Capacity-Achieving Private Information Retrieval Schemes from Uncoded Storage Constrained Servers with Low Sub-packetization

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Abstract

This paper investigates reducing sub-packetization of capacity-achieving schemes for uncoded Storage Constrained Private Information Retrieval (SC-PIR) systems. In the SC-PIR system, a user aims to download one out of K files from N servers while revealing nothing about the identity of the requested file to any individual server, in which the K files are stored at the Nservers in an uncoded form and each server can store up to μK equivalent files, where μ is the normalized storage capacity of each server. We first prove that there exists a capacity-achieving SC-PIR scheme for a given storage design if and only if all the packets are stored exactly at $M \triangleq \mu N$ servers for μ such that $M = \mu N \in \{2, 3, \dots, N\}$. Then, the optimal sub-packetization for capacity-achieving linear SC-PIR schemes is characterized as the solution to an optimization problem, which is typically hard to solve since it involves non-continuous indicator functions. Moreover, a new notion of array called Storage Design Array (SDA) is introduced for the SC-PIR system. With any given SDA, an associated capacity-achieving SC-PIR scheme is constructed. Next, the SC-PIR schemes that have equal-size packets are investigated. Furthermore, the optimal equal-size sub-packetization among all capacity-achieving linear SC-PIR schemes characterized by Woolsey *et al.* is proved to be $\frac{N(M-1)}{\gcd(N,M)}$, which is achieved by a construction of SDA. Finally, by allowing unequal size of packets, a greedy SDA construction is proposed, where the subpacketization of the associated SC-PIR scheme is upper bounded by $\frac{N(M-1)}{\gcd(N,M)}$. Among all capacity-achieving linear SC-PIR schemes, the sub-packetization is optimal when $\min\{M, N-M\}|N$ or M = N, and within a multiplicative gap $\frac{\min\{M, N-M\}}{\operatorname{cred}(N, M)}$ of the optimal one in general. In particular, for the special case $N = d \cdot M \pm 1$ where the positive integer $d \ge 2$, we propose another SDA construction to obtain lower sub-packetization.

Index Terms

Private information retrieval, uncode, storage constrained servers, sub-packetization, capacity-achieving, storage design array.

I. INTRODUCTION

Along with the rapid advancement of Distributed Storage Systems (DSSs), protecting the download privacy of a user against public servers is of vital importance. The problem of Private Information Retrieval (PIR) was first introduced by Chor *et al.* in [9] and has attracted remarkable attention within computer science community subsequently [9], [11], [18], [35]. In the classical framework, a user wishes to retrieve one out of K files from N servers, each of which stores the whole library of K files, while ensuring that any server can not learn any information about the file index being requested. To this end, the user sends a query string to each server. Then the server responds truthfully with an answer string depending on the received query and the contents stored. Finally, the user correctly decodes the requested file from the answers. Note that to prevent each server from obtaining information about which file is being requested, the query distribution has to be marginally independent of the desired file index.

A trivial strategy is to download all the K files in the library no matter which file is requested by the user, but this results in impractical communication cost, especially in a modern DSS, which typically maintains a large number of files. In the seminal work [9] where each file is of one bit size, the communication cost was measured by the sum of upload cost (the total size of query strings) and download cost (the total size of answer strings). In the sense of information-theoretic security, which

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assures privacy even if the servers have unbounded computational power, it was shown in [9] that the naive strategy is the only feasible solution to a single server, whereas low communication cost can be attained by replicating the files at multiple non-colluding servers. To improve the efficiency, single-server PIR has been widely studied in the sense of computational security, whose privacy is guaranteed by some computational hard problems, for examples, the problems related to so-called Φ -hiding number-theory [6], [12], trapdoor permutations [15], [4], or quadratic/composite residuosity [13], [8]. These works improve the efficiency at the cost of non-zero possibility of disclosing information relevant to the identity of the requested file.

Instead of retrieving a single bit, Shannon theory allows the file size to be arbitrarily large, and therefore the upload cost can be neglected compared to the download cost since it does not scale with file size [20], [7], [22], [3], [25]. Then, the communication efficiency is usually measured by *retrieval rate*, defined as the number of bits that the user can privately retrieve per bit of download data across all random realizations of queries. Particularly, the supremum of retrieval rates over all achievable schemes is called *capacity*. To implement a PIR scheme, the files typically need to be partitioned into some non-overlapping packets. The number of packets is referred to as *sub-packetization* in the literature. The sub-packetization reflects the complexity of the scheme in practice and is preferred to be as small as possible. This is because any practical scheme will require each of the packets to include some header information for user to decode [27], especially the header overhead may be non-negligible when there are a large number of packets. This problem has already been noticed in other applications, for example coded caching [32], [33], [24], [27], [21].

In 2014, Shah *et al.* revisited the PIR problem and reported an interesting scheme achieving the PIR rate $1 - \frac{1}{N}$ and requiring sub-packetization N - 1 [20]. Later in the influential work by Sun and Jafar [22], the exact PIR capacity was characterized as $\left(1 + \frac{1}{N} + \ldots + \frac{1}{N^{K-1}}\right)^{-1}$ for any N and K. However, to achieve the capacity, the smallest sub-packetization of the proposed PIR schemes [22] is N^K , which increases exponentially with the number of files K and thus is impractical even for moderate number of files. Soon afterwards, the sub-packetization was decreased to N^{K-1} in [23], which was proved to be optimal under the assumption that the download cost are identical over all random realizations of queries. In a very recent work [28], Tian *et al.* innovatively introduced a new capacity-achieving scheme, which incurs different download cost for distinct realizations of queries. As a result, the sub-packetization was decreased to N - 1, which is independent of K and shown to be optimal among all the capacity-achieving PIR schemes.

A common assumption in the aforementioned results is that each server has sufficiently large storage capacity to store all the files in the library, i.e., a repetition coding is used to store the files across servers. Though repetition coding can offer simplicity in designing PIR schemes and the high immunity against server failures, it suffers from extremely large storage cost. The storage cost in a PIR system has been widely investigated in terms of the coding structures in the storage design, such as specific Maximum Distance Separable (MDS) codes [3], [25], [37], an uncoded storage [26], [1], [31], and other more complicated coding techniques [20], [10], [5], [19], [36], [14], [2]. Moreover, the tradeoff between the storage cost and retrieval rate was considered without any explicit constraints on the storage codes [24], [29], [30].

As the first step toward exactly characterizing the tradeoff between storage cost and retrieval rate, Tandon *et al.* formulated the problem of uncoded Storage Constrained PIR (SC-PIR) in [26], [1]. In this setup, each server can store up to μKL symbols by some storage design, where $\frac{1}{N} \leq \mu \leq 1$ is the normalized storage and *L* is the number of symbols of each file. The capacity of SC-PIR was proved in [1] to be $\left(1 + \frac{1}{M} + \ldots + \frac{1}{M^{K-1}}\right)^{-1}$ for $M \triangleq \mu N \in \{1, \ldots, N\}$. However, the capacity-achieving SC-PIR scheme in [26], [1] has sub-packetization $\binom{N}{M}M^K$. Thus the problem of high sub-packetization shows up again in this SC-PIR model. Recently, Woolsey *et al.* [31] proposed a general construction of SC-PIR schemes by establishing the connection between storage design and Storage Full PIR (SF-PIR, i.e., the case that each server can store all the files). Then, the sub-packetization to achieve the capacity of the SC-PIR system was reduced to NM^{K-1} in [31], which also increases exponentially with K.

In this paper, we are interested in characterizing the optimal sub-packetization to achieve the capacity of SC-PIR systems. Note from the previous work [1], [31] that *linear* schemes are sufficient to achieve the capacity of SC-PIR. Additionally, it was proved in [1] that, for any μ with $\frac{1}{N} \leq \mu \leq 1$, the capacity of SC-PIR system can be achieved by memory-sharing technique between the discrete points such that $M \in \{1, 2, ..., N\}$, where M = 1 is a trivial case since the user has to download all the contents stored at the N servers to assure privacy. Therefore, the problem comes down to the case $M \in \{2, 3, ..., N\}$ for linear SC-PIR schemes, which is the focus of this paper. The contributions of this paper are:

1) We prove that there exists a capacity-achieving SC-PIR scheme for a given storage design if and only if all the packets

are stored exactly at M servers in the storage phase.

- 2) We characterize the optimal sub-packetization of capacity-achieving linear SC-PIR schemes by an optimization problem. Consequently, a general construction of capacity-achieving linear SC-PIR schemes with optimal sub-packetization can be obtained based on the optimal solution of this optimization problem.
- Storage Design Array (SDA) is introduced to obtain feasible solutions to the optimization problem. Any given SDA is
 associated to a practical capacity-achieving linear SC-PIR scheme with low sub-packetization.
- 4) We prove that the optimal *equal-size* sub-packetization is $\frac{N(M-1)}{\gcd(N,M)}$ among all the classes of capacity-achieving linear SC-PIR schemes characterized by Woolsey *et al.* [31].
- 5) In order to further decrease sub-packetization of capacity-achieving SC-PIR schemes, we investigate the problem under a more general assumption, i.e., the sizes of the packets are allowed to be unequal. In particular, a greedy algorithm is proposed to construct SDA for any positive integers N, M such that $1 \le M \le N$. The sub-packetization of the associated SC-PIR scheme is shown to be optimal among all capacity-achieving linear SC-PIR schemes when $\min\{M, N - M\}|N$ or M = N. In the other cases, the sub-packetization is within a multiplicative gap $\frac{\min\{M, N-M\}}{\gcd(N,M)}$ compared to its lower bound. Moreover, for the case $N = d \cdot M \pm 1$ where the integer $d \ge 2$, we propose another construction of SDA to achieve lower sub-packetization compared to the greedy SDA.

The rest of this paper is organized as follows. In Section II, we introduce the system model and problem formulation. In Section III, we establish an information-theoretic lower bound on sub-packetization of capacity-achieving linear SC-PIR schemes. In Section IV, we characterize a generic construction of capacity-achieving linear SC-PIR schemes with optimal sub-packetization. In Section V, we introduce SDA to construct capacity-achieving SC-PIR schemes with low sub-packetization. In Section VI, we present the results under the assumption of equal-size packets. Section VII proposes two SDA constructions and proves the optimality of the resultant sub-packetization. Finally, the paper is concluded in Section VIII.

The following notation is used throughout this paper.

- For any integers n, m, s, N with $n \le m$, [n:m] and $([n:m] + s)_N$ respectively denote the sets $\{n, n+1, \ldots, m\}$ and $\{i + s \pmod{N} : n \le i \le m\}$;
- For a finite set S, |S| denotes its cardinality;
- Denote $A_{1:m}$ a vector (A_1, \ldots, A_m) , and define A_{Γ} as $(A_{\gamma_1}, \ldots, A_{\gamma_k})$ for any index set $\Gamma = \{\gamma_1, \ldots, \gamma_k\} \subseteq [1:m]$ with $\gamma_1 < \ldots < \gamma_k$ or any index vector $\Gamma = (\gamma_1, \ldots, \gamma_k)$;
- Define $\mathbf{1}(x)$ as a function of a logical variable x, i.e., $\mathbf{1}(x) = 1$ if x is true and $\mathbf{1}(x) = 0$ otherwise.

II. SYSTEM MODEL

Let \mathbb{F}_q be the finite field for a prime power q. Consider a non-colluding PIR system with K files $W_1, \ldots, W_K \in \mathbb{F}_q^{L \times 1}$ stored across N servers in an uncoded fashion. Each of files is comprised of L i.i.d. uniform symbols over \mathbb{F}_q , i.e.,

$$H(W_1) = \dots = H(W_K) = L,$$
(1)

$$H(W_1, \dots, W_K) = \sum_{k=1}^{N} H(W_k),$$
(2)

where the entropy function $H(\cdot)$ is measured with logarithm q. Let Z_n ($n \in [1 : N]$) be the contents stored at server n, which is subject to the storage capacity of server n, then the *storage constraint* for each server is

$$H(Z_n) \le \mu KL, \quad \forall n \in [1:N], \tag{3}$$

where μ is the *normalized storage capacity*. Notice that, when $\mu < \frac{1}{N}$, the total storage capacity of N servers is insufficient to store all the K files. For $\mu = 1$, each server can store all the K files. Thus, we are interest in the case $\frac{1}{N} \le \mu \le 1$.

The system operates in the following two phases:

Storage Phase: Each file W_k is partitioned into F disjoint *packets* and thus it will be convenient to label the F packets as $W_{k,1}, W_{k,2}, \ldots, W_{k,F}$, where $W_{k,i}$ is the *i*-th packet of file W_k . By convention, we call F sub-packetization. Then, for any $k \in [1:K]$,

$$W_k = \{W_{k,i} : i \in [1:F]\},$$
(4)

$$H(W_k) = \sum_{i=1}^{F} H(W_{k,i}).$$
(5)

Clearly, each of these packets must be stored at *at least one server* because of the constraint of reliable decoding. In particular, all the files are partitioned and stored in the same manner¹, i.e.,

$$H(W_{1,i}) = H(W_{2,i}) = \dots = H(W_{K,i}), \quad \forall i \in [1:F],$$
(6)

$$Z_n = \{ W_{k,i} : k \in [1:K], i \in \mathcal{Z}_n \}, \quad \forall n \in [1:N],$$
(7)

where Z_n is a subset of [1:F] such that Z_n satisfies (3). In other words, $Z_n \subseteq [1:F]$ consists of the indices of packets stored at server n.

Retrieval Phase: A user selects an index $\theta \in [1:K]$ privately and wishes to retrieve the file W_{θ} from the system without disclosing any information about θ to any individual server. For this purpose, the user generates N queries $Q_{1:N}^{[\theta]}$ and sends $Q_n^{[\theta]}$ to server $n \in [1:N]$. Indeed, the queries are generated independently of file realizations, i.e.,

$$I(Q_{1:N}^{[\theta]}; W_{1:K}) = 0, \quad \forall \theta \in [1:K],$$
(8)

where $I(\cdot)$ is the mutual information function. Upon receiving the query $Q_n^{[\theta]}$, server *n* responds with an answer $A_n^{[\theta]}$, which is determined by the received query and its stored contents. Thus, by the data processing inequality,

$$H(A_n^{[\theta]}|Q_n^{[\theta]}, Z_n) = H(A_n^{[\theta]}|Q_n^{[\theta]}, W_{1:K}) = 0, \quad \forall n \in [1:N].$$
(9)

Finally, from all the answers $A_{1:N}^{[\theta]}$ collected from the N servers, the user must be able to decode the desired file W_{θ} correctly, i.e.,

$$H(W_{\theta}|A_{1:N}^{[\theta]}, Q_{1:N}^{[\theta]}) = 0, \quad \forall \theta \in [1:K].$$
(10)

To ensure the privacy, the strategies for retrieving any two files W_{θ} and $W_{\theta'}$ must be indistinguishable in terms of any individual server, i.e.,

$$(Q_n^{[\theta]}, A_n^{[\theta]}, Z_n) \sim (Q_n^{[\theta']}, A_n^{[\theta']}, Z_n), \quad \forall \, \theta, \theta' \in [1:K], \forall \, n \in [1:N],$$
(11)

where $X \sim Y$ means that the random variables X and Y are identical distribution. Equivalently, the desired index θ must be hidden from all the information available to each server, i.e.,

$$I(Q_n^{[\theta]}, A_n^{[\theta]}, Z_n; \theta) = 0, \quad \forall n \in [1:N].$$

$$\tag{12}$$

Throughout this paper, we refer to this system as a (μ, N, K) Storage Constrained PIR (SC-PIR) system. If $\mu = 1$, the system is also referred to as an (N, K) Storage Full PIR (SF-PIR) system.

In order to measure the performance of SC-PIR systems, the following two quantities are considered:

- 1. The sub-packetization F, which reflects the complexity of the SC-PIR scheme in practical applications, and thus is preferred to be as small as possible.
- 2. The retrieval rate R, which is the number of desired bits that the user can retrieve privately per bit of downloaded data, is defined as

$$R \triangleq \frac{H(W_{\theta})}{\sum_{n=1}^{N} H(A_n^{[\theta]})} = \frac{L}{D},\tag{13}$$

where $D \triangleq \sum_{n=1}^{N} H(A_n^{[\theta]})$ is the average download cost from the N servers over random queries. Obviously, R and D are independent of θ by (1) and (11).

A retrieval rate R is said to be achievable if there exists a design of both storage and retrieval phases satisfying (3)–(12) such that its retrieval rate is greater than or equal to R. The capacity of the SC-PIR system, denoted by C^* , is the supremum over all the achievable rates, i.e.,

$$C^* = \sup \{R : R \text{ is achievable}\}$$

¹To the best of our knowledge, all the previous storage constrained PIR schemes satisfy this assumption [26], [1], [31], which is also a popular storage manner in coded caching [16], [34], [27], [21], [17].

Define the total normalized storage capacity as

$$M \triangleq \mu N \in [1, N]$$

For the case $M \in [1:N]$, the capacity of SC-PIR is exactly characterized in [1] as

$$C^* = \left(1 + \frac{1}{M} + \ldots + \frac{1}{M^{K-1}}\right)^{-1}.$$
(14)

Generally, for other $M \in [1, N]$ (or equivalently $\mu \in [\frac{1}{N}, 1]$), the capacity can be achieved by memory-sharing technique between the integer points $\lceil M \rceil$ and $\lfloor M \rfloor$ (see [1, Claim 1 & Theorem 2]). Thus, in the sequel, we will concentrate our discussion on the case $M \in [2 : N]$ since it is straightforward to prove that the optimal sub-packetization is F = N for the case M = 1.

Moreover, the existing work [26], [1], [31] have shown that linear SC-PIR schemes can achieve the capacity.

Definition 1 (Linear SC-PIR Scheme). For a given scheme of the (μ, N, K) SC-PIR system, let ℓ_n be the answer length² of query $Q_n^{[\theta]}$. It is said to be a linear SC-PIR scheme if the answers $A_n^{[\theta]}$ $(n \in [1 : N])$ are formed by

$$A_n^{[\theta]} = \mathbf{LC}_n^{[\theta]}(Z_n) = \left(\mathbf{LC}_{n,1}^{[\theta]}(Z_n), \dots, \mathbf{LC}_{n,\ell_n}^{[\theta]}(Z_n)\right), \quad \forall n \in [1:N]$$

with each entry $\mathbf{LC}_{n,j}^{[\theta]}(Z_n)$ $(j \in [1 : \ell_n])$ given by a linear combination of the packets stored at server n, i.e.,

$$\mathbf{LC}_{n,j}^{[\theta]}(Z_n) = \sum_{k \in [1:K]} \sum_{i \in \mathcal{Z}_n} \beta_{n,k,i,j}^{[\theta]} \cdot W_{k,i},$$
(15)

where $\beta_{n,k,i,j}^{[\theta]} \in \mathbb{F}_q$ is the coefficient of packet $W_{k,i}$ in the *j*-th entry of $A_n^{[\theta]}$ and is determined completely by the received query $Q_n^{[\theta]}$. Here, it implicitly assumes that each of packets $\{W_{k,i} : k \in [1:K], i \in [1:F]\}$ is represented by a vector over \mathbb{F}_q . If the packets have different dimensions, then the additions are performed by padding the vectors with zeros to the largest dimension.

The objective of the paper is to design (μ, N, K) linear SC-PIR schemes achieving the SC-PIR capacity with the minimum sub-packetization for the case $M \in [2:N]$.

III. A LOWER BOUND ON SUB-PACKETIZATION OF CAPACITY-ACHIEVING LINEAR SC-PIR SCHEMES

To simplify our notations in the following discussion, denote $W_{k,S}$ the set of packets of file W_k that are exclusively stored by servers in $S, S \subseteq [1:N]$, i.e.,

$$W_{k,\mathcal{S}} \triangleq \left\{ W_{k,i} : i \in \left(\bigcap_{n \in \mathcal{S}} \mathcal{Z}_n \right) \setminus \left(\bigcup_{m \in [1:N] \setminus \mathcal{S}} \mathcal{Z}_m \right) \right\}, \quad \forall k \in [1:K].$$
(16)

Obviously, $W_{k,\emptyset} = \emptyset$ and $H(W_{k,\emptyset}) = 0$ due to the constraint of reliable decoding. Then, file W_k and the storage contents at server n can be respectively rewritten as

$$W_k = \bigcup_{\mathcal{S} \subseteq [1:N]} W_{k,\mathcal{S}}, \quad \forall k \in [1:K]$$
(17)

and

$$Z_n = \bigcup_{\substack{k \in [1:K] \\ n \in S}} \bigcup_{\substack{S \subseteq [1:N] \\ n \in S}} W_{k,S}, \quad \forall n \in [1:N].$$
(18)

Notice from (6) and (7) that both the entropy of random variable $W_{k,S}$ and the size of set $W_{k,S}$ are irrespective of k. Thus, for all $S \subseteq [1 : N]$, we can set $H(W_{k,S}) \triangleq \alpha_S L$ and $F_S \triangleq |W_{k,S}|$ where $\alpha_S \in [0, 1]$. In other words, α_S is the normalized file size of $W_{k,S}$ and F_S is the number of packets in $W_{k,S}$. By (1), (3), (17) and (18), the file size, storage size, and sub-packetization F are respectively constrained as

$$\sum_{\mathcal{S}\subseteq[1:N]} \alpha_{\mathcal{S}} = 1,\tag{19}$$

²Throughout this paper, the "length" is counted by the number of packets, thus "answer length" refers to as the number of packets in the answer.

$$\sum_{\substack{S \subseteq [1:N]\\n \in S}} \alpha_S \le \mu, \quad \forall n \in [1:N],$$
(20)

and

$$F = \sum_{\mathcal{S} \subseteq [1:N]} F_{\mathcal{S}}.$$
(21)

In the following, we establish an information-theoretical lower bound on sub-packetization of any capacity-achieving (μ, N, K) linear SC-PIR scheme with $M = \mu N \in [2 : N]$, which is characterized by the following optimization problem.

Definition 2. Given any positive integers N and $M = \mu N \in [2 : N]$, **Problem 1** is defined as

$$\{\alpha_{\mathcal{S}}^{*}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M} = \arg\min \sum_{\substack{\mathcal{S}\subseteq[1:N]\\|\mathcal{S}|=M}} \mathbf{1}(\alpha_{\mathcal{S}} > 0)$$
s.t.
$$\sum_{\substack{\mathcal{S}\subseteq[1:N]\\|\mathcal{S}|=M,n\in\mathcal{S}\\0\leq\alpha_{\mathcal{S}}\leq1,}} \alpha_{\mathcal{S}} = \mu, \quad \forall n \in [1:N]$$
(22)
(23)

where $\{\alpha_{\mathcal{S}}^*\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ is called the optimal solution to Problem 1 and $\eta^* = \sum_{\substack{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}} \mathbf{1}(\alpha_{\mathcal{S}}^*>0)$ is called the optimal value of Problem 1. In addition, the parameters $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ satisfying (22) and (23) are called a feasible solution to Problem 1.

A. Necessary Conditions of Capacity-Achieving Linear SC-PIR Schemes

In this subsection, we derive five necessary conditions (Lemmas 1 and 2 below) for capacity-achieving linear SC-PIR schemes, whose proofs are left in Appendix.

Lemma 1. Given any (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$ and $\{\alpha_{S} : \alpha_{S} \in [0, 1], S \subseteq [1 : N]\}$, the storage design of any capacity-achieving SC-PIR scheme must satisfy:

- P1. All the packets must be stored exactly at M servers, i.e., $\alpha_{\mathcal{S}} = 0$ for all $\mathcal{S} \subseteq [1:N]$ with $|\mathcal{S}| \neq M$;
- P2. The storage capacity at all servers must be used up, i.e., $\sum_{S \subseteq [1:N], n \in S} \alpha_S = \mu$ for all $n \in [1:N]$.

Remark 1. Given any parameters $\{\alpha_{\mathcal{S}} : \alpha_{\mathcal{S}} \in [0, 1], \mathcal{S} \subseteq [1 : N]\}$, P1 along with P2 are equivalent to the constraints (22) and (23) of Problem 1.

For any $\mathcal{K} \subseteq [1:K], \mathcal{S} \subseteq [1:N]$, denote $W_{\mathcal{K},\mathcal{S}} \triangleq \bigcup_{k \in \mathcal{K}} W_{k,\mathcal{S}}$. Given $\theta \in [1:K]$, let $\widetilde{\mathbf{LC}}_n^{[\theta]}(Z_n)$ be the answer of server n when receiving the query realization $\widetilde{Q}_n^{[\theta]}$. Let $\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{\mathcal{K},\mathcal{S}})$ be the part of $\widetilde{\mathbf{LC}}_n^{[\theta]}(Z_n)$ involving the linear combinations of packets in $W_{\mathcal{K},\mathcal{S}}$, i.e.,

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\mathcal{K},\mathcal{S}}) \triangleq \left(\widetilde{\mathbf{LC}}_{n,1}^{[\theta]}(W_{\mathcal{K},\mathcal{S}}), \dots, \widetilde{\mathbf{LC}}_{n,\tilde{\ell}_{n}}^{[\theta]}(W_{\mathcal{K},\mathcal{S}})\right), \quad \forall n \in [1:N],$$
(24)

where $\tilde{\ell}_n$ is the answer length for the query realization $\widetilde{Q}_n^{[\theta]}$, and $\widetilde{\mathbf{LC}}_{n,j}^{[\theta]}(W_{\mathcal{K},\mathcal{S}})$ is given by

$$\widetilde{\mathbf{LC}}_{n,j}^{[\theta]}(W_{\mathcal{K},\mathcal{S}}) = \sum_{k \in \mathcal{K}, W_{k,i} \in W_{\mathcal{K},\mathcal{S}}} \widetilde{\beta}_{n,k,i,j}^{[\theta]} \cdot W_{k,i}, \quad \forall j \in [1:\widetilde{\ell}_n]$$
(25)

in which the coefficient $\tilde{\beta}_{n,k,i,j}^{[\theta]}$ is the realization of $\beta_{n,k,i,j}^{[\theta]}$ in (15) when the query realization $\tilde{Q}_{n}^{[\theta]}$ is received by server n.

Lemma 2. Given any (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$, let $S \subseteq [1 : N]$ and $\theta, \theta' \in [1 : K]$ such that $|S| = M, \theta \neq \theta'$. For every realization of queries $\tilde{Q}_{1:N}^{[\theta]}$ with positive probability, the retrieval phase for any capacity-achieving linear SC-PIR scheme must satisfy:

P3. (Independence of the retrieved data) The M random variables

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,\mathcal{S}}), \quad \forall n \in \mathcal{S}$$
(26)

are independent of each other;

P4. (Independence of the requested data) The M random variables

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]}\left(W_{[1:K]\setminus\{\theta'\},\mathcal{S}}\right), \quad \forall n \in \mathcal{S}$$

$$\tag{27}$$

are independent of each other;

P5. (Identical information for the residuals) The M random variables

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]}\left(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}\right), \quad \forall n \in \mathcal{S}$$

$$(28)$$

are deterministic of each other.

B. Lower Bound on Sub-packetization of Capacity-Achieving Linear SC-PIR Schemes

Lemma 3. Given any capacity-achieving (μ, N, K) linear SC-PIR scheme with $M = \mu N \in [2 : N]$ and $\{\alpha_{\mathcal{S}} : \alpha_{\mathcal{S}} \in [0, 1], \mathcal{S} \subseteq [1 : N]\}$,

$$\begin{cases} F_{\mathcal{S}} \ge M - 1, & \text{if } \alpha_{\mathcal{S}} > 0, \mathcal{S} \subseteq [1:N], |\mathcal{S}| = M \\ F_{\mathcal{S}} = 0, & \text{otherwise} \end{cases}$$

Proof: It is clear that $F_{\mathcal{S}} = 0$ if $\alpha_{\mathcal{S}} = 0$. By Lemma 1, we just need to prove $F_{\mathcal{S}} \ge M-1$ if $\alpha_{\mathcal{S}} > 0, \mathcal{S} \subseteq [1:N], |\mathcal{S}| = M$. Let W_{θ^*} and $\widetilde{Q}_{1:N}^{[\theta^*]}$ be the desired file of the user and a realization of queries with positive probability, respectively. For

any S with $|W_{\theta^*,S}| > 0$, recall from (16) that $W_{\theta^*,S}$ are exclusively stored at servers in S. Thus, in the conditioning of the realization of queries $\widetilde{Q}_{1:N}^{[\theta^*]}$, to ensure that the user can correctly decode W_{θ^*} , there must be a server $n \in S$ such that the coefficients of packets $W_{\theta^*,S}$ in $\widetilde{\mathbf{LC}}_n^{[\theta^*]}(Z_n)$ are not all zeros, i.e.,

$$H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta^{*}]}(W_{\theta^{*},\mathcal{S}})\right) > 0, \quad \forall n \in \mathcal{S}$$

Note that, the random queries for retrieving distinct files at a given server have the identical distribution by the privacy constraint (11). Thus, the following observation holds:

Observation: For any realization of queries $\tilde{Q}_{1:N}^{[\theta^*]}$ with positive probability, the query $\tilde{Q}_n^{[\theta^*]}$, sent to server *n* for retrieving file W_{θ^*} , can also be sent to the same server *n* but for retrieving any distinct file $\theta \neq \theta^*$ in another realization of queries $\tilde{Q}_{1:N}^{[\theta]}$ with positive probability, where $\tilde{Q}_n^{[\theta]} = \tilde{Q}_n^{[\theta^*]}$. As a result, for the two realizations of queries $\tilde{Q}_{1:N}^{[\theta^*]}$ and $\tilde{Q}_{1:N}^{[\theta]}$, server *n* will respond the same answer, i.e.,

$$\widetilde{\mathbf{LC}}_{n}^{[\theta^{*}]}(Z_{n}) = \left(\widetilde{\mathbf{LC}}_{n,1}^{[\theta^{*}]}(Z_{n}), \dots, \widetilde{\mathbf{LC}}_{n,\tilde{\ell}_{n}}^{[\theta^{*}]}(Z_{n})\right) = \left(\widetilde{\mathbf{LC}}_{n,1}^{[\theta]}(Z_{n}), \dots, \widetilde{\mathbf{LC}}_{n,\tilde{\ell}_{n}}^{[\theta]}(Z_{n})\right) = \widetilde{\mathbf{LC}}_{n}^{[\theta]}(Z_{n})$$

That is, for any $\theta \in [1:K] \setminus \{\theta^*\}$, there exists another realization of queries $\widetilde{Q}_{1:N}^{[\theta]}$ with positive probability such that server n will respond with the same answer $\widetilde{\mathbf{LC}}_n^{[\theta]}(Z_n) = \widetilde{\mathbf{LC}}_n^{[\theta^*]}(Z_n)$, where the terms involving $W_{\theta^*,S}$ are identical, i.e.,

$$H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta^{*},\mathcal{S}})\right) = H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta^{*}]}(W_{\theta^{*},\mathcal{S}})\right) > 0.$$

Then, for any $\theta' \in [1:K] \setminus \{\theta, \theta^*\},\$

$$H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right)$$

$$\stackrel{(a)}{\geq} H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}) \middle| W_{[1:K]\setminus\{\theta,\theta',\theta^{*}\},\mathcal{S}}\right)$$

$$\stackrel{(b)}{\equiv} H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta^{*},\mathcal{S}})\right)$$

$$> 0, \qquad (29)$$

where (a) holds because conditioning reduces entropy; (b) follows from the linearity of (25).

Assume that the number of packets in $W_{\theta,S}$ is less than M-1, i.e., $F_S < M-1$. According to (25) and (26), the M random variables $\widetilde{\mathbf{LC}}_{n'}^{[\theta]}(W_{\theta,S})$ $(n' \in S)$ consisting of linear combinations of F_S packets in $W_{\theta,S}$ are independent of each other. Thus, $F_S < M-1$ results in that there must exist two distinct servers $i, j \in S$ such that

$$\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{\theta,\mathcal{S}}) = \widetilde{\mathbf{LC}}_{j}^{[\theta]}(W_{\theta,\mathcal{S}}) = \mathbf{0}, \quad \forall i, j \in \mathcal{S}, i \neq j.$$

$$(30)$$

However, we have

$$0 \stackrel{(a)}{=} I\left(\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{[1:K]\setminus\{\theta'\},\mathcal{S}}); \widetilde{\mathbf{LC}}_{j}^{[\theta]}(W_{[1:K]\setminus\{\theta'\},\mathcal{S}})\right)$$

$$\stackrel{(b)}{=} I\left(\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{[1:K]\setminus\{\theta',\theta\},\mathcal{S}}) + \widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{\theta,\mathcal{S}}); \widetilde{\mathbf{LC}}_{j}^{[\theta]}(W_{[1:K]\setminus\{\theta',\theta\},\mathcal{S}}) + \widetilde{\mathbf{LC}}_{j}^{[\theta]}(W_{\theta,\mathcal{S}})\right)$$

$$\stackrel{(c)}{=} I\left(\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{[1:K]\setminus\{\theta',\theta\},\mathcal{S}}); \widetilde{\mathbf{LC}}_{j}^{[\theta]}(W_{[1:K]\setminus\{\theta',\theta\},\mathcal{S}})\right)$$

$$\stackrel{(d)}{=} H\left(\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{[1:K]\setminus\{\theta',\theta\},\mathcal{S}})\right)$$

$$\stackrel{(e)}{>} 0,$$

where (a) follows by (27); (b) follows from the linearity of (25) again; (c) is due to (30); (d) is because of (28); (e) holds since

$$H\left(\widetilde{\mathbf{LC}}_{i}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right) = H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right) > 0$$

by (28) and (29). Thus, $F_{S} \ge M - 1$ and the proof is completed.

Now, we are ready to characterize a lower bound on the sub-packetization among all capacity-achieving linear SC-PIR schemes.

Theorem 1. For any given (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$, the sub-packetization of any capacity-achieving linear SC-PIR scheme is lower bounded by

$$F \ge \eta^* \cdot (M-1),\tag{31}$$

where η^* is the optimal value to Problem 1.

Proof: Given any capacity-achieving linear SC-PIR scheme with $\{\alpha_{\mathcal{S}} : \alpha_{\mathcal{S}} \in [0, 1], \mathcal{S} \subseteq [1 : N]\}$, the sub-packetization

$$F \stackrel{(a)}{=} \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ \geq \\ |\mathcal{S}| = M}} F_{\mathcal{S}}$$
$$\stackrel{(b)}{\geq} \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M}} \mathbf{1}(\alpha_{\mathcal{S}} > 0) \cdot (M-1)$$
$$\stackrel{(c)}{\geq} \eta^* \cdot (M-1),$$

where (a) follows by (21); (b) holds by Lemma 3; (c) is due to Lemma 1 and Remark 1 that $\{\alpha_{\mathcal{S}} : \alpha_{\mathcal{S}} \in [0,1], \mathcal{S} \subseteq [1:N]\}$ of any capacity-achieving SC-PIR scheme must satisfy (22) and (23).

Definition 3. The sub-packetization F of a capacity-achieving linear SC-PIR scheme is said to be optimal if it achieves the equality in (31).

IV. A GENERIC CAPACITY-ACHIEVING LINEAR SC-PIR SCHEME WITH OPTIMAL SUB-PACKETIZATION

In this section, we present a generic construction of capacity-achieving linear SC-PIR schemes with optimal sub-packetization.

A. SF-SC-PIR Schemes Based on Transformation From SF-PIR Schemes to SC-PIR Schemes

We first introduce a class of SC-PIR schemes in Algorithm 1, which are constructed by a transformation from SF-PIR schemes to SC-PIR schemes, where we term the resultant schemes as *SF-SC-PIR schemes* for convenience. Actually, the transformation was first characterized in [31].

Theorem 2. For any positive integers N, K, M with $M \in [2 : N]$, given any feasible solution $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ to Problem 1 and any capacity-achieving (M, K) linear SF-PIR scheme with sub-packetization F_{SF} , the $(\mu = M/N, N, K)$ linear SF-SC-PIR scheme obtained in Algorithm 1 is capacity-achievable with sub-packetization $F = \eta \cdot F_{SF}$, where $\eta = \sum_{\substack{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}} \mathbf{1}(\alpha_{\mathcal{S}} > 0)$.

Proof: Obviously, the output SF-SC-PIR scheme of Algorithm 1 is linear if the input SF-PIR scheme is linear. Furthermore, by Lines 2-5, the sub-packetization of the output scheme is $F = \eta \cdot F_{SF}$. Consequently, we prove the theorem by showing

Algorithm 1 Capacity-Achieving SF-SC-PIR Schemes

Input: A feasible solution $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ to Problem 1 and a capacity-achieving (M, K) SF-PIR scheme with subpacketization F_{SF}

Output: Capacity-Achieving SF-SC-PIR Scheme with sub-packetization $F = \eta \cdot F_{SF}$, where $\eta = \sum_{S \subseteq [1:N], |S|=M} \mathbf{1}(\alpha_S > 0)$.

- 1: procedure Storage
- 2: for $k \in [1:K]$ do
- 3: Divide W_k into $\{W_{k,S} : S \subseteq [1:N], |S| = M, \alpha_S > 0\}$ such that $H(W_{k,S}) = \alpha_S L$
- 4: Further divide each $W_{k,S}$ into F_{SF} disjointed packets for $S \subseteq [1:N], |S| = M, \alpha_S > 0$
- 5: end for
- 6: for $n \in [1:N]$ do
- 7: $Z_n \leftarrow \{W_{k,\mathcal{S}} : k \in [1:K], \mathcal{S} \subseteq [1:N], |\mathcal{S}| = M, \alpha_{\mathcal{S}} > 0, n \in \mathcal{S}\}$
- 8: end for
- 9: end procedure
- 10: procedure Retrieval
- 11: for each $\mathcal{S} \subseteq [1:N], |\mathcal{S}| = M, \alpha_{\mathcal{S}} > 0$ do
- 12: Employ the (M, K) capacity-achieving SF-PIR scheme³ independently to retrieve $W_{\theta,S}$ privately from the M servers in S and the K packet sets in $\{W_{k,S} : k \in [1:K]\}$

13: end for

14: The user combines $W_{\theta,S}$ ($S \subseteq [1:N], |S| = M, \alpha_S > 0$) to recover W_{θ}

15: end procedure

that the storage design in Algorithm 1 is achievable and the SF-SC-PIR scheme is capacity-achievable while satisfying the constraints of correctness and privacy.

In Line 3, each file can be partitioned into $W_k = \{W_{k,S} : S \subseteq [1:N], |S| = M, \alpha_S > 0\}$ since the feasible solution $\{\alpha_S\}_{S \subseteq [1:N], S=M}$ that satisfies (22) and (23) has

$$\sum_{\substack{S \subseteq [1:N] \\ |S|=M,\alpha_S > 0}} \alpha_S = \sum_{\substack{S \subseteq [1:N] \\ |S|=M}} \alpha_S$$

$$= \frac{1}{M} \sum_{\substack{S \subseteq [1:N] \\ |S|=M}} \alpha_S \sum_{n \in [1:N]} \mathbf{1} (n \in S)$$

$$= \frac{1}{M} \sum_{\substack{n \in [1:N] \\ |S|=M}} \sum_{\substack{S \subseteq [1:N] \\ |S|=M}} \alpha_S \cdot \mathbf{1} (n \in S)$$

$$= \frac{1}{M} \sum_{\substack{n \in [1:N] \\ |S|=M}} \sum_{\substack{S \subseteq [1:N] \\ |S|=M,n \in S}} \alpha_S$$

$$\stackrel{(a)}{=} \frac{\mu N}{M}$$

$$= 1,$$
(32)

where (a) is due to (22). In Line 7, the storage content Z_n at each server is

$$Z_n = \bigcup_{\substack{k \in [1:K] \\ \alpha_{\mathcal{S}} > 0, n \in \mathcal{S}}} \bigcup W_{k,\mathcal{S}}, \quad \forall n \in [1:N].$$

Since all the random variables $W_{k,S}$ are independent of each other, by applying $H(W_{k,S}) = \alpha_S L$ and (22), we get

$$H(Z_n) = KL \sum_{\substack{\mathcal{S} \subseteq [1:N], |\mathcal{S}| = M \\ \alpha_{\mathcal{S}} > 0, n \in \mathcal{S}}} \alpha_{\mathcal{S}} = \mu KL,$$

³It is not necessary to use different SF-PIR schemes as building blocks since the sub-packetization of resultant SC-PIR scheme can be further reduced by adopting the identical SF-PIR scheme with minimum sub-packetization.

which satisfies the storage constraint (3). Thus, the storage design in Algorithm 1 is achievable.

Then, we prove the scheme in Algorithm 1 to be capacity-achievable. By Lines 4 and 7, for any $S \subseteq [1:N]$, |S| = M, $\alpha_S > 0$, each $W_{k,S}$ ($k \in [1:K]$) is partitioned into F_{SF} packets and is stored at M servers in S. Thus, in Lines 11-13, the non-zero $W_{\theta,S}$ can be retrieved from servers in S by employing the capacity-achieving (M, K) SC-PIR schemes independently. Then, according to (13), the download cost for retrieving $W_{\theta,S}$ is

$$D_{\mathcal{S}} = \frac{H(W_{\theta,\mathcal{S}})}{C_{\rm SF}} = \left(1 + \frac{1}{M} + \ldots + \frac{1}{M^{K-1}}\right) \alpha_{\mathcal{S}} L,$$

where $C_{SF} = (1 + 1/M + ... + 1/M^{K-1})^{-1}$ is the capacity of (M, K) SF-PIR scheme. Therefore, the rate for retrieving W_{θ} is

$$R = \frac{L}{D} = \frac{L}{\sum_{\substack{S \subseteq [1:N] \\ |S| = M, \alpha_S > 0}}} = \left(1 + \frac{1}{M} + \dots + \frac{1}{M^{K-1}}\right)^{-1},$$

which achieves the capacity of SC-PIR in (14).

The user can recover W_{θ} by combining all non-zero $W_{\theta,S}$, where privacy is guaranteed because the SF-PIR scheme satisfying the constraint of privacy is independently employed to download the desired packet sets.

We know from Theorem 2 that any parameters $\{\alpha_{\mathcal{S}} : \alpha_{\mathcal{S}} \in [0,1], \mathcal{S} \subseteq [1:N]\}$ satisfying (22) and (23) result in a storage design of a capacity-achieving SC-PIR scheme. Thus, we have the following corollary according to Remark 1.

Corollary 1. Given any parameters $\{\alpha_{S} : \alpha_{S} \in [0,1], S \subseteq [1:N]\}$, P1 and P2 are necessary and sufficient conditions for the storage design of a capacity-achieving SC-PIR scheme.

B. Capacity-Achieving Linear SC-PIR Schemes with Optimal Sub-packetization

From Algorithm 1, we can construct a capacity-achieving SC-PIR scheme by using any feasible solution to Problem 1 and any capacity-achieving SF-PIR scheme as a building block. Such capacity-achieving (M, K) SF-PIR schemes have been found in [22], [23], [28]. If the SF-PIR scheme with sub-packetization M^{K-1} [23] is employed, then we can obtain a capacityachieving SC-PIR scheme with sub-packetization $\eta \cdot M^{K-1}$, which has identical download cost across all random realizations of queries. Whereas, if the scheme with sub-packetization M - 1 in [28] is adopted, the asymmetry of download cost over all realizations of queries will be inherited by the resultant SC-PIR scheme so that the sub-packetization is reduced to $\eta \cdot (M-1)$. In particular, when any optimal solution to Problem 1 is further employed, a capacity-achieving SF-SC-PIR scheme with sub-packetization $\eta^* \cdot (M-1)$ can be obtained.

For the sake of completeness, we summarize the scheme of [28] in Algorithm 2, where Q is defined as

$$\mathcal{Q} \triangleq \left\{ (q_1, \dots, q_K) \in [0: M-1]^K \right\}.$$

Note that the dummy packets in Line 2 are not stored by the servers at all. Let $\Delta = \sum_{i \in [1:K] \setminus \{\theta\}} W_{i,q_i}$, then the user can decode file $W_k = (W_{k,0}, \dots, W_{k,M-2})$ from the answers $(A_0^{[\theta]}, \dots, A_{M-1}^{[\theta]})$ in Line 5 because of $(A_0^{[\theta]}, \dots, A_{M-1}^{[\theta]}) = (\Delta + W_{\theta,q_{\theta}}, \dots, \Delta + W_{\theta,M-2}, \Delta + \mathbf{0}, \Delta + W_{\theta,0}, \dots, \Delta + W_{\theta,q_{\theta}-1}).$

The following result is immediate by Theorems 1 and 2.

Theorem 3. For any positive integers N, K, M with $M \in [2 : N]$, given any optimal solution $\{\alpha_{\mathcal{S}}^*\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ to Problem 1 and the capacity-achieving (M, K) linear SF-PIR scheme in Algorithm 2, Algorithm 1 outputs a capacity-achieving $(\mu = M/N, N, K)$ linear SC-PIR scheme with sub-packetization $F^* = \eta^* \cdot (M - 1)$, where $\eta^* = \sum_{\substack{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}} \mathbf{1}(\alpha_{\mathcal{S}}^* > 0)$. Particularly, the sub-packetization F^* is optimal among all capacity-achieving linear SC-PIR schemes.

V. STORAGE DESIGN ARRAY

According to Theorem 3, in terms of designing capacity-achieving linear SC-PIR schemes with optimal sub-packetization, it is crucial to solve the optimization problem in Problem 1. However, it is not easy because of the involved indicator functions. Thus, in this section, we dedicate to construct concrete capacity-achieving linear schemes with low sub-packetization by finding sub-optimal solutions to Problem 1.

Algorithm 2 Capacity-Achieving (M, K) Linear SF-PIR Scheme with Sub-packetization M - 1

1: Relabel the indices of the M servers as $0, 1, \ldots, M - 1$.

2: For each $k \in [1:K]$, file W_k is uniformly partitioned into M-1 disjoint packets $W_{k,0}, W_{k,1}, \ldots, W_{k,M-2}$. For easy of exploration, each file W_k is appended a dummy packet $W_{k,M-1} \triangleq \mathbf{0}$, i.e.,

$$W_k = (W_{k,0}, \dots, W_{k,M-2}, \mathbf{0}), \quad \forall k \in [1:K].$$

3: Select a vector from the set Q independently and uniformly:

$$\mathbf{q} = (q_1, \ldots, q_{\theta-1}, q_\theta, q_{\theta+1}, \ldots, q_K).$$

4: Query Phase: Based on the vector **q**, the user constructs a query sent to server m as

$$Q_m^{[\theta]} = (q_1, \dots, q_{\theta-1}, (q_\theta + m)_M, q_{\theta+1}, \dots, q_K), \quad \forall m \in [0: M-1].$$

5: Answer Phase: After receiving the query $Q_m^{[\theta]}$, the answer at server m is

$$A_{m}^{[\theta]} = \begin{cases} \text{NULL}, & \text{if } Q_{m}^{[\theta]} = (M - 1, \dots, M - 1) \\ \sum_{i \in [1:K] \setminus \{\theta\}} W_{i,q_{i}} + W_{\theta,(q_{\theta} + m)_{M}}, & \text{else} \end{cases}, \quad \forall m \in [0:M - 1],$$

where the value NULL indicates that the server keeps silence.

6: Decoding Phase: Decode file $W_{\theta} = (W_{\theta,0}, \dots, W_{\theta,M-2})$ from the answers $A_0^{[\theta]}, \dots, A_{M-1}^{[\theta]}$

For clarity, we first introduce Storage Design Array (SDA) to construct feasible solutions of Problem 1.

Definition 4 (Storage Design Array (SDA)). For any positive integers N, M with $M \in [1 : N]$, an (N, M) storage design array is an array of size $N \times \frac{N}{\gcd(N,M)}$ with each entry being either "*" or "NULL" that satisfies

- S1. Each column has M "*"s;
- S2. Each row has $\frac{M}{\gcd(N,M)}$ "*"s.

Definition 5 (Number of Distinct Columns of SDA). Let $\mathbf{P} = [p_{i,j}]_{N \times \frac{N}{\gcd(N,M)}}$ be an (N, M) SDA. For each $j \in [1 : \frac{N}{\gcd(N,M)}]$, let S_j be the set of row indices corresponding to "*"s in column j, i.e.,

$$\mathcal{S}_j \triangleq \{i \in [1:N] : p_{i,j} = *\}, \quad \forall j \in \left[1 : \frac{N}{\gcd(N,M)}\right].$$
(33)

We denote $\eta_{\mathbf{P}}$ as the number of distinct columns in \mathbf{P} , i.e.,

$$\eta_{\mathbf{P}} \triangleq \left| \left\{ \mathcal{S}_j : j \in \left[1 : \frac{N}{\gcd(N, M)} \right] \right\} \right|.$$
(34)

Let $\{S_{i_l}\}_{l=1}^{\eta_{\mathbf{P}}}$ be the $\eta_{\mathbf{P}}$ distinct ones in $\{S_j\}_{j=1}^{\overline{\operatorname{gcd}(N,M)}}$ and s_l be the occurrence that S_{i_l} appears in $\{S_j\}_{j=1}^{\overline{\operatorname{gcd}(N,M)}}$ for $l \in [1:\eta_{\mathbf{P}}]$, *i.e.*,

$$s_{l} \triangleq \left| \left\{ j \in \left[1 : \frac{N}{\gcd(N, M)} \right] : \mathcal{S}_{j} = \mathcal{S}_{i_{l}} \right\} \right|, \quad \forall l \in [1 : \eta_{\mathbf{P}}].$$
(35)

Example 1. An (N = 9, M = 4) SDA P and another (N = 11, M = 5) SDA P' can be written as follows, respectively.

In fact, the SDAs **P** and **P**' are constructed by greedy Algorithm 3, which will be illustrated in Section VII-A in detail. Apparently, there are $\eta_{\mathbf{P}} = 6$ distinct columns in **P** (i.e., columns 1, 5, 6, 7, 8 and 9) with $\{s_1 = 4, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 1, s_6 = 1\}$ and $\eta_{\mathbf{P}'} = 7$ distinct columns in **P**' (i.e., columns 1, 6, 7, 8, 9, 10 and 11) with $\{s_1 = 5, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 1, s_6 = 1, s_7 = 1\}$.

Lemma 4. Given N and $M \in [1:N]$, any (N, M) SDA **P** is associated to a set of parameters $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ that is a feasible solution to Problem 1.

Proof: Notice from S1 and (33) that $|S_{i_l}| = M$ for all $l \in [1 : \eta_{\mathbf{P}}]$. Thus, we can obtain a set of parameters $\{\alpha_{\mathcal{S}}\}_{\mathcal{S} \subseteq [1:N], |\mathcal{S}| = M}$,

$$\alpha_{\mathcal{S}} = \begin{cases} s_l \cdot \frac{\gcd(N,M)}{N}, & \text{if } \mathcal{S} = \mathcal{S}_{i_l} \text{ for some } l \in [1:\eta_{\mathbf{P}}] \\ 0, & \text{otherwise} \end{cases}$$
(36)

where s_l is defined in (35). Then, for any $n \in [1:N]$,

$$\sum_{\substack{S \subseteq [1:N] \\ |S|=M,n \in S}} \alpha_S = \sum_{\substack{l \in [1:\eta_{\mathbf{P}}] \\ n \in S_{i_l}}} \alpha_{S_{i_l}}$$
$$= \frac{\gcd(N, M)}{N} \cdot \sum_{\substack{l \in [1:\eta_{\mathbf{P}}] \\ n \in S_{i_l}}} s_l$$
$$\stackrel{(a)}{=} \frac{\gcd(N, M)}{N} \cdot \sum_{\substack{j \in [1:\frac{N}{\gcd(N,M)}]}} \mathbf{1}(p_{n,j} = *)$$
$$\stackrel{(b)}{=} \frac{\gcd(N, M)}{N} \cdot \frac{M}{\gcd(N, M)}$$
$$= \mu,$$

where (a) follows from (33) and (35), and (b) is due to S2. That is, the parameters $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ satisfy (22) and (23), and thus are feasible for Problem 1.

Obviously, taking the feasible solution $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ and the (M,K) SF-PIR scheme in Algorithm 2 as inputs of Algorithm 1, one can obtain a storage design scheme by Lines 1-9 in Algorithm 1 and a capacity-achieving SF-SC-PIR scheme with sub-packetization $\eta_{\mathbf{P}} \cdot (M-1)$ by Theorem 2, where $\eta_{\mathbf{P}} = \sum_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M} \mathbf{1}(\alpha_{\mathcal{S}} > 0)$ by (36).

Theorem 4. Given any positive integers N, K, M with $M \in [2 : N]$ and any (N, M) SDA P, there is a capacity-achieving $(\mu = M/N, N, K)$ linear SC-PIR scheme with sub-packetization $\eta_{\mathbf{P}} \cdot (M - 1)$.

Example 2. For the (N = 9, M = 4) SDA **P** in Example 1, set

$$\alpha_{\{1,2,3,4\}} = \frac{4}{9}, \ \alpha_{\{5,6,7,8\}} = \alpha_{\{5,6,7,9\}} = \alpha_{\{5,6,8,9\}} = \alpha_{\{5,7,8,9\}} = \alpha_{\{6,7,8,9\}} = \frac{1}{9},$$

and all the other $\alpha_{\mathcal{S}}$ to be zeros. It is easy to see that $\{\alpha_{\mathcal{S}}\}_{\mathcal{S}\subseteq[1:9],|\mathcal{S}|=4}$ is a feasible solution of Problem 1 with $\sum_{\mathcal{S}\subseteq[1:9],|\mathcal{S}|=4} \mathbf{1}(\alpha_{\mathcal{S}} > 0) = \eta_{\mathbf{P}} = 6$. Then, we can generate a capacity-achieving $(\mu = 4/9, N = 9, K)$ linear SC-PIR scheme with sub-packetization 18. Similarly, the (N = 11, M = 5) SDA P' in Example 1 is associated to a capacity-achieving $(\mu = 5/11, N = 11, K)$ linear SC-PIR scheme with sub-packetization 28.

VI. EQUAL-SIZE CAPACITY-ACHIEVING LINEAR SC-PIR SCHEMES

Recall that the setup in Section II allows us to partition each file into unequal-size packets. Actually, the equal-size partition of the files is one of the most important cases in practice, which is also considered by the previous capacity-achieving SC-PIR schemes [26], [1], [31] and SF-PIR schemes [22], [23], [28] in the storage phase. Thus, we first focus on capacity-achieving SC-PIR/SF-PIR schemes with small sub-packetization by imposing the assumption of equal-size packets, i.e.,

$$H(W_{k,1}) = H(W_{k,2}) = \dots = H(W_{k,F}) = \frac{L}{F}, \quad \forall k \in [1:K].$$
(37)

For simplicity, we will refer to the sub-packetization of a scheme satisfying (37) as *equal-size sub-packetization*. In particular, we characterize the optimal equal-size sub-packetization of all capacity-achieving linear SF-SC-PIR schemes in the following theorem.

Theorem 5. Given any (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$, the optimal equal-size sub-packetization of all capacity-achieving linear SF-SC-PIR schemes is given by $\frac{N(M-1)}{\gcd(N,M)}$.

The theorem will be proved by constructing an SDA-based SF-SC-PIR scheme with equal-size sub-packetization and showing the optimality of its sub-packetization separately.

A. SDA-Based SF-SC-PIR Schemes with Equal-size Sub-packetization

In this subsection, we construct an (N, M) SDA **P** with all columns being distinct, i.e., $\eta_{\mathbf{P}} = \frac{N}{\gcd(N,M)}$. Later, it will be shown that such SDA **P** is associated to a capacity-achieving SF-SC-PIR scheme with equal-size sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$. Before that, a simple example is presented.

Example 3. For N = 12 and M = 5, an SDA with all columns being distinct is given by

	*		*		*			*		*		-]
P =	*		*			*		*		*			
	*		*			*		*			*		
	*			*		*		*			*		
	*			*		*			*		*		
		*		*		*			*		*		
		*		*			*		*		*		
		*		*			*		*			*	
		*			*		*		*			*	
		*			*		*			*		*	
			*		*		*			*		*	
			*		*			*		*		* -	$ _{12 \times 12}$

Then it corresponds to a set of non-zero and equal-size parameters

$$\alpha_{\{1,2,3,4,5\}} = \alpha_{\{6,7,8,9,10\}} = \alpha_{\{11,12,1,2,3\}} = \alpha_{\{4,5,6,7,8\}} = \alpha_{\{9,10,11,12,1\}} = \alpha_{\{2,3,4,5,6\}}$$
(38)
= $\alpha_{\{7,8,9,10,11\}} = \alpha_{\{12,1,2,3,4\}} = \alpha_{\{5,6,7,8,9\}} = \alpha_{\{10,11,12,1,2\}} = \alpha_{\{3,4,5,6,7\}} = \alpha_{\{8,9,10,11,12\}} = \frac{1}{12}.$

By employing these parameters and the (M = 5, K) SF-PIR scheme in Algorithm 2 as inputs of Algorithm 1, a capacityachieving SF-SC-PIR scheme is obtained, where each packet has equal size $\frac{1}{48}L$ and sub-packetization 48.

Formally, an (N, M) SDA $\mathbf{P} = [p_{i,j}]_{N \times \frac{N}{\gcd(N,M)}}$ satisfying $\eta_{\mathbf{P}} = \frac{N}{\gcd(N,M)}$ is constructed as

$$p_{i,j} = \begin{cases} *, & \text{if } i \in S_j \\ \text{NULL}, & \text{if } i \notin S_j \end{cases},$$
(39)

where

$$\mathcal{S}_{j} \triangleq \left([0:M-1] + (j-1) \cdot M \right)_{N} + 1, \quad \forall j \in \left[1: \frac{N}{\gcd(N,M)} \right].$$
(40)

It is easy to check that the array P is an (N, M) SDA satisfying $\eta_{\mathbf{P}} = \frac{N}{\gcd(N, M)}$, by the following three facts from S_j $(j \in \left[1 : \frac{N}{\gcd(N,M)}\right]):$

F1. For each $j \in \left[1 : \frac{N}{\gcd(N,M)}\right]$, set \mathcal{S}_j is of size M, i.e., $|\mathcal{S}_j| = M$ for all $j \in [1 : \frac{N}{\gcd(N,M)}]$;

F2. All the sets in (40) are distinct, i.e., $S_i \neq S_j$ for any $i \neq j \in [1 : \frac{N}{\gcd(N,M)}]$; F3. For any given $n \in [1 : N]$, n exactly occurs in $\frac{M}{\gcd(N,M)}$ different sets in (40), i.e., $|\{j \in [1 : \frac{N}{\gcd(N,M)}] : n \in S_j\}| = 1$ $\frac{M}{\operatorname{gcd}(N,M)}$ for all $n \in [1:N]$.

By (36), $\alpha_{\mathcal{S}} = \frac{\gcd(N,M)}{N}$ if $\mathcal{S} = \mathcal{S}_j$ for some $j \in [1 : \frac{N}{\gcd(N,M)}]$, and $\alpha_{\mathcal{S}} = 0$ otherwise. Accordingly, all the non-zero $\{W_{k,S} : k \in [1:K], S \subseteq [1:N], |S| = M, \alpha_S > 0\}$ are of equal size and each is partitioned into M - 1 equal-size packets by Algorithms 1-2. Therefore, its capacity-achieving SF-SC-PIR scheme has equal-size sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$

B. Optimality of Equal-Size Sub-packetization

Recall from Algorithm 1 that any feasible solution $\{\alpha_{\mathcal{S}}\}_{\mathcal{S} \subseteq [1:N], |\mathcal{S}|=M}$ to Problem 1 can support a capacity-achieving $(\mu = M/N, N, K)$ linear SF-SC-PIR scheme by employing any specific capacity-achieving (M, K) linear SF-PIR scheme as a building block. According to Line 4 in Algorithm 1 and (37), each $W_{k,S}$ of size $\alpha_S > 0$ is partitioned into F_{SF} equal-sized disjoint packets. Thus, to design a linear SF-SC-PIR scheme with equal-size sub-packetization, it is necessary that $\alpha_S > 0$ is a constant.

Lemma 5. Given any (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$ and $\{\alpha_S\}_{S \subseteq [1:N]}$, the storage design of any capacity-achieving linear SF-SC-PIR scheme with equal-size sub-packetization must satisfy:

P6. The equal-size partition storage is adopted, i.e., all the non-zero α_S has the same value.

From Theorem 1, the equal-size sub-packetization of any SF-SC-PIR scheme is no less than $\eta_e^* \cdot (M-1)$, where η_e^* is the optimal value of the following problem by Lemmas 1 and 5.

Definition 6. Given any positive integers N and $M = \mu N \in [2:N]$, **Problem 2** is defined as

$$(\{\alpha_{\mathcal{S}}^{*}\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}, \Delta^{*}) = \arg\min \sum_{\substack{\mathcal{S}\subseteq[1:N]\\|\mathcal{S}|=M}} \mathbf{1}(\alpha_{\mathcal{S}} > 0)$$
s.t.
$$\sum_{\substack{\mathcal{S}\subseteq[1:N]\\|\mathcal{S}|=M}} \alpha_{\mathcal{S}} = \mu, \quad \forall n \in [1:N]$$
(41)

$$\alpha_{\mathcal{S}} \in \{\Delta, 0\}, \qquad \forall \mathcal{S} \subseteq [1:N], |\mathcal{S}| = M$$
(42)

$$0 < \Delta < 1 \tag{43}$$

where $(\{\alpha_{\mathcal{S}}^*\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}, \Delta^*)$ is called the optimal solution to Problem 2 and $\eta_e^* = \sum_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M} \mathbf{1}(\alpha_{\mathcal{S}}^* > 0)$ is called the optimal value of Problem 2.

Thus, we just need to prove that the optimal value η_e^* of Problem 2 satisfies $\eta_e^* \ge \frac{N}{\gcd(N,M)}$. From Problem 2,

$$1 \stackrel{(a)}{=} \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M}} \alpha_{\mathcal{S}}^*$$
$$= \Delta^* \cdot \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M}} \mathbf{1}(\alpha_{\mathcal{S}}^* > 0)$$
$$= \Delta^* \cdot \eta_e^*,$$

where (a) follows by (32) and (41)-(43). Thus, we have $\Delta^* = \frac{1}{\eta_*^*}$.

By (41), the storage constraint of any server n is

$$\begin{split} \mu &= \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M, n \in \mathcal{S}}} \alpha_{\mathcal{S}}^* \\ &= \Delta^* \cdot \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M, n \in \mathcal{S}}} \mathbf{1}(\alpha_{\mathcal{S}}^* > 0) \\ &= \frac{1}{\eta_e^*} \nu. \end{split}$$

Note that $\nu = \sum_{S \subseteq [1:N], |S|=M, n \in S} \mathbf{1}(\alpha_S^* > 0)$ is an integer. Then, the above equation indicates that

$$\nu = \mu \cdot \eta_e^* = \frac{M}{N} \cdot \eta_e^*$$

is an integer. Therefore, η_e^* must be greater than or equal to $\frac{N}{\gcd(N,M)}$, which completes the proof of Theorem 5.

VII. CAPACITY-ACHIEVING LINEAR SC-PIR SCHEMES WITH LOWER SUB-PACKETIZATION

The sub-packetization reflects the implementation complexity of a scheme, specifically in a PIR system, low sub-packetization achieves low complexity [28]. In order to further reduce sub-packetization, we allow unequal-size packets in this section. Notably, unequal-size packets are usually unavoidable in such SC-PIR [26], [1], since the memory-sharing technique typically results in schemes with unequal-size packets [1], [16]. Particularly, memory-sharing is often used to achieve the capacity for any storage $M \in [1, N]$ by resorting to the discrete points with M = 1, 2, ..., N.

Next, we first present an example to illustrate that allowing unequal-size packets can further decrease sub-packetization of capacity-achieving SC-PIR schemes.

Example 4. An (N = 12, M = 5) SDA can be also constructed by the form of

	*	*	*	*	*]
P =	- * * * *	*	*	*	*								
	*	*	*	*	*								
	*	*	*	*	*								
	*	*	*	*	*								
						*	*	*	*	*			
						*	*	*	*		*		
						*	*	*	*			*	
						*	*			*	*	*	
						*	*			*	*	*	
								*	*	*	*	*	
	L							*	*	*	*	*	$\Big]_{12 \times 12}$

There are $\eta_{\mathbf{P}} = 6$ distinct columns in **P**, i.e., columns 1, 6, 8, 10, 11 and 12. By Theorem 4, the SDA can be used for constructing a capacity-achieving ($\mu = 5/12, N = 12, K$) linear SC-PIR scheme with sub-packetization 24, which is smaller than 48, the optimal equal-size sub-packetization as illustrated in Example 3.

Based on Theorem 4, we wish to construct an SDA P with $\eta_{\rm P}$ as low as possible for further reducing sub-packetization.

A. Greedy Construction of Storage Design Arrays

In this subsection, we propose a greedy construction of (N, M) SDA P for any N and $M \in [1 : N]$. By convenience, for any positive integers n, m, we use $[*]_{n \times m}$ to denote an array of size $n \times m$ with all the entries being "*"s.

Clearly, when gcd(N, M) > 1, an (N, M) SDA P can be yielded by

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}' \\ \vdots \\ \mathbf{P}' \end{bmatrix}_{N \times \frac{N}{\gcd(N,M)}}^{N}$$
(44)

where \mathbf{P}' is an $\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right)$ SDA of size $\frac{N}{\gcd(N,M)} \times \frac{N}{\gcd(N,M)}$. That is, \mathbf{P} is generated by repeating $\mathbf{P}' \gcd(N,M)$ times. Hereafter, we only concentrate on the construction of (N, M) SDA for $\gcd(N, M) = 1$.

Notice that we aim to construct (N, M) SDA P with $\eta_{\mathbf{P}}$ as small as possible for gcd(N, M) = 1. Intuitively, by the definition of $\eta_{\mathbf{P}}$ in (34), the columns of SDA should be repeated as much as possible to reduce $\eta_{\mathbf{P}}$, i.e., s_l should be as big as possible for every $l \in [1 : \eta_{\mathbf{P}}]$. However, s_l is upper bounded by $\min\{M, N - M\}$ since

- $s_l \leq M$ follows from S2 directly.
- If $s_l > N M$, i.e., some column is repeated more than N M times, without loss of generality, assume that the first M rows and first m (m > N M) columns of P form array $[*]_{M \times m}$. Then by S1, the first m entries of the (M + 1)-th row are NULL and thus the (M + 1)-th row has at most N m < M "*"s, contradicting S2.

Based on the above fact that each column in SDA is repeated at most $\min\{M, N - M\}$ times, Algorithm 3 is proposed to recursively construct (N, M) SDA for gcd(N, M) = 1 as follows:

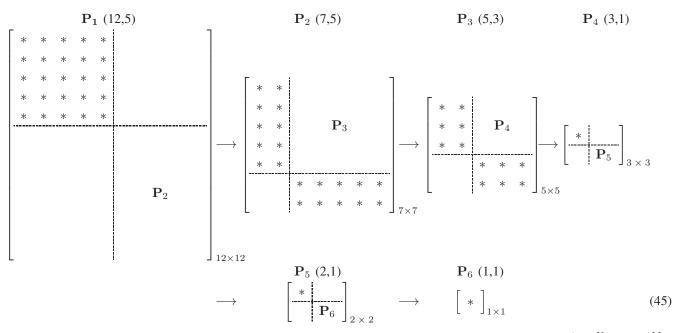
- 1) Case 1 (Lines 6-7): If $N M \ge M$ (i.e., $N \ge 2M$), $\min\{M, N M\} = M$. Thus, greedily generate $[*]_{M \times M}$ and then proceed an (N M, M) SDA.
- 2) Case 2 (Lines 9-10) If N-M < M (i.e., N < 2M), $\min\{M, N-M\} = N-M$. Hence greedily generate $[*]_{M \times (N-M)}$, $[*]_{(N-M) \times M}$, and then proceed an (M, 2M N) SDA.

The two cases above are recursively carried out until N = 1, i.e., $\mathbf{P} = [*]_{1 \times 1}$.

Algorithm 3 Greedy SDA Algorithm (G-SDA)

Input: Positive integers (N, M) with $1 \le M \le N$ and gcd(N, M) = 1; **Output:** An (N, M) array **P** of size $N \times N$; 1: **Procedure** GreedySDA (N, M)2: if N = 1 then $P = [*]_{1 \times 1};$ 3: 4: **else** if $N \geq 2M$ then 5:
$$\begin{split} \mathbf{P}' &= \operatorname{GreedySDA}\left(N-M,M\right);\\ \mathbf{P} &= \left[\begin{array}{c|c} [*]_{M \times M} \\ \hline \end{array}\right]_{N \times N}; \end{split}$$
6: 7: else 8: $\mathbf{P}' = \operatorname{GreedySDA}(M, 2M - N);$ $\mathbf{P} = \begin{bmatrix} [*]_{M \times (N-M)} & \mathbf{P}' \\ & [*]_{(N-M) \times M} \end{bmatrix}_{N \times N};$ 9: 10: end if 11: 12: end if 13: end Procedure

Example 5. The SDA in Example 4 is in fact constructed by the G-SDA algorithm with input parameters (N = 12, M = 5). The recursive processes are illustrated in (45), where \mathbf{P}_i $(1 \le i \le 6)$ is the output array in the *i*-th recursive step with input parameters specified in the brackets to its right.



Theorem 6. Given any positive integers N, M with $1 \le M \le N$, there exists an (N, M) SDA \mathbf{P} with $\eta_{\mathbf{P}} = \eta\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right)$, where $\eta(N, M)$ is recursively defined for any N, M with $1 \le M \le N$ and $\gcd(N, M) = 1$ by

$$\eta(N,M) = \begin{cases} 1, & \text{if } N = 1\\ 1 + \eta (N - M, M), & \text{if } N > 1 \text{ and } N \ge 2M \\ 1 + \eta (M, 2M - N), & \text{if } N > 1 \text{ and } N < 2M \end{cases}$$
(46)

Proof: For any N, M with $1 \le M \le N$, the SDA **P** in (44) has $\eta_{\mathbf{P}} = \eta_{\mathbf{P}'}$, where **P**' is an array output by Algorithm 3 with input parameters $\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right)$. Thus, it is sufficient to prove that the output array of Algorithm 3 is an SDA **P** with $\eta_{\mathbf{P}} = \eta(N, M)$ for any N, M with $1 \le M \le N$ and $\gcd(N, M) = 1$.

First of all, for any input parameters (N, M) with $1 \le M \le N$ and gcd(N, M) = 1, we observe two facts from Lines 6 and 9 of Algorithm 3: (1) During each recursive procedure, the recursive input parameters (N, M) always maintain the property that $1 \le M \le N$ and gcd(N, M) = 1; (2) N strictly decreases and thus eventually decreases to 1. Then, the recursive procedure will terminate at Line 3, i.e., N = 1. Actually, it is also easy to observe that the recursions of Algorithm 3 happen $\eta(N, M)$ times.

Secondly, it is easy to verify from Lines 7 and 10 that P satisfies S1 and S2 with parameters (N, M) if and only if P' satisfies them with parameters (N - M, M) (if $N \ge 2M$) or (M, 2M - N) (if N < 2M). So, we can easily prove that the output P is an SDA with $\eta_{\mathbf{P}}$ satisfying (46) by the induction method.

The corollary below follows from Theorems 4 and 6.

Corollary 2. For any positive integers N, K, M with $M \in [2 : N]$, there exists a capacity-achieving $(\mu = M/N, N, K)$ linear SC-PIR scheme with sub-packetization $\eta(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}) \cdot (M-1)$.

Remark 2. Here, we show that the SDA P constructed in (44) can further decrease the sub-packetization of capacity-achieving SC-PIR schemes compared to the optimal equal-size sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$ in Theorem 5. Since the (N, M) SDA P has $\frac{N}{\gcd(N,M)}$ columns for any $M \in [2:N]$, $\eta_{\mathbf{P}} = \eta(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}) \leq \frac{N}{\gcd(N,M)}$. Remarkably, it is easy to prove from (46) that the equality (i.e., $\eta_{\mathbf{P}} = \frac{N}{\gcd(N,M)}$) holds if and only if $\frac{M}{\gcd(N,M)} = 1$ or $\frac{N-M}{\gcd(N,M)} = 1$. In the other cases, i.e., $\frac{M}{\gcd(N,M)} \neq 1$ and $\frac{N-M}{\gcd(N,M)} \neq 1$, we have $\eta_{\mathbf{P}} < \frac{N}{\gcd(N,M)}$ and thus such SDAs can be used for generating capacity-achieving SC-PIR schemes with sub-packetization strictly smaller than the optimal equal-size sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$ (e.g. Example 4). In addition, when $\frac{M}{\gcd(N,M)} = 1$ or $\frac{N-M}{\gcd(N,M)} = 1$, it will be shown in Theorem 8 that the associated capacity-achieving SC-PIR scheme has the optimal sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$.

B. Improved Construction of Storage Design Arrays

Recall that Algorithm 3 always greedily repeats columns in the current recursive step l, which may lead to many $s_l = 1$ in the latter steps and thus results in large $\eta_{\mathbf{P}}$. Particularly, when N = 2M + 1, it generates SDA with $\{s_1 = M, s_2 = \ldots = 1\}$ $s_{2+M} = 1$. In principle, in order to minimize $\eta_{\mathbf{P}}$ of SDA, it should be better to design repeated columns from a global perspective. For this case, by decreasing s_1 to M-1 and increasing some s_l from 1 to 2, we are able to present an improved construction to decrease the sub-packetization.

Before that, it is worthy to point out the following simple property of SDA.

Lemma 6. For any (N, M) SDA $\mathbf{P} = [p_{i,j}]_{N \times \frac{N}{\gcd(N,M)}}$, its opposite array $\overline{\mathbf{P}} = [\overline{p}_{i,j}]_{N \times \frac{N}{\gcd(N,M)}}$ defined by

$$\overline{p}_{i,j} = \begin{cases} *, & \text{if } p_{i,j} = \text{NULI} \\ \text{NULL}, & \text{if } p_{i,j} = * \end{cases}$$

is an (N, N - M) SDA. Moreover, the number of distinct columns in **P** and $\overline{\mathbf{P}}$ are equal, i.e., $\eta_{\mathbf{P}} = \eta_{\overline{\mathbf{P}}}$.

Firstly, given any positive integer $M \ge 2$, we construct an (N, M) = (2M + 1, M) SDA \mathbf{Q}_M of size $(2M + 1) \times (2M + 1)$ as

• If M is even,

$$\mathbf{Q}_{M} = \begin{bmatrix} [*]_{M \times (M-1)} & & & \\ & & \\ & & \\ \hline \\ \\ & & \\ \hline \\ \\ \hline \\ \\ & & \\ \hline \\ \hline \\ \\ \hline \\ \\ \hline \\ \\ \hline \\ \hline \\ \hline \\ \\ \hline \\ \\ \hline \\ \hline \\ \\ \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \\ \hline \\ \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \hline \\ \\ \hline \\ \hline \\ \\ \hline \\ \\ \hline \\ \\ \hline \\$$

• If M is odd.

where $diag(*)_{n \times n}$ denotes an $n \times n$ array with the entries in diagonal being "*"s and the rest entries being "NULL"s. It is easily checked that \mathbf{Q}_M is a (2M+1, M) SDA with $\eta_{\mathbf{Q}_M} = \left\lceil \frac{M}{2} \right\rceil + 3$.

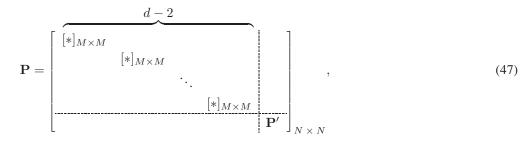
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Example 6. When M = 4 and M = 5, \mathbf{Q}_4 and \mathbf{Q}_5 are the following forms, respectively.

Obviously, $\eta_{\mathbf{Q}_4} = 5$ with $\{s_1 = 3, s_2 = 2, s_3 = 2, s_4 = 1, s_5 = 1\}$ and $\eta_{\mathbf{Q}_5} = 6$ with $\{s_1 = 4, s_2 = 2, s_3 = 2, s_4 = 1, s_5 = 1\}$ 1, $s_6 = 1$ }. Compared to **P** with { $s_1 = 4, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 1, s_6 = 1$ } (resp. **P**' with { $s_1 = 5, s_2 = 1, s_3 = 1, s_4 = 1, s_5 = 1, s_6 = 1$ } $1, s_5 = 1, s_6 = 1, s_7 = 1$) in Example 1, the distribution of repeated columns s_l is more flexible than the greedy algorithm, which leads to a smaller number of distinct columns.

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Notice that \mathbf{Q}_{M-1} is a (2M-1, M-1) SDA with $\eta_{\mathbf{Q}_{M-1}} = \lfloor \frac{M}{2} \rfloor + 3$ for any $M \ge 3$. By Lemma 6, we can obtain a (2M-1, M) SDA $\overline{\mathbf{Q}}_{M-1}$ with $\eta_{\overline{\mathbf{Q}}_{M-1}} = \eta_{\mathbf{Q}_{M-1}} = \lfloor \frac{M}{2} \rfloor + 3$ for any $M \ge 3$. Next, based on \mathbf{Q}_M and $\overline{\mathbf{Q}}_{M-1}$, for any given positive integers M, d such that $M \ge 3, d \ge 2$, we can construct a class of (N, M) SDA with $N = dM \pm 1$ as:



where

$$\mathbf{P}' = \begin{cases} \mathbf{Q}_M, & \text{if } N = dM + 1\\ \overline{\mathbf{Q}}_{M-1}, & \text{if } N = dM - 1 \end{cases}$$

The following result is straightforward.

Theorem 7. Given any positive integer $N = dM \pm 1$ for integers $M \in [3:N]$ and $d \ge 2$, there is an (N, M) SDA **P** with

$$\eta_{\mathbf{P}} = \begin{cases} d + \left\lceil \frac{M}{2} \right\rceil + 1, & \text{if } N = dM + 1\\ d + \left\lfloor \frac{M}{2} \right\rfloor + 1, & \text{if } N = dM - 1 \end{cases}$$

Remark 3. Notice that gcd(N, M) = 1 when $N = dM \pm 1$ for any integer $d \ge 2$. Let (N, M) be the input parameters of the G-SDA algorithm, then the number of distinct columns of the output SDA is equal to $\eta(N, M)$ in (46) at $N = dM \pm 1$, i.e.,

$$\eta(N,M) = \begin{cases} d+M, & \text{if } N = dM+1\\ d+M-1, & \text{if } N = dM-1 \end{cases}$$

Thus, the SDA presented in (47) decreases the number of distinct columns and further reduces the required sub-packetization of capacity-achieving SC-PIR scheme.

Finally, we obtain the following corollary from Theorems 4 and 7.

Corollary 3. Given any positive integers N, K, M with $M \in [3 : N]$ and $N = dM \pm 1$ for some integer $d \ge 2$, there exists a capacity-achieving $(\mu = M/N, N, K)$ linear SC-PIR scheme with sub-packetization $\left(d + \left\lceil \frac{M}{2} \right\rceil + 1\right) \cdot (M - 1)$ if N = dM + 1 or $\left(d + \left\lfloor \frac{M}{2} \right\rfloor + 1\right) \cdot (M - 1)$ if N = dM - 1.

In general, it is difficult to extend the above improvement to all the parameters N and $M \in [1 : N]$, since the modification of s_l becomes very complicated and sometimes even unsolvable under the SDA constraints S1 and S2 if there are many different values of s_l .

C. Optimality on Sub-packetization of Greedy SDA

In this subsection, firstly we establish a lower bound on the optimal value of Problem 1. Next, we discuss the optimality on sub-packetization of the SC-PIR scheme associated to the proposed SDA with respect to this bound.

Lemma 7. For any positive integers N, M with $2 \le M < N$, the optimal value of Problem 1 satisfies

$$\eta^* \ge \max\left\{ \left\lceil \frac{N}{M} \right\rceil, \left\lceil \frac{N}{N-M} \right\rceil \right\}.$$
(48)

Proof: Suppose that $\{\alpha_{\mathcal{S}}^*\}_{\mathcal{S}\subseteq[1:N],|\mathcal{S}|=M}$ is the optimal solution to Problem 1. Let $\{\mathcal{S}_i\}_{i=1}^{\eta^*}$ be the indices of non-zero elements in the solution, i.e., $\alpha_{\mathcal{S}}^* > 0$ if and only if $\mathcal{S} \in \{\mathcal{S}_i\}_{i=1}^{\eta^*}$. By (32), the optimal solution satisfying (22) and (23) implies the following constraint

$$\sum_{\substack{\mathcal{S}\subseteq[1:N]\\|\mathcal{S}|=M}} \alpha_{\mathcal{S}}^* = \sum_{i\in[1:\eta^*]} \alpha_{\mathcal{S}_i}^* = 1.$$
(49)

Hence, there must be an index $j \in [1 : \eta^*]$ such that

$$\alpha_{\mathcal{S}_j}^* \ge \frac{1}{\eta^*}.\tag{50}$$

Assume that $\eta^* < \lceil \frac{N}{M} \rceil$. Then, $\frac{N}{M} - \eta^* > 0$ since η^* is an integer. Accordingly, there must exist a positive number $c = \frac{N}{M} - \eta^* > 0$. By (50),

$$\alpha_{\mathcal{S}_j}^* \ge \frac{1}{\frac{N}{M} - c}$$
$$> \frac{M}{N}$$
$$= \mu.$$

Thus, for any $n \in S_j$,

$$\sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M, n \in \mathcal{S}}} \alpha_{\mathcal{S}}^* \ge \alpha_{\mathcal{S}_j}^* > \mu,$$
(51)

which contradicts (22). Thus, $\eta^* \ge \left\lceil \frac{N}{M} \right\rceil$.

Assume that $\eta^* < \left\lceil \frac{N}{N-M} \right\rceil$. Similarly, we have $\eta^* = \frac{N}{N-M} - c$ for some c > 0. Then,

$$\alpha_{\mathcal{S}_j}^* \ge \frac{1}{\frac{N}{N-M} - c}$$
$$> \frac{N-M}{N}$$
$$= 1 - \mu.$$

Since M < N, there exists $n \in [1 : N] \setminus S_j$ such that

$$\sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = M, n \in \mathcal{S}}} \alpha_{\mathcal{S}}^* = \sum_{\substack{\mathcal{S} \subseteq [1:N], |\mathcal{S}| = M \\ \mathcal{S} \neq \mathcal{S}_j, n \in \mathcal{S}}} \alpha_{\mathcal{S}}^*$$

$$\stackrel{(a)}{\leq} 1 - \alpha_{\mathcal{S}_j}^*$$

$$< \mu,$$

where (a) is due to (49), a contradiction to (22) again. That is, $\eta^* \ge \left\lceil \frac{N}{N-M} \right\rceil$.

Recall from Theorem 6 and Corollary 2 that the capacity-achieving linear SC-PIR scheme associated to the SDA in (44) has sub-packetization

$$F_{\text{G-SDA}}^{(N,M)} \triangleq \eta\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right) \cdot (M-1).$$
(52)

In general, it is difficult to directly compare the sub-packetization $F_{G-SDA}^{(N,M)}$ and the optimal value $F^* = \eta^* \cdot (M-1)$ in Theorem 3 since $\eta(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)})$ is defined in (46) as a *recursive* form. In the following theorem, we obtain a multiplicative gap by relaxing $\eta(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)})$ to an upper bound and η^* to a lower bound. In addition, this theorem also characterize two special optimal cases.

Theorem 8. Given any (μ, N, K) SC-PIR system with $M = \mu N \in [2 : N]$, the sub-packetization $F_{G-SDA}^{(N,M)}$ is within a multiplicative gap $\frac{\min\{M, N-M\}}{\gcd(N,M)}$ of the optimal sub-packetization of capacity-achieving linear SC-PIR schemes. In particular, in the cases $\min\{M, N-M\}|N$ or M = N, the sub-packetization $F_{G-SDA}^{(N,M)} = \frac{N(M-1)}{\gcd(N,M)}$ is optimal.

Proof: From Theorem 3, the optimal sub-packetization is $F^* = \eta^* \cdot (M-1)$. In the case M = N, since $\eta^* \ge 1$, $F_{G-SDA}^{(N,M)} = M - 1 = F^*$ is optimal. When $M \in [2: N-1]$,

$$1 \leq \frac{F_{\text{G-SDA}}^{(N,M)}}{F^*}$$
$$\stackrel{(a)}{=} \frac{\eta\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right)}{\eta^*}$$
(53)

$$\overset{(b)}{\leq} \frac{\frac{N}{\gcd(N,M)}}{\max\left\{\left\lceil\frac{N}{M}\right\rceil, \left\lceil\frac{N}{N-M}\right\rceil\right\}} \\
\leq \frac{\min\{M, N-M\}}{\gcd(N,M)},$$
(54)

where (a) follows by (52); (b) is due to the lower bound in (48) and the upper bound $\eta\left(\frac{N}{\gcd(N,M)}, \frac{M}{\gcd(N,M)}\right) \leq \frac{N}{\gcd(N,M)}$ by Remark 2.

In particular, in the case $\min\{M, N-M\}|N$, we have $gcd(N, M) = \min\{M, N-M\}$. Thus, $F_{G-SDA}^{(N,M)} = F^*$ by (54), i.e., $F_{G-SDA}^{(N,M)} = \frac{N}{gcd(N,M)} \cdot (M-1)$ is the optimal sub-packetization in this case.

Remark 4. The SC-PIR problem straightly degrades to the problem of replicated PIR by setting M = N. Obviously, when M = N, our SC-PIR scheme associated to the SDA in (44) achieves the optimal sub-packetization $F^* = M - 1 = N - 1$, which is the same as that in [28].

VIII. CONCLUSION

In this paper, we investigated the sub-packetization of uncoded Storage Constrained PIR (SC-PIR). We first characterized the optimal sub-packetization of capacity-achieving SC-PIR schemes by an optimization problem, which is hard to solve due to the involved non-continuous indicator functions. We introduced Storage Design Array (SDA) to construct practical operational SC-PIR schemes with low sub-packetization. It turns out that, for the SC-PIR system with N servers and a total normalized storage capacity $M \in \{2, ..., N\}$, the equal-size sub-packetization $\frac{N(M-1)}{\gcd(N,M)}$ is optimal among all capacity-achieving linear SC-PIR schemes characterized by Woolsey *et al.*. Finally, by allowing unequal-size packets, two constructions of SDAs were proposed to further decrease the sub-packetization. The resultant sub-packetizations were shown to be optimal in the cases $\min\{M, N - M\}|N$ or M = N, and within a multiplicative gap of $\frac{\min\{M, N-M\}}{\gcd(N,M)}$ compared to a lower bound on the optimal sub-packetization otherwise.

APPENDIX

In this section, we provide the proof of the five necessary conditions P1–P5 for any capacity-achieving linear SC-PIR scheme. To this end, we first refine the converse proof of SC-PIR capacity given in [1], and accordingly, we have placed an emphasis on the necessary properties of capacity-achieving SC-PIR schemes by constraining the inequalities in the refined proof to be held with equalities. Further, the obtained properties are specialized to linear SC-PIR schemes to complete the proof. Actually, the similar approach was used in [28] and [37] for the setups of replicated PIR and MDS-coded PIR, respectively.

We start by proving two useful lemmas.

A. Preliminary Lemmas

Lemma 8. For any $i \in [1:N], \mathcal{K} \subseteq [1:K]$, and $\mathcal{N} \subseteq [1:N]$,

$$H(A_i^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta]}) = H(A_i^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}), \quad \forall \theta \in [1:K].$$

$$(55)$$

Proof: Notice that

$$\begin{split} 0 &\leq H(A_{i}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) - H(A_{i}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}) \\ &= I(A_{i}^{[\theta]}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) \\ &\leq I(A_{i}^{[\theta]}, W_{[1:K] \setminus \mathcal{K}}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) \\ &= I(W_{[1:K] \setminus \mathcal{K}}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) + I(A_{i}^{[\theta]}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|W_{1:K}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) \\ &\stackrel{(a)}{=} I(W_{[1:K] \setminus \mathcal{K}}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{i}^{[\theta]}) \\ &\leq I(W_{1:K}, Z_{\mathcal{N}}; Q_{[1:N] \setminus \{i\}}^{[\theta]}|Q_{i}^{[\theta]}) \\ &\leq I(W_{1:K}, Z_{\mathcal{N}}; Q_{1:N}^{[\theta]}) \\ &\stackrel{(b)}{=} I(W_{1:K}; Q_{1:N}^{[\theta]}) \\ &\stackrel{(c)}{=} 0, \end{split}$$

where (a) holds because $A_i^{[\theta]}$ is a function of $W_{1:K}$ and $Q_i^{[\theta]}$ by (9); (b) follows from the fact that Z_N is a function of the files $W_{1:K}$ by (7); and (c) follows from the independence of files and queries by (8).

Lemma 9. For any
$$i \in [1:N], \mathcal{K} \subseteq [1:K]$$
, and $\mathcal{N} \subseteq [1:N]$,

$$H(A_i^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta]}) = H(A_i^{[\theta']}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta']}), \quad \forall \theta, \theta' \in [1:K].$$
(56)

Proof: For any $\theta \in [1:K]$,

$$0 \leq I(Q_i^{[\theta]}, A_i^{[\theta]}, W_{\mathcal{K}}, Z_{\mathcal{N}}; \theta)$$

$$\leq I(Q_i^{[\theta]}, A_i^{[\theta]}, W_{1:K}, Z_{\mathcal{N}}; \theta)$$

$$\stackrel{(a)}{=} I(Q_i^{[\theta]}, W_{1:K}; \theta)$$

$$= I(Q_i^{[\theta]}; \theta) + I(W_{1:K}; \theta | Q_i^{[\theta]})$$

$$\stackrel{(b)}{=} I(Q_i^{[\theta]}; \theta)$$

$$\leq I(Q_i^{[\theta]}, A_i^{[\theta]}, Z_i; \theta)$$

$$\stackrel{(c)}{=} 0,$$

where (a) holds because $A_i^{[\theta]}$ and Z_N are determined by $Q_i^{[\theta]}$ and $W_{1:K}$, (b) holds because the files are independent of the desired file index and the query, and (c) follows from the privacy constraint (12).

Notice that, $I(Q_i^{[\theta]}, A_i^{[\theta]}, W_{\mathcal{K}}, Z_{\mathcal{N}}; \theta) = 0$ indicates that $(Q_i^{[\theta]}, A_i^{[\theta]}, W_{\mathcal{K}}, Z_{\mathcal{N}})$ and θ are independent of each other. Therefore, (56) holds.

Next, we present some properties of any SC-PIR scheme.

B. Properties of SC-PIR Schemes

For $n \in [0: N-1]$ and $k \in [0: K-1]$, define

$$T(n,k) \triangleq \frac{1}{NK\binom{K-1}{k}\binom{N-1}{n}} \sum_{\substack{\mathcal{K} \subseteq [1:K] \\ |\mathcal{K}| = k}} \sum_{\substack{\mathcal{N} \subseteq [1:N] \\ |\mathcal{N}| = n}} \sum_{\theta \in [1:K] \setminus \mathcal{K}} \sum_{i \in [1:N] \setminus \mathcal{N}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta]})$$
(57)

and

$$T(n,K) \triangleq 0, \quad \forall n \in [0:N-1].$$
(58)

In addition,

$$\lambda_n \triangleq \frac{1}{K\binom{N}{n}} \sum_{\substack{\mathcal{N} \subseteq [1:N] \\ |\mathcal{N}|=n}} \sum_{\substack{\theta \in [1:K] \\ H(W_{\theta}|Z_{\mathcal{N}}), \quad \forall n \in [0:N-1].}$$
(59)

Lemma 10. For each $n \in [0: N-1]$ and $k \in [0: K-1]$, any SC-PIR scheme must satisfy the following inductive relationship.

$$T(n,k) \ge \frac{1}{N-n} \left[\sum_{n'=n}^{N-1} T(n',k+1) + \lambda_n \right].$$
 (60)

Moreover, to establish the equality in (60), for every realization of queries $\widetilde{Q}_{1:N}^{[\theta]}$ with positive probability,

• For any $\mathcal{K} \subseteq [1:K]$ of size k, $\mathcal{N} \subseteq [1:N]$ of size n, and $\theta \in [1:K] \setminus \mathcal{K}$,

$$\sum_{i \in [1:N] \setminus \mathcal{N}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = H(A_{[1:N] \setminus \mathcal{N}}^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]});$$
(61)

• For any $\mathcal{K} \subseteq [1:K]$, $\theta \in \mathcal{K}$, $\mathcal{N}_1 \subseteq [1:N]$, $\mathcal{N}_2 \subseteq [1:N] \setminus \mathcal{N}_1$ and $i \in [1:N] \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ such that $|\mathcal{K}| = k+1$, $|\mathcal{N}_1| = n$, $|\mathcal{N}_2| = n' - n$ with $n' \in [n+1:N-1]$,

$$I(A_i^{[\theta]}; Z_{\mathcal{N}_2} | W_{\mathcal{K}}, Z_{\mathcal{N}_1}, A_{\mathcal{N}_2}^{[\theta]}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = 0.$$
(62)

Proof: For any $n \in [0: N-1]$ and $k \in [0: K-1]$, T(n, k) in (57) can be lower bounded as

$$\begin{split} T(n,k) &= \frac{1}{NK^{\binom{N-1}{k}} \binom{N-1}{n}} \sum_{\substack{K \subseteq [1:K] \ | N \subseteq [1:N] \ | N \in [1:K] \ | N \subseteq [1:N] \ | N \in [1:K] \ | N \in [1:K] \ | N \subseteq [1:K] \ | N \in [1:K$$

where (a) follows by (55); (b) follows from the property of independence bound on entropy; (c) follows from (9) in which $A_{\mathcal{N}}^{[\theta]}$ are a determined function of $Z_{\mathcal{N}}$ and $Q_{\mathcal{N}}^{[\theta]}$, thus with $(A_{[1:N]\setminus\mathcal{N}}^{[\theta]}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]})$, the file W_{θ} can be decoded by (10), i.e., $H(W_{\theta}|A_{[1:N]\setminus\mathcal{N}}^{[\theta]}, W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}) = 0$; (d) is due to $H(W_{\theta}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}) = H(W_{\theta}|Z_{\mathcal{N}})$ for $\theta \notin \mathcal{K}$ by the independence of the files (2) and the fact that queries are independent of the files (8); (e) and (f) follow by simply changing the summation indices; (g) follows from $\sum_{\substack{\mathcal{K} \subseteq [1:K]\setminus\{\theta\}, |\mathcal{K}|=k}} H(W_{\theta}|Z_{\mathcal{N}}) = \binom{K-1}{k}H(W_{\theta}|Z_{\mathcal{N}})$; (h) follows from the definition of λ_n in (59). Notice that when $n \in [0:N-1]$ and k = K-1, all the files $W_{1:K}$ are presented in the conditions of each entropy function in $\widetilde{T}(n,k)$. Since the answers $A_{[1:N]\setminus\mathcal{N}}^{[\theta]}$ is a function of $Q_{[1:N]\setminus\mathcal{N}}^{[\theta]}$ and $W_{1:K}$ by (9), we have $\widetilde{T}(n, K-1) = 0$. Thus, by (64), for $n \in [0:N-1]$ and k = K-1, where (a) follows by (55); (b) follows from the property of independence bound on entropy; (c) follows from (9) in which

$$T(n, K-1) \ge \frac{1}{N-n} \cdot \lambda_n, \tag{65}$$

This proves (60) for $n \in [0: N-1]$ and k = K-1. We proceed to prove the other cases by deriving a lower bound on $\widetilde{T}(n,k).$

Let \mathcal{P}_N be the set consisting of all possible permutations of [1:N], and $\boldsymbol{\sigma} \triangleq (\sigma_1, \sigma_2, \dots, \sigma_N) \in \mathcal{P}_N$ denote a permutation operation of [1:N]. For any $n \in [0:N-1]$ and $k \in [0:K-2]$, we further lower bound $\widetilde{T}(n,k)$ as follows.

$$\begin{split} \tilde{T}(n,k) &= \frac{1}{NK\binom{K-1}{k}} \sum_{\substack{K \subseteq [1:K] \\ |K| = k+1}} \sum_{\substack{K \in [$$

where (a) holds because for each $\mathcal{N} \subseteq [1:N]$ with $|\mathcal{N}| = n$, there are exactly n!(N-n)! permutation operations $\boldsymbol{\sigma} \in \mathcal{P}_N$ satisfying $H(A_{\sigma_{[n+1:N]}}^{[\theta]}|W_{\mathcal{K}}, Z_{\sigma_{[1:n]}}, Q_{1:N}^{[\theta]}) = H(A_{[1:N]\setminus\mathcal{N}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]})$, i.e., such n!(N-n)! permutation operations $\boldsymbol{\sigma}$ are restricted over \mathcal{N} and $[1:N]\setminus\mathcal{N}$, hence,

$$\sum_{\boldsymbol{\sigma}\in\mathcal{P}_{N}} H(A_{\sigma_{[n+1:N]}}^{[\theta]}|W_{\mathcal{K}}, Z_{\sigma_{[1:n]}}, Q_{1:N}^{[\theta]}) = n!(N-n)! \cdot \sum_{\substack{\mathcal{N}\subseteq[1:N]\\|\mathcal{N}|=n}} H(A_{[1:N]\setminus\mathcal{N}}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]});$$

(b) follows from the chain rule of entropy; (c) holds because conditioning reduces entropy; (d) holds because the answers $A_{\sigma_{[n+1:n']}}^{[\theta]}$ is a function of the stored contents $Z_{\sigma_{[n+1:n']}}$ and the queries $Q_{\sigma_{[n+1:n']}}^{[\theta]}$ by (9); (e) follows by (55); (f) follows from similar arguments to (a); (g) follows from (56); (h) is due to $\sum_{\theta \in \mathcal{K}} H(A_i^{[\theta']}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta']}) = (k+1)H(A_i^{[\theta']}|W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_i^{[\theta']}).$

We prove (60) by replacing (64) with (67).

Remarkably, to establish the equality in (60), the inequalities in (63) and (66) have to hold with equalities.

• The equality in (63) indicates that for any $\mathcal{K} \subseteq [1:K]$ of size $k, \mathcal{N} \subseteq [1:N]$ of size n, and $\theta \in [1:K] \setminus \mathcal{K}$,

$$\sum_{\substack{\boldsymbol{\ell} \in [1:N] \setminus \mathcal{N}}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}) = H(A_{[1:N] \setminus \mathcal{N}}^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}).$$

That is,

$$0 = \sum_{i \in [1:N] \setminus \mathcal{N}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]}) - H(A_{[1:N] \setminus \mathcal{N}}^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]})$$

$$= \sum_{\widetilde{Q}_{1:N}^{[\theta]}} \Pr(Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) \left[\sum_{i \in [1:N] \setminus \mathcal{N}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) - H(A_{[1:N] \setminus \mathcal{N}}^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) \right]. (68)$$

Whereas, for each realization of queries $\widetilde{Q}_{1:N}^{[\theta]}$ with positive probability,

$$\sum_{\boldsymbol{\ell} \in [1:N] \setminus \mathcal{N}} H(A_i^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) \ge H(A_{[1:N] \setminus \mathcal{N}}^{[\theta]} | W_{\mathcal{K}}, Z_{\mathcal{N}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}).$$

That is, the terms in square bracket of (68) are nonnegative. Accordingly, (61) holds for all realizations of queries $\tilde{Q}_{1:N}^{[\theta]}$ with positive probability.

• Similarly, the equality in (66) indicates that for any $\mathcal{K} \subseteq [1 : K], \theta \in \mathcal{K}, \mathcal{N}_1 \subseteq [1 : N], \mathcal{N}_2 \subseteq [1 : N] \setminus \mathcal{N}_1$ and $i \in [1 : N] \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)$ such that $|\mathcal{K}| = k + 1$, $|\mathcal{N}_1| = n$, $|\mathcal{N}_2| = n' - n$ with $n' \in [n + 1 : N - 1]$,

$$0 = H(A_{i}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}_{1}}, A_{\mathcal{N}_{2}}^{[\theta]}, Q_{1:N}^{[\theta]}) - H(A_{i}^{[\theta]}|W_{\mathcal{K}}, Z_{\mathcal{N}_{1}}, Z_{\mathcal{N}_{2}}, A_{\mathcal{N}_{2}}^{[\theta]}, Q_{1:N}^{[\theta]})$$

$$= I(A_{i}^{[\theta]}; Z_{\mathcal{N}_{2}}|W_{\mathcal{K}}, Z_{\mathcal{N}_{1}}, A_{\mathcal{N}_{2}}^{[\theta]}, Q_{1:N}^{[\theta]})$$

$$= \sum_{\widetilde{Q}_{1:N}^{[\theta]}} \Pr(Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]})I(A_{i}^{[\theta]}; Z_{\mathcal{N}_{2}}|W_{\mathcal{K}}, Z_{\mathcal{N}_{1}}, A_{\mathcal{N}_{2}}^{[\theta]}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}).$$
(69)

Notice that the mutual information terms in (69) are nonnegative. Consequently, they have to be zero for all realizations of queries $\tilde{Q}_{1:N}^{[\theta]}$ with positive probability, i.e., (62) holds.

The proof of this lemma is completed.

Define coefficients $\tilde{\alpha}_{\ell}$ and a function $D(\ell)$ as follows:

$$\widetilde{\alpha}_{\ell} \triangleq \frac{1}{\binom{N}{\ell}} \sum_{\substack{\mathcal{S} \subseteq [1:N] \\ |\mathcal{S}| = \ell}} \alpha_{\mathcal{S}}, \quad \forall \ell \in [0:N],$$
(70)

$$\widetilde{D}(\ell) \triangleq 1 + \frac{1}{\ell} + \ldots + \frac{1}{\ell^{K-1}}, \quad \forall \ell \in [1:N+1]$$
(71)

with the boundary conditions

$$\widetilde{\alpha}_{N+1} \triangleq 0, \quad \widetilde{D}(0) \triangleq NK.$$

We next apply $\tilde{\alpha}_{\ell}$ to write the constraint of file size in (19) as

$$\sum_{\ell=1}^{N} \binom{N}{\ell} \widetilde{\alpha}_{\ell} = 1.$$
(72)

Lemma 11. The download cost D for any SC-PIR scheme has the following lower bound:

$$D \ge L \cdot \sum_{\ell=1}^{N} {N \choose \ell} \widetilde{D}(\ell) \widetilde{\alpha}_{\ell}.$$
(73)

Moreover, given any $S \subseteq [1:N]$ and $\theta, \theta' \in [1:K]$ such that $|S| = M \in [2:N]$ and $\theta \neq \theta'$, if the equality in (73) holds, then for every realization of queries $\tilde{Q}_{1:N}^{[\theta]}$ with positive probability, any SC-PIR scheme must satisfy

P3'. The answers at servers in S are independent of each other in the conditioning of $W_{[1:K]\setminus\{\theta\}}$ and $Z_{[1:N]\setminus S}$, i.e.,

$$\sum_{i \in \mathcal{S}} H(A_i^{[\theta]} | W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = H(A_{\mathcal{S}}^{[\theta]} | W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}).$$

P4'. The answers at servers in S are independent of each other in the conditioning of $W_{\theta'}$ and $Z_{[1:N]\setminus S}$, i.e.,

$$\sum_{i \in \mathcal{S}} H(A_i^{[\theta]} | W_{\theta'}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = H(A_{\mathcal{S}}^{[\theta]} | W_{\theta'}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]})$$

P5'. The answer at server *i* is independent of the contents stored at server *j* in the conditioning of $W_{\theta}, W_{\theta'}, Z_{[1:N]\setminus S}$ and $A_j^{[\theta]}$ for any $i, j \in S$, i.e.,

$$I(A_i^{[\theta]}; Z_j | W_{\theta}, W_{\theta'}, Z_{[1:N] \setminus \mathcal{S}}, A_j^{[\theta]}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = 0, \quad \forall i, j \in \mathcal{S}.$$

Proof: We have the following two boundary conditions on T(n,k) in (57) and λ_n in (59):

$$T(0,0) = \frac{1}{NK} \sum_{\theta \in [1:K]} \sum_{i \in [1:N]} H(A_i^{[\theta]} | Q_i^{[\theta]}) \stackrel{(a)}{=} \frac{1}{N} \sum_{i \in [1:N]} H(A_i^{[\theta]} | Q_i^{[\theta]}),$$
(74)

where (a) follows from (56) by setting $\mathcal{N} = \emptyset$ and $\mathcal{K} = \emptyset$, and

$$\lambda_0 = \frac{1}{K} \sum_{\theta \in [1:K]} H(W_\theta) \stackrel{(a)}{=} L,\tag{75}$$

where (a) is due to (1). Therefore,

$$\begin{split} D &\stackrel{(a)}{=} \sum_{i=1}^{N} H(A_{i}^{[\theta]}) \\ &\geq \sum_{i=1}^{N} H(A_{i}^{[\theta]}) Q_{i}^{[\theta]}) \\ &\stackrel{(b)}{=} N \cdot T(0,0) \\ &\stackrel{(c)}{\geq} \lambda_{0} + \sum_{n_{1}=0}^{N-1} T(n_{1},1) \\ &\stackrel{(d)}{=} \lambda_{0} + \sum_{n_{1}=0}^{N-1} \frac{\lambda_{n_{1}}}{N-n_{1}} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \frac{\lambda_{n_{2}}}{(N-n_{1})(N-n_{2})} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \sum_{n_{3}=n_{2}}^{N-1} \frac{\lambda_{n_{3}}}{(N-n_{1})(N-n_{2})(N-n_{3})} \\ &\quad + \dots + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \dots \sum_{n_{K-1}=n_{K-2}}^{N-1} \frac{\lambda_{n_{K-1}} + \sum_{n_{K-1}=n_{K-2}}^{N-1} \frac{\lambda_{n_{K-1}} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \frac{\lambda_{n_{K}}}{(N-n_{1})(N-n_{2}) \times \dots \times (N-n_{K-1})} \\ &\stackrel{(e)}{=} L + \sum_{n_{1}=0}^{N-1} \frac{\lambda_{n_{1}}}{N-n_{1}} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \frac{\lambda_{n_{2}}}{(N-n_{1})(N-n_{2})} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \sum_{n_{3}=n_{2}}^{N-1} \frac{\lambda_{n_{3}}}{(N-n_{1})(N-n_{2})(N-n_{3})} \\ &\quad + \dots + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \dots \sum_{n_{K-1}=n_{K-2}}^{N-1} \frac{\lambda_{n_{K-1}}}{(N-n_{1})(N-n_{2})} + \sum_{n_{1}=0}^{N-1} \sum_{n_{2}=n_{1}}^{N-1} \sum_{n_{3}=n_{2}}^{N-1} \frac{\lambda_{n_{3}}}{(N-n_{1})(N-n_{2})(N-n_{3})} \\ &\quad + \dots + \sum_{n_{1}=0}^{N} \sum_{n_{2}=n_{1}}^{N-1} \dots \sum_{n_{K-1}=n_{K-2}}^{N-1} \frac{\lambda_{n_{K-1}}}{(N-n_{1})(N-n_{2}) \times \dots \times (N-n_{K-1})} \\ \\ &\stackrel{(f)}{=} L + \sum_{n_{1}=1}^{N} \frac{\lambda_{N-n_{1}}}{n_{1}} + \sum_{n_{1}=1}^{N} \sum_{n_{2}=n_{1}}^{N} \frac{\lambda_{N-n_{2}}}{n_{1}n_{2}}} + \sum_{n_{1}=1}^{N} \sum_{n_{2}=n_{1}}^{N} \frac{\lambda_{N-n_{3}}}{n_{3}n_{3}} + \dots + \sum_{n_{1}=1}^{N} \dots \sum_{n_{K-1}=n_{K-2}}^{N} \frac{\lambda_{N-n_{1}}}{n_{1} \times \dots \times n_{K-1}} \\ &\stackrel{(g)}{=} L + \sum_{n_{1}=1}^{N} \left(\underbrace{\frac{1}{n_{1}} + \sum_{n_{2}=n_{1}}^{N} \frac{\lambda_{N-n_{1}}}{n_{1}n_{2}}} + \sum_{n_{1}=1}^{N} \sum_{n_{2}=n_{1}}^{N} \frac{\lambda_{N-n_{1}}}{n_{3}n_{2}n_{2}} + \sum_{n_{1}=n_{2}=n_{1}}^{N} \frac{\lambda_{N-n_{1}}}{n_{3}n_{2}n_{2}} + \sum_{n_{1}=$$

$$\stackrel{(h)}{=} L + \sum_{n_1=1}^{N} \sum_{\ell=1}^{n_1} \binom{n_1}{\ell} S(n_1, K) \widetilde{\alpha}_{\ell} L$$
$$= L \cdot \left(1 + \sum_{\ell=1}^{N} \widetilde{\alpha}_{\ell} \underbrace{\sum_{n_1=\ell}^{N} \binom{n_1}{\ell} S(n_1, K)}_{\triangleq \alpha(\ell, K)} \right)$$

$$\stackrel{(j)}{=} L \cdot \sum_{\ell=1}^{N} \binom{N}{\ell} \widetilde{D}(\ell) \widetilde{\alpha}_{\ell},$$

where (a) follows from (13); (b) follows from (74); (c) and (d) follow by applying (60) K times recursively; (e) follows from (75) and (58); (f) and (g) follow by changing of the summation indices simply; (h) holds because $\lambda_n = \sum_{\ell=1}^{N-n} {N-n \choose \ell} \tilde{\alpha}_{\ell} L$ by the result in [1, Eq. (54)]; (i) follows from the definition of $\tilde{D}(\ell)$ in (71) and $\alpha(\ell, K) = {N \choose \ell} (\tilde{D}(\ell) - 1)$ by the result in [1, Eq. (60)]; (j) follows from (72).

In order to establish the equality in (73), it is easy to see that, in the steps of applying (60), all the inequalities must hold with equalities. This implies that the equality in (60) holds for all $n \in [0 : N - 1]$ and $k \in [1 : K - 1]$. Thus, by Lemma 10, (61) and (62) hold for any $n \in [0 : N - 1]$ and $k \in [1 : K - 1]$.

Accordingly, for any $\theta, \theta' \in [1:K]$ and $S \subseteq [1:N]$ such that $\theta \neq \theta'$ and |S| = N - n = M, we have

• Let $\mathcal{K} = [1:K] \setminus \{\theta\}$ and $\mathcal{N} = [1:N] \setminus S$, then \mathcal{K} is of size k = K - 1 and \mathcal{N} is of size $n \in [0:N-1]$ (i.e., $M \in [1:N]$). Thus, by (61),

$$\sum_{i \in \mathcal{S}} H(A_i^{[\theta]} | W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = H(A_{\mathcal{S}}^{[\theta]} | W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}).$$

• Let $\mathcal{K} = \{\theta'\}$ and $\mathcal{N} = [1:N] \setminus S$, then \mathcal{K} is of size k = 1 and \mathcal{N} is of size $n \in [0:N-1]$. Thus, by (61) again,

$$\sum_{\boldsymbol{\theta}\in\mathcal{S}} H(A_i^{[\boldsymbol{\theta}]}|W_{\boldsymbol{\theta}'}, Z_{[1:N]\setminus\mathcal{S}}, Q_{1:N}^{[\boldsymbol{\theta}]} = \widetilde{Q}_{1:N}^{[\boldsymbol{\theta}]}) = H(A_{\mathcal{S}}^{[\boldsymbol{\theta}]}|W_{\boldsymbol{\theta}'}, Z_{[1:N]\setminus\mathcal{S}}, Q_{1:N}^{[\boldsymbol{\theta}]} = \widetilde{Q}_{1:N}^{[\boldsymbol{\theta}]}).$$

For n ∈ [0 : N − 2] (i.e., M ∈ [2 : N]), let k = 1 and n' = n + 1, then K = {θ, θ'} is of size k + 1, N₁ = [1 : N]\S is of size n, N₂ = {j} is of size n' − n = 1 for j ∈ S. Thus given any i ∈ S\{j}, by (62),

$$I(A_i^{[\theta]}; Z_j | W_{\theta}, W_{\theta'}, Z_{[1:N] \setminus \mathcal{S}}, A_j^{[\theta]}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}) = 0.$$

To conclude, P3'-P5' must hold if the equality in (73) holds.

C. Proof of Lemma 1

For fixed $j \in [0:N]$, the total storage of the N servers (20) is constrained as

$$\mu N \ge \sum_{n \in [1:N]} \sum_{\substack{S \subseteq [1:N] \\ n \in S}} \alpha_S$$

$$= \sum_{\ell=1}^N \ell \binom{N}{\ell} \widetilde{\alpha}_\ell$$

$$= j + \sum_{\ell \in [1:N] \setminus \{j,j+1\}} \binom{N}{\ell} (\ell - j) \widetilde{\alpha}_\ell + \binom{N}{j+1} \widetilde{\alpha}_{j+1},$$
(77)

where the last two equalities follow from (70) and (72), respectively. Hence,

$$\frac{D}{L} \stackrel{(a)}{\geq} \sum_{\ell=1}^{N} \binom{N}{\ell} \widetilde{D}(\ell) \widetilde{\alpha}_{\ell}$$

$$\stackrel{(b)}{=} \widetilde{D}(j) + \sum_{\ell \in [1:N] \setminus \{j,j+1\}} \binom{N}{\ell} (\widetilde{D}(\ell) - \widetilde{D}(j)) \widetilde{\alpha}_{\ell} + \binom{N}{j+1} (\widetilde{D}(j+1) - \widetilde{D}(j)) \widetilde{\alpha}_{j+1}$$

$$\stackrel{(c)}{\geq} \widetilde{D}(j) + \sum_{\ell \in [1:N] \setminus \{j,j+1\}} \binom{N}{\ell} (\widetilde{D}(\ell) - \widetilde{D}(j)) \widetilde{\alpha}_{\ell} + (\widetilde{D}(j+1) - \widetilde{D}(j)) (\mu N - j - \sum_{\ell \in [1:N] \setminus \{j,j+1\}} \binom{N}{\ell} (\ell - j) \widetilde{\alpha}_{\ell}) (79)$$

$$= (\mu N - j) \widetilde{D}(j+1) - (\mu N - j - 1) \widetilde{D}(j) + \sum_{\ell \in [1:N] \setminus \{j,j+1\}} \binom{N}{\ell} (\widetilde{D}(\ell) + (\ell - j - 1) \widetilde{D}(j) - (\ell - j) \widetilde{D}(j + 1)) \widetilde{\alpha}_{\ell}$$

$$\stackrel{(d)}{\geq} (\mu N - j) \widetilde{D}(j+1) - (\mu N - j - 1) \widetilde{D}(j)$$
(80)

$$= (M-j)\widetilde{D}(j+1) - (M-j-1)\widetilde{D}(j),$$
(81)

where (a) follows by (73); (b) is due to (72); (c) follows from (77) and the fact that $\widetilde{D}(j+1) - \widetilde{D}(j)$ is negative for all $j \in [0:N]$; (d) is because $\widetilde{\alpha}_{\ell} \geq 0$ and $\widetilde{D}(\ell) + (\ell - j - 1)\widetilde{D}(j) - (\ell - j)\widetilde{D}(j + 1) \geq 0$ for all $\ell \in [1:N] \setminus \{j, j+1\}$ by [1, Lemma 5].

Therefore, by (13) and (81), we have

$$R \le \left((M-j)\widetilde{D}(j+1) - (M-j-1)\widetilde{D}(j) \right)^{-1}, \quad \forall j \in [0:N].$$
(82)

In particular, for j = M and j = M - 1, (82) results in

$$R \le \left(\widetilde{D}(M)\right)^{-1} = \left(1 + \frac{1}{M} + \ldots + \frac{1}{M^{K-1}}\right)^{-1},\tag{83}$$

which achieves the capacity of the SC-PIR system (see (14)).

Thus, for any capacity-achieving SC-PIR scheme, the inequality in (83) must hold with equality, which implies that the inequalities in (78), (79) and (80) hold with equality for both j = M and j = M - 1. Therefore,

As a result, P1-P2 must be satisfied in the storage design phase of any capacity-achieving SC-PIR scheme. In addition, the equality in (78) indicates $D = L \cdot \sum_{\ell=1}^{N} {N \choose \ell} \widetilde{D}(\ell) \widetilde{\alpha}_{\ell}$, thus we obtain the following corollary by Lemma 11.

Corollary 4. Any capacity-achieving SC-PIR scheme must satisfy P3'-P5'.

D. Proof of Lemma 2

By Corollary 4, P3'-P5' hold for any capacity-achieving SC-PIR scheme. Next, we prove Lemma 2 by specializing P3'-P5' to linear SC-PIR schemes.

According to Lemma 1, in the capacity-achieving SC-PIR scheme, all the servers only store the packets placing in M different servers, thus the stored contents (18) at server n are simplified to

$$Z_n = \bigcup_{\substack{k \in [1:K] \\ |\mathcal{S}| = M, n \in \mathcal{S}}} \bigcup_{\substack{K \in \mathcal{S} \\ |\mathcal{S}| = M, n \in \mathcal{S}}} W_{k,\mathcal{S}}, \quad \forall n \in [1:N].$$
(84)

For any realization of queries $\widetilde{Q}_{1:N}^{[\theta]}$ with positive probability, the answer $A_n^{[\theta]}$ is a deterministic function of the corresponding query realization $\widetilde{Q}_n^{[\theta]}$ and the stored contents Z_n by (9). Therefore, by the stored contents in (84), $A_n^{[\theta]}$ is merely the function of $W_{\theta,S}$ and $\widetilde{Q}_n^{[\theta]}$ conditioned on the files $W_{[1:K]\setminus\{\theta\}}$ and $Z_{[1:N]\setminus S}$. For linear SC-PIR scheme, it is exactly $\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{\theta,S})$ conditioned on $W_{[1:K]\setminus\{\theta\}}$ and $Z_{[1:N]\setminus S}$. Therefore,

$$\sum_{n \in \mathcal{S}} H\left(A_n^{[\theta]} \mid W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}\right)$$

$$= \sum_{n \in \mathcal{S}} H\left(\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{\theta,\mathcal{S}}) \mid W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}\right)$$

$$\stackrel{(a)}{=} \sum_{n \in \mathcal{S}} H\left(\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{\theta,\mathcal{S}})\right), \tag{85}$$

where (a) follows from independence of all the packets (5), and the fact that queries are independent of files by (8). Similarly,

$$H\left(A_{\mathcal{S}}^{[\theta]} \mid W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}\right)$$
$$= H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,\mathcal{S}}) : n \in \mathcal{S} \mid W_{[1:K] \setminus \{\theta\}}, Z_{[1:N] \setminus \mathcal{S}}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}\right)$$

$$= H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,\mathcal{S}}): n \in \mathcal{S}\right).$$
(86)

Substituting the two sides in P3' for (85) and (86), we obtain

$$\sum_{n\in\mathcal{S}} H\Big(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,\mathcal{S}})\Big) = H\Big(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,\mathcal{S}}): n\in\mathcal{S}\Big).$$

Hence the random variables $\{\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{\theta,S}): n \in S\}$ are independent of each other, i.e., P3 holds. By the similar argument to (85), P4' can be rewritten as

$$\sum_{n\in\mathcal{S}} H\Big(\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{[1:K]\setminus\{\theta'\},\mathcal{S}})\Big) = H\Big(\widetilde{\mathbf{LC}}_n^{[\theta]}(W_{[1:K]\setminus\{\theta'\},\mathcal{S}}): n\in\mathcal{S}\Big).$$

Therefore,

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]} (W_{[1:K] \setminus \{\theta'\}, \mathcal{S}}), \quad \forall n \in \mathcal{S}$$

are also independent of each other, i.e., P4 holds.

Let $n, n' \in S$ and $\theta' \neq \theta$. By P5',

$$0 = I\left(A_{n}^{[\theta]}; Z_{n'} \mid W_{\theta}, W_{\theta'}, Z_{[1:N]\setminus\mathcal{S}}, A_{n'}^{[\theta]}, Q_{1:N}^{[\theta]} = \widetilde{Q}_{1:N}^{[\theta]}\right)$$

$$\stackrel{(a)}{=} I\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}); W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}} \mid \widetilde{\mathbf{LC}}_{n'}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right)$$

$$= H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}) \mid \widetilde{\mathbf{LC}}_{n'}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right)$$

$$-H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}} \mid W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}, \widetilde{\mathbf{LC}}_{n'}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right)$$

$$\stackrel{(b)}{=} H\left(\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}) \mid \widetilde{\mathbf{LC}}_{n'}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})\right),$$

where (a) follows from the similar argument to (85) again; (b) is due to the fact that $\widetilde{\mathbf{LC}}_{n}^{[\theta]}(W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}})$ is a function of packets in $W_{[1:K]\setminus\{\theta,\theta'\},\mathcal{S}}$ by (24) and (25). Thus,

$$\widetilde{\mathbf{LC}}_{n}^{[\theta]} \big(W_{[1:K] \setminus \{\theta, \theta'\}, \mathcal{S}} \big), \quad \forall \, n \in \mathcal{S}$$

are deterministic of each other, i.e., P5 holds.

As a result, P3-P5 are necessary conditions of any capacity-achieving linear SC-PIR scheme.

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