{I}(4)| \equiv_2 0$ , we have  $\ell_4 + \ell_5 \equiv_2 \ell_6 + \ell_7$ .
- Since  $\ell_6 + \ell_7 + \ell_{14} + \ell_{15} \equiv_2 |\mathbf{I}(2) \cap \mathbf{I}(3)| \equiv_2 0$ , we have  $\ell_6 + \ell_7 \equiv_2 \ell_{14} + \ell_{15}$ .
- Since  $\ell_{10} + \ell_{11} + \ell_{14} + \ell_{15} \equiv_2 |\mathbf{I}(2) \cap \mathbf{I}(4)| \equiv_2 0$ , we have  $\ell_{10} + \ell_{11} \equiv_2 \ell_{14} + \ell_{15}$ .

s	$\mathcal{P}_s^{(1)}$	$\mathcal{P}_s^{(2)}$						
1	$\{(2,3),(4,5),(6,7)\}$	$\{(8,9),(10,11),(12,13),(14,15)\}$						
2	$\{(1,3),(4,6),(5,7)\}$	$\{(8,10),(9,11),(12,14),(13,15)\}$						
3	$\{(1,2),(4,7),(5,6)\}$	$\{(8,11),(9,10),(12,15),(13,14)\}$						
4	$\{(1,5),(2,6),(3,7)\}$	$\{(8,12),(9,13),(10,14),(11,15)\}$						
5	$\{(1,4),(2,7),(3,6)\}$	$\{(8,13),(9,12),(10,15),(11,14)\}$						
6	$\{(1,7),(2,4),(3,5)\}$	$\{(8,14),(9,15),(10,12),(11,13)\}$						
7	$\{(1,6),(2,5),(3,4)\}$	$\{(8,15),(9,14),(10,13),(11,12)\}$						
8	$\{(1,9),(2,10),(3,11)\}$	$\{(4,12),(5,13),(6,14),(7,15)\}$						
9	$\{(1,8),(2,11),(3,10)\}$	$\{(4,13),(5,12),(6,15),(7,14)\}$						
10	$\{(1,11),(2,8),(3,9)\}$	$\{(4,14),(5,15),(6,12),(7,13)\}$						
11	$\{(1,10),(2,9),(3,8)\}$	$\{(4,15),(5,14),(6,13),(7,12)\}$						
12	$\{(1,13),(2,14),(3,15)\}$	$\{(4,8),(5,9),(6,10),(7,11)\}$						
13	$\{(1,12),(2,15),(3,14)\}$	$\{(4,9),(5,8),(6,11),(7,10)\}$						
14	$\{(1,15),(2,12),(3,13)\}$	$\{(4,10),(5,11),(6,8),(7,9)\}$						
15	$\{(1,14),(2,13),(3,12)\}$	$\{(4,11), (5,10), (6,9), (7,8)\}$						
	Table III.1							

Table III.1 
$$\mathcal{P}_s^{(t)}$$
 FOR  $s=1,2,\ldots,15$  AND  $t=1,2$ 

- Since  $\ell_{12} + \ell_{13} + \ell_{14} + \ell_{15} \equiv_2 |\mathbf{I}(3) \cap \mathbf{I}(4)| \equiv_2 0$ , we have  $\ell_{12} + \ell_{13} \equiv_2 \ell_{14} + \ell_{15}$ .
- Since  $\ell_8 + \ell_9 + \ell_{10} + \ell_{11} \equiv_2 |\mathbf{I}(4)| + |\mathbf{I}(3) \cap \mathbf{I}(4)| \equiv_2 0$ , we have  $\ell_8 + \ell_9 \equiv_2 \ell_{10} + \ell_{11}$ .

This implies the case s=1 since Equation (III.5) can be simply written as

$$\begin{split} \ell_2 + \ell_3 &\equiv_2 \ell_4 + \ell_5 \equiv_2 \ell_6 + \ell_7 \equiv_2 \ell_8 + \ell_9 \\ &\equiv_2 \ell_{10} + \ell_{11} \equiv_2 \ell_{12} + \ell_{13} \equiv_2 \ell_{14} + \ell_{15}. \end{split}$$

The rest cases can be shown in the same manner.

#### Remark III.7.

(1) For  $s \in \{1, 2, \dots, 7\}$ ,

$$\{i \in \mathbb{Z} \mid i \text{ appears in } \mathcal{P}_s^{(1)}\} = \{1, 2, \dots, 7\} \setminus \{s\}.$$

(2) For any  $i\neq j\in\{8,9,\ldots,15\}$ , there exists  $s\in\{1,2,\ldots,7\}$  such that  $(i,j)\in\mathcal{P}_s^{(2)}$ .

Now we are ready to introduce our characterization for selforthogonality.

**Theorem III.8.** Let C be an [n,4] code generated by G. The code C is SO if and only if there is  $s \in \{1,2,\ldots,15\}$  such that for each t=1,2,

$$\ell_i \equiv_2 \ell_j \quad for \ i, j \in \mathcal{I}_s^{(t)}.$$
 (III.6)

Here,  $\mathcal{I}_s^{(t)}$ 's are sets given in Table III.2.

*Proof.* It will be proved in Appendix A.  $\Box$ 

### IV. ALGORITHMS TO CONSTRUCT SHORTEST SO EMBEDDINGS

In this section, considering Lemmas III.4, III.5, and Theorem III.8, we introduce an algorithm which extends an [n,k] code to an SO code by adding the smallest number of columns

s	$\mathcal{I}_s^{(1)}$	$\mathcal{I}_s^{(2)}$
1	{1, 2, 3, 4, 5, 6, 7}	{8, 9, 10, 11, 12, 13, 14, 15}
2	$\{1, 2, 3, 8, 9, 10, 11\}$	$\{4, 5, 6, 7, 12, 13, 14, 15\}$
3	$\{1, 2, 3, 12, 13, 14, 15\}$	$\{4, 5, 6, 7, 8, 9, 10, 11\}$
4	$\{1,4,5,8,9,12,13\}$	$\{2, 3, 6, 7, 10, 11, 14, 15\}$
5	$\{1,4,5,10,11,14,15\}$	$\{2, 3, 6, 7, 8, 9, 12, 13\}$
6	$\{1, 6, 7, 8, 9, 14, 15\}$	$\{2, 3, 4, 5, 10, 11, 12, 13\}$
7	$\{1,6,7,10,11,12,13\}$	$\{2, 3, 4, 5, 8, 9, 14, 15\}$
8	$\{2,4,6,8,10,12,14\}$	$\{1, 3, 5, 7, 9, 11, 13, 15\}$
9	$\{2,4,6,9,11,13,15\}$	$\{1, 3, 5, 7, 8, 10, 12, 14\}$
10	$\{2, 5, 7, 8, 10, 13, 15\}$	$\{1, 3, 4, 6, 9, 11, 12, 14\}$
11	$\{2, 5, 7, 9, 11, 12, 14\}$	$\{1, 3, 4, 6, 8, 10, 13, 15\}$
12	$\{3,4,7,8,11,12,15\}$	$\{1, 2, 5, 6, 9, 10, 13, 14\}$
13	$\{3,4,7,9,10,13,14\}$	$\{1, 2, 5, 6, 8, 11, 12, 15\}$
14	$\{3, 5, 6, 8, 11, 13, 14\}$	$\{1, 2, 4, 7, 9, 10, 12, 15\}$
15	$\{3, 5, 6, 9, 10, 12, 15\}$	{1,2,4,7,8,11,13,14}

Table III.2  $\mathcal{I}_s^{(t)}$  for  $s=1,2,\ldots,15$  and t=1,2

for k = 2, 3, 4. We also introduce an algorithm which extends an [n, 5] code to an SO code.

**Definition IV.1.** Let C be an [n, k] code generated by G.

(1) An SO embedding of  $\mathcal C$  is an SO code whose generator matrix  $\widetilde G$  is obtained by adding a set S of column vectors to G, that is,

$$\widetilde{G} := [G | S].$$

(2) An SO embedding of C is called a *shortest SO embedding* of C if its length is shortest among all SO embeddings of C.

In Example III.3,  $\widetilde{\mathcal{C}}_{10,3}$  is a shortest SO embedding of  $\mathcal{C}_{10,3}$ .

A. Algorithms for dimension 2 and 3

We begin with the following algorithm for two-dimensional linear codes.

#### Algorithm IV.2.

- Input: A generator matrix G of an [n, 2] code.
- Output: A generator matrix  $\widetilde{G}$  for a shortest SO embedding.
- (A1) Put  $\widetilde{G} \leftarrow G$  and  $i \leftarrow 1$ .
- (A2) Let

$$\widetilde{G} \leftarrow \begin{cases} \widetilde{G} & \text{if } \ell_i(\widetilde{G}) \equiv_2 0, \\ \left\lceil \widetilde{G} \right\| \mathbf{h}_i \right\rceil & \text{if } \ell_i(\widetilde{G}) \equiv_2 1, \end{cases}$$

where  $\left[\widetilde{G} \| \mathbf{h}_i \right]$  is the juxtaposition of  $\widetilde{G}$  and  $\mathbf{h}_i$ .

(A3) If i < 3, then put  $i \leftarrow i+1$  and go to (A2). Otherwise, terminate the algorithm.

With the resulting matrix  $\hat{G}$  of Algorithm IV.2, we let

$$\widetilde{\mathcal{C}} := \text{the linear code generated by } \widetilde{G}.$$
 (IV.1)

**Remark IV.3.** We can obtain an SO code from an [n, 2] code by adding at most 3 columns.

More precisely, we have the following.

**Theorem IV.4.** Let C be an [n, 2] code generated by G. Then  $\widetilde{C}$  is a shortest SO embedding of C.

*Proof.* By Lemma III.4, it is obvious.

**Example IV.5.** Let  $C_{7,2}$  be an optimal [7,2,4] code generated by

$$G_{7,2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can construct an SO code from  $C_{7,2}$  using Algorithm IV.2. In Step (A1), we put  $\widetilde{G}_{7,2} \leftarrow G_{7,2}$ . Note that

$$\ell_1(\widetilde{G}_{7,2}) = 3$$
,  $\ell_2(\widetilde{G}_{7,2}) = 3$ , and  $\ell_3(\widetilde{G}_{7,2}) = 1$ .

Therefore, when we apply Step (A2) and (A3), we put the juxtaposition of  $\widetilde{G}_{7,2}$ ,  $h_1$ ,  $h_2$ , and  $h_3$  to new  $\widetilde{G}_{7,2}$ . Thus, we obtain

$$\widetilde{G}_{7,2} = \left[ \begin{array}{cccc|c} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ \end{array} \right] \left[ \begin{array}{cccc|c} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ \end{array} \right].$$

It is easy to check that the [10,2,6] code  $\widetilde{C}_{7,2}$  generated by  $\widetilde{G}_{7,2}$  satisfies the condition in Lemma III.4, and thus it is a shortest SO embedding of  $C_{7,2}$ . Moreover,  $\widetilde{C}_{7,2}$  is optimal SO.

**Remark IV.6.** In [20, Section 2], Kobayashi and Takada proposed an algorithm which extends an [n,k] code  $\mathcal C$  to an SO code by juxtaposing k column vectors to its generator matrix G. In the algorithm, they first find a solution  $a_0$  of the equation  $r_1 \cdot r_1 + x^2 = 0$ . Next, they claim that the equation  $r_1 \cdot r_i + a_0 x = 0$  has a solution for every  $1 < i \le k$ . However, in case where  $a_0 = 0$  and  $r_1 \cdot r_i = 1$  for some  $1 < i \le k$ , there are no solutions to this equation. It makes the entire method proposed in [20] wrong. Indeed, if  $\mathcal C$  is an [n,2] code with a generator matrix  $G = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ , then we cannot extend  $\mathcal C$  to an SO code by adding any two columns to G.

Now, let us consider the three-dimensional case. First, for a  $3 \times n$  matrix G and j=0,1, we set

$$J_j(G) := \{i \in \{1, 2, \dots, 7\} \mid \ell_i \equiv_2 j\}.$$

The basic idea of the following algorithm is that we keep adding the smallest number of columns to G so that all  $\ell_i$ 's have the same parity.

#### Algorithm IV.7.

- Input: A generator matrix G of an [n,3] code.
- Output: A generator matrix  $\widehat{G}$  for a shortest SO embedding.
- **(B1)** Put  $\widetilde{G} \leftarrow G$ .
- **(B2)** If  $J_0(\tilde{G}) = \emptyset$  (i.e.,  $\ell_i \equiv_2 1$  for all i = 1, 2, ..., 7) or  $J_1(\tilde{G}) = \emptyset$  (i.e.,  $\ell_i \equiv_2 0$  for all i = 1, 2, ..., 7), then terminate the algorithm. Otherwise, go to **(B3)**.
- (B3) If  $|J_0(G)| < |J_1(G)|$ , then let  $i_0$  be the smallest integer in  $J_0(\widetilde{G})$ . Otherwise, let  $i_0$  be the smallest integer in  $J_1(\widetilde{G})$ .
- **(B4)** Let

$$\widetilde{G} \leftarrow \left[ \widetilde{G} \| \mathbf{h}_{i_0} \right].$$

Go to (**B**2).

**Remark IV.8.** Note that when we apply Algorithm IV.7 to a generator matrix G of an [n,3] code, we add  $|J_0(G)|$  columns to G if  $|J_0(G)| < |J_1(G)|$  and  $|J_1(G)|$  columns to G otherwise. Thus, we can obtain an SO code from an [n,3] code by adding at most 3 columns because  $\min\{|J_0(G)|, |J_1(G)|\} \le |7/2|$ .

With the resulting matrix  $\widetilde{G}$  of Algorithm IV.7, let

$$\widetilde{\mathcal{C}} :=$$
 the linear code generated by  $\widetilde{G}$ .

Then we have the following theorem.

**Theorem IV.9.** Let C be an [n,3] code generated by G. Then  $\widetilde{C}$  is a shortest SO embedding of C.

*Proof.* Note that if  $\mathcal{C}$  is SO, then  $J_0(\widetilde{G}) = \emptyset$  or  $J_1(\widetilde{G}) = \emptyset$  by Lemma III.5. Thus, we have  $\widetilde{\mathcal{C}} = \mathcal{C}$ . Therefore, we may assume that  $\mathcal{C}$  is not SO. Then we have  $|J_j(G)| < |J_{j'}(G)|$  for some  $j \neq j' \in \{0,1\}$ .

From Step (B2) to Step (B4), one can see that Algorithm IV.7 will stop only when  $J_j(\widetilde{G}) = \emptyset$ . Therefore,  $\widetilde{G}$  satisfies Equation (III.3) and thus  $\widetilde{\mathcal{C}}$  is SO by Lemma III.5.

To prove  $\widetilde{\mathcal{C}}$  is a shortest SO embedding of  $\mathcal{C}$ , we let

$$\widehat{G} := \left[ G \middle\| M \right],$$

where M is a  $3 \times l$  matrix for some  $0 < l < |J_j(G)|$  and  $\widehat{\mathcal{C}}$  be a linear code generated by  $\widehat{G}$ . Since  $0 < l < |J_j(G)|$  and  $|J_j(G)| + |J_{j'}(G)| = 7$ , the inequality

$$0 < |J_j(G)| - l \le |J_r(\widehat{G})| \le |J_{j'}(G)| + l < 7$$

holds for r = 0, 1. Thus, by Lemma III.5,  $\widehat{\mathcal{C}}$  is not SO.  $\square$ 

**Example IV.10.** Let  $C_{10,3}$  be an optimal [10,3,5] code generated by

$$G_{10,3} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

It is feasible to construct an SO code from  $C_{10,3}$  using Algorithm IV.7 as follows.

In Step (**B**1), put  $\widetilde{G}_{10,3} \leftarrow G_{10,3}$ . In Step (**B**2), considering the number of the columns of  $\widetilde{G}_{10,3}$ , we let

$$J_0(\widetilde{G}_{10,3}) = \{1, 2, 4, 6, 7\}$$
 and  $J_1(\widetilde{G}_{10,3}) = \{3, 5\}.$ 

Since both are nonempty sets, proceed to Step (B3). In Step (B3), let  $i_0=3$  since  $|J_1(\widetilde{G}_{10,3})|<|J_0(\widetilde{G}_{10,3})|$ . In Step (B4), put

Next, go to Step (B2) and repeat the steps. Now

$$J_0(\widetilde{G}_{10,3}) = \{1,2,3,4,6,7\} \quad \text{and} \quad J_1(\widetilde{G}_{10,3}) = \{5\}.$$

Since neither  $J_0(\widetilde{G}_{10,3})$  nor  $J_1(\widetilde{G}_{10,3})$  is an empty set, go to Step (B3). In Step (B3), we let  $i_0 = 5$  since  $|J_1(\tilde{G}_{10,3})| <$  $|J_0(G_{10,3})|$ . Applying Step (**B**4), we put

Go to Step (B2) again. Since  $J_1(\widetilde{G}_{10,3}) = \emptyset$ , the algorithm is terminated and we obtain the desired  $\widetilde{G}_{10,3}$ . The code  $\widetilde{\mathcal{C}}_{10,3}$ generated by  $\widetilde{G}_{10,3}$  is the [12, 3, 6] SO code, which is a shortest SO embedding of  $C_{10,3}$ . Moreover,  $\widetilde{C}_{10,3}$  is optimal SO.

#### B. Algorithm for dimension 4

In this subsection, we introduce an algorithm that embeds an [n, 4] code to an SO code by Theorem III.8. We let

$$J_1(G) := \{i \in \{1, 2, \dots, 15\} \mid \ell_i \equiv_2 1\}$$

for a  $4 \times n$  matrix G.

The basic idea of the following algorithm is that we keep adding the smallest number of columns to G so that seven and eight  $\ell_i$ 's each have the same parity.

#### Algorithm IV.11.

- Input: A generator matrix G of an [n, 4] code.
- $\bullet$  Output: A generator matrix G for a shortest SO embed-

(C1) For 
$$s=1,2,\ldots,15,$$
 if  $|\mathcal{I}_s^{(1)}\cap J_1(G)|<4$  (resp.  $|\mathcal{I}_s^{(1)}\cap J_1(G)|<4$  (resp.  $|\mathcal{I}_s^{(1)}\cap J_1(G)|$ )

$$\mathfrak{n}_s^{(1)} \leftarrow |\mathcal{I}_s^{(1)} \cap J_1(G)| \quad \text{(resp. } \mathfrak{n}_s^{(1)} \leftarrow |\mathcal{I}_s^{(1)} \setminus J_1(G)| \text{)}.$$

(C2) For  $s=1,2,\ldots,15$ , if  $|\mathcal{I}_s^{(2)}\cap J_1(G)|\leq 4$  (resp.  $|\mathcal{I}_s|=4$  (resp.  $|\mathcal{I}_s^{(2)}\cap J_1(G)|\leq 4$  (resp.  $|\mathcal{$  $J_1(G)| > 4$ ), then let

$$\mathfrak{n}_s^{(2)} \leftarrow |\mathcal{I}_s^{(2)} \cap J_1(G)| \quad \text{(resp. } \mathfrak{n}_s^{(2)} \leftarrow |\mathcal{I}_s^{(2)} \setminus J_1(G)| \text{)}.$$

(C3) Find the smallest  $s_0 \in \{1, 2, \dots, 15\}$  such that

$$\mathfrak{n}_{s_0}^{(1)} + \mathfrak{n}_{s_0}^{(2)} = \min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \le s \le 15\}.$$

(C4) Put  $\widetilde{G} \leftarrow G$ . (C5) If  $|\mathcal{I}_{s_0}^{(1)} \cap J_1(\widetilde{G})| < 4$  (resp.  $|\mathcal{I}_{s_0}^{(1)} \cap J_1(\widetilde{G})| \ge 4$ ), then

$$\mathcal{I}^{(1)} \leftarrow \mathcal{I}_{s_0}^{(1)} \cap J_1(\widetilde{G}) \quad \text{(resp. } \mathcal{I}^{(1)} \leftarrow \mathcal{I}_{s_0}^{(1)} \setminus J_1(\widetilde{G})\text{)}.$$

(C6) If  $\mathcal{I}^{(1)} \neq \emptyset$ , then take the smallest  $i_0 \in \mathcal{I}^{(1)}$  and put

$$\widetilde{G} \leftarrow \left[ \widetilde{G} \| \mathbf{h}_{i_0} \right].$$

Go to (C5). Otherwise, go to (C7). (C7) If  $|\mathcal{I}_{s_0}^{(2)} \cap J_1(\widetilde{G})| \leq 4$  (resp.  $|\mathcal{I}_{s_0}^{(2)} \cap J_1(\widetilde{G})| > 4$ ), then

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_{s_0}^{(2)} \cap J_1(\widetilde{G}) \quad \text{(resp. } \mathcal{I}^{(2)} \leftarrow \mathcal{I}_{s_0}^{(2)} \setminus J_1(\widetilde{G})\text{)}.$$

(C8) If  $\mathcal{I}^{(2)} \neq \emptyset$ , then take the smallest  $i_0 \in \mathcal{I}^{(2)}$  and put

$$\widetilde{G} \leftarrow \left[ \widetilde{G} \| \mathbf{h}_{i_0} \right].$$

Go to (C7). Otherwise, terminate the algorithm.

For readers' convenience, we give Algorithm IV.11 written in MAGMA in Appendix C. With the resulting matrix G of Algorithm IV.11, we let

 $\widetilde{\mathcal{C}}$  := the linear code generated by  $\widetilde{G}$ .

Then we have the following theorem.

**Theorem IV.12.** Let C be an [n, 4] code generated by G. Then  $\widetilde{\mathcal{C}}$  is a shortest SO embedding of  $\mathcal{C}$ .

*Proof.* It will be proved in Appendix B. 

**Remark IV.13.** In Algorithm IV.11,  $\widetilde{G}$  is obtained by juxtaposing G and h<sub>i</sub>'s for  $i \in \mathcal{I}^{(1)}(G) \cup \mathcal{I}^{(2)}(G)$ , where

$$\begin{split} \mathcal{I}^{(1)}(G) &:= \begin{cases} \mathcal{I}_{s_0}^{(1)} \cap J_1(G) & \text{if } |\mathcal{I}_{s_0}^{(1)} \cap J_1(G)| < 4, \\ \mathcal{I}_{s_0}^{(1)} \setminus J_1(G) & \text{if } |\mathcal{I}_{s_0}^{(1)} \cap J_1(G)| \ge 4, \end{cases} \\ \mathcal{I}^{(2)}(G) &:= \begin{cases} \mathcal{I}_{s_0}^{(2)} \cap J_1(G) & \text{if } |\mathcal{I}_{s_0}^{(1)} \cap J_1(G)| \le 4, \\ \mathcal{I}_{s_0}^{(2)} \setminus J_1(G) & \text{if } |\mathcal{I}_{s_0}^{(1)} \cap J_1(G)| > 4. \end{cases} \end{split}$$

**Proposition IV.14.** A shortest SO embedding of an [n, 4] code can be obtained by adding at most 5 columns.

*Proof.* For each  $s \in \{1, 2, \dots, 15\}$ , denote the sets  $\mathcal{I}_s^{(1)}$  and  $\mathcal{I}_{s}^{(2)}$  given in Table III.2 by

$${a_{s,1} < a_{s,2} < \cdots < a_{s,7}}$$
 and  ${b_{s,1} < b_{s,2} < \cdots < b_{s,8}}$ ,

respectively. Let

$$\mathbf{v}_s = (v_{s,1}, v_{s,2}, \dots, v_{s,15})$$
  
:=  $(a_{s,1}, a_{s,2}, \dots, a_{s,7}, b_{s,1}, b_{s,2}, \dots, b_{s,8}) \in (\mathbb{Z}^+)^{15}$ .

By the definition of  $\mathbf{v}_s$ , one can  $\{v_{s,1}, v_{s,2}, \dots, v_{s,15}\}$  $\{1, 2, \dots, 15\}$  for =  $s \in \{1, 2, ..., 15\}$ . Therefore, for each  $s \in \{1, 2, ..., 15\}$ , there is a permutation  $\sigma_s$  of  $\{1, 2, \dots, 15\}$  such that

$$\sigma_s(\mathbf{v}_s) := (\sigma_s(v_{s,1}), \sigma_s(v_{s,2}), \dots, \sigma_s(v_{s,15}))$$
$$= (1, 2, \dots, 15) = \mathbf{v}_1.$$

It is checked by MAGMA that for each  $s, i \in \{1, 2, ..., 15\}$ , there exists  $j(s,i) \in \{1,2,\ldots,15\}$  such that

$$\sigma_s(\mathcal{I}_i^{(1)}) = \{\sigma_s(a_{i,1}), \dots, \sigma_s(a_{i,7})\} = \mathcal{I}_{j(s,i)}^{(1)},$$
  

$$\sigma_s(\mathcal{I}_i^{(2)}) = \{\sigma_s(b_{i,1}), \dots, \sigma_s(b_{i,8})\} = \mathcal{I}_{j(s,i)}^{(2)}.$$
(IV.2)

Assume that Algorithm IV.11 is applied to a generator matrix G of an [n,4] code. Note that  $s_0 \in \{1,2,\ldots,15\}$ is obtained in Step (C3). For each  $i \in \{1, 2, ..., 15\}$ , Equation (IV.2) gives that

$$\mathcal{I}_{i}^{(t)} \cap J_{1}(G) = \sigma_{s_{0}}^{-1} \left( \mathcal{I}_{j(s_{0},i)}^{(t)} \right) \cap J_{1}(G), 
\mathcal{I}_{i}^{(t)} \setminus J_{1}(G) = \sigma_{s_{0}}^{-1} \left( \mathcal{I}_{j(s_{0},i)}^{(t)} \right) \setminus J_{1}(G)$$
(IV.3)

for t=1,2. Note that  $\sigma_{s_0}$  is a bijection. Then, by Equation (IV.3), for t = 1, 2,

$$|\mathcal{I}_{i}^{(t)} \cap J_{1}(G)| = |\mathcal{I}_{j(s_{0},i)}^{(t)} \cap \sigma_{s_{0}}(J_{1}(G))|,$$
  

$$|\mathcal{I}_{i}^{(t)} \setminus J_{1}(G)| = |\mathcal{I}_{j(s_{0},i)}^{(t)} \setminus \sigma_{s_{0}}(J_{1}(G))|.$$
(IV.4)

Let us consider a new algorithm obtained by replacing

$$\begin{split} & \mathcal{I}_{i}^{(t)} \cap J_{1}(G) \text{ by } \mathcal{I}_{i}^{(t)} \cap \sigma_{s_{0}}(J_{1}(G)), \\ & \mathcal{I}_{i}^{(t)} \setminus J_{1}(G) \text{ by } \mathcal{I}_{i}^{(t)} \setminus \sigma_{s_{0}}(J_{1}(G)), \text{ and} \\ & \mathsf{h}_{i_{0}} \text{ by } \mathsf{h}_{\sigma_{s_{0}}^{-1}(j_{0})} \end{split} \tag{IV.5}$$

in Algorithm IV.11. To emphasize the difference between this new algorithm and Algorithm IV.11, we denote the integer  $\mathfrak{n}_s^{(1)}(\text{resp. }\mathfrak{n}_s^{(2)})$  obtained in Step (C1) (resp. (C2)) of the new algorithm by  $\widehat{\mathfrak{n}}_s^{(1)}(\text{resp. }\widehat{\mathfrak{n}}_s^{(2)})$ ,  $s_0$  in Step (C3) by  $\widehat{s}_0$ , and  $\widetilde{G}$  by  $\widehat{G}$ . Then, by Equation (IV.4), we have

$$\mathfrak{n}_s^{(1)} = \widehat{\mathfrak{n}}_{j(s_0,s)}^{(1)} \quad \text{and} \quad \widehat{\mathfrak{n}}_s^{(2)} = \mathfrak{n}_{j(s_0,s)}^{(2)}.$$

Therefore, the facts that  $j(s_0, s_0) = 1$  and

$$\mathfrak{n}_{s_0}^{(1)} + \mathfrak{n}_{s_0}^{(2)} = \min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \le s \le 15\}$$

imply  $\hat{s}_0 = 1$ .

Note that Remark IV.13 can be modified as follows:  $\widehat{G}$  is obtained by juxtaposing G and  $\mathsf{h}_{\sigma_{s_0}^{-1}(i)}$ 's for  $i \in \widehat{\mathcal{I}}^{(1)}(G) \cup \widehat{\mathcal{I}}^{(2)}(G)$ , where

$$\widehat{\mathcal{I}}^{(1)}(G) := \begin{cases}
\mathcal{I}_{1}^{(1)} \cap \sigma_{s_{0}}(J_{1}(G)) & \text{if } |\mathcal{I}_{1}^{(1)} \cap \sigma_{s_{0}}(J_{1}(G))| < 4, \\
\mathcal{I}_{1}^{(1)} \setminus \sigma_{s_{0}}(J_{1}(G)) & \text{if } |\mathcal{I}_{1}^{(1)} \cap \sigma_{s_{0}}(J_{1}(G))| \ge 4, \\
\widehat{\mathcal{I}}^{(2)}(G) := \begin{cases}
\mathcal{I}_{1}^{(2)} \cap \sigma_{s_{0}}(J_{1}(G)) & \text{if } |\mathcal{I}_{1}^{(2)} \cap \sigma_{s_{0}}(J_{1}(G))| \le 4, \\
\mathcal{I}_{1}^{(2)} \setminus \sigma_{s_{0}}(J_{1}(G)) & \text{if } |\mathcal{I}_{1}^{(2)} \cap \sigma_{s_{0}}(J_{1}(G))| > 4.
\end{cases}$$

One can see that

$$\{\mathsf{h}_i \mid i \in \mathcal{I}^{(t)}(G)\} = \{\mathsf{h}_{\sigma_{s_0}^{-1}(i)} \mid i \in \widehat{\mathcal{I}}^{(t)}(G)\}. \tag{IV.6}$$

for t=1,2. For instance, assume that  $|\mathcal{I}_{s_0}^{(1)}\cap J_1(G)|<4$ . Then, by Equation (IV.4),  $|\mathcal{I}_{j(s_0,i)}^{(t)}\cap\sigma_{s_0}(J_1(G))|<4$ . Thus, we have

$$\mathcal{I}^{(1)}(G) = \mathcal{I}_{s_0}^{(1)} \cap J_1(G)$$
 and  $\widehat{\mathcal{I}}^{(1)}(G) = \mathcal{I}_1^{(1)} \cap \sigma_{s_0}(J_1(G)).$ 

Observe that

$$\begin{split} \{\mathsf{h}_i \mid i \in \mathcal{I}^{(1)}(G)\} &= \{\mathsf{h}_i \mid i \in \mathcal{I}_{s_0}^{(1)} \cap J_1(G)\} \\ &= \left\{\mathsf{h}_{\sigma_{s_0}^{-1}(i)} \mid i \in \sigma_{s_0} \left(\mathcal{I}_{s_0}^{(1)} \cap J_1(G)\right)\right\} \\ &= \left\{\mathsf{h}_{\sigma_{s_0}^{-1}(i)} \mid i \in \mathcal{I}_1^{(1)} \cap \sigma_{s_0}(J_1(G))\right\} \\ &= \left\{\mathsf{h}_{\sigma_{s_0}^{-1}(i)} \mid i \in \widehat{\mathcal{I}}^{(1)}(G)\right\}. \end{split}$$

By Remark IV.13, the above modified remark, and Equation (IV.6),  $\widehat{G}$  and  $\widehat{G}$  are equal up to column permutations. This implies that it suffices to consider all the possible cases only when s=1 to show the extended length by Algorithm IV.11 is at most five.

For the remaining part of the proof, we prove the following two claims.

**Claim 1.** The difference between the length of  $\widetilde{\mathcal{C}}$  and that of  $\mathcal{C}$  cannot be 7.

*Proof.* It is easy to know that  $0 \le \mathfrak{n}_1^{(1)} + \mathfrak{n}_1^{(2)} \le 7$  in Step (C3) of Algorithm IV.11 since  $0 \le \mathfrak{n}_1^{(1)} \le 3$  and  $0 \le \mathfrak{n}_1^{(2)} \le 4$ . Note that  $\mathfrak{n}_1^{(1)} + \mathfrak{n}_1^{(2)} = 7$  only when  $\mathfrak{n}_1^{(1)} = 3$  and  $\mathfrak{n}_1^{(2)} = 4$ . Therefore, the number of all possible cases is  $\binom{7}{3} \cdot \binom{8}{4} = 2450$ .

The values of  $\min\{\mathfrak{n}_s^{(1)}+\mathfrak{n}_s^{(2)}\mid 1\leq s\leq 15\}$  for such cases in Step (C3) are checked by MAGMA to find out that they are at most five.

**Claim 2.** The difference between the length of  $\widetilde{\mathcal{C}}$  and that of  $\mathcal{C}$  cannot be 6.

*Proof.* Similarly,  $\mathfrak{n}_1^{(1)} + \mathfrak{n}_1^{(2)} = 6$  only when  $\mathfrak{n}_1^{(1)} = \mathfrak{n}_1^{(2)} = 3$ , or  $\mathfrak{n}_1^{(1)} = 2$  and  $\mathfrak{n}_1^{(2)} = 4$ . Therefore, the number of possible cases are  $\binom{7}{3} \cdot \binom{8}{3} + \binom{7}{2} \cdot \binom{8}{4} = 1960 + 1470 = 3430$ . The values of  $\min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \le s \le 15\}$  for such cases in Step (C3) are checked by MAGMA to find out that they are at most five.

Hence a shortest SO embedding of an [n, 4] code can be constructed by juxtaposing at most five columns.

#### Example IV.15.

(1) Let  $C_{7,4}$  be the [7,4] Hamming code and take its generator matrix

$$G_{7,4} = \left[ egin{array}{ccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{array} 
ight].$$

Since  $\ell_1 = \ell_2 = \ell_4 = \ell_7 = \ell_8 = \ell_{11} = \ell_{13} = 1$ , we have

$$J_1(G_{7,4}) = \{ i \in \{1, 2, \dots, 15\} \mid \ell_i \equiv_2 1 \}$$
  
= \{1, 2, 4, 7, 8, 11, 13\}.

Applying Steps (C1) and (C2), we have

														14		_
$\mathfrak{n}_s^{(1)}$	3	3	3	3	3	3	3	3	3	3	3	3	3	3	0	
$\mathfrak{n}_s^{(2)}$	3	3	4	4	4	3	3	4	3	3	4	4	4	4	1	

Thus,  $s_0$  is set to be 15 in Step (C3). Recall that

$$\mathcal{I}_{15}^{(1)} = \{3, 5, 6, 9, 10, 12, 15\}$$

and

$$\mathcal{I}_{15}^{(2)} = \{1, 2, 4, 7, 8, 11, 13, 14\}.$$

In Step (C4), we put  $\widetilde{G}_{7,4} \leftarrow G_{7,4}$  and in Step (C5), we let

$$\mathcal{I}^{(1)} \leftarrow \mathcal{I}_{15}^{(1)} \cap J_1(\widetilde{G}_{7,4}) = \emptyset.$$

Thus, we pass Step (C6) and go to Step (C7). In Step (C7), let

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_{15}^{(2)} \setminus J_1(\widetilde{G}_{7,4}) = \{14\}.$$

Applying Step (C8), we put

$$\widetilde{G}_{7,4} \leftarrow \left[ egin{array}{ccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array} \right].$$

Go to Step (C7) and repeat the steps. Since  $J_1(\widetilde{G}_{7,4}) = \emptyset$ ,

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_{15}^{(2)} \cap J_1(\widetilde{G}_{7,4}) = \emptyset$$

and thus, the algorithm is terminated.

The code  $\widetilde{C}_{7,4}$  generated by  $\widetilde{G}_{7,4}$  is a shortest SO embedding of  $\mathcal{C}_{7,4}$ , which is optimal as well. Notice that  $\widetilde{C}_{7,4}$  is the [8,4] extended Hamming code.

(2) Let  $C_{5,4}$  be an optimal [5,4,2] code generated by

$$G_{5,4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that  $\ell_1 = \ell_2 = \ell_4 = \ell_8 = \ell_{15}$ . Therefore,

$$J_1(G_{5,4}) = \{i \in \{1, 2, \dots, 15\} \mid \ell_i \equiv_2 1\}$$
  
= \{1, 2, 4, 8, 15\}.

Applying Step (C1) and Step (C2), we have

s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathfrak{n}_s^{(1)}$	3	3	3	3	3	3	1	3	3	3	1	3	1	1	1
$\mathfrak{n}_s^{(2)}$	2	2	2	2	2	2	4	2	2	2	4	2	4	4	4

Thus,  $s_0$  is set to be 1 in Step (C3). Recall that

$$\mathcal{I}_{1}^{(1)} = \{1, 2, 3, 4, 5, 6, 7\}$$

and

$$\mathcal{I}_{1}^{(2)} = \{8, 9, 10, 11, 12, 13, 14, 15\}.$$

In Step (C4), we put  $\widetilde{G}_{5,4} \leftarrow G_{5,4}$  and in Step (C5), we let

$$\mathcal{I}^{(1)} \leftarrow \mathcal{I}_1^{(1)} \cap J_1(\widetilde{G}_{5,4}) = \{1, 2, 4\}.$$

Applying Step (C6), we put

$$\widetilde{G}_{5,4} \leftarrow \left[ egin{array}{ccc|c} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array} \right].$$

Go to Step (C5) and repeat the steps. Then, we have

$$\widetilde{G}_{5,4} \leftarrow \left[ egin{array}{ccc|c} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array} 
ight].$$

and  $J_1(\widetilde{G}_{5,4})=\{8,15\}$ . Therefore, we have

$$\mathcal{I}^{(1)} = \mathcal{I}_1^{(1)} \cap J_1(\widetilde{G}_{5,4}) = \emptyset$$

and thus, go to Step (C7). In Step (C7), let

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_1^{(2)} \cap J_1(\widetilde{G}_{5,4}) = \{8, 15\}.$$

Applying Step (C8), we put

Go to Step (C7) and repeat the steps. Then we have

and  $J_1(\widetilde{G}_{5,4}) = \emptyset$ . Therefore,

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_1^{(2)} \cap J_1(\widetilde{G}_{5,4}) = \emptyset$$

and thus, we terminate the algorithm in Step (C8).

The code  $\widetilde{C}_{5,4}$  generated by  $\widetilde{G}_{5,4}$  is an [10,4,4] SO code, which is a shortest SO embedding of  $C_{5,4}$ . Moreover,  $\widetilde{C}_{5,4}$  is optimal SO.

#### Remark IV.16.

(1) For simplicity of the algorithm, we chose  $s_0$  to be the smallest among  $1 \le s \le 15$  satisfying the condition

$$\mathfrak{n}_{s_0}^{(1)} + \mathfrak{n}_{s_0}^{(2)} = \min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \le s \le 15\}$$

in Step (C3) of Algorithm IV.11. Notice that even if we choose any element  $1 \le s_0' \le 15$  satisfying

$$\mathfrak{n}_{s'_0}^{(1)} + \mathfrak{n}_{s'_0}^{(2)} = \min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \leq s \leq 15\},$$

Algorithm IV.11 still gives a shortest SO embedding.

(2) For simplicity of the algorithm, when  $|\mathcal{I}_{s_0}^{(2)} \cap J_1(\widetilde{G})| = 4$ , we let

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}^{(2)}_{s_0} \cap J_1(\widetilde{G})$$

in Step (C7) of Algorithm IV.11. However, even if we let

$$\mathcal{I}^{(2)} \leftarrow \mathcal{I}_{s_0}^{(2)} \setminus J_1(\widetilde{G}),$$

Algorithm IV.11 still gives a shortest SO embedding.

#### C. Algorithm for dimension 5

In this subsection, we introduce an algorithm that embeds an [n, 5] code to an SO code using Algorithm IV.11. Recall that for a matrix G, we denote by  $r_i(G)$  the ith row of G.

The basic idea of the following algorithm is as follows.

- We add at most two columns to G to obtain a matrix  $G_1$  satisfying  $r_{j_0}(G_1) \cdot r_i(G_1) \equiv_2 0$  for  $j_0 \in \{1, 2, \dots, 5\}$  and all  $1 \leq i \leq 5$ .
- For the submatrix  $G_2$  which consists of the rows of  $G_1$  except  $r_{j_0}(G_1)$ , apply Algorithm IV.11 (let  $\widetilde{G}_2$  be the resulting matrix).
- Juxtapose  $r_{j_0}(G_1)$  and a zero vector horizontally so that the resulting vector  $\tilde{r}_{j_0}(G_1)$  has the same number of coordinates to the number of columns of  $\tilde{G}_2$ .
- Attach  $\widetilde{r}_{j_0}(G_1)$  and  $\widetilde{G}_2$  vertically to obtain the matrix  $\widetilde{G}$ .

#### Algorithm IV.17.

- Input: A generator matrix G of an [n, 5] code.
- ullet Output: A generator matrix  $\widetilde{G}$  for an SO embedding.
- (**D**1) If there is  $j_0 \in \{1, 2, 3, 4, 5\}$  such that  $r_{j_0}(G) \cdot r_i(G) \equiv_2 0$  for all  $1 \leq i \leq 5$ , then put  $G_1 \leftarrow G$ , let  $G_2$  be the matrix obtained by deleting the  $j_0$ th row of  $G_1$ , and go to Step (**D**3). Otherwise, go to Step (**D**2).
- (**D**2) Do Step (**D**2-1) or Step (**D**2-2).
- (D2-1) Suppose  $r_{j_0}(G) \cdot r_{j_0}(G) \equiv_2 1$  for some  $1 \leq j_0 \leq 5$ . Then put

$$G_1 \leftarrow \begin{bmatrix} r_1(G) & y_1 \\ \vdots & \vdots \\ r_{j_0}(G) & 1 \\ \vdots & \vdots \\ r_5(G) & y_5 \end{bmatrix}.$$

where  $y_i = r_i(G) \cdot r_{j_0}(G)$ . Let  $G_2$  be the matrix obtained by deleting the  $j_0$ th row of  $G_1$ . Go to Step (D3).

(D2-2) Otherwise, that is, if  $r_j(G) \cdot r_j(G) \equiv_2 0$  for all  $1 \leq j \leq 5$ , then let  $j_0 = 1$  so that  $r_{j_0}(G) = r_1(G)$  and put

$$G_{1} \leftarrow \begin{bmatrix} r_{1}(G) & 1 & 1 \\ r_{2}(G) & y_{2} & 0 \\ \vdots & \vdots & \vdots \\ r_{5}(G) & y_{5} & 0 \end{bmatrix}.$$

where  $y_i = r_i(G) \cdot r_1(G)$ . Let  $G_2$  be the matrix obtained by deleting the first row of  $G_1$ . Go to Step (D3).

- (D3) Let  $\widetilde{G}_2$  be the resulting matrix when we input  $G_2$  into Algorithm IV.11 and let l be the difference between the number of columns of  $\widetilde{G}_2$  and  $G_2$ . Go to Step (D4).
- (D4) Let  $\widetilde{r}_{j_0}(G_1)$  be the vector obtained by juxtaposing  $r_{j_0}(G_1)$  and the zero vector  $\mathbf{0}$  of length l. Go to Step (D5)
- (D5) Let  $\widetilde{G}$  be the matrix obtained by putting together  $\widetilde{G}_2$  and  $\widetilde{r}_{i_0}(G_1)$ . Terminate the algorithm.

**Remark IV.18.** We can obtain an SO code from an [n, 5] code by adding at most 2 columns using algorithm IV.17 and adding at most 5 columns using algorithm IV.11. Therefore, we need at most 7 columns to get an SO code from an [n, 5] code.

With the resulting matrix  $\widetilde{G}$  of Algorithm IV.17, we let

 $\widetilde{\mathcal{C}}:=$  the linear code generated by  $\widetilde{G}.$ 

Then we have the following theorem.

**Theorem IV.19.** Let C be an [n,5] code generated by G. Then  $\widetilde{C}$  is an SO embedding of C.

*Proof.* By following the procedure of Algorithm IV.17 we easily see that the code  $\widetilde{C}$  is an SO embedding.

**Example IV.20.** Let  $C_{9,5}$  be an optimal [9,5,3] code generated by

$$G_{9,5} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Since there are no  $i_0 \in \{1,2,3,4,5\}$  such that  $r_{i_0}(G) \cdot r_i(G) \equiv_2 0$  for all  $1 \leq i \leq 5$ , we pass Step (D1). In Step (D2), since  $r_1(G) \cdot r_1(G) \equiv_2 1$ , we apply Step (D2-1). In Step (D2-1), we put

and by deleting  $r_1(G)$ , let

In Step (D3), we let

and l = 1. In Step (**D**4), we let

$$\widetilde{r}_1(G_1) = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1][0].$$

In Step (D5), let

The code  $\widetilde{C}_{9,5}$  generated by  $\widetilde{G}_{9,5}$  is an [11,5,4] SO code, which is an SO embedding of  $C_{9,5}$ . Moreover,  $\widetilde{G}_{9,5}$  is optimal SO.

**Remark IV.21.** One can embed an [n, 6] code to an SO code in the same manner as Algorithm IV.17. In fact, Algorithm IV.17 can be generalized recursively to construct an SO embedding for higher-dimensional linear codes. We leave this generalized SO embedding algorithm written in MAGMA in Appendix D.

## V. THE MINIMUM DISTANCES OF OPTIMAL LINEAR CODES AND OPTIMAL SO CODES

In this section, for k=1,2,3,4,5, we enumerate an upper bound on d(n,k) given by Griesmer [21] and the explicit formula for d(n,k) given by Baumert and McEliece[25]. Furthermore, we calculate explicit formulas of  $d_{\rm so}(n,k)$  when k=1,2,3,4, and  $d_{\rm so}(n,5)$  when  $n\not\equiv_{31}6,13,14,21,22,28,29$ .

First, let us consider the one-dimensional case. It is clear that for each  $n \in \mathbb{Z}^+$ , the only optimal [n,1] code is the repetition code. On the other hand, it is also trivial that an optimal [n,1] SO code for an even integer n is the [n,1,n] repetition code and that for an odd n is the [n,1,n-1] code which is obtained from [n-1,1,n-1] repetition code by attaching a zero coordinate. Thus, we have

$$d(n,1) = n$$
 and  $d_{so}(n,1) = \begin{cases} n & \text{if } n \equiv_2 0, \\ n-1 & \text{if } n \equiv_2 1. \end{cases}$ 

Next, let us consider the two-dimensional case. The optimal [n,2] codes are introduced in [26]. On the other hand, in [19, Section 3], the authors explained that optimal [n,2] codes are SO only for n=6m,6m+1,6m+4 and constructed all optimal [n,2] SO codes. Summing it up, one can immediately obtain the following theorem.

**Theorem V.1.** ([19], [25], [26]) The following explicit formulas hold.

(1) For n > 2, we have

$$d(n,2) = |2n/3|$$
.

(2) For n > 4,

$$d_{so}(n,2) = \begin{cases} \lfloor 2n/3 \rfloor & \text{if } n \equiv_6 0, 1, 4, \\ \lfloor 2n/3 \rfloor - 1 & \text{if } n \equiv_6 2, 5, \\ \lfloor 2n/3 \rfloor - 2 & \text{if } n \equiv_6 3. \end{cases}$$

Theorem V.1 implies the following corollar

**Corollary V.2.** For  $n \geq 4$ ,  $d(n,2) = d_{so}(n,2)$  if and only if  $n \equiv_6 0, 1, 4.$ 

Hereafter, we will also leave our proof for known results for the reader's understanding.

Now, let us consider the three-dimensional case. The Griesmer bound says that all [n, 3, d] codes satisfy that

$$n \ge d + \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{d}{4} \right\rceil = \begin{cases} 7d/4 & \text{if } d \equiv_4 0, \\ (7d+5)/4 & \text{if } d \equiv_4 1, \\ (7d+2)/4 & \text{if } d \equiv_4 2, \\ (7d+3)/4 & \text{if } d \equiv_4 3. \end{cases}$$
(V.1)

Manipulating this bound, one can obtain the following upper bound on d(n,3).

**Lemma V.3.** ([21]) For  $n \ge 3$ ,

$$d(n,3) \le \lfloor 4n/7 \rfloor - \delta(n \equiv_7 2)$$

$$= \begin{cases} \lfloor 4n/7 \rfloor & \text{if } n \not\equiv_7 2, \\ |4n/7| - 1 & \text{if } n \equiv_7 2. \end{cases}$$

*Proof.* From Equation (V.1), we have

$$d(n,3) \leq \max \begin{pmatrix} \left\{ d_1 \in \mathbb{Z}_{\geq 0} \ \middle| \ d_1 \equiv_4 \ 0 \ \text{and} \ \frac{7d_1}{4} \leq n \right\} \\ \cup \left\{ d_2 \in \mathbb{Z}_{\geq 0} \ \middle| \ d_2 \equiv_4 \ 1 \ \text{and} \ \frac{7d_2 + 5}{4} \leq n \right\} \\ \cup \left\{ d_3 \in \mathbb{Z}_{\geq 0} \ \middle| \ d_3 \equiv_4 \ 2 \ \text{and} \ \frac{7d_3 + 2}{4} \leq n \right\} \\ \cup \left\{ d_4 \in \mathbb{Z}_{\geq 0} \ \middle| \ d_4 \equiv_4 \ 3 \ \text{and} \ \frac{7d_4 + 3}{4} \leq n \right\} \\ \cup \left\{ d_4 \in \mathbb{Z}_{\geq 0} \ \middle| \ d_4 \equiv_4 \ 3 \ \text{and} \ \frac{7d_4 + 3}{4} \leq n \right\} \\ = \text{For (2) reorganizing the results in [19] Section 3], we have }$$

$$= \max \left\{ \begin{cases} \left\{ \frac{4t_1}{7} \mid t_1 \equiv_7 0 \text{ and } 0 \leq t_1 \leq n \right\} \\ \cup \left\{ \frac{4t_2 - 5}{7} \mid t_2 \equiv_7 3 \text{ and } 0 \leq t_2 \leq n \right\} \\ \cup \left\{ \frac{4t_3 - 2}{7} \mid t_3 \equiv_7 4 \text{ and } 0 \leq t_3 \leq n \right\} \\ \cup \left\{ \frac{4t_4 - 3}{7} \mid t_4 \equiv_7 6 \text{ and } 0 \leq t_4 \leq n \right\} \end{cases} \right\}, \quad \text{Inat for } n \geq 6,$$

$$d_{\text{so}}(n, 3) = \begin{cases} \lfloor 4n/7 \rfloor & \text{if } n \equiv_7 0, 1, 5 \text{ and } n \neq 5, \\ \lfloor 4n/7 \rfloor - 1 & \text{if } n \equiv_7 2, 3, 6, \\ \lfloor 4n/7 \rfloor - 2 & \text{if } n \equiv_7 4 \text{ or } n = 5. \end{cases}$$
From Theorem V.4, we immediately obtain the follow

by substituting

$$t_1 = \frac{7d_1}{4}, \ t_2 = \frac{7d_2 + 5}{4}, \ t_3 = \frac{7d_3 + 2}{4}, \ \text{and} \ t_4 = \frac{7d_4 + 3}{4}. \quad \text{\textbf{Corollary V.5.} For } n \geq 6, \ d(n,3) = d_{so}(n,3) \ \textit{if and only if } n \equiv_7 0, 1, 2, 5.$$

This gives us that

$$d(n,3) \le \begin{cases} 4m & \text{if } 7m \le n \le 7m+2, \\ 4m+1 & \text{if } n=7m+3, \\ 4m+2 & \text{if } n=7m+4,7m+5, \\ 4m+3 & \text{if } n=7m+6, \end{cases}$$
$$= \lfloor 4n/7 \rfloor - \delta(n \equiv_7 2).$$

In [19], the authors classified all optimal [n, 3] SO codes. It gives the minimum distances of optimal [n, 3] SO codes. Based on their results, one can obtain the following theorem.

**Theorem V.4.** ([19], [25]) The following explicit formulas hold:

(1) *For*  $n \ge 3$ ,

$$d(n,3) = \lfloor 4n/7 \rfloor - \delta(n \equiv_7 2)$$

$$= \begin{cases} \lfloor 4n/7 \rfloor & \text{if } n \not\equiv_7 2, \\ \lfloor 4n/7 \rfloor - 1 & \text{if } n \equiv_7 2. \end{cases}$$

(2) For  $n \ge 6$ ,

$$d_{so}(n,3) = \begin{cases} \lfloor 4n/7 \rfloor & \text{if } n \equiv_7 0, 1, 5, \\ \lfloor 4n/7 \rfloor - 1 & \text{if } n \equiv_7 2, 3, 6, \\ \lfloor 4n/7 \rfloor - 2 & \text{if } n \equiv_7 4. \end{cases}$$

*Proof.* For (1), by Lemma V.3, it suffices to find an [n, 3] code whose minimum distance is  $|4n/7| - \delta(n \equiv_7 2)$ .

For  $3 \le t \le 9$ , such codes are known in [22]. For  $3 \le t \le$ 9, let  $C_t$  be an  $[t, 3, \lfloor 4t/7 \rfloor - \delta(t \equiv_7 2)]$  code and  $G_t$  be a generator matrix of  $C_t$ . For any  $n \geq 10$ , we define

$$G_n := \left[ sH_3 \, \middle\| \, G_t \, \right],$$

$$4s + (\lfloor 4t/7 \rfloor - \delta(n \equiv_7 2)) = \lfloor 4n/7 \rfloor - \delta(n \equiv_7 2)$$

For (2), reorganizing the results in [19, Section 3], we have hat for n > 6,

$$d_{\rm so}(n,3) = \begin{cases} \lfloor 4n/7 \rfloor & \text{if } n \equiv_7 0, 1, 5 \text{ and } n \neq 5 \\ \lfloor 4n/7 \rfloor - 1 & \text{if } n \equiv_7 2, 3, 6, \\ \lfloor 4n/7 \rfloor - 2 & \text{if } n \equiv_7 4 \text{ or } n = 5. \end{cases}$$

From Theorem V.4, we immediately obtain the following

Let us consider the four-dimensional case. The Griesmer bound says that all [n, 4, d] codes satisfy that

$$n \geq d + \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{d}{4} \right\rceil + \left\lceil \frac{d}{8} \right\rceil$$

$$= \begin{cases} 15d/8 & \text{if } d \equiv_8 0, \\ (15d+17)/8 & \text{if } d \equiv_8 1, \\ (15d+10)/8 & \text{if } d \equiv_8 2, \\ (15d+11)/8 & \text{if } d \equiv_8 3, \\ (15d+4)/8 & \text{if } d \equiv_8 4, \\ (15d+13)/8 & \text{if } d \equiv_8 5, \\ (15d+6)/8 & \text{if } d \equiv_8 6, \\ (15d+7)/8 & \text{if } d \equiv_8 7. \end{cases}$$

As the previous cases, one can obtain the following upper bound on d(n, 4).

**Lemma V.6.** ([21]) For  $n \ge 4$ ,

$$d(n,4) \le \lfloor 8n/15 \rfloor - \delta(n \equiv_{15} 2, 3, 4, 6, 10)$$

$$= \begin{cases} \lfloor 8n/15 \rfloor & \text{if } n \not\equiv_{15} 2, 3, 4, 6, 10, \\ \lfloor 8n/15 \rfloor - 1 & \text{if } n \equiv_{15} 2, 3, 4, 6, 10. \end{cases}$$

*Proof.* It can be proved in the same manner as the proof of Lemma V.3.  $\Box$ 

**Theorem V.7.** ([25]) For  $n \ge 4$ ,

$$d(n,4) = \lfloor 8n/15 \rfloor - \delta(n \equiv_{15} 2, 3, 4, 6, 10)$$

$$= \begin{cases} \lfloor 8n/15 \rfloor & \text{if } n \not\equiv_{15} 2, 3, 4, 6, 10, \\ \lfloor 8n/15 \rfloor - 1 & \text{if } n \equiv_{15} 2, 3, 4, 6, 10. \end{cases}$$

*Proof.* By Lemma V.6, it suffices to find an [n, 4] code whose minimum distance is  $\lfloor 8n/15 \rfloor - \delta(n \equiv_{15} 2, 3, 4, 6, 10)$ .

For  $4 \le t \le 18$ , such codes are known in [22]. For  $4 \le t \le 18$ , let  $G_t$  be a generator matrix of an optimal [t,4] code. For any  $n \ge 19$ , we define

$$G_n := \left[ sH_4 \, \middle\| \, G_t \, \right],$$

where n=15s+t for  $4 \le t \le 18$ . Since the minimum distance of the simplex code  $S_4$  is 8,  $G_n$  generates the [n,4] code whose minimum distance is

$$8s + (\lfloor 8t/15 \rfloor - \delta(n \equiv_{15} 2, 3, 4, 6, 10))$$
  
=  $\lfloor 8n/15 \rfloor - \delta(n \equiv_{15} 2, 3, 4, 6, 10).$ 

**Lemma V.8.** For  $n \ge 4$  such that  $n \equiv_{15} 4$ , there are no  $\left[n, 4, \left\lfloor \frac{8n}{15} \right\rfloor - 1\right]$  SO codes.

*Proof.* Let n = 15t + 4 for some  $t \in \mathbb{Z}_{\geq 0}$ . Then we have

$$\left| \frac{8n}{15} \right| - 1 = 8t + 1 \equiv_2 1.$$

Since the minimum distances of SO codes should be even, our assertion follows.  $\Box$ 

In [10], the authors classified all optimal [n,4] SO codes for  $n \equiv_{15} 1, 2, 6, 7, 8, 9, 13, 14$  and proved that  $d_{\rm so}(n,4) < \lfloor 8n/15 \rfloor$  for each  $n \equiv_{15} 5, 12$ . Combining the results in [10], Lemma V.6, and Lemma V.8, we derive the following theorem.

**Theorem V.9.** For  $4 \le n \le 7$ , there are no [n, 4] SO codes and for n > 8,

$$d_{so}(n,4) = \begin{cases} \lfloor 8n/15 \rfloor & \text{if } n \equiv_{15} 0, 1, 8, 9, 13 \text{ and } n \neq 13, \\ \lfloor 8n/15 \rfloor - 1 & \text{if } n \equiv_{15} 2, 3, 6, 7, 10, 11, 14, \\ \lfloor 8n/15 \rfloor - 2 & \text{if } n \equiv_{15} 4, 5, 12 \text{ or } n = 13. \end{cases}$$
 (V.2)

*Proof.* We first note that for  $n \equiv_{15} 1, 2, 6, 7, 8, 9, 13, 14$ , the equation  $d_{\rm so}(n,4) = \left\lfloor \frac{8n}{15} \right\rfloor - \delta(n \in \mathcal{E}_1^{(4)}) - 2\delta(n \in \mathcal{E}_2^{(4)})$  follows from the classification for optimal [n,4] SO codes in [10].

For  $n \equiv_{15} 0, 3, 4, 5, 10, 11, 12$ , combining Theorem V.7 with the inequality  $d_{\rm so}(n,4) < \left\lfloor \frac{8n}{15} \right\rfloor$  from [10, Corollary 3.2] gives an upper bound  $d_{\rm so}(n,4) \leq \left\lfloor \frac{8n}{15} \right\rfloor - \delta(n \in \mathcal{E}_1^{(4)}) - 2\delta(n \in \mathcal{E}_2^{(4)})$ .

Now, let us find an [n,4] SO code whose minimum distance is equal to  $\left\lfloor \frac{8n}{15} \right\rfloor - \delta(n \in \mathcal{E}_1^{(4)}) - 2\delta(n \in \mathcal{E}_2^{(4)})$  for  $n \equiv_{15} 0, 3, 4, 5, 10, 11, 12$ . For  $4 \le n \le 18$ , optimal [n,4] SO codes are given in [19] whose minimum distances are equal to  $\left\lfloor \frac{8n}{15} \right\rfloor - \delta(n \in \mathcal{E}_1^{(4)}) - 2\delta(n \in \mathcal{E}_2^{(4)})$ . For  $4 \le t \le 18$ , let  $G_t^{\text{SO}}$  be a generator matrix of an optimal [t,4] SO code. For any n = 15s + t when  $s \ge 1$  and  $t \in \{4,5,10,11,12,15,18\}$ , define

$$G_n^{SO} := [sH_4 || G_t^{SO}].$$

Since the minimum distance of the simplex code  $S_4$  is 8,  $G_n^{SO}$  generates the [n,4] SO code whose minimum distance is

$$8s + (\lfloor 8t/15 \rfloor - \delta(n \equiv_{15} 3, 10, 11) - 2\delta(n \equiv_{15} 4, 5, 12))$$
  
=  $\lfloor 8n/15 \rfloor - \delta(n \equiv_{15} 3, 10, 11) - 2\delta(n \equiv_{15} 4, 5, 12).$ 

From Theorem V.7 and V.9, we immediately obtain the following corollary.

**Corollary V.10.** For  $n \ge 8$ ,  $d(n,4) = d_{so}(n,4)$  if and only if  $n \equiv_{15} 0, 1, 2, 3, 6, 8, 9, 10, 13$  and  $n \ne 13$ .

#### Remark V.11.

- (1) We have checked that any optimal linear code denoted by BKLC(GF(2), n, k) for k = 2, 3 and  $4 \le n \le 256$  in MAGMA database is embedded to an optimal self-orthogonal code using Algorithms IV.2 and IV.7.
- (2) We have also checked that any optimal linear code denoted by BKLC(GF(2), n, 4) for  $4 \le n \le 256$  in MAGMA database is embedded to an optimal self-orthogonal code using the modified version of Algorithm IV.11 in the sense of Remark IV.16.

**Example V.12.** Let  $C_{4,4}$  be an optimal [4,4,1] code generated by

$$G_{4,4} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

When we apply Algorithm IV.11 to  $G_{4,4}$ , we choose  $s_0 = 1$  in Step (C3) and obtain

$$\widetilde{G}_{4,4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the linear code  $\widetilde{\mathcal{C}}_{4,4}$  generated by  $\widetilde{G}_{4,4}$  is an [8,4,2] code. By Theorem V.9, one can see that  $d_{\rm so}(8,4)=4$  and thus  $\widetilde{\mathcal{C}}_{4,4}$  is not an optimal SO code.

On the other hand, if we modify Algorithm IV.11 by taking  $s_0 = 15$  in Step (C3) and letting  $\mathcal{I}_2 = \mathcal{I}_{s_0}^{(2)} \setminus J_1(\widetilde{G})$  in Step (C7), then we obtain the new matrix

$$\overline{G}_{4,4} = \left[ egin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} 
ight].$$

The code  $\overline{C}_{4,4}$  generated by  $\overline{G}_{4,4}$  is the well-known optimal SO code that is the extended Hamming [8,4,4] code.

Observing Remark V.11 and Example V.12, we conjecture the following.

**Conjecture V.13.** Any optimal [n, 4] code can be embedded to an optimal SO.

Finally, let us consider the five-dimensional case. The Griesmer bound says that all [n,5,d] codes satisfy that

$$n \geq d + \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{d}{4} \right\rceil + \left\lceil \frac{d}{8} \right\rceil + \left\lceil \frac{d}{16} \right\rceil$$

$$\begin{cases} 31d/16 & \text{if } d \equiv_{16} 0, \\ (31d+49)/16 & \text{if } d \equiv_{16} 1, \\ (31d+34)/16 & \text{if } d \equiv_{16} 2, \\ (31d+35)/16 & \text{if } d \equiv_{16} 3, \\ (31d+20)/16 & \text{if } d \equiv_{16} 4, \\ (31d+37)/16 & \text{if } d \equiv_{16} 6, \\ (31d+22)/16 & \text{if } d \equiv_{16} 6, \\ (31d+23)/16 & \text{if } d \equiv_{16} 7, \\ (31d+8)/16 & \text{if } d \equiv_{16} 8, \\ (31d+41)/16 & \text{if } d \equiv_{16} 9, \\ (31d+26)/16 & \text{if } d \equiv_{16} 10, \\ (31d+27)/16 & \text{if } d \equiv_{16} 11, \\ (31d+12)/16 & \text{if } d \equiv_{16} 12, \\ (31d+29)/16 & \text{if } d \equiv_{16} 13, \\ (31d+14)/16 & \text{if } d \equiv_{16} 14, \\ (31d+15)/16 & \text{if } d \equiv_{16} 15. \end{cases}$$

$$(V.3)$$

As the previous cases, one can obtain the following upper bound on d(n, 5).

**Lemma V.14.** ([21]) For  $n \ge 5$ ,

$$d(n,5) \le \lfloor 16n/31 \rfloor - \delta(n \in E_1) - 2\delta(n \in E_2)$$

$$= \begin{cases} \lfloor 16n/31 \rfloor & \text{if } n \notin E_1 \cup E_2, \\ \lfloor 16n/31 \rfloor - 1 & \text{if } n \in E_1, \\ |16n/31| - 2 & \text{if } n \in E_2, \end{cases}$$

where

$$E_1 := \left\{ i \mid i \equiv_{31} 2, 3, 5, 6, 7, 8, 10, 11, 12, \atop 14, 18, 19, 20, 22, 26 \right\},$$
  
$$E_2 := \left\{ i \mid i \equiv_{31} 4 \right\}.$$

*Proof.* It can be proved in the same manner as the proof of Lemma V.3.  $\Box$ 

From the above lemma, one can derive an explicit formula for d(n,5) as the following theorem.

**Theorem V.15.** ([25]) For  $n \ge 5$ ,

$$d(n,5) = \lfloor 16n/31 \rfloor - \delta(n \in \mathcal{E}_1) - 2\delta(n \in \mathcal{E}_2)$$

$$= \begin{cases} \lfloor 16n/31 \rfloor & \text{if } n \notin \mathcal{E}_1 \cup \mathcal{E}_2, \\ \lfloor 16n/31 \rfloor - 1 & \text{if } n \in \mathcal{E}_1, \\ \lfloor 16n/31 \rfloor - 2 & \text{if } n \in \mathcal{E}_2, \end{cases}$$

where  $\mathcal{E}_1 := E_1 \cup \{9, 13\} \setminus \{8, 12\}$  and  $\mathcal{E}_2 := E_2 \cup \{8, 12\}$ .

*Proof.* It can be proved in the same manner as the proof of Theorem V.7.  $\Box$ 

To obtain an explicit formula for  $d_{so}(n, 5)$ , we prove the following lemma.

**Lemma V.16.** For  $n \geq 5$ , if

- (1) n = 13,
- (2)  $n \equiv_{31} 12 \text{ and } n \neq 12, \text{ or }$
- (3)  $n \equiv_{31} 5, 8, 15, 20, 23, 27, 30,$

then there are no [n, 5, d(n, 5)] SO codes.

**Proof.** By Theorem V.15, d(n,5) is odd if n satisfies either (1), (2), or (3) for  $n \ge 5$ . Since the minimum distances of SO codes should be even, our assertion follows.

Now, we obtain the following theorem.

**Theorem V.17.** For  $n \ge 11$ , if n = 13 or  $n \not\equiv_{31} 6, 13, 14, 21, 22, 28, 29$ , then

$$d_{so}(n,5) = \lfloor 16n/31 \rfloor - \delta(n \in \mathcal{E}_{1}^{SO}) - 2\delta(n \in \mathcal{E}_{2}^{SO})$$

$$= \begin{cases} \lfloor 16n/31 \rfloor & \text{if } n \notin \mathcal{E}_{1}^{SO} \cup \mathcal{E}_{2}^{SO}, \\ \lfloor 16n/31 \rfloor - 1 & \text{if } n \in \mathcal{E}_{1}^{SO}, \\ \lfloor 16n/31 \rfloor - 2 & \text{if } n \in \mathcal{E}_{2}^{SO}, \end{cases}$$
(V.4)

where

$$\mathcal{E}_{1}^{SO} := \{ i \mid i \equiv_{31} 2, 3, 7, 10, 11, 15, 18, 19, 23, 26, 27, 30 \},$$

$$\mathcal{E}_{2}^{SO} := \{ i \mid i \equiv_{31} 4, 5, 8, 12, 20 \} \cup \{13\}.$$

*Proof.* Combining Theorem V.15 with Lemma V.16 gives an upper bound  $d_{\rm so}(n,5) \leq \left\lfloor \frac{16n}{31} \right\rfloor - \delta(n \in \mathcal{E}_1^{(5)}) - 2\delta(n \in \mathcal{E}_2^{(5)})$ . Therefore, it suffices to find an [n,5] SO code whose minimum distance is equal to the right-hand side of Equation (V.4).

For  $11 \le t \le 40$ , optimal SO codes are given in [19]. Their minimum distances are equal to the right-hand side of Equation (V.4). For  $10 \le t \le 40$ , let  $G_t^{\rm SO}$  be a generator matrix of an optimal [t,5] SO code.

For any n=31s+t when  $s\geq 1$  and  $t\in \{i\in \{11,12,\ldots,41\}\mid i\neq 13,14,21,22,28,29,37\}$ , we define  $G_n^{\mathrm{SO}}:=\lceil sH_5\, \|\, G_t^{\mathrm{SO}}\, \rceil$ .

Since the minimum distance of the simplex code  $S_5$  is 15,  $G_n^{\rm SO}$  generates the [n,5] SO code whose minimum distance is

$$\begin{split} &15s + (\lfloor 16t/31 \rfloor - \delta(n \in \mathcal{E}_{1}^{\mathrm{SO}}) - 2\delta(n \in \mathcal{E}_{2}^{\mathrm{SO}})) \\ &= \lfloor 16n/31 \rfloor - \delta(n \in \mathcal{E}_{1}^{\mathrm{SO}}) - 2\delta(n \in \mathcal{E}_{2}^{\mathrm{SO}}). \end{split}$$

#### Remark V.18.

- (1) Combining Theorems V.15 and V.17, one can see that there are [n-1,5,d(n,5)-2] SO codes if  $n\equiv_{31}6,13,14,21,22,28,29$  for  $n\geq 10$ . Therefore, one can obtain an [n,5,d(n,5)-2] SO code by attaching a zero coordinate to an [n-1,5,d(n,5)-2] SO code.
- (2) For  $10 \le n \le 100$ ,  $n \equiv_{31} 6, 13, 14, 21, 22, 28, 29$ , and  $n \ne 13$ , we choose  $2^{16}$  random [n, 5] SO codes and calculate their minimum distances using MAGMA [23]. In our calculation, there was no SO code whose minimum distance is equal to d(n, 5).

Based on Table 1 in [19] and Remark V.18, we leave the following conjecture.

**Conjecture V.19.** For  $n \ge 10$ , if  $n \ne 13$  and  $n \equiv_{31} 6, 13, 14, 21, 22, 28, 29, then$ 

$$d_{so}(n,5) = d(n,5) - 2,$$

that is, there are no [n, 5, d(n, 5)] SO codes.

#### VI. CONCLUSION

We have introduced a characterization on self-orthogonality for given binary linear codes in terms of the number of column vectors in its generator matrix. In particular, we have described the characterization explicitly for each k = 2, 3, 4.

As a consequence of our characterizations, for each k=2,3,4, we have proposed algorithms that embed an [n,k] code to a self-orthogonal code by minimum lengthening. We have also suggested an algorithm that embeds an [n,k] code to a self-orthogonal code for k>4.

We have also given new explicit formulas for the minimum distances of optimal linear codes for dimensions 4 and 5 and those of optimal self-orthogonal codes for any length n with dimension 4 and any length  $n \not\equiv 6, 13, 14, 21, 22, 28, 29 \pmod{31}$  with dimension 5.

Using our explicit formulas and MAGMA, we have obtained that the above algorithms embed optimal linear codes into optimal self-orthogonal codes for  $n \le 256$  and k = 2, 3, 4.

Finally, we have suggested two conjectures in Conjecture V.13 and Conjecture V.19.

#### APPENDIX

#### A. Proof of Theorem III.8

*Proof.* ( $\Leftarrow$ ) We prove the case where s=1 since the other cases can be easily shown in the same manner. By the hypothesis, for each  $t=1,2,\ \ell_i\equiv_2\ell_j$  for  $i,j\in\mathcal{I}_1^{(t)}$ , that is,

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 \ell_4 \equiv_2 \ell_5 \equiv_2 \ell_6 \equiv_2 \ell_7 \quad \text{and}$$

$$\ell_8 \equiv_2 \ell_9 \equiv_2 \ell_{10} \equiv_2 \ell_{11} \equiv_2 \ell_{12} \equiv_2 \ell_{13} \equiv_2 \ell_{14} \equiv_2 \ell_{15}.$$
(A.1)

Combining Equations (III.4) and (A.1), we have

$$|\mathbf{I}(1)| = \ell_1 + \ell_3 + \ell_5 + \ell_7 + \ell_9 + \ell_{11} + \ell_{13} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(2)| = \ell_2 + \ell_3 + \ell_6 + \ell_7 + \ell_{10} + \ell_{11} + \ell_{14} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(3)| = \ell_4 + \ell_5 + \ell_6 + \ell_7 + \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(4)| = \ell_8 + \ell_9 + \ell_{10} + \ell_{11} + \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(1) \cap \mathbf{I}(2)| = \ell_3 + \ell_7 + \ell_{11} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(1) \cap \mathbf{I}(3)| = \ell_5 + \ell_7 + \ell_{13} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(1) \cap \mathbf{I}(4)| = \ell_9 + \ell_{11} + \ell_{13} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(2) \cap \mathbf{I}(3)| = \ell_6 + \ell_7 + \ell_{14} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(2) \cap \mathbf{I}(4)| = \ell_{10} + \ell_{11} + \ell_{14} + \ell_{15} \equiv_2 0,$$

$$|\mathbf{I}(3) \cap \mathbf{I}(4)| = \ell_{12} + \ell_{13} + \ell_{14} + \ell_{15} \equiv_2 0.$$

Therefore,  $|\mathbf{I}(j) \cap \mathbf{I}(j')|$  is even for all  $0 < j \le j' \le 4$  and hence  $\mathcal{C}$  is SO by Theorem III.2.

 $(\Rightarrow)$  For j=0,1, let

$$J_j(G) := \{i \in \{1, 2, \dots, 15\} \mid \ell_i \equiv_2 j\}.$$

Note that one of  $|J_0(G)|$  and  $|J_1(G)|$  is larger than or equal to 8. Thus, we can choose  $I:=\{i_1< i_2< \cdots < i_8\}\subset \{1,2,\ldots,15\}$  so that

$$\ell_{i_k} \equiv_2 \ell_{i_{k'}}$$
 for  $k, k' = 1, 2, \dots, 8$ . (A.2)

Note that there exists a subset  $S:=\{j_1,j_2,j_3,j_4\}$  of  $\{i_1,i_2,\ldots,i_8\}$  such that

$$S \subset \{1, 2, \dots, 7\}$$
 or  $S \subset \{8, 9, \dots, 15\}$ .

Suppose that  $S \subset \{1, 2, \dots, 7\}$ .

**Claim A.1.** If there is  $s \in \{1, 2, ..., 7\}$  such that

$$\mathcal{P}_s^{(1)} = \{(j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i')\}$$

for a permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i' \in \{1,2,\ldots,7\}$ , then

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 \ell_4 \equiv_2 \ell_5 \equiv_2 \ell_6 \equiv_2 \ell_7.$$

Note that we consider (i, j) and (j, i) the same.

**Claim A.2.** If there are no  $s \in \{1, 2, ..., 7\}$  such that

$$\mathcal{P}_s^{(1)} = \{ (j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i') \}$$

for any permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i' \in \{1,2,\ldots,7\}$ , then S should be one of the following sets:

$$\{1,2,4,7\}, \{1,2,5,6\}, \{1,3,4,6\}, \{1,3,5,7\}, \{2,3,4,5\}, \{2,3,6,7\}, \{4,5,6,7\}.$$

For Claim A.1, suppose that there is  $s \in \{1, 2, \dots, 7\}$  such that

$$\mathcal{P}_s^{(1)} = \{(j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i')\}$$

for a permutation  $\sigma$  of  $\{1, 2, 3, 4\}$  and  $i, i' \in \{1, 2, ..., 7\}$ . Since  $\mathcal{C}$  is SO, we have the following by Lemma III.6.

$$\ell_{j_{\sigma(1)}} + \ell_{j_{\sigma(2)}} \equiv_2 \ell_{j_{\sigma(3)}} + \ell_i \equiv_2 \ell_{j_{\sigma(4)}} + \ell_{i'}.$$

The following equivalence is obtained by the condition (A.2) since  $S \subset \{i_1, i_2, \dots, i_8\}$ .

$$\ell_{i_1} \equiv_2 \ell_{i_2} \equiv_2 \ell_{i_3} \equiv_2 \ell_{i_4} \equiv_2 \ell_i \equiv_2 \ell_{i'}.$$
 (A.3)

Since  $j_1 \in \{1, 2, ..., 7\}$ , we attain the following by Remark III.7 (1).

$$\{i \in \mathbb{Z} \mid i \text{ appears in } \mathcal{P}_{j_1}^{(1)}\} = \{1, 2, \dots, 7\} \setminus \{j_1\}$$
  
=  $\{s, j_2, j_3, j_4, i, i'\}.$ 

Thus, by Lemma III.6 and Equation (A.3), we have

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 \ell_4 \equiv_2 \ell_5 \equiv_2 \ell_6 \equiv_2 \ell_7.$$

For Claim A.2, suppose that there are no  $s \in \{1, 2, \dots, 7\}$  such that

$$\mathcal{P}_{s}^{(1)} = \{(j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i')\}$$

for any permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i'\in\{1,2,\ldots,7\}$ . Then for all  $s\in\{1,2,\ldots,7\}\setminus S$ ,

$$\mathcal{P}_s^{(1)} = \{ (j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, j_{\sigma(4)}), (i, i') \},\$$

for a permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i' \in \{1,2,\ldots,7\} \setminus S$ . Considering Table III.1, one can see that S should be one of the following sets:

$$\{1, 2, 4, 7\}, \{1, 2, 5, 6\}, \{1, 3, 4, 6\}, \{1, 3, 5, 7\}, \{2, 3, 4, 5\}, \{2, 3, 6, 7\}, \{4, 5, 6, 7\}.$$
 (A.4)

For the case where  $S \subset \{8,9,\ldots,15\}$ , we have similar claims as follows.

**Claim A.3.** If there is  $s \in \{1, 2, ..., 7\}$  such that

$$\mathcal{P}_s^{(2)} = \{ (j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i'), (\alpha, \beta) \}$$

for a permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i',\alpha,\beta \in \{8,9,\ldots,15\}$ , then

$$\ell_8 \equiv_2 \ell_9 \equiv_2 \ell_{10} \equiv_2 \ell_{11} \equiv_2 \ell_{12} \equiv_2 \ell_{13} \equiv_2 \ell_{14} \equiv_2 \ell_{15}.$$

**Claim A.4.** If there are no  $s \in \{1, 2, ..., 7\}$  such that

$$\mathcal{P}_s^{(2)} = \{ (j_{\sigma(1)}, j_{\sigma(2)}), (j_{\sigma(3)}, i), (j_{\sigma(4)}, i'), (\alpha, \beta) \}$$

for any permutation  $\sigma$  of  $\{1,2,3,4\}$  and  $i,i',\alpha,\beta \in \{8,9,\ldots,15\}$ , then S should be one of the following sets:

Claims A.3 and A.4 can be proved in the same manner as the proof of Claims A.1 and A.2. Therefore, we omit their proofs.

Now we consider the following four cases.

Case 1: 
$$|I \cap \{1, 2, ..., 7\}| > 4$$
.

In this case, there should be  $\{j_1, j_2, j_3, j_4\} \subset I$  and  $s \in \{1, 2, \dots, 7\}$  such that

$$\mathcal{P}_s^{(1)} = \{(j_1, j_2), (j_3, i), (j_4, i')\}$$

for some  $i, i' \in \{1, 2, \dots, 7\}$ . Thus, we have the following by Claim A.1.

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_7. \tag{A.6}$$

For any  $j \in \{8,9,\ldots,15\}$ , there exists  $s \in \{1,2,\ldots,7\}$  such that  $(i_8,j) \in \mathcal{P}_s^{(2)}$  by Remark III.7 (2) since  $i_8$  should be in

 $\{8,9,\ldots,15\}$ . Therefore, by Lemma III.6 and Equation (A.6), we have

$$\ell_{i_8} + \ell_j \equiv_2 \ell_{h_1} + \ell_{h_2} \equiv_2 0$$

for any  $(h_1, h_2) \in \mathcal{P}_s^{(1)}$ . Thus, we have

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}.$$

Case 2:  $0 < |I \cap \{1, 2, ..., 7\}| < 4$ .

In this case, one can show

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}$$

in the same manner as Case 1.

Case 3:  $|I \cap \{1, 2, \dots, 7\}| = 0$ , that is,  $I = \{8, 9, \dots, 15\}$ . Suppose that there is  $i \in \{1, 2, \dots, 7\}$  such that  $\ell_i \equiv_2 \ell_{i_1}$ . Let

$$I' := \{i, i_2, i_3, \dots, i_8\}.$$

We obtain the following by Case 2 since  $0 < |I' \cap \{1, 2, \dots, 7\}| = 1 < 4$ .

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}.$$

If  $\ell_i \not\equiv_2 \ell_{i_1}$  for all  $i \in \{1, 2, \dots, 7\}$ , then for t = 1, 2, we have

$$\begin{split} \ell_1 &\equiv_2 \ell_2 \equiv_2 \ell_3 \equiv_2 \ell_4 \equiv_2 \ell_5 \equiv_2 \ell_6 \equiv_2 \ell_7 \quad \text{and} \\ \ell_8 &\equiv_2 \ell_9 \equiv_2 \ell_{10} \equiv_2 \ell_{11} \equiv_2 \ell_{12} \equiv_2 \ell_{13} \equiv_2 \ell_{14} \equiv_2 \ell_{15}, \end{split}$$

that is,

$$\ell_i \equiv_2 \ell_j$$
 for  $i, j \in \mathcal{I}_1^{(t)}$ .

Case 4:  $|I \cap \{1, 2, ..., 7\}| = 4$ , that is

$$\{i_1, i_2, i_3, i_4\} \subset \{1, 2, \dots, 7\}$$
 and  $\{i_5, i_6, i_7, i_8\} \subset \{8, 9, \dots, 15\}.$ 

If there is  $i \in \{1, 2, \dots, 15\} \setminus I$  such that  $\ell_i \equiv_2 \ell_{i_1}$ , then by letting  $I' = \{i, i_2, i_3, \dots, i_8\}$ , we obtain

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}.$$

Assume that

$$\ell_i \not\equiv_2 \ell_{i_1}$$
 for all  $i \in \{1, 2, \dots, 15\} \setminus I$ . (A.7)

Suppose that there is  $s \in \{1, 2, \dots, 7\}$  such that

$$\mathcal{P}_s^{(1)} = \{ (i_{\sigma(1)}, i_{\sigma(2)}), (i_{\sigma(3)}, j), (i_{\sigma(4)}, j') \} \quad \text{or}$$

$$\mathcal{P}_s^{(2)} = \{ (i_{\sigma(1)}, i_{\sigma(2)}), (i_{\sigma(3)}, h), (i_{\sigma(4)}, h'), (\alpha, \beta) \}$$

for a permutation  $\sigma$  of  $\{1,2,3,4\}$ ,  $j,j'\in\{1,2,\ldots,7\}$ , and  $h,h',\alpha,\beta\in\{8,9,\ldots,15\}$ . Then, by Claims A.1 and A.3, we have

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_7$$
 or  $\ell_8 \equiv_2 \ell_9 \equiv_2 \cdots \equiv_2 \ell_{15}$ ,

which contradicts to the assumption (A.7). Thus, by Claims A.2 and A.4,  $\{i_1, i_2, i_3, i_4\}$  is one of the sets in (A.4) and  $\{i_5, i_6, i_7, i_8\}$  is one of the sets in (A.5).

Now, we need to calculate all these cases. For instance, suppose that

$$\{i_1, i_2, i_3, i_4\} = \{1, 2, 4, 7\}$$
 and  $\{i_5, i_6, i_7, i_8\} = \{8, 9, 10, 11\}$ .

By Lemma III.6, we have  $\ell_3 \equiv_2 \ell_i$  for all  $i \in I$  since  $\mathcal{P}_8^{(1)} =$  $\{(1,9),(2,10),(3,11)\}$ . Moreover, since

$$\mathcal{P}_{12}^{(1)} = \{(1,13), (2,14), (3,15)\} \quad \text{and} \quad$$

$$\mathcal{P}_{12}^{(2)} = \{(4,8), (5,9), (6,10), (7,11)\}$$

for any  $i \in I \cup \{3\}$ , we have

$$\ell_5 \equiv_2 \ell_6 \equiv_2 \ell_{13} \equiv_2 \ell_{14} \equiv_2 \ell_{15} \equiv_2 \ell_i.$$
 (A.8)

Since  $\mathcal{P}_8^{(2)} = \{(4,12), (5,13), (6,14), (7,15)\}$ , we have

$$\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}.$$

by Lemma III.6 and Equation (A.8).

In the same manner, one can show that the other cases induce  $\ell_1 \equiv_2 \ell_2 \equiv_2 \cdots \equiv_2 \ell_{15}$  except for the following cases:

$\{i_1, i_2, i_3, i_4\}$	$\{i_5, i_6, i_7, i_8\}$
$\{4, 5, 6, 7\}$	$\{12, 13, 14, 15\}$
${\{4,5,6,7\}}$	$\{8, 9, 10, 11\}$
$\{2, 3, 6, 7\}$	$\{10, 11, 14, 15\}$
$\{2, 3, 6, 7\}$	$\{8, 9, 12, 13\}$
$\{2, 3, 4, 5\}$	$\{10, 11, 12, 13\}$
$\{2, 3, 4, 5\}$	$\{8, 9, 14, 15\}$
$\{1, 3, 5, 7\}$	$\{9, 11, 13, 15\}$
$\{1, 3, 5, 7\}$	$\{8, 10, 12, 14\}$
$\{1, 3, 4, 6\}$	$\{9, 11, 12, 14\}$
$\{1, 3, 4, 6\}$	$\{8, 10, 13, 15\}$
$\{1, 2, 5, 6\}$	$\{9, 10, 13, 14\}$
$\{1, 2, 5, 6\}$	$\{8, 11, 12, 15\}$
$\{1, 2, 4, 7\}$	$\{9, 10, 12, 15\}$
$\{1, 2, 4, 7\}$	$\{8, 11, 13, 14\}$

Thus, we have

$$\ell_i \equiv_2 \ell_j$$
 for  $(i, j \in I)$  or  $(i, j \in \{1, 2, \dots, 15\} \setminus I)$ .

Note that for each exceptional case, there is  $s \in \{2, 3, ..., 15\}$ such that

$$\mathcal{I}_s^{(1)} = \{1,2,\ldots,15\} \setminus I \quad \text{and} \quad \mathcal{I}_s^{(2)} = I.$$

Hence, our assertion holds.

#### B. Proof of Theorem IV.12

*Proof.* One can easily see that if  $\mathcal{C}$  is SO, we have  $\mathfrak{n}_{s_0}^{(1)} + \mathfrak{n}_{s_0}^{(2)} =$ 0 by Theorem III.8. This implies that  $\widetilde{G} = G$  and thus  $\widetilde{C} = C$ . Therefore, we may assume that C is not SO.

Suppose that  $s_0$  is chosen in Step (C3). One can see that Steps (C5) and (C6) (resp. Steps (C7) and (C8)) will run repeatedly until  $\mathcal{I}_{s_0}^{(t)} \cap J_1(\widetilde{G}) = \emptyset$  or  $\mathcal{I}_{s_0}^{(t)} \subset J_1(\widetilde{G})$  for t = 1(resp. t = 2). Therefore,  $s_0$  and  $\widetilde{G}$  satisfy Equation (III.6) and thus  $\widetilde{\mathcal{C}}$  is SO by Theorem III.8.

Let  $\mathfrak{n}$  be the difference between the length of  $\widetilde{\mathcal{C}}$  and  $\mathcal{C}$ ,

$$\widehat{G}:=\left[\right.G\left\|\right.M\left.\right],$$

where M is a  $4 \times l$  matrix for some  $0 < l < \mathfrak{n}$  and  $\widehat{\mathcal{C}}$  is a linear code generated by  $\widehat{G}$ .

Note that, by Theorem III.8,  $\widehat{C}$  is SO if and only if there is  $s \in \{1, 2, ..., 15\}$  such that

$$\mathcal{I}_s^{(t)} \cap J_1(\widehat{G}) = \emptyset$$
 or  $\mathcal{I}_s^{(t)} \cap J_1(\widehat{G}) = \mathcal{I}_s^{(t)}$  (A.9)

for all t = 1, 2.

For  $s \in \{1, 2, \dots, 15\}$ , let  $M_s^{(1)}$  and  $M_s^{(2)}$  be submatrices of M such that  $M_s^{(1)}$  (resp.  $M_s^{(2)}$ ) consists of column vectors  $\mathbf{h}_i$ 's where  $i \in \mathcal{I}_s^{(1)}$  (resp.  $i \in \mathcal{I}_s^{(2)}$ ). There is an  $n \times n$  permutation matrix  $P_s = 1$ .

$$MP = \left[ M_s^{(1)} \middle| M_s^{(2)} \right],$$

that is,  $\left[\left.M_s^{(1)}\right|M_s^{(2)}\right]$  is M with the columns interchanged. For t=1,2, let

$$l_s^{(t)} := \left( ext{the number of columns of } M_s^{(t)} 
ight).$$

We also let

$$\mathfrak{n}_{s,0}^{(1)} := |\mathcal{I}_s^{(1)} \setminus J_1(G)| \text{ and } \mathfrak{n}_{s,1}^{(1)} := |\mathcal{I}_s^{(1)} \cap J_1(G)|$$
 (A.10)

$$\mathfrak{n}_{s,0}^{(2)} := |\mathcal{I}_s^{(2)} \setminus J_1(G)| \text{ and } \mathfrak{n}_{s,1}^{(2)} := |\mathcal{I}_s^{(2)} \cap J_1(G)|.$$
 (A.11)

For t = 1, 2, take  $j^{(t)} \neq j'^{(t)} \in \{0, 1\}$  so that  $\mathfrak{n}_s^{(t)} = \mathfrak{n}_{s, j^{(t)}}^{(t)}$ , where  $\mathfrak{n}_s^{(1)}$  and  $\mathfrak{n}_s^{(2)}$  are the integers defined in Steps (C1) and

(C2), respectively. Since  $l_s^{(1)} + l_s^{(2)} = l < \mathfrak{n} = \min\{\mathfrak{n}_s^{(1)} + \mathfrak{n}_s^{(2)} \mid 1 \le s \le 15\},$ there is  $t_0 \in \{1, 2\}$  such that

$$l_s^{(t_0)} < \mathfrak{n}_s^{(t_0)} \left( = \mathfrak{n}_{s,j^{(t_0)}}^{(t_0)} \right).$$

In case where  $t_0 = 1$ , since  $l_s^{(1)} < \mathfrak{n}_{s,j^{(1)}}^{(1)}$  and  $\mathfrak{n}_{s,j^{(1)}}^{(1)} +$  $\mathfrak{n}_{s,i'(1)}^{(1)} = 7$ , we have

$$0 < \mathfrak{n}_{s,i^{(1)}}^{(1)} - l_s^{(1)} \le |\mathcal{I}_s^{(1)} \cap J_1(\widehat{G})| \le \mathfrak{n}_{s,i^{\prime(1)}}^{(1)} + l_s^{(1)} < 7,$$

by Equation (A.10). Since  $|\mathcal{I}_s^{(1)}| = 7$ ,  $\widehat{\mathcal{C}}$  is not SO by Equation (A.9).

Similarly, in case where  $t_0 = 2$ , since  $l_s^{(2)} < \mathfrak{n}_{s,i^{(2)}}^{(2)}$  and  $\mathfrak{n}_{s,i^{(2)}}^{(2)} + \mathfrak{n}_{s,i^{(2)}}^{(2)} = 8$ , we have

$$0 < \mathfrak{n}_{s,j^{(2)}}^{(2)} - l_s^{(2)} \le |\mathcal{I}_s^{(2)} \cap J_1(\widehat{G})| \le \mathfrak{n}_{s,j^{(2)}}^{(2)} + l_s^{(2)} < 8,$$

by Equation (A.11). Since  $|\mathcal{I}_s^{(2)}|=8,\ \widehat{\mathcal{C}}$  is not SO by Equation (A.9).

C. Algorithm IV.11 in MAGMA: Construction of a shortest SO embedding for dimension four

h\_vector(i) gives the ith column of the generator matrix  $H_k$ of the  $[2^k-1,k]$  simplex code.

Input: The dimension k and a column index  $1 \le i \le 2^k - 1$ Output: the ith column vector of  $H_k$ 

function h\_vector(k,i)

H\_k := ZeroMatrix(IntegerRing(),k,2^ k-1); for i in  $[1..2^k-1]$  do

```
for j in [0..k-1] do
                                                                       end if:
       if Floor(i/2^j) \mod 2 = 1 then
                                                                     end for:
         H_k[k-j,i] := 1;
                                                                    return J_1;
       end if:
                                                                  end function;
     end for;
  end for;
                                                                   SOconst_matrix_dim4(G) gives a generator matrix
  return ColumnSubmatrix(H_k,i,1);
                                                                   for a shortest SO embedding of an \left[n,4\right] linear code
end function;
                                                                   (Algorithm IV.11).
                                                                   Input: A generator matrix G of an [n,4] code
Num_cols(G) gives the list of \ell_i(G).
                                                                  Output: A generator matrix \widetilde{G} for a shortest SO embedding
Input: a generator matrix G of \left[n,k\right] code
Output: the list of \ell_i(G)'s
                                                                  function SOconst matrix dim4(G)
                                                                     BR := BaseRing(G); G := Matrix(IntegerRing(),G);
                                                                  //(C1) and (C2)
function Num cols(G)
  k := Nrows(G); col_mult_set := {**}; ell_s := [];
                                                                    n1 := []; n2 := []; J_1 := J1(G);
  for j in [1..Ncols(G)] do
                                                                     for s in [1..15] do
    ind := 0;
                                                                       if #(I1[s] meet J_1) lt 4 then
    for i in [1..k] do
                                                                          Append(~n1, #(I1[s] meet J_1));
      ind := ind + G[i][j]*2^(k-i);
    end for:
                                                                          Append(~n1, #(I1[s] diff J_1));
  Include(~col_mult_set,ind);
                                                                       end if:
  end for;
                                                                       if \#(I2[s] \text{ meet J\_1}) le 4 then
  for s in [1..2^k - 1] do
                                                                         Append(~n2, #(I2[s] meet J_1));
    Append(~ell_s,Multiplicity(col_mult_set,s));
                                                                          Append(~n2, #(I2[s] diff J 1));
  end for;
  return ell_s;
                                                                       end if;
end function;
                                                                     end for:
/*
                                                                   //(C3)
Sets in Table III.2.
                                                                     Min := Minimum(\{Integers() \mid n1[s]+n2[s]: s in [1..15]\});
*/
                                                                     for i in [1..15] do
I1 := {@
                                                                       if (n1[i] + n2[i]) eq Min then
\{1,2,3,4,5,6,7\}, \{1,2,3,8,9,10,11\},
                                                                          s_0 := i; break;
{1,2,3,12,13,14,15}, {1,4,5,8,9,12,13},
                                                                       end if:
\{1,4,5,10,11,14,15\}, \{1,6,7,8,9,14,15\},
                                                                    end for:
{1,6,7,10,11,12,13}, {2,4,6,8,10,12,14},
                                                                  //(C4)
\{2,4,6,9,11,13,15\},\{2,5,7,8,10,13,15\},
                                                                     tilde_G := G;
{2,5,7,9,11,12,14}, {3,4,7,8,11,12,15},
                                                                  //(C5) and (C6)
{3,4,7,9,10,13,14}, {3,5,6,8,11,13,14},
                                                                     repeat
{3,5,6,9,10,12,15} @};
                                                                       if \#(I1[s_0] \text{ meet } J1(tilde_G)) lt 4 then
                                                                          calI1 := I1[s_0] meet J1(tilde_G);
\{8, 9, 10, 11, 12, 13, 14, 15\}, \{4, 5, 6, 7, 12, 13, 14, 15\},\
                                                                       else
\{4,5,6,7,8,9,10,11\}, \{2,3,6,7,10,11,14,15\},
                                                                         calI1 := I1[s_0] diff J1(tilde_G);
\{2,3,6,7,8,9,12,13\}, \{2,3,4,5,10,11,12,13\},
                                                                       end if;
\{2,3,4,5,8,9,14,15\}, \{1,3,5,7,9,11,13,15\},
                                                                       if (IsEmpty(calI1) eq false) then
{1,3,5,7,8,10,12,14}, {1,3,4,6,9,11,12,14},
                                                                         i_0 := Minimum(calI1);
{1,3,4,6,8,10,13,15}, {1,2,5,6,9,10,13,14},
                                                                         tilde_G := HorizontalJoin(tilde_G, h_vector(4, i_0));
\{1,2,5,6,8,11,12,15\}, \{1,2,4,7,9,10,12,15\},
                                                                       end if;
{1,2,4,7,8,11,13,14} @};
                                                                     until IsEmpty(calI1) eq true;
/*
                                                                   //(C7) and (C8)
J1(G) gives the set J_1(G) defined in Subsection 4.B.
                                                                     repeat
Input: a generator matrix G of an \left[n,4\right] code.
                                                                       if \#(I2[s_0] \text{ meet J1(tilde_G)}) le 4 then
Output: the set J_1(G)
                                                                          calI2 := I2[s_0] meet J1(tilde_G);
*/
                                                                       else
                                                                          calI2 := I2[s_0] diff J1(tilde_G);
function J1(G)
  ell_i_set := Num_cols(G); J_1 := {@@};
                                                                       end if;
  for k in [1..15] do
                                                                       if (IsEmpty(calI2) eq false) then
    if IsEven(ell_i_set[k]) eq false then
                                                                         i_0 := Minimum(calI2);
       Include(~J_1,k);
                                                                          tilde_G := HorizontalJoin(tilde_G, h_vector(4, i_0));
```

G := R[Ind[1]];

Remove (~Ind, 1);

for i in Ind do

for i in [1..k] do

Case := 1; i\_0 := i;

break;

Exclude(~Ind,i\_0);
if Case eq 1 then

for j in Ind do

end if:

end for;

end for;

end for;

Case := 0;

 $i \ 0 := 1;$ 

G := VerticalJoin(G,R[i]);

if InnerProduct(R[i], R[i]) eq 1 then

if InnerProduct(R[i\_0],R[j]) eq 0 then

R[j] := HorizontalJoin(R[j], Zero);

```
end if;
until IsEmpty(calI2) eq true;
tilde_G := Matrix(BR, tilde_G);
return tilde_G;
end function;
/*
SOconst_code_dim4(C) gives a shortest SO embedding of an
[n,4] linear code.
Input: An [n,4] code C
Output: A shortest SO embedding of C
*/
function SOconst_code_dim4(C)
G := GeneratorMatrix(C);
tilde_G := SOconst_matrix_dim4(G);
tilde_C := LinearCode(tilde_G);
return tilde_C;
end function;
```

## D. Algorithm IV.17 in MAGMA: Construction of an SO embedding for higher dimensions

Exclude(~Ind,i\_0);

```
R[j] := HorizontalJoin(R[j],One);
SOconst_matrix_dim_ge5(G) gives a generator matrix for an SO
                                                                             end if;
embedding of an [n,k] linear code for k \geq 5.
                                                                           end for;
Input: A generator matrix G of an [n,5] code for k\geq 5
                                                                          R[i 0] := HorizontalJoin(R[i 0],One);
Output: A generator matrix \widetilde{G} for an SO embedding
                                                                        else
                                                                           for j in Ind do
function SOconst_matrix_dim_ge5(G)
                                                                             if InnerProduct(R[i_0],R[j]) eq 0 then
  T :=[**];
                                                                               R[j] := HorizontalJoin(R[j], ZZero);
  I_0 := [**];
  repeat
                                                                               R[j] := HorizontalJoin(R[j],ZOne);
    n := Ncols(G);
                                                                             end if;
                                                                          end for;
    k := Nrows(G);
    R := [**];
                                                                        R[i_0] := HorizontalJoin(R[i_0],OOne);
    Ind := [1..k];
                                                                        end if:
    One := Matrix(GF(2),[[1]]);
                                                                        RemoveRow(~G, i_0);
    Zero := Matrix(GF(2),[[0]]);
                                                                        G := R[Ind[1]];
    OOne := Matrix(GF(2),[[1,1]]);
                                                                        Remove(~Ind,1);
    ZOne := Matrix(GF(2),[[0,1]]);
                                                                        for i in Ind do
    ZZero := Matrix(GF(2),[[0,0]]);
                                                                          G := VerticalJoin(G,R[i]);
    for i in [1..k] do
                                                                        end for;
       Append(~R,RowSubmatrix(G,i,1));
                                                                      end if;
    end for;
                                                                      Append(~T, R[i_0]);
    i_0 := 0;
                                                                      Append(~I_0, i_0);
    for i in [1..k] do
                                                                   until k eq 5;
       discriminant := 0;
                                                                   pre_G := SOconst_matrix_dim4(G);
       for j in [1..k] do
                                                                   NC := Ncols(pre_G);
         if InnerProduct(R[i],R[j]) eq 1 then
                                                                   for i in [1..#T] do
            discriminant := discriminant + 1;
                                                                     for j in [1..NC-Ncols(T[i])] do
         end if:
                                                                        T[i] := HorizontalJoin(T[i],Zero);
                                                                      end for;
       end for:
       if discriminant eq 0 then
                                                                   end for:
         i_0 := i;
                                                                   for i in [\#I\_0..1 \text{ by } -1] do
         break;
                                                                     NR := Nrows(pre_G);
       end if:
                                                                     tG := ZeroMatrix(GF(2), 1, NC);
    end for;
                                                                     for j in [1..I_0[i]-1] do
    if i_0 ne 0 then
                                                                       tG := VerticalJoin(tG,pre_G[j]);
```

```
tG := VerticalJoin(tG,T[i]);
     for j in [I_0[i]..NR] do
       tG := VerticalJoin(tG,pre_G[j]);
     end for:
     RemoveRow(~tG, 1);
     pre G := tG;
   end for;
  return tG;
end function;
{\tt SOconst\_code\_dim\_ge5(C)} \ {\tt gives} \ {\tt an SO \ embedding \ of \ an} \ [n,k]
linear code for k > 5.
Input: An [n,k] code C for k \geq 5
Output: An SO embedding of {\cal C}
function SOconst_code_dim_ge5(C)
  G := GeneratorMatrix(C);
  tilde_G := SOconst_matrix_dim_ge5(G);
  tilde_C := LinearCode(tilde_G);
  return tilde C:
end function;
```

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