A Class of Optimal Structures for Node Computations in Message Passing Algorithms

Xuan He, Kui Cai, and Liang Zhou

Abstract—Consider the computations at a node in a message passing algorithm. Assume that the node has incoming and outgoing messages $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, respectively. In this paper, we investigate a class of structures that can be adopted by the node for computing y from x, where each $y_j, j = 1, 2, \dots, n$ is computed via a binary tree with leaves x excluding x_i . We make three main contributions regarding this class of structures. First, we prove that the minimum complexity of such a structure is 3n-6, and if a structure has such complexity, its minimum latency is $\delta + \lceil \log(n-2^{\delta}) \rceil$ with $\delta = \lfloor \log(n/2) \rfloor$, where the logarithm always takes base two. Second, we prove that the minimum latency of such a structure is $\lceil \log(n-1) \rceil$, and if a structure has such latency, its minimum complexity is $n\log(n-1)$ when n-1 is a power of two. Third, given (n,τ) with $\tau \geq \lceil \log(n-1) \rceil$, we propose a construction for a structure which we conjecture to have the minimum complexity among structures with latencies at most τ . Our construction method runs in $O(n^3 \log^2(n))$ time, and the obtained structure has complexity at most (generally much smaller than) $n\lceil \log(n) \rceil - 2$.

Index Terms—Binary structure, Complexity, latency, low-density parity-check (LDPC) code, message passing algorithm.

I. INTRODUCTION

Message passing algorithms are widely applied for the decoding of error correction codes such as the low-density parity-check (LDPC) codes [1]-[6]. The algorithms can be considered as working on a graph, in which messages are passing along edges, and each node receives incoming messages from its connecting edges and then computes outgoing messages that will be passed back along the connecting edges. More specifically, consider a node, such as a check/variable node of LDPC codes, which has n connecting edges. (We assume $n \geq 3$ throughout this paper and specify cases for n < 3 separately.) The incoming messages are denoted by $\mathbf{x} = (x_1, x_2, \dots, x_n), \text{ where for } j \in [n] = \{1, 2, \dots, n\}, x_j$ comes from the j-th connecting edge. This node then computes n outgoing messages, denoted by $\mathbf{y} = (y_1, y_2, \dots, y_n)$, where for $j \in [n]$, y_j will be passed back along the j-th connecting edge. The corresponding node computations are to compute each $y_i, j \in [n]$ from x excluding x_i . We remark that the messages need not to be real numbers.

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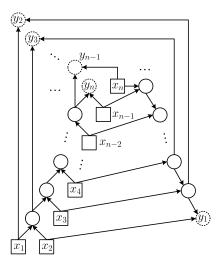


Fig. 1. A structure for realizing the forward-backward computation of y [7], where squares, circles, and dotted circles represent input, internal, and output nodes, respectively.

In this paper, we consider a class of structures, in which each $y_j, j \in [n]$ is computed by using a binary tree with leaves x excluding x_j . For example, assume

$$y_j = \min_{i \in [n] \setminus \{j\}} x_i, \forall j \in [n], \tag{1}$$

which is used in the computation at the check node in the minsum decoding of LDPC codes [3] (messages considered here are real numbers). Fig. 1 shows a classical structure [7] for the computation of (1). This structure realizes the computation of a given y_j based on a binary tree whose leaves correspond to $\{x_i : i \in [n] \setminus \{j\}\}$ and whose internal nodes correspond to the two-input min operations. Taking n = 6 as an example, the six binary trees resulted from Fig. 1 are shown in Fig 2.

The structure in Fig. 1 actually carries out the forward-backward computation [7]. Taking the computation of (1) as an example, the forward and backward computations are given by

$$f_j = \min_{i=1,\dots,j} x_i = \begin{cases} x_1, & j=1, \\ \min\{f_{j-1}, x_j\}, & 1 < j < n, \end{cases}$$
 and

$$b_j = \min_{i=j,\dots,n} x_i = \begin{cases} x_n, & j=n, \\ \min\{b_{j+1}, x_j\}, & 1 < j < n, \end{cases}$$

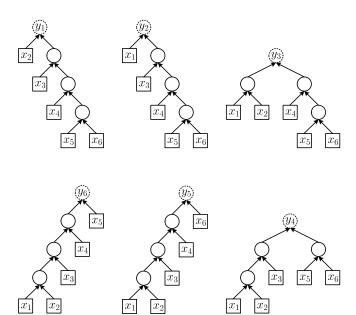


Fig. 2. Examples of directed binary trees (DBTs) used for computing y_j , $\forall j \in [n] = [6]$, where squares, circles, and dotted circles represent leaves, internal nodes, and roots, respectively.

respectively. Then, we have

$$y_j = \min_{i \in [n] \setminus \{j\}} x_i = \begin{cases} b_{j+1}, & j = 1, \\ \min\{f_{j-1}, b_{j+1}\}, & 1 < j < n, \\ f_{j-1}, & j = n. \end{cases}$$

The complexity of this structure is defined as the number of internal nodes (min operations) which is given by 3n-6: each of $\{f_2,\ldots,f_{n-1},b_2,\ldots,b_{n-1},y_2,\ldots,y_{n-1},\}$ takes one min operation. The latency of the structure is defined as the longest distance between any pair of $(x_i,y_j), i\neq j$, which is given by n-2 (e.g., from x_1 to y_n).

It is natural to ask what are the minimum complexity and minimum latency of such a class of structures? Accordingly, this paper derives the following results.

- We prove that the minimum complexity of such a structure is 3n-6. If a structure has such complexity (i.e., complexity-optimal), its minimum latency is $\delta + \lceil \log(n-2^{\delta}) \rceil$ with $\delta = \lfloor \log(n/2) \rfloor$, where the logarithm always takes base two in this paper. We also propose a simple construction for complexity-optimal structures which have such latency.
- We prove that the minimum latency of such a structure is $\lceil \log(n-1) \rceil$. If a structure has such latency (i.e., latency-optimal), its minimum complexity is $n \log(n-1)$ for $n=2^k+1$ with k>0, and we propose a simple construction for this case.
- Given (n,τ) with $\tau \geq \lceil \log(n-1) \rceil$, we propose a construction for a structure $S_{n,\tau}$ which we conjecture to have the minimum complexity among structures with latencies at most τ . Our construction method runs in $O(n^3 \log^2(n))$ time, and the obtained $S_{n,\tau}$ has complexity at most (generally much smaller than) $n\lceil \log(n) \rceil 2$.

The complexities of $S_{n,\tau}$, denoted by $\phi(n,\tau)$, are derived in Section VI, and some typical values of $\phi(n,\tau)$ are presented

in Table I of Section VI. It is worth mentioning that structures that are both complexity-optimal and latency-optimal only exist for n=2,3,4,6. Assume that the min-sum algorithm [3] (or its variants) is applied to decode the 802.11n LDPC code [8] which has check node degrees of 7 and 8. For each degree-7 check node (n=7), using the structure of Fig. 1 to implement (1) leads to complexity 15 and latency 5. On the contrary, Table I shows that there exist a structure of complexity 15 and latency 4, and also a structure of complexity 18 and latency 3. Moreover, for each degree-8 check node (n=8), using the structure of Fig. 1 for implementing (1) results in complexity 18 and latency 6. Table I, however, shows that there exist a structure of complexity 18 and latency 4, and a structure of complexity 22 and latency 3.

We remark that there exist some other structures [9], [10] which are specially designed for the computation of (1). They do not belong to the class of structures considered in this paper. To make a fair comparison in terms of complexity and latency, more factors need to be taken into consideration: the comparators with different bit widths, the multiplexers, the latency of comparators and multiplexers, and so on, which are out of the scope of the current paper. We thus only make two more remarks. First, the structures proposed in [9], [10] can never achieve the minimum latency $\lceil \log(n-1) \rceil$. Second, they are only suitable for the min operations, while the class of structures considered in this paper is always applicable for the node computation no matter what binary operations are involved. For example, the considered class of structures is perfectly suitable for the mutual information-maximizing lookup table (MIM-LUT) decoding [11]–[21], which recently attracts much attention as it takes two-input table lookup operations to eliminate arithmetic operations.

The remainder of this paper is organized as follows. Section II introduces preliminaries regarding graphs and trees. Section III defines the structures considered in this paper for node computation. Sections IV and V investigate complexity-optimal and latency-optimal structures, respectively. Section VI considers the construction of the aforementioned structure $S_{n,\tau}$. Finally, Section VII concludes this paper.

II. PRELIMINARIES

In this section, we introduce preliminaries regarding graphs and trees, mainly based on their definitions in [22, Appendix B].

A directed (resp. undirected) graph G is a pair (G_v, G_e) , where G_v and G_e are the node/vertex set and edge set, respectively, and any element in G_e is called a directed (resp. undirected) edge which is denoted by an ordered (resp. unordered) pair $(a,b) \in G_v \times G_v$. The term "ordered" (resp. "unordered") implies that $(a,b) \neq (b,a)$ (resp. (a,b) = (b,a)). (Note that self-loops are forbidden in this paper, i.e., we have $a \neq b, \forall (a,b) \in G_e$.) When drawing a graph, we use arrows and lines to represent directed and undirected edges, respectively. For convenience, we also consider that $G = G_v \cup G_e$, and accordingly, we also write $a \in G_v$ as $a \in G$ and $(a,b) \in G_e$ as $(a,b) \in G$. A graph G' is called a subgraph of G if $G' \subseteq G$.

In a directed graph G, we say that $(a,b) \in G$ leaves a and enters b; accordingly, (a,b) is a leaving/outgoing edge of a and an entering/incoming edge of b. Instead, in an undirected graph G, we simply say that $(a,b) \in G$ connects a and b; accordingly (a,b) is an edge of a and b. We use the subtraction/addition (i.e., -/+) to describe the operation of removing/adding a node a or an edge (a,b) from/into a graph $G = (G_v, G_e)$, where (a,b) is a directed edge if and only if (iff) G is a directed graph. Specifically, $G - a = (G_v \setminus \{a\}, G_e \setminus \{(a_1, a_2) \in G_e : a_1 = a \text{ or } a_2 = a\})$, $G + a = (G_v \cup \{a\}, G_e)$, $G - (a,b) = (G_v, G_e \setminus \{(a,b)\})$, and $G + (a,b) = (G_v \cup \{a,b\}, G_e \cup \{(a,b)\})$.

A path P of length k from $a \in G$ to $b \in G$ is a node sequence $P = (v_0, v_1, \dots, v_k)$ such that $v_0 = a, v_k = b$, and $(v_{i-1}, v_i) \in G, \forall i \in [k]$. The distance from a to b is the length of the shortest path from a to b (the distance is defined as ∞ if there is no such a path). P is a simple path if $v_i \neq v_j, \forall 0 \leq i < j \leq k$. Moreover, P forms a (simple) cycle if $k \geq 2$, $v_0 = v_k$, $(v_0, v_1) \neq (v_1, v_2)$, and (v_1, v_2, \dots, v_k) is a simple path. A graph with no cycle is acyclic. We refer to a directed acyclic graph by a DAG. If there is a path P from a to b, we say that b is reachable from a (via P), denoted by $a \leadsto b$. For any directed graph G and $a \in G$, we say that $E(a,G) = (\{b \in G : b \leadsto a\}, \{(b',b'') \in G : b'' \leadsto a\})$ is the subgraph entering a in G, and $L(a,G) = (\{b \in G : a \leadsto a)\}$ $b\}, \{(b', b'') \in G : a \leadsto b'\})$ is the subgraph leaving a in G. An undirected graph is connected if every node is reachable from all other nodes.

A tree is a connected, acyclic, undirected graph. For any tree T and any node $a \in T$, a is called an internal node (resp. external node or leaf) if a has more than one (resp. only one) edge. There is a unique simple path between any pair of nodes in T. The diameter of T, denoted by d(T), is the length of the longest simple path in T.

A rooted tree is a tree in which there is a unique node called the root of the tree. Consider a rooted tree T, and denote its root by r(T). The distance between any node $a \in T$ and r(T) is called the depth of a in T. A level of T consists of all nodes at the same depth. The height of T is equal to the largest depth of any node in T. For any edge $(a,b) \in T$, assuming that a has a larger depth (which is equal to one plus the depth of b), then, b is called the parent of a, and a is called a child of b. The directed version of T, say T', is to change each undirected edge, say $(a,b) \in T$ with a being a child of b, into the directed edge $(a,b) \in T'$. T' is called a directed rooted tree (DRT), and we say that T is the undirected version of T'. For any $a \in T'$, E(a,T') is the subtree of T' rooted at a; accordingly, the undirected version of E(a,T') is the subtree of T rooted at a.

A (full) binary tree T is a rooted tree in which each node has either zero or two children (left child and right child). Assume that the height of an arbitrary binary tree T is h_T . T is called a complete binary tree iff for $0 \le i < h_T$, the i-th level of T contains 2^i nodes, and nodes in the h_T -th level of T are as far left as possible. Moreover, T is called a perfect binary tree iff for $0 \le i \le h_T$, the i-th level of T contains 2^i nodes. The subtree rooted at the left (resp. right) child of T is called the left (resp. right) subtree of T. Similar to DRTs,

we have directed binary trees (DBTs). Meanwhile, we refer to the directed version of a complete (resp. perfect) binary tree as a complete (resp. perfect) DBT.

For any graph $G=(G_v,G_e)$, G is labelled iff every node in G is given a unique label, such as $1,2,\ldots,|G_v|$ (as a result, each edge is also given a unique label). Otherwise, G is partially unlabelled (even if no node is labelled). Two labelled graphs G and G' are the same, i.e., G=G', iff G and G' have same labelled nodes and edges (and root for rooted trees). Two partially unlabelled graphs G and G' are the same iff there exists a way to label all unlabelled nodes in G and G' such that G and G' become labelled and the same.

III. STRUCTURES FOR NODE COMPUTATION

Recall that $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ denote the incoming and outgoing messages, respectively. In this paper, we consider the case where for $j\in [n]$, a DBT T_j is used to describe the computation of y_j from \mathbf{x} excluding x_j . More specifically, in T_j , leaves correspond to incoming messages \mathbf{x} excluding x_j , internal nodes correspond to binary operations, and the root $r(T_j)$ corresponds to y_j . Some examples of such DBTs for n=6 are shown in Fig. 2.

Define an input node set $X = \{x_j : j \in [n]\}$ and an output node set $Y = \{y_j : j \in [n]\}$, where node x_j (resp. y_j) is called the j-th input (resp. output) node which corresponds to the j-th incoming message x_j (resp. outgoing message y_j). In this paper, we remark that for any graph G and any node $a \in G$, a is labelled in G iff a is an input node from X or an output node from Y. As a result, G is generally partially unlabelled. We consider to use a structure, defined below, to describe a computation process.

Definition 1: A structure S considered in this paper is a DAG fulfilling the following three properties.

- For any $a \in S$, we have $a \in X$ iff a has no incoming edge in S.
- For any $a \in S$, E(a, S) is a DBT.
- For two different nodes $a, b \in S$, $E(a, S) \neq E(b, S)$ (the inequality corresponds to comparison between two partially unlabelled graphs).

For any $a \in S \cap X$, we also call a an input node of S, and we say that S has input size $|S \cap X|$ (the number of input nodes in S). Any other node in S is called a computation node, and it must have exactly two incoming edges in S. In particular, any computation node with no outgoing edge is also called an output node (may not belong to Y). For any $a \in S$, we call E(a,S) the subtree of S rooted at a. The third property in Definition 1 indicates that S does not have the same subtrees. For convenience, let $E(S) = \{E(a,S) : a \in S\}$ be the set of all subtrees of S. For any two structures S and S', denote the union of S and S' by $S \vee S'$, where only one copy of the same subtrees is kept such that $S \vee S'$ is still a structure. We have $E(S \vee S') = E(S) \cup E(S')$. For example, the six DBTs (structures) in Fig. 2 can be united (under \vee) into the structure shown in Fig. 1 with n=6.

Definition 2: A structure S used for computing y is a structure (see Definition 1) additionally fulfilling the following property.

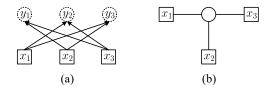


Fig. 3. (a) The only structure $S \in \mathcal{S}_3$ and (b) only T-tree $T \in \mathcal{T}_3$, and we have S = h(T).

• S contains n output nodes, which are exactly $Y = \{y_1, y_2, \ldots, y_n\}$, where for $j \in [n]$, $E(y_j, S)$ is a DBT with leaves $X \setminus \{x_j\}$.

We remark that any structure defined by Definition 2 can be used for computing \mathbf{y} , but that defined by Definition 1 may not. From Definition 2, we have $S = \bigvee_{j \in [n]} E(y_j, S)$, and S has input size n. Let \mathcal{S}_n be the set of all structures used for computing \mathbf{y} (and with input size n). Fig. 1 shows an instance in \mathcal{S}_n . Meanwhile, the only structure in \mathcal{S}_3 is shown by Fig. 3(a). For any $S \in \mathcal{S}_n$, it is easy to see that for any $j \in [n]$, each output node in $Y \setminus \{y_j\}$ is reachable from input node x_j via a unique path in S, but y_j is not reachable from x_j . Moreover, after removing any nodes and/or edges from S, we can no longer have $S \in \mathcal{S}_n$.

Definition 3: For any structure S, the complexity of S, denoted by c(S), is equal to the number of computation nodes in S. The latency of S, denoted by l(S), is equal to the length of the longest simple path in S.

As an example, the complexity and latency of the structure in Fig. 1 are 3n-6 and n-2, respectively. It is reasonable to use complexity and latency as two key criteria for evaluating the performance of a structure. In this paper, one of our main purpose is to discover complexity-optimal and/or latency-optimal structures in \mathcal{S}_n , as defined below.

Definition 4: Let $c_n^{\min} = \min_{S \in \mathcal{S}_n} c(S)$ and $l_n^{\min} = \min_{S \in \mathcal{S}_n} l(S)$. Moreover, let $\mathcal{S}_n^{\text{co}} = \{S \in \mathcal{S}_n : c(S) = c_n^{\min}\}$ and $\mathcal{S}_n^{\text{lo}} = \{S \in \mathcal{S}_n : c(S) = c_n^{\min}\}$. For any structure $S \in \mathcal{S}_n$, S is complexity-optimal (resp. latency-optimal) iff $S \in \mathcal{S}_n^{\text{co}}$ (resp. $S \in \mathcal{S}_n^{\text{lo}}$).

IV. COMPLEXITY-OPTIMAL STRUCTURES

In this section, we first investigate the properties of complexity-optimal structures, including deriving the value of c_n^{\min} . Then, we propose to use a class of trees, called T-trees, to equivalently describe complexity-optimal structures. T-trees make it easy to find the minimum latency of complexity-optimal structures, and also lead to a simple construction for complexity-optimal structures.

A. Properties of Complexity-Optimal Structures

For any directed graph G and any node $a \in G$, converting a into a directed edge (a_1,a_2) is to split a into two new nodes a_1 and a_2 in G such that they keep only the incoming and outgoing edges of a, respectively, and the directed edge (a_1,a_2) is also added into G (a no longer exists in G). An example is shown in Fig. 4, where G_c is the resulting graph after converting a and b in G_d into (a_1,a_2) and (b_1,b_2) ,

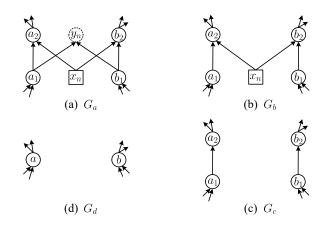


Fig. 4. An example for illustrating functions f and g. (For simplicity, only the subgraphs of interest, but not the whole graphs, are drawn.) $G_d = f(G_a)$: (I) $G_b = G_a - y_n$, (II) $G_c = G_b - x_n$, and (III) G_d is the resulting graph after converting (a_1,a_2) and (b_1,b_2) in G_c into a and b, respectively. (Steps (I)–(III) correspond to steps (A1)–(A3).) On the other hand, $G_a = g(a,b,G_d)$: (i) G_c is the resulting graph after converting a and b in G_d into (a_1,a_2) and (b_1,b_2) , respectively, (ii) $G_b = G_c + (x_n,a_2) + (x_n,b_2)$, and (iii) $G_a = G_b + (a_1,y_n) + (b_1,y_n)$. (Steps (i)–(iii) correspond to steps (B1)–(B3).)

respectively. Conversely, for $(a_1,a_2) \in G$, converting (a_1,a_2) into node a is to merge a_1 and a_2 into the new node a in G such that a keeps all edges of a_1 and a_2 except for the edge (a_1,a_2) (nodes a_1 and a_2 no longer exist in G). An example is also shown in Fig. 4, where G_d is the resulting graph after converting (a_1,a_2) and (b_1,b_2) in G_c into a and b, respectively.

For $n \geq 4$, define a function $f: \mathcal{S}_n \to \mathcal{S}_{n-1}$ which works with the following three steps for any $S \in \mathcal{S}_n$. (An example is shown in Fig. 4 to illustrate how f works.)

- (A1) For any output node $a \in S$ (i.e., a has no outgoing edge) such that $a \notin Y \setminus \{y_n\}$, remove a from S in a recursive manner. Denote the resulting graph by S'.
- (A2) Let $S'' = S' x_n$.
- (A3) For any $(a, b) \in S''$ such that (a, b) is the only incoming edge of b in S'', convert (a, b) into a new node. (Actually, b is a computation node in S with incoming edges (a, b) and (x_n, b) .) Denote the resulting graph by f(S).

Lemma 1: For $n \geq 4$ and any $S \in \mathcal{S}_n$, we have $f(S) \in \mathcal{S}_{n-1}$ and $c(S) \geq c(f(S)) + 3$.

Proof: Assume $n \geq 4$ and $S \in \mathcal{S}_n$. We can easily verify that $f(S) \in \mathcal{S}_{n-1}$. In step (A1) of f(S), at least the computation node y_n is removed from S. The number of additional computation nodes removed from S in steps (A2) and (A3) is equal to the number of edges of x_n in S. Therefore, to prove $c(S) \geq c(f(S)) + 3$, we only need to prove that x_n has at least two outgoing edges in S.

Since each output node in $Y \setminus \{y_n\}$ is reachable from x_n in S, x_n must have at least one outgoing edge, say $(x_n,a) \in S$. Note that a is a computation node in S, indicating that a is reachable from at least an input node $x_i, i \neq n$ in S. Since y_i is not reachable from x_i in S, then y_i must not be reachable from a. Therefore, x_n must have another outgoing edge such that y_i can be reachable from x_n in S. This completes the proof.

Theorem 1: We have $c_n^{\min} = \min_{S \in \mathcal{S}_n} c(S) = 3n - 6$. **Proof:** We have $c_n^{\min} = 3n - 6$ for n = 3, since \mathcal{S}_3 contains only one structure, as shown in Fig. 3(a). Then, according to Lemma 1, we have $c_n^{\min} \geq 3n - 6$ for $n \geq 4$. Further noting that the structure in Fig. 1 has complexity 3n-6, the theorem is proved.

According to the discussions on Lemma 1 and Theorem 1, we know that for $n \geq 4$ and any $S \in \mathcal{S}_n^{co}$, we have c(S) = c(f(S)) + 3. More specifically, only one computation node, i.e., y_n , is removed in step (A1) of f(S), x_n has exactly two outgoing edges in S, and we have $f(S) \in \mathcal{S}_{n-1}^{\text{co}}$. This motivates us to construct another function, which works like the inverse process of f, to convert a structure in S_{n-1}^{co} to a structure in $\mathcal{S}_n^{\text{co}}$.

For any $a, b \in S \in \mathcal{S}_n$, the unordered pair $\langle a, b \rangle$ is called a complement pair of S iff $x_j \in E(a,S) \iff x_j \notin$ $E(b,S), \forall j \in [n]$. For example, $\langle x_3, y_3 \rangle = \langle y_3, x_3 \rangle$ is a complement pair of S in Fig. 3(a). Let P(S) denote the set of all complement pairs of S. For $n \geq 4$, $S \in \mathcal{S}_{n-1}^{\text{co}}$, and $\langle a,b\rangle\in P(S)$, define g(a,b,S) as the graph obtained by the three steps described as follows. (An example is shown in Fig. 4 to illustrate how g works.)

- (B1) Convert a and b into directed edges (a_1, a_2) and (b_1, b_2) , respectively. Denote the resulting graph by S'.
- (B2) Let $S'' = S' + (x_n, a_2) + (x_n, b_2)$.
- (B3) Let $g(a, b, S) = S'' + (a_1, y_n) + (b_1, y_n)$.

Theorem 2: Structures in S_n^{co} fulfill the following properties.

- (C1) For any $S \in \mathcal{S}_n^{\text{co}}$, any non-output node in S has exactly two outgoing edges.
- (C2) For any $S \in \mathcal{S}_n^{co}$, $a \in S$, and $j \in [n]$, we have $x_j \in$ $E(a,S) \iff y_j \notin L(a,S).$
- (C3) For any $S \in \mathcal{S}_n^{\text{co}}$ and $a \in S$, there exists a unique $b \in S$ such that $\langle a, b \rangle \in P(S)$.
- (C4) For $n \geq 4$, $S_n^{\text{co}} = \{g(a,b,S) : S \in S_{n-1}^{\text{co}}, \langle a,b \rangle \in S_n^{\text{co}}\}$ P(S).
- (C5) $|S_n^{\text{co}}| = (2n-5)!! = (2n-5) \times (2n-7) \times \cdots \times 1.$ Proof: See Appendix A.

B. T-Trees

Definition 5: A T-tree T used for computing v is a (undirected) tree fulfilling the following two properties.

- T has n leaves, which are exactly X.
- Each internal node in T has exactly three edges.

The letter 'T' in "T-tree" actually comes from the second property above ('T' is short for "Triplet", and it also looks like an internal node with three edges). Denote \mathcal{T}_n as the set of all T-trees. In particular, the only T-tree in \mathcal{T}_3 is shown in Fig. 3(b). For any $T \in \mathcal{T}_n$, T has n-2 internal nodes and 2n-3 edges. For any $(a,b) \in T$, let $D(a,b,T) = E(a,T_b)$, where T_b is the directed version of the tree resulted by making T as a rooted tree with root b. Obviously, D(a, b, T) is a DBT with root a. Let $D(T) = \{D(a, b, T) : (a, b) \in T\}$. We have |D(T)| = 4n - 6, since $D(a_1, b_1, T) \neq D(a_2, b_2, T)$ for any $(a_1, b_1), (a_2, b_2) \in T$ with $a_1 \neq a_2$ or $b_1 \neq b_2$.

Theorem 3: For any $T \in \mathcal{T}_n$, let

$$h(T) = \bigvee_{i \in [n], (a, x_i) \in T} D(a, x_i, T).$$

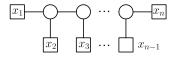


Fig. 5. The T-tree $T \in \mathcal{T}_n$ with $h(T) \in \mathcal{S}_n^{\text{co}}$ given by Fig. 1.

Then h is a bijection from \mathcal{T}_n to $\mathcal{S}_n^{\text{co}}$.

Proof: See Appendix B.

According to Theorem 3, it suffices to investigate \mathcal{T}_n when $\mathcal{S}_n^{\text{co}}$ is of interest. In particular, for any $T \in \mathcal{T}_n$, T is a much simpler graph than $h(T) \in \mathcal{S}_n^{\text{co}}$, with respect to that i) T is a simple tree as described in Definition 5 and ii) T contains 2n-2 nodes and 2n-3 edges while h(T) contains 4n-6nodes and 6n-12 edges. As an example, let S denote the structure in Fig. 1, and we have $S \in \mathcal{S}_n^{\text{co}}$. The T-tree T with h(T) = S is shown in Fig. 5. A simpler example for h is shown in Fig. 3.

Lemma 2: For any $T \in \mathcal{T}_n$, we have l(h(T)) = d(T) - 1, where d(T) is the diameter of T.

Proof: Note that d(T) must be equal to the distance between a certain pair of leaves in T. Without loss of generality, assume that d(T) is equal to the distance between x_i and x_j in T. As a result, $D(a, x_i, T)$ with $(a, x_i) \in T$ has the largest height among D(T). Therefore, l(h(T)) is equal to the height of $D(a, x_i, T)$, i.e., l(h(T)) = d(T) - 1.

Lemma 3: Let $\delta = \lfloor \log(n/2) \rfloor$. We have

$$d_n^{\min} = \min_{T \in \mathcal{T}_n} d(T) = \delta + \lceil \log(n - 2^{\delta}) \rceil + 1.$$

Proof: See Appendix C.

Theorem 4: Let $\delta = \lfloor \log(n/2) \rfloor$. We have

$$\min_{S \in \mathcal{S}_{\infty}^{\text{oo}}} l(S) = d_n^{\min} - 1 = \delta + \lceil \log(n - 2^{\delta}) \rceil.$$

Proof: This is the result by combining Theorem 3, Lemma 2, and Lemma 3.

The proof of Lemma 3 in Appendix C also leads to the following construction for complexity-optimal structures with latency $d_n^{\min} - 1$.

Construction 1: Let $T \in \mathcal{T}_n$ be a T-tree, in which there exists an edge $(a,b) \in T$ such that D(a,b,T) and D(b,a,T)are two complete DBTs with leaves $\{x_1, x_2, \dots, x_{2^{\delta}}\}$ and $\{x_{2^{\delta}+1}, x_{2^{\delta}+2}, \dots, x_n\}$, respectively, where $\delta = \lfloor \log(n/2) \rfloor$. Return h(T) as the constructed structure.

As an example, for n = 6, Construction 1 may lead to the T-tree $T \in \mathcal{T}_6$ in Fig. 6(a). We have $d(T) = d_6^{\min} = 4$. The constructed structure $h(T) \in \mathcal{S}_6^{\text{co}}$ is shown in Fig. 6(b), which has the optimal complexity $c(h(T)) = c_6^{\min} = 12$ and the minimum latency $l(h(T)) = d_6^{\min} - 1 = 3$ among $\mathcal{S}_6^{\text{co}}$. It is worth mentioning that h(T) in Fig. 6(b) was used in [20] and [21] to implement check node update for decoding regular LDPC codes with variable node degree 3 and check node degree 6.

V. LATENCY-OPTIMAL STRUCTURES

In this section, we first derive the value of l_n^{\min} . Then, for $n=2^k+1$ with k>0, we propose an optimal construction for an $S \in \mathcal{S}_n^{\text{lo}}$ such that $c(S) = \min_{S' \in \mathcal{S}_n^{\text{lo}}} c(S')$. (The

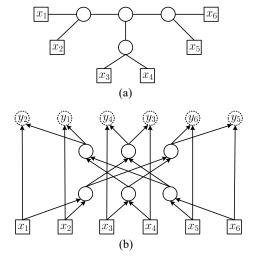


Fig. 6. An example of Construction 1 for n=6. (a) A resulted T-tree $T\in\mathcal{T}_6$. (b) The constructed structure $h(T)\in\mathcal{S}_6^{\text{co}}$ corresponding to T.

construction of latency-optimal structures with other values of n will be addressed later in Construction 4 of Section VI.)

Theorem 5: We have $l_n^{\min} = \min_{S \in \mathcal{S}_n} l(S) = \lceil \log(n-1) \rceil$. Proof: For any $S \in \mathcal{S}_n$ and $j \in [n]$, $E(y_j, S)$ is a DBT

Proof: For any $S \in \mathcal{S}_n$ and $j \in [n]$, $E(y_j, S)$ is a DBT with leaves $X \setminus \{x_j\}$. As a result, the minimum height of $E(y_j, S)$ is $\lceil \log(n-1) \rceil$ which is achievable when $E(y_j, S)$ is a complete DBT. Since $S = \bigvee_{j \in [n]} E(y_j, S)$, we have $l(S) \geq \lceil \log(n-1) \rceil$, where the equality holds when each $E(y_j, S)$ is a complete DBT. This completes the proof.

According to Theorems 1 and 5, the structure S in Fig. 6(b) is both complexity-optimal and latency-optimal, i.e., $S \in \mathcal{S}_6^{\text{co}} \cap \mathcal{S}_6^{\text{lo}}$. However, structures that are both complexity-optimal and latency-optimal rarely exist. In fact, according to Theorems 4 and 5, we can easily derive the following corollary.

Corollary 1: For $n \geq 3$, we have $S_n^{co} \cap S_n^{lo} \neq \emptyset$ iff n = 3, 4, 6.

We now propose a simple construction for latency-optimal structures when n-1 is a power of two.

Construction 2 (For $n=2^k+1$ with k>0): Let $S=(\{v_{0,j}: j\in [n]\},\emptyset)$ with $v_{0,j}=x_j$. For $i=1,2,\ldots,k$ and $j\in [n]$, create a new node $v_{i,j}\notin S$, and let $S=S+(v_{i-1,j},v_{i,j})+(v_{i-1,j+2^{i-1}},v_{i,j})$, where $v_{i-1,j+2^{i-1}}=v_{i-1,j+2^{i-1}-n}$ if $j+2^{i-1}>n$. Return S as the constructed structure.

Theorem 6: Assume $n = 2^k + 1$ with k > 0. S returned by Construction 2 belongs to S_n^{lo} , and we have

$$c(S) = n\log(n-1) = nk = \min_{S' \in \mathcal{S}_n^{\text{lo}}} c(S').$$

Proof: See Appendix D.

Note that Construction 2 is deterministic, i.e., the result of Construction 2 is unique for any $n=2^k+1$ with k>0. An example of Construction 2 for n=5 is shown in Fig. 7, which has latency 2 and complexity 10.

VI. TRADEOFF BETWEEN COMPLEXITY AND LATENCY

A general problem is to find the minimum complexity of structures in S_n that have latencies at most τ for any given (n, τ) . We give a solution to this problem in this section.

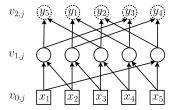


Fig. 7. An example of Construction 2 for n = 5.

For n=2, outgoing messages are given by $y_1=x_2$ and $y_2=x_1$. Accordingly, the graph that only consists of nodes $\{x_1,x_2\}$, say G, can be considered as a valid (and the only) structure used for computing \mathbf{y} for n=2. Moreover, G is both complexity-optimal and latency-optimal. For convenience, we let $S_2=\{G\}$.

For any $S \in \mathcal{S}_n$, recall that P(S) is the set of all complement pairs of S. Note that we must have $P(S) \neq \emptyset$. Let $\pi(S) = \min_{\langle a,b \rangle \in P(S)} \pi(a,b)$ and $P_{\pi}(S) = \{\langle a,b \rangle \in P(S) : \pi(a,b) = \pi(S)\}$, where $\pi(a,b)$ is equal to one plus the maximum height of E(a,S) and E(b,S). As a result, we have $\pi(S) > \lceil \log(n) \rceil$.

Lemma 4: For $n \ge 3$, we have $\pi(S) = \lceil \log(n) \rceil$ with S returned by Construction 1.

Proof: Let S be returned by Construction 1 and let $T = h^{-1}(S)$, where h^{-1} is the inverse function of h defined in Theorem 3. We have $d(T) = d_n^{\min} = l(S) + 1 =$ $\delta + \lceil \log(n-2^{\delta}) \rceil + 1$ with $\delta = \lceil \log(n/2) \rceil$. There exist two leaves, say $x_i, x_i \in T$, such that the distance between x_i and x_j is d(T). Moreover, given x_i and x_j , there exists a unique node a (resp. b) such that a (resp. b) is contained in the path from x_i to x_i and the distance between x_i and a (resp. b) is |d(T)/2| (resp. |d(T)/2|+1). Note that $(a,b) \in T$. As a result, there exists a complement pair $\langle a', b' \rangle \in P(S)$ such that D(a,b,T) = E(a',S) and D(b,a,T) = E(b',S). Accordingly, the heights of E(a', S) and E(b', S) are |d(T)/2| and d(T) - |d(T)/2| - 1, respectively. We then have $\pi(a', b') =$ $1 + \max\{\lfloor d(T)/2 \rfloor, d(T) - \lfloor d(T)/2 \rfloor - 1\} = 1 + \lfloor d(T)/2 \rfloor.$ Note that $2^{\delta+1} \le n < 2^{\delta+2}$. If $n = 2^{\delta+1}$, we have $\pi(a',b') = 1 + \left \lfloor \overline{d(T)/2} \right \rfloor = 1 + \left \lfloor (\delta + \left \lceil \log(n-2^{\delta}) \right \rceil + 1)/2 \right \rfloor = 1 + \delta = \left \lceil \log(n) \right \rceil.$ If $2^{\delta+1} < n < 2^{\delta+2}$, we have $\pi(a', b') = 1 + \delta + 1 = \lceil \log(n) \rceil$. As $\pi(S) \ge \lceil \log(n) \rceil$, we finally have $\pi(S) = \lceil \log(n) \rceil$.

For two integers i and j, let $[i,j] = \{i,i+1,\ldots,j\}$, where $[i,j] = \emptyset$ if i > j. We now propose a method to construct larger (in terms of input size) structures based on smaller structures.

Construction 3: If there exist (m, n_0, \ldots, n_m) such that $m \in [n-1]$, $n_0 \ge m$, and $\sum_{i \in [0,m]} n_i = n+m$ with $n_i \in [2,n-1]$, we can construct an $S \in \mathcal{S}_n$ from any $S_i \in \mathcal{S}_{n_i}, \forall i \in [0,m]$ with the following steps.

- (D1) For each $i \in [0,m]$ and $j \in [n_i]$, refer to $x_j \in S_i$ and $y_j \in S_i$ by $a_{i,j}$ and $b_{i,j}$, respectively. (Note that if $n_i = 2$, we have $a_{i,1} = b_{i,2}$ and $a_{i,2} = b_{i,1}$.)
- (D2) Let S be the joint graph of all $S_i, i \in [0, m]$. (Simply put all $S_i, i \in [0, m]$ together into S without extra operations, such as merging nodes or edges.)

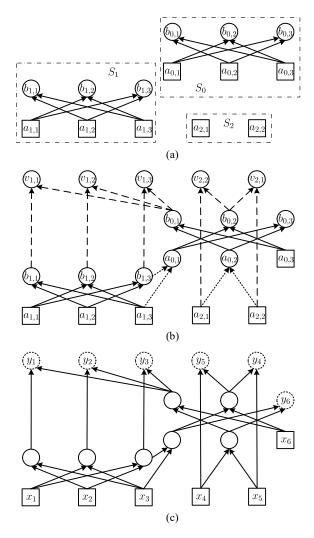


Fig. 8. An example of Construction 3 for $(n, m, n_0, n_1, n_2) = (6, 2, 3, 3, 2)$. (a) S formed in steps (D1) and (D2). (Note that $a_{2,1} = b_{2,2}$ and $a_{2,2} = b_{2,1}$.) (b) S formed in steps (D3) and (D4), where for easy reference, the edges added into S in steps (D3) and (D4) are represented by dotted and dashed arrows, respectively. (c) S formed in step (D5), which is also the constructed structure returned in step (D6).

- (D3) For each $i \in [m]$ and an arbitrary complement pair $\langle u_i, v_i \rangle \in P_{\pi}(S_i)$, let $S = S + (u_i, a_{0,i}) + (v_i, a_{0,i})$.
- (D4) For each $i \in [m]$ and $j \in [n_i]$, create a new node $v_{i,j} \notin S$, and let $S = S + (b_{i,j}, v_{i,j}) + (b_{0,i}, v_{i,j})$.
- (D5) For the nodes in S, label those with no incoming edges by x_1, x_2, \ldots, x_n ; label those with no outgoing edges by y_1, y_2, \dots, y_n such that for any $j \in [n], E(y_j, S)$ has leaves $\{x_1, x_2, \dots, x_n\} \setminus \{x_i\}$; unlabel all other nodes.
- (D6) Return S as the constructed structure.

An example of Construction 3 for $(n, m, n_0, n_1, n_2) =$ (6,2,3,3,2) is shown in Fig. 8. Moreover, we have the following result.

Lemma 5: Use the notations in Construction 3 and let S be

the returned structure. We have $S \in \mathcal{S}_n$. Moreover, we have

$$\begin{split} c(S) &= c(S_0) + \sum_{i \in [m]} \left(c(S_i) + n_i + 1 \right), \\ \pi(S) &\leq \pi(S_0) + \max_{i \in [m]} \pi(S_i), \text{ and} \end{split}$$

$$\pi(S) \le \pi(S_0) + \max_{i \in [m]} \pi(S_i), \text{ and }$$

$$l(S) \le \max \left\{ \max_{i \in [m]} l(S_i) + 1, \ l(S_0) + 1 + \max_{i \in [m]} \pi(S_i) \right\}.$$

Proof: Note that at the end of step (D1) (in Construction 3), for each $i \in [0, m]$ and $j \in [n_i]$, $E(b_{i,j}, S_i)$ is a DBT with leaves $\{a_{i,j'}: j' \in [n_i]\} \setminus \{a_{i,j}\}$. At the end of step (D3), for each $i \in [m]$, $E(a_{0,i}, S)$ is a DBT of height $\pi(S_i)$ and with leaves $\{a_{i,j'}: j' \in [n_i]\}$. At the end of step (D4), for each $i \in [m]$ and $j \in [n_i]$, $E(v_{i,j}, S)$ is a DBT of height at most $\max \{l(S_i) + 1, l(S_0) + \max_{i' \in [m]} \pi(S_{i'}) + 1\}$ and with leaves $\{a_{i',j'}: i' \in [m], j' \in [n_{i'}]\} \setminus \{a_{i,j}\}$. We can then easily verify the correctness of Lemma 5.

For any non-negative integer τ , let

$$S_{n,\tau} = \{ S \in S_n : l(S) \le \tau, \pi(S) = \lceil \log(n) \rceil \}.$$

We remark that $S_{n,\tau} \subseteq S_{n,\tau'}$ for $\tau \leq \tau'$. Moreover, for n=2, we have $S_{2,0} = S_2$. For $n \geq 3$, we have $S \in S_{n,d_n^{\min}-1}$ if S is returned by Construction 1 (according to Lemma 4), and have $S \in \mathcal{S}_{n,\log(n-1)}$ if S is returned by Construction 2. Given these observations and motivated by Construction 3, we have the following construction for a structure $S_{n,\tau} \in \mathcal{S}_{n,\tau}$ if $S_{n,\tau} \neq \emptyset$; otherwise, we say that $S_{n,\tau}$ does not exist.

Construction 4: Let \hat{n} and $\hat{\tau}$ be the maximum values of nand τ , respectively. For $n=2,3,\ldots,\hat{n}$ and $\tau=0,1,\ldots,\hat{\tau}$, we construct a structure $S_{n,\tau}$ based on the following cases.

- (E1) If n=2, let $S_{n,\tau}$ be the only structure in S_2 .
- (E2) Else if $\tau \geq d_n^{\min} 1$, construct $S_{n,\tau}$ via Construction 1.
- (E3) Else if $\tau < \lceil \log(n-1) \rceil$, $S_{n,\tau}$ does not exist.
- (E4) Else if $2^{\tau} = n 1$, construct $S_{n,\tau}$ via Construction 2.
- (E5) Otherwise, let $\phi(n,\tau) = \infty$. For any $(m, n_0, \dots, n_m, \dots, n_m)$ τ_0,\ldots,τ_m) such that

$$\begin{cases} m \in [n-1], n_0 \ge m, \\ \sum_{i \in [0,m]} n_i = n + m, n_i \in [2, n-1], \\ \forall i \in [0,m], S_{n_i,\tau_i} \text{ exists}, \\ \max_{i \in [m]} \lceil \log(n_i) \rceil \le \lceil \log(n) \rceil - \lceil \log(n_0) \rceil, \\ \max_{i \in [m]} \lceil \log(n_i) \rceil \le \tau - 1 - \tau_0, \\ \max_{i \in [m]} \tau_i \le \tau - 1, \end{cases}$$
 (2)

construct S from $S_{n_i,\tau_i}, \forall i \in [0,m]$ via Construction 3. If $c(S) < \phi(n, \tau)$, let $\phi(n, \tau) = c(S)$ and $S_{n,\tau} = S$.

Theorem 7: Iff $n \ge 2$ and $\tau \ge \lceil \log(n-1) \rceil$, Construction 4 can obtain a structure $S_{n,\tau} \in \mathcal{S}_{n,\tau} \neq \emptyset$. Moreover, if $S_{n,\tau}$ exists, we have

$$c(S_{n,\tau}) \le n\lceil \log(n)\rceil - 2. \tag{3}$$

Proof: See Appendix E.

We remark that in case (E5) of Construction 4, we try to reuse the same subtrees (same intermediate computation results) as often as possible. This implies that for $n \geq 2$ and $\tau \geq \lceil \log(n-1) \rceil$, we likely have

$$c(S_{n,\tau}) = \min_{S \in \mathcal{S}_{n,\tau}} c(S),$$

which is guaranteed to be true for cases (E1), (E2), and (E4). On the other hand, there likely exists a structure $S \in \{S' \in \mathcal{S}_n : l(S') \leq \tau\}$ such that $c(S) = \min_{S' \in \mathcal{S}_n, l(S') \leq \tau} c(S')$ and $\pi(S) = \lceil \log(n) \rceil$ (i.e., $S \in \mathcal{S}_{n,\tau}$). As a result, we likely have

$$c(S_{n,\tau}) = \min_{S \in \mathcal{S}_n, l(S) < \tau} c(S), \tag{4}$$

which is guaranteed to be true for cases (E1), (E2), and (E4). However, we currently are not able to prove this result. Formally, we give the following conjecture.

Conjecture 1: Let $S_{n,\tau}$ be returned by Construction 4. Then, $S_{n,\tau}$ satisfies (4).

In general, it is not possible to enumerate $(m, n_0, \ldots, n_m, \tau_0, \ldots, \tau_m)$ by using the brute-force method in case (E5) of Construction 4. However, finding a $(m, n_0, \ldots, n_m, \tau_0, \ldots, \tau_m)$ to minimize $c(S_{n,\tau})$ is of great interest to practice. In the rest of this section, we illustrate how to efficiently find such a $(m, n_0, \ldots, n_m, \tau_0, \ldots, \tau_m)$.

For $n \geq 2$, let $\phi(n,\tau) = c(S_{n,\tau})$, where $\phi(n,\tau) = \infty$ if $S_{n,\tau}$ does not exist. For any $i_1, i_2 \in [0, \hat{n}], i_3 \in [0, \lceil \log(\hat{n}) \rceil]$ and $i_4 \in [0, \hat{\tau}]$, let $\eta(i_1, i_2, i_3, i_4)$ denote the minimum value of $\sum_{j \in [i_2]} (\phi(n_j, i_4) + n_j + 1)$, where $n_j \in [1, 2^{i_3}]$ and $\sum_{j' \in [i_2]} n_{j'} = i_1$. We remark that $\phi(n_j, i_4) + n_j + 1$ actually corresponds to the complexity related to S_{n_i,i_4} when it is used to construct a larger structure with input size i_1 . For example, the complexity related to S_1 in Fig. 8 is $\phi(3,1) + 3 + 1$. We further remark that in (E5) of Construction 4, we require $n_i \geq 2$. However, to simplify the computation in Algorithm 1, here we allow $n_j = 1$ and define $\phi(1, \tau) = -2$ for any τ to make $\phi(1,\tau)+1+1=0$. Taking Fig. 8 as an example, we have $(n_1, n_2, n_3) = (3, 2, 1)$, where $n_3 = 1$ is associated with the single node $a_{0,3}$. Note that we have $\phi(n,\tau) \geq \phi(n,\tau')$ for $\tau \leq \tau'$, and we also have $\eta(i_1, i_2, i_3, i_4) \geq \eta(i_1, i_2, i_3', i_4')$ for $i_3 \leq i_3'$ and $i_4 \leq i_4'$. We can compute ϕ and η by using the proposed Algorithm 1.

In Algorithm 1, line 1 is to initialize η by using $\phi(1,\tau), \forall \tau \geq 0$. Lines 13–18 correspond to case (E5) in Construction 4. More specifically, for any $n_0 \in [2,n-1]$ and $\tau_0 \in [0,\tau-1]$ such that $\omega < \infty$, without loss of generality, assume that $\eta(n,n_0,\theta,\tau-1) = \sum_{i \in [n_0]} (\phi(n_i,\tau-1)+n_i+1)$ with $n_1 \geq n_2 \geq \cdots \geq n_m \geq 2 > n_{m+1} = n_{m+1} = \cdots = n_{n_0} = 1$. Then, $(m,n_0,\ldots,n_m,\tau_0,\ldots,\tau_m)$ satisfy (2), and we have $c(S) = \omega$ with S constructed from $S_{n_i,\tau_i}, \forall i \in [0,m]$ via Construction 3. As a result, according to the definition of η , $\phi(n,\tau)$ computed via lines 13–18 is equal to $c(S_{n,\tau})$ with $S_{n,\tau}$ given in case (E5) of Construction 4. Moreover, lines 21–24 are to update η by using $\phi(n,\tau)$, so as to keep $\eta(i_1,i_2,i_3,\tau)$ to be the minimum value of $\sum_{j \in [i_2]} (\phi(n_j,\tau)+n_j+1)$, where $n_j \in [1,n]$ and $\sum_{j' \in [i_2]} n_{j'} = i_1$.

Theorem 8: For $n \ge 2$ and $\tau \ge 0$, let $\phi(n,\tau)$ be computed via Algorithm 1, and let $S_{n,\tau}$ be returned by Construction 4. We have $\phi(n,\tau) = c(S_{n,\tau})$ if $\tau \ge \lceil \log(n-1) \rceil$, and $\phi(n,\tau) = \infty$ otherwise.

Proof: The statement is true according to the above discussions regarding Algorithm 1.

Note that $\phi(n,\tau)=3n-6$ for $\tau \geq d_n^{\min}-1$, where $d_n^{\min} \leq 2\log(n)+1$ according to Lemma 3. We only need to compute $\phi(n,\tau)$ for $\tau < d_n^{\min}-1$. As a result, the complexity of

Algorithm 1 Computation of ϕ and η

```
Input: \hat{n} and \hat{\tau}.
Output: \phi and \eta.
  1: For any i_1, i_2 \in [0, \hat{n}], i_3 \in [0, \lceil \log(\hat{n}) \rceil] and i_4 \in [0, \hat{\tau}], set
      \eta(i_1, i_2, i_3, i_4) as 0 if i_1 == i_2 and as \infty otherwise.
 2: for n = 2, 3, ..., \hat{n} and \tau = 0, 1, ..., \hat{\tau} do
          //Compute \phi(n,\tau)
 3:
          if n == 2 then
 4:
 5:
             \phi(n,\tau)=0.
          else if \tau \geq d_n^{\min} - 1 then
              \phi(n,\tau) = 3n - 6.
 7:
          else if \tau < \lceil \log(n-1) \rceil then
 8:
 9:
             \phi(n,\tau)=\infty.
          else if 2^{\tau} == n-1 then
10:
             \phi(n,\tau) = n\tau.
11:
12:
13:
             \phi(n,\tau)=\infty.
14:
             for n_0 = 2, 3, ..., n-1 and \tau_0 = 0, 1, ..., \tau-1 do
                 \theta = \min \{ \lceil \log(n) \rceil - \lceil \log(n_0) \rceil, \tau - 1 - \tau_0 \}.
15:
                 \omega = \phi(n_0, \tau_0) + \eta(n, n_0, \theta, \tau - 1).
16:
17:
                  \phi(n,\tau) = \min\{\phi(n,\tau), \omega\}.
18:
19:
          end if
20:
          //Update \eta by using \phi(n,\tau)
          for i_1 = n, n + 1, \dots, \hat{n}, i_2 = 1, 2, \dots, \hat{n}, \text{ and } i_3 = 1, \dots, \hat{n}
21:
          \lceil \log(n) \rceil, \lceil \log(n) \rceil + 1, \dots, \lceil \log(\hat{n}) \rceil do
22:
              \lambda = \eta(i_1 - n, i_2 - 1, i_3, \tau) + \phi(n, \tau) + n + 1.

\eta(i_1, i_2, i_3, \tau) = \min\{\eta(i_1, i_2, i_3, \tau), \lambda\}.

23:
24:
          end for
25: end for
```

Algorithm 1 for computing $\phi(n,\tau)$ is $O(n^3 \log^2(n))$. For easy reference, we present $\phi(n,\tau)$ for some typical (n,τ) in Table I. We can see that $\phi(n,\tau)$ is generally much smaller than $n\lceil\log(n)\rceil-2$, the upper bound given by (3).

To find a $(m,n_0,\ldots,n_m,\tau_0,\ldots,\tau_m)$ to minimize $c(S_{n,\tau})$ in case (E5) of Construction 4, we only need to record the solutions to $\phi(n,\tau)$ and $\eta(i_1,i_2,i_3,\tau)$ in lines 17 and 23 of Algorithm 1, respectively. More specifically, record (n_0,τ_0) such that $\phi(n,\tau)=\phi(n_0,\tau_0)+\eta(n,n_0,\theta,\tau-1)$, and record (n,i_3) such that $\eta(i_1,i_2,i_3,\tau)=\eta(i_1-n,i_2-1,i_3,\tau)+\phi(n,\tau)+n+1$. In this case, we can find a $(m,n_0,\ldots,n_m,\tau_0,\ldots,\tau_m)$ by traceback.

We remark that for the computation at a variable node of LDPC codes, there exists a unique incoming message, say x_1 , which corresponds to the received channel message. In this case, y_1 is used for hard decision of the corresponding transmitted bit and should be computed from \mathbf{x} without excluding x_1 . We can slightly modify the structure $S_{n,\tau}$ returned by Construction 4 to perfectly match the aforementioned variable node computation. More specifically, for an arbitrary complement pair $\langle a,b\rangle\in P_\pi(S_{n,\tau})$, let $S=S_{n,\tau}-y_1+(a,y_1)+(b,y_1)$. Then, $E(y_1,S)$ is a DBT of height $\lceil\log(n)\rceil$ and with leaves $X=\{x_1,x_2,\ldots,x_n\}$, and $E(y_j,S)=E(y_j,S_{n,\tau}), \forall j=2,3,\ldots,n$. This indicates that S can be used to implement the aforementioned variable node computation. Moreover, we have $c(S)=c(S_{n,\tau})$, and $l(S)=l(S_{n,\tau})+1$ if $n=2^{\tau}+1$; otherwise, $l(S)=l(S_{n,\tau})$.

VII. CONCLUSION

Let $S \in \mathcal{S}_n$ be an arbitrary structure satisfying Definition 2. First, we have proved that the minimum complexity of S

TABLE I

Some Typical Values and Upper Bounds (UBs) of $\phi(n,\tau)$: $n\geq 2, \tau=\lceil\log(n-1)\rceil+i, i=0,1,\ldots; \phi(n,\tau)$ Is the Complexity of the Structure $S_{n,\tau}$ Obtained from Construction 4; $S_{n,\tau}$ Has n Input Nodes and Its Latency Is at Most τ ; for $\tau<\underline{\tau}=\lceil\log(n-1)\rceil,S_{n,\tau}$ Does Not Exist; for $\tau\geq \overline{\tau}=\delta+\lceil\log(n-2^\delta)\rceil$ with $\delta=\lfloor\log(n/2)\rfloor$, We Have $\phi(n,\tau)=3n-6$; the UB is $n\lceil\log(n)\rceil-2$ Given by (3).

$n, \ \underline{\tau}, \ \overline{\tau} \setminus i$	0	1	2	3	UB	$n, \underline{\tau}, \overline{\tau} \setminus i$	0	1	2	3	4	UB
1, -, -	_	_	_	_	_	33, 5, 9	165	114	99	94	93	196
2, 0, 0	0	0	0	0	0	34, 6, 9	118	102	97	96	96	202
3, 1, 1	3	3	3	3	4	35, 6, 9	122	105	100	99	99	208
4, 2, 2	6	6	6	6	6	36, 6, 9	126	108	103	102	102	214
5, 2, 3	10	9	9	9	13	37, 6, 9	133	115	106	105	105	220
6, 3, 3	12	12	12	12	16	38, 6, 9	137	118	109	108	108	226
7, 3, 4	18	15	15	15	19	39, 6, 9	141	122	112	111	111	232
8, 3, 4	22	18	18	18	22	40, 6, 9	145	125	115	114	114	238
9, 3, 5	27	22	21	21	34	41, 6, 9	159	132	122	117	117	244
10, 4, 5	25	24	24	24	38	42, 6, 9	163	135	125	120	120	250
11, 4, 5	32	27	27	27	42	43, 6, 9	168	139	128	123	123	256
12, 4, 5	36	30	30	30	46	44, 6, 9	172	142	131	126	126	262
13, 4, 6	45	36	33	33	50	45, 6, 9	179	146	135	129	129	268
14, 4, 6	50	39	36	36	54	46, 6, 9	183	149	138	132	132	274
15, 4, 6	57	43	39	39	58	47, 6, 9	188	153	141	135	135	280
16, 4, 6	62	46	42	42	62	48, 6, 9	192	156	144	138	138	286
17, 4, 7	68	51	46	45	83	49, 6, 10	243	176	153	144	141	292
18, 5, 7	54	49	48	48	88	50, 6, 10	250	180	156	147	144	298
19, 5, 7	61	52	51	51	93	51, 6, 10	259	184	159	150	147	304
20, 5, 7	65	55	54	54	98	52, 6, 10	266	188	162	153	150	310
21, 5, 7	72	62	57	57	103	53, 6, 10	277	192	167	156	153	316
22, 5, 7	76	65	60	60	108	54, 6, 10	284	196	170	159	156	322
23, 5, 7	80	69	63	63	113	55, 6, 10	293	200	173	162	159	328
24, 5, 7	84	72	66	66	118	56, 6, 10	300	204	176	165	162	334
25, 5, 8	108	81	72	69	123	57, 6, 10	325	210	182	169	165	340
26, 5, 8	114	84	75	72	128	58, 6, 10	332	214	185	172	168	346
27, 5, 8	122	89	78	75	133	59, 6, 10	341	218	189	175	171	352
28, 5, 8	128	92	81	78	138	60, 6, 10	348	222	192	178	174	358
29, 5, 8	138	98	85	81	143	61, 6, 10	359	226	196	181	177	364
30, 5, 8	144	102	88	84	148	62, 6, 10	366	230	199	184	180	370
31, 5, 8	152	106	91	87	153	63, 6, 10	375	234	203	187	183	376
32, 5, 8	158	110	94	90	158	64, 6, 10	382	238	206	190	186	382

is 3n-6, and if S has such complexity, its minimum latency is $\delta+\lceil\log(n-2^\delta)\rceil$ with $\delta=\lfloor\log(n/2)\rfloor$. Next, we have proved that the minimum latency of S is $\lceil\log(n-1)\rceil$, and if S has such latency, its minimum complexity is $n\log(n-1)$ for $n=2^k+1$ with k>0. Finally, given (n,τ) with $\tau\geq\lceil\log(n-1)\rceil$, we have proposed a construction, i.e., Construction 4, for a structure $S_{n,\tau}$ which we conjecture to have the minimum complexity among structures with latencies at most τ . Construction 4 can run in $O(n^3\log^2(n))$ time, and the obtained $S_{n,\tau}$ has complexity at most (generally much smaller than) $n\lceil\log(n)\rceil-2$. One left problem is to verify whether $S_{n,\tau}$ returned by Construction 4 achieves the minimum complexity among structures with latencies at most τ , i.e., to prove/disprove Conjecture 1.

APPENDIX A PROOF OF THEOREM 2

For n=3, the only structure in S_n^{co} , as shown in Fig. 3(a), fulfills properties (C1)–(C5). Assume that for $n=k-1\geq 3$,

structures in $\mathcal{S}_{k-1}^{\text{co}}$ fulfill properties (C1)–(C5). We now prove for n=k, structures in $\mathcal{S}_k^{\text{co}}$ also fulfill properties (C1)–(C5). Let $S\in\mathcal{S}_k^{\text{co}}$ be an arbitrary structure.

Proof of (C1): y_k has exactly two incoming edges in S, say $(a_1, y_k), (b_1, y_k) \in S$. Moreover, as discussed earlier, x_k has exactly two outgoing edges in S, say $(x_k, a_2), (x_k, b_2) \in S$, and we have $f(S) \in S_{k-1}^{co}$ such that f(S) fulfills properties (C1)–(C3). As a result, nodes $y_k, x_k, a_1, b_1, a_2, b_2$ and their edges must form a subgraph of S exactly the same as that in Fig. 4(a) (with n = k), and this subgraph changes to a subgraph in f(S) exactly the same as that in Fig. 4(d). Note that we have $S - y_k - x_k - a_1 - b_1 - a_2 - b_2 = f(S) - a - b$. Hence, S fulfills property (C1).

Proof of (C2): Note that in the following proof, the definitions of (a_1,b_1,a_2,b_2) are from inside the proof of (C1) and Fig. 4(a). We first have $E(a_1,S)=E(a,f(S))$, $E(b_1,S)=E(b,f(S))$, $L(a_2,S)=L(a,f(S))$, and $L(b_2,S)=L(b,f(S))$. As a result, we have $\langle a,b\rangle\in P(f(S))$, since $X\setminus\{x_k\}=\{x_1,x_2,\ldots,x_{k-1}\}\subseteq E(y_k,S)$. This

indicates that E(a,f(S))-a, E(b,f(S))-b, L(a,f(S))-a, and L(b,f(S))-b pairwise do not share the same node in f(S), and we have $Y\setminus\{y_k\}=\{y_1,y_2,\ldots,y_{k-1}\}\subseteq L(a,f(S))\cup L(b,f(S))$ since f(S) fulfills property (C2). On the other hand, E(a,f(S)) and E(b,f(S)) are two DBTs. Meanwhile, since f(S) fulfills property (C1), the undirected versions of L(a,f(S)) and L(b,f(S)) are two binary trees rooted at a and b in f(S), respectively. Therefore, E(a,f(S))-a,E(b,f(S))-b,L(a,f(S))-a, and L(b,f(S))-b contain 4(k-1)-8 nodes, which are exactly all the nodes in f(S)-a-b. Accordingly, $E(y_k,S)$ and $L(x_k,S)$ contain 4k-6 nodes, which are exactly all the nodes in S.

For any $v \in E(y_k, S)$, if $v = y_k$, we obviously have $x_j \in E(v, S) \iff y_j \notin L(v, S), \forall j \in [k]$. Assume $v \neq y_k$. As discussed above, there exists a unique $v' \in E(a, f(S)) \cup E(b, f(S))$ such that E(v', f(S)) = E(v, S). Meanwhile, we have $y_j \in L(v', f(S)) \iff y_j \in L(v, S), \forall j \in [k-1]$. Since f(S) fulfills property (C2), we have $x_j \in E(v', f(S)) \iff y_j \notin L(v', f(S)), \forall j \in [k-1]$. Therefore, we have $x_j \in E(v, S) \iff y_j \notin L(v, S), \forall j \in [k]$ by further noting that $x_k \notin E(v, S)$ and $y_k \in L(v, S)$.

On the other hand, for any $v \in L(x_k, S)$, if $v = x_k$, we obviously have $x_j \in E(v, S) \iff y_j \notin L(v, S), \forall j \in [k]$. For $v \neq x_k$, let $v' \in L(a, f(S)) \cup L(b, f(S))$ such that L(v', f(S)) = L(v, S). We can similarly derive $x_j \in E(v, S) \iff y_j \notin L(v, S), \forall j \in [k]$. As a result, S fulfills property (C2).

Proof of (C3): For any $v \in E(y_k, S)$, if $v = y_k$, we obviously have $\langle v, \bar{v} \rangle \in P(S) \iff \bar{v} = x_k$. Assume $v \neq y_k$. There exists a unique $v' \in E(a, f(S)) \cup E(b, f(S))$ such that E(v', f(S)) = E(v, S). Since f(S) fulfills property (C3), there exists a unique $\bar{v}' \in f(S)$ such that $\langle v', \bar{v}' \rangle \in P(f(S))$. On the one hand, we must have $\bar{v}' \in L(a, f(S)) \cup L(b, f(S))$. As a result, there exists a unique $\bar{v} \in L(x_k, S)$ such that $L(\bar{v}, S) = L(\bar{v}', f(S))$. On the other hand, since f(S) fulfills property (C2), we have for $j \in [k-1], x_i \in E(v', f(S)) \iff$ $x_j \notin E(\bar{v}', f(S)) \iff y_j \in L(\bar{v}', f(S))$. Therefore, we have for $j \in [k]$, $x_j \in E(v,S) \iff y_j \in L(\bar{v},S) \iff x_j \notin$ $E(\bar{v},S)$ by further noting that $x_k \notin E(v,S), y_k \notin L(\bar{v},S)$ and S fulfills property (C2). This indicates that $\langle v, \bar{v} \rangle \in P(S)$. Note that $E(y_k, S)$ contains half nodes of S, $L(x_k, S)$ contains another half nodes of S, and each $v \in E(y_k, S)$ leads to a unique $\bar{v} \in L(x_k, S)$ such that $\langle v, \bar{v} \rangle \in P(S)$. Hence, S fulfills property (C3), and we also have |P(S)| = 2k - 3.

Proof of (C4): On the one hand, for any $S' \in \mathcal{S}_{k-1}^{\text{co}}$ and $\langle a,b \rangle \in P(S')$, we have $g(a,b,S') \in \mathcal{S}_k^{\text{co}}$. This implies $\{g(a,b,S'):S' \in \mathcal{S}_{k-1}^{\text{co}}, \langle a,b \rangle \in P(S')\} \subseteq \mathcal{S}_k^{\text{co}}$. On the other hand, for any $S \in \mathcal{S}_k^{\text{co}}$, we have $f(S) \in \mathcal{S}_{k-1}^{\text{co}}$, and there exists $\langle a,b \rangle \in P(f(S))$ such that S=g(a,b,f(S)). This implies $\mathcal{S}_k^{\text{co}} \subseteq \{g(a,b,S'):S' \in \mathcal{S}_{k-1}^{\text{co}}, \langle a,b \rangle \in P(S')\}$. As a result, we have $\mathcal{S}_k^{\text{co}} = \{g(a,b,S'):S' \in \mathcal{S}_{k-1}^{\text{co}}, \langle a,b \rangle \in P(S')\}$, indicating that $\mathcal{S}_k^{\text{co}}$ fulfills property (C4).

Proof of (C5): For any $S' \in \mathcal{S}^{\text{co}}_{k-1}$, on the one hand, we have $g(a,b,S') \neq g(a',b',S')$ for any $\langle a,b \rangle, \langle a',b' \rangle \in P(S')$ with $\langle a,b \rangle \neq \langle a',b' \rangle$. This implies $|\{g(a,b,S'): \langle a,b \rangle \in P(S')\}| = |P(S')| = 2(k-1)-3$, since S' fulfills property (C3). On the other hand, for any $S' \neq S'' \in \mathcal{S}^{\text{co}}_{k-1}$, we have $\{g(a,b,S'): \langle a,b \rangle \in P(S')\} \cap \{g(a,b,S''): \langle a,b \rangle \in P(S')\}$

 $P(S'')\} = \emptyset$. Therefore, we have $|\mathcal{S}_k^{\text{co}}| = |\{g(a,b,S'): S' \in \mathcal{S}_{k-1}^{\text{co}}, \langle a,b \rangle \in P(S')\}| = |\mathcal{S}_{k-1}^{\text{co}}| \cdot (2(k-1)-3) = (2k-5)!!,$ indicating that $\mathcal{S}_k^{\text{co}}$ fulfills property (C5).

APPENDIX B PROOF OF THEOREM 3

For any $T \in \mathcal{T}_n$ and $j \in [n]$, x_j is a leaf in T. Let $(a,x_j) \in T$ be the only edge of x_j . $D(a,x_j,T)$ is a DBT with root y_j and leaves $X \setminus \{x_j\}$. As a result, we have $h(T) \in \mathcal{S}_n$. On the other hand, we have $E(h(T)) = \bigcup_{j \in [n], (a,x_j) \in T} E(D(a,x_j,T)) = D(T)$, leading to c(h(T)) = |E(h(T))| - n = |D(T)| - n = 3n - 6. Therefore, we have $h(T) \in \mathcal{S}_n^{\text{co}}$. Moreover, for any $T \neq T' \in \mathcal{T}_n$, we obviously have $h(T) \neq h(T')$, indicating that h is an injection from \mathcal{T}_n to $\mathcal{S}_n^{\text{co}}$. In the following, we prove $|\mathcal{T}_n| = |\mathcal{S}_n^{\text{co}}| = (2n - 5)!!$ such that h is surjective and the proof is completed.

Assume $n \geq 4$. For any $T' \in \mathcal{T}_{n-1}$ and $(a,b) \in T'$, let $\beta(a,b,T') = T' - (a,b) + (x_n,v) + (a,v) + (b,v)$, where v is a new internal node (unlabelled) added into T'. We have $\beta(a,b,T') \in \mathcal{T}_n$. This implies $\{\beta(a,b,T'): T' \in \mathcal{T}_{n-1}, (a,b) \in T'\} \subseteq \mathcal{T}_n$. On the other hand, for any $T \in \mathcal{T}_n$, let $\alpha(T) = T - x_n - v + (a,b)$, where v,a,b fulfill $(x_n,v),(a,v),(b,v) \in T$ (note that v and (a,b) are unique given T). We have $\alpha(T) \in \mathcal{T}_{n-1}$ and $T = \beta(a,b,\alpha(T))$. This implies $\mathcal{T}_n \subseteq \{\beta(a,b,T'): T' \in \mathcal{T}_{n-1}, (a,b) \in T'\}$. As a result, we have $\mathcal{T}_n = \{\beta(a,b,T'): T' \in \mathcal{T}_{n-1}, (a,b) \in T'\}$.

Moreover, note that for any $T' \in \mathcal{T}_{n-1}$, we have $\beta(a,b,T') \neq \beta(a',b',T')$ for $(a,b),(a',b') \in T'$ with $(a,b) \neq (a',b')$. This implies $|\{\beta(a,b,T'):(a,b) \in T'\}| = 2(n-1)-3$. Meanwhile, we have $\{\beta(a,b,T'):(a,b) \in T'\} \cap \{\beta(a,b,T''):(a,b) \in T''\} = \emptyset$ for any $T' \neq T'' \in \mathcal{T}_{n-1}$. As a result, we have $|\mathcal{T}_n| = (2(n-1)-3)\cdot |\mathcal{T}_{n-1}| = (2(n-1)-3)\cdot (2(n-2)-3)\cdot |\mathcal{T}_{n-2}| = (2n-5)!!$, since $|\mathcal{T}_3| = 1 = (2\cdot 3-5)!!$. This completes the proof.

APPENDIX C PROOF OF LEMMA 3

Let $\delta=\lfloor\log(n/2)\rfloor$. We have $2^{\delta+1}\leq n<2^{\delta+2}$. Let T_1 and T_2 be two complete binary trees with leaves $\{x_1,x_2,\ldots,x_{2^\delta}\}$ and $\{x_{2^\delta+1},x_{2^\delta+2},\ldots,x_n\}$, respectively. Since T_1 is a perfect binary tree, T_1 has height $h_1=\delta$ and diameter $d(T_1)=2\delta$. Meanwhile, T_2 has height $h_2=\lceil\log(n-2^\delta)\rceil\leq\delta+2$, since $n-2^\delta<3\cdot 2^\delta$. More specifically, the left subtree of T_2 has height h_2-1 , and the right subtree of T_2 has height at most δ , leading to $d(T_2)\leq \max\{2(h_2-1),2\delta,h_2-1+\delta+2\}=h_2+\delta+1$. Furthermore, there exists a $T\in\mathcal{T}_n$ and an edge $(a,b)\in T$ such that D(a,b,T) and D(b,a,T) are the directed versions of T_1 and T_2 , respectively. We have $d_n^{\min}\leq d(T)=\max\{d(T_1),d(T_2),h_1+h_2+1\}=h_1+h_2+1=\delta+\lceil\log(n-2^\delta)\rceil+1$.

On the other hand, for any $T \in \mathcal{T}_n$, assume that the distance between x_i and x_j is equal to d(T). The height of $D(v, x_i, T)$ with $(v, x_i) \in T$ is d(T) - 1 such that $D(v, x_i, T)$ contains at most $2^{d(T)-1}$ leaves. We must have $2^{d(T)-1} \geq n-1$, leading to $d(T) \geq \lceil \log(n-1) \rceil + 1 \geq \delta + 1$. As a result, there exists a unique node a (resp. b) such that a (resp. b) is contained in the path from x_i to x_j and the distance between x_i and a

(resp. b) is δ (resp. $\delta+1$). Note that we have $(a,b)\in T$. The height of D(a,b,T) is δ and hence D(a,b,T) contains at most 2^{δ} leaves. Meanwhile, the height of D(b,a,T) is $d(T)-\delta-1$ and hence D(b,a,T) contains at most $2^{d(T)-\delta-1}$ leaves. Therefore, we must have $2^{\delta}+2^{d(T)-\delta-1}\geq n$, leading to $d(T)\geq \delta+\lceil\log(n-2^{\delta})\rceil+1$. This implies $d_n^{\min}\geq \delta+\lceil\log(n-2^{\delta})\rceil+1$. Combining with the previous result $d_n^{\min}\leq \delta+\lceil\log(n-2^{\delta})\rceil+1$, the proof is completed.

APPENDIX D PROOF OF THEOREM 6

Use the notations in Construction 2 and let S be the returned structure. For each $j \in [n]$, $E(v_{k,j},S)$ is a perfect DBT of height k and with leaves $X \setminus \{x_{j-1}\}$, where we let $x_0 = x_n$. This indicates $y_j = v_{k,j}$ and $S \in \mathcal{S}_n^{\text{lo}}$. On the other hand, the computation nodes in S are $\{v_{i,j}: i \in [k], j \in [n]\}$. We thus have $c(S) = n \log(n-1) = nk$. We are now to prove $\min_{S' \in \mathcal{S}_n^{\text{lo}}} c(S') = nk$.

Given an arbitrary structure $S' \in \mathcal{S}_n^{\text{lo}}$. For any $j \in [n]$, $E(y_j, S')$ must be a perfect DBT of height k. This also implies that for any $a \in S'$, E(a, S') is a perfect DBT. For $i \in [k]$, let $A_i = \{a \in S' : E(a, S') \text{ has height } i\}$. Accordingly, we have $A_k = Y = \{y_1, y_2, \ldots, y_n\}$. Our idea is to prove $|A_i| \geq n, \forall i \in [k]$ such that $c(S') = \sum_{i \in [k]} |A_i| \geq nk$, which can complete the proof.

For any $a \in S'$, let $\Gamma(a) = (\gamma_1, \gamma_2, \dots, \gamma_n)$, where for any $j \in [n]$, $\gamma_j = 1$ if $x_j \in E(a, S')$ and $\gamma_j = 0$ otherwise. Meanwhile, for any $i \in [k]$ and $A \subseteq A_i$, let $\Gamma(A) = \bigoplus_{a \in A} \gamma(a)$, where \oplus is the component-wise XOR operation. If $A = \emptyset$, let $\Gamma(A) = (0, 0, \dots, 0)$ (n zeros in total). Moreover, let $\Gamma(i) = \{\Gamma(A) : A \subseteq A_i, |A| \text{ is even}\}, \forall i \in [k].$

On the one hand, we have $\Gamma(a) = \Gamma(a_1) \oplus \Gamma(a_2)$ for any $(a_1,a), (a_2,a) \in S'$. This leads to $\Gamma(k) \subseteq \Gamma(k-1) \subseteq \cdots \subseteq \Gamma(1)$. On the other hand, for any $A \subseteq A_k = Y$ with even |A|, we have $\Gamma(A) = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, where for any $j \in [n]$, $\gamma_j = 1$ if $y_j \in A$ and $\gamma_j = 0$ otherwise. This leads to $|\Gamma(k)| = 2^{n-1}$. As a result, we have $|\Gamma(i)| \geq 2^{n-1}, \forall i \in [k]$, indicating that $|A_i| \geq n$. This completes the proof.

APPENDIX E PROOF OF THEOREM 7

First of all, we have $S_{n,\tau} \neq \emptyset$ iff $n \geq 2$ and $\tau \geq \lceil \log(n-1) \rceil$. Assume $n \geq 2$ and $\tau \geq \lceil \log(n-1) \rceil$. If (n,τ) fulfill case (E1) (in Construction 4), we have $S_{n,\tau} \in S_{n,0} \subseteq S_{n,\tau}$ and (3) holds. Else if (n,τ) fulfill case (E2), we have $S_{n,\tau} \in S_{n,d_n^{\min}-1} \subseteq S_{n,\tau}$ and $c(S_{n,\tau}) = 3n-6 \leq n\lceil \log(n) \rceil - 2$. Else if (n,τ) fulfill case (E4), we have $S_{n,\tau} \in S_{n,\tau}$ and $c(S_{n,\tau}) = n\log(n-1) \leq n\lceil \log(n) \rceil - 2$. Otherwise, (n,τ) fulfill case (E5) and we have $n \geq 4$ and $\tau \geq \lceil \log(n) \rceil$. We continue the proof for this case.

For any $(m,n_0,\ldots,n_m,\tau_0,\ldots,\tau_m)$ fulfilling (2), construct S from $S_{n_i,\tau_i}, \forall i \in [0,m]$ via Construction 3. According to Lemma 5, and further noting that $\pi(S) \geq \lceil \log(n) \rceil$, we have $S \in \mathcal{S}_{n,\tau}$. Let $(m,n_0,n_1,n_2,\tau_0,\tau_1,\tau_2) = (2,2,\lceil n/2\rceil,\lfloor n/2\rfloor,0,\tau-1,\tau-1)$. Since $n\geq 4$ and $\tau\geq \lceil \log(n) \rceil$, for any $i\in [0,2]$, we have $n_i\in [2,n-1]$ and $\tau_i\geq \lceil \log(n_i-1) \rceil$. To continue proof by induction, we

assume that for any $i \in [0,2]$, S_{n_i,τ_i} exists and fulfills (3), which must be true for $n_i < 4$ as discussed previously. We can then easily verify that $(m,n_0,n_1,n_2,\tau_0,\tau_1,\tau_2) = (2,2,\lceil n/2\rceil,\lfloor n/2\rfloor,0,\tau-1,\tau-1)$ fulfill (2). Construct S from $S_{n_i,\tau_i}, \forall i \in [0,2]$ via Construction 3. As a result, we have $S \in \mathcal{S}_{n,\tau}$. Moreover, according to Lemma 5, we have $c(S) = c(S_{n_0,\tau_0}) + \sum_{i \in [2]} \left(c(S_{n_i,\tau_i}) + n_i + 1\right) \leq \sum_{i \in [2]} \left(n_i \lceil \log(n_i) \rceil + n_i - 1\right) \leq n \lceil \log(n) \rceil - 2$. Since S is a candidate for $S_{n,\tau}$, we have $S_{n,\tau} \in \mathcal{S}_{n,\tau}$ and $c(S_{n,\tau}) \leq n \lceil \log(n) \rceil - 2$. This completes the proof.

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