Optimal additive quaternary codes of low dimension

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Abstract

An additive quaternary [n, k, d]-code (length n, quaternary dimension k, minimum distance d) is a 2k-dimensional \mathbb{F}_2 -vector space of n-tuples with entries in $Z_2 \times Z_2$ (the 2-dimensional vector space over \mathbb{F}_2) with minimum Hamming distance d. We determine the optimal parameters of additive quaternary codes of dimension $k \leq 3$. The most challenging case is dimension k = 2.5. We prove that an additive quaternary [n, 2.5, d]-code where d < n - 1 exists if and only if $3(n - d) \geq \lceil d/2 \rceil + \lceil d/4 \rceil + \lceil d/8 \rceil$. In particular we construct new optimal 2.5-dimensional additive quaternary codes. As a by-product we give a direct proof for the fact that a binary linear

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 $[3m, 5, 2e]_2$ -code for e < m-1 exists if and only if the Griesmer bound $3(m-e) \ge \lceil e/2 \rceil + \lceil e/4 \rceil + \lceil e/8 \rceil$ is satisfied.

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1 Introduction

The concept of additive codes is a far-reaching and natural generalization of linear codes, see [2], Chapter 18. Here we restrict to the quaternary case.

Definition 1. Let k be such that 2k is a positive integer. An additive quaternary [n, k]-code C (length n, dimension k) is a 2k-dimensional subspace of \mathbb{F}_2^{2n} where the coordinates come in pairs of two. We view the codewords as n-tuples where the coordinate entries are elements of \mathbb{F}_2^2 and use the Hamming distance.

We write the parameters of the code as [n, k, d] where d is the minimum Hamming distance. Here k is the quaternary dimension. As an example, in case k = 2.5 the code is a 5-dimensional vector space over \mathbb{F}_2 . Additive codes are particularly interesting because of a link to quantum stabilizer codes, see [4, 5, 9]. We will also use the geometric construction of additive quaternary codes. In fact, a quaternary [n, k, d]-code is equivalent to a multiset of n lines in PG(2k-1,2) such that each hyperplane of PG(2k-1,2) contains at most s = n - d of those lines, in the multiset sense. Blokhuis and Brouwer [8] first suggested the problem of determining the optimum parameters of additive quaternary codes. In earlier work we determined all such optimal parameters when $n \leq 13$, see [2], Chapter 18 and [6]. For further results concerning larger lengths see [1, 7]. In the present work we determine all optimal parameters when the quaternary dimension is $k \leq 3$. Dimensions $k \leq 2$ are degenerate cases, see Section 2. Dimension 3 is easily dealt with as well, see Section 3. Our main result is Theorem 2 in Section 4 where the optimal parameters of 2.5-dimensional additive quaternary codes are determined. For k > 1we prefer to work with the species s = n - d instead of the minimum distance d. Define $n_k(s)$ to be the maximal length n such that an additive [n, k, n-s]-code exists. For integer k, let $n_{k,lin}(s)$ be the maximal n such that a linear quaternary $[n, k, n - s]_4$ -code exists. In the present paper we determine $n_k(s)$ for $k \leq 3$ and all s. The following obvious lemma will be used to prove nonexistence results:

Lemma 1. The concatenation of a quaternary additive [n, k, d]-code and the binary linear $[3, 2, 2]_2$ -code is a binary linear $[3n, 2k, 2d]_2$ -code.

2 Dimensions $k \leq 2$.

Clearly dimension k=1 is a trivial case, the optimal parameters being [n,1,n]. Dimension k=1.5 is degenerate as well. The ambient space is the Fano plane and the optimal choice is to use each of its seven lines with multiplicity s. This shows $n_{1.5}(s)=7s$. The corresponding codes have parameters [7s,1.5,6s]. Dimension k=2 still is degenerate. In the linear case we have $n_{2,lin}(s)=5s$. In fact we work in the projective line PG(1,4) and the optimal choice is to use each of its points with multiplicity s.

Proposition 1. We have $n_2(s) = n_{2,lin}(s) = 5s$ for all s.

Proof. Assume $n_2(s) > 5s$. We would have a [5s+1, 2, 4s+1]-code. Lemma 1 would yield a binary linear $[15s+3, 4, 8s+2]_2$ -code. This contradicts the Griesmer bound.

3 The case of dimension k = 3.

The optimal parameters of linear quaternary 3-dimensional codes are of course known:

Proposition 2. We have
$$n_{3,lin}(2) = 6$$
, $n_{3,lin}(3) = 9$, $n_{3,lin}(4) = 16$, $n_{3,lin}(5i) = 21i$, $n_{3,lin}(5i+1) = 21i+1$ and $n_{3,lin}(5i+\sigma) = 21i+1+5(\sigma-1)$ for $i > 1$, $\sigma \in \{2,3,4\}$.

Proof. For d < 9 this is easy to check. For larger d we can invoke a result by Hamada-Tamari [10] stating that linear $[n, 3, d]_q$ -codes for $d \ge (q - 1)^2$ exist if and only if the parameters satisfy the Griesmer bound (see [2], Theorem 17.7). This coincides with the statement of our proposition.

Theorem 1. We have $n_3(s) = n_{3,lin}(s)$ for all s.

Proof. Assume there is an additive 3-dimensional code with larger n and the same species. We illustrate with case s=5i. We would have a [21i+1,3,16i+1]-code. Lemma 1 yields a linear $[63i+3,6,32i+2]_2$ -code, which contradicts the Griesmer bound . The other cases are analogous.

4 The case of dimension 2.5.

Our main result is the following:

Theorem 2. An additive quaternary [n, 2.5, d]-code where d < n - 1 exists if and only if $3(n - d) \ge \lceil d/2 \rceil + \lceil d/4 \rceil + \lceil d/8 \rceil$.

In the present section we prove Theorem 2. In the sequel use the abbreviation $d_l = \lceil d/l \rceil$. The necessity is obvious. In fact, Lemma 1 applied to an additive quaternary [n, 2.5, d]-code yields a binary linear $[3n, 5, 2d]_2$ -code. The condition of Theorem 2 is the Griesmer bound as applied to this binary code. It remains to prove sufficiency: given d, n satisfying the condition of the theorem we need to construct an additive quaternary [n, 2.5, d]-code. As before, let s = n - d. For each s consider the pair $D_s = (s, m_s)$ where m_s is the maximal n such that n, d = n - s satisfy the condition in Theorem 2. We need to prove the existence of an $[m_s, 2.5, m_s - s]$ -code, for all $s \ge 2$. When such a code exists we say that we represented D_s . Here are some examples:

$$D_2 = (2, 8), D_3 = (3, 11), D_4 = (4, 16), D_5 = (5, 21), D_6 = (6, 26), D_7 = (7, 31).$$

Let C be an [n, 2.5, d]-code and C' the code obtained by increasing each line multiplicity of C by 1. As PG(4,2) has 155 lines and PG(3,2) has 35 lines we see that C' is an [n+155, 2.5, d+120]-code. Concerning the bound of the theorem we observe that $3(n-d)-d_2-d_4-d_8$ is invariant under the substitution $n\mapsto n+155, d\mapsto d+120$. This shows that we need prove the existence of an [n, 2.5, d]-code only for n<155. This means that it suffices to construct $D_2, D_3, \ldots, D_{35}=(35,155)$. Observe that there is an obvious sum construction which shows that the existence of codes $[m_1, 2.5, l_1]$ and $[m_2, 2.5, l_2]$ implies the existence of an $[m_1 + m_2, 2.5, l_1 + l_2]$ -code. This shows that if D_{s_1} and D_{s_2} can be constructed then also $D_{s_1} + D_{s_2}$ can be constructed. We see now that it suffices to construct D_2, \ldots, D_7 as the remaining $D_s, s \leq 35$ follow from the sum construction. Here are some examples:

$$D_8 = D_6 + D_2, D_9 = D_7 + D_2, D_{10} = D_5 + D_5, D_{11} = D_9 + D_2, D_{12} = D_6 + D_6.$$

It remains to construct D_2, \ldots, D_7 . Now D_2 implies D_4 as $D_2 + D_2 = D_4$ and $D_5 = (5, 21)$ is constructed as there is even a linear $[21, 3, 16]_4$ -code (corresponding to the points of PG(2, 4)). We are reduced to construct D_2, D_3, D_6, D_7 . Now $D_2 = (2, 8)$ corresponds to a [8, 2.5, 6]-code. This is

the Blokhuis-Brouwer construction [8, 3]. In the same context an [11, 2.5, 8]-code was constructed. This is a representation of $D_3 = (3, 11)$. We are finally reduced to construct D_6 and D_7 .

A construction

Consider a chain

$$l_0 \subset E_0 \subset H_0 \subset PG(4,2)$$

where l_0 is a line, E_0 a plane and H_0 a solid (hyperplane) in PG(4,2). Let \mathcal{V} be a set of 8 lines such that each point in $E_0 \setminus l_0$ is on precisely two lines of \mathcal{V} , each point outside H_0 is on precisely one line of \mathcal{V} . Also, let \mathcal{E} be a set of 8 lines partitioning the points outside E_0 (Blokhuis-Brouwer construction).

Definition 2. Let C(g, h, v, e) be the additive 2.5-dimensional quaternary code described by the following multiset of lines: line l_0 with multiplicity g, the remaining lines of E_0 each with multiplicity h, the lines of V with multiplicity h and the lines of E with multiplicity h.

Clearly C(g,h,v,e) has length n=g+6h+8v+8e. Let m(P) be the number of codelines (including multiplicities) that contain point P. If $P \in l_0$, then m(P)=g+2h, if $P \in E_0 \setminus l_0$ then m(P)=3h+2v. If $P \in H_0 \setminus E_0$ then m(P)=e whereas points P outside H_0 have m(P)=v+e. For each hyperplane H let $m(H)=\sum_{P\in H}m(P)$. By double counting we obtain

$$s(H) = (m(H) - n)/2$$

where s(H) (the species of H) is the number of codelines contained in H. It follows that the numbers n - s(H) are the nonzero weights of our code. The numbers m(H) and s(H) are easy to determine:

Lemma 2. If $l_0 \not\subset H$ then m(H) = g + 8h + 12v + 12e. If $l_0 \subset H$ but $E_0 \not\subset H$ then m(H) = 3g + 6h + 8v + 12e. If $E_0 \subset H \neq H_0$ then m(H) = 3g + 18h + 16v + 8e. Finally $m(H_0) = 3g + 18h + 8v + 8e$.

Proof. This is a trivial calculation. In the first case above H has one point of l_0 , two further points in E_0 , four further points in H_0 and finally 8 affine points for a grand total m(H) = g + 8h + 4v + 4e + 8(v + e). In the second case H contains three points on l_0 , no further point on E_0 , four further points on H_0 and eight affine points: m(H) = 3g + 6h + 4e + 8(v + e). The remaining two cases are analogous.

Our basic formula yields:

Corollary 1. The nonzero weights of the codewords of C(g, h, v, e) are

$$g + 5h + 6(v + e), 6h + 8v + 6e, 4v + 8e, 8(v + e).$$

C(g, h, v, e) is an $[g + 6h + 8(v + e), 2.5, d]_4$ -code where

$$d = Min(w_1 = g + 5h + 6(v + e), w_2 = 6h + 8v + 6e, w_3 = 4v + 8e).$$

We see that C(2,0,1,2) is a [26,2.5,20]-code and C(1,1,0,3) is a [31,2.5,24]-code. This completes the proof of Theorem 2. Lemma 1 yields

Corollary 2. A binary linear $[3m, 5, 2e]_2$ -code for e < m-1 exists if and only if the Griesmer bound $3(m-e) \ge e_2 + e_4 + e_8$ is satisfied.

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