Invertible Low-Divergence Coding

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Abstract—Several applications in communication, control, and learning require approximating target distributions to within small informational divergence (I-divergence). The additional requirement of invertibility usually leads to using encoders that are one-to-one mappings, also known as distribution matchers. However, even the best one-to-one encoders have I-divergences that grow logarithmically with the block length in general. To improve performance, an encoder is proposed that has an invertible one-to-many mapping and a low-rate resolution code. Two algorithms are developed to design the mapping by assigning strings in either a most-likely first or least-likely first order. Both algorithms give information rates approaching the entropy of the target distribution with exponentially decreasing I-divergence and with vanishing resolution rate in the block length.

I. INTRODUCTION

Approximating target distributions has applications such as energy-efficient communication, random number generators, distributed control, coordination, learning, stealth, and others. We are motivated by applications that require both good distribution matching and invertibility. For example, informational divergence (I-divergence)-minimization and invertibility are useful for variational inference [1], [2] and image processing [3]. Another example is the stealth communication problem [4]–[6] where two parties try to hide communication from a "warden". The model has two possible states: one party sends either "noise" with per-letter statistics Q_A or it sends a string $a^n = a_1, \ldots, a_n$ of symbols that carries a message but resembles the noise. The warden observes a^n and makes a hypothesis test. One finds that if the I-divergence of the noise and message statistics is zero, then the best that the warden can do is to guess.

We are interested in block codes and encoders that:

- 1) map uniformly-distributed messages to strings a^n ;
- 2) transmit messages at rate near the entropy $\mathbb{H}(Q_A)$;
- 3) exhibit vanishing I-divergence in the block length n;
- 4) permit recovering the transmitted message from a^n .

The last requirement suggests that the encoder should be a one-to-one mapping. However, invertibility makes the problem trickier than usual. For example, we find that:



Fig. 1: (a) Distribution matching (DM); (b) resolution coding (RC); and (c) invertible low-divergence (ILD) coding.

- distribution matching (DM) encoders such as constant composition distribution matching (CCDM) [7], [8] or shell mapping [9], [10] have rates that approach H (Q_A) from *below* and are one-to-one mappings, see Fig. 1a; decoding is therefore invertible but the I-divergence grows with n [11];
- random number generators (RNGs) [12], [13] or resolution codes (RCs) [14], [15] have rates that approach H (Q_A) from *above* and I-divergences that vanish with n; however, the encoders are many-to-one mappings, see Fig. 1b, and decoding is not invertible.

We refer to [13], [16] for more discussion and references on the relations between DM and RC/RNG and their applications to, e.g., shaping for communication. Results for learning, stealth, control, and coordination are developed and reviewed in, e.g., [1]–[6], [17]–[20].

The above discussion suggest that invertible low-divergence (ILD) coding might be impossible. There is, however, one more option. Observe that one-to-many mappings are invertible if the images of any pair of inputs are disjoint, see Fig. 1c. This opens the possibility to combine DM and RC/RNG to create an *invertible one-to-many* mapping. To ensure that the RC is efficient, we add the requirement that

5) the RC rate vanishes with n.

Our main contribution is to construct invertible one-to-many encoders with rates approaching $\mathbb{H}(Q_A)$, exponentially decaying I-divergence, and vanishing RC rate in n.

This paper is organized as follows. Sec. II introduces notation and bounds and Sec. III specifies the model and requirements. Sec. IV develops an encoder with a one-tomany mapping and a RC. Sec. V treats DM and generalizes results of [11]. Sec. VI introduces the most-likely first (MLF) and least-likely first (LLF) algorithms for encoder design. Sec. VII develops lower bounds on the I-divergence. Sec. VIII provides numerical results and compares them to the bounds. Sec. IX concludes the paper. Appendixes A-C provide proofs of Lemmas and Theorems.

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II. PRELIMINARIES

A. Notation

Sets are written with calligraphic letters \mathcal{A} and the empty set with \emptyset . The cardinality of \mathcal{A} is $|\mathcal{A}|$ and the *n*-fold Cartesian product of \mathcal{A} is \mathcal{A}^n .

Random variables (RVs) are written with uppercase letters such as A, their realizations with corresponding lowercase letters a, and their alphabets as \mathcal{A} . A probability mass function (pmf) of a RV A is denoted by P_A or Q_A . We use Q_A for target pmfs and P_A for synthesized pmfs. We discard the subscripts when referring to generic pmfs. A pmf or function is sometimes written as a vector, e.g., pmf Q_A with alphabet $\mathcal{A} = \{1, \ldots, |\mathcal{A}|\}$ is written as $[Q_A(1), \ldots, Q_A(|\mathcal{A}|)]$. The uniform pmf over a set of K elements is denoted by U_K .

A random string is denoted by $A^n = A_1 A_2 \dots A_n$ and its realizations by $a^n = a_1 a_2 \dots a_n \in \mathcal{A}^n$. We write $Q_{A^n} = Q_A^n$ for the pmf of a string of independent and identically distributed (iid) RVs. The probability of a set $S \subseteq \mathcal{A}^n$ of strings with respect to Q_A^n is written as

$$Q^n_{\mathsf{A}}(\mathcal{S}) := \sum_{a^n \in \mathcal{S}} Q^n_{\mathsf{A}}(a^n) \tag{1}$$

and as $q_S = Q^n_A(S)$ for short. Probabilities conditioned on the event S are written as

$$Q_{\mathsf{A}|\mathcal{S}}^{n}(a^{n}) := \begin{cases} \frac{Q_{\mathsf{A}}^{n}(a^{n})}{Q_{\mathsf{A}}^{n}(\mathcal{S})}, & a^{n} \in \mathcal{S} \\ 0, & a^{n} \notin \mathcal{S}. \end{cases}$$
(2)

Let $n_i = n_i(a^n)$ be the number of occurrences of the letter i in a^n for $i = 1, ..., |\mathcal{A}|$. The *empirical* pmf (or *type*) of a^n is $\pi_{a^n} = \frac{1}{n}[n_1, ..., n_{|\mathcal{A}|}]$. Let \mathcal{P}_n be the set of empirical pmfs with denominator n (the *n*-types). The string a^n is called *typical* with respect to P and ϵ if (see [21, Ch. 2.4])

$$|\pi_{a^n}(i) - P(i)| \le \epsilon P(i) \tag{3}$$

for all $i \in A$. The set of typical strings is denoted $\mathcal{T}_{\epsilon}(P)$.

The *expectation* of a real-valued function f of a random variable A with respect to P is

$$\mathbb{E}_{P}\left[f(\mathsf{A})\right] = \sum_{a \in \operatorname{supp}(P)} P(a)f(a) \tag{4}$$

where $\operatorname{supp}(P) \subseteq \mathcal{A}$ is the support of P, i.e., the set of $a \in \mathcal{A}$ with P(a) > 0. For example, the *variance* of f(A) is

$$\mathbb{V}_P\left[f(A)\right] = \mathbb{E}_P\left[f(\mathsf{A})^2\right] - \mathbb{E}_P\left[f(\mathsf{A})\right]^2.$$
 (5)

The *self-information* of a with respect to a pmf P is

$$\iota_P(a) = -\log_2 P(a). \tag{6}$$

The entropy of P is

$$\mathbb{H}(P) = \mathbb{E}_P[\iota_P(\mathsf{A})] = \mathbb{E}_P[-\log_2 P(\mathsf{A})].$$
(7)

The binary entropy function is $h(p) = -p \log_2(p) - (1 - p) \log_2(1 - p)$ for 0 and <math>h(p) = 0 otherwise. The average conditional entropy is written as

$$\mathbb{H}\left(P_{\mathsf{A}|\mathsf{W}}\right) = \sum_{w \in \text{supp}(P_{\mathsf{W}})} P_{\mathsf{W}}(w) \,\mathbb{H}\left(P_{\mathsf{A}|\mathsf{W}}(\cdot|w)\right). \tag{8}$$

The cross entropy of two pmfs P and Q is

$$\mathbb{X}(P||Q) = \mathbb{E}_P\left[-\log_2 Q(\mathsf{A})\right].$$
(9)

For example, we have $\iota_{Q_A^n}(a^n) = n\mathbb{X}(\pi_{a^n} || Q_A)$. The I-divergence of two pmfs P and Q is

$$\mathbb{D}(P||Q) = \mathbb{E}_P\left[\log_2 \frac{P(\mathsf{A})}{Q(\mathsf{A})}\right]$$
(10)

and we have

$$\mathbb{X}(P||Q) = \mathbb{H}(P) + \mathbb{D}(P||Q).$$
(11)

I-divergence is also known as relative entropy and Kullback-Leibler divergence [22, Ch. 2.3].

The ℓ_1 distance between two pmfs P and Q on \mathcal{A} is

$$d_1(P,Q) = \sum_{a \in \mathcal{A}} |P(a) - Q(a)| \tag{12}$$

where $d_1(P,Q) \leq 2$ with equality if and only if the supports of P and Q are disjoint. For sequences f(n) and g(n), $n = 1, 2, \ldots$, the little-o notation f(n) = o(g(n)) means that $\lim_{n\to\infty} f(n)/g(n) \to 0$, see [23, p. 61].

B. Bounds for Entropy and I-divergence

We state several results that we need below. The shorthand d_1 refers to $d_1(P,Q)$.

Lemma 1 ([22, Ch. 2.6]). $\mathbb{D}(P||Q) \ge 0$ and $\mathbb{X}(P||Q) \ge \mathbb{H}(P)$, both with equality if and only if P = Q.

Lemma 2 ([22, Ch. 2.7]). $\mathbb{D}(P||Q)$ is convex in the pmfpair (P,Q). $\mathbb{X}(P||Q)$ is linear in P and convex in Q. $\mathbb{H}(P)$ is concave in P.

Lemma 3 ([22, Sec. 17.3]). If $d_1 \le 1/2$ then

$$|\mathbb{H}(P) - \mathbb{H}(Q)| \le -d_1 \log_2 \frac{d_1}{|\mathcal{A}|}.$$
(13)

Lemma 4. Let p_{\min} and p_{\max} be the respective minimum and maximum probabilities of P. If $d_1 \leq 2p_{\min}$ then

$$\mathbb{H}(P) - \mathbb{H}(Q) \le \frac{d_1}{2} \log_2 \frac{p_{\max} + d_1/2}{p_{\min} - d_1/2}.$$
 (14)

Proof. See Appendix A and [11, Lemma 1].

Lemma 5 (Pinsker Inequalities [22, Ch. 11], [24, Eq. (23)]).

$$\frac{1}{2\ln 2} d_1^2 \le \mathbb{D}\left(P \| Q\right) \le \frac{1}{q_{\min} \ln 2} d_1^2 \tag{15}$$

where $q_{\min} = \min_{a \in \text{supp}(P)} Q(a)$. For instance, the righthand side of (15) is ∞ if Q(a) = 0 when P(a) > 0.

We remark that one advantage of Lemma 4 over Lemmas 3 and 5 is that its bound is effectively linear in d_1 for small d_1 rather than behaving as $-d_1 \log d_1$ or d_1^2 .

C. Probability Bounds for Sums and Sets

The following Lemma specializes a result of [25] to iid discrete random variables.

Lemma 6 (Hoeffding [25, Thm. 1]). Consider the iid string A^n of random variables where $A := A_1$ has pmf P with alphabet A. Let $S_n = \frac{1}{n} \sum_{i=1}^n f(A_i)$ for a real-valued function f satisfying $0 \le f(a) \le 1$ for all $a \in A$. We have

$$\Pr\left[\mathsf{S}_n - \mathbb{E}_P\left[f(\mathsf{A})\right] \ge t\right] \le e^{-2nt^2} \tag{16}$$

for $t \geq 0$.

The next lemma gives basic bounds for typical strings.

Lemma 7 ([21, Ch. 2.4]). Consider the pmf P with alphabet \mathcal{A} . Let $a^n \in \mathcal{T}_{\epsilon}(P)$ and let $p_{\min} = \min_{a \in \operatorname{supp}(\mathcal{A})} P(a)$. We have

$$1 - \delta_{\epsilon} \le P\left(\mathcal{T}_{\epsilon}(P)\right) \le 1 \tag{17}$$

$$2^{-n\mathbb{H}(P)(1+\epsilon)} \le P^n(a^n) \le 2^{-n\mathbb{H}(P)(1-\epsilon)} \tag{18}$$

$$(1 - \delta_{\epsilon})2^{n\mathbb{H}(P)(1-\epsilon)} \le |\mathcal{T}_{\epsilon}(P)| \le 2^{n\mathbb{H}(P)(1+\epsilon)}$$
(19)

where $\delta_{\epsilon} = 2|\mathcal{A}| \exp\left(-2n p_{\min}^2 \epsilon^2\right)$.

Proof. The left-hand side of (17) follows by Lemma 6 and the union bound, the bounds (18) by the definition of typical strings, and the bounds (19) by (17) and (18).

D. Bounds for Binomial and Multinomial Coefficients

We state several results for binomial coefficients.

Lemma 8 (see [26, p. 166]). For a non-negative integer k and a positive integer n with k < n we have

$$\sum_{i=0}^{k} \binom{n}{i} \left(\frac{n}{2} - i\right) = \frac{k+1}{2} \binom{n}{k+1} = \frac{n-k}{2} \binom{n}{k}.$$
 (20)

Lemma 9. For 0 we have [27, p. 530]

$$\frac{2^{nh(p)}}{\sqrt{8np(1-p)}} \le \binom{n}{np} \le \frac{2^{nh(p)}}{\sqrt{2\pi np(1-p)}}.$$
 (21)

Lemma 10. For $0 \le p = k/n < 1/2$ we have [28, Eq. (25)] (see also [11, Lemma 3])

$$\alpha \beta \binom{n}{np} \le \sum_{i=0}^{np} \binom{n}{i} \le \alpha \binom{n}{np}$$
(22)

where

$$\alpha = \frac{1 - p + 1/n}{1 - 2p + 1/n}, \quad \beta = \frac{(1 - 2p)^2}{(1 - 2p)^2 + 1/n}.$$
 (23)

Multinomial coefficients are written as

$$\binom{n}{n_1 \dots n_{|\mathcal{A}|}} = \frac{n!}{\prod_{i=1}^{|\mathcal{A}|} n_i!}$$
(24)

with the integers $0 \le n_i \le n$ and $n = \sum_{i=1}^{|\mathcal{A}|} n_i$. An analog of Lemma 9 is as follows.



Fig. 2: Transmission experiment.

Lemma 11. For the pmf $P = [p_1, \ldots, p_{|\mathcal{A}|}]$ and $0 < p_i = n_i/n < 1$ for all *i*, we have

$$\frac{2^{n\mathbb{H}(P)}}{\left[(8n)^{|\mathcal{A}|-1}\prod_{i=1}^{|\mathcal{A}|}p_i\right]^{1/2}} \leq \binom{n}{np_1\dots np_{|\mathcal{A}|}} \\
\leq \frac{2^{n\mathbb{H}(P)}}{\left[(2\pi n)^{|\mathcal{A}|-1}\prod_{i=1}^{|\mathcal{A}|}p_i\right]^{1/2}}.$$
(25)

Proof. Use the binomial expansion

$$\binom{n}{np_1\dots np_{|\mathcal{A}|}} = \prod_{i=1}^{|\mathcal{A}|-1} \binom{n\left(1-\sum_{j=1}^{i-1} p_j\right)}{np_i}$$
(26)

and apply the bounds (21) to (26).

III. MODEL AND REQUIREMENTS

Consider the model depicted in Fig. 2. The source generates a message W with pmf $P_W = U_K$. The information rate is

$$R_{\rm info} = \frac{\mathbb{H}(P_{\rm W})}{n} = \frac{\log_2 K}{n}.$$
 (27)

To permit randomization, the encoder is given a RNG that generates an index Z(w) with pmf $P_{Z|W}(\cdot|w)$ given W = w. For example, one may choose $Z(w) = A^n(w)$. We consider two types of RNGs, namely idealized RNGs and RNGs based on RCs. One can measure the RNG rates in two ways: with the average conditional entropy $\mathbb{H}(P_{Z|W})$ and with the number B_{rng} of RC bits. The resulting rates are

$$H_{\rm rng} = \frac{\mathbb{H}\left(P_{\rm Z|W}\right)}{n}, \quad R_{\rm rng} = \frac{B_{\rm rng}}{n}$$
(28)

and $H_{\text{mg}} \leq R_{\text{mg}}$ because Z(w) is a function of the RC bits for all w.

The encoder output is $A^n = f(W, Z)$ for some function f. The resolution quality is measured via the I-divergence

$$\mathbb{D}\left(P_{\mathsf{A}^n} \| Q_{\mathsf{A}}^n\right) \tag{29}$$

for a specified pmf Q_A . For instance, for the stealth problem a warden knows the target pmf Q_A , the code statistics P_{A^n} , and the RNG statistics $P_{Z|W}$. Given a^n the warden must decide whether a code word was transmitted or not. One can show [4], [5] that the best the warden can do is to guess if (29) vanishes with the block length n.

The problem requirements are thus as follows: the decoder must recover W without error, $\mathbb{D}(P_{A^n} || Q_A^n)$ must vanish with growing *n*, and R_{rng} must vanish with growing *n*. The rate R_{info} is said to be achievable if these requirements are met. We wish to maximize the achievable rate. In fact, these requirements are coupled, as shown below. For example, vanishing $\frac{1}{n}\mathbb{D}(P_{\mathsf{A}^n} \| Q_{\mathsf{A}}^n)$ and R_{rng} imply that $R_{\mathsf{info}} \to \mathbb{H}(Q_{\mathsf{A}})$.

Observe that if $Q_A = U_{|\mathcal{A}'|}$ for any $\mathcal{A}' \subseteq \mathcal{A}$ then one achieves zero I-divergence at maximal rate $R_{info} = \log_2 |\mathcal{A}'|$ without a RNG by choosing $K = |\mathcal{A}'|^n$ and putting out the $|\mathcal{A}'|$ -ary representation of w - 1. We hence focus on Q_A that are not uniformly distributed over any subset.

A. Rate Bounds

We use the bounding approach of [5, Sec. 1.3.3]. The linearity of cross entropy gives

$$\mathbb{X}\left(P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n}\right) = n \,\mathbb{X}\left(\bar{P}_{\mathsf{A}} \| Q_{\mathsf{A}}\right) \tag{30}$$

where $\bar{P}_{A} = \frac{1}{n} \sum_{i=1}^{n} P_{A_i}$ is the average letter pmf of A^n . Lemma 2 and (11) further give

$$\mathbb{H}(P_{\mathsf{A}^n}) \le \sum_{i=1}^n H(P_{\mathsf{A}_i}) \le n \,\mathbb{H}\left(\bar{P}_{\mathsf{A}}\right) \tag{31}$$

$$(P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n}) = n \mathbb{X} \left(\bar{P}_{\mathsf{A}} \| Q_{\mathsf{A}} \right) - \mathbb{H} \left(P_{\mathsf{A}^{n}} \right)$$

$$\geq n \mathbb{D} \left(\bar{P}_{\mathsf{A}} \| Q_{\mathsf{A}} \right).$$
(32)

We have the following lemmas.

 \mathbb{D}

Lemma 12. Vanishing $\frac{1}{n}\mathbb{D}(P_{\mathsf{A}^n} || Q_{\mathsf{A}}^n)$ requires

$$R_{info} \le \mathbb{H}\left(Q_{\mathsf{A}}\right). \tag{33}$$

Moreover, if the decoder can recover both the message W and the RNG index Z, then we have the stronger bound

$$R_{info} + H_{rng} \le \mathbb{H}\left(Q_{\mathsf{A}}\right). \tag{34}$$

Proof. Consider $0 \le \xi \le \frac{1}{2}$ and

$$\frac{1}{n}\mathbb{D}\left(P_{\mathsf{A}^{n}}\|Q_{\mathsf{A}}^{n}\right) \leq \frac{\xi^{2}}{2\ln 2}.$$
(35)

The bound (32) and Lemmas 3 and 5 give

$$\left|\mathbb{H}\left(\bar{P}_{\mathsf{A}}\right) - \mathbb{H}\left(Q_{\mathsf{A}}\right)\right| \leq -\xi \log_{2} \frac{\xi}{|\mathcal{A}|}$$
(36)

and therefore

$$\mathbb{H}\left(\bar{P}_{\mathsf{A}}\right) \leq \mathbb{H}\left(Q_{\mathsf{A}}\right) - \xi \, \log_2 \frac{\xi}{|\mathcal{A}|}.\tag{37}$$

We further have $\mathbb{H}(P_W) \leq \mathbb{H}(P_{A^n})$ since W is a function of A^n . Combining this bound with (31) and (37) proves (33) for $\xi \to 0$. To prove (34), note that if W and Z are functions of A^n then $\mathbb{H}(P_{WZ}) \leq \mathbb{H}(P_{A^n})$.

A reverse bound to (34) holds more generally.

Lemma 13. Vanishing $\frac{1}{n}\mathbb{D}(P_{\mathsf{A}^n}||Q_{\mathsf{A}}^n)$ requires

$$R_{info} + H_{rng} \ge \mathbb{H}\left(Q_{\mathsf{A}}\right). \tag{38}$$

Proof. Consider the bound (35). We have

$$R_{\text{info}} + H_{\text{rng}} \stackrel{(a)}{\geq} \frac{1}{n} \mathbb{H} \left(P_{\mathsf{A}^{n}} \right) + \left(\frac{1}{n} \mathbb{D} \left(P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n} \right) - \frac{\xi^{2}}{2 \ln 2} \right)$$

$$\stackrel{(b)}{\equiv} \mathbb{X} \left(\bar{P}_{\mathsf{A}} \| Q_{\mathsf{A}} \right) - \frac{\xi^{2}}{2 \ln 2}$$

$$\stackrel{(c)}{\geq} \mathbb{H} \left(\bar{P}_{\mathsf{A}} \right) - \frac{\xi^{2}}{2 \ln 2}$$

$$\stackrel{(d)}{\geq} \mathbb{H} \left(Q_{\mathsf{A}} \right) + \xi \log_{2} \frac{\xi}{|\mathcal{A}|} - \frac{\xi^{2}}{2 \ln 2}$$

$$(39)$$

where (a) follows because A^n is a function of W and Z and by hypothesis (35), (b) follows by (11) and (30), (c) follows by Lemma 1, and (d) follows by (36). Finally, let $\xi \to 0$.

Lemma 13 is valid for DM, RC, and for one-to-many mappings. For example, if $R_{info} \rightarrow 0$ as $n \rightarrow \infty$ then we asymptotically require $R_{rng} \geq H_{rng} \geq \mathbb{H}(Q_A)$. Finally, we remark that the inequalities (34) and (38) are usually strict for finite n and hence it is not clear whether ILD coding is possible.

B. Discussion

Sec. I reviews two approaches to approximate target pmfs, namely DM and RC. DM uses a one-to-one mapping which is a special case of the above model without a RNG. Vanishing normalized (or un-normalized) I-divergence thus implies that R_{info} is asymptotically *upper* bounded by $\mathbb{H}(Q_A)$, see (33). In fact, the I-divergence of the best binary DM grows as $\frac{1}{2}\log_2 n$ with n [11]. Applications of DM, such as probabilistic shaping for energy-efficient communication, usually require only vanishing *normalized* I-divergence. Algorithms for DM that have $R_{info} \to \mathbb{H}(Q_A)$ for large n include CCDM [7], [8] and shell mapping [9], [10], [29], [30].

RC uses a many-to-one mapping and the RC rate for vanishing I-divergence is asymptotically *lower* bounded by $\mathbb{H}(Q_A)$, see (39). To approach the lower bound, one can, e.g., apply random coding arguments [15], [31], interval algorithms [13], fixed-to-variable length codes [32], variable-to-fixed length codes [33], or fixed-to-fixed length codes [16], [34] such as polar codes [35], [36]. These algorithms use deterministic many-to-one mappings that are not invertible in general.

ILD coding uses a one-to-many mapping that combines DM and RC, see Fig. 1c. This is similar to randomized encoding which is a common tool in multi-user information theory, e.g., for RC/RNG and wiretap channels [37]. The differences between the approaches are subtle. In particular, we require zero error while the randomization for wiretap and other problems permits small error. Also, we must carefully design the DM encoder and RC because $\mathbb{D}(P_{A^n} || Q_A^n)$ should vanish.

IV. ENCODER DESIGN

An ILD encoder is a one-to-many mapping into disjoint sets, see Fig. 1c. All strings a^n assigned to message w are collected in the set S_w and we require $S_v \cap S_w = \emptyset$ for $v \neq w$. We denote the set of all strings under consideration as

$$S = \bigcup_{w} S_{w} \tag{40}$$

where $\{S_w : w = 1, ..., K\}$ partitions S. A basic choice for S is $\operatorname{supp}(Q_A)^n$.

A. Two-Step Encoding

Encoding involves two-steps. First, the message w chooses the set S_w . The encoder then requests an index $Z(w) = A^n(w)$ from the RNG $P_{Z|W}(\cdot|w)$ that uniquely identifies a string $a^n = f(w, z)$ from S_w . For this a^n , we have

$$P_{\mathsf{A}^n|\mathsf{W}}\left(a^n|w\right) = P_{\mathsf{Z}|\mathsf{W}}(z|w) \tag{41}$$

and for each $a^n \in \mathcal{S}_w$ we have

$$P_{\mathsf{A}^{n}}(a^{n}) = P_{\mathsf{A}^{n}W}(a^{n}, w) = \frac{1}{K} P_{\mathsf{A}^{n}|\mathsf{W}}(a^{n}|w).$$
(42)

Suppose that n and Q_A are given. Encoder design involves choosing the:

- number K of messages;
- code: the set S of strings;
- encoder map: sets S_w , $w = 1, \ldots, K$, that partition S;
- RNG: pmfs $P_{Z|W}(\cdot|w), w = 1, ..., K$.

The encoder output is $A^n = f(W, Z)$ and f is invertible, so we are in the case described for the bound (34) and with

$$\mathbb{H}(P_{\mathsf{A}^n}) = \mathbb{H}(P_{\mathsf{WZ}}) = n\left(R_{\mathsf{info}} + H_{\mathsf{rng}}\right).$$
(43)

One might, therefore, consider the transmission rate to be $R_{info} + H_{rng}$ rather than R_{info} . However, Z is non-uniform and generated by a many-to-one mapping in general so that one cannot necessarily recover the $B_{rng} = nR_{rng}$ bits that generate Z. Thus, we consider the information rate to be R_{info} . At the same time, the encoder does "share randomness" via Z.

B. Idealized RNG

Recall that $q_{\mathcal{S}} = Q_{\mathcal{A}}^n(\mathcal{S})$ and $Q_{\mathcal{A}|\mathcal{S}}^n(a^n) = Q_{\mathcal{A}}^n(a^n)/q_{\mathcal{S}}$ for $a^n \in \mathcal{S}$, see (2). We expand (29) by using (42) as follows:

$$\sum_{w=1}^{K} \sum_{\substack{a^n \in \mathcal{S}_w \cap \operatorname{supp}(P_{\mathsf{A}^n}) \\ W}} \frac{P_{\mathsf{A}^n | \mathsf{W}}(a^n | w)}{K} \log_2 \frac{\frac{1}{K} P_{\mathsf{A}^n | \mathsf{W}}(a^n | w)}{Q_{\mathsf{A}}^n(a^n)}$$
$$= \mathbb{D} \left(U_K \| [q_{\mathcal{S}_1}, \dots, q_{\mathcal{S}_K}] \right)$$
$$+ \sum_{w=1}^{K} \frac{1}{K} \mathbb{D} \left(P_{\mathsf{A}^n | \mathsf{W}}(\cdot | w) \| Q_{\mathsf{A} | \mathcal{S}_w}^n \right).$$
(44)

The effects of the two-step encoding are apparent in (44): the first term accounts for the choice of set S_w and the second term accounts for the RNG. We will study the I-divergences

$$\mathbb{D}\left(U_K \| [q_{\mathcal{S}_1}, \dots, q_{\mathcal{S}_K}]\right) \tag{45}$$

$$\mathbb{D}\left(P_{\mathsf{A}^{n}|\mathsf{W}}(\cdot|w)\|Q_{\mathsf{A}|\mathcal{S}_{w}}^{n}\right) \tag{46}$$

separately. The identity (44) and Lemma 1 give the following result.

Proposition 1. The encoder RNG with

$$P_{\mathsf{A}^n|\mathsf{W}}(\cdot|w) = Q^n_{\mathsf{A}|\mathcal{S}_w} \tag{47}$$

for all w gives the smallest I-divergence

$$\mathbb{D}\left(P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n}\right) = \mathbb{D}\left(U_{K} \| [q_{\mathcal{S}_{1}}, \dots, q_{\mathcal{S}_{K}}]\right).$$
(48)

Proposition 1 gives intuition on how to choose the partition $\{S_1, \ldots, S_K\}$: the pmf $[q_{S_1}, \ldots, q_{S_K}]$ should be close to uniform. Sec. VI develops algorithms that separate the a^n into approximately equally likely sets with respect to Q^n_A .

C. RNG via RC

The idealized RNG of (47) cannot be implemented in general. To approximate it, various authors have developed theory and algorithms for RCs with vanishing I-divergence (46), see Sec. III-B. We study fixed-to-fixed length encoders generated by Algorithm 2 in [16]. Consider the subset S_w and suppose we are given $nR_{\rm rng}$ independent and uniformly-distributed random bits. Proposition 4 in [16] specifies that if $|S_w| \leq 2^{nR_{\rm rng}}$ then fixed-to-fixed length encoding gives

$$\mathbb{D}\left(P_{\mathsf{A}^{n}|\mathsf{W}}(\cdot|w)\|Q_{\mathsf{A}|\mathcal{S}_{w}}^{n}\right) \leq \log_{2}\left(1 + \frac{|\mathcal{S}_{w}|}{2\,q_{\min}(w)\,2^{2nR_{\mathrm{mg}}}}\right)$$
$$\leq \frac{|\mathcal{S}_{w}|}{(2\ln 2)\,q_{\min}(w)\,2^{2nR_{\mathrm{mg}}}} \tag{49}$$

where $q_{\min}(w) = \min_{a^n \in \operatorname{supp}(P_{\mathsf{A}^n | \mathsf{W}}(\cdot | w))} Q^n_{\mathsf{A} | \mathcal{S}_w}(a^n).$

It remains to bound $|S_w|$ and $q_{\min}(w)$ and this is done in Theorem 2 and Appendix C below. The result is that R_{rng} can be made to vanish with growing n, and the I-divergence (46) can be made to decay exponentially in n for all $w = 1, \ldots, K$.

D. Code for Minimum I-divergence

We next consider code design for the I-divergence (45).

Proposition 2. The code $S = \text{supp}(Q_A)^n$ gives the smallest *I*-divergence (45).

Proof. Suppose $S \subseteq \operatorname{supp}(Q_A)^n$ so that $q_S = Q_A^n(S) < 1$. The encoder has sets S_w with probabilities q_{S_w} . Now assign the unassigned strings with positive probability to obtain new sets S'_w with probabilities $q_{S'_w}$ satisfying $q_{S'_w} \ge q_{S_w}$ and where at least one inequality is strict. We thus have

$$\mathbb{D}\left(U_{K}\|[q_{\mathcal{S}'_{1}},\ldots,q_{\mathcal{S}'_{K}}]\right) = \sum_{w} \frac{1}{K} \log \frac{1/K}{q_{\mathcal{S}'_{w}}}$$
$$< \mathbb{D}\left(U_{K}\|[q_{\mathcal{S}_{1}},\ldots,q_{\mathcal{S}_{K}}]\right). \quad (50)$$

Proposition 2 shows that one should use all strings with positive probability if an ideal RNG is available. Moreover, inflating S by strings outside $\operatorname{supp}(Q_A)^n$ does not change the I-divergence.

E. Code Empirical Distribution and I-divergence

The I-divergence (29) simplifies by applying (30) that one can interpret in terms of the code empirical pmf. Let

$$\bar{n}_i = \sum_{a^n \in \mathcal{S}} P_{\mathsf{A}^n}(a^n) \, n_i(a^n) \tag{51}$$

be the average number of occurrences of letter i in S and define the code empirical pmf as

$$\bar{P} = \sum_{a^n \in \mathcal{S}} P_{\mathsf{A}^n}(a^n) \, \pi_{a^n} = \frac{1}{n} [\bar{n}_1, \dots, \bar{n}_{|\mathcal{A}|}].$$
(52)

Using (11) and (30), we have (see (32))

$$\mathbb{D}\left(P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n}\right) = n \mathbb{X}\left(\bar{P} \| Q_{\mathsf{A}}\right) - \mathbb{H}\left(P_{\mathsf{A}^{n}}\right).$$
(53)

We next use (53) to analyze the performance of DMs.

V. DISTRIBUTION MATCHING

This section generalizes results of [11] to non-binary alphabets. Recall that DM is a special case of the model in Sec. III where (46) is zero because there is no RNG. Furthermore, from (44) and (53) we have

$$\mathbb{D} (P_{\mathsf{A}^{n}} \| Q_{\mathsf{A}}^{n}) = \mathbb{D} (U_{K} \| Q_{\mathsf{A}}^{n})$$

= $n \mathbb{X} (\bar{P} \| Q_{\mathsf{A}}) - \log_{2} K$
= $n (\mathbb{X} (\bar{P} \| Q_{\mathsf{A}}) - R_{info})$
= $n (\mathbb{H} (\bar{P}) + \mathbb{D} (\bar{P} \| Q_{\mathsf{A}}) - R_{info}).$ (54)

A. CCDM Performance

Consider the target pmf $Q_A = [q_1, \ldots, q_{|\mathcal{A}|}]$ and a CCDM where all a^n have the empirical pmf $P = [p_1, \ldots, p_{|\mathcal{A}|}]$ where np_i is an integer for all *i*. We clearly have $\overline{P} = P$ and

$$K = \binom{n}{np_1 \dots np_{|\mathcal{A}|}}, \quad Q^n_{\mathsf{A}}(a^n) = \prod_{i=1}^{|\mathcal{A}|} q_i^{np_i} \qquad (55)$$

for all $a^n \in \mathcal{S}$ and the rate is

$$R_{\rm info} = \frac{1}{n} \log_2 \binom{n}{np_1 \dots np_{|\mathcal{A}|}}.$$
 (56)

The bounds (25) imply

$$\frac{|\mathcal{A}| - 1}{2n} \log_2(2\pi n c) \le \mathbb{H}(P) - R_{\text{info}} \le \frac{|\mathcal{A}| - 1}{2n} \log_2(8nc)$$
(57)

where

$$c = \left(\prod_{i=1}^{|\mathcal{A}|} p_i\right)^{1/(|\mathcal{A}|-1)}$$
(58)

and hence $R_{info} \to \mathbb{H}(P)$ for large *n*. Moreover, we obtain $\mathbb{H}(P) \to \mathbb{H}(Q_A)$ by choosing *P* appropriately. For example, Algorithm 1 of [16] gives a *P* with $d_1(P, Q_A) \le |\mathcal{A}|/(2n)$ (for $|\mathcal{A}| = 2$, we obtain $p = \lfloor nq \rfloor/n$). Lemma 3 (or Lemma 4) gives the desired rate but by combining (54) and (57) we have

$$\frac{|\mathcal{A}| - 1}{2} \log_2(2\pi n c) + \mathbb{D}(P || Q_{\mathsf{A}})$$

$$\leq \mathbb{D}(P_{\mathsf{A}^n} || Q_{\mathsf{A}}^n) \leq \frac{|\mathcal{A}| - 1}{2} \log_2(8nc) + \mathbb{D}(P || Q_{\mathsf{A}}).$$
(59)

The I-divergence thus grows as $\frac{1}{2}(|\mathcal{A}| - 1)\log n$ with n if $\mathbb{D}(P||Q_{\mathsf{A}}) \to 0$ or $\mathbb{H}(P) \to \mathbb{H}(Q_{\mathsf{A}})$.

B. Improving CCDM

We consider two classes of pmfs for which the CCDM prelog factor $\frac{1}{2}(|\mathcal{A}| - 1)$ is suboptimal. 1) Product Distributions: Suppose the target pmf Q_A splits into a product of pmfs:

$$Q_{\mathsf{A}}(a) = Q_{\mathsf{A}}(f(a', a'')) = Q_{\mathsf{A}'}(a')Q_{\mathsf{A}''}(a'')$$
(60)

where f is an invertible function.

Example 1. Consider $A = [A_1, A_2]$ where A_1 and A_2 are independent with pmfs [p, 1-p] and [q, 1-q], respectively. The 4-ary pmf is [pq, p(1-q), (1-p)q, (1-p)(1-q)].

For pmfs (60) one can use product distribution matching (PDM) [38]–[40, Sec. III] that operates two or more component DMs in parallel. The I-divergence (54) is then the sum of the I-divergences of the component DMs, e.g., the PDM prelog factor for Example 1 is $\frac{1}{2} + \frac{1}{2} = 1$ while a 4-ary CCDM has the pre-log factor $\frac{3}{2}$.

2) Unique Probabilities: The best DM for the target pmf $Q_A = U_{|\mathcal{A}|}$ has zero I-divergence by putting out the $|\mathcal{A}|$ -ary representation of w-1. We extend this observation to sources where Q_A and the empirical pmfs P have the following form. Let $\mathcal{U} = \{1, \ldots, |\mathcal{U}|\}$ enumerate the unique probabilities in

$$P = [\underbrace{p_1, \dots, p_1}_{\nu_1 \text{ times}}, \dots, \underbrace{p_{|\mathcal{U}|}, \dots, p_{|\mathcal{U}|}}_{\nu_{|\mathcal{U}|} \text{ times}}]$$
(61)

where $p_j \neq p_k$ for $j \neq k$. The entropy is

$$\mathbb{H}(P) = \mathbb{H}\left([r_1, \dots, r_{|\mathcal{U}|}]\right) + \sum_{j=1}^{|\mathcal{U}|} r_j \log_2 \nu_j \qquad (62)$$

where $r_j = \nu_j p_j$ for $j = 1, \ldots, |\mathcal{U}|$.

The key step now is as follows. Consider a^n with empirical pmf P, and consider the nr_j positions where there are letters with empirical probability p_j . For these positions, we expand the CCDM set S to include all a^n with any of the $\nu_j^{r_j n}$ patterns of ν_j letters. These new strings all have the same probability $Q^n_A(a^n)$. The new DM again has $\bar{P} = P$ but now

$$K = \binom{n}{nr_1 \dots nr_{|\mathcal{U}|}} \cdot \prod_{j=1}^{|\mathcal{U}|} \nu_j^{r_j n}, \quad Q_{\mathsf{A}}^n(a^n) = \prod_{j=1}^{|\mathcal{U}|} q_j^{nr_j}$$
(63)

for all $a^n \in \mathcal{S}$ and therefore

$$R_{\text{info}} = \frac{1}{n} \log_2 \left(\frac{n}{nr_1 \dots nr_{|\mathcal{U}|}} \right) + \sum_{j=1}^{|\mathcal{U}|} r_j \log_2 \nu_j.$$
(64)

Equations (57) and (58) are therefore updated as follows:

$$\frac{|\mathcal{U}| - 1}{2n} \log_2(2\pi n \, c) \le \mathbb{H}\left(P\right) - R_{\text{info}} \le \frac{|\mathcal{U}| - 1}{2n} \log_2(8nc) \tag{65}$$

where

$$c = \left(\prod_{j=1}^{|\mathcal{U}|} r_j\right)^{1/(|\mathcal{U}|-1)}.$$
(66)

Hence we again have $R_{info} \to \mathbb{H}(P)$ for large n and we can make $\mathbb{H}(P) \to \mathbb{H}(Q_A)$ by choosing P appropriately as for CCDM. Using the same approach as for (59) we further obtain

$$\frac{|\mathcal{U}| - 1}{2} \log_2(2\pi n c) + \mathbb{D} \left(P \| Q_{\mathsf{A}} \right)$$

$$\leq \mathbb{D} \left(P_{\mathsf{A}^n} \| Q_{\mathsf{A}}^n \right) \leq \frac{|\mathcal{U}| - 1}{2} \log_2(8nc) + \mathbb{D} \left(P \| Q_{\mathsf{A}} \right). \quad (67)$$

The I-divergence now grows as $\frac{1}{2}(|\mathcal{U}|-1)\log n$ with n rather than $\frac{1}{2}(|\mathcal{A}|-1)\log n$ if $\mathbb{D}(P||Q_{\mathsf{A}}) \to 0$ or $\mathbb{H}(P) \to \mathbb{H}(Q_{\mathsf{A}})$.

Example 2. Consider the pmf $Q_A = [0.6, 0.2, 0.2]$ and strings of length 5. The code S has all strings with empirical pmfs [3, 2, 0]/5, [3, 1, 1]/5, and [3, 0, 2]/5. We compute

$$K = \binom{5}{3,2} \cdot 2^2 = 40.$$
 (68)

The code size of the corresponding CCDM is instead

$$\binom{5}{3,1,1} = 20. \tag{69}$$

For large *n*, the bounds (59) show that the I-divergence of the new DM scales as $\log_2 n$ rather than $\frac{3}{2} \log_2 n$ as for CCDM.

C. Optimal DM Codes

The following result generalizes [11, Lemma 5] to nonbinary alphabets.

Proposition 3. The DM code S that minimizes $\mathbb{D}(U_K || Q_A^n)$ has all a^n with at least a specified probability with respect to Q_A^n , i.e., S has all a^n satisfying $Q_A^n(a^n) \ge 2^{-nI}$ for some I. Alternatively, S has all strings a^n satisfying

$$\frac{1}{n}\iota_{Q_{\mathsf{A}}^{n}}\left(a^{n}\right) = \mathbb{X}\left(\pi_{a^{n}} \| Q_{\mathsf{A}}\right) \le I.$$
(70)

Proof. Consider some values \hat{I} and I with $\hat{I} < I$. Define $S = S' \cup S''$ where $S' = \{a^n : \mathbb{X}(\pi_{a^n} || Q_A) \leq \hat{I}\}$ and S'' has ℓ strings a^n with $\mathbb{X}(\pi_{a^n} || Q_A) = I$. We thus have $K = |S| = |S'| + \ell$ and

$$\mathbb{D}\left(U_{K} \| Q_{\mathsf{A}}^{n}\right) = -\log_{2}(|\mathcal{S}'| + \ell) + \frac{n|\mathcal{S}'|}{|\mathcal{S}'| + \ell}\bar{I} + \frac{n\ell}{|\mathcal{S}'| + \ell}I$$
(71)

where $\bar{I} = \frac{1}{|S'|} \sum_{a^n \in S'} \mathbb{X}(\pi_{a^n} || Q_A)$ and therefore $\bar{I} < I$. Now consider ℓ as a continuous variable and compute

$$\frac{\partial}{\partial \ell} \mathbb{D} \left(U_K \| Q_A^n \right) = -\frac{1}{(\ln 2)(|\mathcal{S}'| + \ell)} + \frac{n|\mathcal{S}'|}{(|\mathcal{S}'| + \ell)^2} \Delta I \quad (72)$$

$$\frac{\partial^2}{\partial \ell} \mathbb{P} \left(U_K \| Q_A^n \right) = \frac{1}{(\ln 2)(|\mathcal{S}'| + \ell)} + \frac{n|\mathcal{S}'|}{(|\mathcal{S}'| + \ell)^2} \Delta I \quad (72)$$

$$\frac{\partial}{\partial \ell^2} \mathbb{D}\left(U_K \| Q_\mathsf{A}^n\right) = \frac{1}{(\ln 2)(|\mathcal{S}'| + \ell)^2} - \frac{2\hbar|\mathcal{S}|}{(|\mathcal{S}'| + \ell)^3} \Delta I \quad (73)$$

with $\Delta I = I - \overline{I} > 0$. The first derivative is zero only at

$$\ell_0 = |\mathcal{S}'| \left((\ln 2)n\Delta I - 1 \right) \tag{74}$$

which means that there is only one extreme point. Note that ℓ_0 can be negative but is larger than -|S'|. The second derivative at $\ell = \ell_0$ evaluates to

$$\frac{\partial^2}{\partial \ell^2} \mathbb{D}\left(U_K \| Q_{\mathsf{A}}^n\right) \Big|_{\ell=\ell_0} = -\frac{1}{(\ln 2)^3 |\mathcal{S}'|^2 n^2 \Delta I^2}$$
(75)



Fig. 3: I-divergence (45) vs. |S| for n = 4 and various q.

which is negative and therefore the I-divergence (assuming ℓ is continuous) is maximum at $\ell = \ell_0$.

We now find the integer $\hat{\ell} \in \{0, 1, \dots, \ell_{\max}\}$ that minimizes $\mathbb{D}(U_K || Q_A^n)$ where ℓ_{\max} is the number of length *n* code strings that have cross entropy *I*. We distinguish three cases.

- $\ell_0 \in [0, \ell_{\max}]$: $\mathbb{D}(U_K || Q_A^n)$ increases with ℓ for $0 \le \ell \le \ell_0$ and decreases with ℓ for $\ell_0 \le \ell \le \ell_{\max}$.
- $\ell_0 < 0$: $\mathbb{D}(U_K || Q_A^n)$ decreases with ℓ for $0 \le \ell \le \ell_{\max}$.
- $\ell_0 > \ell_{\max}$: $\mathbb{D}(U_K || Q_A^n)$ increases with ℓ for $0 \le \ell \le \ell_{\max}$.

In all cases we have $\hat{\ell} = 0$ or $\hat{\ell} = \ell_{\text{max}}$. Thus, the best code has all strings up to cross entropy \hat{I} or I.

Proposition 3 is certainly not obvious, e.g., it implies that optimal DM codes have *all* strings of any empirical pmf that they contain.

Example 3. For $|\mathcal{A}| = 2$ there are only n + 1 possible optimal codes although K ranges from 1 to 2^n . Fig. 3 shows the I-divergence behavior for a binary alphabet, block length n = 4, and various q. The minimal I-divergence is achieved at one of the n + 1 = 5 values K = 1, 5, 11, 15, 16.

Proposition 3 helps to prove the following basic result for binary strings.

Theorem 1 ([11]). Binary DM codes and encoders that minimize the I-divergence have $\mathbb{D}(P_{A^n} || Q_A^n)$ that grows as $\frac{1}{2} \log_2 n$ with n. Moreover, for binary alphabets CCDM achieves this growth.

1

VI. MLF AND LLF ALGORITHMS

Since DM cannot achieve low divergence in general, we now study one-to-many mappings. We propose two algorithms that generate encoder sets S_w , w = 1, ..., K. Consider a code S and initialize $S_w = \emptyset$ for all w. Order the strings in S from the most likely to the least likely. We consider two greedy approaches to populate the S_w :

• *Most-Likely First (MLF):* Successively insert the *most-likely* string into the set that has accumulated the least probability.

Algorithm 1 MLF and LLF Algorithms

1: **procedure** PARTITION($S, K, Q_A, Algo$) 2: $\mathcal{S}_w \leftarrow \emptyset, \quad w = 1, \dots, K$ Sort $\mathcal{S} = \{a_1^n, \dots, a_{|\mathcal{S}|}^n\}$ so that $Q_{\mathsf{A}}^n(a_i^n) \geq Q_{\mathsf{A}}^n(a_i^n)$ 3: for $i \leq j$ while $S \neq \emptyset$ do 4: if Algo = MLF then 5: $a^n \leftarrow \text{first_string}(\mathcal{S})$ 6: 7: else $a^n \leftarrow \text{last_string}(\mathcal{S})$ 8: end if 9: $w \leftarrow \arg\min_k Q^n_{\mathsf{A}}(\mathcal{S}_k)$ 10: $\mathcal{S}_w \leftarrow \mathcal{S}_w \cup \{a^n\}$ 11: $\mathcal{S} \leftarrow \mathcal{S} \setminus \{a^n\}$ 12: 13: end while return $\mathcal{S}_1, \ldots, \mathcal{S}_K$ 14: 15: end procedure

• *Least-Likely First (LLF):* Successively insert the *least*-likely string into the set that has accumulated the least probability.

The MLF and LLF approaches are specified in Algorithm 1 where the choice of algorithm is reflected in steps 5 to 9.

A. One Bit of Information per String

Consider transmitting one bit of information so that (45) is $\mathbb{D}(U_2 || [q_{S_1}, q_{S_2}])$. The MLF and LLF algorithms are not optimal in general.

Example 4. Suppose the string probabilities are $[\frac{3}{10}, \frac{2}{10}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}]$. Both algorithms arrive at the set probabilities $[q_{S_1}, q_{S_2}] = [11/20, 9/20]$ but here it is best to group the strings to obtain $[q_{S_1}, q_{S_2}] = [1/2, 1/2]$.

The LLF algorithm suggests a simple upper bound on $\mathbb{D}(U_2 \| [q_{S_1}, q_{S_2}])$. The worst case has both sets equally likely just before inserting the last (most probable) string. For $S = \text{supp}(Q_A)^n$, this string has *n* occurrences of the most probable letter(s), i.e., its probability is q_{\max}^n where q_{\max} is the largest probability of any letter. The worst case pmf is thus $[(1 + q_{\max}^n)/2, (1 - q_{\max}^n)/2]$ and we have

$$\mathbb{D}\left(U_{2}\|[q_{\mathcal{S}_{1}}, q_{\mathcal{S}_{2}}]\right) = \frac{1}{2}\log_{2}\frac{1}{1-q_{\max}^{2n}} \leq \frac{1}{2\ln 2}\frac{q_{\max}^{2n}}{1-q_{\max}^{2n}}$$
(76)

where the bound follows by $\ln(1+x) \leq x$. The relation (76) means that we can encode one bit of information with exponentially decreasing I-divergence in n. However, from (38) and $R_{\text{info}} = 1/n$ we find that the RNG rate must satisfy $R_{\text{rmg}} \geq H_{\text{rmg}} \geq \mathbb{H}(Q_{\text{A}}) - 1/n$.

B. MLF Encoder Properties

We first develop a special property of the MLF algorithm.

Definition 1 (Pareto-optimal sets). *The assignment of strings* to sets is Pareto-optimal if moving any individual string from one set to another does not decrease the I-divergence (45).

Proposition 4. The MLF algorithm generates Pareto-optimal sets. The LLF algorithm does not generate Pareto-optimal sets in general.

Proof. Consider first the LLF algorithm with the ordered string probabilities [0.8, 0.1, 0.1] and K = 2. LLF assigns the third and first strings to one set and the second string to the other. By moving the third string (i.e., the first string that LLF assigns) to the second set we obtain a better encoder.

Consider next the MLF algorithm. Moving a^n from S_v to S_w is Pareto efficient for (45) if and only if

$$\log_2 \frac{1}{(q_{\mathcal{S}_v} - Q^n_{\mathsf{A}}(a^n))(q_{\mathcal{S}_w} + Q^n_{\mathsf{A}}(a^n))} \le \log_2 \frac{1}{q_{\mathcal{S}_v} q_{\mathcal{S}_w}}$$
(77)

which is equivalent to $q_{S_v} - q_{S_w} \ge Q^n_A(a^n)$. Assuming $q_{S_v} > q_{S_w}$, the difference $q_{S_v} - q_{S_w}$ is at most the probability of the last string \tilde{a}^n that was assigned to S_v . Otherwise \tilde{a}^n would have been assigned to S_w . Therefore, moving \tilde{a}^n from S_v to S_w does not improve I-divergence (45). All other strings in S_v have at least the same probability as \tilde{a}^n .

Next, consider the code S and let p_i , i = 1, ..., |S|, be the probabilities of the ordered strings in S, i.e., we have $p_i \ge p_j$ for $i \le j$. Let Δ_i be the difference in probability of the most likely set and least likely set after the *i*th (most likely) string from S has been assigned to a message.

Lemma 14. *MLF* has $\Delta_i \leq p_1$ for all $i \geq 0$.

Proof. We have $\Delta_0 = 0 \leq p_1$ and proceed by induction. Suppose that $\Delta_{i-1} \leq p_1$ and $i \geq 1$.

Consider first the case $p_i \ge \Delta_{i-1}$ so that the set to which string *i* is assigned will have the most accumulated probability. We thus have $\Delta_i \le p_i$ with equality if the probability of the two least likely sets was the same before assigning string *i*. But then $\Delta_i \le p_1$ by the string ordering.

Consider next the case $p_i < \Delta_{i-1}$ so that the most likely set did not change. Now we have $\Delta_i \leq \Delta_{i-1}$ with equality if the two least likely sets was the same before assigning string *i*. But then we have $\Delta_i \leq p_1$ by the inductive hypothesis. \Box

C. LLF Encoder Properties

We begin with an observation concerning the LLF encoder.

Proposition 5. *LLF assigns the* (K-i)*-th string of the ordered list to the set* $S_{(i \mod K)+1}$.

Proof. At any step of the LLF Algorithm, the difference of the most likely set probability and the least likely set probability is at most the probability of the next string to assign. After the assignment, the least probable set becomes (one of) the most probable set(s). In case of a tie, we order the new set last among the most probable sets.

We remark that LLF lets the decoder calculate the position in the ordered list and apply a modulo operation on the list. Algorithms that can accomplish this task include enumerative source encoding [41] and shell mapping [9], [10].

Let Δ_i again be the difference in probability of the most likely set and least likely set after the *i*th (least likely) string from S has been assigned to a message.

Lemma 15. LLF has $\Delta_i \leq p_{K-i+1}$ for all $i \geq 0$. In particular, LLF has $\Delta_i \leq p_1$ for all $i \geq 0$.

Proof. Define $p_{K+1} = 0$. We have $\Delta_0 = 0 \leq p_{K+1}$ and proceed by induction. Suppose $\Delta_{i-1} \leq p_{K-i+2}$ and $i \geq 1$.

Consider first the case $p_{K-i+1} \geq \Delta_{i-1}$ so that the set to which string K - i + 1 is assigned will have the most accumulated probability. We thus have $\Delta_i \leq p_{K-i+1}$ with equality if the probability of the two least likely sets was the same before assigning string *i*.

Consider next the case $p_{K-i+1} < \Delta_{i-1}$ so that the most likely set did not change. Now we have $\Delta_i \leq \Delta_{i-1}$ with equality if the two least likely sets was the same before assigning string i. But then we have $\Delta_i \leq p_{K-i+1}$ by the string ordering.

D. Achievable Rates

We next analyze the information rate and I-divergence (45) of the MLF and LLF algorithms. This section treats binary alphabets for simplicity and Appendix C treats general alphabets. Suppose that $Q_{A}(1) = q = 1 - Q_{A}(0) < 1/2$.

To prove our main result in Theorem 2 below, we will use the code $\mathcal{S} = \mathcal{T}_{\epsilon}(Q_{A})$. However, to facilitate the development and to gain insight, consider first the code S of binary strings with at most n - k zeros. This means that the most likely string has probability $p_1 = q^k (1-q)^{n-k}$. We remark that Proposition 2 lets one reduce the I-divergence (45) by later assigning the remaining strings in $\operatorname{supp}(Q_A)^n$. The LLF algorithm fits naturally into this framework. The MLF algorithm does not fit, strictly speaking, because if we begin the MLF assignment with the strings with n - k zeros and later add the remaining strings in $supp(Q_A)^n$ then we do not have an MLF algorithm. We will see, however, that the remaining strings can have an accumulated probability that vanishes exponentially in n, so the distinction makes little difference.

Consider the code S as specified and the iid string A^n where A_1 has pmf Q_A . We may write

$$q_{\mathcal{S}} = \Pr\left[\mathsf{S}_n \le \frac{n-k}{n}\right] \tag{78}$$

where $S_n = \sum_{i=1}^n (1 - A_i)/n$. Lemma 6 with (k-1)/n < qgives the exponentially decaying bound

$$1 - q_{\mathcal{S}} = \Pr\left[\mathsf{S}_n - (1 - q) \ge q - \frac{k - 1}{n}\right]$$
$$\le \exp\left(-2n\left(q - \frac{k - 1}{n}\right)^2\right). \tag{79}$$

By Lemmas 14 and 15, we have $\Delta_{|S|} \leq q^k (1-q)^{n-k}$ and

$$K\left(\max_{w} q_{\mathcal{S}_{w}} - q^{k}(1-q)^{n-k}\right)$$

$$\leq q_{\mathcal{S}} \leq K\left(\min_{w} q_{\mathcal{S}_{w}} + q^{k}(1-q)^{n-k}\right).$$
(80)

We thus have

$$\frac{q_{\mathcal{S}}}{K} - q^k (1-q)^{n-k} \le q_{\mathcal{S}_w} \le \frac{q_{\mathcal{S}}}{K} + q^k (1-q)^{n-k}$$
(81)

for all $w = 1, \ldots, K$ and

$$\mathbb{D}\left(U_{K}\|[q_{S_{1}},\ldots,q_{S_{K}})\right] \leq \sum_{w} \frac{1}{K} \log_{2} \frac{1/K}{q_{S}/K - q^{k}(1-q)^{n-k}} = \log_{2} \frac{1}{1 - [(1-q_{S}) + Kq^{k}(1-q)^{n-k}]}$$
(82)

$$\stackrel{\scriptstyle \leftarrow}{\leq} 2\left[(1-q_{\mathcal{S}}) + Kq^k(1-q)^{n-k}\right] \tag{83}$$

where (a) follows if the term in square brackets is at most 1/2, since $-\log_2(1-x) \le 2x$ if $0 \le x \le 1/2$.

Consider the two summands in (83). We have already seen that the term $1 - q_S$ vanishes exponentially in n as long as q > (k-1)/n. In particular, neglecting quantization issues, we set $k = n(q - \epsilon)$ for small ϵ and (79) gives

$$1 - q_{\mathcal{S}} \le \exp\left(-2n\epsilon^2\right). \tag{84}$$

Next, consider a δ with $0 < \delta < 1$ and choose K so that

$$Kq^{k}(1-q)^{n-k} = (1-\delta)^{n}.$$
(85)

Taking logarithms and normalizing, one obtains

$$R_{\text{info}} = \mathbb{X} \left(Q_{\mathsf{A}} + \left[-\epsilon, \epsilon \right] \| Q_{\mathsf{A}} \right) + \log_2(1 - \delta).$$
(86)

We combine (83)-(86), choose small positive ϵ and δ , and choose n sufficiently large so that the term in square brackets in (83) is at most 1/2.

More generally, we have the following result for binary and non-binary alphabets, but with a different code $S = T_{\epsilon}(Q_{A})$. The reason for the change is to show that R_{rng} can be made to vanish with n.

Theorem 2. The MLF and LLF algorithms generate encoders with $R_{info} \to \mathbb{H}(Q_A)$, exponentially decaying $\mathbb{D}(P_{A^n} || Q_A^n)$, and $R_{rng} \to 0$ in *n* by using $S = \mathcal{T}_{\epsilon}(Q_A)$.

E. Discussion

The two key steps to show that the MLF and LLF algorithms have $R_{info} \to \mathbb{H}(Q_A)$ and $\mathbb{D}(P_{A^n} || Q_A^n) \to 0$ for large n are establishing that q_{S} is close to one, see (79), and that the $q_{\mathcal{S}_w}$ are close to 1/K, see (81). There are several codes and encoders that meet these requirements. For example, for binary alphabets one may choose the code above that has all strings with at most n - k zeros. Alternatively, one may use S = $\mathcal{T}_{\epsilon}(Q_{\mathsf{A}})$ as for Theorem 2. In both cases, one satisfies (81) by choosing any partition of S into K subsets with $q_{S_m} \approx 1/K$ for all w. The MLF and LLF algorithms are two methods to accomplish this task.

VII. I-DIVERGENCE LOWER BOUNDS

Consider the target pmf Q_A with q_{max} as the largest letter probability. Fig. 4 shows an example of K bins where $q_{\max}^n > 1/K$. The N_{\uparrow} blue bins have exactly one string whose probability is at least 1/K; the $N_{\downarrow} = K - N_{\uparrow}$ red bins have accumulated at most 1/K in probability and have one or more



Fig. 4: Example bins after after applying the MLF Algorithm. The bin heights represent the bin probabilities $q_{S_w} = Q_A^n(S_w)$.



Fig. 5: Equalized bins.

strings. Let the subsets S_{\uparrow} and S_{\downarrow} collect all strings of the blue and red bins, respectively.

Define the pmf

$$\bar{Q} = [\underbrace{q_{\uparrow}, \dots, q_{\uparrow}}_{N_{\uparrow} \text{ times}}, \underbrace{q_{\downarrow}, \dots, q_{\downarrow}}_{N_{\downarrow} \text{ times}}]$$
(87)

obtained by spreading the probability $q_{S\uparrow}$ equally over the $N\uparrow$ bins with $a^n \in S\uparrow$, and similarly for the $N\downarrow$ bins with $a^n \in S\downarrow$. The convexity of I-divergence (see Lemma 2) implies

$$\mathbb{D}\left(U_{K}\|[q_{\mathcal{S}_{1}},\ldots,q_{\mathcal{S}_{K}}]\right) \geq \mathbb{D}\left(U_{K}\|\bar{Q}\right).$$
(88)

We expand the right-hand side of (88) as

$$\mathbb{D}\left(U_{K}\|\bar{Q}\right) = \frac{N_{\uparrow}}{K}\log_{2}\frac{\frac{1}{K}}{q_{\uparrow}} + \frac{N_{\downarrow}}{K}\log_{2}\frac{\frac{1}{K}}{q_{\downarrow}}$$
$$= \mathbb{D}\left(\left[\frac{N_{\uparrow}}{K}, \frac{N_{\downarrow}}{K}\right]\|\left[q_{S_{\uparrow}}, q_{S_{\downarrow}}\right]\right).$$
(89)

In fact, it is not necessary to group the first N_{\uparrow} bins; any number N' of grouped sets with $N' \leq N_{\uparrow}$ and with accumulated probabilities $q'_{S_{\uparrow}}$, $q'_{S_{\uparrow}}$ works for the following result.

Theorem 3. The I-divergence (45) generated with one-tomany mappings into disjoint sets satsifies

$$\mathbb{D}\left(U_{K} \| [q_{\mathcal{S}_{1}}, \dots, q_{\mathcal{S}_{K}}]\right) \\ \geq \max_{N' \leq N_{\uparrow}} \mathbb{D}\left(\left[\frac{N'}{K}, 1 - \frac{N'}{K}\right] \| \left[q'_{\mathcal{S}_{\uparrow}}, q'_{\mathcal{S}_{\downarrow}}\right]\right).$$
(90)

A. Binary Alphabet

Consider $|\mathcal{A}| = 2$, $\mathcal{S} = \sup(Q_A)^n$, and q < 1/2. Let k be the maximum integer for which $q^k(1-q)^{n-k} \ge 1/K = 2^{-nR_{info}}$, or equivalently

$$k = \left\lfloor n \cdot \frac{\log_2(1-q) + R_{\text{info}}}{\log_2(1-q) - \log_2 q} \right\rfloor.$$
 (91)



Fig. 6: I-divergence (45) vs. R_{info} for $Q_A = [0.11, 0.89]$ and different block lengths n. Note that $\mathbb{H}(Q_A) \approx 0.5$.

We then have

$$N_{\uparrow} = \sum_{i=0}^{k} \binom{n}{i} \tag{92}$$

$$q_{\mathcal{S}_{\uparrow}} = \sum_{i=0}^{k} \binom{n}{i} q^{i} (1-q)^{n-i}.$$
(93)

Suppose $R_{info} = h(q)$ which implies $k = \lfloor nq \rfloor$ according to (91). We use Lemmas 9 and 10 to obtain

$$\frac{N_{\uparrow}}{K} \le \frac{1 - q + \frac{2}{n}}{1 - 2q + \frac{1}{n}} \cdot \frac{1}{\sqrt{2\pi nq(1 - q)}}$$
(94)

which decreases as $1/\sqrt{n}$ in n so that

$$\lim_{n \to \infty} \frac{N_{\uparrow}}{K} = 0.$$
(95)

For the probability $q_{S_{\uparrow}}$, observe that the median of a binomial distribution is either $\lfloor nq \rfloor$ or $\lceil nq \rceil$ so $q_{S_{\uparrow}}$ converges to 1/2. The I-divergence (89) thus evaluates to 1 for large n because the first pmf converges to [0,1] and the second pmf converges to [1/2, 1/2]. This means that the code cannot have I-divergence (45) below 1 bit for large n. For $R_{info} \leq -\log_2(1-q)$ the lower bound is zero because $S_{\uparrow} = \emptyset$.

VIII. NUMERICAL RESULTS

We evaluate the performance of the MLF and LLF algorithms with $S = \text{supp}(Q_A)^n$. Fig. 6 plots the I-divergence (45) against R_{info} , as well as an upper bound based on (82) and (86), and the lower bound (90). For all simulations, the target pmf is $Q_A = [0.11, 0.89]$. We evaluate the lower bound (90) for $N' = \sum_{i=0}^{k'} {n \choose i}$, where k' is integer and $k' \leq k$.

Note that the MLF and LLF algorithms sort binary strings of length n so their complexity grows exponentially in n. The

simulation results are restricted to string lengths with $n \le 20$. As a reference, we plot the I-divergence of the optimal DMs for n = 10 and n = 16. Observe that MLF outperforms LLF and has the same I-divergence as the lower bound for small rates.

IX. CONCLUSIONS AND OUTLOOK

We showed that ILD coding is possible at rates approaching the entropy of a target pmf with exponentially decaying I-divergence and vanishing RNG rate in the block length. The key step was to introduce invertible one-to-many mappings. For such mappings, an encoder was proposed that first chooses a subset of strings followed by an RNG that chooses a string from the subset. The first step uses subsets that are generated by either an MLF or LLF algorithm. The second step uses a good RC.

An interesting direction for future work is designing practical algorithms that approach the performance predicted by the theory.

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APPENDIX A Proof of Lemma 4

Consider the pmfs P and $Q = P + \Delta$ where $\sum_i \Delta(i) = 0$ and $\sum_i |\Delta(i)| = d_1$. Define

$$\Delta_{+} = \sum_{i: \Delta(i) > 0} \Delta(i), \quad \Delta_{-} = \sum_{i: \Delta(i) < 0} \Delta(i)$$
(96)

and observe that $\Delta_+ = -\Delta_- = d_1/2$. We expand

$$\mathbb{H}(P) - \mathbb{H}(Q)$$

= $-\mathbb{D}(P||Q) + \sum_{i} \Delta(i) \log_2(P(i) + \Delta(i))$ (97)

and using $p_{\min} \leq P(i) \leq p_{\max}$ and $|\Delta(i)| \leq d_1/2$ we have

$$\sum_{i:\Delta(i)>0} \Delta(i) \underbrace{\log_2(P(i) + \Delta(i))}_{\leq \log_2(p_{\max} + d_1/2)} \leq \frac{d_1}{2} \log_2(p_{\max} + d_1/2)$$
(98)

$$\sum_{i:\Delta(i)<0} \Delta(i) \underbrace{\log_2(P(i) + \Delta(i))}_{\geq \log_2(p_{\min} - d_1/2)} \leq -\frac{d_1}{2} \log_2(p_{\min} - d_1/2).$$
(99)

Now insert (98) and (99) into (97) and apply Lemma 1.

APPENDIX B PROOF OF THEOREM 1

This appendix reviews results on binary DM from [11]. Consider $\mathcal{A} = \{0, 1\}$ and observe that Proposition 3 lets one restrict attention to the n + 1 code books \mathcal{S} consisting of all strings with weight at most k for $0 \le k \le n$. We have

$$K = |\mathcal{S}| = \sum_{i=0}^{k} \binom{n}{i}.$$

The fraction of 1s in S is

$$\bar{p} = \frac{\sum_{i=0}^{k} \binom{n}{i}i}{|\mathcal{S}|n} \tag{100}$$

which increases monotonically in k and reaches its maximum $\bar{p} = 1/2$ for k = n. Let p = k/n. The next lemma shows that $\bar{p} \rightarrow p$ for large n as long as p < 1/2.

Lemma 16. For every positive integer n and every integer k, $0 \le k < n/2$, we have

$$0 \le \frac{k}{n} - \bar{p} \le \frac{1 - k/n}{n(1 - 2k/n)} + \frac{1}{2n^2(1 - 2k/n)^2}.$$
 (101)

Proof. The lower bound is trivial. For the upper bound, we use Lemma 8 to write

$$\bar{p} = \frac{1}{2} - \frac{\sum_{i=0}^{k} \binom{n}{i} (\frac{n}{2} - i)}{\sum_{j=0}^{k} \binom{n}{j} n}$$

$$= \frac{1}{2} - \frac{\frac{n-k}{2} \binom{n}{k}}{\sum_{j=0}^{k} \binom{n}{j} n}$$

$$= \frac{1}{2} - \left(\frac{1}{2} - \frac{k}{2n}\right) \frac{\binom{n}{k}}{\sum_{j=0}^{k} \binom{n}{j}}.$$
(102)

Let p = k/n and insert the lower bound in (22) to obtain

$$\bar{p} \ge \frac{1}{2} - \left(\frac{1}{2} - \frac{p}{2}\right) \frac{1 - 2p + 1/n}{1 - p + 1/n} \left(1 + \frac{1}{n(1 - 2p)^2}\right)$$
$$\ge \frac{1}{2} - \frac{1 - 2p + 1/n}{2} \left(1 + \frac{1}{n(1 - 2p)^2}\right)$$
$$= p - \frac{1 - p}{n(1 - 2p)} - \frac{1}{2n^2(1 - 2p)^2}$$
(103)

which establishes the upper bound of (101).

Let $\overline{P} = [\overline{p}, 1 - \overline{p}]$ and $Q_A = [q, 1 - q]$ where 0 < q < 1/2. Recall that (54) gives

$$\mathbb{D}\left(U_{K} \| Q_{\mathsf{A}}^{n}\right) = n \mathbb{X}\left(\bar{P} \| Q_{\mathsf{A}}\right) - \log_{2} |\mathcal{S}|.$$
(104)

For small n, the best k may have $k \ge n/2$. For example, Fig. 3 shows that k = 4 gives the maximum $R_{info} = 1$ and the minimum $\mathbb{D}(U_K || Q_A^n)$ for q = 0.23 and n = 4. However, the following lemma shows that $k \ge n/2$ is not interesting for large n.

Lemma 17. $\mathbb{D}(U_K || Q_A^n)$ grows linearly with n if

$$\frac{k}{n} > p_1 := \frac{1 + \log_2(1-q)}{-\log_2 q + \log_2(1-q)}$$
(105)

and p_1 satisfies $q < p_1 < 1/2$.

Proof. As already stated, \bar{p} increases with p = k/n. Now choose p so that $\bar{p} = p_1$ so that $\mathbb{X}(\bar{P}||Q_A) = 1$. For this p and large n, we have $\frac{1}{n}\log_2 |\mathcal{S}| < h(p_1) < 1$ and the I-divergence (104) grows linearly with n. Increasing p further gives $\mathbb{X}(\bar{P}||Q_A) > 1$ and (104) also grows linearly with n. Moreover, if $p_1 then Lemma 16 shows that <math>\bar{p} \rightarrow p$ and (104) grows linearly in n. The bounds $q < p_1 < 1/2$ follow by using 1 > h(q) and by showing that p_1 increases with q to $p_1 = 1/2$ when q = 1/2.

Recall that CCDM achieves $\frac{1}{2}\log_2 n$ growth, see Sec. V-A. Lemma 17 thus implies that we can focus on $k < np_1 < n/2$ for large n. We remark that the bounds (34) and (38) imply that for $\frac{1}{n}\mathbb{D}(U_K||Q_A) \to 0$ we must have $\frac{1}{n}\log_2|\mathcal{S}| \to h(q)$ and therefore $p \to q$ for large n.

Now for p < 1/2, we obtain the following bounds from (21) and (22):

$$|\mathcal{S}| \le \binom{n}{np} \frac{1-p}{1-2p} \le \frac{2^{nh(p)}}{\sqrt{2\pi np(1-p)}} \cdot \frac{1-p}{1-2p}.$$
 (106)

Inserting into (104), we have

$$\mathbb{D}\left(U_{K} \| Q_{\mathsf{A}}^{n}\right) \geq \frac{1}{2} \log_{2} n - n \left[h(p) - h(\bar{p})\right] + n \mathbb{D}\left(\bar{P} \| Q_{\mathsf{A}}\right) - \frac{1}{2} \log_{2} \frac{1 - p}{2\pi p(1 - 2p)^{2}}.$$
(107)

Define

$$\epsilon(n) = \frac{1-p}{n(1-2p)} + \frac{1}{2n^2(1-2p)^2}.$$
 (108)

For sufficiently large n, Lemmas 4 and 16 give

$$n [h(p) - h(\bar{p})] \le \left(\frac{1-p}{1-2p} + \frac{1}{2n(1-2p)^2}\right) \log_2 \frac{1-p+\epsilon(n)}{p-\epsilon(n)}.$$
 (109)

Since $\epsilon(n) \to 0$ for $n \to \infty$, we have

$$\liminf_{n \to \infty} \left(\mathbb{D} \left(U_K \| Q_A^n \right) - \frac{1}{2} \log_2 n - n \mathbb{D} \left([p, 1-p] \| Q_A \right) \right)$$

$$\geq -\frac{1-p}{1-2p} \log_2 \frac{1-p}{p} - \frac{1}{2} \log_2 \frac{1-p}{2\pi p(1-2p)^2}.$$
(110)

The I-divergence thus grows at least as $\frac{1}{2}\log_2 n$ with n. Moreover, CCDM achieves this growth by choosing $p = \lfloor nq \rfloor/n$ so that $d_1([p, 1-p], Q_A) \le 1/n$ and $n\mathbb{D}([p, 1-p] || Q_A) \to 0$ for large n, see Sec. V-A. Note that $R_{info} \to h(q)$ for large n.

APPENDIX C Proof of Theorem 2

This appendix extends the analysis of Sec. VI-D to nonbinary discrete alphabets. The key steps are to choose a code S with probability close to one and to show that all subset probabilities q_{S_w} are close to 1/K.

Consider the code $S = T_{\epsilon}(Q_A)$. The left-hand side of (17) in Lemma 7 gives

$$1 - q_{\mathcal{S}} \le 2|\mathcal{A}| \exp\left(-2n \, q_{\min}^2 \, \epsilon^2\right). \tag{111}$$

By Lemmas 14 and 15, we have $\Delta_{|S|} \leq Q^n_A(a^n)$ for the a^n with the largest probability in the typical set. For this a^n , we can bound (see (81))

$$\frac{q_{\mathcal{S}}}{K} - Q_{\mathsf{A}}^n(a^n) \le q_{\mathcal{S}_w} \le \frac{q_{\mathcal{S}}}{K} + Q_{\mathsf{A}}^n(a^n).$$
(112)

Following the same steps as in (83), we have

$$\mathbb{D}\left(U_{K}\|\left[q_{\mathcal{S}_{1}},\ldots,q_{\mathcal{S}_{K}}\right]\right) \leq 2\left[\left(1-q_{\mathcal{S}}\right)+KQ_{\mathsf{A}}^{n}(a^{n})\right] \quad (113)$$

if the scalar in square brackets is at most 1/2.

Consider the two summands in (113). We have already seen that the term $1-q_S$ vanishes exponentially in n. Next, consider a δ with $0 < \delta < 1$ and choose K so that

$$KQ^{n}_{\mathsf{A}}(a^{n}) = (1-\delta)^{n}.$$
 (114)

Taking logarithms and normalizing, we have

$$\frac{1}{n}\log_2 K = -\frac{1}{n}\log_2 Q_{\mathsf{A}}^n(a^n) + \log_2(1-\delta)$$

$$\geq \mathbb{H}\left(Q_{\mathsf{A}}\right)(1-\epsilon) + \log_2(1-\delta) \tag{115}$$

where the inequality follows by the right-hand side of (18) in Lemma 7. We thus choose

$$R_{\rm info} = \mathbb{H}\left(Q_{\rm A}\right)\left(1-\epsilon\right) - \gamma \tag{116}$$

where $\gamma = -\log_2(1 - \delta)$, and (113)-(115) guarantee that this rate gives vanishing I-divergence (113). Note that the term in square brackets in (113) is less than 1/2 for large *n*. Finally, choose small positive ϵ and δ and large *n* to complete the first part of the proof.

Next, consider the RNG and the bound (49). The bounds (18) in Lemma 7 and $q_{S_w} = \sum_{a^n \in S_w} Q^n_A(a^n)$ give

$$q_{\mathcal{S}_w} 2^{n \mathbb{H}(Q_{\mathsf{A}})(1-\epsilon)} \le |\mathcal{S}_w| \le q_{\mathcal{S}_w} 2^{n \mathbb{H}(Q_{\mathsf{A}})(1+\epsilon)}.$$
 (117)

The left-hand side of (18) also gives

$$q_{\min}(w) \ge \frac{2^{-n\mathbb{H}(Q_{\mathsf{A}})(1+\epsilon)}}{q_{\mathcal{S}_w}}.$$
(118)

Inserting (117) and (118) into (49), we have

$$\mathbb{D}\left(P_{\mathsf{A}^{n}|\mathsf{W}}(\cdot|w)\|Q_{\mathsf{A}|\mathcal{S}_{w}}^{n}\right) \leq \frac{q_{\mathcal{S}_{w}}^{2}2^{2n\mathbb{H}(Q_{\mathsf{A}})(1+\epsilon)}}{(2\ln 2)2^{2nR_{\mathsf{mg}}}}$$

$$\stackrel{(a)}{\leq} \frac{2^{4n\epsilon\mathbb{H}(Q_{\mathsf{A}})+2n\gamma+2}}{(2\ln 2)2^{2nR_{\mathsf{mg}}}} \qquad (119)$$

where step (a) follows by applying (112), the right-hand side of (18), and (115) to bound

$$q_{\mathcal{S}_w} \leq \frac{1}{K} + 2^{-n\mathbb{H}(Q_{\mathsf{A}})(1-\epsilon)}$$
$$\leq 2 \cdot 2^{-n\mathbb{H}(Q_{\mathsf{A}})+n\epsilon\mathbb{H}(Q_{\mathsf{A}})+n\gamma}.$$
(120)

We may thus choose

$$R_{\rm rng} = 2\epsilon \mathbb{H}\left(Q_{\rm A}\right) + 2\gamma \tag{121}$$

which may vanish with n because we can choose ϵ and γ to vanish with n. Note that the rate (121) suffices for each w, i.e., one need not average over w to achieve small R_{rng} .

Finally, both the I-divergences on the left-hand sides of (113) and (119) decay exponentially with n if R_{info} and R_{rng} are given by (116) and (121), respectively. This implies hat $\mathbb{D}(P_{A^n} || Q_A^n)$ decays exponentially with n.