# QUATERNARY LINEAR CODES AND RELATED BINARY SUBFIELD CODES 

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#### Abstract

In this paper, we mainly study quaternary linear codes and their binary subfield codes. First we obtain a general explicit relationship between quaternary linear codes and their binary subfield codes in terms of generator matrices and defining sets. Second, we construct quaternary linear codes via simplicial complexes and determine the weight distributions of these codes. Third, the weight distributions of the binary subfield codes of these quaternary codes are also computed by employing the general characterization. Furthermore, we present two infinite families of optimal linear codes with respect to the Griesmer Bound, and a class of binary almost optimal codes with respect to the Sphere Packing Bound. We also need to emphasize that we obtain at least 9 new quaternary linear codes.


## 1. Introduction

Let $\mathbb{F}_{q^{m}}$ be the finite field with $q^{m}$ elements, where $q$ is a power of a prime and $m$ is a positive integer. Given an $[n, k]$ linear code $\mathcal{C}$ over $\mathbb{F}_{q^{m}}$, Ding and Heng [4] recently constructed a new linear code $\mathcal{C}^{(q)}$ over $\mathbb{F}_{q}$ with respect to $\mathcal{C}$, which is called a subfield code. In the paper, the authors mainly developed the general theory of subfield codes, investigated subfield codes of two families of ovoid codes, and presented some new and optimal subfield codes. After that, there have been literature on subfield codes of combinatorial codes ( $[7,9]$ ); MDS codes ([8, 16]), and some other linear codes ([10, 17, 18, 21, 22]). We record a table here (Table 1) for the convenience of the reader. In the table, we list some optimal linear subfield codes with respect to the Griesmer Bound or the Sphere Packing Bound.

Based on the generic construction for linear codes, Hyun et al. [14 constructed some infinite families of binary optimal linear codes by choosing the defining set as the complement of some simplicial complexes. A more general situation was considered by Hyun et al. [13] by using posets, and they presented some optimal and minimal binary linear codes not satisfying the condition of Ashikhmin-Barg [1]. Recently, Zhu and Wei [23] constructed quaternary linear codes via simplicial complexes and presented an infinite family of minimal optimal quaternary linear codes with respect to the Griesmer bound.

[^0]Table 1. Optimal linear subfield codes

| Reference | $q$-Ary | [ $n, k, d]$ Code | \#Weight | Bound | Result |
| :---: | :---: | :---: | :---: | :---: | :---: |
| [4] | $q$-ary | $\left[q^{2}+1,4, q^{2}-q\right]$ | 2 | Griesmer bound | Thm.1.1 |
| [7] | binary | $\left[2^{m}+2,2^{m}-m, 4\right]$ |  | Sphere Packing | Thm. 11 |
|  | $p$-ary | $\left[p^{m}+1, p^{m}-2 m, 4\right]$ |  | Sphere Packing | Thm. 16 |
| [8] | binary | $\left[2^{m}+1,2^{m}-m, 3\right]$ |  | Sphere Packing | Thm.VII. 4 |
|  | $p$-ary | $\left[p^{m}+1, p^{m}+1-2 m, 3\right]$ |  | Sphere Packing | Thm.VI. 7 |
|  |  | $\left[p^{m}+1, p^{m}-m, 3\right]$ |  | Sphere Packing | Thm.V. 1 |
|  |  | $\left[p^{m}+1, m+1, p^{m-1}(p-1)\right]$ | 3 | Griesmer bound |  |
| [9] | binary | $\begin{gathered} {\left[2^{m+s}+2^{s}-2^{m}, 2^{m+s}+2^{s}-2^{m}-\right.} \\ 2 m-2-m, 4] \end{gathered}$ |  | Sphere Packing | Thm. 12 |
|  |  | $\left[2^{m}+2,2^{m}-2 m, 6\right]$ |  | Sphere Packing | Thm. 18 |
| [10] | $q$-ary | $\left[q^{2}-1,4, q^{2}-q-2\right]$ | 5 | Griesmer bound | Thm. 6 |
|  |  | $\left[q^{2}-1, q^{2}-5,4\right]$ |  | Sphere Packing |  |
|  |  | $\left[q^{2}, 4, q^{2}-q-1\right]$ | 4 | Griesmer bound | Thm. 9 |
|  |  | $\left[q^{2}, q^{2}-4,4\right]$ |  | Sphere Packing |  |
| [21] | binary | $\left[2^{2 m}+2,2 m+1,2^{2 m-1}\right]$ | 4 | Griesmer bound | Thm. 1 |
|  |  | $\left[2^{2 m}+2,2^{2 m}-2 m+1,3\right]$ |  | Sphere Packing |  |
|  |  | $\left[2^{2 m}+1,2 m+1,2^{2 m-1}\right]$ | 3 | Griesmer bound | Thm. 2 |
|  |  | $\left[2^{2 m}+1,2^{2 m}-2 m, 3\right]$ |  | Sphere Packing |  |

From the argument of Xiang in [20] and the generic construction for linear codes, each linear code can be expressed as the defining code $\mathcal{C}_{D}$ with a defining set $D$. Due to the key role in estimating the error-correcting capability of codes, weight distributions is an important research topic in coding theory. Motivated by the above work, we ask the following questions:

1) What is the relationship between quaternary linear codes and their binary subfield codes? Namely for the generic construction of linear codes, what is the relationship between defining sets of quaternary linear codes and the binary subfield codes?
2) Can we obtain more optimal quaternary linear codes and determine their weight distributions?
3) Can we obtain more optimal the binary subfield linear codes of quaternary linear codes and determine their weight distributions?

The basic questions above are the major motivation of this paper. First, we find a direct approach to deal with the relationship between quaternary linear codes and their binary subfield codes. Second, we will follow the idea in [14, and use simplicial complexes to construct quaternary linear codes. Weight distributions of these quaternary codes are determined when those simplicial complexes are generated by one or two maximal elements. Third, we also compute weight distributions of the binary subfields code of these quaternary codes. In addition, we present several classes of
optimal and almost optimal linear codes and some examples of linear codes with optimal parameters. By Magma, we obtain at least 9 new quaternary linear codes.

The rest of this paper is organized as follows. In Section 2, we recall some bounds on linear codes, some concepts of simplicial complexes, and generating functions. In Section 3, we will deal with the relationship between quaternary linear codes and their binary subfield codes. We will compute the weight distributions of some quaternary codes and their binary subfield codes in Sections 4 and 5. Furthermore, we obtain two classes of optimal linear codes and a class of almost optimal binary codes. In Section 6, we present some new quaternary linear codes and conclude the paper.

## 2. Preliminaries

### 2.1. Two bounds of linear codes.

Let $\mathcal{C}$ be an $[n, k, d]$ linear code over $\mathbb{F}_{q}$. Assume that there are $A_{i}$ codewords in $\mathcal{C}$ with Hamming weight $i$ for $1 \leq i \leq n$. Then $\mathcal{C}$ has weight distribution $\left(1, A_{1}, \ldots, A_{n}\right)$ and weight enumerator $1+A_{1} z+\cdots+A_{n} z^{n}$. Moreover, if the number of nonzero $A_{i}$ 's in the sequence $\left(A_{1}, \ldots, A_{n}\right)$ is exactly equal to $t$, then the code is called $t$-weight. The $[n, k, d]$ code $\mathcal{C}$ is called distance optimal if there is no $[n, k, d+1]$ code (that is, this code has the largest minimum distance for given length $n$ and dimension $k$ ), and it is called almost optimal if an $[n, k, d+1]$ code is distance optimal (refer to [11, Chapter 2]).

Next we recall two well-known bounds on linear codes.
Lemma 2.1. (Griesmer Bound [6]) For a given $[n, k, d]$ linear code over $\mathbb{F}_{q}$, there is a bound as follows:

$$
\sum_{i=0}^{k-1}\left\lceil\frac{d}{q^{i}}\right\rceil \leq n
$$

where $\lceil\cdot\rceil$ is the ceiling function.
We say that a linear code is a Griesmer code if it meets the Griesmer bound with equality. One can verify that Griesmer codes are distance optimal.

Lemma 2.2. (Sphere Packing Bound [11]) For a given $[n, k, d]$ linear code over $\mathbb{F}_{q}$, there is a bound as follows:

$$
\sum_{i=0}^{\left\lfloor\frac{d-1}{2}\right\rfloor}\binom{n}{i}(q-1)^{i} \leq q^{n-k}
$$

where $\lfloor\cdot\rfloor$ is the floor function.
When we have a code for which equality in the above bound is ture, the code is called perfect. One can verify that perfect codes are also distance optimal.

### 2.2. The generic construction of linear codes.

Let $m$ be a positive integer, $q$ be a prime power, and $\left(V_{m}, \cdot\right)$ be an $m$-dimensional vector space over $\mathbb{F}_{q}$, where $\cdot$ denotes an inner product on $V_{m}$. For a linear code of length $n$ over $\mathbb{F}_{q}$, there is a generic construction as follows:

$$
\begin{equation*}
\mathcal{C}_{D}=\left\{\left(x \cdot d_{1}, x \cdot d_{2}, \ldots, x \cdot d_{n}\right): x \in V_{m}\right\} \tag{2.1}
\end{equation*}
$$

where $D=\left\{d_{1}, \ldots, d_{n}\right\} \subseteq V_{m}$. The set $D$ is called the defining set of the code $\mathcal{C}_{D}$. If the set $D$ is properly chosen, the code $\mathcal{C}_{D}$ may have good parameters. The following two situations are common:
(1) When $V_{m}=\mathbb{F}_{q^{m}}, x \cdot y=\operatorname{Tr}_{q^{m} / q}(x y)$ for $x, y \in \mathbb{F}_{q^{m}}$ and $\operatorname{Tr}_{q^{m} / q}$ is the trace function from $\mathbb{F}_{q^{m}}$ to $\mathbb{F}_{q}$. In this case, the corresponding code $\mathcal{C}_{D}$ in Equation (2.1) is called a trace code over $\mathbb{F}_{q}$. This generic construction was first introduced by Ding et al. 3.
(2) When $V_{m}=\mathbb{F}_{q}^{m}, \mathbf{x} \cdot \mathbf{y}=\sum_{i=1}^{m} x_{i} y_{i}$ for $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right), \mathbf{y}=\left(y_{1}, \ldots, y_{m}\right) \in \mathbb{F}_{q}^{m}$. This standard construction in Equation (2.1) can be also found in [11.

### 2.3. Simplicial complexes and generating functions.

Let $\mathbb{F}_{2}$ be the finite field with two elements. Assume that $m$ is a positive integer. The support $\operatorname{supp}(\mathbf{v})$ of a vector $\mathbf{v} \in \mathbb{F}_{2}^{m}$ is defined by the set of nonzero coordinates. The Hamming weight $w t(\mathbf{v})$ of $\mathbf{v} \in \mathbb{F}_{2}^{m}$ is defined by the size of $\operatorname{supp}(\mathbf{v})$. For two subsets $A, B \subseteq[m]$, the set $\{x: x \in A$ and $x \notin B\}$ and the number of elements in the set $A$ are denoted by $A \backslash B$ and $|A|$, respectively.

For two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_{2}^{m}$, we say $\mathbf{v} \subseteq \mathbf{u}$ if $\operatorname{supp}(\mathbf{v}) \subseteq \operatorname{supp}(\mathbf{u})$. We say that a family $\Delta \subseteq \mathbb{F}_{2}^{m}$ is a simplicial complex if $\mathbf{u} \in \Delta$ and $\mathbf{v} \subseteq \mathbf{u}$ imply $\mathbf{v} \in \Delta$. For a simplicial complex $\Delta$, a maximal element of $\Delta$ is one that is not properly contained in any other element of $\Delta$. Let $\mathcal{F}=\left\{F_{1}, \ldots, F_{l}\right\}$ be the family of maximal elements of $\Delta$. For each $F \subseteq[m]$, the simplicial complex $\Delta_{F}$ generated by $F$ is defined to be the family of all subsets of $F$.

Let $X$ be a subset of $\mathbb{F}_{2}^{m}$. Hyun et al. [2] introduced the following $m$-variable generating function associated with the set $X$ :

$$
\mathcal{H}_{X}\left(x_{1}, x_{2}, \ldots, x_{m}\right)=\sum_{\mathbf{u} \in X} \prod_{i=1}^{m} x_{i}^{u_{i}} \in \mathbb{Z}\left[x_{1}, x_{2}, \ldots, x_{m}\right]
$$

where $\mathbf{u}=\left(u_{1}, u_{2}, \ldots, u_{m}\right) \in \mathbb{F}_{2}^{m}$ and $\mathbb{Z}$ is the ring of integers.
The following lemma plays an important role in determining the weight distributions of the quaternary codes defined in Equation (2.1).

Lemma 2.3. [2, Theorem 1] Let $\Delta$ be a simplicial complex of $\mathbb{F}_{2}^{m}$ with the set of maximal elements $\mathcal{F}$. Then

$$
\mathcal{H}_{\Delta}\left(x_{1}, x_{2}, \ldots, x_{m}\right)=\sum_{\emptyset \neq S \subseteq \mathcal{F}}(-1)^{|S|+1} \prod_{i \in \cap S}\left(1+x_{i}\right)
$$

where $\cap S$ denotes the intersection of all elements in $S$. In particular, we also have

$$
|\Delta|=\sum_{\emptyset \neq S \subseteq \mathcal{F}}(-1)^{|S|+1} 2^{|\cap S|}
$$

There is a bijection between $\mathbb{F}_{2}^{m}$ and $2^{[m]}$ being the power set of $[m]=\{1, \ldots, m\}$, defined by $\mathbf{v} \mapsto \operatorname{supp}(\mathbf{v})$. Throughout this paper, we will identify a vector in $\mathbb{F}_{2}^{m}$ with its support.

Example 2.4. Let $\Delta$ be a simplicial complex of $\mathbb{F}_{2}^{4}$ with the set of maximal elements $\mathcal{F}=\{(1,1,0,0),(0,1,1,0),(0,0,1,1)\}$. Then
$\mathcal{H}_{\Delta}\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\prod_{i \in\{1,2\}}\left(1+x_{i}\right)+\prod_{i \in\{3,4\}}\left(1+x_{i}\right)+\prod_{i \in\{2,3\}}\left(1+x_{i}\right)-\left(1+x_{2}\right)-\left(1+x_{3}\right)$
and $|\Delta|=8$.

## 3. Relationship between quaternary codes and the subfield codes

For the finite field $\mathbb{F}_{4}$, as we known $\mathbb{F}_{4} \cong \mathbb{F}_{2}[x] /\left\langle x^{2}+x+1\right\rangle$, where $x^{2}+x+1$ is the only irreducible polynomial of degree two in $\mathbb{F}_{2}[x]$. Let $w$ be an element in some extend field of $\mathbb{F}_{2}$ such that $w^{2}+w+1=0$. Then $\mathbb{F}_{4}=\mathbb{F}_{2}(w)$ and for each $u \in \mathbb{F}_{4}$ there is a unique representation $u=a+w b$, where $a, b \in \mathbb{F}_{2}$. Let $m$ be a positive integer, and $\mathbb{F}_{4}^{m}$ be the set of $m$-tuples over $\mathbb{F}_{4}$. Any vector $\mathbf{x} \in \mathbb{F}_{4}^{m}$ can be written as $\mathbf{x}=\mathbf{a}+w \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{F}_{2}^{m}$.

From the argument of Xiang in [20], any quaternary linear code of length $n$ can be also expressed as the code $\mathcal{C}_{D}$ in Equation (2.1), where $D=\left\{\mathbf{d}_{1}, \mathbf{d}_{2}, \ldots, \mathbf{d}_{n}\right\} \subseteq \mathbb{F}_{4}^{m}$ and $m$ is some positive integer.

The following result plays an important role in the research of the subfield codes.
Lemma 3.1. [4, Theorem 2.4] Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{F}_{q^{m}}$ with generator matrix

$$
G=\left(\begin{array}{cccc}
g_{11} & g_{12} & \ldots & g_{1 n} \\
g_{21} & g_{22} & \ldots & g_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
g_{k 1} & g_{k 3} & \ldots & g_{k n}
\end{array}\right)
$$

Let $\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m}\right\}$ be a basis of $\mathbb{F}_{q^{m}}$ over $\mathbb{F}_{q}$. Then the subfield code $\mathcal{C}^{(q)}$ with respect to $\mathcal{C}$ has a generator matrix

$$
G^{(q)}=\left(\begin{array}{c}
G_{1}^{(q)} \\
G_{2}^{(q)} \\
\vdots \\
G_{k}^{(q)}
\end{array}\right)
$$

where each $G_{i}^{(q)}$ is defined as

$$
\left(\begin{array}{cccc}
\operatorname{Tr}_{q^{m} / q}\left(g_{i 1} \alpha_{1}\right) & \operatorname{Tr}_{q^{m} / q}\left(g_{i 2} \alpha_{1}\right) & \ldots & \operatorname{Tr}_{q^{m} / q}\left(g_{i n} \alpha_{1}\right) \\
\operatorname{Tr}_{q^{m} / q}\left(g_{i 1} \alpha_{2}\right) & \operatorname{Tr}_{q^{m} / q}\left(g_{i 2} \alpha_{2}\right) & \ldots & \operatorname{Tr}_{q^{m} / q}\left(g_{i n} \alpha_{2}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\operatorname{Tr}_{q^{m} / q}\left(g_{i 1} \alpha_{m}\right) & \operatorname{Tr}_{q^{m} / q}\left(g_{i 2} \alpha_{m}\right) & \ldots & \operatorname{Tr}_{q^{m} / q}\left(g_{i n} \alpha_{m}\right)
\end{array}\right) .
$$

In Lemma 3.1, let $q=2, m=2$ and $\left\{\alpha_{1}=1, \alpha_{2}=w\right\}$ be a basis of $\mathbb{F}_{4}$ over $\mathbb{F}_{2}$. Assume that $g_{i j}=g_{i j}^{(0)}+w g_{i j}^{(1)}$, where $g_{i j}^{(0)}, g_{i j}^{(1)} \in \mathbb{F}_{2}$. Hence $\operatorname{Tr}_{4 / 2}\left(g_{i j} \alpha_{1}\right)=g_{i j}^{(1)}$ and $\operatorname{Tr}_{4 / 2}\left(g_{i j} \alpha_{2}\right)=g_{i j}^{(0)}+g_{i j}^{(1)}$. Then we have the following theorem.

Theorem 3.2. Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{F}_{4}$ with generator matrix $G=$ $G_{1}+w G_{2}$, where $w \in \mathbb{F}_{4}$ with $w^{2}+w+1=0$ and $G_{1}, G_{2}$ are two matrices over $\mathbb{F}_{2}$. Then the binary subfield code $\mathcal{C}^{(2)}$ with respect to $\mathcal{C}$ has a generator matrix

$$
G^{(2)}=\binom{G_{2}}{G_{1}+G_{2}} .
$$

Moreover, if the quaternary code $\mathcal{C}$ has the defining set $D=D_{1}+w D_{2}$ with $D_{1}, D_{2} \subseteq$ $\mathbb{F}_{2}^{m}$, then the binary subfield code $\mathcal{C}^{(2)}$ with respect to $\mathcal{C}$ has defining set:

$$
D^{(2)}=\left\{\left(\mathbf{d}_{2}, \mathbf{d}_{1}+\mathbf{d}_{2}\right): \mathbf{d}_{1} \in D_{1}, \mathbf{d}_{2} \in D_{2}\right\} .
$$

Remark 3.3. There is a well-known the Plotkin construction, for linear codes from old codes which is documented in [15]. By Theorem 3.2, the subfield construction of quaternary codes includes the Plotkin construction.

We give the following example to illustrate Theorem 3.2.
Example 3.4. Let $\mathcal{C}$ be a $[4,2]$ linear code over $\mathbb{F}_{4}$ with defining set $D=D_{1}+w D_{2}$, where $D_{1}=\{(0,1),(1,0)\}$ and $D_{2}=\{(0,1),(1,1)\}$. Then its generator matrix is

$$
\left(\begin{array}{ccll}
w & 0 & 1+w & 1 \\
1+w & 1+w & w & w
\end{array}\right)=\left(\begin{array}{cccc}
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0
\end{array}\right)+w\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1
\end{array}\right) .
$$

By Lemma 3.1, the binary subfield code $\mathcal{C}^{(2)}$ with respect to $\mathcal{C}$ has a generator matrix

$$
G^{(2)}=\left(\begin{array}{cccc}
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}\right)
$$

By Theorem 3.2, $D^{(2)}=\{(1,1,1,0),(0,1,0,0),(1,1,0,1),(0,1,1,1)\}$.

## 4. Weight distributions of quaternary codes

In this section, we will construct some quaternary codes via simplicial complexes and determine their weight distributions.

Let $D_{1}, D_{2}$ be two subsets of $\mathbb{F}_{2}^{m}$ and $D=D_{1}+w D_{2} \subseteq \mathbb{F}_{4}^{m *}$, where $\mathbb{F}_{4}^{m *}$ is the set of non-zero element of $\mathbb{F}_{4}^{m}$ and $w \in \mathbb{F}_{4}$ such that $w^{2}+w+1=0$. We define a quaternary code as follows:

$$
\begin{equation*}
\mathcal{C}_{D}=\left\{c_{D}(\mathbf{a})=(\mathbf{a} \cdot \mathbf{d})_{\mathbf{d} \in D}: \mathbf{a} \in \mathbb{F}_{4}^{m}\right\} . \tag{4.1}
\end{equation*}
$$

First of all, from Equation (4.1), it is easy to check that the code $\mathcal{C}_{D}$ is a quaternary linear code. The length of the code $\mathcal{C}_{D}$ is $|D|$. If $\mathbf{a}=\mathbf{0}$, then the Hamming weight of the codeword $c_{D}(\mathbf{a})$ is equal to $\operatorname{wt}\left(c_{D}(\mathbf{a})\right)=0$. Next we assume that $\mathbf{a} \neq \mathbf{0}$. Suppose that $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}, \mathbf{d}=\mathbf{d}_{1}+w \mathbf{d}_{2}$, where $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{m}\right), \boldsymbol{\beta}=\left(\beta_{1}, \ldots, \beta_{m}\right) \in \mathbb{F}_{2}^{m}$, $\mathbf{d}_{1} \in D_{1}$, and $\mathbf{d}_{2} \in D_{2}$. Then

$$
\begin{aligned}
\mathrm{wt}\left(c_{D}(\mathbf{a})\right) & =\operatorname{wt}\left(\left((\boldsymbol{\alpha}+w \boldsymbol{\beta}) \cdot\left(\mathbf{d}_{1}+w \mathbf{d}_{2}\right)\right)_{\mathbf{d}_{1} \in D_{1}, \mathbf{d}_{2} \in D_{2}}\right) \\
& =\operatorname{wt}\left(\left(\boldsymbol{\alpha} \mathbf{d}_{1}+w\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta} \mathbf{d}_{1}\right)+w^{2} \boldsymbol{\beta} \mathbf{d}_{2}\right)_{\mathbf{d}_{1} \in D_{1}, \mathbf{d}_{2} \in D_{2}}\right) \\
& =\operatorname{wt}\left(\left(\boldsymbol{\alpha} \mathbf{d}_{1}+\boldsymbol{\beta} \mathbf{d}_{2}+w\left(\boldsymbol{\beta} \mathbf{d}_{2}+\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta} \mathbf{d}_{1}\right)\right)_{\mathbf{d}_{1} \in D_{1}, \mathbf{d}_{2} \in D_{2}}\right) .
\end{aligned}
$$

By the definition of Hamming weight of vector $\mathbf{x}=\mathbf{y}+w \mathbf{z} \in \mathbb{F}_{4}^{m}$ with $\mathbf{y}, \mathbf{z} \in \mathbb{F}_{2}^{m}$, $\mathrm{wt}(\mathbf{x})=0$ if and only if $\mathbf{y}=\mathbf{z}=\mathbf{0}$. Hence

$$
\begin{aligned}
\operatorname{wt}\left(c_{D}(\mathbf{a})\right) & =|D|-\sum_{\mathbf{d}_{1} \in D_{1}} \sum_{\mathbf{d}_{2} \in D_{2}}\left(\frac{1}{2} \sum_{y \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{1}+\boldsymbol{\beta} \mathbf{d}_{2}\right) y}\right)\left(\frac{1}{2} \sum_{z \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right) z}\right) \\
& =|D|-\frac{1}{4} \sum_{\mathbf{d}_{1} \in D_{1}} \sum_{\mathbf{d}_{2} \in D_{2}}\left(1+(-1)^{\boldsymbol{\alpha \mathbf { d } _ { 1 } + \boldsymbol { \beta }} \mathbf{d}_{2}}\right)\left(1+(-1)^{\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)}\right) \\
& =\frac{3}{4}|D|-\frac{1}{4}\left(\sum_{\mathbf{d}_{1} \in D_{1}}(-1)^{\alpha \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in D_{2}}(-1)^{\left.\boldsymbol{\beta \mathbf { d } _ { 2 }}\right)}\right. \\
& -\frac{1}{4}\left(\sum_{\mathbf{d}_{1} \in D_{1}}(-1)^{\beta \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in D_{2}}(-1)^{(\alpha+\beta) \mathbf{d}_{2}}\right) \\
& -\frac{1}{4}\left(\sum_{\mathbf{d}_{1} \in D_{1}}(-1)^{(\alpha+\boldsymbol{\beta}) \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in D_{2}}(-1)^{\left.\boldsymbol{\alpha \mathbf { d } _ { 2 }}\right) .}\right.
\end{aligned}
$$

For a subset $P$ of $\mathbb{F}_{2}^{m}$ and $\mathbf{u} \in \mathbb{F}_{2}^{m}$, let us define $\chi_{\mathbf{u}}(P)=\sum_{\mathbf{v} \in P}(-1)^{\mathbf{u v}}$. Then

$$
\begin{array}{r}
\operatorname{wt}\left(c_{D}(\mathbf{a})\right)=\frac{3}{4}|D| \\
-\frac{1}{4}\left[\chi_{\boldsymbol{\alpha}}\left(D_{1}\right) \chi_{\boldsymbol{\beta}}\left(D_{2}\right)+\chi_{\boldsymbol{\beta}}\left(D_{1}\right) \chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\left(D_{2}\right)+\chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\left(D_{1}\right) \chi_{\boldsymbol{\alpha}}\left(D_{2}\right)\right] . \tag{4.2}
\end{array}
$$

Let $D^{c}=\mathbb{F}_{4}^{m *} \backslash D$ and $\delta$ be the Kronecker delta function. Then

$$
\begin{array}{r}
\mathrm{wt}\left(c_{D^{c}}(\mathbf{a})\right)=\frac{3}{4}\left(\left|D^{c}\right|-2^{2 m} \delta_{\mathbf{0}, \mathbf{a}}\right) \\
+\frac{1}{4}\left[\chi_{\boldsymbol{\alpha}}\left(D_{1}\right) \chi_{\boldsymbol{\beta}}\left(D_{2}\right)+\chi_{\boldsymbol{\beta}}\left(D_{1}\right) \chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\left(D_{2}\right)+\chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}\left(D_{1}\right) \chi_{\boldsymbol{\alpha}}\left(D_{2}\right)\right] .
\end{array}
$$

By Equation (4.2), we have

$$
\begin{equation*}
\mathrm{wt}\left(c_{D^{c}}(\mathbf{a})\right)=3 \times 2^{2 m-2}\left(1-\delta_{\mathbf{0}, \mathbf{a}}\right)-\operatorname{wt}\left(c_{D}(\mathbf{a})\right) . \tag{4.3}
\end{equation*}
$$

Recall that there is a bijection between $\mathbb{F}_{2}^{m}$ and $2^{[m]}$ being the power set of $[m]=$ $\{1, \ldots, m\}$, defined by $\mathbf{v} \mapsto \operatorname{supp}(\mathbf{v})$. For a subset $D \subseteq \mathbb{F}_{4}^{m}$, we use $D^{*}$ to denote the set $D \backslash\{\mathbf{0}\}$.

From the proof of [13, Theorem 5.3], we derive the following lemma, which is need in computing weight distributions of the codes.

Lemma 4.1. [13] For two subsets $A, B$ of $[m]$, we set

$$
\begin{gathered}
\mathcal{U}_{1}=\left\{u \in \mathbb{F}_{2}^{m}: u \cap(A \cup B)=\emptyset\right\}, \\
\mathcal{U}_{2}=\left\{u \in \mathbb{F}_{2}^{m}: u \cap A=\emptyset, u \cap(B \backslash A) \neq \emptyset\right\}, \\
\mathcal{U}_{3}=\left\{u \in \mathbb{F}_{2}^{m}: u \cap B=\emptyset, u \cap(A \backslash B) \neq \emptyset\right\}, \\
\mathcal{U}_{4}=\left\{u \in \mathbb{F}_{2}^{m}: u \cap(A \backslash B) \neq \emptyset, u \cap(A \cap B)=\emptyset, u \cap(B \backslash A) \neq \emptyset\right\}, \\
\mathcal{U}_{5}=\left\{u \in \mathbb{F}_{2}^{m}: u \cap(A \cap B) \neq \emptyset\right\} .
\end{gathered}
$$

Then we have

$$
\begin{gathered}
\left|\mathcal{U}_{1}\right|=2^{m-|A \cup B|},\left|\mathcal{U}_{2}\right|=2^{m-|A|}-2^{m-|A \cup B|}=2^{m-|A \cup B|}\left(2^{|B \backslash A|}-1\right) \\
\left|\mathcal{U}_{3}\right|=2^{m-|B|}-2^{m-|A \cup B|}=2^{m-|A \cup B|}\left(2^{|A \backslash B|}-1\right) \\
\left|\mathcal{U}_{4}\right|=2^{m-|A \cup B|}\left(2^{|A \backslash B|}-1\right)\left(2^{|A \backslash B|}-1\right) \\
\left|\mathcal{U}_{5}\right|=2^{m-|A \cap B|}\left(2^{|A \cap B|}-1\right)
\end{gathered}
$$

Proposition 4.2. Let $A, B$ be two subsets of $[m]$ and $D=\Delta_{A}+w \Delta_{B} \subset \mathbb{F}_{4}^{m}$. Then $\mathcal{C}_{D^{*}}$ in Equation (4.1) is a $\left[2^{|A|+|B|}-1,|A \cup B|, 2^{|A|+|B|-1}\right]$ quaternary code and its weight distribution is presented in Table 2.

Table 2. Weight distribution of the code in Proposition 4.2

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{\|A\|+\|B\|-1}$ | $3\left(2^{\|A \backslash B\|+\|B \backslash A\|}-1\right)$ |
| $3 \times 2^{\|A\|+\|B\|-2}$ | $4^{\|A \cup B\|}-1-3\left(2^{\|A \backslash B\|+\|B \backslash A\|}-1\right)$ |

Proof. It is easy to check that the length of the code $\mathcal{C}_{D^{*}}$ is $\left|D^{*}\right|=2^{|A|+|B|}-1$. To compute the weights and frequencies, we need to introduce the following notation.

For $X$ a subset of $\mathbb{F}_{2}^{m}$, we use $\psi(\mathbf{u} \mid X)$ to denote a Boolean function in $m$-variable, and $\psi(\mathbf{u} \mid X)=1$ if and only if $\mathbf{u} \bigcap X=\emptyset$. For a vector $\mathbf{u}=\left(u_{1}, \ldots, u_{m}\right) \in \mathbb{F}_{2}^{m}$ and a nonempty simplicial complex $\Delta_{A}$, by Lemma 2.3 we have

$$
\begin{align*}
\sum_{\mathbf{x} \in \Delta_{A}}(-1)^{\mathbf{u} \cdot \mathbf{x}}=\mathcal{H}_{\Delta_{A}}\left((-1)^{u_{1}},\right. & \left.(-1)^{u_{2}}, \ldots,(-1)^{u_{m}}\right)=\prod_{i \in A}\left(1+(-1)^{u_{i}}\right) \\
& =\prod_{i \in A}\left(2-2 u_{i}\right)=2^{|A|} \prod_{i \in A}\left(1-u_{i}\right)=2^{|A|} \psi(\mathbf{u} \mid A) \tag{4.4}
\end{align*}
$$

Suppose that $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$. By Equations (4.2) and (4.4)

$$
\begin{aligned}
& \mathrm{wt}\left(c_{D^{*}}(\mathbf{a})\right)=\mathrm{wt}\left(c_{D}(\mathbf{a})\right)=\frac{3}{4}|D| \\
& -2^{|A|+|B|-2}[\psi(\boldsymbol{\alpha} \mid A) \psi(\boldsymbol{\beta} \mid B)+\psi(\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)+\psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha} \mid B)] .
\end{aligned}
$$

Let $T=\psi(\boldsymbol{\alpha} \mid A) \psi(\boldsymbol{\beta} \mid B)+\psi(\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)+\psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha} \mid B)$. We divide the proof into the following cases:
(1) $T=3$. In this case we have $\operatorname{wt}\left(c_{D^{*}}(\mathbf{a})\right)=0$ and

$$
\psi(\boldsymbol{\alpha} \mid A)=\psi(\boldsymbol{\alpha} \mid B)=\psi(\boldsymbol{\beta} \mid A)=\psi(\boldsymbol{\beta} \mid B)=\psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid A)=\psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)=1
$$

which is equivalent to

$$
\boldsymbol{\alpha} \cap(A \cup B)=\emptyset \text { and } \boldsymbol{\beta} \cap(A \cup B)=\emptyset .
$$

By Lemma 4.1, the number of such $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$ is $4^{m-|A \cup B|}$.
(2) $T=2$. Without loss of generality, suppose that $\psi(\boldsymbol{\alpha} \mid A) \psi(\boldsymbol{\beta} \mid B)=0$. We have

$$
\left\{\begin{array} { l } 
{ \psi ( \boldsymbol { \beta } | A ) \psi ( \boldsymbol { \alpha } + \boldsymbol { \beta } | B ) = 1 } \\
{ \psi ( \boldsymbol { \alpha } + \boldsymbol { \beta } | A ) \psi ( \boldsymbol { \alpha } | B ) = 1 }
\end{array} \Longleftrightarrow \left\{\begin{array}{l}
\boldsymbol{\beta} \cap A=(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap B=\emptyset \\
\boldsymbol{\alpha} \cap B=(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A=\emptyset
\end{array}\right.\right.
$$

We have $\boldsymbol{\alpha} \cap A \neq \emptyset$ or $\boldsymbol{\beta} \cap B \neq \emptyset$. Note that the support of the vector $\boldsymbol{\alpha}+\boldsymbol{\beta}$ is equal to $(\operatorname{supp}(\boldsymbol{\alpha}) \cup \operatorname{supp}(\boldsymbol{\beta})) \backslash(\operatorname{supp}(\boldsymbol{\alpha}) \cap \operatorname{supp}(\boldsymbol{\beta}))$. From $\boldsymbol{\alpha} \cap A \neq \emptyset$ and $\boldsymbol{\beta} \cap A=\emptyset$, we have $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A \neq \emptyset$, which is a contradiction with $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap A=\emptyset$. Similarly, we derive $(\boldsymbol{\alpha}+\boldsymbol{\beta}) \cap B \neq \emptyset$ from $\boldsymbol{\beta} \cap B \neq \emptyset$. Hence there is no $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$ such that $T=2$.
(3) $T=1$. In this case we have $\operatorname{wt}\left(c_{D^{*}}(\mathbf{a})\right)=2^{|A|+|B|-1}$. If $\boldsymbol{\alpha} \in \mathcal{U}_{1}$ in Lemma 4.1, then $T=1$ if and only if $\boldsymbol{\beta} \in \mathcal{U}_{2} \cup \mathcal{U}_{3}$. If $\boldsymbol{\alpha} \in \mathcal{U}_{2}$ and $\boldsymbol{\beta} \in \mathcal{U}_{1} \cup \mathcal{U}_{3}$, then $T=1$. If $\boldsymbol{\alpha} \in \mathcal{U}_{2}$ and $\boldsymbol{\beta} \in \mathcal{U}_{2}$, then $T=\psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)=1$ if and only if $\boldsymbol{\alpha}+\boldsymbol{\beta} \cap B=\emptyset$, which is equivalent to $\operatorname{supp}(\boldsymbol{\alpha}) \cap B=\operatorname{supp}(\boldsymbol{\beta}) \cap B$. If $\boldsymbol{\alpha} \in \mathcal{U}_{3}$ and $\boldsymbol{\beta} \in \mathcal{U}_{1}$, then $T=1$. If $\boldsymbol{\alpha} \in \mathcal{U}_{3}$ and $\boldsymbol{\beta} \in \mathcal{U}_{3} \cup \mathcal{U}_{4}$, then $T=1$ if and only if $\operatorname{supp}(\boldsymbol{\alpha}) \cap A=\operatorname{supp}(\boldsymbol{\beta}) \cap A$. If $\boldsymbol{\alpha} \in \mathcal{U}_{4}$ and $\boldsymbol{\beta} \in \mathcal{U}_{2}$, then $T=1$ if and only if $\operatorname{supp}(\boldsymbol{\alpha}) \cap B=\operatorname{supp}(\boldsymbol{\beta}) \cap B$. By Lemma 4.1, the number of such $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$ is

$$
3 \times 4^{m-|A \cup B|}\left(2^{|A \backslash B|+|B \backslash A|}-1\right)
$$

(4) $T=0$. In this case we have $\operatorname{wt}\left(c_{D^{*}}(\mathbf{a})\right)=3 \times 2^{|A|+|B|-2}$. By Lemma 4.1, the number of such $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$ is $4^{m-|A \cup B|}\left[4^{|A \cup B|}-1-3\left(2^{|A \backslash B|+|B \backslash A|}-1\right)\right]$.

This completes the proof.

Corollary 4.3. Let $A$ be a subset of $[m]$ and $D=\mathbb{F}_{2}^{m}+w \Delta_{A} \subset \mathbb{F}_{4}^{m}$ or $D=$ $\Delta_{A}+w \mathbb{F}_{2}^{m} \subset \mathbb{F}_{4}^{m}$. Then $\mathcal{C}_{D^{*}}$ is a $\left[2^{m+|A|}-1, m, 2^{m+|A|-1}\right]$ two-weight quaternary code and its weight distribution is presented in Table 3.

Table 3. Weight distribution of the code in Corollary 4.3

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{m+\|A\|-1}$ | $3 \times\left(2^{m-\|A\|}-1\right)$ |
| $3 \times 2^{m+\|A\|-2}$ | $2^{2 m}-1-3 \times\left(2^{m-\|A\|}-1\right)$ |

Theorem 4.4. Let $A, B$ be two subsets of $[m]$ and $D=\Delta_{A}+w \Delta_{B} \subset \mathbb{F}_{4}^{m}$. Then $\mathcal{C}_{D^{c}}$ in (4.1) is a $\left[4^{m}-2^{|A|+|B|}, m, 3 \times 2^{2 m-2}-3 \times 2^{|A|+|B|-2}\right]$ quaternary code and its weight distribution is presented in Table 4. Moreover, the code $\mathcal{C}_{D^{c}}$ is a Griesmer code.

Table 4. Weight distribution of the code in Theorem 4.4

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $3 \times 2^{2 m-2}-3 \times 2^{\|A\|+\|B\|-2}$ | $4^{m-\|A \cup B\|}\left[4^{\|A \cup B\|}-1-3\left(2^{\|A \backslash B\|+\|B \backslash A\|}-1\right)\right]$ |
| $3 \times 2^{2 m-2}-2^{\|A\|+\|B\|-1}$ | $4^{m-\|A \cup B\|} \times 3\left(2^{\|A \backslash B\|+\|B \backslash A\|}-1\right)$ |
| $3 \times 2^{2 m-2}$ | $4^{m-\|A \cup B\|}-1$ |

Proof. By Equation (4.3), we have the weight distribution of the code.
By the Griesmer bound, we have

$$
\begin{aligned}
& \sum_{i=0}^{m-1}\left[\left.\frac{3\left(2^{2 m-2}-2^{|A|+|B|-2}\right)}{4^{i}} \right\rvert\,\right. \\
& =\sum_{i=0}^{m-1} \frac{3 \times 2^{2 m-2}}{4^{i}}-\sum_{i=0}^{m-1}\left\lfloor\frac{3 \times 2^{|A|+|B|-2}}{4^{i}}\right] \\
& =3 \times 2^{2 m-2}+3 \times 2^{2 m-4}+\cdots+3 \\
& -\left(3 \times 2^{|A|+|B|-2}+3 \times 2^{|A|+|B|-4}+\cdots+X+Y\right)
\end{aligned}
$$

where $X=3$ and $Y=0$ if $|A|+|B|-2$ is even; and $X=6$ and $Y=1$ if $|A|+|B|-2$ is odd. Then

$$
\begin{aligned}
& \sum_{i=0}^{m-1}\left[\frac{3\left(2^{2 m-2}-2^{|A|+|B|-2}\right)}{4^{i}}\right] \\
& =\frac{3 \times 2^{2 m-2}-3 \times \frac{1}{4}}{1-\frac{1}{4}}-\frac{3 \times 2^{|A|+|B|-2}-X \times \frac{1}{4}}{1-\frac{1}{4}}-Y \\
& =2^{2 m}-1-\left(2^{|A|+|B|}-1\right)=4^{m}-2^{|A|+|B|} .
\end{aligned}
$$

This completes the proof.

The following are examples of Proposition 4.2 and Theorem 4.4.
Example 4.5. Let $m=4, A=\{1,2,3\}$, and $B=\{3,4\}$.
(1) The code $\mathcal{C}_{D^{*}}$ in Proposition 4.2 is a two-weight quaternary [31,4,16] linear code with weight enumerator $1+21 z^{16}+234 z^{24}$. In fact, the optimal quaternary linear code has parameter [31, 4, 22], according to [5].
(2) The code $\mathcal{C}_{D^{c}}$ in Theorem 4.4 is a two-weight quaternary [224, 4, 168] linear code with weight enumerator $1+21 z^{168}+234 z^{176}$. According to [5], the code is optimal.

Example 4.6. Let $m=4, A=\{2,3\}$, and $B=\{3,4\}$.
(1) The code $\mathcal{C}_{D^{*}}$ in Proposition 4.2 is a two-weight quaternary $[15,3,8]$ linear code with weight enumerator $1+9 z^{8}+54 z^{12}$. In fact, the optimal quaternary linear code has parameter $[15,3,11]$, according to [5].
(2) The code $\mathcal{C}_{D^{c}}$ in Theorem 4.4 is a three-weight quaternary [240, 4, 180] linear code with weight enumerator $1+216 z^{180}+36 z^{184}+3 z^{192}$. According to [5], the code is optimal.

Next we consider the case of a simplicial complex with two maximal elements.
Proposition 4.7. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ $[m]$. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$. Then $\mathcal{C}_{D^{*}}$ in (4.1) is a $\left[\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-1,|A \cup B|\right]$ quaternary code and its weight distribution is presented in Table 5.

Table 5. Weight distribution of the code in Proposition 4.7

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{\|A\|}\left(3 \times 2^{\|A\|-2}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $3\left(2^{\|A \backslash B\|}-1\right)$ |
| $2^{\|B\|}\left(2^{\|A\|}+3 \times 2^{\|B\|-2}-2^{\|A \cap B\|}\right)$ | $3\left(2^{\|B \backslash A\|}-1\right)$ |
| $\left(2^{\|A\|}+2^{\|B\|}\right)\left(3 \times 2^{\|A\|-2}+3 \times 2^{\|B\|-2}-2^{\|A \cap B\|}\right)$ | $3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)$ |
| $\frac{3}{4} 2^{\|A\|}\left(2^{\|A\|}+2^{\|B\|+1}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|A \backslash B\|}-2\right)$ |
| $\frac{3}{4} 2^{\|B\|}\left(2^{\|A\|+1}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|B \backslash A\|}-1\right)\left(2^{\|B \backslash A\|}-2\right)$ |
| $\begin{gathered} \frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-\frac{1}{4}\left(2^{\|A\|}-2^{\|A \cap B\|}\right)\left(2^{\|B\|}-\right. \\ \left.2^{\|A \cap B\|}\right)+\frac{1}{4} 2^{\|A \cap B\|}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|+1}\right) \\ \hline \end{gathered}$ | $6\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)$ |
| $\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{4} 2^{\|A \cap B\|}\left(2^{\|A\|+1}-3 \times 2^{\|A \cap B\|}\right)$ | $\begin{gathered} 3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-\right. \\ 1)\left(2^{\|B \backslash A\|}-2\right) \\ \hline \end{gathered}$ |
| $\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{4} 2^{\|A \cap B\|}\left(2^{\|B\|+1}-3 \times 2^{\|A \cap B\|}\right)$ | $\begin{gathered} 3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-\right. \\ 1)\left(2^{\|A \backslash B\|}-2\right) \\ \hline \end{gathered}$ |
| $\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}\right)\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\begin{aligned} &\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-\right. \\ & \text { 1) }\left(2^{\|A \backslash B\|}-2\right)\left(2^{\|B \backslash A\|}-2\right) \end{aligned}$ |
| $\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}$ | $4^{\|A \cup B\|}-4^{\|A \backslash B\|+\|B \backslash A\|}$ |

Proof. It is easy to check that the length of the code $\mathcal{C}_{D^{*}}$ is $\left|D^{*}\right|=\left(2^{|A|}+2^{|B|}-\right.$ $\left.2^{|A \cap B|}\right)^{2}-1$.

By Lemma 2.3

$$
\begin{align*}
& \chi_{\mathbf{u}}(\Delta)=\sum_{\mathbf{x} \in \Delta}(-1)^{\mathbf{u} \cdot \mathbf{x}}=\mathcal{H}_{\Delta}\left((-1)^{u_{1}},(-1)^{u_{2}}, \ldots,(-1)^{u_{m}}\right) \\
& =\prod_{i \in A}\left(1+(-1)^{u_{i}}\right)+\prod_{i \in B}\left(1+(-1)^{u_{i}}\right)-\prod_{i \in A \cap B}\left(1+(-1)^{u_{i}}\right) \\
& \quad=2^{|A|} \psi(\mathbf{u} \mid A)+2^{|B|} \psi(\mathbf{u} \mid B)-2^{|A \cap B|} \psi(\mathbf{u} \mid A \cap B) . \tag{4.5}
\end{align*}
$$

Suppose that $\mathbf{a}=\boldsymbol{\alpha}+w \boldsymbol{\beta}$. By Equations (4.2) and (4.5)

$$
\begin{aligned}
& \operatorname{wt}\left(c_{D^{*}}(\mathbf{a})\right)=\operatorname{wt}\left(c_{D}(\mathbf{a})\right)=\frac{3}{4}|D| \\
& -\frac{1}{4}\left[\chi_{\boldsymbol{\alpha}}(\Delta) \chi_{\boldsymbol{\beta}}(\Delta)+\chi_{\boldsymbol{\beta}}(\Delta) \chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\Delta)+\chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\Delta) \chi_{\boldsymbol{\alpha}}(\Delta)\right] .
\end{aligned}
$$

By Lemma 4.1,

$$
\chi_{\boldsymbol{\alpha}}(\Delta)=\left\{\begin{array}{l}
2^{|A|}+2^{|B|}-2^{|A \cap B|}, \text { if } \boldsymbol{\alpha} \in \mathcal{U}_{1}, \\
2^{|A|}-2^{|A \cap B|}, \text { if } \boldsymbol{\alpha} \in \mathcal{U}_{2}, \\
2^{|B|}-2^{|A \cap B|}, \text { if } \boldsymbol{\alpha} \in \mathcal{U}_{3}, \\
-2^{|A \cap B|}, \text { if } \boldsymbol{\alpha} \in \mathcal{U}_{4}, \\
0, \text { if } \boldsymbol{\alpha} \in \mathcal{U}_{5} .
\end{array}\right.
$$

Due to the above value distribution, we determine the location of $\boldsymbol{\alpha}+\boldsymbol{\beta}$ in the following table.

| $\boldsymbol{\alpha}+\boldsymbol{\beta} \backslash \boldsymbol{\beta}$ | $\mathcal{U}_{1}$ | $\mathcal{U}_{2}$ | $\mathcal{U}_{3}$ | $\mathcal{U}_{4}$ | $\mathcal{U}_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\boldsymbol{\alpha}$ |  |  |  | $\mathcal{U}_{4}$ | $\mathcal{U}_{5}$ |
| $\mathcal{U}_{1}$ | $\mathcal{U}_{1}$ | $\mathcal{U}_{2}$ | $\mathcal{U}_{3}$ | $\mathcal{U}_{3}$ or $\mathcal{U}_{4}$ | $\mathcal{U}_{5}$ |
| $\mathcal{U}_{2}$ | $\mathcal{U}_{2}$ | $\mathcal{U}_{1}$ or $\mathcal{U}_{2}$ | $\mathcal{U}_{4}$ | $\mathcal{U}_{2}$ or $\mathcal{U}_{4}$ | $\mathcal{U}_{5}$ |
| $\mathcal{U}_{3}$ | $\mathcal{U}_{3}$ | $\mathcal{U}_{4}$ | $\mathcal{U}_{1}$ or $\mathcal{U}_{3}$ | $\mathcal{U}^{2}$ | $\mathcal{U}_{5}$ |
| $\mathcal{U}_{4}$ | $\mathcal{U}_{4}$ | $\mathcal{U}_{3}$ or $\mathcal{U}_{4}$ | $\mathcal{U}_{2}$ or $\mathcal{U}_{4}$ | $\mathcal{U}_{1}$ or $\mathcal{U}_{2}$ or $\mathcal{U}_{3}$ or $\mathcal{U}_{4}$ | $\mathcal{U}_{5}$ |
| $\mathcal{U}_{5}$ | $\mathcal{U}_{5}$ | $\mathcal{U}_{5}$ | $\mathcal{U}_{5}$ |  |  |

It needs to be further pointed out that if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{U}_{4}$, then
$\boldsymbol{\alpha}+\boldsymbol{\beta} \in\left\{\begin{array}{l}\mathcal{U}_{1}, \text { if } \operatorname{supp}(\boldsymbol{\alpha}) \cap A=\operatorname{supp}(\boldsymbol{\beta}) \cap A, \operatorname{supp}(\boldsymbol{\alpha}) \cap B=\operatorname{supp}(\boldsymbol{\beta}) \cap B, \\ \mathcal{U}_{2}, \text { if } \operatorname{supp}(\boldsymbol{\alpha}) \cap A=\operatorname{supp}(\boldsymbol{\beta}) \cap A, \operatorname{supp}(\boldsymbol{\alpha}) \cap B \neq \operatorname{supp}(\boldsymbol{\beta}) \cap B, \\ \mathcal{U}_{2}, \text { if } \operatorname{supp}(\boldsymbol{\alpha}) \cap A \neq \operatorname{supp}(\boldsymbol{\beta}) \cap A, \operatorname{supp}(\boldsymbol{\alpha}) \cap B=\operatorname{supp}(\boldsymbol{\beta}) \cap B, \\ \mathcal{U}_{4}, \text { if } \operatorname{supp}(\boldsymbol{\alpha}) \cap A \neq \operatorname{supp}(\boldsymbol{\beta}) \cap A, \operatorname{supp}(\boldsymbol{\alpha}) \cap B \neq \operatorname{supp}(\boldsymbol{\beta}) \cap B .\end{array}\right.$
Let $T=\chi_{\boldsymbol{\alpha}}(\Delta) \chi_{\boldsymbol{\beta}}(\Delta)+\chi_{\boldsymbol{\beta}}(\Delta) \chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\Delta)+\chi_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(\Delta) \chi_{\boldsymbol{\alpha}}(\Delta)$. Suppose that $\boldsymbol{\alpha} \in \mathcal{U}_{1}$. If $\boldsymbol{\beta} \in \mathcal{U}_{1}$, then $T=3\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}$. If $\boldsymbol{\beta} \in \mathcal{U}_{2}$, Then $T=2\left(2^{|A|}+2^{|B|}-\right.$ $\left.2^{|A \cap B|}\right)\left(2^{|A|}-2^{|A \cap B|}\right)+\left(2^{|A|}-2^{|A \cap B|}\right)^{2}$. If $\boldsymbol{\beta} \in \mathcal{U}_{3}$, Then $T=2\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)\left(2^{|B|}-\right.$ $\left.2^{|A \cap B|}\right)+\left(2^{|B|}-2^{|A \cap B|}\right)^{2}$. If $\boldsymbol{\beta} \in \mathcal{U}_{4}$, Then $T=2\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)\left(-2^{|A \cap B|}\right)+\left(2^{|A \cap B|}\right)^{2}$. If $\boldsymbol{\beta} \in \mathcal{U}_{5}$, Then $T=0$.

Similarly, we can determine the values of $T$ under the condition of $\boldsymbol{\alpha} \in \mathcal{U}_{i}, i=$ $2,3,4,5$. Then the results follow from Lemma 4.1.

Remark 4.8. By massive computation, weight distributions of quaternary codes can be also determined in the case of $D=D_{1}+w D_{2}$, where $D_{1}$ is generated by two maximal elements $A, B \subseteq[m]$ and $D_{2}$ is generated by two maximal elements $C, F \subseteq[m]$.

For some special sets $A, B$, we obtain some few-weight quaternary codes.
Corollary 4.9. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ $[m]$. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$.
(i) If $A \cap B=\emptyset$ and $|A|=|B|=1$, then $\mathcal{C}_{D^{*}}$ in Proposition 4.7 is a three-weight $[8,2,5]$ quaternary code and its weight enumerator $1+6 z^{5}+6 z^{7}+3 z^{8}$.
(ii) If $A \cap B=\emptyset$ and $|A|=|B|>1$, then $\mathcal{C}_{D^{*}}$ in Proposition 4.7 is a six-weight $\left[\left(2^{|A|+1}-1\right)^{2}-1,2|A|\right]$ quaternary code and its weight distribution is presented in Table 6.

Table 6. Weight distribution of the code in Corollary 4.9 (ii)

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{\|A\|}\left(3 \times 2^{\|A\|-2}+2^{\|A\|}-1\right)$ | $6\left(2^{\|A\|}-1\right)$ |
| $2^{\|A\|+1}\left(3 \times 2^{\|A\|-1}-1\right)$ | $3\left(2^{\|A\|}-1\right)^{2}$ |
| $3 \times 2^{\|A\|-1}\left(3 \times 2^{\|A\|-1}-1\right)$ | $3\left(2^{\|A\|}-1\right)\left(2^{\|A\|}-2\right)$ |
| $11 \times 2^{2\|A\|-2}-2^{\|A\|+1}$ | $6\left(2^{\|A\|}-1\right)^{2}$ |
| $3 \times 2^{2\|A\|}-3 \times 2^{\|A\|}+2^{\|A\|-1}$ | $6\left(2^{\|A\|}-1\right)^{2}\left(2^{\|A\|}-2\right)$ |
| $3 \times 2^{\|A\|}\left(2^{\|A\|}-1\right)$ | $\left(2^{\|A\|}-1\right)^{2}\left(2^{\|A\|}-2\right)^{2}$ |

(iii) If $A \cap B \neq \emptyset,|A|=|B|$, and $|A \backslash B|=|B \backslash A|=1$, then $\mathcal{C}_{D^{*}}$ in Proposition 4.7 is a four-weight $\left[9 \times 2^{2|A|-2}-1,|A|+1\right]$ quaternary code and its weight distribution is presented in Table 7.

Table 7. Weight distribution of the code in Corollary 4.9 (iii)

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{2\|A\|-2}+2^{2\|A\|}$ | 6 |
| $2^{2\|A\|+1}$ | 3 |
| $26 \times 2^{2\|A\|-4}$ | 6 |
| $27 \times 2^{2\|A\|-4}$ | $4^{\|A\|+1}-16$ |

Similar to Theorem 4.4, we have the following theorem.

Theorem 4.10. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ $[m]$. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$. Then $\mathcal{C}_{D^{c}}$ in (4.1) is a $\left[4^{m}-\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}, m\right]$ quaternary code and its weight distribution is presented in Table 8.

Table 8. Weight distribution of the code in Theorem 4.10

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $3 \times 2^{2 m-2}-2^{\|A\|}\left(3 \times 2^{\|A\|-2}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $3\left(2^{\|A \backslash B\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $3 \times 2^{2 m-2}-2^{\|B\|}\left(2^{\|A\|}+3 \times 2^{\|B\|-2}-2^{\|A \cap B\|}\right)$ | $3\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $\begin{gathered} 3 \times 2^{2 m-2}-\left(2^{\|A\|}+2^{\|B\|}\right)\left(3 \times 2^{\|A\|-2}+3 \times\right. \\ \left.2^{\|B\|-2}-2^{\|A \cap B\|}\right) \\ \hline \end{gathered}$ | $3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $3 \times 2^{2 m-2}-\frac{3}{4} 2^{\|A\|}\left(2^{\|A\|}+2^{\|B\|+1}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|A \backslash B\|}-2\right) 4^{m-\|A \cup B\|}$ |
| $3 \times 2^{2 m-2}-\frac{3}{4} 2^{\|B\|}\left(2^{\|A\|+1}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|B \backslash A\|}-1\right)\left(2^{\|B \backslash A\|}-2\right) 4^{m-\|A \cup B\|}$ |
| $\begin{gathered} \hline 3 \times 2^{2 m-2}-\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{4}\left(2^{\|A\|}-\right. \\ \left.2^{\|A \cap B\|}\right)\left(2^{\|B\|}-2^{\|A \cap B\|}\right)-\left.\frac{1}{4}\right\|^{\|A \cap B\|}\left(2^{\|A\|}+2^{\|B\|}-\right. \\ \left.2^{\|A \cap B\|+1}\right) \\ \hline \end{gathered}$ | $6\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $\begin{gathered} 3 \times 2^{2 m-2}-\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}- \\ \frac{1}{4} 2^{\|A \cap B\|}\left(2^{\|A\|+1}-3 \times 2^{\|A \cap B\|}\right) \\ \hline \end{gathered}$ | $\begin{gathered} 3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)\left(2^{\|B \backslash A\|}-\right. \\ 2) 4^{m-\|A \cup B\|} \end{gathered}$ |
| $\begin{gathered} 3 \times 2^{\frac{4}{2 m-2}-\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-} \\ \frac{1}{4} 2^{\|A \cap B\|}\left(2^{\|B\|+1}-3 \times 2^{\|A \cap B\|}\right) \\ \hline \end{gathered}$ | $\begin{gathered} 3\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)\left(2^{\|A \backslash B\|}-\right. \\ 2) 4^{m-\|A \cup B\|} \end{gathered}$ |
| $3 \times 2^{2 m-2}-\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}\right)\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\begin{gathered} \left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)\left(2^{\|A \backslash B\|}-\right. \\ 2)\left(2^{\|B \backslash A\|}-2\right) 4^{m-\|A \cup B\|} \\ \hline \end{gathered}$ |
| $3 \times 2^{2 m-2}-\frac{3}{4}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}$ | $\left(4^{\|A \cup B\|}-4^{\|A \backslash B\|+\|B \backslash A\|}\right) 4^{m-\|A \cup B\|}$ |
| $3 \times 2^{2 m-2}$ | $4^{m-\|A \cup B\|}-1$ |

The following is an example of Proposition 4.7 and Theorem 4.10.
Example 4.11. Let $m=4, A=\{1,2,3\}$, and $B=\{3,4\}$.
(1) The code $\mathcal{C}_{D^{*}}$ in Proposition 4.7 is a seven-weight quaternary [99, 4, 36] linear code with weight enumerator

$$
1+3 z^{36}+9 z^{64}+6 z^{72}+192 z^{75}+18 z^{76}+9 z^{84}+18 z^{88}
$$

In fact, the optimal quaternary linear code has parameter [99, 4, 73] and its dual code has parameters [99, 95, 2], according to [5].
(2) The code $\mathcal{C}_{D^{c}}$ in Theorem 4.10 is a seven-weight quaternary $[156,4,104]$ linear code with weight enumerator

$$
1+18 z^{104}+9 z^{108}+18 z^{116}+192 z^{117}+6 z^{120}+9 z^{128}+3 z^{156}
$$

In fact, the optimal quaternary linear code has parameter [156, 4, 116] and its dual code has parameters [156, 152, 2], according to [5].

By Corollary 4.9 and Theorem 4.10, we obtain some few-weight quaternary codes.

Corollary 4.12. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ $[m]$. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$.
(i) If $A \cap B=\emptyset$ and $|A|=|B|=1$, then $\mathcal{C}_{D^{c}}$ in Theorem 4.10 is a four-weight $\left[4^{m}-9, m\right]$ quaternary code and its weight enumerator
$1+6 \times 4^{m-2} z^{3 \times 2^{2 m-2}-5}+6 \times 4^{m-2} z^{3 \times 2^{2 m-2}-7}+3 \times 4^{m-2} z^{3 \times 2^{2 m-2}-8}+\left(4^{m-2}-1\right) z^{3 \times 2^{2 m-2}}$.
(ii) If $A \cap B=\emptyset$ and $|A|=|B|>1$, then $\mathcal{C}_{D^{c}}$ in Theorem 4.10 is a seven-weight $\left[4^{m}-\left(2^{|A|+1}-1\right)^{2}, m\right]$ quaternary code and its weight distribution is presented in Table 9.

TABLE 9. Weight distribution of the code in Corollary 4.12 (ii)

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $3 \times 2^{2 m-2}-2^{\|A\|}\left(3 \times 2^{\|A\|-2}+2^{\|A\|}-1\right)$ | $6\left(2^{\|A\|}-1\right) 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}-2^{\|A\|+1}\left(3 \times 2^{\|A\|-1}-1\right)$ | $3\left(2^{\|A\|}-1\right)^{2} 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}-3 \times 2^{\|A\|-1}\left(3 \times 2^{\|A\|-1}-1\right)$ | $3\left(2^{\|A\|}-1\right)\left(2^{\|A\|}-2\right) 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}-11 \times 2^{2\|A\|-2}-2^{\|A\|+1}$ | $6\left(2^{\|A\|}-1\right)^{2} 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}-3 \times 2^{2\|A\|}-3 \times 2^{\|A\|}+2^{\|A\|-1}$ | $6\left(2^{\|A\|}-1\right)^{2}\left(2^{\|A\|}-2\right) 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}-3 \times 2^{\|A\|}\left(2^{\|A\|}-1\right)$ | $\left(2^{\|A\|}-1\right)^{2}\left(2^{\|A\|}-2\right)^{2} 4^{m-2\|A\|}$ |
| $3 \times 2^{2 m-2}$ | $4^{m-2\|A\|}-1$ |

(iii) If $A \cap B \neq \emptyset,|A|=|B|$, and $|A \backslash B|=|B \backslash A|=1$, then $\mathcal{C}_{D^{c}}$ in Theorem 4.10 is a five-weight $\left[4^{m}-9 \times 2^{2|A|-2}, m\right.$ ] quaternary code and its weight distribution is presented in Table 10.

Table 10. Weight distribution of the code in Corollary 4.12 (iii)

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $3 \times 2^{2 m-2}-2^{2\|A\|-2}+2^{2\|A\|}$ | $6 \times 4^{m-\|A\|-1}$ |
| $3 \times 2^{2 m-2}-2^{2\|A\|+1}$ | $3 \times 4^{m-\|A\|-1}$ |
| $3 \times 2^{2 m-2}-26 \times 2^{2\|A\|-4}$ | $6 \times 4^{m-\|A\|-1}$ |
| $3 \times 2^{2 m-2}-27 \times 2^{2\|A\|-4}$ | $\left(4^{\|A\|+1}-16\right) 4^{m-\|A\|-1}$ |
| $3 \times 2^{2 m-2}$ | $4^{m-\|A\|-1}-1$ |

## 5. WEIGHT DISTRIBUTIONS OF THE SUBFIELD CODES

In this section, we will determine weight distributions of the binary subfield codes of those quaternary codes obtained in Section 4.

Proposition 5.1. Let $A, B$ be two subsets of $[m]$ and $D=\Delta_{A}+w \Delta_{B} \subset \mathbb{F}_{4}^{m}$. Then the subfield code $\mathcal{C}_{D^{*}}^{(2)}$ with respect to $\mathcal{C}_{D^{*}}$ in Proposition 4.2 is a $\left[2^{|A|+|B|}-1,|A|+\right.$ $\left.|B|, 2^{|A|+|B|-1}\right]$ one-weight binary linear code.

Proof. By Theorem 3.2, $\mathcal{C}_{D^{*}}^{(2)}$ can be generated by

$$
\mathcal{C}_{D^{*}}^{(2)}=\left\{c_{D}(\boldsymbol{\alpha}, \boldsymbol{\beta})=\left(\boldsymbol{\alpha} \cdot \mathbf{d}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right)_{\mathbf{d}_{1} \in \Delta_{A}, \mathbf{d}_{2} \in \Delta_{B}}: \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{2}^{m}\right\}
$$

Hence

$$
\begin{aligned}
\mathrm{wt}\left(c_{D^{*}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right)=\mathrm{wt}\left(c_{D}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) & =|D|-\sum_{\mathbf{d}_{1} \in \Delta_{A}} \sum_{\mathbf{d}_{2} \in \Delta_{B}}\left(\frac{1}{2} \sum_{y \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right) y}\right) \\
& =|D|-\frac{1}{2} \sum_{\mathbf{d}_{1} \in \Delta_{A}} \sum_{\mathbf{d}_{2} \in \Delta_{B}}\left(1+(-1)^{\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)}\right) \\
& =\frac{1}{2}|D|-\frac{1}{2}\left(\sum_{\mathbf{d}_{1} \in \Delta_{A}}(-1)^{\beta \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in \Delta_{B}}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathrm{d}_{2}}\right) \\
& =2^{|A|+|B|-1}(1-\psi(\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)) .
\end{aligned}
$$

This completes the proof.
Theorem 5.2. Let $A, B$ be two subsets of $[m]$ and $D=\Delta_{A}+w \Delta_{B} \subset \mathbb{F}_{4}^{m}$. Then the subfield code $\mathcal{C}_{D^{c}}^{(2)}$ with respect to $\mathcal{C}_{D^{c}}$ in Theorem 4.4 is a $\left[2^{2 m}-2^{|A|+|B|}, 2 m, 2^{2 m-1}-\right.$ $\left.2^{|A|+|B|-1}\right]$ two-weight binary linear code and its weight distribution is given by

$$
1+\left(4^{m}-2^{2 m-|A|-|B|}\right) z^{2^{2 m-1}-2^{|A|+|B|-1}}+\left(2^{2 m-|A|-|B|}-1\right) z^{2^{2 m-1}}
$$

Moreover, the code $\mathcal{C}_{D^{c}}^{(2)}$ is a Griesmer code.
Proof. Note that $D^{c}=\left(\Delta_{A}^{c}+w \mathbb{F}_{2}^{m}\right) \cup\left(\Delta_{A}+w \Delta_{B}^{c}\right)$. By Theorem 3.2, $\mathcal{C}_{D^{c}}^{(2)}$ can be generated by

$$
\mathcal{C}_{D^{c}}^{(2)}=\left\{c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta}): \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{2}^{m}\right\}
$$

where

$$
c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta})=\left(\left(\boldsymbol{\alpha} \cdot \mathbf{d}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)_{\mathbf{d}_{1} \in \Delta_{A}^{c}, \mathbf{d}_{2} \in \mathbb{F}_{2}^{m}} \mid\left(\boldsymbol{\alpha} \cdot \mathbf{f}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{f}_{1}+\mathbf{f}_{2}\right)_{\mathbf{f}_{1} \in \Delta_{A}, \mathbf{f}_{2} \in \Delta_{B}^{c}}\right)\right) .\right.
$$

Hence

$$
\begin{aligned}
\mathrm{wt}\left(c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) & =\left|D^{c}\right|-\sum_{\mathbf{d}_{1} \in \Delta_{A}^{c}} \sum_{\mathbf{d}_{2} \in \mathbb{F}_{2}^{m}} \frac{1}{2} \sum_{y \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right) y} \\
& -\sum_{\mathbf{f}_{1} \in \Delta_{A}} \sum_{\mathbf{f}_{2} \in \Delta_{B}^{c}} \frac{1}{2} \sum_{z \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{f}_{2}+\boldsymbol{\beta}\left(\mathbf{f}_{1}+\mathbf{f}_{2}\right)\right) z} \\
& =\frac{1}{2}\left|D^{c}\right|-\frac{1}{2}\left(\sum_{\mathbf{d}_{1} \in \Delta_{A}^{c}}(-1)^{\boldsymbol{\beta} \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in \mathbb{F}_{2}^{m}}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{d}_{2}}\right) \\
& -\frac{1}{2}\left(\sum_{\mathbf{f}_{1} \in \Delta_{A}}(-1)^{\boldsymbol{\beta} \mathbf{f}_{1}}\right)\left(\sum_{\mathbf{f}_{2} \in \Delta_{B}^{c}}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{f}_{2}}\right) \\
& =2^{2 m-1}\left(1-\delta_{\mathbf{0}, \boldsymbol{\beta}} \delta_{\mathbf{0}, \boldsymbol{\alpha}+\boldsymbol{\beta}}\right)-2^{|A|+|B|-1}(1-\psi(\boldsymbol{\beta} \mid A) \psi(\boldsymbol{\alpha}+\boldsymbol{\beta} \mid B)) .
\end{aligned}
$$

Then we have the weight distribution of the code by Proposition 5.1.
By the Griesmer bound, we have

$$
\begin{aligned}
& \sum_{i=0}^{2 m-1}\left[\left.\frac{2^{2 m-1}-2^{|A|+|B|-1}}{2^{i}} \right\rvert\,\right. \\
= & \left.\left.\sum_{i=0}^{2 m-1} \frac{2^{2 m-1}}{2^{i}}-\sum_{i=0}^{2 m-1} \right\rvert\, \frac{2^{|A|+|B|-1}}{2^{i}}\right] \\
= & \left(2^{2 m}-1\right)-\left(2^{|A|+|B|}-1\right)=2^{2 m}-2^{|A|+|B|} .
\end{aligned}
$$

This completes the proof.

Proposition 5.3. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ $[m]$. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$. Then the subfield code $\mathcal{C}_{D^{*}}^{(2)}$ with respect to $\mathcal{C}_{D^{*}}$ in Proposition 4.7 is a $\left[\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-1,2|A \cup B|\right]$ binary code and its weight distribution is presented in Table 11.

Table 11. Weight distribution of the code in Proposition 5.3

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{\|A\|-1}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right)$ |
| $2^{\|B\|-1}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|B \backslash A\|}-1\right)$ |
| $\left(2^{\|A\|-1}+2^{\|B\|-1}\right)\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)$ |
| $2^{\|B\|-1}\left(2^{\|A\|+1}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|B \backslash A\|}-1\right)^{2}$ |
| $2^{\|A\|-1}\left(2^{\|A\|}+2^{\|B\|+1}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|A \backslash B\|}-1\right)^{2}$ |
| $\left(2^{\|A\|-1}+2^{\|B\|-1}\right)\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|B \backslash A\|}-1\right)^{2}\left(2^{\|B \backslash A\|}-1\right)^{2}$ |
| $\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-\frac{1}{2}\left(2^{\|A\|}-2^{\|A \cap B\|}\right)\left(2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)$ |
| $\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{2}\left(2^{\|A\|}-2^{\|A \cap B\|}\right) 2^{\|A \cap B\|}$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)^{2}$ |
| $\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{2}\left(2^{\|B\|}-2^{\|A \cap B\|}\right) 2^{\|A \cap B\|}$ | $2\left(2^{\|A \backslash B\|}-1\right)^{2}\left(2^{\|B \backslash A\|}-1\right)$ |
| $\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}$ | $4^{\|A \cup B\|}-4^{\|A \backslash B\|+\|B \backslash A\|}$ |

Proof. By Theorem 3.2, $\mathcal{C}_{D^{*}}^{(2)}$ can be generated by

$$
\mathcal{C}_{D^{*}}^{(2)}=\left\{c_{D}(\boldsymbol{\alpha}, \boldsymbol{\beta})=\left(\boldsymbol{\alpha} \cdot \mathbf{d}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right)_{\mathbf{d}_{1}, \mathbf{d}_{2} \in \Delta}: \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{2}^{m}\right\} .
$$

Hence

$$
\begin{aligned}
\mathrm{wt}\left(c_{D}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) & =|D|-\sum_{\mathbf{d}_{1} \in \Delta} \sum_{\mathbf{d}_{2} \in \Delta}\left(\frac{1}{2} \sum_{y \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right) y}\right) \\
& =|D|-\frac{1}{2} \sum_{\mathbf{d}_{1} \in \Delta} \sum_{\mathbf{d}_{2} \in \Delta}\left(1+(-1)^{\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)}\right) \\
& =\frac{1}{2}|D|-\frac{1}{2}\left(\sum_{\mathbf{d}_{1} \in \Delta}(-1)^{\beta \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in \Delta}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{d}_{2}}\right) \\
& =\frac{1}{2}\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-\frac{1}{2} T_{\boldsymbol{\beta}} T_{\boldsymbol{\alpha}+\boldsymbol{\beta}},
\end{aligned}
$$

where $T_{\mathbf{u}}=2^{|A|} \psi(\mathbf{u} \mid A)+2^{|B|} \psi(\mathbf{u} \mid B)-2^{|A \cap B|} \psi(\mathbf{u} \mid A \cap B)$.
By the proof of Proposition 4.7, we prove the results.

Remark 5.4. It is noted that the subfield code $\mathcal{C}_{D^{*}}^{(2)}$ with respect to $\mathcal{C}_{D^{*}}$ in Theorem 5.3 has at most seven weights when the sets $A, B$ have the same size.

Theorem 5.5. The dual code of the subfield code $\mathcal{C}_{D^{*}}^{(2)}$ in Proposition 5.3 has minimum distance three and it is an almost optimal binary code with respect to the Sphere Packing Bound.

Proof. By Theorem 3.2, the subfield code $\mathcal{C}_{D^{*}}^{(2)}$ in Proposition 5.3 has the following defining set:

$$
D^{(2)}=\left\{\left(\mathbf{d}_{2}, \mathbf{d}_{1}+\mathbf{d}_{2}\right): \mathbf{d}_{1}, \mathbf{d}_{2} \in \Delta\right\}=\left\{\mathbf{g}_{1}, \mathbf{g}_{2}, \ldots, \mathbf{g}_{t}\right\}
$$

where $t=\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}$. Let $G$ be the $2 m \times t$ matrix as follows:

$$
G=\left[\begin{array}{lll}
\mathbf{g}_{1}^{T} & \mathbf{g}_{2}^{T} & \cdots
\end{array} \mathbf{g}_{t}^{T}\right]
$$

where the column vector $\mathbf{g}_{i}^{T}$ denotes the transpose of a row vector $\mathbf{g}_{i}$. Let $\mathbf{e}_{k}=$ $\left(e_{1}, e_{2}, \ldots, e_{m}\right) \in \mathbb{F}_{2}^{m}$, where $e_{k}=1$ and $e_{l}=0$ if $l \neq k$. Suppose that $i, j \in A \cup B$. Then it is easy to check that $\left(\mathbf{e}_{i}^{T}, \mathbf{e}_{i}^{T}\right),\left(\mathbf{e}_{j}^{T}, \mathbf{e}_{j}^{T}\right)$, and $\left(\mathbf{e}_{i}^{T}+\mathbf{e}_{j}^{T}, \mathbf{e}_{i}^{T}+\mathbf{e}_{j}^{T}\right)$ are three different columns of $G$; therefore, the minimum distance of $\left(\mathcal{C}_{D^{*}}^{(2)}\right)^{\perp}$ is 3 .

By Proposition 5.3, $\left(\mathcal{C}_{D^{*}}^{(2)}\right)^{\perp}$ has parameters

$$
\left[\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-1,\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-1-2|A \cup B|, 3\right]
$$

By Sphere Packing Bound, let $n=\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}-1$, then it is easy to check that

$$
\sum_{i=0}^{2}\binom{n}{i}=1+n+\frac{n(n-1)}{2}>2^{2|A \cup B|}
$$

This completes the proof.

Proposition 5.6. Let $\Delta$ be a simplicial complex with two maximal elements $A, B \subseteq$ [m]. Let $D=\Delta+w \Delta \subset \mathbb{F}_{4}^{m}$. Then the subfield code $\mathcal{C}_{D^{c}}^{(2)}$ with respect to $\mathcal{C}_{D^{c}}$ in Theorem 4.8 is a

$$
\left[4^{m}-\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}, 2 m, 2^{2 m-1}-\left(2^{|A|-1}+2^{|B|-1}\right)\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)\right]
$$

binary code and its weight distribution is presented in Table 12.

Table 12. Weight distribution of the code in Proposition 5.6

| Weight | Frequency |
| :---: | :---: |
| 0 | 1 |
| $2^{2 m-1}-2^{\|A\|-1}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-2^{\|B\|-1}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-\left(2^{\|A\|-1}+2^{\|B\|-1}\right)\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-2^{\|B\|-1}\left(2^{\|A\|+1}+2^{\|B\|}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|B \backslash A\|}-1\right)^{2} 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-2^{\|A\|-1}\left(2^{\|A\|}+2^{\|B\|+1}-2^{\|A \cap B\|+1}\right)$ | $\left(2^{\|A \backslash B\|}-1\right)^{2} 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-\left(2^{\|A\|-1}+2^{\|B\|-1}\right)\left(2^{\|A\|}+2^{\|B\|}-\right.$ | $\left(2^{\|B \backslash A\|}-1\right)^{2}\left(2^{\|B \backslash A\|}-1\right)^{2} 4^{m-\|A \cup B\|}$ |
| $\left.2^{\|A \cap B\|+1}\right)$ |  |
| $2^{2 m-1}-\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}+\frac{1}{2}\left(2^{\|A\|}-\right.$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $\left.2^{\|A \cap B\|}\right)\left(2^{\|B\|}-2^{\|A \cap B\|}\right)$ | $2\left(2^{\|A \backslash B\|}-1\right)\left(2^{\|B \backslash A\|}-1\right)^{2} 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-\frac{1}{2}\left(2^{\|A\|}-\right.$ | $\left.2^{\|A \cap B\|}\right)\left.\right\|^{\|A \cap B\|}$ |
| $2^{2 m-1}-\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-\frac{1}{2}\left(2^{\|B\|}-\right.$ | $2\left(2^{\|A \backslash B\|}-1\right)^{2}\left(2^{\|B \backslash A\|}-1\right) 4^{m-\|A \cup B\|}$ |
| $\left.2^{\|A \cap B\|}\right) 2^{\|A \cap B\|}$ | $\left(4^{\|A \cup B\|}-4^{\|A \backslash B\|+\|B \backslash A\|}\right) 4^{m-\|A \cup B\|}$ |
| $2^{2 m-1}-\frac{1}{2}\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}$ | $4^{m-\|A \cup B\|}-1$ |
| $2^{2 m-1}$ |  |

Proof. Note that $D^{c}=\left(\Delta^{c}+w \mathbb{F}_{2}^{m}\right) \cup\left(\Delta+w \Delta^{c}\right)$. By Theorem 3.2, $\mathcal{C}_{D^{c}}^{(2)}$ can be generated by

$$
\mathcal{C}_{D^{c}}^{(2)}=\left\{c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta}): \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{2}^{m}\right\},
$$

where

$$
c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta})=\left(\left(\boldsymbol{\alpha} \cdot \mathbf{d}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)_{\mathbf{d}_{1} \in \Delta^{c}, \mathbf{d}_{2} \in \mathbb{F}_{2}^{m}} \mid\left(\boldsymbol{\alpha} \cdot \mathbf{f}_{2}+\boldsymbol{\beta} \cdot\left(\mathbf{f}_{1}+\mathbf{f}_{2}\right)_{\mathbf{f}_{1} \in \Delta, \mathbf{f}_{2} \in \Delta^{c}}\right)\right) .\right.
$$

Hence

$$
\begin{aligned}
\mathrm{wt}\left(c_{D^{c}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\right) & =\left|D^{c}\right|-\sum_{\mathbf{d}_{1} \in \Delta^{c}} \sum_{\mathbf{d}_{2} \in \mathbb{F}_{2}^{m}} \frac{1}{2} \sum_{y \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{d}_{2}+\boldsymbol{\beta}\left(\mathbf{d}_{1}+\mathbf{d}_{2}\right)\right) y} \\
& -\sum_{\mathbf{f}_{1} \in \Delta \mathbf{f}_{2} \in \Delta^{c}} \frac{1}{2} \sum_{z \in \mathbb{F}_{2}}(-1)^{\left(\boldsymbol{\alpha} \mathbf{f}_{2}+\boldsymbol{\beta}\left(\mathbf{f}_{1}+\mathbf{f}_{2}\right)\right) z} \\
& =\frac{1}{2}\left|D^{c}\right|-\frac{1}{2}\left(\sum_{\mathbf{d}_{1} \in \Delta_{A}^{c}}(-1)^{\boldsymbol{\beta} \mathbf{d}_{1}}\right)\left(\sum_{\mathbf{d}_{2} \in \mathbb{F}_{2}^{m}}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{d}_{2}}\right) \\
& -\frac{1}{2}\left(\sum_{\mathbf{f}_{1} \in \Delta}(-1)^{\beta \mathbf{f}_{1}}\right)\left(\sum_{\mathbf{f}_{2} \in \Delta^{c}}(-1)^{(\boldsymbol{\alpha}+\boldsymbol{\beta}) \mathbf{f}_{2}}\right) \\
& =2^{2 m-1}\left(1-\delta_{\mathbf{0}, \boldsymbol{\beta}} \delta_{\mathbf{0}, \boldsymbol{\alpha}+\boldsymbol{\beta}}\right)-\frac{1}{2}\left(2^{|A|}+2^{|B|}-2^{|A \cap B|}\right)^{2}+\frac{1}{2} T_{\boldsymbol{\beta}} T_{\boldsymbol{\alpha}+\boldsymbol{\beta}},
\end{aligned}
$$

where $T_{\mathbf{u}}=2^{|A|} \psi(\mathbf{u} \mid A)+2^{|B|} \psi(\mathbf{u} \mid B)-2^{|A \cap B|} \psi(\mathbf{u} \mid A \cap B)$. Then the result follows from Proposition 5.3.

Remark 5.7. By massive computation, weight distributions of the binary subfield codes of these quaternary codes can be also determined in the case of $D=D_{1}+w D_{2}$, where $D_{1}$ is generated by two maximal elements $A, B \subseteq[m]$ and $D_{2}$ is generated by two maximal elements $C, F \subseteq[m]$.

Remark 5.8. It is noted that the subfield code $\mathcal{C}_{D^{c}}^{(2)}$ with respect to $\mathcal{C}_{D^{c}}$ in Proposition 5.6 has at most eight weights when the sets $A, B$ have the same size.

The following is an example of Proposition 5.6.
Example 5.9. Let $m=4$.
(1) If $A=\{1,2\}$ and $B=\{2,3\}$, then the code $\mathcal{C}_{D^{c}}^{(2)}$ in Proposition 5.6 is a fiveweight quaternary $[220,8,104]$ linear code with weight enumerator

$$
1+8 z^{104}+195 z^{110}+20 z^{112}+16 z^{116}+16 z^{118}
$$

In fact, the optimal binary linear code has parameter [220, 8, 109], according to [5].
(2) If $A=\{1,2\}$ and $B=\{3,4\}$, then the code $\mathcal{C}_{D^{c}}^{(2)}$ in Proposition 5.6 is a fiveweight quaternary $[207,8,100]$ linear code with weight enumerator

$$
1+18 z^{100}+108 z^{102}+81 z^{104}+36 z^{108}+12 z^{114}
$$

In fact, the optimal binary linear code has parameter [207, 8, 102], according to [5].

## 6. Code comparisons and concluding remarks

To show significant advantages of our codes, in this section, we present two tables.
In Table 13, we list recent works on linear codes over finite fields constructed from simplicial complexes for the convenience of the reader. Compare with known results, some quaternary codes and their subfield codes obtained in this work have flexible
and new parameters. To the best of our knowledge, this is the first paper on the construction of linear codes over a non-prime field and their subfield codes by using simplicial complexes.

Table 14 presents optimal quaternary linear codes from two simplicial complexes $\Delta_{A}$ and $\Delta_{B}$ in Theorem 4.4. In Table 14, * indicates that the corresponding codes are optimal codes, and "new" are also indicated according to the current data base [6]. In fact, according to the current data base [6] we find as least 9 new optimal codes (lengths: $12,48,56,60,192,224,240,248,252$ ); even though their parameters are not new but they are inequivalent to currently best-known linear codes. We confirmed those results by Magma.

The main contributions of this paper are the following

- A general explicit relationship between quaternary linear codes and their binary subfield codes in terms of generator matrices and defining sets (Theorem 3.2).
- The determination of weight distributions of four classes of quaternary codes when these simplicial complexes are all generated by a single maximal element or two maximal elements (Propositions 4.2, 4.7, and Theorems 4.4, 4.10).
- The determination of weight distributions of four classes of the subfield codes of those quaternary codes (Propositions 5.1, 5.3, 5.6 and Theorem 5.2).
- Two infinite families of optimal linear codes meeting the Griesmer Bound (Theorems 4.4, 5.2) and a class of binary almost optimal linear codes with respect to Sphere Packing Bound (Theorem 5.5).
- At least 9 new optimal quaternary linear codes (Table 14).

Very recently, Hyun et al. [13] extended the construction of linear codes to posets. It would be interesting to find more optimal quaternary codes by employing posets.

On the other hand, the quaternary linear code in Example 3.4 is

$$
\begin{array}{r}
\mathcal{C}=\{(0,0,0,0),(w, 0,1+w, 1),(1+w, 0,1, w),(1,0, w, 1+w) \\
(1+w, 1+w, w, w),(1,1,1+w, 1+w),(w, w, 1,1) \\
(1,1+w, 1,1+w),(w, 1, w, 1),(1+w, w, 1+w, w) \\
(1+w, 1,0, w),(1, w, 0,1+w),(w, 1+w, 0,1) \\
(0, w, w, 0),(0,1+w, 1+w, 0),(0,1,1,0)\}
\end{array}
$$

It is easy to check that its binary subfield subcode

$$
\mathcal{C} \mid \mathbb{F}_{2}=\mathcal{C} \cap \mathbb{F}_{2}^{4}=\{(0,0,0,0),(0,1,1,0)\}
$$

We just wonder that whether there is a direct way to compute the binary subfield subcodes of these quaternary codes obtained in this paper.

Table 13. Linear codes constructed from simplicial complexes

| Reference | $q$-Ary | Defining Set | [ $n, k, d]$ Code | \#Weight | Bound | Result |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| [14] | binary | $\Delta^{*}$ | $\left[2^{\|A\|}-1,\|A\|, 2^{\|A\|-1}\right]$ | 1 | Griesmer | Lem. 7 |
|  |  | $\mathbb{F}_{2}^{m} \backslash \Delta$ | $\left[2^{m}-2^{\|A\|}, m, 2^{m-1}-2^{\|A\|-1}\right]$ | 2 | Griesmer |  |
|  |  |  | $\left[2^{m-1}, m, 4\right]$ |  | Sphere <br> Packing | Ex. 10 |
|  |  |  | $\begin{aligned} & {\left[2^{m}-2 \sum_{i=1}^{s} 2^{\left\|A_{i}-1\right\|}+s-\right.} \\ & \left.1, m, 2^{m-1}-\sum_{i=1}^{s} 2^{\left\|A_{i}\right\|-1}\right] \end{aligned}$ |  | Griesmer | Coro. 21 |
|  |  | $\Delta^{*}$ |  | 3 |  | Lem. 26 |
|  |  |  | $\begin{gathered} {\left[2^{\left\|A_{1}\right\|+2^{\left\|A_{2}\right\|}}-2^{\left\|A_{1} \cap A_{2}\right\|}-1, \mid A_{1} \cup\right.} \\ \left.A_{2} \mid, 2^{\left\|A_{1}\right\|-1}\right] \\ \hline \end{gathered}$ | 4 |  |  |
|  |  | $\mathbb{F}_{2}^{m} \backslash \Delta$ | $\begin{gathered} {\left[2^{m}-2^{\left\|A_{1}\right\|}-2^{\left\|A_{2}\right\|}+\right.} \\ \left.1, m, 2^{m-1}-2^{\left\|A_{1}\right\|-1}-2^{\left\|A_{2}-1\right\|}\right] \end{gathered}$ | 3 or 4 | Griesmer | Thm. 27 |
|  |  |  | $\begin{gathered} {\left[2^{m}-2^{\left\|A_{1}\right\|}-2^{\left\|A_{2}\right\|}+2^{\left\|A_{1} \cap A_{2}\right\|}-\right.} \\ \left.1, m, 2^{m-1}-2^{\left\|A_{1}\right\|-1}-2^{\left\|A_{2}\right\|-1}\right] \end{gathered}$ | 4 or 5 |  |  |
| [19] | binary | $\Delta_{A} \backslash \Delta_{B}$ | $\left[2^{\|A\|}-2^{\|B\|},\|A\|, 2^{\|A\|-1}-2^{\|B\|-1}\right]$ | 2 | Griesmer | Thm. 5 |
|  |  |  | $\begin{gathered} {\left[2^{\|A\|}-2^{\|B\|},\|A\|, 2^{\|A\|}-2^{\|B\|}-\right.} \\ \|A\|, 3 \text { or } 4] \\ \hline \end{gathered}$ |  |  | Thm. 6 |
| [23] | 4-ary | $\mathbb{F}_{4}^{m} \backslash \Delta_{A}+w \Delta_{B}$ | $\begin{gathered} {\left[\left(2^{m}-2^{\|A\|}\right) 2^{\|B\|}, m, 3\left(2^{m+\|B\|-2}-\right.\right.} \\ \left.\left.2^{\|A\|+\|B\|-2}\right)\right] \\ \hline \end{gathered}$ | 5 |  | Thm.3.1 |
|  |  | $\mathbb{F}_{2}^{m}+w \Delta_{B}$ | $\left[2^{m+\|B\|}, m, 2^{m+\|B\|-1}\right]$ | 2 |  | Coro.3.2 |
|  |  | $\mathbb{F}_{2}^{m} \backslash \Delta_{A}+w \mathbb{F}_{2}^{m}$ | $\begin{gathered} {\left[\left(2^{m}-2^{\|A\|}\right) 2^{m}, m, 3\left(2^{2 m-2}-\right.\right.} \\ \left.\left.2^{\|A\|+m-2}\right)\right] \end{gathered}$ | 2 | Griesmer | Coro.3.3 |
| [12] | p-ary | $\mathbb{F}_{p}^{m} \backslash \Delta$ | $\left[p^{m}-r-1, m,(p-1) p^{m-1}-r\right]$ | 2 | Griesmer | Thm.4.1 |
|  |  |  | $\left[p^{m}-2 r-2, m,(p-1) p^{m-1}-2 r-1\right]$ | 4 | Griesmer | Thm.4.4 |
|  |  |  | $\begin{gathered} {\left[p^{m}-3(r+1), m,(p-1) p^{m-1}-\right.} \\ 3 r-2] \\ \hline \end{gathered}$ | 5 | Griesmer | Thm.4.7 |
|  |  |  | $\begin{aligned} & {\left[p^{m}-(r+1)(p-1), m,(p-\right.} \\ & \left.1) p^{m-1}-(r+1) p+2 r+1\right] \end{aligned}$ | 4 | Griesmer | Thm.4.11 |
|  |  |  | $\begin{aligned} & {\left[p^{m}-(r+1)(p-2), m,(p-\right.} \\ & \text { 1) } \left.p^{m-1}-(r+1) p+3 r+1\right] \end{aligned}$ | 5 | Griesmer | Thm.4.14 |
| This paper | 4-ary | $\Delta_{A}+w \Delta_{B}$ | $\left[2^{\|A\|+\|B\|}-1,\|A \cup B\|, 2^{\|A\|+\|B\|-1}\right]$ | 2 |  | Prop.4.2 |
|  |  |  | $\left[\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-1,\|A \cup B\|\right]$ | $\leq 10$ |  | Prop.4.7 |
|  |  | $\mathbb{F}_{4}^{m} \backslash \Delta_{A}+w \Delta_{B}$ | $\begin{gathered} {\left[4^{m}-2^{\|A\|+\|B\|}, m, 3 \times 2^{2 m-2}-\right.} \\ \left.3 \times 2^{\|A\|+\|B\|-2}\right] \\ \hline \end{gathered}$ | 3 | Griesmer | Thm.4.4 |
|  |  |  | $\left[4^{m}-\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}, m\right]$ | $\leq 11$ |  | Thm.4.10 |
|  | binary | $\Delta_{A}+w \Delta_{B}$ | $\left[2^{\|A\|+\|B\|}-1,\|A\|+\|B\|, 2^{\|A\|+\|B\|-1}\right]$ | 1 |  | Prop.5.1 |
|  |  |  | $\left[2^{2 m}-2^{\|A\|+\|B\|}, 2 m\right]$ | 2 | Griesmer | Thm.5.2 |
|  |  | $\Delta+w \Delta$ | $\left[\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-1,2\|A \cup B\|\right]$ | $\leq 10$ |  | Prop.5.3 |
|  |  |  | $\begin{aligned} & {\left[\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-1,\left(2^{\|A\|}+\right.\right.} \\ & \left.\left.2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}-1-2\|A \cup B\|, 3\right] \\ & \hline \end{aligned}$ |  | Sphere <br> Packing | Thm.5.5 |
|  |  |  | $\left[4^{m}-\left(2^{\|A\|}+2^{\|B\|}-2^{\|A \cap B\|}\right)^{2}, 2 m\right]$ | $\leq 11$ |  | Prop.5.6 |

Table 14. Optimal quaternary linear codes from Theorem 4.4

| $m$ | A | B | $[n, k, d]$ Code | Remark |
| :---: | :---: | :---: | :---: | :---: |
| 2 | $(1,0)$ | $(1,0)$ | [12, 2, 9]* | new |
|  |  | $(0,1)$ | [12, 2, 9]* |  |
|  |  | $(1,1)$ | $[8,2,6]^{*}$ |  |
| 3 | $(1,0,0)$ | $(1,0,0)$ | [60, 3, 45]* | new |
|  |  | $(0,1,1)$ | [ $56,3,42]^{*}$ | new |
|  |  | $(1,1,1)$ | [ $48,3,36]^{*}$ |  |
|  | $(0,1,1)$ | ( $0,1,0$ ) | [ $56,3,42]^{*}$ | new |
|  |  | $(0,1,1)$ | $[48,3,36]^{*}$ | new |
|  |  | $(1,1,1)$ | [32, 3, 24] ${ }^{*}$ |  |
|  | $(1,1,1)$ | $(0,1,0)$ | $[48,3,36]^{*}$ |  |
|  |  | $(0,1,1)$ | $[32,3,24]^{*}$ |  |
| 4 | (1, 0, 0, 0) | $(1,0,0,0)$ | [252, 4, 189]* | new |
|  |  | $(1,1,0,0)$ | [248, 4, 186]* | new |
|  |  | $(1,1,1,0)$ | [240, 4, 180]* | new |
|  | (0, 0, 1, 1) | (0, $0,1,1)$ | [240, 4, 180]* | new |
|  |  | ( $0,1,1,1$ ) | $[224,4,168]^{*}$ | new |
|  |  | $(1,1,1,1)$ | [192, 4, 144]* | new |
|  | (1, 1, 1, 0) | (0, 1, 1, 0) | [224, 4, 168]* | new |
|  |  | (0, 1, 0, 0) | [240, 4, 180]* | new |
|  |  | $(1,1,1,0)$ | [192, 4, 144]* |  |
|  | (1, 1, 1, 1) | ( $0,1,0,0$ ) | [224, 4, 168]* | new |
|  |  | $(0,0,1,1)$ | [192, 4, 144]* | new |

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