# Shortened linear codes from APN and PN functions 

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#### Abstract

Linear codes generated by component functions of perfect nonlinear (PN) and almost perfect nonlinear (APN) functions and the first-order Reed-Muller codes have been an object of intensive study in coding theory. The objective of this paper is to investigate some binary shortened codes of two families of linear codes from APN functions and some $p$-ary shortened codes associated with PN functions. The weight distributions of these shortened codes and the parameters of their duals are determined. The parameters of these binary codes and $p$-ary codes are flexible. Many of the codes presented in this paper are optimal or almost optimal. The results of this paper show that the shortening technique is very promising for constructing good codes.


## Index Terms

Linear code, shortened code, PN function, APN function, $t$-design

## I. Introduction

Let $\operatorname{GF}(q)$ denote the finite field with $q=p^{m}$ elements, where $p$ is a prime and $m$ is a positive integer. A $[v, k, d]$ linear code $\mathcal{C}$ over $\operatorname{GF}(q)$ is a $k$-dimensional subspace of $\mathrm{GF}(q)^{v}$ with minimum (Hamming) distance $d$. Let $A_{i}$ denote the number of codewords with Hamming weight $i$ in a code $\mathcal{C}$ of length $v$. The weight enumerator of $C$ is defined by $1+A_{1} z+A_{2} z^{2}+\cdots+A_{v} z^{v}$. The sequence $\left(1, A_{1}, \ldots, A_{v}\right)$ is called the weight distribution of $\mathcal{C}$ and is an important research topic in coding theory, as it contains crucial information about the error correcting capability of the code. Thus the study of the weight distribution has attracted much attention in coding theory and much work focuses on the determination of the weight distributions of linear codes (see, for example, [14], [15], [16], [17], [35], [33], [39], [40], [46], [471). Denote by $C^{\perp}$ and $\left(A_{0}^{\perp}, A_{1}^{\perp}, \ldots, A_{v}^{\perp}\right)$ the dual code of a linear code $\mathcal{C}$ and its weight distribution, respectively. The Pless power moments [28], i.e.,

$$
\begin{equation*}
\sum_{i=0}^{v} i^{t} A_{i}=\sum_{i=0}^{t}(-1)^{i} A_{i}^{\perp}\left[\sum_{j=i}^{t} j!S(t, j) q^{k-j}(q-1)^{j-i}\binom{v-i}{v-j}\right] \tag{1}
\end{equation*}
$$

play an important role in calculating the weight distributions of linear codes, where $A_{0}=1,0 \leq t \leq v$ and $S(t, j)=\frac{1}{j!} \sum_{i=0}^{j}(-1)^{j-i}\left(\begin{array}{c}j \\ i \\ i\end{array}\right) i^{t}$. A code $\mathcal{C}$ is said to be a $t$-weight code if the number of nonzero $A_{i}$ in the sequence $\left(A_{1}, A_{2}, \cdots, A_{v}\right)$ is equal to $t$. A $[v, k, d]$ code over $\mathrm{GF}(q)$ is said to be distance-optimal if no $\left[v, k, d^{\prime}\right]$ code over $\mathrm{GF}(q)$ with $d^{\prime}>d$ exists, dimension-optimal if no $\left[v, k^{\prime}, d\right]$ code over $\mathrm{GF}(q)$ with $k^{\prime}>k$ exists, and length-optimal if no $\left[v^{\prime}, k, d\right]$ code over $\mathrm{GF}(q)$ with $v^{\prime}<v$ exists. A linear code is said to be optimal if it is distance-optimal, or dimension-optimal, or length-optimal, or meets a bound for linear codes.

[^0]Let $\mathcal{C}$ be a $[\mathrm{v}, k, d]$ linear code over $\mathrm{GF}(q)$ and $T$ a set of $t$ coordinate positions in $\mathcal{C}$. We use $\mathcal{C}^{T}$ to denote the code obtained by puncturing $\mathcal{C}$ on $T$, which is called the punctured code of $\mathcal{C}$ on $T$. Let $\mathcal{C}(T)$ be the set of codewords of $\mathcal{C}$ which are 0 on $T$. We now puncture $\mathcal{C}(T)$ on $T$, and obtain a linear code $\mathcal{C}_{T}$, which is called the shortened code of $\mathcal{C}$ on $T$. The following lemma plays an important role in determining the parameters of the punctured and shortened codes of $\mathcal{C}$.

Lemma 1. [28] Theorem 1.5.7] Let $\mathcal{C}$ be a $[v, k, d]$ linear code over $\mathrm{GF}(q)$ and $d^{\perp}$ the minimum distance of $\mathcal{C}^{\perp}$. Let $T$ be any set of $t$ coordinate positions. Then the following hold:

- $\left(\mathcal{C}_{T}\right)^{\perp}=\left(C^{\perp}\right)^{T}$ and $\left(C^{T}\right)^{\perp}=\left(C^{\perp}\right)_{T}$.
- If $t<\min \left\{d, d^{\perp}\right\}$, then the codes $\mathcal{C}_{T}$ and $\mathcal{C}^{T}$ have dimension $k-t$ and $k$, respectively.

The shortening and puncturing techniques are two important approaches to constructing new linear codes. Very recently, Tang et al. obtained some ternary linear codes with few weights by shortening and puncturing a class of ternary codes in [37]. Afterwards, they presented a general theory for punctured and shortened codes of linear codes supporting t-designs and generalized the Assmus-Mattson theorem in [38]. Liu et al. studied some shortened linear codes over finite fields in [32]. However, till now not much work about shortened codes has been done and it is in general hard to determine the weight distributions of shortened codes. Motivated by these facts, we investigate some shortened codes of linear codes from almost perfect nonlinear (APN) and perfect nonlinear (PN) functions, and determine their parameters in this paper. Many of these shortened codes are optimal or almost optimal.

The rest of this paper is arranged as follows. Section $\Pi$ introduces some notation and results related to group characters, Gauss sums, $t$-designs and linear codes from APN and PN functions. Section $\Pi$ III gives some general results about shortened codes. Section IV investigates some shortened codes of binary linear codes from APN functions. Section $\square$ studies some shortened codes of two classes of special linear codes from PN functions. Section VI concludes this paper and makes concluding remarks.

## II. Preliminaries

In this section, we briefly recall some results on group characters, Gauss sums, $t$-designs, and linear codes from APN and PN functions. These results will be used later in this paper. We begin this section by fixing some notation throughout this paper.

- $p$ is a prime and $p^{*}=(-1)^{(p-1) / 2} p$ for odd prime $p$.
- $\zeta_{p}=e^{\frac{2 \pi \sqrt{-1}}{p}}$ is the primitive $p$-th root of unity.
- $q$ is a power of $p$.
- $\mathrm{GF}(q)^{*}=\mathrm{GF}(q) \backslash\{0\}$.
- $\mathrm{Tr}_{q / p}$ is the trace function from $\mathrm{GF}(q)$ to $\mathrm{GF}(p)$.
- SQ and NSQ denote the set of all squares and nonsquares in $\operatorname{GF}(p)^{*}$, respectively.
- $\eta$ and $\bar{\eta}$ are the quadratic characters of $\operatorname{GF}(q)^{*}$ and $\operatorname{GF}(p)^{*}$, repsectively. We extend these quadratic characters by letting $\eta(0)=0$ and $\bar{\eta}(0)=0$.


## A. Group characters and Gauss sums

An additive character of $\mathrm{GF}(q)$ is a nonzero function $\chi$ from $\mathrm{GF}(q)$ to the set of nonzero complex numbers such that $\chi(x+y)=\chi(x) \chi(y)$ for any pair $(x, y) \in \operatorname{GF}(q)^{2}$. For each $b \in \operatorname{GF}(q)$, the function

$$
\begin{equation*}
\chi_{b}(x)=\zeta_{p}^{\operatorname{Tr}_{q / p}(b x)} \tag{2}
\end{equation*}
$$

defines an additive character of $\mathrm{GF}(q)$. When $b=0, \chi_{0}(x)=1$ for all $x \in \mathrm{GF}(q)$, and $\chi_{0}$ is called the trivial additive character of $\operatorname{GF}(q)$. The character $\chi_{1}$ in (2) is called the canonical additive character
of $\operatorname{GF}(q)$. It is well known that every additive character of $\operatorname{GF}(q)$ can be written as $\chi_{b}(x)=\chi_{1}(b x)$ [31, Theorem 5.7]. The orthogonality relation of additive characters is given by

$$
\sum_{x \in \mathrm{GF}(q)} \chi_{1}(a x)= \begin{cases}q & \text { for } a=0 \\ 0 & \text { for } a \in \mathrm{GF}(q)^{*}\end{cases}
$$

The Gauss sum $G\left(\eta, \chi_{1}\right)$ over $\operatorname{GF}(q)$ is defined by

$$
\begin{equation*}
G\left(\eta, \chi_{1}\right)=\sum_{x \in \operatorname{GF}(q)^{*}} \eta(x) \chi_{1}(x)=\sum_{x \in \operatorname{GF}(q)} \eta(x) \chi_{1}(x) \tag{3}
\end{equation*}
$$

and the Gauss sum $G\left(\bar{\eta}, \bar{\chi}_{1}\right)$ over $\operatorname{GF}(p)$ is defined by

$$
\begin{equation*}
G\left(\bar{\eta}, \bar{\chi}_{1}\right)=\sum_{x \in \operatorname{GF}(p)^{*}} \bar{\eta}(x) \bar{\chi}_{1}(x)=\sum_{x \in \operatorname{GF}(p)} \bar{\eta}(x) \bar{\chi}_{1}(x) \tag{4}
\end{equation*}
$$

where $\bar{\chi}_{1}$ is the canonical additive character of $\operatorname{GF}(p)$.
The following four lemmas are proved in [31, Theorems 5.15, 5.33, Corollary 5.35] and [16, Lemma 7], respectively.
Lemma 2. [31] Let $q=p^{m}$ and $p$ be an odd prime. Then

$$
\begin{aligned}
G\left(\eta, \chi_{1}\right) & =(-1)^{m-1}(\sqrt{-1})^{\left(\frac{p-1}{2}\right)^{2} m} \sqrt{q} \\
& =\left\{\begin{array}{lll}
(-1)^{m-1} \sqrt{q} & \text { for } p \equiv 1 & (\bmod 4) \\
(-1)^{m-1}(\sqrt{-1})^{m} \sqrt{q} & \text { for } p \equiv 3 & (\bmod 4)
\end{array}\right.
\end{aligned}
$$

and

$$
G\left(\bar{\eta}, \bar{\chi}_{1}\right)=\sqrt{-1}^{\left(\frac{p-1}{2}\right)^{2}} \sqrt{p}=\sqrt{p *}
$$

Lemma 3. [31] Let $\chi$ be a nontrivial additive character of $\mathrm{GF}(q)$ with $q$ odd, and let $f(x)=a_{2} x^{2}+a_{1} x+$ $a_{0} \in \operatorname{GF}(q)[x]$ with $a_{2} \neq 0$. Then

$$
\sum_{x \in \operatorname{GF}(q)} \chi(f(x))=\chi\left(a_{0}-a_{1}^{2}\left(4 a_{2}\right)^{-1}\right) \eta\left(a_{2}\right) G(\eta, \chi)
$$

Lemma 4. [31] Let $\chi_{b}$ be a nontrivial additive character of $\mathrm{GF}(q)$ with $q$ even and $f(x)=a_{2} x^{2}+a_{1} x+a_{0} \in$ $\operatorname{GF}(q)[x]$, where $b \in \operatorname{GF}(q)^{*}$. Then

$$
\sum_{x \in \operatorname{GF}(q)} \chi_{b}(f(x))= \begin{cases}\chi_{b}\left(a_{0}\right) q & \text { if } a_{2}=b a_{1}^{2} \\ 0 & \text { otherwise }\end{cases}
$$

Lemma 5. [16] Let $p$ be an odd prime. If $m \geq 2$ is even, then $\eta(x)=1$ for each $x \in \mathrm{GF}(p)^{*}$. If $m \geq 1$ is odd, then $\eta(x)=\bar{\eta}(x)$ for each $x \in \operatorname{GF}(p)$.

Let $e$ be a positive integer and $(a, b) \in \operatorname{GF}(q)^{2}$, define the exponential sum

$$
\begin{equation*}
S_{e}(a, b)=\sum_{x \in \mathrm{GF}(q)} \chi_{1}\left(a x^{p^{e}+1}+b x\right) \tag{5}
\end{equation*}
$$

Then we have the following five known results.
Lemma 6. [12] Let $e$ be a positive integer and $m$ be even with $\operatorname{gcd}(m, e)=1$. Let $p=2, q=2^{m}$ and $a \in \operatorname{GF}(q)^{*}$. Then

$$
S_{e}(a, 0)= \begin{cases}(-1)^{\frac{m}{2}} 2^{\frac{m}{2}} & \text { if } a \neq \alpha^{3 t} \text { for any } t \\ -(-1)^{\frac{m}{2}} 2^{\frac{m}{2}+1} & \text { if } a=\alpha^{3 t} \text { for some } t\end{cases}
$$

where $\alpha$ is a generator of $\operatorname{GF}(q)^{*}$.

Lemma 7. [8] Let $e, h$ be positive integers and $m$ be even with $\operatorname{gcd}(m, e)=1$. Let $p=2, q=2^{m}$ and $a \in \mathrm{GF}(q)^{*}$. Then

$$
\sum_{b \in \operatorname{GF}(q)^{*}}\left(S_{e}(a, b)\right)^{h}= \begin{cases}\left(2^{m}-1\right) 2^{\frac{m}{2} \cdot h} & \text { if } h \text { is even and } a \neq \alpha^{3 t} \text { for any } t \\ \left(2^{m-2}-1\right) 2^{\left(\frac{m}{2}+1\right) \cdot h} & \text { if } h \text { is even and } a=\alpha^{3 t} \text { for some } t\end{cases}
$$

where $\alpha$ is a generator of $\operatorname{GF}(q)^{*}$.
Lemma 8. [9] Let $p$ be an odd prime, $q=p^{m}$, and $e$ be any positive integer such that $m / \operatorname{gcd}(m, e)$ is odd. Suppose $a \in \mathrm{GF}(q)^{*}$ and $b \in \mathrm{GF}(q)^{*}$. Let $x_{a, b}$ be the unique solution of the equation

$$
a^{p^{e}} x^{p^{2 e}}+a x+b^{p^{e}}=0
$$

Then

$$
S_{e}(a, b)= \begin{cases}(-1)^{m-1} \sqrt{q} \eta(-a) \chi_{1}\left(-a x_{a, b}^{p^{e}+1}\right), & \text { if } p \equiv 1 \bmod 4 \\ (-1)^{m-1} \sqrt{-1}^{3 m} \sqrt{q} \eta(-a) \chi_{1}\left(-a x_{a, b}^{p^{e}+1}\right), & \text { if } p \equiv 3 \bmod 4\end{cases}
$$

Lemma 9. [44] Let the notation and assumptions be the same as those in the previous lemma. Write $\Delta=\sum_{c \in \operatorname{GF}(p) *} S_{e}(a c, b c)$. Then we have the following results.

- If $m$ is odd, then

$$
\Delta= \begin{cases}0, & \text { if } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)=0 \\ \eta(a) \eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right) \sqrt{q} \sqrt{p^{*}}, & \text { if } p \equiv 1 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0 \\ \eta(a) \eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right) \sqrt{-1}^{3 m} \sqrt{q} \sqrt{p^{*}}, & \text { if } p \equiv 3 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0\end{cases}
$$

- If $m$ is even, then

$$
\Delta= \begin{cases}-(p-1) \eta(a) \sqrt{q}, & \text { if } p \equiv 1 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)=0, \\ \eta(a) \sqrt{q}^{q}, & \text { if } p \equiv 1 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0, \\ -\sqrt{-1}^{m}(p-1) \eta(a) \sqrt{q}, & \text { if } p \equiv 3 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)=0, \\ \sqrt{-1}^{m} \eta(a) \sqrt{q}, & \text { if } p \equiv 3 \bmod 4 \text { and } \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0\end{cases}
$$

Lemma 10. [44] Let $p$ be an odd prime, $m$ and $e$ be positive integers such that $m / \operatorname{gcd}(m, e)$ is odd. Let $q=p^{m}$. Define

$$
\hat{N}_{0}(a, b)=\sharp\left\{x \in \mathrm{GF}(q): \operatorname{Tr}_{q / p}\left(a x^{p^{p}+1}+b x\right)=0\right\} .
$$

Then we have the following results.

- If $a=0$ and $b=0$, then $\hat{N}_{0}(a, b)=q$.
- If $a=0$ and $b \neq 0$, then $\hat{N}_{0}(a, b)=p^{m-1}$.
- If $a \neq 0$ and $b=0$, then

$$
\hat{N}_{0}(a, b)= \begin{cases}p^{m-1} & \text { if } m \text { is odd, } \\ \frac{1}{p}(q-(p-1) \eta(a) \sqrt{q}), & \text { if } m \text { is even and } p \equiv 1 \bmod 4 \\ \frac{1}{p}\left(q-\sqrt{-1}^{m}(p-1) \eta(a) \sqrt{q}\right), & \text { if } m \text { is even and } p \equiv 3 \bmod 4\end{cases}
$$

- If $a \neq 0$ and $b \neq 0$, then $\hat{N}_{0}(a, b)=\frac{1}{p}(q+\Delta)$, where $\Delta$ was given in Lemma 9


## B. Combinatorial t-designs and related results

Let $k, t$ and $v$ be positive integers with $1 \leq t \leq k \leq v$. Let $\mathcal{P}$ be a set of $v \geq 1$ elements, and let $\mathcal{B}$ be a set of $k$-subsets of $\mathcal{P}$. The incidence structure $\mathbb{D}=(\mathcal{P}, \mathcal{B})$ is said to be a $t-(v, k, \lambda)$ design if every $t$-subset of $\mathcal{P}$ is contained in exactly $\lambda$ elements of $\mathcal{B}$. The elements of $\mathcal{P}$ are called points, and those of $\mathcal{B}$ are referred to as blocks. We usually use $b$ to denote the number of blocks in $\mathcal{B}$. A $t$-design is called simple if $\mathcal{B}$ has no repeated blocks. A $t$-design is called symmetric if $v=b$ and trivial if $k=t$ or $k=v$. When $t \geq 2$ and $\lambda=1$, a $t$-design is called a Steiner system and traditionally denoted by $S(t, k, v)$.

Linear codes and $t$-designs are companions. A $t$-design $\mathbb{D}=(\mathcal{P}, \mathcal{B})$ induces a linear code over $\operatorname{GF}(p)$ for any prime $p$. Let $\mathcal{P}=\left\{p_{1}, \ldots, p_{v}\right\}$. For any block $B \in \mathcal{B}$, the characteristic vector of $B$ is defined by the vector $\mathbf{c}_{B}=\left(c_{1}, \ldots, c_{v}\right) \in\{0,1\}^{V}$, where

$$
c_{i}= \begin{cases}1, & \text { if } p_{i} \in B \\ 0, & \text { if } p_{i} \notin B\end{cases}
$$

For a prime $p$, a linear code $C_{p}(\mathbb{D})$ over the prime field $\operatorname{GF}(p)$ from the design $\mathbb{D}$ is spanned by the characteristic vectors of the blocks of $\mathbb{B}$, which is the subspace $\operatorname{Span}\left\{\mathbf{c}_{B}: B \in \mathcal{B}\right\}$ of the vector space $\operatorname{GF}(p)^{v}$. Linear codes $C_{p}(\mathbb{D})$ from designs $\mathbb{D}$ have been studied and documented in the literature (see, for example, [1], [17], [41], [42]).

On the other hand, a linear code $\mathcal{C}$ may induce a $t$-design under certain conditions, which is formed by supports of codewords of a fixed Hamming weight in $\mathcal{C}$. Let $\mathcal{P}(\mathcal{C})$ be the set of the coordinate positions of $\mathcal{C}$, where $\# \mathcal{P}(\mathcal{C})=v$ is the length of $\mathcal{C}$. For a codeword $\mathbf{c}=\left(c_{i}\right)_{i \in \mathcal{P}(\mathcal{C})}$ in $\mathcal{C}$, the support of $\mathbf{c}$ is defined by

$$
\operatorname{Supp}(\mathbf{c})=\left\{i: c_{i} \neq 0, i \in \mathcal{P}(\mathcal{C})\right\} .
$$

Let $\mathcal{B}_{w}(\mathcal{C})=\{\operatorname{Supp}(\mathbf{c}): w t(\mathbf{c})=w$ and $\mathbf{c} \in \mathcal{C}\}$. For some special $\mathcal{C},\left(\mathcal{P}(\mathcal{C}), \mathcal{B}_{w}(\mathcal{C})\right)$ is a $t$-design. In this way, many $t$-designs are derived from linear codes (see, for example, [1], [18], [19], [20], [25], [26], [29], [36], [41], [42]). A major approach to constructing $t$-designs from linear codes is the use of linear codes with $t$-homogeneous or $t$-transitive automorphism groups (see [14, Theorem 4.18]). Another major approach to constructing $t$-designs from codes is the use of the Assmus-Mattson Theorem [3], [28]. The following Assmus-Mattson Theorem for constructing simple $t$-designs was developed in [2].
Theorem 11. Let $\mathcal{C}$ be a linear code over $\operatorname{GF}(q)$ with length $v$ and minimum weight d. Let d denote the minimum weight of the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}$. Let $t\left(1 \leq t<\min \left\{d, d^{\perp}\right\}\right)$ be an integer such that there are at most $d^{\perp}-t$ weights of $\mathcal{C}$ in the range $\{1,2, \ldots, v-t\}$. Then the following hold:

- $\left(\mathcal{P}(\mathcal{C}), \mathcal{B}_{k}(\mathcal{C})\right)$ is a simple $t$-design provided that $A_{k} \neq 0$ and $d \leq k \leq w$, where $w$ is defined to be the largest integer satisfying $w \leq v$ and

$$
w-\left\lfloor\frac{w+q-2}{q-1}\right\rfloor<d .
$$

- $\left(\mathcal{P}\left(\mathcal{C}^{\perp}\right), \mathcal{B}_{k}\left(\mathcal{C}^{\perp}\right)\right)$ is a simple $t$-design provided that $A_{k}^{\perp} \neq 0$ and $d^{\perp} \leq k \leq w^{\perp}$, where $w^{\perp}$ is defined to be the largest integer satisfying $w^{\perp} \leq v$ and

$$
w^{\perp}-\left\lfloor\frac{w^{\perp}+q-2}{q-1}\right\rfloor<d^{\perp} .
$$

We will need the following results about the punctured and shortened codes of $\mathcal{C}$ documented in [38, Lemma 3.1,Theorem 3.2] .

Lemma 12. [38] Let $\mathcal{C}$ be a linear code of length $v$ and minimum distance $d$ over $\operatorname{GF}(q)$ and $d^{\perp}$ the minimum distance of $\mathcal{C}^{\perp}$. Let $t$ and $k$ be two positive integers with $0<t<\min \left\{d, d^{\perp}\right\}$ and $1 \leq k \leq v-t$.

Let $T$ be a set of $t$ coordinate positions in $\mathcal{C}$. Suppose that $\left(\mathcal{P}(\mathcal{C}), \mathcal{B}_{i}(\mathcal{C})\right)$ is a $t$-design for all $i$ with $k \leq i \leq k+t$. Then

$$
A_{k}\left(C^{T}\right)=\sum_{i=0}^{t} \frac{\binom{v-t}{k}\binom{k+i}{t}\binom{t}{i}}{\binom{v-t}{k-t+i}\binom{v}{t}} A_{k+i}(\mathcal{C})
$$

Theorem 13. [38] Let $\mathcal{C}$ be a $[v, \bar{k}, d]$ linear code over $\mathrm{GF}(q)$ and $d^{\perp}$ be the minimum distance of $\mathcal{C}^{\perp}$. Let t be a positive integer with $0<t<\min \left\{d, d^{\perp}\right\}$. Let $T$ be a set of $t$ coordinate positions in $\mathcal{C}$. Suppose that $\left(\mathcal{P}(\mathcal{C}), \mathcal{B}_{i}(\mathcal{C})\right)$ is a $t$-design for any $i$ with $d \leq i \leq v-t$. Then the shortened code $\mathcal{C}_{T}$ is a linear code of length $v-t$ and dimension $\bar{k}-t$. The weight distribution $\left(A_{k}\left(\mathcal{C}_{T}\right)\right)_{k=0}^{v-t}$ of $\mathcal{C}_{T}$ is independent of the specific choice of the elements in T. Specifically,

$$
A_{k}\left(\mathcal{C}_{T}\right)=\frac{\binom{k}{t}\binom{v-t}{k}}{\binom{v}{t}\binom{v-t}{k-t}} A_{k}(\mathcal{C})
$$

## C. Linear codes from APN and PN functions

Let $m, \tilde{m}$ be two positive integers with $m \geq \tilde{m}$ and $F$ be a mapping from $\operatorname{GF}\left(p^{m}\right)$ to $\operatorname{GF}\left(p^{\tilde{m}}\right)$. Define

$$
\delta_{F}=\max \left\{\delta_{F}(a, b): a \in \operatorname{GF}\left(p^{m}\right)^{*}, b \in \operatorname{GF}\left(p^{\tilde{m}}\right)\right\},
$$

where $\delta_{F}(a, b)=\#\left\{x \in \operatorname{GF}\left(p^{m}\right): F(x+a)-F(x)=b\right\}, a \in \operatorname{GF}\left(p^{m}\right)$ and $b \in \operatorname{GF}\left(p^{\tilde{m}}\right)$. The function $F(x)$ is called PN function if $\delta_{F}=p^{m-\tilde{m}}$, and it is called APN function if $m=\tilde{m}$ and $\delta_{F}=2$. From the above definition one immediately sees that $F(x)$ is PN if and only if $F(x+a)-F(x)$ is balanced for each $a \in \operatorname{GF}\left(p^{m}\right)^{*}$. Currently, all known PN and APN functions over $\mathrm{GF}\left(p^{m}\right)$ are summarized in [4], [6], [7], [10], [11], [13], [14], [22], [45]. It is known that PN and APN functions are very important functions for constructing linear codes with good parameters (see, for example, [5], [34], [43], [44]).

Let $q=p^{m}$ and let $\mathcal{C}$ denote the linear code of length $q$ defined by

$$
\begin{equation*}
\mathcal{C}=\left\{\left(\operatorname{Tr}_{q / p}(a f(x)+b x+c)\right)_{x \in \operatorname{GF}(q)}: a, b, c \in \mathrm{GF}(q)\right\} \tag{6}
\end{equation*}
$$

where $f(x)$ is a polynomial over $\mathrm{GF}(q)$. Then we can regard $\mathrm{GF}(q)$ as the set of the coordinate positions $\mathcal{P}(\mathcal{C})$ of $\mathcal{C}$. It is known that $\mathcal{C}$ has dimension $2 m+1$ and the weight distribution in Table $\square$ when $p=2$, $m \geq 5$ is odd and $f(x)=x^{s}$ is an APN function, where $s$ takes the following values [14].

- $s=2^{e}+1$, where $\operatorname{gcd}(e, m)=1$ and $e$ is a positive integer.
- $s=2^{2 e}-2^{e}+1$, where $e$ is a positive integer and $\operatorname{gcd}(e, m)=1$.
- $s=2^{(m-1) / 2}+3$.
- $s=2^{(m-1) / 2}+2^{(m-1) / 4}-1$, where $m \equiv 1(\bmod 4)$.
- $s=2^{(m-1) / 2}+2^{(3 m-1) / 4}-1$, where $m \equiv 3(\bmod 4)$.

When $f(x)=x^{2^{e}+1}, p=2$ and $m \geq 4$ is even with $\operatorname{gcd}(e, m)=1$, the code $\mathcal{C}$ defined in (6) has dimension $2 m+1$ and the weight distribution in Table [II [14].

TABLE I
The weight distribution of $\mathcal{C}$ For $m$ odd

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $\left(2^{m}-1\right) 2^{m-1}$ |
| $2^{m-1}$ | $\left(2^{m}-1\right)\left(2^{m+1}-2^{m}+2\right)$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $\left(2^{m}-1\right) 2^{m-1}$ |
| $2^{m}$ | 1 |

It is known that the code $\mathcal{C}$ defined in (6) has dimension $2 m+1$ and a few weights when $p$ is an odd prime and $f(x)=x^{s}$ is a PN function. If $s$ takes the following values [14], [30]

TABLE II
The weight distribution of $\mathcal{C}$ For $m$ EVEN

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{m / 2}$ | $\left(2^{m}-1\right) 2^{m-2} / 3$ |
| $2^{m-1}-2^{(m-2) / 2}$ | $\left(2^{m}-1\right) 2^{m+1} / 3$ |
| $2^{m-1}$ | $2\left(2^{m}-1\right)\left(2^{m-2}+1\right)$ |
| $2^{m-1}+2^{(m-2) / 2}$ | $\left(2^{m}-1\right) 2^{m+1} / 3$ |
| $2^{m-1}+2^{m / 2}$ | $\left(2^{m}-1\right) 2^{m-2} / 3$ |
| $2^{m}$ | 1 |

- $s=2$,
- $s=p^{e}+1$, where $m / \operatorname{gcd}(m, e)$ is odd,
- $s=\left(3^{e}+1\right) / 2$, where $p=3, e$ is odd and $\operatorname{gcd}(m, e)=1$,
then $f(x)=x^{s}$ is a PN and also planar function, $\left.\operatorname{Tr}_{q / p}(\beta f(x))\right)$ is a weakly regular bent function [23], [27] for any $\beta \in \mathrm{GF}(q)^{*}$, and the code $\mathcal{C}$ defined in (6) has four or six weights [30].

Let $f(x)$ be a function from $\operatorname{GF}(q)$ to $\mathrm{GF}(p)$, the Walsh transform of $f$ at a point $\beta \in \mathrm{GF}(q)$ is defined by

$$
\mathcal{W}_{f}(\beta)=\sum_{x \in \operatorname{GF}(q)} \zeta_{p}^{f(x)-\operatorname{Tr}_{q / p}(\beta x)}
$$

The function $f(x)$ is said to be a $p$-ary bent function, if $\left|\mathcal{W}_{f}(\beta)\right|=p^{\frac{m}{2}}$ for any $\beta \in \mathbb{F}_{q}$. A bent function $f(x)$ is weakly regular if there exists a complex $u$ with unit magnitude satisfying $\mathcal{W}_{f}(\beta)=u p^{\frac{m}{2}} \zeta_{p}^{f^{*}(\beta)}$ for some function $f^{*}(x)$. Such function $f^{*}(x)$ is called the dual of $f(x)$. A weakly regular bent function $f(x)$ satisfies

$$
\mathcal{W}_{f}(\beta)=\varepsilon{\sqrt{p^{*}}}^{m} \zeta_{p}^{f^{*}(\beta)},
$$

where $\varepsilon= \pm 1$ is called the sign of the Walsh Transform of $f(x)$. Let $\mathcal{R} \mathcal{F}$ be the set of $p$-ary weakly regular bent functions with the following two properties:

- $f(0)=0$; and
- $f(a x)=a^{h} f(x)$ for any $a \in \mathrm{GF}(p)^{*}$ and $x \in \mathrm{GF}(q)$, where $h$ is a positive even integer with $\operatorname{gcd}(h-$ $1, p-1)=1$.
We will need the following results about $p$-ary weakly regular bent functions in [39].
Lemma 14. [39] Let $\beta \in \operatorname{GF}(q)^{*}$ and $f(x) \in \mathcal{R} \mathcal{F}$ with $\mathcal{W}_{f}(0)=\varepsilon{\sqrt{p^{*}}}^{m}$. Define

$$
N_{f, \beta}=\#\left\{x \in \operatorname{GF}(q): f(x)=0 \text { and } \operatorname{Tr}_{q / p}(\beta x)=0\right\}
$$

If $f^{*}(\beta)=0$, then

$$
N_{f, \beta}= \begin{cases}p^{m-2}+\varepsilon \bar{\eta}^{m / 2}(-1)(p-1) p^{(m-2) / 2}, & \text { if } m \text { is even } ; \\ p^{m-2}, & \text { if } m \text { is odd } .\end{cases}
$$

Lemma 15. [39] Let $\beta \in \operatorname{GF}(q)^{*}$ and $f(x) \in \mathcal{R} \mathcal{F}$ with $\mathcal{W}_{f}(0)=\varepsilon{\sqrt{p^{*}}}^{m}$. Let

$$
N_{s q, \beta}=\#\left\{x \in \mathrm{GF}(q): f(x) \in S Q \text { and } \operatorname{Tr}_{q / p}(\beta x)=0\right\},
$$

and

$$
N_{n s q, \beta}=\#\left\{x \in \mathrm{GF}(q): f(x) \in N S Q \text { and } \operatorname{Tr}_{q / p}(\beta x)=0\right\}
$$

We have the following results.

- If $m$ is even and $f^{*}(\beta)=0$, then

$$
N_{s q, \beta}=N_{n s q, \beta}=\frac{p-1}{2}\left(p^{m-2}-\varepsilon \bar{\eta}^{m / 2}(-1) p^{(m-2) / 2}\right) .
$$

- If $m$ is odd and $f^{*}(\beta)=0$, then

$$
N_{s q, \beta}=\frac{p-1}{2}\left(p^{m-2}+\varepsilon \sqrt{p *}^{m-1}\right)
$$

and

$$
N_{n s q, \beta}=\frac{p-1}{2}\left(p^{m-2}-\varepsilon \sqrt{p *}^{m-1}\right) .
$$

## III. Shortened binary linear codes with special weight distributions

In this section, we give some general results on the shortened codes of linear codes with the weight distributions in Tables [ and II.

Let $T$ be a set of $t$ coordinate positions in $\mathcal{C}$ (i.e., $T$ is a $t$-subset of $\mathcal{P}(\mathcal{C})$ ). Define

$$
\Lambda_{T, w}(\mathcal{C})=\{\operatorname{Supp}(\mathbf{c}): \mathbf{c} \in \mathcal{C}, w t(\mathbf{c})=w, \text { and } T \subseteq \operatorname{Supp}(\mathbf{c})\}
$$

and $\lambda_{T, w}(\mathcal{C})=\# \Lambda_{T, w}(\mathcal{C})$.
We will consider some shortened $\operatorname{code} \mathcal{C}_{T}$ of $C$ for the case $m \geq 4$ and $t \geq 1$.

## A. Shortened linear codes holding $t$-designs

Let $p=2$ and $q=2^{m}$. Notice that if a binary code $\mathcal{C}$ has length $2^{m}$ and the weight distribution in Table (resp. Table 【II), then the code $\mathcal{C}$ holds 3 -designs (resp. 2-designs) ([14], [21]). The following two theorems are easily derived from Theorem [13, Tables I and II, and we omit their proofs.
Theorem 16. Let $m \geq 5$ be odd, and $\mathcal{C}$ be a binary linear code with length $2^{m}$ and the weight distribution in Table $\mathbb{\square}$ Let $T$ be a $t$-subset of $\mathcal{P}(\mathcal{C})$. We have the following results.

- If $t=1$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-1,2 m, 2^{m-1}-2^{(m-1) / 2}\right]$ binary linear code with the weight distribution in Table III
- If $t=2$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-2,2 m-1,2^{m-1}-2^{(m-1) / 2}\right]$ binary linear code with the weight distribution in Table IV
- If $t=3$, then the shortened code $\mathcal{C}_{T}$ is $a\left[2^{m}-3,2 m-2,2^{m-1}-2^{(m-1) / 2}\right]$ binary linear code with the weight distribution in Table $\square$

TABLE III
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ ODD AND $t=1$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $2^{(m-5) / 2}\left(2^{m}-1\right)\left(2+2^{(1+m) / 2}\right)$ |
| $2^{m-1}$ | $-1+2^{m-1}+2^{2 m-1}$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $2^{(m-5) / 2}\left(2^{m}-1\right)\left(-2+2^{(1+m) / 2}\right)$ |

Example 17. Let $m=5$ and $T$ be a 1 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 16 is a $[31,10,12]$ binary linear code with the weight enumerator $1+310 z^{12}+527 z^{16}+186 z^{20}$. The code $\mathcal{C}_{T}$ is optimal. The dual code of $\mathcal{C}_{T}$ has parameters $[31,21,5]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

Example 18. Let $m=5$ and $T$ be a 2-subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 16 is a $[30,9,12]$ linear code with the weight enumerator $1+190 z^{12}+255 z^{16}+66 z^{20}$. The code $\mathcal{C}_{T}$ is optimal. The dual code of $\mathcal{C}_{T}$ has parameters $[30,21,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

TABLE IV
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ ODD AND $t=2$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $2^{(m-7) / 2}\left(-4+2^{2+m}+2^{(1+3 m) / 2}\right)$ |
| $2^{m-1}$ | $-1+2^{2 m-2}$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $2^{(m-7) / 2}\left(4-2^{2+m}+2^{(1+3 m) / 2}\right)$ |

TABLE V
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ ODD AND $t=3$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $-2^{(m-3) / 2}+3 \cdot 2^{(3 m-7) / 2}+2^{m-3}+2^{2 m-4}$ |
| $2^{m-1}$ | $\left(-1+2^{m-2}\right)\left(1+2^{m-1}\right)$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $2^{(m-3) / 2}-3 \cdot 2^{(3 m-7) / 2}+2^{m-3}+2^{2 m-4}$ |

Example 19. Let $m=5$ and $T$ be a 3-subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 16 is a $[29,8,12]$ binary linear code with the weight enumerator $1+114 z^{12}+119 z^{16}+22 z^{20}$. The code $\mathcal{C}_{T}$ is optimal. The dual code of $\mathcal{C}_{T}$ has parameters $[29,21,3]$ and is almost optimal according to the tables of best known codes maintained at http://www.codetables.de

Theorem 20. Let $m \geq 4$ be even, and $\mathcal{C}$ be a binary linear code with length $2^{m}$ and the weight distribution in Table II Let $T$ be a $t$-subset of $\mathcal{P}(\mathcal{C})$. We have the following results.

- If $t=1$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-1,2 m, 2^{m-1}-2^{m / 2}\right]$ binary linear code with the weight distribution in Table VI]
- If $t=2$, then the shortened code $\mathcal{C}_{T}$ is $a\left[2^{m}-2,2 m-1,2^{m-1}-2^{m / 2}\right]$ binary linear code with the weight distribution in Table VII.

TABLE VI
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ EVEN AND $t=1$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{m / 2}$ | $1 / 3 \cdot 2^{-3+m / 2}\left(2+2^{m / 2}\right)\left(-1+2^{m}\right)$ |
| $2^{m-1}-2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2}\left(-1+2^{m / 2}\right)\left(1+2^{m / 2}\right)^{2}$ |
| $2^{m-1}$ | $\left(2^{m}-1\right)\left(1+2^{m-2}\right)$ |
| $2^{m-1}+2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2}\left(-1+2^{m / 2}\right)^{2}\left(1+2^{m / 2}\right)$ |
| $2^{m-1}+2^{m / 2}$ | $1 / 3 \cdot 2^{-3+m / 2}\left(-2+2^{m / 2}\right)\left(-1+2^{m}\right)$ |

Example 21. Let $m=4$ and $T$ be a 1 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 20 is a $[15,8,4]$ linear code with the weight enumerator $1+15 z^{4}+100 z^{6}+75 z^{8}+60 z^{10}+5 z^{12}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[15,7,5]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

TABLE VII
The weight distribution of $\mathcal{C}_{T}$ FOR $t=2$ AND $m$ EVEN

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-4}\left(2+2^{m / 2}\right)\left(2^{m}+2^{1+m / 2}-2\right)$ |
| $2^{m-1}-2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2-1}\left(1+2^{m / 2}\right)\left(2^{m}+2^{m / 2}-2\right)$ |
| $2^{m-1}$ | $\left(2^{m-1}-1\right)\left(1+2^{m-2}\right)$ |
| $2^{m-1}+2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2-1}\left(-1+2^{m / 2}\right)\left(2^{m}-2^{m / 2}-2\right)$ |
| $2^{m-1}+2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-4}\left(4+2^{1+m / 2}+2^{3 m / 2}-2^{2+m}\right)$ |

Example 22. Let $m=4$ and $T$ be a 2-subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 20 is a $[14,7,4]$ binary linear code with the weight enumerator $1+11 z^{4}+60 z^{6}+35 z^{8}+20 z^{10}+z^{12}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[14,7,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

## B. Several general results on shortened codes

Lemma 23. Let $m \geq 5$ be odd (resp., $m \geq 4$ be even), and $\mathcal{C}$ be a binary linear code with the length $2^{m}$ and the weight distribution in Table $\square$ (resp., Table III). Then the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}$ has parameters $\left[2^{m}, 2^{m}-2 m-1,6\right]$.
Proof. The weight distribution in Table (or III) means that the dimension of $\mathcal{C}$ is $2 m+1$. Thus, the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}$ has dimension $2^{m}-2 m-1$. Since the code length of $\mathcal{C}$ is $2^{m}$, from the weight distribution in Table【 (or III) and the first seven Pless power moments in (1), it is easily obtain that $A_{6}\left(C^{\perp}\right)>0$ and $A_{i}\left(C^{\perp}\right)=0$ for any $i \in\{1,2,3,4,5\}$. The desired conclusions then follow .

Theorem 24. Let $m \geq 4$, and $\mathcal{C}$ be a binary linear code with length $2^{m}$ and the weight distribution in Table $\rrbracket$ for odd $m$ and Table 【I] for even $m$. Let $T$ be a 4 -subset of $\mathcal{P}(\mathcal{C})$ and $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\lambda$, then $\lambda=0$ or 1. Furthermore, we have the following results.
(I) If $m \geq 5$ is odd and $\lambda=0$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-4,2 m-3,2^{m-1}-2^{(m-1) / 2}\right]$ binary linear code with the weight distribution in Table VIII
(II) If $m \geq 5$ is odd and $\lambda=1$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-4,2 m-3,2^{m-1}-2^{(m-1) / 2}\right]$ binary linear code with the weight distribution in Table IX

TABLE VIII
The weight distribution of $\mathcal{C}_{T}$ For $\lambda=0$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $-2^{(m-3) / 2}+2^{m-3}+2^{2 m-5}+2^{(3 m-5) / 2}$ |
| $2^{m-1}$ | $-1-2^{m-2}+4^{m-2}$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $2^{(m-3) / 2}+2^{m-3}+2^{2 m-5}-2^{(3 m-5) / 2}$ |

Proof. By the definition of $\Lambda_{T, 6}\left(C^{\perp}\right)$, we have

$$
\lambda=\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\#\left\{\operatorname{Supp}(\mathbf{c}): \mathbf{c} \in \mathcal{C}^{\perp}, w t(\mathbf{c})=6 \text { and } T \subseteq \operatorname{Supp}(\mathbf{c})\right\}
$$

TABLE IX
The weight distribution of $\mathcal{C}_{T}$ FOR $\lambda=1$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{(m-1) / 2}$ | $3 \times 2^{m-4}-2^{(-3+m) / 2}+2^{-5+2 m}+2^{(-5+3 m) / 2}$ |
| $2^{m-1}$ | $2^{-4}\left(-8+2^{m}\right)\left(2+2^{m}\right)$ |
| $2^{m-1}+2^{(m-1) / 2}$ | $3 \times 2^{m-4}+2^{(-3+m) / 2}+2^{-5+2 m}-2^{(-5+3 m) / 2}$ |

If $\lambda \geq 2$, there would be $\operatorname{Supp}\left(\mathbf{c}_{1}\right), \operatorname{Supp}\left(\mathbf{c}_{2}\right) \in \Lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)$. Then $\mathbf{c}_{1}+\mathbf{c}_{2} \in \mathcal{C}^{\perp}$ and the weight $w t\left(\mathbf{c}_{1}+\mathbf{c}_{2}\right) \leq 4$. This is a contradiction to the minimum distance 6 of $\mathcal{C}^{\perp}$ in Lemma 23, Thus, $\lambda=0$ or 1 .

We treat the weight distribution of $\mathcal{C}_{T}$ according to the value of $\lambda$ as follows.
(I) The case that $\lambda=0$ and $m$ is odd.

By Lemma 23, the minimum distance of $C^{\perp}$ is 6 . Thus,

$$
\begin{equation*}
A_{1}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=A_{2}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=0, A_{1}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=A_{2}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=0 \tag{7}
\end{equation*}
$$

and the shortened code $\mathcal{C}_{T}$ has length $n=2^{m}-4$ and dimension $k=2 m-3$ from $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=0$ and Lemma 1. By definition and Lemma [23, we have $A_{i}\left(C_{T}\right)=0$ for $i \notin\left\{0, i_{1}, i_{2}, i_{3}\right\}$, where $i_{1}=2^{m-1}-2^{(m-1) / 2}$, $i_{2}=2^{m-1}$ and $i_{3}=2^{m-1}+2^{(m-1) / 2}$. Therefore, from (7) and (1), the first three Pless power moments

$$
\left\{\begin{array}{l}
A_{i_{1}}+A_{i_{2}}+A_{i_{3}}=2^{2 m-3}-1, \\
i_{1} A_{i_{1}}+i_{2} A_{i_{2}}+i_{3} A_{i_{3}}=2^{2 m-3-1}\left(2^{m}-4\right), \\
i_{1}^{2} A_{i_{1}}+i_{2}^{2} A_{i_{2}}+i_{3}^{2} A_{i_{3}}=2^{2 m-3-2}\left(2^{m}-4\right)\left(2^{m}-4+1\right)
\end{array}\right.
$$

yield the weight distribution in Table VIII, This completes the proof of (I).
(II) The case that $\lambda=1$ and $m$ is odd.

The proof is similar to that of (I). Since $\lambda_{T, 6}\left(C^{\perp}\right)=1$ and the minimum distance of $C^{\perp}$ is 6 , from Lemma 1 we have

$$
\begin{align*}
& A_{1}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=0, A_{2}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=1 \\
& A_{1}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=0, \quad A_{2}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=1 \tag{8}
\end{align*}
$$

Then the desired conclusions follow from (8), the definitions and the first three Pless power moments of (1). This completes the proof.

Lemma 25. Let $m \geq 4$ be even, and $C$ be a binary linear code with length $2^{m}$ and the weight distribution in Table [II Let $T$ be a 3-subset of $\mathcal{P}(\mathcal{C})$. Suppose $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\lambda$, then $A_{1}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=A_{2}\left(\left(\mathcal{C}^{\perp}\right)^{T}\right)=0$, $A_{3}\left(\left(C^{\perp}\right)^{T}\right)=\lambda$ and $A_{4}\left(\left(C^{\perp}\right)^{T}\right)=2 \cdot\left(2^{m-2}-1\right)^{2}-3 \lambda$.
Proof. By Lemma 23, the minimum distance of $\mathcal{C}^{\perp}$ is 6 . Thus, from $\# T=3$ and the definition of $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)$, we have $A_{1}\left(\left(C^{\perp}\right)^{T}\right)=A_{2}\left(\left(C^{\perp}\right)^{T}\right)=0$ and $A_{3}\left(\left(C^{\perp}\right)^{T}\right)=\lambda$. Note that the code $\mathcal{C}$ has length $2^{m}$ and dimension $2 m+1$. By Lemma 23, Table II and the first seven Pless power moments of (1), we have

$$
A_{6}\left(C^{\perp}\right)=\frac{1}{45} \cdot 2^{m-4}\left(2^{m}-4\right)^{2}\left(2^{m}-1\right)
$$

Further, from Theorem 11, Lemmas 12] and 23, we conclude deduce that $\left(\mathcal{P}\left(\mathcal{C}^{\perp}\right), \mathcal{B}_{6}\left(\mathcal{C}^{\perp}\right)\right)$ is a 2-design and

$$
A_{4}\left(\left(C^{\perp}\right)^{\left\{t_{1}, t_{2}\right\}}\right)=\frac{\binom{6}{2}}{\binom{q}{2}} \cdot A_{6}\left(C^{\perp}\right)=\frac{2}{3} \cdot\left(2^{m-2}-1\right)^{2}
$$

for any $\left\{t_{1}, t_{2}\right\} \subseteq \mathcal{P}(\mathcal{C})$. Let $T=\left\{t_{1}, t_{2}, t_{3}\right\}$. Since $\# T=3$ and the minimum distance of $\mathcal{C}^{\perp}$ is 6 , an easy computation shows that

$$
\begin{aligned}
A_{4}\left(\left(C^{\perp}\right)^{T}\right) & =\sum_{1 \leq i<j \leq 3}\left(A_{4}\left(\left(C^{\perp}\right)^{\left\{t_{i}, t_{j}\right\}}\right)-\lambda_{T, 6}\left(C^{\perp}\right)\right) \\
& =\binom{3}{2}\left(A_{4}\left(\left(C^{\perp}\right)^{\left\{t_{1}, t_{2}\right\}}\right)-\lambda\right) .
\end{aligned}
$$

Then the desired conclusions follow.
Theorem 26. Let $m \geq 4$ be even, and $\mathcal{C}$ be a binary linear code with length $2^{m}$ and the weight distribution in Table [I] Let $T$ be a 3-subset of $\mathcal{P}(\mathcal{C})$. Suppose $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\lambda$, then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-3,2 m-2,2^{m-1}-2^{m / 2}\right]$ binary linear code with the weight distribution in Table $X$

Proof. The proof is similar to that of Theorem 24. From Lemma 1 and $\# T=3$, the shortened code $\mathcal{C}_{T}$ has length $n=2^{m}-3$ and dimension $k=2 m-2$. By definition and the weight distribution in Table 【, we have $A_{i}\left(\mathcal{C}_{T}\right)=0$ for $i \notin\left\{0, i_{1}, i_{2}, i_{3}, i_{4}, i_{5}\right\}$, where $i_{1}=2^{m-1}-2^{m / 2}, i_{2}=2^{m-1}-2^{(m-2) / 2}, i_{3}=2^{m-1}$, $i_{4}=2^{m-1}+2^{(m-2) / 2}$ and $i_{5}=2^{m-1}+2^{m / 2}$. Moreover, from Lemmas 1 and 25 we have $A_{1}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=$ $A_{2}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=0, A_{3}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=\lambda$ and $A_{4}\left(\left(\mathcal{C}_{T}\right)^{\perp}\right)=2 \cdot\left(2^{m-2}-1\right)^{2}-3 \lambda$. Therefore, the first five Pless power moments of (1) yield the weight distribution in Table $\mathbb{X}$. This completes the proof.

TABLE X
The weight distribution of $\mathcal{C}_{T}$ FOR $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\lambda$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-5}\left(8+2^{3+m / 2}+2^{3 m / 2}+2^{2+m}+12 \lambda\right)$ |
| $2^{m-1}-2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2-3}\left(\left(2+2^{m / 2}\right)\left(-8+3 \cdot 2^{m / 2}+2^{1+m}\right)-6 \lambda\right)$ |
| $2^{m-1}$ | $-1+4^{m-2}$ |
| $2^{m-1}+2^{(m-2) / 2}$ | $1 / 3 \cdot 2^{m / 2-3}\left(\left(-2+2^{m / 2}\right)\left(-8-3 \cdot 2^{m / 2}+2^{1+m}\right)+6 \lambda\right)$ |
| $2^{m-1}+2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-5}\left(-8+2^{3+m / 2}+2^{3 m / 2}-2^{2+m}-12 \lambda\right)$ |

## IV. Shortened linear codes from APN functions

Let $p=2$ and $q=2^{m}$. In this section, we study some shortened codes $\mathcal{C}_{T}$ of linear codes $\mathcal{C}$ defined by (6) and determine their parameters for the case that $f(x)$ is an APN monomial function $x^{2^{e}+1}$. It is known that $\mathcal{C}$ has the weight distribution in Tables (resp. Tables III) when $m$ is odd (resp. $m$ is even).

Let $T$ be a $t$-subset of $\mathcal{P}(\mathcal{C}):=\mathrm{GF}(q)$. We will consider some shortened codes $\mathcal{C}_{T}$ of $\mathcal{C}$ for the cases $t=3$ or 4 .

## A. Some shortened codes for the case $t=4$ and $m$ odd

We notice that it is difficult to determine the value of $\lambda_{T, 6}\left(C^{\perp}\right)$ in Theorem 24 for general APN function $f$. We will determine $\lambda_{T, 6}\left(C^{\perp}\right)$ for APN function $f(x)=x^{2^{e}+1}$. To this end, the following lemma will be needed.

Lemma 27. Let $e$ and $m \geq 4$ be positive integers with $\operatorname{gcd}(m, e)=1$. Let $q=2^{m}$ and $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ a 4 -subset of $\mathrm{GF}(q)$. Denote $S_{i}=x_{1}^{i}+x_{2}^{i}+x_{3}^{i}+x_{4}^{i}$. Let $N$ be the number of solutions $(x, y) \in \mathrm{GF}(q)^{2}$ of the system of equations

$$
\left\{\begin{array}{cl}
x+y & =S_{1}  \tag{9}\\
x^{2^{e}+1}+y^{2^{e}+1} & =S_{2^{e}+1} \\
\#\left\{x_{1}, x_{2}, x_{3}, x_{4}, x, y\right\} & =6
\end{array}\right.
$$

Then $N=2$ if $S_{1} \neq 0$ and $\operatorname{Tr}_{q / 2}\left(\frac{S_{2 e}+1}{S_{1}^{2 e+1}}+1\right)=0$, and $N=0$ otherwise.
Proof. Let us denote by $\mathcal{C}$ the linear code from $f(x)=x^{2^{e}+1}$ given in (6). By Lemma 23, the minimum weight of the dual code $\mathcal{C}^{\perp}$ is equal to 6 . Consequently, (9) is equivalent to the following system of equation

$$
\left\{\begin{array}{cl}
x+y & =S_{1}  \tag{10}\\
x^{2^{e}+1}+y^{2^{e}+1} & =S_{2^{e}+1}
\end{array}\right.
$$

Substituting $y=x+S_{1}$ into the second equation of (10) leads to

$$
\begin{align*}
& x^{2^{e}+1}+\left(x+S_{1}\right)^{2^{e}+1}+S_{2^{e}+1} \\
& =x^{2^{e}+1}+\left(x^{2^{e}}+S_{1}^{2^{e}}\right)\left(x+S_{1}\right)+S_{2^{e}+1} \\
& =S_{1} x^{2^{e}}+S_{1}^{2^{e}} x+S_{1}^{2^{e}+1}+S_{2^{e}+1} \\
& =0 \tag{11}
\end{align*}
$$

We claim that $S_{1} \neq 0$ if $N \neq 0$. On the contrary, suppose that $N \neq 0$ and $S_{1}=0$. Now (11) clearly forces $S_{2^{e}+1}=0$. It follows that the four coordinate positions $x_{1}, x_{2}, x_{3}, x_{4}$ give rise to a codeword of weight 4 of $\mathcal{C}^{\perp}$. This contradicts the fact that the minimum weight of $\mathcal{C}^{\perp}$ equals 6 . Therefore $S_{1} \neq 0$ if $N \neq 0$. In particular, $N=0$ if $S_{1}=0$.

When $S_{1} \neq 0$, Equation (11) is equivalent to

$$
\begin{equation*}
\frac{S_{1} x^{2^{e}}+S_{1}^{2^{e}} \cdot x+S_{1}^{2^{e}+1}+S_{2^{e}+1}}{S_{1}^{2^{e}+1}}=\left(\frac{x}{S_{1}}\right)^{2^{e}}+\frac{x}{S_{1}}+1+\frac{S_{2^{e}+1}}{S_{1}^{2^{e}+1}}=0 . \tag{12}
\end{equation*}
$$

If $S_{1} \neq 0$ and $\operatorname{Tr}_{q / 2}\left(\frac{S_{2} e+1}{S_{1}^{2 e+1}}+1\right) \neq 0$, it may be concluded that there is no solution in $\operatorname{GF}(q)$ to Equation (12). Thus $N=0$.

If $S_{1} \neq 0$ and $\operatorname{Tr}_{q / 2}\left(\frac{S_{2} e+1}{S_{1}^{2 e+1}}+1\right)=0$, we see that Equation (12) has exactly two different solutions $x, x+S_{1} \in \mathrm{GF}(q)$ from $\operatorname{gcd}(m, e)=1$. This means that Equation (10) has exactly two different solutions $\left(x, x+S_{1}\right)$ and $\left(x+S_{1}, x\right)$ in $\operatorname{GF}(q)^{2}$. Therefore $N=2$. This completes the proof.
Lemma 28. Let $e$ and $m \geq 4$ be positive integers with $\operatorname{gcd}(m, e)=1$. Let $q=2^{m}, f(x)=x^{2^{e}+1}$ and $\mathcal{C}$ be defined in (6). Let $T=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ be a 4-subset of $\mathcal{P}(\mathcal{C})$. Then $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=1$ if $\sum_{i=1}^{4} x_{i} \neq 0$ and $\operatorname{Tr}_{q / 2}\left(\sum_{i=1}^{4} x_{i}^{2^{e}+1} /\left(\sum_{i=1}^{4} x_{i}\right)^{2^{e}+1}+1\right)=0$, and $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=0$ otherwise.
Proof. By definition, the dual code $\mathcal{C}^{\perp}$ of $\mathcal{C}$ has minimum distance 6. By definition we have $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=\frac{N}{2!}$, where $N$ was defined in Lemma 27, Then the desired conclusions follow from Lemma 27.

By Theorem 24 and Lemma 28, we have the following theorem, which is one of the main results in this paper.

Theorem 29. Let $m \geq 5$ be odd and $e$ be a positive integer with $\operatorname{gcd}(m, e)=1$. Let $q=2^{m}, f(x)=x^{2^{e}+1}$ and $\mathcal{C}$ be defined in (6). Let $T=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ be a 4 -subset of $\mathcal{P}(\mathcal{C})$. Then $\mathcal{C}_{T}$ has the weight distribution of Table VIII if $\sum_{i=1}^{4} x_{i} \neq 0$ and $\operatorname{Tr}_{q / 2}\left(\sum_{i=1}^{4} x_{i}^{2^{e}+1} /\left(\sum_{i=1}^{4} x_{i}\right)^{2^{e}+1}\right)=1$, and Table IX otherwise.
Example 30. Let $m=5, q=2^{5}$ and $\alpha$ be a primitive element of $\operatorname{GF}(q)$ with minimum polynomial $\alpha^{5}+\alpha^{2}+1=0$. Let $e=1$ and $T=\left\{\alpha^{1}, \alpha^{2}, \alpha^{4}, \alpha^{5}\right\}$. Then $\gamma=\alpha^{17}, \operatorname{Tr}_{q / 2}\left(\frac{\bar{\gamma}}{\gamma^{3}}\right)=0$ and $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=0$, where $\gamma=\alpha^{1}+\alpha^{2}+\alpha^{4}+\alpha^{5}$ and $\bar{\gamma}=\alpha^{3}+\alpha^{6}+\alpha^{12}+\alpha^{15}$. The shortened code $\mathcal{C}_{T}$ in Theorem 29 is $a$ $[28,7,12]$ binary linear code with the weight enumerator $1+66 z^{12}+55 z^{16}+6 z^{20}$. The code $\mathcal{C}_{T}$ is optimal according to the tables of best known codes maintained at http://www.codetables.de
Example 31. Let $m=5, q=2^{5}$ and $\alpha$ be a primitive element of $\operatorname{GF}(q)$ with minimal polynomial $\alpha^{5}+$ $\alpha^{2}+1=0$. Let $e=1$ and $T=\left\{\alpha^{1}, \alpha^{2}, \alpha^{3}, \alpha^{4}\right\}$. Then $\gamma=\alpha^{24}, \operatorname{Tr}_{q / 2}\left(\frac{\bar{\gamma}}{\gamma^{3}}\right)=1$, and $\lambda_{T, 6}\left(C^{\perp}\right)=1$, where $\gamma=\alpha^{1}+\alpha^{2}+\alpha^{3}+\alpha^{4}$ and $\bar{\gamma}=\alpha^{3}+\alpha^{6}+\alpha^{9}+\alpha^{12}$. The shortened code $\mathcal{C}_{T}$ in Theorem 29 is a $[28,7,12]$ binary linear code with the weight enumerator $1+68 z^{12}+51 z^{16}+8 z^{20}$. The code $\mathcal{C}_{T}$ is optimal according to the tables of best known codes maintained at http://www.codetables.de.
B. Some shortened codes for the case $t=3$ and $m$ even

For any $T=\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq \mathcal{P}(\mathcal{C})$, we notice that it is difficult to determine the value of $\lambda$ in Theorem 26. We will study a class of special linear codes $\mathcal{C}$ from the APN monomial functions $x^{2^{e}+1}$ defined by (6). In the following, we will determine the value of $\lambda$ and the parameters of the shortened code $\mathcal{C}_{T}$. We need the result in the following lemma.

Lemma 32. Let e be a positive integer, $m$ be even with $\operatorname{gcd}(m, e)=1$, and $q=2^{m}$. Let $\hat{N}$ be the number of solutions $\left(x_{1}, x_{2}, x_{3}\right) \in \mathrm{GF}(q)^{3}$ of the system of equations

$$
\left\{\begin{array}{c}
x_{1}+x_{2}+x_{3}=a  \tag{13}\\
x_{1}^{2^{e}+1}+x_{2}^{2^{2}+1}+x_{3}^{2^{e}+1}=b
\end{array}\right.
$$

where $a, b \in \operatorname{GF}(q)$ and $a^{2^{e}+1} \neq b$. Then $\hat{N}=2^{m}+(-2)^{m / 2}-2$ if $a^{2^{e}+1}+b$ is not a cubic residue, and $\hat{N}=2^{m}+(-2)^{m / 2+1}-2$ if $a^{2^{e}+1}+b$ is a cubic residue.
Proof. Replacing $x_{1}$ with $x+a, x_{2}$ with $y+a$ and $x_{3}$ with $z+a$, we have

$$
\left\{\begin{array}{ccc}
x+y+z & =0  \tag{14}\\
x^{2^{e}+1}+y^{2^{e}+1}+z^{z^{e}+1} & =a^{2^{e}+1}+b
\end{array}\right.
$$

Substituting $z=x+y$ into the second equation of (14) yields to

$$
\begin{equation*}
x^{2^{e}} y+y^{2^{e}} x=a^{2^{e}+1}+b \tag{15}
\end{equation*}
$$

Thus, $\hat{N}$ equals the number of solutions $(x, y) \in \mathrm{GF}(q)^{2}$ to Equation (15).
Since $a^{2^{e}+1}+b \neq 0$, replacing $y$ with $x y^{\prime}$ in Equation (15) gives

$$
\begin{equation*}
x^{2^{e}} y^{\prime}+y^{\prime 2^{e}} x=x^{2^{e}+1} y^{\prime}+y^{\prime 2^{e}} x^{2^{e}+1}=x^{2^{e}+1}\left(y^{\prime}+y^{\prime 2^{e}}\right)=a^{2^{e}+1}+b . \tag{16}
\end{equation*}
$$

A rearrangement of Equation (16) yields

$$
\begin{equation*}
y^{\prime}+y^{\prime 2^{e}}=\left(a^{2^{e}+1}+b\right) x^{-\left(2^{e}+1\right)} . \tag{17}
\end{equation*}
$$

Then

$$
\begin{equation*}
\operatorname{Tr}\left(\left(a^{2^{e}+1}+b\right) x^{-\left(2^{e}+1\right)}\right)=0 \tag{18}
\end{equation*}
$$

Further, from Lemma 6 we get

$$
\begin{align*}
& \#\left\{x \in \operatorname{GF}(q)^{*}: \operatorname{Tr}\left(\left(a^{2^{e}+1}+b\right) x^{-\left(2^{e}+1\right)}\right)=0\right\} \\
& =\#\left\{x \in \operatorname{GF}(q)^{*}: \operatorname{Tr}\left(\left(a^{2^{e}+1}+b\right) x^{\left(2^{e}+1\right)}\right)=0\right\} \\
& =\frac{1}{2} \cdot \sum_{u \in \operatorname{GF}(2)} \sum_{x \in \operatorname{GF}(q)^{*}}(-1)^{\operatorname{Tr}\left(u\left(a^{2^{e}+1}+b\right) x^{\left(2^{e}+1\right)}\right)} \\
& =\frac{1}{2}\left(q-2+\sum_{x \in \operatorname{GF}(q)}(-1)^{\operatorname{Tr}\left(\left(a^{a^{e}+1}+b\right) x^{\left(2^{e}+1\right)}\right)}\right) \\
& = \begin{cases}\frac{1}{2}\left(q-2+(-1)^{m / 2} 2^{m / 2}\right), & \text { if } a^{2^{e}+1}+b \text { is not a cubic residue, } \\
\frac{1}{2}\left(q-2-(-1)^{m / 2} 2^{m / 2+1}\right), & \text { if } a^{2^{e}+1}+b \text { is a cubic residue. }\end{cases} \tag{19}
\end{align*}
$$

Since $\operatorname{gcd}(m, e)=1$, it follows easily that if $\operatorname{Tr}\left(\left(a^{2^{e}+1}+b\right) x^{-\left(2^{e}+1\right)}\right)=0$, where $x \in \operatorname{GF}(q)^{*}$, then there exactly exist two $y^{\prime}$ in $\operatorname{GF}(q)$ satisfying $y^{\prime}+y^{2^{e}}=\left(a^{2^{e}+1}+b\right) x^{-\left(2^{e}+1\right)}$. Then the desired conclusions follow from Equation (19).
Theorem 33. Let e be a positive integer and $m \geq 4$ be even with $\operatorname{gcd}(m, e)=1$. Let $q=2^{m}, f(x)=x^{2^{e}+1}$ and $\mathcal{C}$ be defined in (6). Let $T=\left\{x_{1}, x_{2}, x_{3}\right\}$ be a 3-subset of $\mathcal{P}(\mathcal{C})$. Then

$$
\lambda=\lambda_{T, 6}\left(C^{\perp}\right)= \begin{cases}\frac{1}{6}\left(q-2+(-1)^{m / 2} 2^{m / 2}\right)-1, & \text { if } a^{2^{e}+1}+b \text { is not a cubic residue }  \tag{20}\\ \frac{1}{6}\left(q-2-(-1)^{m / 2} 2^{m / 2+1}\right)-1, & \text { if } a^{2^{e}+1}+b \text { is a cubic residue }\end{cases}
$$

where $a=\sum_{i=1}^{3} x_{i}$ and $b=\sum_{i=1}^{3} x_{i}^{2^{e}+1}$. Moreover, the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-3,2 m-2,2^{m-1}-2^{m / 2}\right]$ binary linear code with the weight distribution in Table $\mathbb{X}$
Proof. By definition, the code $C$ has parameters $\left[2^{m}, 2 m+1,2^{m-1}-2^{m / 2}\right]$ and the weight distribution in Table [I] By Lemma 23, the minimum distance of $\mathcal{C}^{\perp}$ is 6 . Consider the system of equations given by

$$
\left\{\begin{array}{cc}
x+y+z=a  \tag{21}\\
x^{2^{e}+1}+y^{2^{e}+1}+z^{2^{e}+1}=b
\end{array}\right.
$$

Since $\left(x_{1}, x_{2}, x_{3}\right)$ must be the solution $(x, y, z) \in \operatorname{GF}(q)^{3}$ of Equation (21), by definition we have $\lambda_{T, 6}\left(C^{\perp}\right)=$ $\frac{\hat{N}}{3!}-1$, where $\hat{N}$ was defined in Lemma 32. Then the desired conclusions follow from Lemma 32 and Theorem 26,
Example 34. Let $m=4, q=2^{4}$ and $\alpha$ be a primitive element of $\operatorname{GF}(q)$ with minimal polynomial $\alpha^{4}+$ $\alpha+1=0$. Let $e=1$ and $T=\left\{\alpha^{1}, \alpha^{2}, \alpha^{4}\right\}$. Then $\left(\alpha^{1}+\alpha^{2}+\alpha^{4}\right)^{3}+\left(\alpha^{3}+\alpha^{6}+\alpha^{12}\right)=1, \lambda=0$ and the shortened code $\mathcal{C}_{T}$ in Theorem 26 is a $[13,6,4]$ binary linear code with the weight enumerator $1+7 z^{4}+36 z^{6}+15 z^{8}+4 z^{10}+z^{12}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[13,7,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.
Example 35. Let $m=4, q=2^{4}$ and $\alpha$ be a primitive element of $\operatorname{GF}(q)$ with minimal polynomial $\alpha^{4}+$ $\alpha+1=0$. Let $e=1$ and $T=\left\{\alpha^{2}, \alpha^{5}, \alpha^{7}\right\}$. Then $\left(\alpha^{2}+\alpha^{5}+\alpha^{7}\right)^{3}+\left(\alpha^{6}+\alpha^{15}+\alpha^{21}\right)=\alpha^{11}, \lambda=2$ and the shortened code $\mathcal{C}_{T}$ in Theorem 26 is a [13,6,4] binary linear code with the weight enumerator $1+8 z^{4}+34 z^{6}+15 z^{8}+6 z^{10}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[13,7,3]$ and is almost optimal according to the tables of best known codes maintained at http://www.codetables.de.
C. Some shortened codes for the case $t=4$ and $m$ even

Let $T=\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ be a 4 -subset of $\mathcal{P}(\mathcal{C})$. Magma programs show that the weight distributions of $\mathcal{C}_{T}$ for the codes $\mathcal{C}$ from APN functions are very complex. Thus, it is difficult to determine their parameters in general. In this subsection, we will study a class of special linear codes $C$ with the weight distribution in Table $\Pi$ and determine the parameters of $\mathcal{C}_{T}$ for certain 4-subsets $T$ in Theorem 38 ,

In order to determine the parameters of $\mathcal{C}_{T}$, we need the next two lemmas.
Lemma 36. Let $m \geq 4$ be even and $q=2^{m}$. Define

$$
\begin{aligned}
& R_{(3, i)}=\#\left\{x \in \operatorname{GF}(q)^{*}: \operatorname{Tr}_{q / 2}(x)=i \text { and } x \text { is a cubic residue }\right\} \\
& \bar{R}_{(3, i)}=\#\left\{x \in \operatorname{GF}(q)^{*}: \operatorname{Tr}_{q / 2}(x)=i \text { and } x \text { is not a cubic residue }\right\},
\end{aligned}
$$

where $i=0$ or $i=1$. Then

$$
\left\{\begin{array}{l}
R_{(3,0)}=\frac{1}{6}\left(2^{m}-2+(-2)^{m / 2+1}\right) \\
R_{(3,1)}=\frac{2^{m}-1}{3}-\frac{1}{6}\left(2^{m}-2+(-2)^{m / 2+1}\right) \\
\bar{R}_{(3,0)}=\left(2^{m-1}-1\right)-\frac{1}{6}\left(2^{m}-2+(-2)^{m / 2+1}\right) \\
\bar{R}_{(3,1)}=\frac{2^{m}}{3}-\frac{(-2)^{m / 2}}{3}
\end{array}\right.
$$

Proof. By definition, we get

$$
\begin{aligned}
R_{(3,0)} & =\frac{1}{3} \cdot \#\left\{x \in \operatorname{GF}(q)^{*}: \operatorname{Tr}_{q / 2}\left(x^{3}\right)=0\right\} \\
& =\frac{1}{6} \sum_{z \in \mathrm{GF}(2)} \sum_{x \in \operatorname{GF}(q)^{*}}(-1)^{z \operatorname{Tr}_{q / 2}\left(x^{3}\right)} \\
& =\frac{1}{6}\left(q-2+\sum_{x \in \operatorname{GF}(q)}(-1)^{\operatorname{Tr}_{q / 2}\left(x^{3}\right)}\right) .
\end{aligned}
$$

Then the value of $R_{(3,0)}$ follows from Lemma 6. Note that there are $\frac{q-1}{3}$ cubic residues in $\mathrm{GF}(q)^{*}$. This gives

$$
\#\left\{x \in \mathrm{GF}(q)^{*}: \operatorname{Tr}_{q / 2}(x)=0\right\}=\frac{q}{2}-1
$$

Then the desired conclusions follow from

$$
\left\{\begin{array}{l}
R_{(3,0)}+\bar{R}_{(3,0)}=\frac{q}{2}-1 \\
R_{(3,0)}+R_{(3,1)}=\frac{q-1}{3}, \\
\bar{R}_{(3,0)}+\bar{R}_{(3,0)}=\frac{2(q-1)}{3} .
\end{array}\right.
$$

This completes the proof.
Lemma 37. Let $m \geq 4$ be even, e be a positive integer with $\operatorname{gcd}(m, e)=1$, and $q=2^{m}$. Let $N_{(0,1)}$ be the number of solutions $(x, y, z, u) \in \mathrm{GF}(q)^{4}$ of the system of equations

Then $N_{(0,1)}=q\left(q-2-(-1)^{m / 2} 2^{m / 2+1}\right)$.

Proof. Let us first observe that (22) is equivalent to the following system of equations

$$
\left\{\begin{array}{lllll}
x & +y & +z & +u & =0  \tag{23}\\
x^{2^{e}+1} & +y^{2^{e}+1} & +z^{2^{e}+1} & +u^{2^{e}+1} & =1
\end{array}\right.
$$

Set $S_{j}=x^{j}+y^{j}+z^{j}+u^{j}$, where $x, y, z, u \in \mathrm{GF}(q)$ and $j$ is a positive integer. An easy computation shows that

$$
\begin{aligned}
N_{(0,1)}= & \frac{1}{q^{2}} \sum_{a, b, x, y, y, z, u \in \operatorname{GF}(q)} \chi_{1}\left(b S_{1}\right) \chi_{1}\left(a\left(S_{2^{e}+1}-1\right)\right) \\
= & \frac{1}{q^{2}} \sum_{a, b \in \mathrm{GF}(q)} \chi_{1}(-a)\left(\sum_{x \in \operatorname{GF}(q)} \chi_{1}\left(a x^{2^{e}+1}+b x\right)\right)^{4} \\
= & \frac{1}{q^{2}}\left(\sum_{a \in \operatorname{GF}(q)} \chi_{1}(-a)\left(\sum_{x \in \operatorname{GF}(q)} \chi_{1}\left(a x^{2^{e}+1}\right)\right)^{4}+\right. \\
& \left.\sum_{a \in \operatorname{GF}(q)^{*} b \in \operatorname{GF}(q)^{*}} \chi_{1}(-a)\left(\sum_{x \in \operatorname{GF}(q)} \chi_{1}\left(a x^{2^{e}+1}+b x\right)\right)^{4}\right) \\
= & \frac{1}{q^{2}}\left(q^{4}+\left(R_{(3,0)}-R_{(3,1)}\right) 2^{m+4}+\left(\bar{R}_{(3,0)}-\bar{R}_{(3,1)}\right) 2^{m}+\right. \\
& \left.\left(R_{(3,0)}-R_{(3,1)}\right)\left(2^{3 m+2}-2^{2 m+4}\right)+\left(\bar{R}_{(3,0)}-\bar{R}_{(3,1)}\right)\left(2^{m}-1\right) 2^{2 m}\right),
\end{aligned}
$$

where $R_{(3,0)}, R_{(3,1)}, \bar{R}_{(3,0)}$ and $\bar{R}_{(3,1)}$ were defined in Lemma 36 and the last equality holds due to Lemmas 6 and 7, Then the desired conclusions follow from Lemma 36 ,
Theorem 38. Let $m \geq 4$ be even and e be a positive integer with $\operatorname{gcd}(m, e)=1$. Let $q=2^{m}, f(x)=x^{2^{e}+1}$ and $\mathcal{C}$ be defined in (6). Let $T=\mathrm{GF}(4)=\left\{0,1, w, w^{2}\right\} \subseteq \mathrm{GF}(q)$, where $w$ is a generator of $\mathrm{GF}(4)^{*}$. Then the shortened code $\mathcal{C}_{T}$ is a $\left[2^{m}-4,2 m-3,2^{m-1}-2^{m / 2}\right]$ binary linear code with the weight distribution in Table XI

TABLE XI
The weight distribution of $\mathcal{C}_{T}$ in Theorem 38

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $2^{m-1}-2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-6}\left(-16+2^{3 m / 2}-2^{m+1}\left(-4+(-1)^{m / 2}\right)-2^{4+m / 2}\left(-1+(-1)^{m / 2}\right)\right)$ |
| $2^{m-1}-2^{(m-2) / 2}$ | $1 / 24 \cdot\left(2^{m / 2+2}+2^{m}\right)\left(2^{m}+(-1)^{m / 2} 2^{m / 2}-2\right)$ |
| $2^{m-1}$ | $-1+2^{2 m-5}-(-1)^{m / 2} 2^{3 m / 2-4}$ |
| $2^{m-1}+2^{(m-2) / 2}$ | $1 / 24 \cdot\left(-2^{m / 2+2}+2^{m}\right)\left(2^{m}+(-1)^{m / 2} 2^{m / 2}-2\right)$ |
| $2^{m-1}+2^{m / 2}$ | $1 / 3 \cdot 2^{m / 2-6}\left(16+2^{3 m / 2}-2^{m+1}\left(4+(-1)^{m / 2}\right)+2^{4+m / 2}\left(1+(-1)^{m / 2}\right)\right)$ |

Proof. By Lemma [23, $\mathcal{C}$ is a $\left[2^{m}, 2 m+1,2^{m-1}-2^{m / 2}\right]$ binary code with the weight distribution in Table III and the minimum distance of $\mathcal{C}^{\perp}$ is 6 . Note that $\lambda_{T, 6}\left(\mathcal{C}^{\perp}\right)=0$ by Lemma 28 and the definition of $T$. Thus, the minimum distance of $\left(C^{\perp}\right)^{T}$ is at least 3. This means that

$$
\begin{equation*}
A_{1}\left(\left(C^{\perp}\right)^{T}\right)=A_{2}\left(\left(C^{\perp}\right)^{T}\right)=0 \tag{24}
\end{equation*}
$$

Further, from Theorem 33 and the definition of $T$, we have

$$
\begin{equation*}
\lambda_{\hat{T}, 6}\left(C^{\perp}\right)=\frac{1}{6}\left(q-2-(-1)^{m / 2} 2^{m / 2+1}\right)-1 \tag{25}
\end{equation*}
$$

for any $\hat{T}=\left\{\hat{x}_{1}, \hat{x}_{2}, \hat{x}_{3}\right\} \subseteq T$. Therefore,

$$
\begin{equation*}
A_{3}\left(\left(C^{\perp}\right)^{T}\right)=\binom{4}{3} \cdot\left(\frac{1}{6}\left(q-2-(-1)^{\frac{m}{2}} 2^{\frac{m}{2}+1}\right)-1\right) \tag{26}
\end{equation*}
$$

Note that the solutions of the system (22) have symmetrical property and $(x, y, z, u)=\left(0,1, w, w^{2}\right)$ is a solution of the system (22). From Lemma 37 we get

$$
\begin{equation*}
\lambda_{T, 8}\left(C^{\perp}\right)=\frac{N_{(0,1)}}{4!}-1-\binom{4}{3} \cdot \lambda_{\hat{T}, 6}\left(C^{\perp}\right) \tag{27}
\end{equation*}
$$

where $N_{(0,1)}$ was defined in Lemma 37 and $\lambda_{\hat{T}, 6}\left(\mathcal{C}^{\perp}\right)$ was given in (25). By the proof of Lemma 25, we have

$$
\begin{equation*}
A_{4}\left(\left(C^{\perp}\right)^{\left\{\bar{x}_{1}, \bar{x}_{2}\right\}}\right)=\frac{2}{3} \cdot\left(2^{m-2}-1\right)^{2} \tag{28}
\end{equation*}
$$

where $\left\{\bar{x}_{1}, \bar{x}_{2}\right\} \subseteq T$. It is obvious that for any $\left\{\bar{x}_{1}, \bar{x}_{2}\right\} \subseteq T$ there exist only two 3 -subsets $\bar{T}$ of $T$ such that $\left\{\bar{x}_{1}, \bar{x}_{2}\right\} \subseteq \overline{\bar{T}} \subseteq T$. By definition we have

$$
\begin{equation*}
A_{4}\left(\left(C^{\perp}\right)^{T}\right)=\binom{4}{2} \cdot\left(A_{4}\left(\left(C^{\perp}\right)^{\{0,1\}}\right)-2 \lambda_{\hat{T}, 6}\left(C^{\perp}\right)\right)+\lambda_{T, 8}\left(C^{\perp}\right) \tag{29}
\end{equation*}
$$

where $\hat{T}=\{0,1, w\}$. Combining Equations (25), (27) and (28) with Equation (29) yields

$$
\begin{equation*}
A_{4}\left(\left(C^{\perp}\right)^{T}\right)=4\left(2^{m-2}-1\right)^{2}-\frac{8}{3}\left(2^{m}-2-(-1)^{m / 2} 2^{m / 2+1}\right)+\frac{N_{(0,1)}}{24}+15 \tag{30}
\end{equation*}
$$

Note that the shortened code $\mathcal{C}_{T}$ has length $2^{m}-4$ and dimension $2 m-3$ due to $\# T=4$ and Lemma 1 . By definition and the weight distribution in Table II, we have that $A_{i}\left(\mathcal{C}_{T}\right)=0$ for $i \notin\left\{0, i_{1}, i_{2}, i_{3}, i_{4}, i_{5}\right\}$, where $i_{1}=2^{m-1}-2^{m / 2}, i_{2}=2^{m-1}-2^{(m-2) / 2}, i_{3}=2^{m-1}, i_{4}=2^{m-1}+2^{(m-2) / 2}$ and $i_{5}=2^{m-1}+2^{m / 2}$. Using Lemma 1 and Equations (24), (26) and (30) and applying the first five Pless power moments of (1) yields the weight distribution in Table XI. This completes the proof.

Example 39. Let $m=4$ and $e=1$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 38 is a $[12,5,4]$ binary linear code with the weight enumerator $1+3 z^{4}+24 z^{6}+3 z^{8}+z^{12}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[12,7,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

## V. Shortened linear codes from PN functions

In this section, we study some shortened linear codes from certain PN functions and determine their parameters.

Let $p$ be odd prime and $q=p^{m}$. Let $f(x)=x^{2}$ and $\mathcal{C}$ be defined by (6). Note that we index the coordinators of the codewords in $\mathcal{C}$ with the elements in $\mathrm{GF}(q)$. It is known that the code $\mathcal{C}$ is a $[q, 2 m+1]$ linear code with the the weight distribution given in [30]. Note that the code $\mathcal{C}$ is affine invariant, and thus holds 2 -designs. Then the following theorem is easily derived from the parameters of the codes $\mathcal{C}$ in [30] and Theorem [13, and we omit its proof.
Theorem 40. Let $p$ be an odd prime, $m$ and $t$ be positive integers. Let $q=p^{m}, f(x)=x^{2}$ and $\mathcal{C}$ be defined in (6). Suppose $T$ is a $t$-subset of $\mathcal{P}(\mathcal{C}):=\mathrm{GF}(q)$. We have the following results.

- It $t=1$, then the shortened code $\mathcal{C}_{T}$ is a $\left[p^{m}-1,2 m\right]$ linear code with the weight distribution in Table XII (resp., Table XIV) when $m$ is odd (resp., even).

TABLE XII
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ ODD AND $t=1$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $p^{m-1}(p-1)$ | $\left(p^{m}-1\right)\left(1+p^{m-1}\right)$ |
| $p^{m-1}(p-1)-p^{\frac{m-1}{2}}$ | $1 / 2 \cdot(p-1) p^{(m-3) / 2}\left(p^{m}-1\right)\left(p+p^{(1+m) / 2}\right)$ |
| $p^{m-1}(p-1)+p^{\frac{m-1}{2}}$ | $1 / 2 \cdot(p-1) p^{(m-3) / 2}\left(p^{m}-1\right)\left(-p+p^{(1+m) / 2}\right)$ |

TABLE XIII
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ ODD AND $t=2$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $p^{m-1}(p-1)$ | $p^{2 m-2}-1$ |
| $p^{m-1}(p-1)-p^{\frac{m-1}{2}}$ | $(p-1)\left(-p^{(m-1) / 2}+p^{2 m-2}+2 p^{(3 m-3) / 2}\right) / 2$ |
| $p^{m-1}(p-1)+p^{\frac{m-1}{2}}$ | $(p-1)\left(p^{(m-1) / 2}+p^{2 m-2}-2 p^{(3 m-3) / 2}\right) / 2$ |

- It $t=2$, then the shortened code $\mathcal{C}_{T}$ is $a\left[p^{m}-2,2 m-1\right]$ linear code with the weight distribution in Table XIII (resp., Table XV) when $m$ is odd (resp., even).

Example 41. Let $m=3, p=3$ and $T$ be a 1 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 40 is a $[26,6,15]$ linear code with the weight enumerator $1+312 z^{15}+260 z^{18}+156 z^{21}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[26,20,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

Example 42. Let $m=4, p=3$ and $T$ be a 1 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 40 is a $[80,8,48]$ linear code with the weight enumerator $1+1320 z^{48}+2400 z^{51}+80 z^{54}+1920 z^{57}+840 z^{60}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[80,72,4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de

Example 43. Let $m=5, p=3$ and $T$ be a 2 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 40 is a $[241,9,153]$ linear code with the weight enumerator $1+8010 z^{153}+6560 z^{162}+5112 z^{171}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[241,232,3]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

Example 44. Let $m=4, p=3$ and $T$ be a 2 -subset of $\mathcal{P}(\mathcal{C})$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 40 is $a[79,7,48]$ linear code with the weight enumerator $1+528 z^{48}+870 z^{51}+26 z^{54}+552 z^{57}+210 z^{60}$. This code $\mathcal{C}_{T}$ is almost optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[79,72,3]$ and is almost optimal according to the tables of best known codes maintained at http://www.codetables.de.

In the following, we will consider the shortened code $\mathcal{C}_{T}$ of $\mathcal{C}$ for the case $T=\mathrm{GF}(p)$. In order to determine the parameters of $\mathcal{C}_{T}$, we need the next lemmas.

Lemma 45. [24] Let $q=p^{m}$ with $p$ an odd prime. Then

$$
\begin{aligned}
& \sharp\left\{a \in \operatorname{GF}(q)^{*}: \eta(a)=1 \text { and } \operatorname{Tr}_{q / p}(a)=0\right\} \\
& \\
& =\left\{\begin{array}{cl}
\frac{p^{m-1}-1-(p-1) p^{\frac{m-2}{2}}(\sqrt{-1})^{\frac{(p-1) m}{2}}}{\frac{p^{m-1}-1}{2},} & \text { if } m \text { is even } \\
& \text { if } m \text { is odd } .
\end{array}\right.
\end{aligned}
$$

TABLE XIV
The weight distribution of $\mathcal{C}_{T}$ FOR $m$ EVEN and $t=1$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $(p-1)\left(p^{m-1}-p^{(m-2) / 2}\right)$ | $p^{m / 2-1}\left(p+p^{m / 2}-1\right)\left(p^{m}-1\right) / 2$ |
| $(p-1) p^{m-1}-p^{(m-2) / 2}$ | $(p-1) p^{m / 2-1}\left(p^{m / 2}-1\right)\left(1+p^{m / 2}\right)^{2} / 2$ |
| $(p-1) p^{m-1}$ | $p^{m}-1$ |
| $(p-1) p^{m-1}+p^{(m-2) / 2}$ | $(p-1) p^{m / 2-1}\left(p^{m / 2}-1\right)^{2}\left(1+p^{m / 2}\right) / 2$ |
| $(p-1)\left(p^{m-1}+p^{(m-2) / 2}\right)$ | $p^{m / 2-1}\left(-p+p^{m / 2}+1\right)\left(p^{m}-1\right) / 2$ |

TABLE XV
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ EVEN AND $t=2$

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $(p-1)\left(p^{m-1}-p^{(m-2 /) 2}\right)$ | $p^{m / 2-2}\left(p^{m / 2}-1\right)\left(-1+p+p^{m / 2}\right)\left(p+p^{m / 2}\right) / 2$ |
| $(p-1) p^{m-1}-p^{(m-2) / 2}$ | $(p-1) p^{m / 2-2}\left(1+p^{m / 2}\right)\left(-p+p^{m / 2}+p^{m}\right) / 2$ |
| $(p-1) p^{m-1}$ | $p^{m-1}-1$ |
| $(p-1) p^{m-1}+p^{(m-2) / 2}$ | $(p-1) p^{m / 2-2}\left(-1+p^{m / 2}\right)\left(-p-p^{m / 2}+p^{m}\right) / 2$ |
| $(p-1)\left(p^{m-1}+p^{(m-2) / 2}\right)$ | $p^{m / 2-2}\left(p^{m / 2}+1\right)\left(1-p+p^{m / 2}\right)\left(p^{m / 2}-p\right) / 2$ |

and

$$
\begin{aligned}
& \sharp\left\{a \in \operatorname{GF}(q)^{*}: \eta(a)=-1 \text { and } \operatorname{Tr}_{q / p}(a)=0\right\} \\
& \quad=\left\{\begin{array}{cl}
\frac{p^{m-1}-1+(p-1) p^{\frac{m-2}{2}}(\sqrt{-1})^{\frac{(p-1) m}{2}}}{\frac{p^{m-1}-1}{2}}, & \text { if } m \text { is even }, \\
& \text { if } m \text { is odd } .
\end{array}\right.
\end{aligned}
$$

Lemma 46. [24] Let $q=p^{m}$ with $p$ an odd prime and $(a, b) \in \operatorname{GF}(q)^{2}$. Denote

$$
\begin{equation*}
N_{0}(a, b)=\sharp\left\{x \in \mathrm{GF}(q): \operatorname{Tr}_{q / p}\left(a x^{2}+b x\right)=0\right\} . \tag{31}
\end{equation*}
$$

Then the following results follow.

- If $m$ is odd, then

$$
\begin{align*}
& N_{0}(a, b)= \\
& \left\{\begin{array}{cl}
p^{m}, & \text { if }(a, b)=(0,0), \\
p^{m-1}, & \text { if } a=0, b \neq 0, \text { or } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)=0 \\
p^{m-1}+p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4},}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right) \neq 0, \eta(a) \bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=1 \\
p^{m-1}+p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)+4}{4}}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right) \neq 0, \eta(a) \bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=-1
\end{array}\right. \tag{32}
\end{align*}
$$

- If $m$ is even, then

$$
N_{0}(a, b)=\left\{\begin{array}{cl}
p^{m} & \text { if }(a, b)=(0,0),  \tag{33}\\
p^{m-1}, & \text { if } a=0, b \neq 0, \\
p^{m-1}+(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4},}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)=0, \eta(a)=1, \\
p^{m-1}+(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}, & \text { if } \left.a \neq 0, \operatorname{Tr}_{q / p} \frac{b^{2}}{4 a}\right)=0, \eta(a)=-1, \\
p^{m-1}+p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}, & \text { if } \left.a \neq 0, \operatorname{Tr}_{q / p} \frac{b^{2}}{4 a}\right) \neq 0, \eta(a)=1, \\
p^{m-1}+p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}}, & \text { if } \left.a \neq 0, \operatorname{Tr}_{q / p} \frac{b^{2}}{4 a}\right) \neq 0, \eta(a)=-1
\end{array}\right.
$$

Lemma 47. Let $q=p^{m}$ with $p$ an odd prime and $a \in \mathrm{GF}(q)^{*}$. Define $f(x)=\operatorname{Tr}_{q / p}\left(-\frac{1}{4 a} x^{2}\right)$. Then the dual of $f(x)$ is $f^{*}(x)=\operatorname{Tr}_{q / p}\left(a x^{2}\right)$ and the sign of the Walsh transform of $f(x)$ is

$$
\varepsilon=\eta(a)(-1)^{m-1+\frac{q-1}{2}}
$$

Proof. Note that $f(x)$ is a weakly regular bent function. By definition and Lemma 3, we have

$$
\mathcal{W}_{f}(\beta)=\sum_{x \in \operatorname{GF}(q)} \zeta_{p}^{\operatorname{Tr}_{q / p}\left(-\frac{1}{4 a} x^{2}\right)-\operatorname{Tr}_{q / p}(\beta x)}=\zeta_{p}^{\operatorname{Tr}_{q / p}\left(a \beta^{2}\right)} \cdot \eta\left(-\frac{1}{4 a}\right) G(\eta, \chi)=\zeta_{p}^{f^{*}(\beta)} \cdot \varepsilon \cdot{\sqrt{p^{*}}}^{m}
$$

Then the desired conclusions follow from Lemma 2 .
Lemma 48. Let $q=p^{m}$ with $p$ an odd prime, $a \in \operatorname{GF}(q)^{*}$ and $e$ be any positive integer such that $m / \operatorname{gcd}(m, e)$ is odd. Define $f(x)=\operatorname{Tr}_{q / p}\left(a x^{p^{e}+1}\right)$. Then the dual of $f(x)$ is $f^{*}(\beta)=\operatorname{Tr}_{q / p}\left(-a\left(x_{a,-\beta}\right)^{p^{e}+1}\right)$ and the sign of the Walsh transform of $f(x)$ is

$$
\varepsilon= \begin{cases}(-1)^{m-1+\frac{q-1}{2}} \eta(a), & \text { if } p \equiv 1 \bmod 4 \\ (-1)^{\frac{q-1}{2}+1} \eta(a), & \text { if } p \equiv 3 \bmod 4\end{cases}
$$

where $\beta \in \mathrm{GF}(q)$ and $x_{a,-\beta}$ is the unique solution of the equation

$$
a^{p^{e}} x^{p^{2 e}}+a x+(-\beta)^{p^{e}}=0
$$

Proof. The proof is similar to that of Lemma 47, and we omit it here. Note that $f(x)$ is a weakly regular bent function. Then the desired conclusions follow from the definitions and Lemma 8 .

Lemma 49. Let $q=p^{m}$ with $p$ an odd prime, $a \in \mathrm{GF}(q)^{*}, \gamma \in \mathrm{GF}(p)$ and $e$ be any positive integer such that $m / \operatorname{gcd}(m, e)$ is odd. Define

$$
\tilde{N}_{\gamma}=\#\left\{b \in \mathrm{GF}(q): \operatorname{Tr}_{q / p}(b)=0 \text { and } \operatorname{Tr}_{q / p}\left(a x_{a, b}^{p^{e}+1}\right)=\gamma\right\}
$$

where $x_{a, b}$ is the unique solution of the equation $a^{p^{e}} x^{p^{2 e}}+a x+b^{p^{e}}=0$. If $\operatorname{Tr}_{q / p}(a)=0$, then

$$
\tilde{N}_{\gamma}=\left\{\begin{array}{cl}
p^{m-2}+\varepsilon \bar{\eta}^{m / 2}(-1)(p-1) p^{(m-2) / 2}, & \text { if } \gamma=0 \text { and } m \text { is even, } \\
p^{m-2}, & \text { if } \gamma=0 \text { and } m \text { is odd, } \\
\frac{p-1}{2}\left(p^{m-2}-\varepsilon \bar{\eta}^{m / 2}(-1) p^{(m-2) / 2}\right), & \text { if } \gamma \neq 0 \text { and } m \text { is even, } \\
\frac{p-1}{2}\left(p^{m-2}+\varepsilon \sqrt{p *}^{m-1}\right), & \text { if } \gamma \in \text { SQ and } m \text { is odd, } \\
\frac{p-1}{2}\left(p^{m-2}-\varepsilon \sqrt{p *}^{m-1}\right) . & \text { if } \gamma \in \text { NSQ and } m \text { is odd, }
\end{array}\right.
$$

where $\varepsilon$ was given in Lemma 48
Proof. By the definition of $x_{a, b}$, we have

$$
a^{p^{e}} x_{a, b}^{p^{2 e}}+a x_{a, b}+b^{p^{e}}=\left(a^{p^{-e}} x_{a, b}\right)^{p^{2 e}}+a x_{a, b}+b^{p^{e}}=0 .
$$

This gives

$$
\operatorname{Tr}_{q / p}\left(\left(a^{p^{-e}}+a\right) x_{a, b}+b\right)=0
$$

Since $a^{p^{e}} x^{p^{2 e}}+a x$ is a linear permutation polynimial over $\operatorname{GF}(q)$, we have

$$
\begin{equation*}
\tilde{N}_{\gamma}=\#\left\{x \in \operatorname{GF}(q): \operatorname{Tr}_{q / p}\left(a x^{p^{e}+1}\right)=\gamma \text { and } \operatorname{Tr}_{q / p}\left(\left(a^{p^{-e}}+a\right) x\right)=0\right\} \tag{34}
\end{equation*}
$$

Note that $x=1$ is the unique solution of the equation

$$
a^{p^{e}}(x)^{p^{2 e}}+a x+\left(-a^{p^{-e}}-a\right)^{p^{e}}=0
$$

By Lemma 48, we get

$$
\begin{equation*}
f^{*}\left(a^{p^{-e}}+a\right)=\operatorname{Tr}_{q / p}(-a)=0 \tag{35}
\end{equation*}
$$

where $f^{*}$ is the dual of $\operatorname{Tr}_{q / p}\left(a x^{p^{e}+1}\right)$. By Equations (34) and (35), the desired conclusions follow from Lemmas 14, 15 and 48 .
Theorem 50. Let $p$ be an odd prime and $m$ be a positive integer. Let $q=p^{m}, f(x)=x^{2}$ and $\mathcal{C}$ be defined in (6). Let $T=\operatorname{GF}(p)$. Then the shortened code $\mathcal{C}_{T}$ is $a\left[p^{m}-p, 2 m-2\right]$ linear code. If $m$ is odd, the weight distribution of $\mathcal{C}_{T}$ is given in Table XVI where $B=(-1)^{\frac{q-1}{2}+\frac{(p-1)(m-1)}{4}} p^{(m-1) / 2}$; if $m \geq 2$ is even, the weight distribution of $\mathcal{C}_{T}$ is given in Table XVII where $B_{1}=\frac{p^{m-1}-1-(p-1) p^{\frac{m-2}{2}}(\sqrt{-1}) \frac{(p-1) m}{2}}{2}$ and $B_{2}=(-1)^{\frac{q+1}{2}+\frac{m(p-1)}{4}}(p-1) p^{(m-2) / 2}$.

TABLE XVI
THE WEIGHT DISTRIBUTION OF $\mathcal{C}_{T}$ FOR $m$ OdD

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $p^{m-1}(p-1)$ | $\left(p^{m-1}-1\right)\left(p^{m-2}+1\right)$ |
| $p^{m-1}(p-1)-p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4}}$ | $\frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}+B\right)$ |
| $p^{m-1}(p-1)+p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4}}$ | $\frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}-B\right)$ |

TABLE XVII
The weight distribution of $\mathcal{C}_{T}$ FOR $m$ EVEN

| Weight | Multiplicity |
| :---: | :---: |
| 0 | 1 |
| $p^{m-1}(p-1)$ | $p^{m-1}-1$ |
| $p^{m-1}(p-1)-(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}}$ | $B_{1} \cdot\left(p^{m-2}+B_{2}\right)$ |
| $p^{m-1}(p-1)-(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}$ | $\left(p^{m-1}-1-B_{1}\right) \cdot\left(p^{m-2}-B_{2}\right)$ |
| $p^{m-1}(p-1)-p^{\frac{m-2}{2}}(-1)^{\frac{m p-1)}{4}}$ | $B_{1} \cdot\left(p^{m-1}-p^{m-2}-B_{2}\right)$ |
| $p^{m-1}(p-1)-p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}}$ | $\left(p^{m-1}-1-B_{1}\right) \cdot\left(p^{m-1}-p^{m-2}+B_{2}\right)$ |

Proof. Denote

$$
\begin{equation*}
H=\left\{(a, b) \in \operatorname{GF}(q)^{2}: \operatorname{Tr}_{q / p}(a)=\operatorname{Tr}_{q / p}(b)=0\right\} \tag{36}
\end{equation*}
$$

By definition, the shortened code $\mathcal{C}_{T}$ has length $p^{m}-p$. The desired conclusion on the dimension of $\mathcal{C}_{T}$ will be clear after the weight distribution of $\mathcal{C}_{T}$ is settled below.

Since $T=\operatorname{GF}(p)$, the weight distribution of $\mathcal{C}_{T}$ is the same as the subcode

$$
\begin{equation*}
\mathcal{C}(T)=\left\{\left(\operatorname{Tr}_{q / p}\left(a x^{2}+b x\right)\right)_{x \in \operatorname{GF}(q)}:(a, b) \in H\right\} . \tag{37}
\end{equation*}
$$

For each $(a, b) \in H$, define the corresponding codeword

$$
\mathbf{c}(a, b)=\left(\operatorname{Tr}_{q / p}\left(a x^{2}+b x\right)\right)_{x \in \operatorname{GF}(q)} \in \mathcal{C}(T)
$$

Then the Hamming weight of $\mathbf{c}(a, b)$ is

$$
\operatorname{wt}(\mathbf{c}(a, b))=q-N_{0}(a, b)
$$

where $N_{0}(a, b)$ was defined in Equation (31). We discuss the value of $\mathrm{wt}(\mathbf{c}(a, b))$ in the following two cases.
(I) The case that $m$ is odd.

From Equation (32), we get

$$
\begin{aligned}
& \text { wt }(\mathbf{c}(a, b))=q-N_{0}(a, b) \\
& = \begin{cases}0, & \text { if }(a, b)=(0,0) \\
p^{m-1}(p-1), & \text { if } a=0 \text { and } b \neq 0, \text { or } a \neq 0 \text { and } \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)=0 \\
p^{m-1}(p-1)-p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4},}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right) \neq 0, \eta(a) \bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=1 \\
p^{m-1}(p-1)+p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4},}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right) \neq 0, \eta(a) \bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=-1\end{cases} \\
& = \begin{cases}0, & \text { with } 1 \text { time, } \\
p^{m-1}(p-1), & \text { with }\left(p^{m-1}-1\right)\left(p^{m-2}+1\right) \text { times, } \\
p^{m-1}(p-1)-p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4},}, & \text { with } \frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}+B\right) \text { times, }, \\
p^{m-1}(p-1)+p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4},}, & \text { with } \frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}-B\right) \text { times, },\end{cases}
\end{aligned}
$$

when $(a, b)$ runs through $H$, where

$$
B=(-1)^{m-1+\frac{q-1}{2}} \cdot \sqrt{p^{*}}{ }^{m-1}=(-1)^{\frac{q-1}{2}+\frac{(p-1)(m-1)}{4}} p^{(m-1) / 2}
$$

and the frequency is obtained from Lemmas 14, 15, 45] and 47, We first compute the frequency $A_{w}$ of the nonzero weight $w$, where

$$
w=p^{m-1}(p-1)-p^{\frac{m-1}{2}}(-1)^{\frac{(p-1)(m+1)}{4}} .
$$

Then

$$
A_{w}=\sharp\left\{(a, b) \in H: a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right) \neq 0, \eta(a) \bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=1\right\}
$$

Clearly, the number of $a \in \operatorname{GF}(q)^{*}$ such that $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=1$, is $\bar{n}_{a}=\frac{p^{m-1}-1}{2}$ by Lemma 45, if we fix $a$ with $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=1$, the number of $b$ such that $\operatorname{Tr}_{q / p}(b)=0$ and $\bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=1$, is $\bar{n}_{b}=\frac{p-1}{2}\left(p^{m-2}+(-1)^{m-1+\frac{q-1}{2}} \cdot \sqrt{p^{*}}{ }^{m-1}\right)$ by Lemmas 15 and 47. Meanwhile, the number of $a \in \operatorname{GF}(q)^{*}$ such that $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=-1$, is $\hat{n}_{a}=\frac{p^{m-1}-1}{2}$ by Lemma 45, if we fix $a$ with $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=-1$, the number of $b$ such that $\operatorname{Tr}_{q / p}(b)=0$ and $\bar{\eta}\left(-\operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)\right)=-1$, is $\hat{n}_{b}=\frac{p-1}{2}\left(p^{m-2}+\right.$ $(-1)^{m-1+\frac{q-1}{2}} \cdot \sqrt{p^{*}}{ }^{m-1}$ ) by Lemmas 15 and 47. Hence,

$$
A_{w}=\bar{n}_{a} \bar{n}_{b}+\hat{n}_{a} \hat{n}_{b}=\frac{p^{m-1}-1}{2}(p-1)\left(p^{m-2}+(-1)^{m-1+\frac{q-1}{2}} \cdot{\sqrt{p^{*}}}^{m-1}\right)
$$

The frequencies of other nonzero weights can be similarly derived.
(II) The case that $m$ is even.

From Equation (33), we get

$$
\begin{aligned}
& \text { wt }(\mathbf{c}(a, b))=q-N_{0}(a, b) \\
& = \begin{cases}0, & \text { if }(a, b)=(0,0) \\
p^{m-1}(p-1), & \text { if } a=0, b \neq 0 \\
p^{m-1}(p-1)-(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)=0, \eta(a)=1 \\
p^{m-1}(p-1)-(p-1) p^{\frac{m-2}{2}(-1)^{\frac{m(p-1)}{4}},}, & \text { if } a \neq 0, \operatorname{Tr}_{q / p}\left(\frac{b^{2}}{4 a}\right)=0, \eta(a)=-1 \\
p^{m-1}(p-1)-p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4},}, & \text { if } \left.a \neq 0, \operatorname{Tr}_{q / p} \frac{b^{2}}{4 a}\right) \neq 0, \eta(a)=1 \\
p^{m-1}(p-1)-p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}}, & \text { if } \left.a \neq 0, \operatorname{Tr}_{q / p} \frac{b^{2}}{4 a}\right) \neq 0, \eta(a)=-1\end{cases} \\
& = \begin{cases}0, & \text { with } 1 \text { time, }, \\
p^{m-1}(p-1), & \text { with } p^{m-1}-1 \text { times, }, \\
p^{m-1}(p-1)+(p-1) p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}, & \text { with } B_{1} \cdot\left(p^{m-2}+B_{2}\right) \text { times, } \\
p^{m-1}(p-1)-(p-1) p^{\frac{m-2}{2}(-1)^{\frac{m(p-1)}{4}},}, & \text { with }\left(p^{m-1}-1-B_{1}\right) \cdot\left(p^{m-2}-B_{2}\right) \text { times, }, \\
p^{m-1}(p-1)-p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}, & \text { with } B_{1} \cdot\left(p^{m-1}-p^{m-2}-B_{2}\right) \text { times, }, \\
p^{m-1}(p-1)+p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)}{4}}, & \text { with }\left(p^{m-1}-1-B_{1}\right) \cdot\left(p^{m-1}-p^{m-2}+B_{2}\right) \text { times, },\end{cases}
\end{aligned}
$$

where $(a, b)$ runs through $H, B_{1}=\frac{p^{m-1}-1-(p-1) p^{\frac{m-2}{2}}\left(\sqrt{-1} \frac{(p-1) m}{2}\right.}{2}$,

$$
B_{2}=(-1)^{m-1+\frac{q-1}{2}} \bar{\eta}^{m / 2}(-1) \cdot(p-1) p^{(m-2) / 2}=(-1)^{\frac{q+1}{2}+\frac{m(p-1)}{4}}(p-1) p^{(m-2) / 2}
$$

and the frequency is easy to obtain from Lemmas 14, 15, 45 and 47. The weight distribution of $\mathcal{C}_{T}$ can be handled in much the same way as the case that $m$ is odd. Details are omitted here.

By the above two cases, the weight distributions in Tables XVI and XVII follow. This completes the proof.
Example 51. Let $m=3$ and $p=3$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 50 is $a[24,4,15]$ linear code with the weight enumerator $1+48 z^{15}+32 z^{18}$. This code $\mathcal{C}_{T}$ is optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[24,20,3]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de

Example 52. Let $m=4$ and $p=3$. Then the shortened code $\mathcal{C}_{T}$ in Theorem 50 is a $[78,6,48]$ linear code with the weight enumerator $1+240 z^{48}+240 z^{51}+26 z^{54}+192 z^{57}+30 z^{60}$. This code $\mathcal{C}_{T}$ is almost optimal. Its dual $\mathcal{C}_{T}^{\perp}$ has parameters $[78,72,2]$ and is almost optimal according to the tables of best known codes maintained at http://www.codetables.de.

Theorem 53. Let $p$ be an odd prime, $m$ and $e$ be positive integers such that $m / \operatorname{gcd}(m, e)$ is odd. Let $q=p^{m}, f(x)=x^{p^{e}+1}$ and $\mathcal{C}$ be defined in (6). Let $T=\mathrm{GF}(p)$. Then the parameters of the shortened code $\mathcal{C}_{T}$ are the same as that of $\mathcal{C}_{T}$ in Theorem 50
Proof. The proof is similar to that of Theorem 50, Recall that the code $\mathcal{C}$ has length $q$ and dimension $2 m+1$. By definition, the shortened code $\mathcal{C}_{T}$ has length $p^{m}-p$. The desired conclusion on the dimension of $\mathcal{C}_{T}$ will be clear after the weight distribution of $\mathcal{C}_{T}$ is settled below. Since $T=\mathrm{GF}(p)$, the weight distribution of $\mathcal{C}_{T}$ is the same as the code

$$
\begin{equation*}
\mathcal{C}(T)=\left\{\left(\operatorname{Tr}_{q / p}\left(a x^{p^{e}+1}+b x\right)\right)_{x \in \operatorname{GF}(q)}:(a, b) \in H\right\}, \tag{38}
\end{equation*}
$$

where $H$ was defined by (36).

For each $(a, b) \in H$, define the corresponding codeword

$$
\mathbf{c}(a, b)=\left(\operatorname{Tr}_{q / p}\left(a x^{p^{e}+1}+b x\right)\right)_{x \in \operatorname{GF}(q)} \in \hat{\mathcal{C}} .
$$

Then the Hamming weight of $\mathbf{c}(a, b)$ is

$$
\begin{equation*}
\operatorname{wt}(\mathbf{c}(a, b))=q-\hat{N}_{0}(a, b), \tag{39}
\end{equation*}
$$

where $\hat{N}_{0}(a, b)$ was defined in Lemma 10 .
We determine the value of $\mathrm{wt}(\mathbf{c}(a, b))$ and its frequencies according to the parity of $m$ and the residue of $p$ modulo 4 as follows.

- (I) $m$ is odd and $p \equiv 1 \bmod 4$.
- (II) $m$ is odd and $p \equiv 3 \bmod 4$.
- (III) $m$ is even and $p \equiv 1 \bmod 4$.
- (IV) $m$ is even and $p \equiv 3 \bmod 4$.

Next we only give the proof for the case (I) and omit the proofs for the other there cases whose proofs are similar.

Suppose that $m$ is odd and $p \equiv 1 \bmod 4$. From Equation (39) and Lemma 10, we get

$$
\begin{aligned}
& \text { wt }(\mathbf{c}(a, b))=q-\hat{N}_{0}(a, b) \\
& = \begin{cases}0, & \text { if }(a, b)=(0,0) \\
p^{m-1}(p-1), & \text { if } a b=0,(a, b) \neq(0,0) \\
p^{m-1}(p-1)-p^{m / 2-1} \sqrt{p^{*}} & \text { or } a b \neq 0, \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p+1}\right)=0 \\
p^{m-1}(p-1)+p^{m / 2-1} \sqrt{p^{*}} & \text { if } a b \neq 0, \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0, \eta\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0, \eta(a) \eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right)=1\end{cases} \\
& = \begin{cases}0, & \text { with } 1 \text { time, }, \\
\left.p_{q / p}^{m-1}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right)=-1 \\
p^{m-1}(p-1), & \text { with }\left(p^{m-1}-1\right)\left(p^{m-2}+1\right) \text { times, }, \\
p^{m-1}(p-1)+p^{\frac{m-1}{2}}, & \text { with } \frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}+B\right) \text { times, }, \\
\text { with } \frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}-B\right) \text { times, },\end{cases}
\end{aligned}
$$

when $(a, b)$ runs through $H$, where $B=(-1)^{m-1+\frac{q-1}{2}} \cdot \sqrt{p^{*}}{ }^{m-1}$ and the frequency is obtained by Lemmas 45 and 48. As an example, we just compute the frequency $A_{w}$ of the nonzero weight $w$, where

$$
w=p^{m-1}(p-1)-p^{\frac{m-1}{2}} .
$$

Thus

$$
A_{w}=\sharp\left\{(a, b) \in H: a b \neq 0, \operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right) \neq 0, \eta(a) \eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right)=1\right\} .
$$

Clearly, the number of $a \in \operatorname{GF}(q)^{*}$ such that $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=1$, is $\bar{n}_{a}=\frac{p^{m-1}-1}{2}$ by Lemma 45, if we fix $a$ with $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=1$, the number of $b$ such that $\operatorname{Tr}_{q / p}(b)=0$ and $\eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right)=$ 1 , is $\bar{n}_{b}=\frac{p-1}{2}\left(p^{m-2}+(-1)^{m-1+\frac{q-1}{2}} \cdot \sqrt{p *}^{m-1}\right)$ by Lemma 49. Meanwhile, the number of $a \in \operatorname{GF}(q)^{*}$ such that $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=-1$, is $\hat{n}_{a}=\frac{p^{m-1}-1}{2}$ by Lemma 45, if we fix $a$ with $\operatorname{Tr}_{q / p}(a)=0$ and $\eta(a)=-1$, the number of nonzero $b$ such that $\operatorname{Tr}_{q / p}(b)=0$ and $\eta\left(\operatorname{Tr}_{q / p}\left(a\left(x_{a, b}\right)^{p^{e}+1}\right)\right)=-1$, is $\hat{n}_{b}=\frac{p-1}{2}\left(p^{m-2}+(-1)^{m-1+\frac{q-1}{2}} \sqrt{p *}^{m-1}\right)$ by Lemma 49. Hence,

$$
A_{w}=\bar{n}_{a} \bar{n}_{b}+\hat{n}_{a} \hat{n}_{b}=\frac{p^{m-1}-1}{2} \cdot(p-1)\left(p^{m-2}+(-1)^{m-1+\frac{q-1}{2}} \cdot{\sqrt{p^{*}}}^{m-1}\right) .
$$

The frequencies of other nonzero weights can be similarly derived. This completes the proof of the weight distribution of Table in XVI for the case $m$ odd and $p \equiv 1 \bmod 4$.

The proofs of the other three cases are similar. The desired conclusions follow from Equation (39), Lemmas 10, 45 and 49. This completes the proof.

## VI. Concluding remarks

In this paper, we mainly investigated some shortened codes of linear codes from PN and APN functions and determined their parameters. The obtained codes have a few weights and many of these codes are optimal or almost optimal. Specifically, the main contributions are summarized below.

- For any binary linear code $\mathcal{C}$ with length $q=2^{m}$ and the weight distribution in Table Il we determined the weight distributions of the shortened codes $\mathcal{C}_{T}$ for $\# T \in\{1,2,3\}$ (see Theorem 16) and gave a general result on the shortened codes $\mathcal{C}_{T}$ with $\# T=4$ in Theorem 24. Meanwhile, when $m$ is odd, the parameters of the shortened codes $\mathcal{C}_{T}$ of a class of binary linear codes from APN functions were determined in Theorem 29.
- For any binary linear code $\mathcal{C}$ with length $q=2^{m}$ and the weight distribution in Table II we settled the weight distributions of the shortened codes $\mathcal{C}_{T}$ with $\# T \in\{1,2\}$ (see Theorem 20) and developed a general result on the shortened codes $\mathcal{C}_{T}$ with $\# T=3$ in Theorem 26, Further, the parameters of the shortened codes $C_{T}$ from certain APN functions were determined for $\# T=3$ and $\# T=4$ in Theorems 33 and 38 ,
- Two classes of $p$-ary shortened codes $\mathcal{C}_{T}$ from PN functions were presented and their parameters were also determined in Theorems 50 and 53, where $p$ is an odd prime.
Furthermore, the parameters of the shortened codes look new.
In addition to the works in [32] and this paper, other linear codes with good parameters may be produced with the shortening technique. However, it seems hard to determine the weight distributions of shortened and punctured codes in general. The reader is cordially invited to join the adventure in this direction.


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