# Mean Estimation from One-Bit Measurements 

Alon Kipnis and John C. Duchi


#### Abstract

We consider the problem of estimating the mean of a symmetric log-concave distribution under the constraint that only a single bit per sample from this distribution is available to the estimator. We study the mean squared error as a function of the sample size (and hence the number of bits). We consider three settings: first, a centralized setting, where an encoder may release $n$ bits given a sample of size $n$, and for which there is no asymptotic penalty for quantization; second, an adaptive setting in which each bit is a function of the current observation and previously recorded bits, where we show that the optimal relative efficiency compared to the sample mean is precisely the efficiency of the median; lastly, we show that in a distributed setting where each bit is only a function of a local sample, no estimator can achieve optimal efficiency uniformly over the parameter space. We additionally complement our results in the adaptive setting by showing that one round of adaptivity is sufficient to achieve optimal mean-square error.


## I. Introduction

We consider estimation of parameters from data collected by multiple units under communication constraints between the units. Such scenarios arise in sensor arrays, where sensor motes collect information, which they transmit to a central estimation unit [2], [3]. More generally, communication is substantially more expensive than computation in modern computing infrastructure [4]. It is thus of interest to understand the extent to which communication constraints induce fundamental accuracy and efficiency limits in parametric estimation problems.

We answer this question in a sylized version of this problem: the estimation of the mean $\theta$ of a symmetric log-concave distribution under the constraint that only a single bit can be communicated about each observation

[^0]

Fig. 1. Three encoding settings: (i) Centralized - an encoder sends $n$ bits after observing $n$ samples. (ii) Adaptive (sequential) - the $i$ th encoder sends the bit $B_{i}$ depending on its private sample $X_{i}$ and previous bits $B_{1}, \ldots, B_{i-1}$. (iii) Distributed - each encoder send the bit $B_{i}$ based on its private sample $X_{i}$ only.
from this distribution. Different information sharing schemes strongly affect the performance of estimators for $\theta$; we illustrate the three main settings we consider in Figure 1.
(i) Centralized encoding: all $n$ encoders confer and produce a single message consists of $n$ bits.
(ii) Adaptive or sequential encoding: The $n$th encoder observes the $n$th sample and the $n-1$ previous bits.
(iii) Distributed encoding: The $n$th message is only a function of the $n$th sample.

The distributed setting (iii) is the most restrictive; as it turns out, (ii) is slightly more restrictive than the fully centralized setting (i), and in our setting, a variant of the adaptive setting (ii) in which there is only one round of adaptivity-as we make formal later-is enough to achieve the same efficiency as the fully sequential setting (ii). Each setting has natural applications:

- Signal acquisition (i): A quantity is measured $n$ times at different instances. The results are averaged in order to reduce measurement noise and the averaged result is then stored or communicated using $n$ bits.
- Analog-to-digital conversion (ii): A sigma-delta modulator (SDM) converts an analog signal into a sequence of bits by sampling the signal at a very high rate and then using one-bit threshold detector combined with a feedback loop to update an accumulated error state [5]. Therefore, the expected error in tracking an analog signal using an SDM falls under our setting (ii) when we assume that the signal at the input to the modulator is a constant (direct current) corrupted by, say, thermal noise [6]. Since the sampling rates in SDM are usually many times more than the bandwidth of its input, analyzing SDM under a constant input provides meaningful lower bound even for non-constant signals.
- Privacy (ii)-(iii): A business entity is interested in estimating the average income of its clients. In order to keep this information as confidential as possible, each client independently provides an answer to a yes/no question related to its income [7].
Let us provide an informal description of our results and setting. For an estimator $\theta_{n}$ with finite quadratic risk (mean squared error (MSE)) $R_{n}=\mathbb{E}_{\theta}\left[\left(\theta_{n}-\theta\right)^{2}\right]$, we study the limit

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} n R_{n} . \tag{1}
\end{equation*}
$$

By comparing this quantity to achievable rates of convergence without communication constraints, we can evaluate the efficiency losses-asymptotic relative efficiency-of the estimator to appropriately optimal (unconstrained) estimators. (We shall be more formal in the sequel.) By lower bounding the quantity (1), we also provide limits on estimation of single-bit-per-measurement constrained signals in more general settings [8], [9], [10], [11], [12].

In setting (i), the estimator can evaluate any optimal estimator of location (e.g., the sample mean if the data is Gaussian), then quantize it using $n$ bits. As the accuracy in describing the empirical mean decreases exponentially in the number of bits, the quantization error is negligible compared to the statistical error in mean estimation [13], [14]. That is, centralized encoding induces no asymptotic efficiency loss. The story is different in settings (ii) and (iii). Precisely, we show that in the adaptive setting (ii), the optimal efficiency of a one-bit scheme is (asypmtotically) precisely that of the sample median, and that this efficiency is achievable. As a concrete example, when $X_{i}$ are i.i.d. Gaussian, we necessarily lose a factor of $\pi / 2 \approx 1.57$ in the asymptotic risk; the one-bit constraint decreases the effective sample size by a factor of $\pi / 2$ compared to estimating it without the bit constraint. It turns out that, in the settings we consider, only a single round of adaptivity (see Fig. 3 for an illustration) is sufficient to achieve optimal convergence rates. In distinction from setting (ii), in setting (iii) when the messages must be independent, there is no distributed estimation scheme that achieves the efficiency of the sample median uniformly over $\theta$. We establish this result via Le Cam's local asyptotic normality theory, allowing us to provide exact characterizations of the asymptotic efficiency of suitably regular encoding schemes.

Our asymptotic setting is important in that it allows us to elide difficulties present in finite sample settings. For example, in setting (i), developing an optimal quantizer at finite $n$ requires choosing a $2^{n}$ level scalar quantizer, which is non-trivial [15]. In interactive and sequential settings (e.g. (ii)), the situation is more challenging, as it is unclear whether any type of compositionality applies, in that an $n-1$-step optimal estimator may be only vaguely related to the $n$-step optimal estimator. Thus, to provide our lower bounds, we rely on stronger information-based inequalities, including the Van Trees inequality [16] and Le Cam's local asymptotic normality theory [17], [18], [19].

## Related Work

The many challenges of estimation under communication constraints have given rise to a large literature investigating different aspects of constrained estimation. While our setting-in which we observe a single bit per signal $X_{i}$-is restrictive, it inspires substantial work. Perhaps the most related is that of Wong and Gray [6], who study one-bit analog-to-digital conversion of a constant input corrupted by Gaussian noise using a Sigma-Delta Modulator (SDM). They show almost sure convergence, but provide no rate (and no rates follow from their analysis); in contrast, we provide an optimal procedure and matching lower bound achieving risk $\frac{\pi}{2} \sigma^{2}$ in the limit (1) when $X_{i} \stackrel{\mathrm{iid}}{\sim} \mathcal{N}\left(\theta, \sigma^{2}\right)$. A growing literature on one-bit measurements in high-dimensional problems [8], [20], [21] shows how to reconstruct sparse signals, where Baraniuk et al. [8] show that in noiseless settings, exponential decay in MSE is possible; our results make precise the penalty for noise under one-bit sensing, showing that the error can decay (under Gaussian noise) at best as $\frac{\pi}{2} \frac{\sigma^{2}}{n}$.

In fully distributed settings (iii), the challenges are different, and there is also a substantial literature with one-bit (quantized) measurements [22], [23], [24], [25], [26]. We complement these results by providing precise lower bounds and optimality results; previous performance bounds are suboptimal. Work on the remote multiterminal source coding problem, or CEO problem [27], [28], [29], [30], provides lower bounds on the MSE in setting (iii); because of the somewhat distinct setting, these bounds are looser than ours (which have optimal constants). In settings more similar to our statistical estimation scenario-such as estimation of parameters in a multi-dimensional linear model-a line of work provides lower bounds on statistical estimation [31], [32], [33], [34], [35], [36], [37], [38]. These results are finite sample and apply more broadly than ours, but as a consequence, they have unusable constants, while our stylized model allows precise identification of exact constants. Work subsequent to the initial draft of this paper [39] uses an approach similar to ours-bounding quantized Fisher information-to derive lower bounds on the error in parametric estimation problems from quantized measurements in non-adaptive settings.

Testing (and discrete estimation) problems also enjoy a robust literature, though as a consequence of our results to come, the results for testing, i.e., when the parameter space $\Theta$ is finite, are quite different from those for estimation, as it is possible to construct optimal decision (testing) rules in a completely distributed fashion. In this context, Longo et al. [40] propose procedures for distributed testing based on optimizing a Bhattacharyya distance. Tsitsiklis [41] shows that when the cardinality of $\Theta$ is at most $M$ and the probability of error criterion is used, then no more than $M(M-1) / 2$ different detection rules are necessary in order to attain probability of error with optimal exponent. Moreover, in a distributed setting, feedback is unnecessary for optimal testing/detection [42], in strong distinction to the estimation case we consider.

The remainder of this paper is organized as follows. In Section II we describe the problem, notation, and our basic assumptions. In Section III we provide two simple bounds on the efficiency and MSE. Our main results for the adaptive and distributed cases are given in Sections IV and V, respectively. In Section VI we provide concluding remarks.

## II. Problem Formulation and Notation

Let $f: \mathbb{R} \rightarrow \mathbb{R}_{+}$be a symmetric and log-concave probability density, which necessarily has finite second moment $\sigma^{2}$, and let $\Theta \subset \mathbb{R}$ be closed and convex. For $\theta \in \mathbb{R}$, let $P_{\theta}$ be the probability distribution with density $f(x-\theta)$, so that $\theta$ indexes the location family $\left\{P_{\theta}\right\}_{\theta \in \Theta}$. The log-concavity and symmetry $f(x)$ imply that $P_{\theta}$ has a unique mean and median at $\theta$ [43]. We observe a sample $X_{1}, \ldots, X_{n} \stackrel{\text { iid }}{\sim} P_{\theta}$, where $\theta$ is unknown, and wish to estimate $\theta$ given only binary messages $B_{1}, \ldots, B_{n} \in\{0,1\}$ about each $X_{i}$. We study this under three distinct computational scenarios, which we illustrate in Figure 1:
(i) Centralized, where $B_{i}=B_{i}\left(X_{1}, \ldots, X_{n}\right), i=1, \ldots, n$.
(ii) Adaptive, where $B_{i}=B_{i}\left(X_{i}, B_{1}, \ldots, B_{i-1}\right), i=2, \ldots, n$.
(iii) Distributed, where $B_{i}=B_{i}\left(X_{i}\right), i=1, \ldots, n$.

We also consider a hybrid of the fully distributed setting (where the bits $B_{i}$ are independent) and the adaptive setting (where each bit $B_{i}$ may depend on the previous bits) to a one-step adaptive setting, where the quantization scheme may be modified to depend on one fixed function of the previous information.
(ii') One-step adaptive, where for some function $g$ and a (fixed) $t$, if $i \leq t$ then $B_{i}=B_{i}\left(X_{i}\right)$ while if $i>t$, then $B_{i}=B_{i}\left(X_{i}, g\left(B_{1}, \ldots, B_{t}\right)\right)$.

We measure the performance of an estimator $\theta_{n} \triangleq \theta_{n}\left(B_{1}, \ldots, B_{n}\right)$ by one of a few notions. In the simplest case, we assume a prior $\pi$ on $\theta$ (which may be a point mass) and consider the quadratic risk

$$
\begin{equation*}
R_{n}=R_{n}(\pi) \triangleq \int \mathbb{E}_{\theta}\left(\theta_{n}-\theta\right)^{2} d \pi(\theta) \tag{2}
\end{equation*}
$$

where the expectation is taken with respect to the distribution of $X_{1}, \ldots, X_{n} \stackrel{\text { iid }}{\sim} P_{\theta}$. The main problems we consider in this paper are the minimal value of the risk (2) as a function of the sample size $n$ and the density $f$, under different choices of the encoding functions in cases (i)-(iii). The quadratic risk (2) may be infinite in some cases; we defer discussion of this case to later sections, as it is technically demanding and detracts from the presentation here.

Now, let $\sigma_{f}^{2} \triangleq \mathbb{E}\left[\frac{f^{\prime}(X)^{2}}{f(X)^{2}}\right]$ be the Fisher information for the location in the family $\left\{P_{\theta}\right\}$, which is finite when $f$ is log-concave and symmetric. We give particular attention to the asymptotic relative efficiency (ARE) of estimators with respect to asymptotically normal efficient estimators achieving the information bound [19]. In this case, if $\{m(n), n \in \mathbb{N}\}$ is a sequence such that

$$
\sqrt{m(n)}\left(\theta_{n}-\theta\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, \sigma_{f}^{2}\right),
$$

then the ARE of the estimator is [44, Def. 6.6.6]

$$
\begin{equation*}
\operatorname{ARE}\left(\theta_{n}\right) \triangleq \liminf _{n \rightarrow \infty} \frac{m(n)}{n} \tag{3}
\end{equation*}
$$

In the special case where there exists $V \in \mathbb{R}$ such that

$$
m(n) R_{n}=m(n) \mathbb{E}_{\theta}\left(\theta_{n}-\theta\right)^{2}=V+o(1),
$$

the ARE of $\theta_{n}$ is $\sigma_{f}^{2} / V$, so that $\theta_{n}$ requires a sample $V / \sigma_{f}^{2}$-times larger than that of an efficient estimator for comparable accuracy to the (information) efficient estimator.

## Notation and basic assumptions

To describe our results and make them formal, we require some additional notation and one main assumption, which restricts the class of distributions we consider. We use the typical notation that $F(x)=\int_{-\infty}^{x} f(t) d t$ is the cumulative distribution function of the $X_{i}$, and we let

$$
h(x) \triangleq \frac{f(x)}{1-F(x)}=\frac{f(x)}{F(-x)}
$$

be the hazard function (or the failure rate or force of mortality), which is monotone increasing as $f$ is logconcave [45]. Given the centrality of the median to our efficiency bounds, it is unsurprising that the quantity

$$
\begin{equation*}
\eta(x) \triangleq \frac{f^{2}(x)}{F(x)(1-F(x))} \stackrel{(\star)}{=} \frac{f(x) f(-x)}{F(x) F(-x)} \tag{4}
\end{equation*}
$$

appears throughout our development (equality $(\star)$ is immediate by the symmetry of $f$ ). For $p \in(0,1)$ and $x=$ $F^{-1}(p)$,

$$
\begin{equation*}
\frac{1}{\eta(x)}=\frac{1}{\eta\left(F^{-1}(p)\right)}=\frac{p(1-p)}{f\left(F^{-1}(p)\right)^{2}} \tag{5}
\end{equation*}
$$

is of course the familiar asymptotic variance of the $p$ th quantile of the sample $X_{1}, \ldots, X_{n}$ (cf. [19], Ch. 21).
For $f$ the normal density, classical results [46], [47] show that $\eta(x)$ is a strictly decreasing function of $|x|$, as we illustrate in Fig. 2. We consider log-concave symmetric distributions sharing this property. Specifically, we require the following.

Assumption A1: The density $f$ is log-concave and symmetric. Additionally, the origin $x=0$ uniquely maximizes $\eta(x)$, and $\eta(x)$ is non-increasing in $|x|$.
Under this assumption,

$$
4 f^{2}(x) \leq \eta(x) \leq \eta(0)
$$



Fig. 2. The function $\eta(x)=f^{2}(x) / F(x) F(-x)$ for $f(x)=\phi(x)$ the standard normal density.
where $\eta(0)=4 f^{2}(0)$ is the asymptotic variance of the sample median (Eq. (5) at $p=1 / 2$ ). Combined with log-concavity of $f(x)$, Assumption A1 implies that $\eta(x)$ vanishes as $|x| \rightarrow \infty$. Several distributions satisfy Assumption A1, including the generalized normal distributions with a shape parameter between 1 and 2 (including the normal and Laplace distributions). Symmetric log-concave distributions failing Assumption A1 include the uniform distribution and the generalized normal distribution with shape parameter greater than 2 . Some restriction on the class of distributions is necessary to develop our results; indeed, in Appendix VII we provide a brief discussion on the uniform distribution, where a one-step adaptive estimator with single bit observations can achieve convergence rates faster than the familiar $\sqrt{n}$ paramateric rate.

## III. Consistent Estimation and Off-the-shelf Bounds

We begin our technical treatment by deriving a few bounds on the efficiency of estimators in setting (iii). These bounds establish the following facts:

1. A consistent estimator with an asymptotically normal distribution always exists in setting (iii), and hence in the adaptive settings (ii) and (ii').
2. For the normal distribution, the asymptotic relative efficiency (3) in the distributed setting (iii) is at most $3 / 4$. No estimator can be as efficient as the sample mean.

## A. Consistent Estimation

The simplest estimator is simply to invert a quantile. Indeed, fix $\theta_{0} \in \mathbb{R}$ and define the $i$ th message by

$$
B_{i}=1\left\{X_{i}<\theta_{0}\right\},
$$

where $1\{A\}$ is the indicator of the event $A$. We have

$$
\bar{B}_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} B_{i} \xrightarrow{\text { a.s. }} F\left(\theta_{0}-\theta\right),
$$

so that

$$
\begin{equation*}
\theta_{n}=\theta_{0}-F^{-1}\left(\bar{B}_{n}\right) \tag{6}
\end{equation*}
$$

is a consistent estimator for $\theta$ in the distributed setting of Figure 1-(iii), where we note that $F$ is invertible over the support of $f$. As the variance of $\bar{B}_{n}$ is $F\left(\theta_{0}-\theta\right)\left(1-F\left(\theta_{0}-\theta\right)\right)$, a delta method calculation [19, Ch. 23] implies that $\theta_{n}$ is asymptotically normal with variance

$$
\frac{F\left(\theta_{0}-\theta\right)\left(1-F\left(\theta_{0}-\theta\right)\right)}{f^{2}\left(\theta_{0}-\theta\right)}=\frac{1}{\eta\left(\theta_{0}-\theta\right)} .
$$

In the Gaussian case where the $X_{i} \stackrel{\mathrm{iid}}{\sim} \mathcal{N}\left(\theta, \sigma^{2}\right)$, the ARE of $\theta_{n}$ is $\eta\left(\theta_{0}-\theta\right) \sigma^{2}$.
Assumption A1 implies that the optimal asymptotic variance for an estimator of the form (6) is $1 / \eta(0)$, the asymptotic of the sample median. Unfortunately, as $\theta$ is (by definition) a priori unknown and $\eta(x)$ monotonically decreases in $|x|$, this naive estimator $\theta_{n}$ may be very inefficient when $\theta$ is far from the initial guess $\theta_{0}$. As an example, when $f$ is a the normal density, the ARE of $\theta_{n}$ is less than 0.15 when $\left|\theta_{0}-\theta\right| \geq 2 \sigma$, and more broadly, $\operatorname{ARE}\left(\theta_{n}\right)$ asymptotes to $\left|\theta_{0}\right| \exp \left(-\theta_{0}^{2} / 2\right) / \sqrt{2 \pi}$ as $\left|\theta_{0}-\theta\right|$ gets large. Yet that $\theta_{0}=\theta$ minimizes this asymptotic variance, and $\eta$ is continuous, is suggestive: if we can use a suitably good initial estimate $\theta_{n}^{\text {init }}$ for $\theta$, it is possible that a one-step adaptive estimator (recall (ii')) may be asymptotically strong, as we see in Section IV.

## B. Multiterminal Source Coding

A related problem is the CEO problem, which considers the estimation of a sequence $\theta_{1}, \theta_{2} \ldots$, where a noisy version of each $\theta_{j}$ is available at $n$ terminals. At each terminal $i$, an encoder observes the $k$ noisy samples

$$
X_{i, j}=\theta_{j}+Z_{i, j}, \quad j=1, \ldots, k, \quad i=1, \ldots, n,
$$

and transmits $r_{i} k$ bits to a central estimator [27]. The central estimator produces estimates $\hat{\theta}_{1}, \ldots, \hat{\theta}_{k}$ with the goal of minimizing the quadratic risk:

$$
R_{\mathrm{CEO}}=\frac{1}{k} \sum_{j=1}^{k} \mathbb{E}\left[\left(\hat{\theta}_{j}-\theta_{j}\right)^{2}\right] .
$$

Note that any distributed encoding scheme using one-bit per sample can be replicated $k$ times and thus leads to a legitimate encoding and estimation scheme for the CEO problem with $r_{1}=\ldots=r_{n}=1$. It follows that, assuming that $\theta$ is drawn once from the prior $\pi$, our mean estimation problem from one-bit samples under distributed encoding corresponds to the CEO setting with $k=1$ realization of $\theta$ observed under noise at $n$ different locations, and communicated at each location using an encoder sending a single bit. Consequently, a lower bound on the MSE in estimating $\theta$ in the distributed encoding setting is given by the minimal MSE in the CEO setting as $k \rightarrow \infty$. Note that the difference between the CEO setting and ours lays in the privilege of each of the encoders to describe $k$ realizations of $\theta$ using $k$ bits with MSE averaged over these realizations, rather than a single realization using a single bit in ours.

When the prior on $\theta$ and the noise corrupting it at each location are Gaussian, Prabhakaran et al. [30] characterize the optimal encoding and its asymptotic risk as $k \rightarrow \infty$. Chen et al. [48] also provide an expression for the quadratic risk in the CEO setting under Gaussian priors. Adapting to our setting, this expression provides the following proposition:

Proposition 1: Assume that $\Theta=\mathbb{R}$ and $\pi(\theta)=\mathcal{N}\left(0, \sigma_{\theta}^{2}\right)$ where $\sigma_{\theta}^{2} \in \mathbb{R}$ is arbitrary. Then any estimator $\theta_{n}$ of $\theta$ in the distributed setting satisfies

$$
\begin{equation*}
n \cdot \mathbb{E}\left[\left(\theta-\theta_{n}\right)^{2}\right] \geq \frac{4}{3} \sigma^{2}+O\left(n^{-1}\right) \tag{7}
\end{equation*}
$$

where the expectation is with respect to $\theta$ and $X_{1}, \ldots, X_{n}$.
See Appendix VII-A for a proof.
As we shall see, this bound is loose: the difference between the MSE lower bound (7) and the actual MSE in the distributed setting (case (iii)) occurs because in the CEO setting, each encoder may encode an arbitrary number of $k$ independent realizations of $\theta$ using $k$ bits; in our situation, $k=1$. That blocking allows more efficient encoding and exploiting the high-dimensional geometry of the product probability space in the CEO problem is perhaps unsurprising, and our goal in the sequel will be to characterize the performance degradation one bit encoding engenders.

## IV. Adaptive Estimation

The first main result of this paper (Theorem 2) gives that the asymptotic variance of any adaptive estimator must be at least $\eta(0) \sigma^{2}$, which is precisely the efficiency of the median of the sample $X_{1}, \ldots, X_{n}$. Conveniently, the stochastic (sub)gradient estimator for the median-which minimizes $\mathbb{E}[|X-\theta|]$-is a sequence of signs (single bits), so that we can exhibit an asymptotically optimal adaptive estimation scheme.

We begin with our first theorem, whose proof we provide in Appendix VIII.
Theorem 2 (Fundamental limits): Let Assumption A1 hold. Let $\theta_{n}$ be any estimator of $\theta$ in the adaptive setting of Figure 1(ii). Assume that the prior density $\pi(\cdot)$ on $\theta$ converges to zero at the endpoints of the interval $\Theta$ and define the prior Fisher information $I_{0} \triangleq \mathbb{E}_{\pi}\left[\left(\pi^{\prime}(\theta) / \pi(\theta)\right)^{2}\right]$. Then

$$
\mathbb{E}\left[\left(\theta-\theta_{n}\right)^{2}\right] \geq \frac{1}{4 f^{2}(0) n+I_{0}}
$$

We now turn to asymptotically optimal estimators, first showing how a simple stochastic gradient scheme is asymptotically optimal (in the fully adaptive setting), after which we show that a one-round adaptive scheme can also achieve this optimal efficiency.

## A. Asymptotically optimal estimator

The starting point for our first estimator is to note that the median of a distribution minimizes $\mathbb{E}[|X-\theta|]$ over $\theta \in \mathbb{R}$, and moreover, we have the familiar result (cf. [19], Ch. 21) that given a sample $X_{1}, \ldots, X_{n} \stackrel{\text { iid }}{\sim} P$, if $\theta=\operatorname{med}(P)$ and $P$ has continuous density $f(\cdot-\theta)$ near $\theta$, then

$$
\sqrt{n}\left(\operatorname{med}\left(X_{1}^{n}\right)-\theta\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, \frac{1}{4 f(0)^{2}}\right),
$$

which is precisely the variance lower bound in Theorem 2. Thus, it is natural to consider a stochastic gradient procedure for minimizing $\mathbb{E}[|X-\theta|]$. To that end, let $\left\{\gamma_{n}\right\}_{n \in \mathbb{N}}$ be a strictly positive sequence of stepsizes, and define the sequence

$$
\begin{equation*}
\theta_{n}=\theta_{n-1}+\gamma_{n} B_{n}, \quad n=1,2, \ldots, \tag{8}
\end{equation*}
$$

where

$$
B_{n}=\operatorname{sgn}\left(X_{n}-\theta_{n-1}\right) .
$$

We make one of two assumptions on the stepsizes $\gamma_{n}$, which are relatively standard: we always have $\gamma_{n}$ nonincreasing, and For some $0<\lambda \leq 1$,

$$
\begin{array}{ll}
\frac{\gamma_{n}-\gamma_{n+1}}{\gamma_{n}^{2}} \rightarrow 0, & \sum_{n} \frac{\gamma_{n}^{\frac{1+\lambda}{2}}}{\sqrt{n}}<\infty \quad \text { or } \\
\gamma_{n}=o\left(n^{-2 / 3}\right), & \sum_{n} \gamma_{n}=\infty \tag{9b}
\end{array}
$$

Then we can adapt the results of Polyak and Juditsky [49] on the asymptotic normality of averaged stochastic gradient estimators to establish the following theorem.

Theorem 3: Define the average $\bar{\theta}_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} \theta_{i}$. Assume that in a neighborhood of $\theta=\operatorname{med}(P)$, the distribution $P$ has a Lipschitz continuous density $f$. Then
(i) Assume that $\left\{\gamma_{n}\right\}_{n \in \mathbb{N}}$ satisfies condition (9a). Then

$$
\sqrt{n}\left(\bar{\theta}_{n}-\theta\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, \frac{1}{4 f(0)^{2}}\right) .
$$

(ii) Let $\left\{P_{\theta}\right\}_{\theta \in \mathbb{R}}$ be the family of distributions with density $f(\cdot-\theta)$, where $f$ has median 0 . Let $h_{n} \rightarrow h \in \mathbb{R}$, and define the distributions $P_{n}=P_{\theta+h_{n} / \sqrt{n}}^{n}$. Then

$$
\sqrt{n}\left(\bar{\theta}_{n}-\theta-h_{n} / \sqrt{n}\right) \underset{P_{n}}{\underset{\leftrightarrow}{d}} \mathcal{N}\left(0, \frac{1}{4 f(0)^{2}}\right),
$$

and for any bounded, symmetric, and quasi-convex function $L$,

$$
\begin{gather*}
\sup _{c<\infty} \limsup _{n \rightarrow \infty} \sup _{\tau:|\theta-\tau| \leq \frac{c}{\sqrt{n}}} \mathbb{E}_{\tau}\left[L\left(\sqrt{n}\left(\bar{\theta}_{n}-\tau\right)\right)\right] \\
=\mathbb{E}[L(Z / 2 f(0))] \tag{10}
\end{gather*}
$$

where $Z \sim \mathcal{N}(0,1)$.
(iii) Assume the stepsizes $\gamma_{n}$ satisfy both conditions (9a) and (9b). Let $\pi$ be a distribution on $\mathbb{R}$ with a finite second moment. Then

$$
\begin{equation*}
\int \mathbb{E}\left[\left(\bar{\theta}_{n}-\theta\right)^{2}\right] \pi(d \theta)=\frac{1}{4 n f(0)^{2}}+o\left(n^{-1}\right) \tag{11}
\end{equation*}
$$

We provide the proofs of items (i)-(iii) in Appendices IX-A, IX-B, IX-C, respectively.
As an immediate corollary to Theorem 3, we obtain the following asymptotic optimality results of the averaged stochastic gradient sequence. Specifically, the average of the stochastic gradient iterates (8) is locally asymptotically minimax, and they achieve the lower bound of Theorem 2.

Corollary 4: Let the conditions of Theorem 2 hold and $\theta_{n}$ be defined by the iteration (8). Let $\left\{P_{\theta}\right\}_{\theta \in \mathbb{R}}$ be the family of distributions with densities $f(\cdot-\theta)$.


Fig. 3. Distributed encoding with one round of threshold adaptation. The estimation obtained from the first $n_{1}$ bits in a distributed manner is utilized in obtaining another $n-n_{1}$ bits in a distributed manner.
(i) Define the shorthand $P_{n}=P_{\theta+h_{n} / \sqrt{n}}^{n}$. If the stepsizes satisfy condition (9a), then

$$
\sqrt{n}\left(\bar{\theta}_{n}-\theta-h_{n} / \sqrt{n}\right) \underset{P_{n}}{\underset{\longrightarrow}{\rightarrow}} \mathcal{N}\left(0, \frac{1}{\eta(0)}\right) .
$$

(ii) If in addition the stepsizes satisfy condition (9b), then they achieve the lower bound of Theorem 2 for any prior $\pi$ on $\mathbb{R}$.

## B. Maximal Efficiency using One Round of Threshold Adaptation

In the encoding and estimating procedure (8), each one-bit message $B_{n}$ depends on its private sample as well as the current gradient descent estimate $\theta_{n-1}$. In this sense, each encoder in this algorithm interacts with previous one by using the current estimate. This amount of adaptivity is unnecessary: as we now consider, a similar encoding yields an asymptotically normal estimator attaining the lower variance bound $1 / \eta(0)$, provided we allow one adaptive update to the threshold value $\theta_{0}$ based on previously observed bits. In this procedure we separate the sample into the disjoint sets $X_{1}, \ldots, X_{n_{1}}$ and $X_{n_{1}+1}, \ldots, X_{n}$ for some $n_{1}<n$. We first use the estimator (6) to obtain an estimate $\theta_{n_{1}}$ based on $B_{1}, \ldots, B_{n_{1}}$, and then use $\theta_{n_{1}}$ as the new threshold value to obtain messages $B_{n_{1}+1}, \ldots, B_{n}$. Figure 3 illustrates a diagram of this procedure.

More formally, we consider the following estimation scheme. Given $n_{1} \in\{1, \ldots, n\}$, set the individual bits

$$
B_{i}= \begin{cases}1\left\{X_{i} \leq \theta_{0}\right\} & i=1, \ldots, n_{1}, \\ 1\left\{X_{i} \leq T_{n}\right\} & i=n_{1}+1, \ldots, n\end{cases}
$$

where

$$
\begin{aligned}
T_{n} \triangleq \theta_{0}-F^{-1}\left(\frac{1}{n_{1}} \sum_{i=1}^{n_{1}} B_{i}\right) \\
\theta_{n} \triangleq T_{n}-F^{-1}\left(\frac{1}{n-n_{1}} \sum_{i=n_{1}}^{n} B_{i}\right) .
\end{aligned}
$$

The intuition here is that the estimator $\theta_{n}$ is a one-step correction (cf. [44, Thm. 6.4.3]) of the initial estimator $T_{n}$, which approximately estimates $\theta_{0}-F^{-1}\left(F\left(\theta_{0}-\theta\right)\right)=\theta$. We then have the following convergence result.

Theorem 5: Assume that $X_{i}=Z_{i}+\theta$, where $Z_{i}$ are i.i.d. with density $f$ and $\operatorname{CDF} F$ and $\operatorname{med}\left(Z_{i}\right)=0$. Assume that $f$ is continuous at 0 , and that as $n \rightarrow \infty, n_{1}(n) \rightarrow \infty$ and $n_{1} / n \rightarrow 0$. Then

$$
\sqrt{n}\left(\theta_{n}-\theta\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, \frac{1}{4 f(0)^{2}}\right) .
$$

That is, under Assumption A1, the method is asymptotically optimal.

Proof: We abuse notation and instead of assuming we receive $n$ observations, assume we receive the $n+n_{1}$ observations $X_{-n_{1}}, \ldots, X_{-1}$ and $X_{1}, \ldots, X_{n}$, defining $T_{n}=\theta_{0}-F^{-1}\left(\frac{1}{n_{1}} \sum_{i=-n_{1}}^{-1} B_{i}\right)$ and $B_{i}=1\left\{X_{i} \leq T_{n}\right\}$ for $i \geq 1$. Letting $X_{i}=Z_{i}+\theta$ for $Z_{i}$ i.i.d. with fixed density $f=F^{\prime}$, we have $\mathbb{E}\left[B_{i}\right] \xrightarrow{\text { a.s. }} F\left(\theta_{0}-\theta\right)$, so that


Now let $E_{n}=\mathbb{E}\left[B_{i} \mid T_{n}\right]=P\left(X_{i} \leq T_{n}\right)$, so that $\operatorname{Var}\left(B_{i} \mid T_{n}\right)=E_{n}\left(1-E_{n}\right)$. Define also the random variable

$$
Y_{n} \triangleq \sqrt{n} \frac{1}{\sqrt{E_{n}\left(1-E_{n}\right)}}\left[\frac{1}{n} \sum_{i=1}^{n} B_{i}-E_{n}\right]
$$

and let $F_{n}\left(\cdot \mid T_{n}\right)$ be its cumulative distribution function. Then because

$$
\mathbb{E}\left[\left|B_{i}-E_{n}\right|^{3} \mid T_{n}\right] \leq E_{n}\left(1-E_{n}\right)
$$

we have

$$
\mathbb{E}\left[\left.\frac{\left|B_{i}-E_{n}\right|^{3}}{\left(E_{n}\left(1-E_{n}\right)\right)^{3 / 2}} \right\rvert\, T_{n}\right] \leq \frac{1}{\sqrt{E_{n}\left(1-E_{n}\right)}}
$$

The Berry-Esseen theorem implies that there exists a constant $C \leq 1$ such that

$$
\sup _{t}\left|F_{n}\left(t \mid T_{n}\right)-\Phi(t)\right| \leq \frac{C}{\sqrt{E_{n}\left(1-E_{n}\right)} \sqrt{n}} \wedge 2
$$

where $\Phi$ is the standard Gaussian CDF. As $E_{n}\left(1-E_{n}\right) \xrightarrow{\text { a.s. }} \frac{1}{4}$ by definition of the median, we have that (with probability 1)

$$
\sup _{t}\left|F_{n}\left(t \mid T_{n}\right)-\Phi(t)\right| \leq \frac{C}{\sqrt{n}} \text { eventually. }
$$

By dominated convergence and Jensen's inequality we thus obtain

$$
\sup _{t}\left|\mathbb{P}\left(Y_{n} \leq t\right)-\Phi(t)\right| \leq \mathbb{E}\left[\sup _{t}\left|F_{n}\left(t \mid T_{n}\right)-\Phi(t)\right|\right] \rightarrow 0
$$

which gives that $Y_{n} \stackrel{d}{\rightsquigarrow} \mathcal{N}(0,1)$. Now, Slutsky's lemmas imply

$$
\begin{align*}
& \sqrt{n} \cdot \frac{2}{n} \sum_{i=1}^{n}\left(B_{i}-E_{n}\right)  \tag{12}\\
& \quad=\frac{1+o_{P}(1)}{\sqrt{n E_{n}\left(1-E_{n}\right)}} \sum_{i=1}^{n}\left(B_{i}-E_{n}\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}(0,1) \tag{13}
\end{align*}
$$

where $o_{P}(1)$ denotes sequence of random variables converging to zero in probability as $n$ goes to infinity. With $\bar{B}_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} B_{i}$ and using that $E_{n}=\mathbb{E}\left[B_{i} \mid T_{n}\right]=F\left(T_{n}-\theta\right)$, we may use the delta method to write

$$
\begin{aligned}
& \sqrt{n}\left(\theta_{n}-\theta\right)=\sqrt{n}\left(T_{n}-F^{-1}\left(\bar{B}_{n}\right)-\theta\right) \\
& \quad=\sqrt{n}\left[T_{n}-F^{-1}\left(F\left(T_{n}-\theta\right)+\bar{B}_{n}-F\left(T_{n}-\theta\right)\right)-\theta\right] \\
& \quad=\sqrt{n}\left[T_{n}-\left(T_{n}-\theta\right)\right. \\
& \left.+\left(F^{-1}\right)^{\prime}\left(T_{n}-\theta+o_{P}(1)\right) \cdot\left(\bar{B}_{n}-E_{n}\right)-\theta\right] \\
& \quad=\sqrt{n}\left(F^{-1}\right)^{\prime}(0)\left(\bar{B}_{n}-E_{n}\right)+o_{P}(1) \\
& \quad \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, \frac{1}{4 f(0)^{2}}\right)
\end{aligned}
$$

where we have used the limiting distribution (13).

Figure 4 illustrates the empirical risks of the estimator (8) and an estimator obtained using one round of threshold adaptation under a series of Monte Carlo simulations when $f(x)$ is the standard normal desnity.


Fig. 4. Normalized empirical risk versus number of samples $n$ for 10,000 Monte Carlo trials with $f(x)$ the standard normal density. In each trial, $\theta$ is chosen uniformly over the interval $(-1.64,1.64)$. The one round threshold adaptation strategy uses $n_{1}=\lfloor\sqrt{n}\rfloor$ samples before adapting the threshold.

## V. Distributed Estimation

We now consider the distributed encoding setting in Figure 1-(iii) where each one-bit message $B_{i}$ is a function only of its private sample $X_{i}$. In this case, the $i$ th encoder is of the form $B_{i}=1\left\{X_{i} \in A_{i}\right\}$, where the detection region $A_{i}$ is a Borel set independent of $X_{1}, X_{2}, \ldots$

## A. Optimal Efficiency

We begin by making a few restrictions on the collections of the sets $A_{i}$, which we believe not unreasonable, but which allow us to develop fundamental limits for estimation. We require a bit of notation to define the assumptions. As we work with a location family based on a density $f$ with associated probability distribution $P$ on variables $Z$, we define

$$
P_{\theta}(A) \triangleq P(Z-\theta \in A)
$$

for $Z$ with density $f$. Whenever $A$ is a collection of disjoint intervals $A=\cup_{i}\left[t_{i}^{-}, t_{i}^{+}\right]$, we may define

$$
\dot{P}_{\theta}(A) \triangleq \frac{\partial}{\partial \theta} P_{\theta}(A)=\sum_{i}\left(f\left(t_{i}^{-}-\theta\right)-f\left(t_{i}^{+}-\theta\right)\right)
$$

and similarly we define the score function $\dot{\ell}_{\theta}(A) \triangleq \dot{P}_{\theta}(A) / P_{\theta}(A)$. For $B=1\{X \in A\}$, we abuse notation and also write $\dot{\ell}_{\theta}(B)=\dot{\ell}_{\theta}(A)$ and similarly for $\dot{P}_{\theta}$. With this, we may define the variance of the scores $\dot{\ell}_{\theta}\left(B_{i}\right)$ under $P_{\theta}$ via

$$
\begin{equation*}
L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right) \triangleq \frac{1}{n} \sum_{i=1}^{n} \frac{\dot{P}_{\theta}\left(A_{i}\right)^{2}}{P_{\theta}\left(A_{i}\right)\left(1-P_{\theta}\left(A_{i}\right)\right)} \tag{14}
\end{equation*}
$$

We then make the following assumption.
Assumption A2: The density and detection regions satisfy
(i) The density function $f$ of $X_{n}-\theta$ is Lipschitz continuous.
(ii) Each set $A_{i}$ is the finite union of $k_{i}$ disjoint intervals (which may include $\pm \infty$ ), where

$$
\frac{1}{n} \cdot \max _{i \leq n} \frac{k_{i}^{3}}{P_{\theta}\left(A_{i}\right)^{4}\left(1-P_{\theta}\left(A_{i}\right)\right)^{4}} \rightarrow 0
$$

(iii) The limit

$$
\begin{equation*}
\kappa(\theta) \triangleq \lim _{n \rightarrow \infty} L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right) \tag{15}
\end{equation*}
$$

exists and is finite.
Roughly speaking, (ii) above holds whenever the intervals consisting each $A_{i}$ are appropriately seperated and their number is relatively small. For example, it applies when each set $A_{i}$ is a half-bounded interval $\left(t_{i}, \infty\right)$ with $\min \left\{P_{\theta}\left(\left(t_{i}, \infty\right)\right), P_{\theta}\left(\left(-\infty, t_{i}\right]\right)\right\}=\omega(1 / n)$ as we dicscuss in more detail below. More generally, let $\Delta_{i}$ the minimal distance between any two interval endpoints in $A_{i}$. Then, if $A_{i}=\cup_{j=1}^{k_{i}}\left[t_{i, j}^{-}, t_{i, j}^{+}\right]$, we have that $P_{\theta}\left(A_{i}\right) \geq$ $\Delta_{i} \sum_{j=1}^{k_{i}} F\left(t_{i, j}^{-}\right)$and $1-P_{\theta}\left(A_{i}\right) \geq \Delta_{i} \sum_{j=1}^{k_{i}} F\left(t_{i, j}^{+}\right)$. Therefore, A2(ii) holds whenever $\max _{i \leq n} k_{i}^{3} \Delta_{i}^{-4}=o(n)$ as long as $\sum_{j=1}^{k_{i}} F\left(t_{i, j}^{-}\right)$and $\sum_{j=1}^{k_{i}} F\left(t_{i, j}^{+}\right)$are bounded away of zero.

Under Assumption A2, we have the following theorem, which provides a local asymptotic minimax lower bound on the efficiency of any non-adaptive estimator.

Theorem 6: Let Assumption A2 hold, and let $\theta_{n}$ be an estimator of $\theta \in \Theta$ from observations $B_{i}=1\left\{X_{i} \in A_{i}\right\}$. Then for $Z \sim \mathcal{N}(0,1)$ and any symmetric and quasi-convex function $L$,

$$
\begin{gathered}
\liminf _{c \rightarrow \infty} \liminf _{n \rightarrow \infty} \sup _{\tau:|\theta-\tau| \leq \frac{c}{\sqrt{n}}} \mathbb{E}\left[L\left(\sqrt{n}\left(\theta_{n}-\tau\right)\right)\right] \\
\geq \mathbb{E}[L(Z / \sqrt{\kappa(\theta)})]
\end{gathered}
$$

See Appendix X for a proof.
Theorem 6 shows that the limiting variance term $\kappa(\theta)$ provides a strong lower bound on the efficiency of any non-adaptive estimator, and moreover, that this bound necessarily depends on $\theta$. As a particular consequence, for the squared error $L(x)=x^{2}$, for any $\delta>0$ and $\theta$, there exists a $c<\infty$ such that $\sup _{|\tau-\theta| \leq c / \sqrt{n}} \mathbb{E}_{\tau}\left[\left(\theta_{n}-\tau\right)^{2}\right] \geq$ $\frac{(1-\delta)}{n \kappa(\theta)}+o(1 / n)$. Consequently, attaining any type of good (uniform) efficiency with non-adaptive estimators will be challenging.

Yet, Theorem 6 limits non-adaptive strategies in stronger ways. Under the density models we have considered, with the additional Assumption A1, we can show stronger optimality results that adaptivity is essential for achieving optimal convergence guarantees. Recall the transformation (4) of the hazard rate function, $\eta(x)=\frac{f^{2}(x)}{F(x)(1-F(x))}$, which has unique maximum at $x=0$ under Assumption A1. When each detection region $A_{n}$ consists of a bounded number of intervals, the next theorem shows that the minimal risk $1 / \eta(0)$ can only be attained at finitely many points within $\Theta$. In particular, distinct from the adaptive setting, no distributed estimation scheme can achieve asymptotic variance $\eta(0)$ uniformly in $\theta \in \Theta$.

Theorem 7: Let Assumptions A1 and A2 hold. Additionally, assume that $A_{i}$ is the union of at most $K$ intervals. The number of points $\theta \in \Theta$ satisfying $\kappa(\theta)=\eta(0)$ is at most $2 K$.
See Appendix XI for a proof.

## B. Threshold Detection

We now consider a restricted case where each detection region is a half-open interval, i.e., the $i$ th message is obtained by comparing $X_{i}$ against a single threshold. Under the adaptive signal acquisition setting, this is sufficient for asymptotic optimality; in non-adaptive settings, it is not sufficient, though we may characterize a few additional optimality results. Assume now that each $B_{i}$ is of the form

$$
B_{i}=\operatorname{sgn}\left(t_{i}-X_{i}\right)= \begin{cases}1 & X_{i}<t_{i}  \tag{16}\\ -1 & X_{i} \geq t_{i}\end{cases}
$$

where $t_{i} \in \mathbb{R}$ is the threshold of the $i$ th encoder. In other words, the detection region of $B_{i}$ is $A_{i}=\left(t_{i}, \infty\right)$ and $\mathbb{P}\left(X_{i} \in A_{i}\right)=F\left(B_{i}\left(t_{i}-\theta\right)\right)$. It follows that

$$
\begin{align*}
L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right) & =\frac{1}{n} \sum_{i=1}^{n} \frac{\left(f\left(t_{i}-\theta\right)\right)^{2}}{F\left(t_{i}-\theta\right) F\left(\theta-t_{i}\right)}  \tag{17}\\
& =\frac{1}{n} \sum_{i=1}^{n} \eta\left(t_{i}-\theta\right) . \tag{18}
\end{align*}
$$

A natural condition for the existence of the limit (18) as $n \rightarrow \infty$ is that the empirical distribution of the threshold values converges to a probability measure. Specifically, for an interval $I \subset \mathbb{R}$, define

$$
\lambda_{n}(I)=\frac{\operatorname{card}\left(I \cap\left\{t_{1}, t_{2}, \ldots\right\}\right)}{n} .
$$

Then an investigation of the proof of Theorem 6 in Section X, specifically Sec. X-B and the bounds (51), show that as $\eta(t) \leq \eta(0)$ for all $t \in \mathbb{R}$ under Assumption A1, the following corollary follows. (The corollary relies on local asymptotic normality [19, Ch. 7]; see Appendix IX-B for some brief discussion of such conditions.)

Corollary 8: Let $\left\{t_{n}\right\}_{n=1}^{\infty}$ be a sequence of threshold values such that $\lambda_{n}$ converges (weakly) to a probability measure $\lambda$ on $\mathbb{R}$. Then the conclusions of Theorem 6 apply with

$$
\kappa(\theta)=\int_{\mathbb{R}} \eta(t-\theta) \lambda(d t) .
$$

Moreover, the family of laws of $\left\{B_{i}=\operatorname{sgn}\left(X_{i}-t_{i}\right)\right\}_{i=1}^{n}$ under $\left\{P_{\theta}\right\}_{\theta \in \Theta}$ is locally asymptotically normal with information $\kappa(\theta)$.

The condition that $\lambda_{n}$ converges to a probability measure is satisfied, for example, whenever $t_{1}, \ldots, t_{n}$ are drawn independently from a probability distribution $\lambda(d t)$ on $\mathbb{R}$.

When the conclusions of Corollary 8 hold, local asymptotic normality of $\left\{B_{n}\right\}_{n=1}^{\infty}$ implies that the maximum likelihood estimator (ML) of $\theta$ from $B_{1}, \ldots, B_{n}$, denoted here by $\theta_{n}^{M L}$, is local asymptotic minimax in the sense that

$$
\sqrt{n}\left(\theta_{n}^{M L}-\theta\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}(0,1 / \kappa(\theta)) .
$$

We note that $\theta_{n}^{M L}$ solves

$$
\begin{equation*}
0=\sum_{i=1}^{n} B_{i} \frac{f\left(t_{i}-\theta\right)}{F\left(B_{i}\left(t_{i}-\theta\right)\right)} . \tag{19}
\end{equation*}
$$

If the collection $\left\{t_{1}, t_{2} \ldots\right\}$ is bounded (for example $\left\{t_{1}, t_{2} \ldots\right\} \subset \Theta$ ), then

$$
\lim _{n \rightarrow \infty} n \cdot \mathbb{E}\left[\left(\theta_{n}^{M L}-\theta\right)^{2}\right]=1 / \kappa(\theta)
$$

so that the ML estimator attains the local asymptotic MSE of Theorem 6.
By Assumption A1, $\eta(x)$ attains its maximum at the origin, so we conclude that

$$
\kappa(\theta) \leq \sup _{t \in \mathbb{R}} \eta(t-\theta)=\eta(0) .
$$

Moreover, this upper bound on $\kappa(\theta)$ is attained only when $\lambda$ is the point mass at $\theta$. Since $\theta$ is a priori unknown, estimation in the distributed setting using threshold detection is strictly suboptimal compared to the adaptive setting; the ability to choose the thresholds $t_{i}$ adaptively conditional on previous messages is necessary for optimal efficiency.

## C. Minimax Threshold Density

We conclude this section by considering the distribution of the threshold values that maximizes the worst-case information $\inf _{\theta} \kappa(\theta)=\kappa_{\lambda}(\theta)$ where $\kappa_{\lambda}(\theta)=\int \eta(t-\theta) \lambda(d t)$. The optimal distribution $\lambda^{\star}$ solves the optimization problem

$$
\begin{array}{ll}
\operatorname{maximize} & \inf _{\theta \in \Theta} \int \eta(t-\theta) \lambda(d t)  \tag{20}\\
\text { subject to } & \lambda(d t) \geq 0, \quad \int \lambda(d t) \leq 1
\end{array}
$$

The objective function (20) is concave in $\lambda(d t)$ and continuous in the weak topology over measures on $\Theta$, so that by discretizing, we can approximately solve this problem using convex optimization. We let $\kappa^{\star}$ denote the maximal value of problem (20) and $\lambda^{\star}(d t)$ be the density achieving the maximum. By drawing thresholds $t_{i} \stackrel{\text { iid }}{\sim} \lambda^{\star}$, Corollary 8 guarantees that for any $\theta \in \Theta$, the maximum likelihood estimator using $\left\{B_{i}=\operatorname{sgn}\left(X_{i}-t_{i}\right)\right\}_{i \in \mathbb{N}}$ is at least $\kappa^{\star}$.

Figure 5 illustrates an approximation to $\lambda^{\star}(d t)$ obtained by solving a discretized version of (20) for the case when $f(x)$ is the normal density with variance $\sigma^{2}$ and $\Theta=[-1 / 2,1 / 2]$. The minimax asymptotic precision parameter $\kappa^{\star}$ obtained this way is illustrated in Fig. 6 as a function of $\sigma$. Also illustrated in these figures is $\kappa_{\text {unif }}$, the precision parameter corresponding to threshold values uniformly distribution over $\Theta$,

$$
\begin{align*}
& \kappa_{\text {unif }} \triangleq \min _{\theta \in[-T, T]} \frac{1}{2 T} \int_{-T}^{T} \eta(t-\theta) d t \\
& =\frac{1}{2 T} \int_{-T}^{T} \eta(t \pm T) d t=\frac{1}{2 T} \int_{0}^{2 T} \eta(t) d t . \tag{21}
\end{align*}
$$



Fig. 5. Optimal threshold density under distributed encoding. The threshold density $\lambda^{\star}(d t)$ (blue) that maximizes the asymptotic relative efficiency for $f(x)$ the normal density with variance $\sigma^{2}$ and $\Theta=[-1 / 2,1 / 2]$. The continuous curve (red) is the ARE for each $\theta \in[-1 / 2,1 / 2]$ under the optimal density, hence the minimax ARE is the minimal value of this curve. The dashed curve (green) is the ARE when the threshold values are uniformly distributed over $[-1 / 2,1 / 2]$; its minimal value is $\kappa_{\text {unif }}(21)$.


Fig. 6. Minimax relative efficiency under distributed encoding. ARE versus $\sigma$ for $f(x)$ the standard normal density with variance $\sigma^{2}$ and parameter space $\Theta=[-1 / 2,1 / 2]$. The dashed curve (green) is the ARE under a uniform threshold density over $\Theta$ given by $K_{\text {unif }} \sigma^{2}$ of (21). The line $\pi / 2$ is attained under adaptive encoding uniformly over the parameter space for any $\sigma$.

## VI. CONCLUSIONS

We considered the risk and efficiency in estimating the mean of a symmetric and log-concave distribution from a sequence of bits, where each bit is obtained by encoding a single sample from this distribution. In an adaptive encoding setting, we showed that, asymptotically, no estimator can be more efficient than the median of the samples. We also showed that this bound is tight by presenting two adaptive encoding and estimation procedures that are as efficient as the median. Furthermore, we showed that only one round of adaptivity is required to attain optimal efficiency. In the distributed setting we provided conditions for local asymptotic normality of the encoded samples, which implies asymptotic minimax bound on both the risk and efficiency relative to the mean. Under local asymptotic normality, the optimal estimation performance derived for the adaptive case can only be attained over a finite number of points, i.e., no scheme is uniformly optimal in this setting. We further considered the special case where the sequence of bits is obtained in a distributed manner by comparing against a prescribed sequence of thresholds. We
characterized the performance of the optimal estimator from such bit-sequence using the density of the thresholds and considered the density that minimizes the minimax risk.
Natural extensions of this work include situations when the communication bit-budget $b$ is larger than one and when each sample is a $d$-dimensional vector. Bounds on rate of convergence of the MSE in this general case follow from several recent works (e.g. [31], [50], [51], [52], [53], [14]), that in particular imply that in some cases the MSE decreases in the regular parametric rate of $1 / n$ when $b$ and $d$ are held fixed in the sample size $n$. Nevertheless, the coefficient of the leading $1 / n$ term corresponding to the ARE, which we characterized here in the case $b=1$ and $d=1$, is still unknown in the general case.

## Appendices

## VII. FAST CONVERGENCE OF UNIFORM ESTIMATORS UNDER BIT CONSTRAINTS

Here we consider the uniform distribution as our location family, demonstrating that in the adaptive setting (ii) or even the one-step adaptive setting (ii'), constrained estimators can attain rates faster than the $1 / \sqrt{n}$ rates regular estimands allow. Indeed, define $c(x)=-\log 2$ for $x \in[-1,1]$ and $c(x)=-\infty$ for $x \notin[-1,1]$. Then $f(x)=e^{-c(x)}$ is log-concave and symmetric, and we may consider the location family with densities $f(x-\theta)$. For notational simplicity, we assume we have a sample of size $2 n$. We provide a proof sketch that there is a one-step adaptive estimator $\theta_{n}$ such that

$$
\begin{equation*}
\sup _{|\theta| \leq \log n} P_{\theta}\left(\left|\theta_{n}-\theta\right| \geq \frac{16 \log n}{n^{3 / 4}}\right) \leq \frac{2}{n^{2}} \tag{22}
\end{equation*}
$$

for all large $n$, and so (by the Borel-Cantelli lemmas), for any $\theta \in \mathbb{R}$ we have $P_{\theta}\left(\left|\theta_{n}-\theta\right| \leq\right.$ $16 \log n / n^{3 / 4}$ eventually) $=1$. This is of course faster than the $1 / \sqrt{n}$ rates we prove throughout.

To prove inequality (22), we proceed in two steps, both quite similar. First, we define an initial estimator $\theta_{n}^{\text {init }}$. Let $\epsilon>0$, which we will determine presently, though we will take $n \epsilon \rightarrow \infty$ as $n \rightarrow \infty$, so that we may assume w.l.o.g. that $\theta \in[-n \epsilon / 2, n \epsilon / 2]$. Take the interval $[-n \epsilon, n \epsilon]$, and construct $m$ thresholds at intervals of size $2 n \epsilon / m$; let the $j$ th such threshold be

$$
t_{j} \triangleq-n \epsilon+\frac{2 n(j-1) \epsilon}{m}
$$

Then we "assign" observations to each pair of thresholds, so that threshold $j$ corresponds to observations $I_{j} \triangleq$ $\left\{\frac{n(j-1)}{m}+1, \ldots, \frac{n j}{m}\right\}$, of which there are $n / m$. For each index $i \in I_{j}$, we set

$$
B_{i}= \begin{cases}1 & \text { if } X_{i}-1 \geq t_{j} \\ 0 & \text { otherwise }\end{cases}
$$

Then we simply set $\theta_{n}^{\text {init }}$ to be the minimal threshold for which $B_{i}=0$ for all observations $X_{i}$ corresponding to that threshold. Denote by $j^{*}$ the index of the threshold corresponding to $\theta_{n}^{\text {init }}$.

Let us now consider the probability that $\theta_{n}^{\text {init }}$ is substantially wrong. Set $\theta_{M} \equiv \max _{i \in I_{j}^{*}} X_{i}-1$. Note that we always have $\theta_{n}^{\text {init }} \geq \theta_{M}$ because no observations will be above $t_{j}^{*}+1$, and that $\theta_{n}^{\text {init }} \leq \theta_{M}+2 n \epsilon / m$. In addition,

$$
P_{\theta}\left(\left|\theta_{M}-\theta\right| \geq \frac{2 n \epsilon}{m}\right)=\left(1-\frac{2 n \epsilon}{m}\right)^{n / m}
$$

Putting it all together using the triangle inequality, we have

$$
P_{\theta}\left(\left|\theta_{n}^{\text {init }}-\theta\right| \geq \frac{4 n \epsilon}{m}\right) \leq\left(1-\frac{2 n \epsilon}{m}\right)^{n / m} \leq e^{-2 \frac{n^{2}}{m^{2}} \epsilon}
$$

Therefore, setting the number of bins $m=\sqrt{n}$ and the resolution $\epsilon=\log n / n$,

$$
\begin{equation*}
\sup _{|\theta| \leq \log n} P_{\theta}\left(\left|\theta_{n}^{\text {init }}-\theta\right| \geq \frac{4 \log n}{\sqrt{n}}\right) \leq \frac{1}{n^{2}} \tag{23}
\end{equation*}
$$

The second stage estimator follows roughly the same strategy, except that the resolution of the bins is tighter. In particular, let us assume that $\left|\theta_{n}^{\text {init }}-\theta\right| \leq \frac{8 \log n}{\sqrt{n}}$, which happens eventually by inequality (23). (We will assume this tacitly for the remainder of the argument.) Consider the interval $\Theta_{n} \triangleq \theta_{n}^{\text {init }}+\left[-\frac{16 \log n}{\sqrt{n}}, \frac{16 \log n}{\sqrt{n}}\right]$ centered at $\theta_{n}^{\text {init }}$; we know that the interval includes $\left[\theta-\frac{8 \log n}{\sqrt{n}}, \theta+\frac{8 \log n}{\sqrt{n}}\right]$. Without loss of generality we assume $\theta_{n}^{\text {init }}=0$. Following precisely the same discretization strategy as that for $\theta_{n}^{\text {init }}$, we divide $\Theta_{n}$ into $m$ equal intervals, with thresholds $t_{j}=-\frac{16 \log n}{\sqrt{n}}+\frac{32(j-1) \log n}{m \sqrt{n}}$; let $\epsilon_{n}=\frac{32 \log n}{m \sqrt{n}}$ be the width of these intervals. Then following exactly the same reasoning as above, we assign indices $I_{j}=\left\{\frac{n(j-1)}{m}+1, \ldots, \frac{n j}{m}\right\}$ and for $i \in I_{j}$, set $B_{i}=1$ if $X_{i}-1 \geq t_{j}$. We
define $\theta_{n}$ to be the minimal threshold $t_{j}$ for which $B_{i}=0$ for all observations $X_{i} \in I_{j}$. Then following precisely the reasoning above, we have (on the event that $\left|\theta_{n}^{\text {init }}-\theta\right| \leq \frac{8 \log n}{\sqrt{n}}$ )

$$
\begin{aligned}
& P_{\theta}\left(\left|\theta_{n}-\theta\right| \geq 2 \epsilon_{n}\right) \leq\left(1-\epsilon_{n}\right)^{\frac{n}{m}} \leq \exp \left(-\frac{n \epsilon_{n}}{m}\right) \\
& \quad=\exp \left(-\frac{32 \sqrt{n} \log n}{m^{2}}\right) .
\end{aligned}
$$

Set $m=4 n^{1 / 4}$ to obtain the claimed result (22).

## A. Proof of Proposition 1

Denote by $D^{\star}$ the optimal MSE in the Gaussian CEO with $L$ observers and under a total sum-rate $r=r_{1}+\ldots+r_{L}$. An expression for $D^{\star}$ as a function of $r$ is give as [48, Eq. 10]:

$$
\begin{equation*}
r=\frac{1}{2} \log ^{+}\left[\frac{\sigma_{\theta}^{2}}{D^{\star}}\left(\frac{D^{\star} L}{D^{\star} L-\sigma^{2}+D^{\star} \sigma^{2} / \sigma_{\theta}^{2}}\right)^{L}\right] . \tag{24}
\end{equation*}
$$

For the special case where $r=n$ and $L=n$, we have

$$
\begin{equation*}
n=\frac{1}{2} \log _{2}\left[\frac{\sigma_{\theta}^{2}}{D^{\star}}\left(\frac{D^{\star} n}{D^{\star} n-\sigma^{2}+D^{\star} \sigma^{2} / \sigma_{\theta}^{2}}\right)^{n}\right] . \tag{25}
\end{equation*}
$$

Consider the distributed encoding setting (iii) in the case where $f(x)=\mathcal{N}\left(0, \sigma^{2}\right)$ and the prior on $\Theta$ is $\pi=$ $\mathcal{N}\left(0, \sigma_{\theta}^{2}\right)$. The Gaussian CEO problem of [28] with a unit bitrate $r_{1}=\ldots=r_{n}=1$ at each terminal and blocklength $k=1$ reduces to our distributed setting (iii). Since $D^{\star}$ satisfying (25) describes the MSE in the CEO setting under an optimal allocation of the sum-rate $r=n$ among $n$ encoders, it provides a lower bound to the minimal MSE in estimating $\theta$ in the distributed setting. By noting that $1 / D^{\star}$ grows no faster than a polynomial in $n$ [28], we rely on the expansion

$$
\left(\frac{\sigma_{\theta}^{2}}{D^{\star}}\right)^{1 / n}=1+\frac{\log \left(\frac{\sigma_{\theta}^{2}}{D^{\star}}\right)}{n}+\frac{\log ^{2}\left(\frac{\sigma_{\theta}^{2}}{D^{\star}}\right)}{2 n^{2}}+O\left(n^{-3}\right)
$$

to obtain that, in limit $n \rightarrow \infty$, (25) behaves as

$$
D^{\star}=\frac{4 \sigma^{2}}{3 n}+\frac{16 \sigma^{2}}{9 n^{2} \sigma_{\theta}^{2}}-\frac{4 \sigma^{2} \log \left(\frac{\sigma_{\theta}^{2}}{D^{\star}}\right)}{9 n^{2}}+O\left(n^{-3}\right)
$$

This implies Proposition 1.

## VIII. PRoof of Theorem 2

We begin with two technical lemmas.
Lemma 9: Let $f$ be a log-concave and symmetric density function for which Assumption A1 holds. For any $x_{1} \geq \ldots \geq x_{n} \in \mathbb{R}$,

$$
\begin{align*}
& \frac{\left|\sum_{k=1}^{n}(-1)^{k+1} f\left(x_{k}\right)\right|^{2}}{\left(\sum_{k=1}^{n}(-1)^{k+1} F\left(x_{k}\right)\right)\left(1-\sum_{k=1}^{n}(-1)^{k+1} F\left(x_{k}\right)\right)} \\
& \leq 4 f(0)^{2} . \tag{26}
\end{align*}
$$

Lemma 10: Let $X$ be a random variable with a symmetric, log-concave, and continuously differentiable density function $f(x)$ such that Assumption A1 holds. For a Borel measurable set $A$, define

$$
B(x) \triangleq \begin{cases}1 & \text { if } x \in A \\ -1 & \text { if } x \notin A .\end{cases}
$$

The Fisher information of $B$ with respect to $\theta$ is bounded from above by $\eta(0)$.
Lemma 9 is the special case $\delta=0$ of Lemma 11 to come in Section VIII-A. We now prove Lemma 10.

Proof of Lemma 10: We first note that in the special case where $f$ is a normal density, Lemma 10 follows from [39, Thm. 3]. The proof below, valid for any log-concave symmetric density satisfying Assumption A1, is based on a different techique than that of [39].

Write the Fisher information of $B$ with respect to $\theta$ as

$$
\begin{align*}
I_{\theta} & =\mathbb{E}\left[\left.\left(\frac{d}{d \theta} \log P(B \mid \theta)\right)^{2} \right\rvert\, \theta\right] \\
& =\frac{\left(\frac{d}{d \theta} P(B=1 \mid \theta)\right)^{2}}{P(B=1 \mid \theta)}+\frac{\left(\frac{d}{d \theta} P(B=-1 \mid \theta)\right)^{2}}{P(B=-1 \mid \theta)} \\
& =\frac{\left(\frac{d}{d \theta} \int_{A} f(x-\theta) d x\right)^{2}}{P(B=1 \mid \theta)}+\frac{\left(\frac{d}{d \theta} \int_{A} f(x-\theta) d x\right)^{2}}{P(B=-1 \mid \theta)} \\
& \stackrel{(a)}{=} \frac{\left(-\int_{A} f^{\prime}(x-\theta) d x\right)^{2}}{P(B=1 \mid \theta)}+\frac{\left(-\int_{A} f^{\prime}(x-\theta) d x\right)^{2}}{P(B=-1 \mid \theta)} \\
& =\frac{\left(\int_{A} f^{\prime}(x-\theta) d x\right)^{2}}{P(B=1 \mid \theta)(1-P(B=1 \mid \theta))}, \\
& =\frac{\left(\int_{A} f^{\prime}(x-\theta) d x\right)\left(\int_{A} f^{\prime}(x-\theta) d x\right)}{\left(\int_{A} f(x-\theta) d x\right)\left(1-\int_{A} f(x-\theta) d x\right)}, \tag{27}
\end{align*}
$$

where differentiation under the integral sign in $(a)$ is justified since $f$ is log-concave hence a.e. differentiable (cf. [54]) with a.e. derivative $f^{\prime}(x)$. By regularity of the Lebesgue, for any $\epsilon>0$ there exists a finite number $k$ of disjoint open intervals $I_{1}, \ldots I_{k}$ such that

$$
\int_{A \backslash \cup_{j=1}^{k} I_{j}} d x<\epsilon
$$

It follows that for any $\epsilon^{\prime}>0$, the set $A$ in (27) can be replaced by a finite union of disjoint intervals without increasing $I_{\theta}$ by more than $\epsilon^{\prime}$. Consequently, we may proceed assuming that $A$ is of the form

$$
A=\cup_{j=1}^{k}\left(t_{j}^{+}, t_{j}^{-}\right)
$$

with $\infty \leq t_{1}^{-} \leq \ldots \leq t_{k}^{-}, t_{1}^{+} \leq \ldots \leq t_{k}^{+} \leq \infty$ and $t_{j}^{-} \leq t_{j}^{+}$for $j=1, \ldots, k$. Under this assumption,

$$
P_{\theta}\left(B_{n}=1\right)=\sum_{j=1}^{k}\left(F\left(t_{j}^{+}-\theta\right)-F\left(t_{j}^{-}-\theta\right)\right)
$$

so we may rewrite Eq. (27) as

$$
\begin{aligned}
I_{\theta} & =\frac{\left(\sum_{j=1}^{k}\left[f\left(t_{j}^{+}-\theta\right)-f\left(t_{j}^{-}-\theta\right)\right]\right)^{2}}{\left(\sum_{j=1}^{k}\left[F\left(t_{j}^{+}-\theta\right)-F\left(t_{j}^{-}-\theta\right)\right]\right)} \\
& \times \frac{1}{1-\left(\sum_{j=1}^{k}\left[F\left(t_{j}^{+}-\theta\right)-F\left(t_{j}^{-}-\theta\right)\right]\right)}
\end{aligned}
$$

It follows from Lemma 9 that for any $\theta \in \mathbb{R}$ and any choice of the intervals' endpoints,

$$
I_{\theta} \leq \max _{t \in\left\{t_{1}^{ \pm}, \ldots, t_{k}^{ \pm}\right\}} 4 f(t)^{2} \leq 4 f(0)^{2}
$$

We now prove Theorem 2. Write the Fisher information of $B_{1}, \ldots, B_{n}$ with respect to $\theta$ as

$$
I_{\theta}\left(B_{1}, \ldots, B_{n}\right)=\sum_{i=1}^{n} I_{\theta}\left(B_{i} \mid B_{1}, \ldots, B_{i-1}\right)
$$

where $I_{\theta}\left(B_{i} \mid B_{i-1}, \ldots, B_{1}\right)$ is the Fisher information of the distribution of $B_{i}$ given $B_{1}, \ldots, B_{i-1}$. By the definition of the adaptive setting, $P_{\theta}\left(B_{i} \mid B_{1}, \ldots, B_{i-1}\right)=P_{\theta}\left(X_{i} \in A_{i}\right)$ for some Borel measurable $A_{i}$. Consequently, Lemma 10 applies, leading to the bound

$$
I_{\theta}\left(B_{i} \mid B_{i-1}, \ldots, B_{1}\right) \leq 4 f(0)^{2}
$$

We conclude

$$
\begin{equation*}
I_{\theta}\left(B_{1}, \ldots, B_{n}\right) \leq 4 f(0)^{2} n \tag{28}
\end{equation*}
$$

The Van Trees inequality in the version of [55] holds under the regularity conditions on $\pi(\cdot)$, which implies

$$
\mathbb{E}\left[\left(\theta_{n}-\theta\right)^{2}\right] \geq \frac{1}{\mathbb{E}_{\pi}\left[I_{\theta}\left(B_{1}, \ldots, B_{n}\right)\right]+I_{0}}
$$

Combining the last display with (28), we get

$$
\mathbb{E}\left[\left(\theta_{n}-\theta\right)^{2}\right] \geq \frac{1}{4 f(0)^{2} n+I_{0}}
$$

## A. Isoperimetric Lemma

The following lemma is essential to the proofs of Theorems 2 and 7.
Lemma 11: Let $f$ be a log-concave and symmetric density function. Let $\delta \geq 0$. Assume that the function

$$
\eta_{\delta}(x) \triangleq \eta^{1+\delta}(x) / f^{\delta}(x)=\frac{(f(x))^{2+\delta}}{(F(x)(1-F(x)))^{1+\delta}}
$$

is non-increasing in $|x|$. Then for any $x_{1} \geq \ldots \geq x_{n} \in \mathbb{R}$,

$$
\begin{align*}
& \frac{\left|\sum_{i=1}^{n}(-1)^{i+1} f\left(x_{i}\right)\right|^{2+\delta}}{\left|\sum_{i=1}^{n}(-1)^{i+1} F\left(x_{i}\right)\right|^{1+\delta}\left|1-\sum_{k=1}^{n}(-1)^{i+1} F\left(x_{i}\right)\right|^{1+\delta}} \\
& \leq \max _{i} \eta_{\delta}\left(x_{i}\right) . \tag{29}
\end{align*}
$$

In particular,

$$
\begin{aligned}
& \frac{\left|\sum_{i=1}^{n}(-1)^{i+1} f\left(x_{i}\right)\right|^{2+\delta}}{\left|\sum_{i=1}^{n}(-1)^{i+1} F\left(x_{i}\right)\right|^{1+\delta}\left|1-\sum_{i=1}^{n}(-1)^{i+1} F\left(x_{i}\right)\right|^{1+\delta}} \\
& \leq \eta_{\delta}(0)=4^{1+\delta} f^{2+\delta}(0) .
\end{aligned}
$$

Proof of Lemma 11: Denote

$$
\begin{aligned}
& \delta_{n}\left(x_{1}, \ldots, x_{n}\right) \triangleq \sum_{i=1}^{n} s_{i} f\left(x_{i}\right), \\
& \Delta_{n}\left(x_{1}, \ldots, x_{n}\right) \triangleq \sum_{i=1}^{n} s_{i} F\left(x_{i}\right),
\end{aligned}
$$

where $s_{i} \triangleq(-1)^{i+1}$. We use induction on $n \in \mathbb{N}$ to show that

$$
\begin{align*}
& \frac{\left|\delta_{n}\left(x_{1}, \ldots, x_{n}\right)\right|^{2+\delta}}{\left|\Delta_{n}\left(x_{1}, \ldots, x_{n}\right)\left(1-\Delta_{n}\left(x_{1}, \ldots, x_{n}\right)\right)\right|^{1+\delta}} \\
& \leq \max _{i} \eta_{\delta}\left(x_{i}\right) . \tag{30}
\end{align*}
$$

Since

$$
\eta_{\delta}(x)=\frac{\left|\delta_{1}(x)\right|^{2+\delta}}{\left|\Delta_{1}(x)\left(1-\Delta_{1}(x)\right)\right|^{1+\delta}},
$$

The case $n=1$ is trivial. Assume that (30) holds for all integers up to $n=N$ and for any $x_{1} \geq \ldots \geq x_{N}$. Consider the case $n=N+1$. Let $i^{*}$ be the index such that $x_{i^{*}}$ has minimal absolute value among $x_{1}, \ldots, x_{N}$. The assumption on $\eta_{\delta}(x)$ implies that

$$
\eta_{\delta}\left(x_{i^{*}}\right)=\max _{i} \eta_{\delta}\left(x_{i}\right) .
$$

Since the LHS of (29) is invariant to a sign flip of all $x_{1}, \ldots, x_{N+1}$, we may assume that $x_{i^{*}}$ is positive without loss of generality. Set $x^{*}=x_{i^{*}}$ and let $k=i^{*}-1$. Consider the function

$$
\begin{align*}
& g\left(y_{1}, \ldots, y_{N}\right) \triangleq g\left(y_{1}, \ldots, y_{N} \mid x^{*}, k\right)  \tag{31}\\
& \triangleq \frac{\left|\delta_{N+1}\left(y_{1}, \ldots, y_{k}, x^{*}, y_{k+1} \ldots, y_{N}\right)\right|^{2+\delta}}{\left|\Delta_{N+1}\left(y_{1}, \ldots, y_{k}, x^{*}, y_{k+1} \ldots, y_{N}\right)\right|} \\
& \times \frac{1}{\left|1-\Delta_{N+1}\left(y_{1}, \ldots, y_{k}, x^{*}, y_{k+1} \ldots, y_{N}\right)\right|^{1+\delta}}
\end{align*}
$$

The LHS of (30) is obtained by taking $y_{i}=x_{k_{i}}$ where $k_{i}$ is the $i$ th element in $\{1, \ldots, N+1\} \backslash\left\{i^{*}\right\}$. It is therefore enough to prove that

$$
\max _{\left(y_{1}, \ldots, y_{N}\right) \in A_{N}\left(x^{*}, k\right)} g\left(y_{1}, \ldots, y_{N}\right) \leq \eta_{\delta}\left(x^{*}\right)
$$

where

$$
\begin{aligned}
A_{N}\left(x^{*}, k\right) \triangleq & \left\{\left(y_{1}, \ldots, y_{N}\right) \in \mathbb{R}^{N}\right. \\
& \left.: y_{1} \geq \ldots \geq y_{k} \geq x^{*} \geq-x^{*} \geq y_{k+1} \ldots \geq y_{N}\right\}
\end{aligned}
$$

Since $f(x)$ is log-concave and symmetric, we may write $f(x)=e^{c(x)}$ where $c(x)$ is concave, symmetric, and superdifferentiable on the interior of its domain with supergradient set $\partial c(x)=\{v \in \mathbb{R} \mid c(y) \leq c(x)+v(y-$ $x)$ for all $y\} ; c$ is also differentiable a.e. with derivative

$$
c^{\prime}(x) \triangleq \frac{f^{\prime}(x)}{f(x)}
$$

(when it exists), and we otherwise simply treat $f^{\prime}(x) / f(x)=c^{\prime}(x) \in \partial c(x)$ as an arbitrary element of the superdifferential. The supergradient sets $\partial c(x)$ are increasing, in that $v_{0} \in \partial c\left(x_{0}\right)$ and $v_{1} \in \partial c\left(x_{1}\right)$ implies that $\left(v_{1}-v_{0}\right)\left(x_{1}-x_{0}\right) \leq 0$. We first prove the lemma under the assumption that $c$ is strictly concave, or, equivalently, that $v_{i} \in \partial c\left(x_{i}\right)$ implies that $\left(v_{1}-v_{0}\right)\left(x_{1}-x_{0}\right)<0$ whenever $x_{1} \neq x_{0}$; that is, $c^{\prime}$ is strictly decreasing.
The maximal value of $g\left(y_{1}, \ldots, y_{N}\right)$ is attained for the same $\left(y_{1}, \ldots, y_{N}\right) \in A_{N}\left(x^{*}, k\right)$ that maximizes

$$
\begin{aligned}
& \log (g)\left(y_{1}, \ldots, y_{N}\right) \\
& =(2+\delta) \log \left(\delta_{N}\right)-(1+\delta) \log \left(\Delta_{N}\right)-(1+\delta) \log \left(1-\Delta_{N}\right),
\end{aligned}
$$

where in the last display and henceforth we suppress the arguments $y_{1}, \ldots, y_{k}, x^{*}, y_{k+1}, \ldots, y_{N}$ of the functions $\delta_{N}$ and $\Delta_{N}$. Within the interior of $A_{N}\left(x^{*}, k\right)$, all three expressions in (31) within an absolute value are positive. It follows that partial derivative of $\log (g)\left(y_{1}, \ldots, y_{N}\right)$ with respect to $y_{i}$ within the interior of $A_{N}\left(x^{*}, k\right)$ is

$$
\frac{\partial \log (g)}{\partial y_{i}}=\frac{(2+\delta) s_{i} f^{\prime}\left(x_{i}\right)}{\delta_{N}}-\frac{(1+\delta) s_{i} f\left(x_{i}\right)}{\Delta_{N}}+\frac{(1+\delta) s_{i} f\left(y_{i}\right)}{1-\Delta_{N}} .
$$

We conclude that the gradient of $\log (g)$ vanishes if and only if

$$
\begin{equation*}
c^{\prime}\left(y_{i}\right)=\frac{f^{\prime}\left(y_{i}\right)}{f\left(y_{i}\right)}=\frac{1+\delta}{2+\delta} \frac{\delta_{N}}{2}\left(\frac{1}{\Delta_{N}}-\frac{1}{1-\Delta_{N}}\right), \tag{32}
\end{equation*}
$$

for $i=1, \ldots, N$. Since we assumed that $\partial c(x)$ is injective, equality (32) holds if and only if $y_{1}=\ldots=y_{N}$. In this case, $g\left(y_{1}, \ldots, y_{N}\right)=\eta_{\delta}\left(x^{*}\right)$ if $N$ is even. If $N$ is odd and $y_{1}=\ldots=y_{N}>x^{*}$, then

$$
\begin{aligned}
& g\left(y_{1}, \ldots, y_{N}\right) \\
& \quad=\frac{\left|f\left(y_{1}\right)-f\left(x^{*}\right)\right|^{2+\delta}}{\left|F\left(y_{1}\right)-F\left(x^{*}\right)\right|^{1+\delta}\left|1-\left(F\left(y_{1}\right)-F\left(x^{*}\right)\right)\right|^{1+\delta}}
\end{aligned}
$$

which is bounded from above by $\eta_{\delta}\left(x^{*}\right)$ by the induction hypothesis. The case where $N$ is odd and $-x^{*} \leq y_{1}=$ $\ldots=y_{N}$ is similar. We now consider the possibility that the maximum of $g\left(y_{1}, \ldots, y_{N}\right)$ is attained at the boundaries of $A_{N}\left(x^{*}, k\right)$. At boundary points for which $y_{i}=y_{i+1}$ for some $i$, the contribution of $y_{i}$ and $y_{i+1}$ to $g\left(y_{1}, \ldots, y_{N}\right)$ is zero and the induction assumption for $n=N-1$ implies that

$$
g\left(y_{1}, \ldots, y_{N}\right) \leq \eta_{\delta}\left(x^{*}\right)
$$

The remaining boundary points of $A_{N}\left(x^{*}, k\right)$ are covered by the following cases:
(i) $y_{N}=-\infty$.
(ii) $y_{1}=\infty$.
(iii) $y_{k}=x^{*}$.
(iv) $y_{k+1}=-x^{*}$.

For case (i),

$$
\begin{aligned}
& g\left(y_{1}, \ldots, y_{N}\right) \\
& \rightarrow \frac{\left|\sum_{i=1}^{k} s_{i} f\left(y_{i}\right)+s_{i^{*}} f\left(x^{*}\right)-\sum_{i=k+1}^{N-1} s_{i} f\left(y_{i}\right)\right|^{2+\delta}}{\left|\sum_{i=1}^{k} s_{i} F\left(y_{i}\right)+s_{i^{*}} F\left(x^{*}\right)-\sum_{i=k+1}^{N-1} s_{i} F\left(y_{i}\right)\right|^{1+\delta}} \\
& \times \frac{1}{\left|1-\sum_{i=1}^{k} s_{i} F\left(y_{i}\right)-s_{i^{*}} F\left(x^{*}\right)+\sum_{i=k+1}^{N-1} s_{i} F\left(y_{i}\right)\right|^{1+\delta}},
\end{aligned}
$$

which is smaller than $\eta_{\delta}\left(x^{*}\right)$ by the induction hypothesis. Similarly, under case (ii),

$$
\begin{aligned}
& g\left(y_{1}, \ldots, y_{N}\right) \\
& \rightarrow \frac{\left|\sum_{i=2}^{k} s_{i} f\left(y_{i}\right)+s_{i^{*}} f\left(x^{*}\right)-\sum_{i=k+1}^{N} s_{i} f\left(y_{i}\right)\right|^{2+\delta}}{\left|1+\sum_{i=2}^{k} s_{i} F\left(y_{i}\right)+s_{i^{*}} F\left(x^{*}\right)-\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right|^{1+\delta}} \\
& \times \frac{1}{\left|-\left(\sum_{i=2}^{k} s_{i} F\left(y_{i}\right)+s_{i^{*}} F\left(x^{*}\right)-\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right)\right|^{1+\delta}} \\
& =\frac{\left|-\sum_{i=2}^{k} s_{i} f\left(y_{i}\right)-s_{i^{*}} f\left(x^{*}\right)+\sum_{i=k+1}^{N} s_{i} f\left(y_{i}\right)\right|^{2+\delta}}{\left|1-\left(-\sum_{i=2}^{k} s_{i} F\left(y_{i}\right)-s_{i^{*}} F\left(x^{*}\right)+\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right)\right|^{1+\delta}} \\
& \times \frac{1}{\left|-\sum_{i=2}^{k} s_{i} F\left(y_{i}\right)-s_{i^{*}} F\left(x^{*}\right)+\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right|^{1+\delta}},
\end{aligned}
$$

which is smaller than $\eta_{\delta}\left(x^{*}\right)$ by the induction hypothesis. Under case (iii), the terms in $\delta_{N}$ and $\Delta_{N}$ corresponding to $y_{k}$ and $x^{*}$ cancel each other. As a result, $g\left(y_{1}, \ldots, y_{N}\right)$ reduces to an expression with $N-2$ variables hence this case is handled by the induction hypothesis. Finally, under case (iv), set

$$
\begin{gathered}
d \triangleq s_{k} F\left(-x^{*}\right)+s_{i^{*}} F\left(x^{*}\right)=s_{i^{*}}\left(1-2 F\left(-x^{*}\right)\right), \\
\sigma \triangleq \sum_{i=1}^{k-1} s_{i} f\left(y_{i}\right)-\sum_{i=k+1}^{N} s_{i} f\left(y_{i}\right) .
\end{gathered}
$$

and

$$
\Sigma \triangleq \sum_{i=1}^{k-1} s_{i} F\left(y_{i}\right)-\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)
$$

We have

$$
\begin{aligned}
& g\left(y_{1}, \ldots, y_{N}\right)= \\
& =\frac{\left|\sum_{i=1}^{k-1} s_{i} f\left(y_{i}\right)-\sum_{i=k+1}^{N} s_{i} f\left(y_{i}\right)\right|^{2+\delta}}{\left|\sum_{i=1}^{k-1} s_{i} F\left(y_{i}\right)+d\left(x^{*}\right)-\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right|^{1+\delta}} \\
& \frac{1}{\left|1-\sum_{i=1}^{k-1} s_{i} F\left(y_{i}\right)-d\left(x^{*}\right)+\sum_{i=k+1}^{N} s_{i} F\left(y_{i}\right)\right|^{1+\delta}} \\
& =\frac{|\sigma|^{2+\delta}}{|\Sigma+d|^{1+\delta}|1-\Sigma-d|^{1+\delta}} \\
& =\frac{|\sigma|^{2+\delta}}{|\Sigma|^{1+\delta}|1-\Sigma|^{1+\delta}}\left|\frac{\Sigma(1-\Sigma)}{\Sigma(1-\Sigma)+d(1-2 \Sigma)-d^{2}}\right|^{1+\delta}
\end{aligned}
$$

By the induction hypothesis,

$$
\frac{|\sigma|^{2+\delta}}{|\Sigma|^{1+\delta}|1-\Sigma|^{1+\delta}} \leq \eta_{\delta}\left(x^{*}\right)
$$

hence it is left to show that

$$
\frac{\Sigma(1-\Sigma)}{\Sigma(1-\Sigma)+d(1-2 \Sigma)-d^{2}} \leq 1
$$

Whenever $d>0$,

$$
\frac{\Sigma(1-\Sigma)+d(1-2 \Sigma)-d^{2}}{\Sigma(1-\Sigma)} \geq 1 \Leftrightarrow 1-2 \Sigma \geq d
$$

while for $d<0$,

$$
\frac{\Sigma(1-\Sigma)+d(1-2 \Sigma)-d^{2}}{\Sigma(1-\Sigma)} \geq 1 \Leftrightarrow 1-2 \Sigma \leq d
$$

Therefore, it is enough to show that $\Sigma \leq F\left(-x^{*}\right)$ if $s_{i^{*}}=1$ and $\Sigma \geq F\left(-x^{*}\right)$ if $s_{i^{*}}=-1$. Indeed, if $s_{i^{*}}=1$, then $s_{k+1}=-1$ and monotonicity of $F(x)$ implies that

$$
\Sigma+d \leq F\left(y_{1}\right)-F\left(y_{k}\right)+F\left(x^{*}\right)-F\left(-x^{*}\right)+F\left(y_{k+2}\right)-F\left(y_{N}\right)
$$

and hence

$$
\Sigma \leq 1-F\left(x^{*}\right)=F\left(-x^{*}\right)
$$

Similarly, if $s_{i^{*}}=-1$ then

$$
1-\Sigma \leq 1-F\left(-x^{*}\right)
$$

This conclude the proof in the case where $c^{\prime}(x)$ is an injection.
In the case where $c(x)$ is not strictly concave, so that $c^{\prime}$ does not strictly decrease, we approximate $c$ using another concave symmetric function with decreasing derivative. We assume w.l.o.g. that $c(0)=0$ maximizes $c$. For $\alpha>0$ consider the function $f_{\alpha}(x)=\kappa(\alpha) e^{-|c(x)|^{1+\alpha}}$, where $\kappa(\alpha)$ normalizes $f_{\alpha}$. Then $c_{\alpha}(x)$ is concave, symmetric, and a.e. differentiable with

$$
c_{\alpha}^{\prime}(x) \triangleq \frac{f_{\alpha}^{\prime}(x)}{f_{\alpha}(x)}=(1+\alpha)|c(x)|^{\alpha} c^{\prime}(x)
$$

Now $c_{\alpha}^{\prime}(x)$ is non-increasing since it is the derivative of a concave function. Furthermore, since $c(x)$ is non-constant on any interval and $c^{\prime}(x)$ is non-increasing, $c_{\alpha}^{\prime}(x)$ is non-constant on any interval hence an injection. It follows from the first part of the proof that, for any $\alpha>0$,

$$
\begin{equation*}
\frac{\left(\delta_{n, \alpha}\right)^{2+\delta}}{\left(\Delta_{n, \alpha}\left(1-\Delta_{n, \alpha}\right)\right)^{1+\delta}} \leq \max _{i} \eta_{\delta, \alpha}\left(x_{i}\right) \tag{33}
\end{equation*}
$$

where

$$
\begin{aligned}
& \delta_{n, \alpha} \triangleq \sum_{k=1}^{n}(-1)^{k+1} f_{\alpha}\left(x_{k}\right), \\
& \Delta_{n, \alpha} \triangleq \sum_{k=1}^{n}(-1)^{k+1} F_{\alpha}\left(x_{k}\right),
\end{aligned}
$$

and

$$
\eta_{\delta, \alpha}(x) \triangleq \frac{\left(f_{\alpha}(x)\right)^{2+\delta}}{\left(F_{\alpha}(x)(1-F(x))\right)^{1+\delta}} .
$$

The proof is completed by noting that

$$
\lim _{\alpha \rightarrow 0} \frac{\left(\delta_{n, \alpha}\right)^{2+\delta}}{\left(\Delta_{n, \alpha}\left(1-\Delta_{n, \alpha}\right)\right)^{1+\delta}}=\frac{\left(\delta_{n}\right)^{2+\delta}}{\left(\Delta_{n}\left(1-\Delta_{n}\right)\right)^{1+\delta}}
$$

and, since the maximum is over a finite set,

$$
\lim _{\alpha \rightarrow 0} \max _{i} \eta_{\delta, \alpha}\left(x_{i}\right)=\max _{i} \eta_{\delta}\left(x_{i}\right)
$$

## IX. Proof of Theorem 3

The estimation algorithm (8) is a special case of the stochastic gradient procedures in the papers [49], [56]. We rely on several of their results. Throughout this proof, we assume without loss of generality that the median $\theta=\operatorname{med}(P)=0$.

## A. Proof of Theorem 3(i)

Consider the following simplified version of [49, Thm. 4]:
Corollary 12: [49, Thms. $3 \& 4]$ Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ and $\left\{Z_{i}\right\}$ be i.i.d. zero-mean random variables, and

$$
X_{i}=\theta+Z_{i} .
$$

Define

$$
\begin{align*}
\theta_{i} & =\theta_{i-1}+\gamma_{i} \varphi\left(X_{i}-\theta_{i-1}\right), \\
\bar{\theta}_{n} & =\frac{1}{n} \sum_{i=0}^{n-1} \theta_{i} \tag{34}
\end{align*}
$$

where in addition,
(i) There exists $K_{1}$ such that $|\varphi(x)| \leq K_{1}(1+|x|)$ for all $x \in \mathbb{R}$.
(ii) The sequence $\left\{\gamma_{i}\right\}_{i=1}^{\infty}$ satisfies condition (9a).
(iii) The function $\psi(x) \triangleq \mathbb{E}\left[\varphi\left(x+Z_{1}\right)\right]$ satisfies $\psi(0)=0$ and $x \psi(x)>0$ for $x \neq 0$. Moreover, $\psi$ is differentiable at 0 with $\psi^{\prime}(0)>0$ and there exists $K_{2}, 0<\lambda \leq 1$, and $r>0$, such that

$$
\begin{equation*}
\left|\psi(x)-\psi^{\prime}(0) x\right| \leq K_{2}|x|^{1+\lambda} \tag{35}
\end{equation*}
$$

for all $|x|<r$.
(iv) The function $\chi(x) \triangleq \mathbb{E}\left[\varphi^{2}\left(x+Z_{1}\right)\right]$ is continuous at zero.

Then $\bar{\theta}_{n} \xrightarrow{\text { a.s. }} \theta$ and $\sqrt{n}\left(\theta_{n}-\theta\right) \xrightarrow{d} \mathcal{N}(0, V)$ for $V=\frac{\chi(0)}{\psi^{\prime}(0)^{2}}$.
Using the notation in Corollary 12, we set $\varphi(x)=\operatorname{sgn}(x)$ and $Z_{i}=X_{i}-\theta$, where $\theta=\operatorname{med}(P)$. Without loss of generality and for notational convenience, we assume for the remainder of this derivation that $\theta=0$. As a consequence, we have $\operatorname{med}(Z)=0$, and $\chi(x)=\mathbb{E}\left[\operatorname{sgn}^{2}\left(x+Z_{1}\right)\right]=1$, so $\chi(0)=1$. In addition,

$$
\begin{aligned}
\psi(x) & =\mathbb{E}\left[\operatorname{sgn}\left(x+Z_{1}\right)\right]=P(Z \geq-x)-P(Z<-x) \\
& =1-2 P(Z \leq-x)
\end{aligned}
$$

Using that $P$ has a density $f$ near its median, it follows that $\psi^{\prime}(x)=2 f(-x)$ and thus $\psi^{\prime}(0)=2 f(0)>0$. We may now verify that the conditions in Corollary 12 hold for $\lambda=1$. Condition (i) is obvious, and the convexity of $|\cdot|$ gives most of condition (iii) excepting inequality (35). For that, note that as $f$ is Lipschitz near 0 with constant $\operatorname{Lip}_{0}(f)$, we have for small $x$ that

$$
\begin{aligned}
\psi(x) & =2 \int_{0}^{x} f(-t) d t \leq 2 \int_{0}^{x}\left[f(0)+\operatorname{Lip}_{0}(f) t\right] d t \\
& =2 f(0) x+\operatorname{Lip}_{0}(f) x^{2}=\psi^{\prime}(0) x+\operatorname{Lip}_{0}(f) x^{2} \\
\psi(x) & =2 \int_{0}^{x} f(-t) d t \geq 2 \int_{0}^{x}\left[f(0)-\operatorname{Lip}_{0}(f) t\right] d t \\
& =2 f(0) x-\operatorname{Lip}_{0}(f) x^{2}=\psi^{\prime}(0) x-\operatorname{Lip}_{0}(f) x^{2}
\end{aligned}
$$

so that condition (iii) holds. As evidently $\chi(0) / \psi^{\prime}(0)^{2}=\frac{1}{4 f(0)^{2}}$, Corollary 12 gives Theorem 3(i).

## B. Proof of Theorem 3(ii)

This proof requires somewhat more technicality than the first part of the theorem, including a brief detour into local asymptotic normality theory, regular estimators, and quadratic-mean differentiability [19, cf.]. We assume without loss of generality that the median of the density $f$ is 0 , so that if $P_{\theta}$ has density $f(\cdot-\theta)$, the median of $P_{\theta}$ is $\theta$. We begin by recalling the statistical concepts we require.

Definition 1: A sequence of estimators $T_{n}$ for a parameter $\theta$ in the parametric family $\left\{P_{\theta}\right\}_{\theta \in \Theta}$ is regular at $\theta$ if there exists a distribution $Q$ such that for any bounded sequence $h_{n}$,

$$
\sqrt{n}\left(T_{n}-\left(\theta+h_{n} / \sqrt{n}\right)\right) \underset{P_{\theta+h_{n} / \sqrt{n}}}{\stackrel{d}{\rightsquigarrow}} Q .
$$

Definition 2: Let $\left\{P_{\theta}\right\}_{\theta \in \Theta}$ have densities $p_{\theta}$ with respect to a base measure $\mu$. The family is quadratic mean differentiable (QMD) at $\theta$ with score $\dot{\ell}_{\theta}$ if

$$
\begin{equation*}
\int\left(\sqrt{p_{\theta+h}}-\sqrt{p_{\theta}}-\frac{1}{2} h^{\top} \dot{\ell}_{\theta} \sqrt{p_{\theta}}\right)^{2} d \mu=o\left(\|h\|^{2}\right) \tag{36}
\end{equation*}
$$

as $h \rightarrow 0$.
Definition 3: A family of distributions $\left\{P_{\theta}\right\}_{\theta \in \Theta}$ is locally asymptotically normal with information matrix $I_{\theta}$ (LAN) at $\theta$ if there exists a sequence of random vectors $\left\{Z_{n}\right\}$ such that for all $h_{n} \rightarrow h$,

$$
\sum_{i=1}^{n} \log \frac{d P_{\theta+h_{n} / \sqrt{n}}}{d P_{\theta}}\left(X_{i}\right)=h^{\top} Z_{n}-\frac{1}{2} h^{\top} I_{\theta} h+o_{P}(1)
$$

where $Z_{n} \stackrel{d}{\rightsquigarrow} \mathcal{N}\left(0, I_{\theta}\right)$ under $P_{\theta}$, where $X_{i} \stackrel{\text { iid }}{\sim} P_{\theta}$.
These three definitions are linked in our case by a few important results. First [19, Theorem 7.2], if $\left\{P_{\theta}\right\}$ is QMD (Def. 2) at the point $\theta$, then it is locally asymptotically normal with $Z_{n}=\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(X_{i}\right)$ and information matrix $I_{\theta}=\mathbb{E}_{\theta}\left[\dot{\ell}_{\theta} \dot{\ell}_{\theta}^{\top}\right]$. Moreover, in any family $\left\{P_{\theta}\right\}$ that is LAN (Def. 3) at $\theta$, if $T_{n}$ is a regular estimator (Def. 1) at $\theta$ with limiting distribution $Q$, then for any bounded, symmetric, quasi-convex loss $L$ and $c<\infty$,

$$
\begin{gather*}
\limsup _{n} \sup _{\|h\| \leq c} \mathbb{E}_{P_{\theta+h / \sqrt{n}}}\left[L\left(\sqrt{n}\left(T_{n}-\theta-h / \sqrt{n}\right)\right)\right] \\
=\mathbb{E}[L(W)] \text { for } W \sim Q \tag{37}
\end{gather*}
$$

(see Beran [57], Eq. (4.2)). Thus, we show two results: first, that the family $\left\{P_{\theta}\right\}$ of distributions defined by the shifted densities $\{f(\cdot-\theta)\}_{\theta \in \mathbb{R}}$ is quadratic-mean-differentiable at any $\theta$, and second, that $\bar{\theta}_{n}$ is regular and asymptotically normal. The combination evidently gives the theorem.

For quadratic mean differentiability, we have the following lemma, somewhat more general than we need; we defer proof to Sec. IX-E.

Lemma 13 (Extension of [19], Lemma 7.6): Let $p_{\theta}$ be a density with respect to $\mu$, and assume that $\theta \mapsto$ $s_{\theta}(x) \triangleq \sqrt{p_{\theta}(x)}$ is absolutely continuous for all $x$. Let $\dot{p}_{\theta}(x)=\nabla_{\theta} p_{\theta}(x)$ (when it exists), and assume that

$$
\mu\left(\left\{x: \dot{p}_{\theta}(x) \text { fails to exist }\right\}\right)=0 .
$$

Assume that $I_{\theta} \triangleq \mathbb{E}_{P_{\theta}}\left[\dot{p}_{\theta} \dot{p}_{\theta}^{\top} / p_{\theta}^{2}\right]$ is continuous at $\theta_{0}$. Then $P_{\theta}$ is QMD (Definition 2) at $\theta=\theta_{0}$ with $\dot{\ell}_{\theta}=\dot{p}_{\theta} / p_{\theta}$.
By the assumption in Theorem 3 that the density $f$ is Lipschitz continuous, $f$ is absolutely continuous hence $\sqrt{f}$ is absolutely continuous. We see that the location family $\left\{P_{\theta}\right\}_{\theta \in \mathbb{R}}$ defined by $d P_{\theta}(x)=f(x-\theta)$ satisfies the conditions of Lemma 13.

It remains to show that the average $\bar{\theta}_{n}$ is regular at $\theta$ with the limiting distribution $\mathcal{N}\left(0,\left(4 f(0)^{2}\right)^{-1}\right)$ :
Lemma 14: Let $h_{n} \rightarrow h \in \mathbb{R}$, and define $P_{n, h}=P_{\theta+h_{n} / \sqrt{n}}^{n}$. Then

$$
\begin{equation*}
\sqrt{n}\left(\bar{\theta}_{n}-\theta\right) \underset{P_{n, h}}{\stackrel{d}{\leadsto}} \mathcal{N}\left(h, \frac{1}{4 f(0)^{2}}\right) . \tag{38}
\end{equation*}
$$

Proof: To show the convergence (38) we use the following refinement of Corollary 12, which provides a generalized convergence result for iteratively defined $\theta_{n}$, and whose proof we defer to Section IX-D.

Corollary 15: Let the conditions of Corollary 12 hold, meaning that $\theta_{i}=\theta_{i-1}+\gamma_{i} \varphi\left(X_{i}-\theta_{i-1}\right)$ for $X_{i}=\theta+Z_{i}$, where $\left\{Z_{i}\right\}$ are i.i.d. with $\mathbb{E}\left[Z_{1}\right]=0$ and $\mathbb{E}\left[\varphi\left(Z_{1}\right)\right]=0$. Additionally assume the local smoothness condition that there exist $0<\lambda \leq 1$ and $K<\infty$ such that

$$
\begin{equation*}
\mathbb{E}\left[\left|\varphi\left(x+Z_{1}\right)-\varphi\left(Z_{1}\right)\right|^{2}\right] \leq K\left(|x|^{\lambda}+x^{2}\right) . \tag{39}
\end{equation*}
$$

Set $\Delta_{i} \triangleq \theta_{i}-\theta$ and $\bar{\Delta}_{n} \triangleq \frac{1}{n} \sum_{i=1}^{n} \Delta_{i}$. Then
(i) The sequence $\left\{\Delta_{i}\right\}$ is regular, that is,

$$
\begin{equation*}
\sqrt{n} \bar{\Delta}_{n}=-\frac{1}{\sqrt{n}} \frac{1}{\psi^{\prime}(0)} \sum_{i=1}^{n-1} \varphi\left(Z_{i}\right)+o_{P, n}(1) \tag{40}
\end{equation*}
$$

(ii) Let $\left\{Z_{i}\right\}$ as in Corollary 12 have absolutely continuous density $p$ with median 0 , define $\dot{\ell}_{h}(z)=\frac{p^{\prime}(z-h)}{p(z-h)}$, and assume that $I_{h} \triangleq \mathbb{E}_{p}\left[\dot{\ell}_{h}\left(Z_{1}\right)^{2}\right]$ is continuous in $h$ near 0 . Then for any converging sequence $h_{n} \rightarrow h$,

$$
\sqrt{n} \bar{\Delta}_{n} \underset{P_{\theta+h_{n} / \sqrt{n}}^{n}}{\stackrel{d}{\rightsquigarrow}} \mathcal{N}\left(\frac{-h}{\psi^{\prime}(0)} \mathbb{E}_{p}\left[\varphi\left(Z_{1}\right) \dot{\ell}_{0}\left(Z_{1}\right)\right], \frac{\chi(0)}{\psi^{\prime 2}(0)}\right) .
$$

We now verify that the setting of Theorem 3 (and Lemma 14) satisfies the conditions of Corollary 15. First, we have the obvious fact that

$$
|\operatorname{sgn}(z)-\operatorname{sgn}(x+z)| \leq 2 \cdot 1\{|x| \geq|z|\}
$$

Recalling that the density $f$ is Lipschitz with median 0 , for $\varphi(z)=\operatorname{sgn}(z)$, and $Z=X-\theta$ distributed with density $f$, we have

$$
\begin{aligned}
& \mathbb{E}[|\varphi(Z)-\varphi(x+Z)|] \leq 2 \mathbb{P}\left(\left|Z_{1}\right| \leq|x|\right) \\
& \quad=2 \int_{-|x|}^{|x|} f(t) d t \leq 4 f(0)|x|+2 \int_{-|x|}^{|x|} \operatorname{Lip}(f) t d t \\
& \quad=4 f(0)|x|+2 \operatorname{Lip}(f) x^{2}
\end{aligned}
$$

where $\operatorname{Lip}(f)$ is the Lipschitz constant of $f$. It follows that condition (39) holds. In addition, we have

$$
\begin{aligned}
& \mathbb{E}_{p}\left[\varphi\left(Z_{1}\right) \dot{\ell}_{0}\left(Z_{1}\right)\right]=\int_{\mathbb{R}} \varphi(x) f^{\prime}(x) d x=\int_{\mathbb{R}} \operatorname{sgn}(x) f^{\prime}(x) d x \\
& \quad=\int_{0}^{\infty} f^{\prime}(x) d x-\int_{-\infty}^{0} f^{\prime}(x) d x=-2 f(0)=-\psi^{\prime}(0)
\end{aligned}
$$

Corollary 15 now implies the convergence (38).
Combining Lemmas 13 and 14 with the limit (37) gives Theorem 3(ii).

## C. Proof of Theorem 3(iii)

We begin with the following result from [56]:
Corollary 16 ([56], Theorem 2): Define the iteration

$$
\begin{cases}U_{n}=U_{n-1}-\gamma_{n} \varphi\left(Y_{n}\right), & Y_{n}=g^{\prime}\left(U_{n-1}\right)+Z_{n}  \tag{41}\\ \bar{U}_{n}=\frac{1}{n} \sum_{i=1}^{n} U_{n}, & n=1,2, \ldots\end{cases}
$$

Assume that the function $g$ is $\mathcal{C}^{2}$, strictly convex, has Lipschitz derivative, and is minimized by $x^{\star}$. Moreover, assume that the noises $\left\{Z_{n}\right\}$ are i.i.d. with density $p$ and that the Fisher information $\mathbb{E}\left[\left(p^{\prime}\left(Z_{1}\right)\right)^{2} / p\left(Z_{1}\right)^{2}\right]$ exists and is finite. Let $\psi(x)$ and $\chi(x)$ be defined as in Corollary 12 and satisfy the conditions in the corollary. Assume in addition that $\chi(0)>0$, condition (35) with $\lambda=1$, and there exits $K_{3}$ such that

$$
\mathbb{E}\left[\left|\varphi\left(x+Z_{1}\right)\right|^{4}\right] \leq K_{3}\left(1+|x|^{4}\right)
$$

Finally, assume that the sequence $\left\{\gamma_{n}\right\}$ satisfies conditions (9a) and (9b). Then

$$
V_{n} \triangleq \mathbb{E}\left[\left(\bar{U}_{n}-x^{\star}\right)^{2}\right]=n^{-1} \frac{\chi(0)}{\left(\psi^{\prime}(0)\right)^{2}\left(g^{\prime \prime}\left(x^{\star}\right)\right)^{2}}+o\left(n^{-1}\right) .
$$

Fix $\theta \in \mathbb{R}$. Apply Corollary 16 with $g(x)=0.5(x-\theta)^{2}, \varphi(x)=\operatorname{sgn}(x), Z_{n}=\theta-X_{n}$. The update (41) gives

$$
U_{n}=U_{n-1}+\gamma_{n} \operatorname{sgn}\left(X_{n}-U_{n-1}\right),
$$

so the estimator $\bar{U}_{n}$ is identical to the stochastic gradient estimator (8) with $\bar{\theta}_{n}=\frac{1}{n} \sum_{i=1}^{n} \theta_{i}$. We have $\mathbb{E}\left[\varphi(x+Z)^{4}\right]=$ 1 and by assumption the Fisher information $\mathbb{E}\left[\left(f^{\prime}(Z)\right)^{2} / f(Z)^{2}\right]$ exists, and the functions $\psi$ and $\chi$ have the desired conditions of Corollary 12 (as we verify in Section IX-A). Finally, the function $\theta \mapsto \mathbb{E}_{P_{\theta}}\left[\left(\bar{\theta}_{n}-\theta\right)^{2}\right]$ is continuous in $\theta$, so that for $x^{\star}=\theta$ and $g^{\prime \prime} \triangleq 1$, we may apply Corollary 16 to obtain

$$
\mathbb{E}\left[\left(\bar{\theta}_{n}-\theta\right)^{2}\right]=\frac{1}{4 n f(0)^{2}}+o\left(n^{-1}\right)
$$

From here, existence of the second moment of $\pi$ implies (11).

## D. Proof of Corollary 15

Proof of Corollary 15(i): The proof of part (i) requires two additional lemmas of Polyak and Juditsky [49].
Lemma 17 ([49], Lemma 2): Define the process $\Delta_{i}^{1}=\Delta_{i-1}^{1}-\gamma_{i}\left(A \Delta_{i-1}^{1}+\xi_{i}\right)$ for $i=1,2, \ldots$. Assume that $A>0$ and the stepsizes $\gamma_{i}$ satisfy condition (9a). Then for $\bar{\Delta}_{n}^{1-1}=\frac{1}{n} \sum_{i=1}^{n} \Delta_{i}^{1}$, we have

$$
\begin{equation*}
\sqrt{n} \bar{\Delta}_{n}^{1}=\frac{\alpha_{n} \Delta_{0}^{1}}{\sqrt{n} \gamma_{0}}+\frac{1}{\sqrt{n} A} \sum_{i=1}^{n-1} \xi_{i}+\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} w_{i}^{n} \xi_{i} \tag{42}
\end{equation*}
$$

where $\alpha_{n}$ and $w_{i}^{n}$ are real numbers such that $\left|\alpha_{n}\right| \leq K$ and $\left|w_{i}^{n}\right| \leq K$ for some $K<\infty$, and $\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n-1}\left|w_{i}^{n}\right|=0$.

Lemma 18 ([49]): Under the conditions of Corollary 15, with probability 1,

$$
\sum_{i=1}^{\infty} \frac{\left|\Delta_{i}\right|^{1+\lambda}}{\sqrt{i}}<\infty
$$

Lemma 18 follows from the proof of Theorem 2 in [49, page 851].
We separate the proof of part (i) into two lemmas, which mirror the proofs of Polyak and Juditsky [49]; together they immediately give the result.

Lemma 19: The expansion (40) holds for the process $\bar{\Delta}_{n}^{1}$ defined by the iteration

$$
\begin{align*}
& \Delta_{i}^{1}=\Delta_{i-1}^{1}-\gamma_{i} \psi^{\prime}(0) \Delta_{i-1}^{1}-\gamma_{i} \varphi\left(Z_{i}\right), \quad \Delta_{0}^{1}=\Delta_{0} \\
& \bar{\Delta}_{n}^{1}=\frac{1}{n} \sum_{i=0}^{n-1} \Delta_{i}^{1} \tag{43}
\end{align*}
$$

Proof: To prove this claim, use Lemma 17 with $A=\psi^{\prime}(0)$ and $\xi_{i}=-\varphi\left(Z_{i}\right)$, which by condition (iii) in Corollary 12 gives that $\mathbb{E}\left[\xi_{i}\right]=0$ and that the $\xi_{i}$ are independent. The first term $\alpha_{n} \Delta_{0}^{1} / \gamma_{0} \sqrt{n} \rightarrow 0$ in Eq. (42). In addition, by independence and that the $\xi_{i}$ are mean-zero, we have

$$
\begin{aligned}
\mathbb{E} & {\left[\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} w_{i}^{n} \xi_{i}\right)^{2}\right] } \\
& =\frac{1}{n} \sum_{i=1}^{n}\left(w_{i}^{n}\right)^{2} \mathbb{E}\left[\xi_{i}^{2}\right]+\frac{1}{n} \sum_{i \neq j}^{n} w_{i}^{n} w_{j}^{n} \mathbb{E}\left[\xi_{i} \xi_{j}\right] \\
& =\frac{1}{n} \sum_{i=1}^{n}\left(w_{i}^{n}\right)^{2} \mathbb{E}\left[\varphi\left(Z_{i}\right)^{2}\right]=\chi(0) \frac{1}{n} \sum_{i=1}^{n}\left(w_{i}^{n}\right)^{2} \rightarrow 0
\end{aligned}
$$

by Lemma 17. Thus, the expansion (42) in Lemma 17 gives

$$
\sqrt{n} \bar{\Delta}_{n}^{1}=-\frac{1}{\sqrt{n}} \frac{1}{\psi^{\prime}(0)} \sum_{i=1}^{n-1} \varphi\left(Z_{i}\right)+o_{P, n}(1)
$$

as desired.
We then have the following asymptotic equivalence.
Lemma 20: The sequences $\bar{\Delta}_{n}$ and $\bar{\Delta}_{n}^{1}$ are asymptotically equivalent, meaning that $\sqrt{n}\left(\bar{\Delta}_{n}-\bar{\Delta}_{n}^{1}\right) \xrightarrow{p} 0$.
Proof: From the recursions (34) and (43), the difference $\delta_{i}=\Delta_{i}-\Delta_{i}^{1}$ satisfies

$$
\begin{aligned}
\delta_{i}= & \delta_{i-1}-\gamma_{i} \psi^{\prime}(0) \delta_{i-1} \\
& +\gamma_{i}\left(\psi^{\prime}(0) \Delta_{i-1}+\varphi\left(Z_{i}\right)-\varphi\left(\Delta_{i-1}+Z_{i}\right)\right),
\end{aligned}
$$

where $\delta_{0}=0$. Applying Lemma 17 with the choices $\xi_{i}=\psi^{\prime}(0) \Delta_{i-1}+\varphi\left(Z_{i}\right)-\varphi\left(\Delta_{i-1}+Z_{i}\right)$ yields

$$
\begin{align*}
& \sqrt{n} \bar{\delta}_{n}=\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1}\left(\frac{1}{\psi^{\prime}(0)}+w_{i}^{n}\right) \xi_{i} \\
& =\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1}\left(\frac{1}{\psi^{\prime}(0)}+w_{i}^{n}\right)\left(\psi^{\prime}(0) \Delta_{i-1}-\psi\left(\Delta_{i-1}\right)\right)  \tag{44}\\
& +\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1}\left(\frac{1}{\psi^{\prime}(0)}+w_{i}^{n}\right)  \tag{45}\\
& \quad \times\left(\psi\left(\Delta_{i-1}\right)+\varphi\left(Z_{i}\right)-\varphi\left(\Delta_{i-1}+Z_{i}\right)\right)
\end{align*}
$$

For the term (44), the assumption (35) that $\left|\psi(x)-\psi^{\prime}(0) x\right|=O\left(x^{1+\lambda}\right)$ and that $\sup _{i, n}\left|w_{i}^{n}\right|<\infty$ by Lemma 17 give that there exists $K<\infty$ such that $\left|\psi^{\prime}(0)^{-1}+w_{i}^{n}\right|\left|\psi^{\prime}(0) \Delta_{i-1}-\psi\left(\Delta_{i-1}\right)\right| \leq K\left|\Delta_{i}\right|^{1+\lambda}$. Lemma 18 gives that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}}\left|\Delta_{i}\right|^{1+\lambda}<\infty$, and so the Kronecker lemma gives that

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n-1}\left(\frac{1}{\psi^{\prime}(0)}+w_{i}^{n}\right)\left(\psi^{\prime}(0) \Delta_{i-1}-\psi\left(\Delta_{i-1}\right)\right) \xrightarrow{\text { a.s. }} 0 .
$$

The term (45) is somewhat more challenging to control. We define

$$
\epsilon_{i} \triangleq \psi\left(\Delta_{i-1}\right)+\varphi\left(Z_{i}\right)-\varphi\left(\Delta_{i-1}+Z_{i}\right),
$$

and let $\mathcal{F}_{i}=\sigma\left(Z_{1}, \ldots, Z_{i}\right)$ be the $\sigma$-field of the randomness through time $i$. We use a square integrable martingale convergence theorem [58, Exercise 5.3.35]. Noting that $\Delta_{i} \in \mathcal{F}_{i}$, we have

$$
\begin{align*}
& \mathbb{E}\left[\epsilon_{i}^{2} \mid \mathcal{F}_{i-1}\right] \\
& \quad=\mathbb{E}\left[\left(\psi\left(\Delta_{i-1}\right)+\varphi\left(Z_{i}\right)-\varphi\left(\Delta_{i-1}+Z_{i}\right)\right)^{2} \mid \mathcal{F}_{i-1}\right] \\
& \quad \leq 2 \psi\left(\Delta_{i-1}\right)^{2}+2 \mathbb{E}\left[\left(\varphi\left(\Delta_{i-1}+Z_{i}\right)-\varphi\left(Z_{i}\right)\right)^{2} \mid \mathcal{F}_{i-1}\right] \\
& \quad \leq K\left[\left|\Delta_{i-1}\right|^{1+\lambda}+\left|\Delta_{i-1}\right|^{\lambda}+\Delta_{i-1}^{2}\right] \tag{46}
\end{align*}
$$

where inequality (46) follows by the conditions (35) and (39), and $\mathbb{E}\left[\varepsilon_{i} \mid \mathcal{F}_{i-1}\right]=0$ for all $i$ by definition of $\psi(x)=\mathbb{E}[\varphi(x+Z)]$ and that $\psi(0)=0$. We now control the expectations of these quantities. For $R<\infty$, define the the stopping time $\tau_{R} \triangleq \inf \left\{i:\left|\Delta_{i}\right|>R\right\}$, which satisfies $\left\{\tau_{R} \leq i\right\} \in \mathcal{F}_{i}$ for each $i$. Then using [49, Eq. (A13-A14)], we have

$$
\mathbb{E}\left[\Delta_{i}^{2} 1\left\{\tau_{R}>i\right\}\right] \leq K \gamma_{i},
$$

and so inequality (46) gives that

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{i=1}^{\infty} \frac{1}{i}\left|\varepsilon_{i}\right|^{2} 1\left\{\tau_{R}>n\right\}\right] \leq K \sum_{i=1}^{\infty} \frac{\gamma_{i}^{\lambda}}{i}<\infty \\
& \text { so } \sum_{i=1}^{\infty} \frac{1}{i} \epsilon_{i}^{2} 1\left\{\tau_{R}>n\right\}<\infty \text { a.s. }
\end{aligned}
$$

by Condition (9a). As in the proof of Theorems 2 and 4 in [49], the Robbins-Siegmund Theorem [59] applied to the increment of $\left|\Delta_{t}\right|^{2}$ implies that for every $\epsilon>0$ there exists some $R^{\prime}>0$ such that

$$
\begin{equation*}
\mathbb{P}\left(\sup _{i}\left|\Delta_{i}\right| \leq R^{\prime}\right) \geq 1-\epsilon \tag{47}
\end{equation*}
$$

Consequently, there exists some $R^{\prime \prime}<\infty$ such that $\tau_{R^{\prime \prime}}=\infty$. We obtain that

$$
\sum_{i=1}^{\infty} \frac{1}{i} \varepsilon_{i}^{2}<\infty \quad \text { a.s.. }
$$

Applying the square integrable martingale convergence theorem of [58, Ex. 5.3.35], we have

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n}\left(\frac{1}{\psi^{\prime}(0)}+w_{i}^{n}\right) \epsilon_{i} \xrightarrow{\text { a.s. }} 0,
$$

so that both equations (44) and (45) converge almost surely to 0 .
Proof of Corollary 15(ii): This is essentially an immediate consequence of Le Cam's third lemma [19, Example 6.7]. Recall [19, Thm. 7.2] that if a family $\left\{P_{\theta}\right\}_{\theta \in \Theta}$ is quadratic mean differentiable at $\theta$ with score $\dot{\ell}_{\theta}$, then it is LAN at $\theta$ (Definition 3) with information matrix $I_{\theta}=\mathbb{E}\left[\dot{\theta}_{\theta} \dot{\dot{\theta}}_{\theta}^{\top}\right]$.

The regularity result (40) gives

$$
\sqrt{n} \bar{\Delta}_{n}=-\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\varphi\left(Z_{i}\right)}{\psi^{\prime}(0)}+o_{P, n}(1)
$$

The conditions in Corollary 15(ii) imply that the Fisher information $I_{h}=\mathbb{E}_{h}\left[\dot{\ell}_{h}\left(Z_{1}\right)^{2}\right]$ exists and is continuous for $\dot{\ell}_{h}(z)=\frac{p^{\prime}(z-h)}{p(z-h)}$, and the asymptotic expansion Definitions 2 and 3 combined with the preceding display, give the joint convergence

$$
\left(\sqrt{n} \bar{\Delta}_{n}, \sum_{i=1}^{n} \log \frac{P_{h_{n}} / \sqrt{n}}{P_{0}}\left(Z_{i}\right)\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}(\mu, \Sigma),
$$

where

$$
\begin{aligned}
\mu & =\left(0,-\frac{h^{2}}{2} I_{0}\right), \quad \text { and } \\
\Sigma & =\left(\begin{array}{cc}
\frac{\chi(0)}{\psi^{\prime}(0)^{2}} & \frac{-h}{\psi^{\prime}(0)} \mathbb{E}_{p}\left[\varphi\left(Z_{1}\right) \dot{\ell}_{0}\left(Z_{1}\right)\right] \\
\frac{-h}{\psi^{\prime}(0)} \mathbb{E}_{p}\left[\varphi\left(Z_{1}\right) \dot{\ell}_{0}\left(Z_{1}\right)\right] & h^{2} I_{0}
\end{array}\right) .
\end{aligned}
$$

Le Cam's third lemma [19, Exm. 6.7] then implies the convergence

$$
\sqrt{n} \bar{\Delta}_{n} \underset{P_{h_{n} / \sqrt{n}}^{n}}{\stackrel{d}{n}} \mathcal{N}\left(\frac{-h}{\psi^{\prime}(0)} \mathbb{E}_{p}[\varphi(Z) \dot{\ell}(Z)], \frac{\chi(0)}{\psi^{\prime}(0)^{2}}\right)
$$

under the alternatives $P_{h_{n} / \sqrt{n}}^{n}$, which gives Corollary 15(ii).

## E. Proof of Lemma 13

The proof is essentially completely parallel to that of [19, Lemma 7.6]. Define $\dot{s}_{\theta}=\frac{1}{2} \dot{p}_{\theta} \sqrt{p_{\theta}}$, which exists $\mu$-almost surely, so that $\int \dot{s}_{\theta} \dot{s}_{\theta}^{\top} d \mu$ is well-defined (though it may be infinite). By Lebesgue's integration theorem, we have

$$
s_{\theta+h}(x)-s_{\theta}(x)=\int_{0}^{1} h^{\top} \dot{s}_{\theta+t h}(x) d t,
$$

and so By Jensen's inequality (or Cauchy-Schwartz) we have

$$
\left(s_{\theta+h}(x)-s_{\theta}(x)\right)^{2} \leq \int_{0}^{1} h^{\top} \dot{s}_{\theta+t h}(x) \dot{s}_{\theta+t h}(x)^{\top} h d t .
$$

Thus, for any $h_{t}$ we have

$$
\begin{aligned}
\int & \left(\frac{s_{\theta+t h_{t}}(x)-s_{\theta}(x)}{t}\right)^{2} d \mu(x) \\
& \leq \iint_{0}^{1}\left(h_{t}^{\top} \dot{s}_{\theta+u t h_{t}}\right)^{2} d u d \mu \\
& =\int_{0}^{1} h_{t}^{\top} \int \dot{s}_{\theta+u t h_{t}} \dot{s}_{\theta+u t h_{t}}^{\top} d \mu(x) h_{t} d u \\
& =\frac{1}{4} h_{t}^{\top}\left(\int_{0}^{1} I_{\theta+u t h_{t}} d u\right) h_{t} .
\end{aligned}
$$

By continuity, as $h_{t} \rightarrow h$ and $t \rightarrow 0$ the assumed continuity of $\theta \mapsto I_{\theta}$ gives that the final display converges to $h^{\top} I_{\theta} h$.

Now, we note that

$$
\lim _{t \downarrow 0}\left(\frac{s_{\theta+t h_{t}}(x)-s_{\theta}(x)}{t}-h^{\top} \dot{s}_{\theta}(x)\right)^{2}=0
$$

for all $x$ excepting a $\mu$-null set, and the variant of the dominated convergence theorem in [19, Prop. 2.29] implies that

$$
\begin{aligned}
& \lim _{t \rightarrow 0} \frac{1}{t^{2}} \int\left(s_{\theta+t h_{t}}(x)-s_{\theta}(x)-t h^{\top} \dot{s}_{\theta}(x)\right)^{2} d \mu(x) \\
& \quad=\lim _{t \rightarrow 0} \int\left(\frac{s_{\theta+t h_{t}}(x)-s_{\theta}(x)}{t}-h^{\top} \dot{s}_{\theta}(x)\right)^{2} d \mu(x)=0,
\end{aligned}
$$

completing the proof.

## X. Proof of Theorem 6

We follow a similar outline to the optimality results we establish in the proof of Theorem 3(ii) in Sec. IX-B. Roughly, we establish that the family $P_{\theta}$ of distributions on the bits $B_{i}$ is locally asymptotically normal (Definition 3) via a quadratic mean differentiability argument. After this, the result follows by standard local asymptotic minimax theory.

We begin with an argument on the smoothness properties of the densities, which is important for our Taylor expansions to come.

Lemma 21: Let Assumption A2(i) hold. Then for any $A=\cup_{i=1}^{k}\left[a_{i}, b_{i}\right]$ and $h \in \mathbb{R}$,

$$
\begin{equation*}
\left|P_{\theta+h}(A)-P_{\theta}(A)-\dot{P}_{\theta}(A) h\right| \leq k \cdot \operatorname{Lip}(f) h^{2}, \tag{48}
\end{equation*}
$$

where

$$
\dot{P}_{\theta}(A)=\sum_{i=1}^{k} f\left(a_{i}-\theta\right)-f\left(b_{i}-\theta\right)
$$

Additionally, we have the bounds

$$
\begin{align*}
& |f(b)-f(a)| \leq 2 \sqrt{\operatorname{Lip}(f) P([a, b])}  \tag{49}\\
& \quad \text { and }\left|\dot{P}_{\theta}(A)\right| \leq 2 \sqrt{k \operatorname{Lip}(f)}
\end{align*}
$$

See Section X-A for a proof.
The second lemma provides the local asymptotic normality we require.
Lemma 22: Let Assumption A2(i) and (ii) hold, and let $B_{i}=1\left\{X_{i} \in A_{i}\right\}$. Let $h_{n} \rightarrow h \in \mathbb{R}$. Then for any $\theta \in \operatorname{int} \Theta$,

$$
\begin{aligned}
\sum_{i=1}^{n} \log & \frac{P_{\theta+h_{n} / \sqrt{n}}\left(B_{i}\right)}{P_{\theta}\left(B_{i}\right)} \\
= & \frac{h}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)-\frac{h^{2}}{4 n} \sum_{i=1}^{n} \operatorname{Var}\left(\dot{\ell}_{\theta}\left(B_{i}\right)\right) \\
& \quad-\frac{h^{2}}{4 n} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)^{2}+o_{P}(1)
\end{aligned}
$$

If additionally Assumption A2(iii) holds, then

$$
\sum_{i=1}^{n} \log \frac{P_{\theta+h_{n} / \sqrt{n}}\left(B_{i}\right)}{P_{\theta}\left(B_{i}\right)}=\frac{h}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)-\frac{h^{2}}{2} \kappa(\theta)+o_{P}(1)
$$

The proof of Lemma 22 is quite technical, so we defer it to Section X-B.
With this lemma, it is not too challenging to demonstrate the local asymptotic normality (Definition 3) of the family $\left\{P_{\theta}\right\}$. Indeed, Lemma 21 guarantees that $\left|\dot{P}_{\theta}\left(A_{n}\right)\right| \leq 2 \sqrt{k_{n} \operatorname{Lip}(f)}$ for all $n$, so that $\mathbb{E}_{\theta}\left[\left|\dot{\theta}_{\theta}\left(B_{i}\right)\right|^{3}\right] \leq$ $C \frac{k_{i}^{3 / 2} \operatorname{Lip}(f)^{3 / 2}}{P_{\theta}\left(A_{i}\right)^{2}\left(1-P_{\theta}\left(A_{i}\right)\right)^{2}}$, while Assumption A2(ii) guarantees that $\frac{1}{n^{3}} \sum_{i=1}^{n} \mathbb{E}_{\theta}\left[\left|\dot{\varphi}_{\theta}\left(B_{i}\right)\right|^{3}\right] \rightarrow 0$. Because $\mathbb{E}\left[\dot{\varphi}_{\theta}\left(B_{i}\right)\right]=0$, the Lyapunov central limit theorem applies to give

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \dot{e}_{\theta}\left(B_{i}\right) \stackrel{d}{\rightsquigarrow} \mathcal{N}(0, \kappa(\theta))
$$

under Assumption A2(iii), so that the family $\left\{P_{\theta}\right\}$ is locally asymptotically normal (Def. 3).
We now recall the familiar Hájek-Le-Cam local asymptotic minimax result [19, Thm. 8.11]: if the family $\left\{P_{\theta}\right\}$ is LAN with precision $\kappa(\theta)$, then

$$
\begin{gathered}
\liminf _{c \rightarrow \infty} \liminf _{n} \sup _{\|\tau-\theta\| \leq c / \sqrt{n}} \mathbb{E}_{\tau}\left[L\left(\sqrt{n}\left(\theta_{n}-\tau\right)\right)\right] \\
\geq \mathbb{E}[L(Z / \sqrt{\kappa(\theta)})]
\end{gathered}
$$

for any symmetric quasi-convex loss $L$, where $Z \sim \mathcal{N}(0,1)$. This immediately gives Theorem 6 .

## A. Proof of Lemma 21

To see the first claim of the lemma, we consider the simpler special case that $A=[a, b]$. Then as $f$ is Lipschitz (and hence absolutely continuous and a.e. differentiable with $\left\|f^{\prime}\right\|_{\infty} \leq \operatorname{Lip}(f)$ ), we have

$$
\begin{aligned}
P_{\theta+h} & (A)-P_{\theta}(A)=\int_{a}^{b}(f(z-\theta-h)-f(z-\theta)) d z \\
& =-\int_{a}^{b} \int_{0}^{h} f^{\prime}(z-\theta-u) d u d z \\
& =-\int_{0}^{h} \int_{a}^{b} f^{\prime}(z-\theta-u) d z d u \\
& =\int_{0}^{h} f(a-\theta-u)-f(b-\theta-u) d u \\
& \lesseqgtr \int_{0}^{h}(f(a-\theta)-f(b-\theta)) d u \pm 2 \int_{0}^{h} \operatorname{Lip}(f) u d u \\
& =[f(a-\theta)-f(b-\theta)] h \pm \operatorname{Lip}(f) h^{2}
\end{aligned}
$$

This gives the first two claims of the lemma.
For the second, we require a bit more work. Let $L=\operatorname{Lip}(f)$ for shorthand. Let $a<b$. Then we always have

$$
\begin{align*}
& P([a, b]) \geq \int_{a}^{b} f(z) d z  \tag{50}\\
& \quad \geq \int_{a}^{b} \max \{f(b)-L(b-z), f(a)-L(z-a), 0\} d z
\end{align*}
$$

If $f(a)+f(b) \geq L(b-a)$, then the point $\hat{z}=\frac{a+b}{2}-\frac{f(b)-f(a)}{2 L}$ satisfies both $f(b)-L(b-\hat{z}) \geq 0$ and $f(a)-L(\hat{z}-a) \geq$ 0 . The integral (50) then becomes

$$
\begin{aligned}
& \int_{a}^{\hat{z}}(f(a)-L(z-a) d z)+\int_{\hat{z}}^{b}(f(b)-L(b-z)) d z \\
& =\frac{f(a)+f(b)}{2}\left(\frac{b-a}{2}\right)-L\left(\frac{b-a}{2}\right)^{2}+\frac{(f(b)-f(a))^{2}}{4 L}
\end{aligned}
$$

and using the assumption that $\frac{f(a)+f(b)}{2} \geq L(b-a)$, we obtain

$$
\begin{aligned}
\frac{(f(b)-f(a))^{2}}{4 L} \leq & \frac{f(b)+f(b)}{2} \frac{b-a}{2} \\
& -L\left(\frac{b-a}{2}\right)^{2}+\frac{(f(b)-f(a))^{2}}{4 L} \\
\leq & P([a, b])
\end{aligned}
$$

That is, $|f(b)-f(a)| \leq 2 \sqrt{\operatorname{Lip}(f) P([a, b])}$. In the converse case that $f(a)+f(b) \leq L(b-a)$, then the integral (50) becomes

$$
\begin{aligned}
P([a, b]) \geq & \int_{a}^{a+\frac{f(a)}{L}}(f(a)-L(z-a)) d z \\
& \quad+\int_{b-\frac{f(b)}{L}}^{b}(f(b)-L(b-z)) d z \\
= & \frac{f(a)^{2}}{L}-\frac{f(a)^{2}}{2 L}+\frac{f(b)^{2}}{L}-\frac{f(b)^{2}}{2 L}
\end{aligned}
$$

so that

$$
\frac{f(a)+f(b)}{\sqrt{2}} \leq \sqrt{f(a)^{2}+f(b)^{2}} \leq \sqrt{2 \operatorname{Lip}(f) P([a, b])}
$$

where the left inequality follows from concavity of $\sqrt{ }$. In sum, we have demonstrated that always the first bound (49) holds. To show the second inequality in expression (49), note that $\sum_{i} P\left(\left[a_{i}, b_{i}\right]\right) \leq 1$, and apply Cauchy-Schwarz.

## B. Proof of Lemma 22

Our proof follows that of [19, Thm. 7.2] closely. We first demonstrate a type of uniform quadratic mean differentiability (Definition 2) for sets $A$ that are finite unions of intervals. By a Taylor approximation and concavity of $\sqrt{ } \cdot$, we have

$$
\sqrt{a}+\frac{b}{2 \sqrt{a}}-\frac{b^{2}}{4 a^{3 / 2}} \leq \sqrt{a+b} \leq \sqrt{a}+\frac{b}{2 \sqrt{a}}
$$

for any $a>0$ and $|b| \leq 3 a / 4$. Consequently, recalling that $\dot{\ell}_{\theta}(A)=\dot{P}_{\theta}(A) / P_{\theta}(A)$, for any $h \in \mathbb{R}$ and $A=$ $\cup_{i=1}^{k}\left[t_{i}^{-}, t_{i}^{+}\right]$the union of $k$ intervals, the expansion (48) yields

$$
\begin{aligned}
& \left(\sqrt{P_{\theta+h}(A)}-\sqrt{P_{\theta}(A)}-\frac{1}{2} h \dot{\ell}_{\theta}(A) \sqrt{P_{\theta}(A)}\right)^{2} \\
& \quad \leq\left(\frac{k \operatorname{Lip}(f)}{2 \sqrt{P_{\theta}(A)}} h^{2}+\frac{\left(\left|\dot{P}_{\theta}(A) h\right|+h^{2} \operatorname{Lip}(f)\right)^{2}}{P_{\theta}(A)^{3 / 2}}\right)^{2}
\end{aligned}
$$

valid for $h$ such that $\left|\dot{P}_{\theta}(A) h\right| \leq P_{\theta}(A) / 4$ and $k h^{2} \operatorname{Lip}^{2}(f) \leq P_{\theta}(A) / 4$. Thus, under Assumption A2(ii), there exists a numerical constant $C<\infty$ such that

$$
\begin{align*}
& \left(\sqrt{P_{\theta+h}(A)}-\sqrt{P_{\theta}(A)}-\frac{1}{2} h \dot{\ell}_{\theta}(A) \sqrt{P_{\theta}(A)}\right)^{2} \\
& \quad \leq\left(\frac{h^{2} k \cdot \operatorname{Lip}(f)}{2 \sqrt{P_{\theta}(A)}}+\frac{\left(\left|\dot{P}_{\theta}(A) h\right|+k h^{2} \operatorname{Lip}(f)\right)^{2}}{P_{\theta}(A)^{3 / 2}}\right)^{2} \\
& \quad \leq \frac{C}{P_{\theta}(A)}\left[k^{2} \operatorname{Lip}^{2}(f)+\dot{\ell}_{\theta}(A)^{2}+\frac{k^{4} h^{4} \operatorname{Lip}^{4}(f)}{P_{\theta}(A)^{2}}\right] \cdot h^{4} \tag{51a}
\end{align*}
$$

valid whenever $\left|\dot{P}_{\theta}(A) h\right| \leq P_{\theta}(A) / 4$ and $k h^{2} \operatorname{Lip}^{2}(f) \leq P_{\theta}(A) / 4$, and similarly, we have

$$
\begin{align*}
& \left(\sqrt{P_{\theta+h}\left(A^{c}\right)}-\sqrt{P_{\theta}\left(A^{c}\right)}-\frac{1}{2} h \dot{\ell}_{\theta}\left(A^{c}\right) \sqrt{P_{\theta}\left(A^{c}\right)}\right)^{2} \\
& \leq \frac{C}{P_{\theta}\left(A^{c}\right)}\left[k^{2} \operatorname{Lip}^{2}(f)+\dot{\ell}_{\theta}\left(A^{c}\right)^{2}+\frac{k^{4} h^{4} \operatorname{Lip}^{4}(f)}{P_{\theta}\left(A^{c}\right)^{2}}\right] \cdot h^{4} . \tag{51b}
\end{align*}
$$

That is, the family $\left\{P_{\theta}\right\}$ with bit observations $B_{n}$ satisfies a uniform type of quadratic-mean differentiability (Def. 2).
For shorthand, define $P_{n}=P_{\theta+h_{n} / \sqrt{n}}$ and $P=P_{\theta}$, and let $p_{n}, p$ be shorthand for the p.m.f.s of the two distributions. For the sets $A_{i}$ we recall that $B_{i}=1\left\{X_{i} \in A_{i}\right\}$. The random variables

$$
W_{n, i} \triangleq 2\left(\sqrt{\frac{p_{n}}{p}}\left(B_{i}\right)-1\right)
$$

are with $P$-probability 1 well-defined, and by the inequalities (51), we have that

$$
\begin{align*}
\operatorname{Var} & \left(W_{n, i}-\frac{h_{n}}{\sqrt{n}} \dot{\ell}_{\theta}\left(B_{i}\right)\right)  \tag{52}\\
\leq & C \frac{k_{i}^{2} \operatorname{Lip}^{2}(f)+\dot{\ell}_{\theta}\left(A_{i}\right)^{2}+\dot{\ell}_{\theta}\left(A_{i}^{c}\right)^{2}}{P_{\theta}\left(A_{i}\right) P_{\theta}\left(A_{i}^{c}\right)} \cdot \frac{h_{n}^{4}}{n^{2}} \\
& +C \frac{k^{4} \operatorname{Lip}^{4}(f)}{P_{\theta}\left(A_{i}\right)^{3} P_{\theta}\left(A_{i}^{c}\right)^{3}} \frac{h_{n}^{8}}{n^{4}} \\
\leq & C \frac{k_{i}^{2} \operatorname{Lip}^{2}(f)+\dot{\ell}_{\theta}\left(A_{i}\right)^{2}+\dot{\ell}_{\theta}\left(A_{i}^{c}\right)^{2}}{P_{\theta}\left(A_{i}\right) P_{\theta}\left(A_{i}^{c}\right)} \cdot \frac{h_{n}^{4}}{n^{2}}  \tag{53}\\
& +C \frac{k^{4} \operatorname{Lip}^{4}(f)}{P_{\theta}\left(A_{i}\right)^{3} P_{\theta}\left(A_{i}^{c}\right)^{3}} \frac{h_{n}^{8}}{n^{4}} \tag{54}
\end{align*}
$$

whenever

$$
\begin{aligned}
& \frac{h}{\sqrt{n}} \max \left\{\dot{\ell}_{\theta}\left(A_{i}\right), \dot{\ell}_{\theta}\left(A_{i}^{c}\right)\right\} \leq \frac{1}{4} \\
& \text { and } \frac{k_{i} h_{n}^{2}}{n} \operatorname{Lip}^{2}(f) \leq \frac{\min \left\{P_{\theta}\left(A_{i}\right), P_{\theta}\left(A_{i}^{c}\right)\right\}}{4}
\end{aligned}
$$

Now, we use Assumption A2(ii), coupled with Lemma 21 to show that the summed variances converge to zero. Indeed, Lemma 21 and inequality (52) give that

$$
\begin{aligned}
& \operatorname{Var}\left(W_{n, i}-\frac{h_{n}}{\sqrt{n}} \dot{\varepsilon}_{\theta}\left(B_{i}\right)\right) \leq C \cdot\left[\frac{k_{i}^{2}}{P_{\theta}\left(A_{i}\right) P_{\theta}\left(A_{i}^{c}\right)} \frac{1}{n}\right. \\
& \left.\quad+\frac{k_{i}}{P_{\theta}\left(A_{i}\right) P_{\theta}\left(A_{i}^{c}\right)} \frac{1}{n}+\frac{k_{i}^{4}}{P_{\theta}\left(A_{i}\right)^{3} P_{\theta}\left(A_{i}^{c}\right)^{3}} \frac{1}{n^{3}}\right] \frac{1}{n}
\end{aligned}
$$

where $C<\infty$ depends only on $\operatorname{Lip}(f)$ and $h_{n}$ (both of which are uniformly bounded) whenever

$$
\frac{k_{i}}{P_{\theta}\left(A_{i}\right) P_{\theta}\left(A_{i}^{c}\right)} \frac{1}{n} \leq \frac{1}{C} .
$$

Assumption A2(ii) thus implies that $\mathbb{E}\left[\dot{\ell}_{\theta}\left(B_{i}\right)\right]=0$ and

$$
\begin{align*}
\operatorname{Var} & \left(\sum_{i=1}^{n} W_{n, i}-\frac{h_{n}}{\sqrt{n}} \dot{\dot{l}}_{\theta}\left(B_{i}\right)\right) \\
& =\sum_{i=1}^{n} \operatorname{Var}\left(W_{n, i}-\frac{h_{n}}{\sqrt{n}} \dot{\ell}_{\theta}\left(B_{i}\right)\right) \rightarrow 0 . \tag{55}
\end{align*}
$$

We now control the expectation of the $W_{n, i}$. Defining $\mu_{i}$ to be the induced counting measure on $B_{i}=1\left\{X_{i} \in A_{i}\right\}$,

$$
\begin{aligned}
& \sum_{i=1}^{n} \mathbb{E}\left[W_{n, i}\right]=2 \sum_{i=1}^{n}\left(\int \sqrt{p_{n}(b)} \sqrt{p(b)} d \mu_{i}(b)-1\right) \\
& =-\sum_{i=1}^{n} \int\left(\sqrt{p_{n}(b)}-\sqrt{p(b)}\right)^{2} d \mu_{i}(b) \\
& =-\frac{h_{n}^{2}}{4 n} \sum_{i=1}^{n} \mathbb{E}\left[\dot{\ell}_{\theta}\left(B_{i}\right)^{2}\right] \\
& -\sum_{i=1}^{n} \int\left(\sqrt{p_{n}(b)}-\sqrt{p(b)}-\frac{h_{n}}{\sqrt{n}} \dot{\ell}_{\theta}(b) \sqrt{p(b)}\right)^{2} d \mu_{i}(b) \\
& -\sum_{i=1}^{n} \int\left(\sqrt{p_{n}(b)}-\sqrt{p(b)}-\frac{h_{n}}{\sqrt{n}} \dot{\dot{Q}}_{\theta}(b) \sqrt{p(b)}\right) \frac{h_{n}}{\sqrt{n}} \dot{\theta}_{\theta}(b) \sqrt{p(b)} d \mu_{i}(b) \\
& =-\left(\frac{h^{2}}{4 n} \sum_{i=1}^{n} \mathbb{E}\left[\dot{\ell}_{\theta}\left(B_{i}\right)^{2}\right]\right)-o(1)
\end{aligned}
$$

uniformly in $h$, with a derivation completely paralleling that above. Therefore, we obtain

$$
\begin{aligned}
\sum_{i=1}^{n} W_{n, i} & =\sum_{i=1}^{n}\left(W_{n, i}-\frac{h_{n}}{\sqrt{n}} \dot{\ell}_{\theta}\left(B_{i}\right)\right)+\frac{h_{n}}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right) \\
& =-\frac{h^{2}}{4 n} \sum_{i=1}^{n} \mathbb{E}\left[\dot{\ell}_{\theta}\left(B_{i}\right)^{2}\right]+\frac{h}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)+o_{P}(1)
\end{aligned}
$$

where we have used that $h_{n} \rightarrow h$.

Now, we write the log-likelihood ratio. We have

$$
\begin{aligned}
& \sum_{i=1}^{n} \log \frac{p_{n}\left(B_{i}\right)}{p\left(B_{i}\right)}=2 \sum_{i=1}^{n} \log \left(1+\frac{1}{2} W_{n, i}\right) \\
& =\sum_{i=1}^{n} W_{n, i}-\frac{1}{4} \sum_{i=1}^{n} W_{n, i}^{2}+\frac{1}{2} \sum_{i=1}^{n} W_{n, i}^{2} R\left(W_{n, i}\right)
\end{aligned}
$$

where the remainder $\left|R\left(W_{n, i}\right)\right| \leq\left|W_{n, i}\right|$ for $\left|W_{n, i}\right| \leq 1$. Using the Taylor expansions of $\sqrt{ }$. and Lemma 21, we have

$$
\begin{align*}
& \left|\frac{1}{2} W_{n, i}\right| \leq \frac{1}{2}\left|\dot{e}_{\theta}\left(B_{i}\right)\right| \frac{h_{n}}{\sqrt{n}} \\
& \quad+\left|\frac{h_{n}^{2}}{n} \frac{k_{i} \operatorname{Lip}(f)}{p\left(B_{i}\right)}+\frac{h_{n}^{2}}{n} \dot{\ell}_{\theta}\left(B_{i}\right)^{2}+\frac{h_{n}^{4}}{n^{2}} \frac{k_{i}^{2} \operatorname{Lip}(f)^{2}}{p\left(B_{i}\right)^{2}}\right| \\
& =\frac{1}{2} \dot{\theta}_{\theta}\left(B_{i}\right) \frac{h_{n}}{\sqrt{n}} \\
& \quad+C\left|\frac{\sqrt{k_{i}}}{\sqrt{n} p\left(B_{i}\right)}+\frac{k_{i}}{n p\left(B_{i}\right)}+\frac{\sqrt{k_{i}}}{p\left(B_{i}\right)^{2} n}+\frac{k_{i}^{2}}{p\left(B_{i}\right)^{2} n^{2}}\right| \tag{56}
\end{align*}
$$

where $|C|<\infty$ depends only on $\operatorname{Lip}(f)$ and $h_{n}$ and so is uniformly bounded. From Assumption A2(ii) we get

$$
C\left|\frac{\sqrt{k_{i}}}{\sqrt{n} p\left(B_{i}\right)}+\frac{k_{i}}{n p\left(B_{i}\right)}+\frac{\sqrt{k_{i}}}{p\left(B_{i}\right)^{2} n}+\frac{k_{i}^{2}}{p\left(B_{i}\right)^{2} n^{2}}\right| \rightarrow 0
$$

Consequently $\max _{i} W_{n, i} \rightarrow 0$, so that

$$
\begin{gather*}
\sum_{i=1}^{n} \log \frac{p_{n}\left(B_{i}\right)}{p\left(B_{i}\right)}=\frac{h_{n}}{\sqrt{n}} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)-\frac{1}{4} \sum_{i=1}^{n} \mathbb{E}\left[\dot{\ell}_{\theta}\left(B_{i}\right)^{2}\right]  \tag{57}\\
-\frac{1}{4} \sum_{i=1}^{n} W_{n, i}^{2}+o_{P}(1) .
\end{gather*}
$$

It remains to compute $\mathbb{E}\left[W_{n, i}^{2}\right]$. Using the bounds that $\left|\dot{\mathscr{\theta}}_{\theta}\left(B_{i}\right)\right| \leq C \sqrt{k_{i}} / p\left(B_{i}\right)$ from Lemma 21, the expansion (56) yields

$$
\begin{aligned}
\mid \mathbb{E}[ & \left.W_{n, i}^{2}-\frac{h_{n}^{2}}{2 n} \dot{\ell}_{\theta}\left(B_{i}\right)^{2}\right] \mid \\
\leq \frac{C}{n} & {\left[\frac{k_{i}^{3 / 2}}{p\left(A_{i}\right)\left(1-p\left(A_{i}\right)\right) \sqrt{n}}+\frac{k_{i}^{3 / 2}}{p\left(A_{i}\right)^{2}\left(1-p\left(A_{i}\right)\right)^{2} \sqrt{n}}\right.} \\
& \left.+\frac{k_{i}^{2}}{p\left(A_{i}\right)\left(1-p\left(A_{i}\right)\right)} \frac{1}{n^{3 / 2}}\right] \\
& +\frac{C}{n}\left[\frac{k_{i}^{2}}{p\left(A_{i}\right)\left(1-p\left(A_{i}\right)\right)} \frac{1}{n}+\frac{k_{i}}{p\left(A_{i}\right)^{3}\left(1-p\left(A_{i}\right)\right)^{3}} \frac{1}{n}\right. \\
& \left.+\frac{k_{i}^{4}}{p\left(A_{i}\right)^{3}\left(1-p\left(A_{i}\right)\right)^{3}} \frac{1}{n^{3}}\right]
\end{aligned}
$$

where $C$ depends only on $h$ and $\operatorname{Lip}(f)$. Thus

$$
\sum_{i=1}^{n} W_{n, i}^{2}=\frac{h_{n}^{2}}{n} \sum_{i=1}^{n} \dot{\ell}_{\theta}\left(B_{i}\right)^{2}+o(1)
$$

giving Lemma 22.

## XI. Proof of Theorem 7

Let $\Xi$ be the set of points $\theta \in \Theta$ for which $\kappa(\theta)=\eta(0)$. Since $B_{1}, B_{2}, \ldots$ satisfy the conditions in Theorem 6 , $\theta$ is in $\Xi$ if and only if $\lim _{n \rightarrow \infty} L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right)=\eta(0)$. By assumption, we have $B_{i}=1\left\{X_{i} \in A_{i}\right\}, A_{i}=$ $\cup_{k=1}^{K}\left(t_{i, k}^{-}, t_{i, k}^{+}\right)$, where $t_{i, 1}^{-} \leq t_{i, 1}^{+} \leq \ldots \leq t_{i, K}^{-} \leq t_{i, K}^{+}$, and $t_{i, 1}^{-}$and $t_{i, K}^{+}$may take the values $-\infty$ and $\infty$, respectively. Denote the set of endpoints

$$
E_{i}=\bigcup_{k=1}^{K}\left\{t_{i, k}^{-}, t_{i, k}^{+}\right\},
$$

and for $\theta$ and $\epsilon>0$, define

$$
S_{n}(\theta, \epsilon) \triangleq\left\{i \leq n \text { s.t. }(\theta-\epsilon, \theta+\epsilon) \cap E_{i} \neq \emptyset\right\}
$$

In words, $S_{n}$ contains all integers smaller than $n$ in which an $\epsilon$-ball around $\theta$ contains an endpoint of one of the intervals defining $A_{i}$. We now claim that if $\theta \in \Xi$ then $\operatorname{card}\left(S_{n}(\theta, \epsilon)\right) / n \rightarrow 1$. Indeed, for such $\theta$ we have

$$
\begin{align*}
& L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right) \\
& \begin{array}{l}
=\frac{1}{n} \sum_{i \in S_{n}(\epsilon, \theta)} \frac{\left(\sum_{k=1}^{K} f\left(\theta-t_{i, k}^{+}\right)-f\left(\theta-t_{i, k}^{-}\right)\right)^{2}}{\sum_{k=1}^{K}\left(F\left(\theta-t_{i, k}^{+}\right)-F\left(\theta-t_{i, k}^{-}\right)\right)} \\
\quad \times \frac{1}{\left(1-\sum_{k=1}^{K}\left(F\left(\theta-t_{i, k}^{+}\right)-F\left(\theta-t_{i, k}^{-}\right)\right)\right)} \\
+\frac{1}{n} \sum_{i \notin S_{n}(\epsilon, \theta)} \frac{\left(\sum_{k=1}^{K} f\left(t_{i, k}^{+}-\theta\right)-f\left(t_{i, k}^{-}-\theta\right)\right)^{2}}{\sum_{k=1}^{K}\left(F\left(\theta-t_{i, k}^{+}\right)-F\left(\theta-t_{i, k}^{-}\right)\right)} \\
\quad \times \frac{1}{\left(1-\sum_{k=1}^{K}\left(F\left(\theta-t_{i, k}^{+}\right)-F\left(\theta-t_{i, k}^{-}\right)\right)\right)} \\
\leq \frac{\operatorname{card}\left(S_{n}(\theta, \epsilon)\right)}{n} \eta(0)+\frac{n-\operatorname{card}\left(S_{n}(\theta, \epsilon)\right)}{n} \eta(\epsilon)
\end{array}
\end{align*}
$$

where the last transition follows from Lemma 11 with $\delta=0$ and the fact that for $i \in S_{n}(\theta, \epsilon)$,

$$
\max \left\{\max _{k} \eta\left(t_{i, k}^{+}-\theta\right), \max _{k} \eta\left(t_{i, k}^{-}-\theta\right)\right\} \leq \eta(\epsilon)<\eta(0)
$$

Unless card $\left(S_{n}(\theta, \epsilon)\right) / n \rightarrow 1$, we get that (58), hence $L_{n}\left(A_{1}, \ldots, A_{n} ; \theta\right)$, are bounded from above by a constant that is smaller then $\eta(0)$ in contradiction to the fact that $\theta \in \Xi$.

Assume for the sake of contradiction that there exists $N \geq 2 K+1$ distinct elements $\theta_{1}, \ldots, \theta_{N} \in \Xi$. Since each $A_{i}$ consists of at most $K$ intervals, we have that

$$
\begin{equation*}
\operatorname{card}\left(\bigcup_{i=1}^{n} A_{i}\right) \leq 2 n K \tag{59}
\end{equation*}
$$

Fix $\epsilon>0$ such that

$$
\epsilon<\frac{1}{2} \min _{i \neq j}\left|\theta_{i}-\theta_{j}\right| .
$$

Since for each $\theta \in \Theta$ we have $\operatorname{card}\left(S_{n}(\theta, \epsilon)\right) \rightarrow 1$, there exists $n$ large enough such that

$$
\operatorname{card}\left(S_{n}\left(\theta_{i}, \epsilon\right)\right) \geq n\left(1-\frac{1}{2 N}\right)
$$

for all $i=1, \ldots, N$. However, $S_{n}\left(\theta_{1}, \epsilon\right), \ldots S_{n}\left(\theta_{N}, \epsilon\right)$ are disjoint, so the cardinality of their union is at least $n\left(1-\frac{1}{2 N}\right) N>2 n K+n / 2$, a contradiction to inequality (59).

## References

[1] A. Kipnis and J. C. Duchi, "Mean estimation from adaptive one-bit measurements," in 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Oct 2017, pp. 1000-1007.
[2] V. Lesser, C. Ortiz, and M. Tambe, Eds., Distributed Sensor Networks: A Multiagent Perspective. Kluwer Academic Publishers, 2003, vol. 9.
[3] D. Li, K. Wong, Y. Hu, and A. Sayeed, "Detection, classification and tracking of targets in distributed sensor networks," in IEEE Signal Processing Magazine, 2002, pp. 17-29.
[4] S. Fuller and L. Millett, The Future of Computing Performance: Game Over or Next Level? National Academies Press, 2011.
[5] J. Candy, "A use of limit cycle oscillations to obtain robust analog-to-digital converters," IEEE Transactions on Communications, vol. 22, no. 3, pp. 298-305, Mar 1974.
[6] P. W. Wong and R. M. Gray, "Sigma-delta modulation with i.i.d. Gaussian inputs," IEEE Transactions on Information Theory, vol. 36, no. 4, pp. 784-798, Jul 1990.
[7] J. C. Duchi, M. I. Jordan, and M. J. Wainwright, "Minimax optimal procedures for locally private estimation (with discussion),"Journal of the American Statistical Association, vol. 113, no. 521, pp. 182-215, 2018.
[8] R. G. Baraniuk, S. Foucart, D. Needell, Y. Plan, and M. Wootters, "Exponential decay of reconstruction error from binary measurements of sparse signals," IEEE Transactions on Information Theory, vol. 63, no. 6, pp. 3368-3385, 2017.
[9] L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, "Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors," IEEE Transactions on Information Theory, vol. 59, no. 4, pp. 2082-2102, 2013.
[10] Y. Plan and R. Vershynin, "One-bit compressed sensing by linear programming," Communications on Pure and Applied Mathematics, vol. 66, no. 8, pp. 1275-1297, 2013.
[11] Y. Li, C. Tao, G. Seco-Granados, A. Mezghani, A. L. Swindlehurst, and L. Liu, "Channel estimation and performance analysis of one-bit massive mimo systems," IEEE Trans. Signal Process, vol. 65, no. 15, pp. 4075-4089, 2017.
[12] J. Choi, J. Mo, and R. W. Heath, "Near maximum-likelihood detector and channel estimator for uplink multiuser massive mimo systems with one-bit adcs," IEEE Transactions on Communications, vol. 64, no. 5, pp. 2005-2018, 2016.
[13] T. Han and S. Amari, "Statistical inference under multiterminal data compression," IEEE Transactions on Information Theory, vol. 44, no. 6, pp. 2300-2324, Oct 1998.
[14] T. T. Cai and H. Wei, "Distributed gaussian mean estimation under communication constraints: Optimal rates and communicationefficient algorithms," arXiv preprint arXiv:2001.08877, 2020.
[15] R. Gray and D. Neuhoff, "Quantization," IEEE Transactions on Information Theory, vol. 44, no. 6, pp. 2325-2383, Oct 1998.
[16] A. B. Tsybakov, Introduction to Nonparametric Estimation. Springer, 2009.
[17] L. Le Cam, Asymptotic Methods in Statistical Decision Theory. Springer-Verlag, 1986.
[18] L. Le Cam and G. L. Yang, Asymptotics in Statistics: Some Basic Concepts. Springer, 2000.
[19] A. W. van der Vaart, Asymptotic Statistics, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998.
[20] M. A. Davenport, Y. Plan, E. van den Berg, and M. Wootters, "One-bit matrix completion," Information and Inference, p. to appear, 2015.
[21] Y. Plan and R. Vershynin, "Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach," IEEE Transactions on Information Theory, vol. 59, no. 1, pp. 482-494, 2013.
[22] W. Shi, T. W. Sun, and R. D. Wesel, "Quasi-convexity and optimal binary fusion for distributed detection with identical sensors in generalized Gaussian noise," IEEE Transactions on Information Theory, vol. 47, no. 1, pp. 446-450, Jan 2001.
[23] P. Venkitasubramaniam, L. Tong, and A. Swami, "Quantization for maximin are in distributed estimation," IEEE Transactions on Signal Processing, vol. 55, no. 7, pp. 3596-3605, July 2007.
[24] A. Vempaty, H. He, B. Chen, and P. K. Varshney, "On quantizer design for distributed bayesian estimation in sensor networks," IEEE Transactions on Signal Processing, vol. 62, no. 20, pp. 5359-5369, Oct 2014.
[25] H. Chen and P. K. Varshney, "Performance limit for distributed estimation systems with identical one-bit quantizers," IEEE Transactions on Signal Processing, vol. 58, no. 1, pp. 466-471, 2010.
[26] _-,"Performance limit for distributed estimation systems with identical one-bit quantizers," IEEE Transactions on Signal Processing, vol. 58, no. 1, pp. 466-471, Jan 2010.
[27] T. Berger, Z. Zhang, and H. Viswanathan, "The CEO problem [multiterminal source coding]," IEEE Transactions on Information Theory, vol. 42, no. 3, pp. 887-902, 1996.
[28] H. Viswanathan and T. Berger, "The quadratic Gaussian CEO problem," IEEE Transactions on Information Theory, vol. 43, no. 5, pp. 1549-1559, 1997.
[29] Y. Oohama, "The rate-distortion function for the quadratic Gaussian CEO problem," IEEE Transactions on Information Theory, vol. 44, no. 3, pp. 1057-1070, 1998.
[30] V. Prabhakaran, D. Tse, and K. Ramachandran, "Rate region of the quadratic Gaussian CEO problem," in Information Theory, 2004. ISIT 2004. Proceedings. International Symposium on. IEEE, 2004, p. 119.
[31] Y. Zhang, J. Duchi, M. I. Jordan, and M. J. Wainwright, "Information-theoretic lower bounds for distributed statistical estimation with communication constraints," in Advances in Neural Information Processing Systems, 2013, pp. 2328-2336.
[32] J. C. Duchi, M. I. Jordan, M. J. Wainwright, and Y. Zhang, "Optimality guarantees for distributed statistical estimation," arXiv preprint arXiv:1405.0782, 2014.
[33] A. Garg, T. Ma, and H. L. Nguyen, "On communication cost of distributed statistical estimation and dimensionality," in Advances in Neural Information Processing Systems 27, 2014.
[34] M. Braverman, A. Garg, T. Ma, H. L. Nguyen, and D. P. Woodruff, "Communication lower bounds for statistical estimation problems via a distributed data processing inequality," in Proceedings of the Forty-Eighth Annual ACM Symposium on the Theory of Computing, 2016. [Online]. Available: https://arxiv.org/abs/1506.07216
[35] Y. Han, A. Özgür, and T. Weissman, "Geometric lower bounds for distributed parameter estimation under communication constraints," CoRR, vol. abs/1802.08417, 2018. [Online]. Available: http://arxiv.org/abs/1802.08417
[36] Z. Zhang and T. Berger, "Estimation via compressed information," IEEE Transactions on Information Theory, vol. 34, no. 2, pp. 198-211, 1988.
[37] Y. Han, P. Mukherjee, A. Ozgur, and T. Weissman, "Distributed statistical estimation of high-dimensional and nonparametric distributions," in 2018 IEEE International Symposium on Information Theory (ISIT). IEEE, 2018, pp. 506-510.
[38] A. Xu and M. Raginsky, "Information-theoretic lower bounds on Bayes risk in decentralized estimation," IEEE Transactions on Information Theory, vol. 63, no. 3, pp. 1580-1600, 2017.
[39] L. Barnes, Y. Han, and A. Ozgur, "A geometric characterization of fisher information from quantized samples with applications to distributed statistical estimation," in $201856 s t$ Annual Allerton Conference on Communication, Control, and Computing (Allerton), Oct 2018.
[40] M. Longo, T. D. Lookabaugh, and R. M. Gray, "Quantization for decentralized hypothesis testing under communication constraints," IEEE Transactions on Information Theory, vol. 36, no. 2, pp. 241-255, Mar 1990.
[41] J. N. Tsitsiklis, "Decentralized detection by a large number of sensors," Mathematics of Control, Signals, and Systems (MCSS), vol. 1, no. 2, pp. 167-182, 1988.
[42] W. P. Tay and J. N. Tsitsiklis, "The value of feedback for decentralized detection in large sensor networks," in International Symposium on Wireless and Pervasive Computing, Feb 2011, pp. 1-6.
[43] I. A. Ibragimov, "On the composition of unimodal distributions," Theory of Probability \& Its Applications, vol. 1, no. 2, pp. 255-260, 1956.
[44] E. L. Lehmann and G. Casella, Theory of Point Estimation, Second Edition. Springer, 1998.
[45] M. Bagnoli and T. Bergstrom, "Log-concave probability and its applications," Economic theory, vol. 26, no. 2, pp. 445-469, 2005.
[46] M. R. Sampford, "Some inequalities on mill's ratio and related functions," The Annals of Mathematical Statistics, vol. 24, no. 1, pp. 130-132, 1953.
[47] J. Hammersley, "On estimating restricted parameters," Journal of the Royal Statistical Society. Series B (Methodological), vol. 12, no. 2, pp. 192-240, 1950.
[48] J. Chen, X. Zhang, T. Berger, and S. Wicker, "An upper bound on the sum-rate distortion function and its corresponding rate allocation schemes for the CEO problem," Selected Areas in Communications, IEEE Journal on, vol. 22, no. 6, pp. 977-987, Aug 2004.
[49] B. T. Polyak and A. B. Juditsky, "Acceleration of stochastic approximation by averaging," SIAM Journal on Control and Optimization, vol. 30, no. 4, pp. 838-855, 1992.
[50] O. Shamir, "Fundamental limits of online and distributed algorithms for statistical learning and estimation," in Advances in Neural Information Processing Systems, 2014, pp. 163-171.
[51] M. Braverman, A. Garg, T. Ma, H. L. Nguyen, and D. P. Woodruff, "Communication lower bounds for statistical estimation problems via a distributed data processing inequality," in Proceedings of the forty-eighth annual ACM symposium on Theory of Computing, 2016, pp. 1011-1020.
[52] Y. Han, A. Özgür, and T. Weissman, "Geometric lower bounds for distributed parameter estimation under communication constraints," in Conference On Learning Theory. PMLR, 2018, pp. 3163-3188.
[53] L. P. Barnes, Y. Han, and A. Ozgur, "Lower bounds for learning distributions under communication constraints via fisher information," Journal of Machine Learning Research, vol. 21, no. 236, pp. 1-30, 2020.
[54] D. P. Bertsekas, "Stochastic optimization problems with nondifferentiable cost functionals," Journal of Optimization Theory and Applications, vol. 12, no. 2, pp. 218-231, 1973.
[55] R. D. Gill and B. Y. Levit, "Applications of the van Trees inequality: a Bayesian Cramér-Rao bound," Bernoulli, pp. 59-79, 1995.
[56] B. T. Polyak, "New stochastic approximation type procedures," Automat. i Telemekh, vol. 7, no. 98-107, p. 2, 1990.
[57] R. Beran, "The role of Hájek's convolution theorem in statistical theory," Kybernetika, vol. 31, no. 3, pp. 221-237, 1995.
[58] A. Dembo, "Lecture notes on probability theory: Stanford statistics 310," 2016, accessed October 1, 2016. [Online]. Available: http://statweb.stanford.edu/~adembo/stat-310b/lnotes.pdf
[59] H. Robbins and D. Siegmund, "A convergence theorem for non negative almost supermartingales and some applications," in Optimizing methods in statistics. Elsevier, 1971, pp. 233-257.

Alon Kipnis is a Senior Lecturer (Assistant Professor) at the School of Computer Science at Reichman University. Previously, he was a postdoctoral research scholar at the Department of Statistics at Stanford University, advised by David Donoho. He completed his Ph.D. in electrical engineering from Stanford University in 2017. His research is in the areas of mathematical statistics and information theory.

John Duchi is an associate professor of Statistics and Electrical Engineering and (by courtesy) Computer Science at Stanford University. His work spans statistical learning, optimization, information theory, and computation, with a few driving goals. (1) To discover statistical learning procedures that optimally trade between real-world resources-computation, communication, privacy provided to study participants-while maintaining statistical efficiency. (2) To build efficient large-scale optimization methods that address the spectrum of optimization, machine learning, and data analysis problems we face, allowing us to move beyond bespoke solutions to methods that robustly work. (3) To develop tools to assess and guarantee the validity of-and confidence we should have in-machine-learned systems.

He has won several awards and fellowships. His paper awards include the SIAM SIGEST award for "an outstanding paper of general interest" and best papers at the Neural Information Processing Systems conference, the International Conference on Machine Learning, and an INFORMS Applied Probability Society Best Student Paper Award (as advisor). He has also received the Society for Industrial and Applied Mathematics (SIAM) Early Career Prize in Optimization, an Office of Naval Research (ONR) Young Investigator Award, an NSF CAREER award, a Sloan Fellowship in Mathematics, the Okawa Foundation Award, the Association for Computing Machinery (ACM) Doctoral Dissertation Award (honorable mention), and U.C. Berkeley's C.V. Ramamoorthy Distinguished Research Award.


[^0]:    Alon Kipnis is with the School of Computer Science at Reichman University, Herzliya, Israel (alon.kipnis@idc.ac.il).
    J. Duchi is with the Department of Statistics and the Department of Electrical Engineering at Stanford University, Stanford, CA, 94035 (jduchi@stanford.edu).

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