Vectorized Hankel Lift: A Convex Approach for Blind Super-Resolution of Point Sources

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Abstract

We consider the problem of resolving r point sources from n samples at the low end of the spectrum when point spread functions (PSFs) are not known. Assuming that the spectrum samples of the PSFs lie in low dimensional subspace (let s denote the dimension), this problem can be reformulated as a matrix recovery problem, followed by location estimation. By exploiting the low rank structure of the vectorized Hankel matrix associated with the target matrix, a convex approach called Vectorized Hankel Lift is proposed for the matrix recovery. It is shown that $n \gtrsim rs \log^4 n$ samples are sufficient for Vectorized Hankel Lift to achieve the exact recovery. For the location retrieval from the matrix, applying the single snapshot MUSIC method within the vectorized Hankel lift framework corresponds to the spatial smoothing technique proposed to improve the performance of the MMV MUSIC for the direction-of-arrival (DOA) estimation.

Keywords. blind super-resolution, vectorized Hankel lift, low rank, MUSIC

1 Introduction

1.1 Problem formulation

In this paper, we study the super-resolution of point sources when point spread functions (PSFs) are not known. More specifically, consider a point source signal x(t) of the form

$$x(t) = \sum_{k=1}^{r} d_k \delta(t - \tau_k),$$
(1.1)

where $\delta(\cdot)$ is the Dirac function, $\{\tau_k\}$ and $\{d_k\}$ are the locations and amplitudes of the point source signal, respectively. Let y(t) be its convolution with unknown point spread functions,

$$y(t) = \sum_{k=1}^{r} d_k \delta(t - \tau_k) * g_k(t) = \sum_{k=1}^{r} d_k \cdot g_k(t - \tau_k),$$
(1.2)

where $\{g_k\}_{k=1}^r$ are the point spread functions depending on the locations of the point sources.

Taking the Fourier transform on both sides of (1.2) yields

$$\widehat{y}(f) = \int_{-\infty}^{+\infty} y(t) e^{-2\pi i f t} dt = \sum_{k=1}^{r} d_k e^{-2\pi i f \tau_k} \widehat{g}_k(f).$$
(1.3)

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The goal in blind super-resolution is to recover $\{d_k, \tau_k\}_{k=1}^r$ from the low end of the spectrum

$$\mathbf{y}[j] = \sum_{k=1}^{r} d_k e^{-2\pi i \tau_k \cdot j} \mathbf{g}_k[j] \quad \text{for } j = 0, \cdots, n-1$$
(1.4)

when $g_k = [\hat{g}_k(0), \dots, \hat{g}_k(n-1)]^\mathsf{T}$, $k = 1, \dots, r$, are not known. Here we assume the index $j \in \{0, 1, \dots, n-1\}$ rather than $j \in \{-\lfloor n/2 \rfloor, \dots, \lfloor n/2 \rfloor\}$ only for convenience of notation. In addition to blind super-resolution, the observation model (1.4) also arises from many other important applications, such as 3D single-molecule microscopy [47], multi-user communication system [43] and nuclear magnetic resonance spectroscopy [46].

It is evident that the blind super-resolution problem is ill-posed without any further assumptions. To address this issue, we assume that the set of vectors $\{g_k\}_{k=1}^r$ corresponding to the unknown point spread functions belong to a common and known low-dimensional subspace represented by $B \in \mathbb{C}^{n \times s}$, i.e.,

$$\boldsymbol{g}_k = \boldsymbol{B}\boldsymbol{h}_k,\tag{1.5}$$

where $h_k \in \mathbb{C}^s$ is the unknown orientation of g_k in this subspace. As is pointed out in [60], the subspace assumption is reasonable in several application scenarios. Moreover, it has been extensively used in the literature, see for example [1, 18, 60, 33, 37].

For any $\tau \in [0, 1)$, define the vector $\boldsymbol{a}_{\tau} \in \mathbb{C}^n$ as

$$\boldsymbol{a}_{\tau} = \begin{bmatrix} 1 & e^{-2\pi i \tau \cdot 1} & \cdots & e^{-2\pi i \tau \cdot (n-1)} \end{bmatrix}^{\mathsf{T}}.$$
(1.6)

Let $b_j \in \mathbb{C}^s$ be the *j*th column vector of B^* . If we define the matrix $X^{\natural} \in \mathbb{C}^{s \times n}$ as

$$\boldsymbol{X}^{\natural} = \sum_{k=1}^{r} d_k \boldsymbol{h}_k \boldsymbol{a}_{\tau_k}^{\mathsf{T}}, \qquad (1.7)$$

then under the subspace assumption (1.5) and using the lifting trick [1, 12, 19, 18, 60, 38, 64, 42, 37], the observation model (1.4) can be reformulated as a linear measurement of X^{\natural} :

$$\boldsymbol{y}[j] = \left\langle \boldsymbol{b}_{j} \boldsymbol{e}_{j}^{\mathsf{T}}, \sum_{k=1}^{r} d_{k} \boldsymbol{h}_{k} \boldsymbol{a}_{\tau_{k}}^{\mathsf{T}} \right\rangle \text{ for } j = 0, \cdots, n-1,$$
(1.8)

where the inner product of two matrices is given by $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace}(\mathbf{A}^*\mathbf{B}), \mathbf{e}_j$ is (j+1)th column of the $n \times n$ identity matrix \mathbf{I}_n , and throughout this paper vectors and matrices are indexed starting with zero. Moreover, we can further rewrite (1.8) in the following compact form,

$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{X}^{\natural}), \tag{1.9}$$

where $\mathcal{A} : \mathbb{C}^{s \times n} \to \mathbb{C}^n$ is a linear operator defined by $[\mathcal{A}(\mathbf{X})]_j = \langle \mathbf{b}_j \mathbf{e}_j^\mathsf{T}, \mathbf{X} \rangle$. The adjoint of the operator $\mathcal{A}(\cdot)$, denoted $\mathcal{A}^*(\cdot)$, is defined as $\mathcal{A}^*(\mathbf{y}) = \sum_{j=0}^{n-1} \mathbf{y}[j] \mathbf{b}_j \mathbf{e}_j^\mathsf{T}$. Based on the above reformulation of blind super-resolution under the subspace assumption, it can be

Based on the above reformulation of blind super-resolution under the subspace assumption, it can be seen that the key is to recover X^{\natural} from the linear measurement vector y. Once X^{\natural} is reconstructed, the frequency components can be extracted from X^{\natural} by the subspace methods which will be detailed in Section 2.2. After the frequency components are obtained, $\{d_k, h_k\}$ can be recovered by solving a least squares system. Moreover, due to the multiplicative form of d_k and h_k in (1.7), we only expect to recover them separately up to a scaling ambiguity. Thus, we will assume that $||h_k||_2 = 1$ without loss of generality.

Note that the formulations in (1.4) and (1.9) are by no means new and they have been utilized in [60]. Moreover, when the point spread function g is shared among all point sources (i.e., the stationary case), (1.4) reduces to the blind sparse spikes deconvolution model considered in [18]. To recover the target matrix X^{\natural} from the linear measurements y, following the approach developed in [53] for spectrally sparse signal recovery, a similar atomic norm minimization method is proposed in [60],

$$\min_{\mathbf{X}} \|\mathbf{X}\|_{\mathcal{B}} \text{ subject to } y = \mathcal{A}(\mathbf{X}), \tag{1.10}$$

where the atomic norm $\|X\|_{\mathcal{B}}$ is defined as

$$\|\boldsymbol{X}\|_{\mathcal{B}} := \inf\{t > 0 : \boldsymbol{X} \in t \cdot \operatorname{conv}(\mathcal{B})\} = \inf_{d_k, \tau_k, \|\boldsymbol{h}_k\|_2 = 1} \left\{ \sum_{k=1}^r d_k : \boldsymbol{X} = \sum_{k=1}^r d_k \boldsymbol{h}_k \boldsymbol{a}_{\tau_k}^*, d_k > 0 \right\},$$

The successful recovery guarantee of (1.10) is studied in [60], while the robust analysis is provided separately in [33]. Note that for spectrally sparse signal recovery, in addition to atomic norm minimization, there are also methods which exploit the low rank property of the structured matrix formed from the signal [15, 6, 7]. This motivates us to develop a low rank approach for blind super-resolution.

1.2 Exploiting the low rank structure: Vectorized Hankel Lift

We start with a brief view of spectrally sparse signal recovery based on the hidden low rank structure. Let x(t) be a spectrally sparse signal consisting of r complex sinusoids,

$$x(t) = \sum_{k=1}^{r} d_k e^{-2\pi i t \tau_k}.$$

Let $\boldsymbol{x} = [x(0), \dots, x(n-1)]^{\mathsf{T}}$ be a vector of length *n* which is obtained by sampling x(t) at *n* contiguous, equally-spaced points. In a nutshell, spectrally sparse signal recovery is about reconstructing the signal \boldsymbol{x} from its partial samples. Recalling the definition of \boldsymbol{a}_{τ} in (1.6), we can represent \boldsymbol{x} as

$$\boldsymbol{x} = \sum_{k=1}^{r} d_k \boldsymbol{a}_{\tau_k}^{\mathsf{T}}.$$
 (1.11)

Let \mathcal{H} be a linear operator which maps a vector \boldsymbol{x} into an $n_1 \times n_2$ Hankel matrix,

$$\mathcal{H}(\boldsymbol{x}) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n_2-1} \\ x_1 & x_2 & \cdots & x_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_1-1} & x_{n_1} & \cdots & x_{n-1} \end{bmatrix} \in \mathbb{C}^{n_1 \times n_2},$$
(1.12)

where x_i is the *i*th entry of x and $n_1 + n_2 = n + 1$. Without loss of generality, we assume $n_1 = n_2 = (n+1)/2$ in this paper. Due to the particular expression of x in (1.11), it is not hard to see that the rank of $\mathcal{H}(x)$ is at most r according to the Vandermonde decomposition of $\mathcal{H}(x)$ [15].

Note that the expression for the data matrix \mathbf{X}^{\natural} in (1.7) is overall similar to that for the spectrally sparse vector \mathbf{x} in (1.11), except that the weights $d_k \mathbf{h}_k$ in front of $\mathbf{a}_{\tau_k}^{\mathsf{T}}$ in (1.7) are vectors and consequently \mathbf{X}^{\natural} is a matrix rather than a vector. Intuitively, if we treat each column of \mathbf{X}^{\natural} as a single element and form a matrix in the same fashion as in (1.12), it can be expected that the resulting matrix is also low rank. This is indeed true. Specifically, let \mathcal{H} be the vectorized Hankel lifting operator which maps a matrix $\mathbf{X} \in \mathbb{C}^{s \times n}$ with columns $\{\mathbf{x}_i\}$ into an $sn_1 \times n_2$ matrix,

$$\mathcal{H}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_{0} & \boldsymbol{x}_{1} & \cdots & \boldsymbol{x}_{n_{2}-1} \\ \boldsymbol{x}_{1} & \boldsymbol{x}_{2} & \cdots & \boldsymbol{x}_{n_{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{x}_{n_{1}-1} & \boldsymbol{x}_{n_{1}} & \cdots & \boldsymbol{x}_{n-1} \end{bmatrix} \in \mathbb{C}^{sn_{1} \times n_{2}},$$
(1.13)

where $n_1 + n_2 = n + 1$. To distinguish the matrix $\mathcal{H}(\mathbf{X})$ in (1.13) from the one in (1.12), we refer to $\mathcal{H}(\mathbf{X})$ as the vectorized Hankel matrix associated with \mathbf{X} . Then a simple algebra yields that the vectorized Hankel matrix $\mathcal{H}(\mathbf{X}^{\natural})$ associated with \mathbf{X}^{\natural} appearing in the blind super-resolution problem admits the following decomposition:

$$\mathcal{H}(\boldsymbol{X}^{\natural}) = \boldsymbol{E}_{\boldsymbol{h},L} \operatorname{diag}(d_1, \cdots, d_r) \boldsymbol{E}_R^{\mathsf{T}}, \qquad (1.14)$$

where the matrices $E_{h,L}$ and E_R are given by

$$\boldsymbol{E}_{\boldsymbol{h},L} = \begin{bmatrix} \boldsymbol{h}_{1} & \boldsymbol{h}_{2} & \cdots & \boldsymbol{h}_{r} \\ \boldsymbol{h}_{1}e^{-2\pi i \tau_{1} \cdot 1} & \boldsymbol{h}_{2}e^{-2\pi i \tau_{2} \cdot 1} & \cdots & \boldsymbol{h}_{r}e^{-2\pi i \tau_{r} \cdot 1} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{h}_{1}e^{-2\pi i \tau_{1} \cdot (n_{1}-1)} & \boldsymbol{h}_{2}e^{-2\pi i \tau_{2} \cdot (n_{1}-1)} & \cdots & \boldsymbol{h}_{r}e^{-2\pi i \tau_{r} \cdot (n_{1}-1)} \end{bmatrix} \in \mathbb{C}^{sn_{1} \times r}$$
(1.15)

and

$$\boldsymbol{E}_{R} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-2\pi i \tau_{1}} & e^{-2\pi i \tau_{2}} & \cdots & e^{-2\pi i \tau_{r}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-2\pi i \tau_{1} \cdot (n_{2}-1)} & e^{-2\pi i \tau_{2} \cdot (n_{2}-1)} & \cdots & e^{-2\pi i \tau_{r} \cdot (n_{2}-1)} \end{bmatrix} \in \mathbb{C}^{n_{2} \times r}.$$
 (1.16)

It follows immediately that the rank of $\mathcal{H}(\mathbf{X}^{\natural})$ is at most r and thus it is a low rank matrix when r is smaller than $\min(sn_1, n_2)$.

In this paper we adopt the popular nuclear norm minimization to exploit the low rank structure of $\mathcal{H}(X^{\natural})$, yielding a convex approach for the reconstruction of X^{\natural} which is also referred to *Vectorized Hankel Lift*. Exact recovery guarantee will be established based on certain assumptions on the subspace matrix B in (1.5).

1.3 Other Related Work

In this section, we give a brief introduction of other related work in addition to [18, 60, 33]. When the point spread functions are known and do not depend on the locations of the point sources, the measurement model (1.4) reduces to

$$\boldsymbol{y}[j] = \sum_{k=1}^{r} d_k e^{-2\pi i \tau_k \cdot j} \text{ for } j = 0, \cdots, n-1.$$
(1.17)

In this case, estimating the locations τ_k and amplitudes d_k from y is typically known as super-resolution or line spectrum estimation. This problem arises in many areas of science and engineering, such as array imaging [31, 52], Direction-of-Arrival (DOA) estimation [51], and inverse scattering [26]. The solution to this problem can date back to Prony [45]. In the Prony's method, the locations are retrieved from the roots of a polynomial whose coefficients form an annihilating filter for the observation vector. Nevertheless, the Prony's method is numerical unstable despite that in the noiseless setting successful retrieval is guaranteed in exact arithmetic. As alternatives, several subspace methods have been developed, including MUSIC [50], ESPRIT [48], and the matrix pencil method [29]. In the absence of noise, the subspace methods are also able to identify the locations of the point sources. When there is noise, the stability of these methods has been studied in [41, 40, 35, 44] in the regime when $\Delta > C/n$, where Δ is the minimum (wraparound) separation between any two locations, and C > 1 is a proper numerical constant. The analysis essentially relies on the lower bound on the smallest singular value of the Vandermonde matrix. The super-resolution limits of MUSIC and ESPRIT have been discussed in [34, 35], which is about the noise level that can be tolerated in order for the algorithms to achieve super-resolution when $\Delta < 1/n$. In this regime, it is difficult to obtain a general and nontrivial lower bound on the smallest singular value of the Vandermonde matrix. Thus, the super-resolution limits in [34, 35] are established for point sources whose locations obey certain configurations.

Inspired by compressed sensing and low rank matrix reconstruction, various optimization based methods have also been developed for super-resolution and related problems. In [9], the total variation (TV) minimization method is used to resolve the locations of the point sources. It is shown that when $\Delta > C/n$, exact recovery of the locations can be guaranteed. Moreover, the solution to the TV minimization problem can be computed by solving a semidefinite programming (SDP). Note, in the discrete setting, super-resolution can be interpreted within the framework of compressed sensing. However, since the measurement model in superresolution considers the low end spectrum, and hence is deterministic, the typical successful recovery guarantee for compressed sensing [11] cannot sufficiently explain the success of the TV norm minimization method for super-resolution. The robustness of TV norm minimization is studied in [8], and the super-resolution problem of non-negative point sources is considered in [22, 49, 20, 21, 23]. Moreover, super-resolution from time domain samples has been investigated in [2, 4, 23].

When only partial entries of y are observed in (1.17), filling in the missing entries is indeed the spectrally sparse signal recovery problem. Motivated by the work in [13], an atomic norm minimization method (ANM) is proposed for this problem. It is shown that y can be reconstructed from $\mathcal{O}(r \log r \log n)$ random samples provided the frequencies are well separated. ANM has been extended in [39, 63] to handle the case when multiple measurement vector (MMV) are available. In the setting of MMV, multiple snapshots of observations are collected and they share the same frequencies information. As already mentioned previously, the Hankel matrix corresponding to y is a low rank matrix. Consequently, spectrally sparse signal recovery can be reformulated as a low rank Hankel matrix completion problem, and replacing the rank objective with the nuclear norm yields a recovery method known as EMaC. It has been shown that EMaC is able to reconstruct a spectrally sparse signal with high probability provided the number of observed entries is $O(r \log^4 n)$. In [61], a formulation of EMaC for the multi-snapshots scenario is presented. Additionally, based on the low rank property of the Hankel matrix, provable non-convex algorithms have been developed in [6, 7] to reconstruct spectrally sparse signals. Later, Zhang et.al. [65] extend one of the non-convex algorithms to complete an MMV matrix, and in this work the same vectorized Hankel lift technique is used to exploit the hidden low rank structure. Recently, a matrix completion problem based on the low dimensional structure in the transform domain is studied in [14]. More precisely, it is assumed that after applying the Fourier transform to each column of the target matrix, each row of the resulting matrix will be a spectrally sparse signal. Since it does not require the spectrally signals share the same frequency information, a block-diagonal low rank structure is adopted to exploit the low dimensional structure. Exact recovery guarantee is also established provided the sampling complexity is nearly optimal.

Apart from super-resolution and spectrally sparse signal recovery, our work is also related to blind deconvolution. After the reparametrization of the signal and blurring kernel under the subspace assumption [1], the goal in blind deconvolution is to recover the vectors x^{\natural} and h^{\natural} simultaneously from the measurement vector in the form of

$$y = \operatorname{diag}(Bh^{\natural})Ax^{\natural}.$$

Noting that the above measurement model can be reformulated as a linear operation on a rank-1 matrix, a nuclear norm minimization method is proposed for blind deconvolution. The performance guarantee of the method has been established in the case when B is a partial Fourier matrix and A is a Gaussian matrix. A non-convex gradient descent approach for blind deconvolution is developed and analyzed in [37], and the identifiability problem is studied in [38, 19].

1.4 Notation and Organization

Throughout this work, vectors, matrices and operators are denoted by bold lowercase letters, bold uppercase letters and calligraphic letters, respectively. Note that vectors and matrices are indexed starting with zero. The letter \mathcal{I} denotes the identity operator. We use G_i to denote the matrix defined by

$$G_{i} = \frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i\\0 \le j \le n_{1}-1\\0 \le k \le n_{2}-1}} e_{j} e_{k}^{\mathsf{T}}, \qquad (1.18)$$

where w_i is a constant defined as

$$w_i = \#\{(j,k)|j+k=i, 0 \le j \le n_1 - 1, 0 \le k \le n_2 - 1\}.$$
(1.19)

In fact, $\{G_i\}_{i=0}^{n-1}$ forms an orthonormal basis of the space of $n_1 \times n_2$ Hankel matrices.

We use $\boldsymbol{x}[i]$ to denote the *i*th entry of \boldsymbol{x} and $\boldsymbol{X}_{j,k}$ or $\boldsymbol{X}[j,k]$ to denote the (j,k)th entry of \boldsymbol{X} . Additionally, the *i*th row and *j*th column of \boldsymbol{X} are denoted by $\boldsymbol{X}_{i,.}$ and $\boldsymbol{X}_{.,j}$, respectively. Furthermore, we use the MATLAB notation $\boldsymbol{X}(i:j,k)$ to denote a vector of size j-i+1, with entries $\boldsymbol{X}_{i,k}, \dots, \boldsymbol{X}_{j,k}$, i.e.,

$$oldsymbol{X}(i:j,k) = ig[oldsymbol{X}_{i,k},\cdots,oldsymbol{X}_{j,k}ig]^{\mathsf{T}}$$
 .

For any matrix X, trace(X), X^* , X^{T} and vec(X) are used to denote the trace, conjugate transpose, transpose and column vectorization of X, respectively. Also, ||X||, $||X||_{\mathsf{F}}$ and $||X||_*$ denote its spectral norm, Frobenius norm and nuclear norm, respectively.

We use diag(a) to denote the diagonal matrix specified by the vector a. For a natural number n, we use [n] to denote the set $\{0, \dots, n-1\}$. For any two matrices A, B of the same size, their inner product is defined as $\langle A, B \rangle = \text{trace}(A^*B)$. Moreover, we will refer to $A \circ B, A \otimes B, A \odot B$ as the Hadamard, Kronecker product and Khatri-Rao product respectively. More precisely, the Hadamard product is the element-wise product of two matrices and the Kronecker product between A and B is given by

$$oldsymbol{A}\otimesoldsymbol{B}=egin{bmatrix} oldsymbol{A}_{11}oldsymbol{B}&oldsymbol{A}_{12}oldsymbol{B}&\cdots&oldsymbol{A}_{1r}oldsymbol{B}\ oldsymbol{A}_{21}oldsymbol{B}&oldsymbol{A}_{22}oldsymbol{B}&\cdots&oldsymbol{A}_{2r}oldsymbol{B}\ dotsymbol{dotsymbol{dotsymbol{B}}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{B}}dotsymbol{B}=egin{bmatrix}oldsymbol{A}_{21}oldsymbol{B}&oldsymbol{A}_{22}oldsymbol{B}&\cdots&oldsymbol{A}_{2r}oldsymbol{B}\ dotsymbol{dotsymbol{dotsymbol{B}}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{B}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{eta}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{B}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{A}}dotsymbol{dotsymbol{dotsymbol{A}}dotsymbol{dotsym$$

and the Khatri-Rao product is given by

$$oldsymbol{A} \odot oldsymbol{B} = egin{bmatrix} oldsymbol{a}_1 \otimes oldsymbol{b}_1 & \cdots & oldsymbol{a}_r \otimes oldsymbol{b}_r \end{bmatrix} \in \mathbb{C}^{sn_1 imes r},$$

where a_i , b_i denote the *i*th column of A and B, respectively. By the application of the Khatri-Rao product, we can rewrite $E_{h,L}$ in (1.15) as $E_{h,L} = E_L \odot H$, where E_L and H are matrices given by

$$\boldsymbol{E}_{L} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-2\pi i \tau_{1}} & e^{-2\pi i \tau_{2}} & \cdots & e^{-2\pi i \tau_{r}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-2\pi i \tau_{1} \cdot (n_{1}-1)} & e^{-2\pi i \tau_{2} \cdot (n_{1}-1)} & \cdots & e^{-2\pi i \tau_{r} \cdot (n_{1}-1)} \end{bmatrix} \in \mathbb{C}^{n_{1} \times r}$$
(1.20)

and $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{h}_1 & \cdots & \boldsymbol{h}_r \end{bmatrix} \in \mathbb{C}^{s \times r}.$

Throughout this paper, c, c_1, c_2, \cdots denote absolute positive numerical constants whose values may vary from line to line. The notation $n \gtrsim f(m)$ means that there exists an absolute constant c > 0 such that $n \ge c \cdot f(m)$. Similarly, the notation $n \lesssim f(m)$ means that there exists an absolute constant c > 0 such that $n \le c \cdot f(m)$.

The rest of this paper is organized as follows. Section 2 begins with the presentation of Vectorized Hankel Lift and its recovery guarantee, followed by the retrieval of the point source locations. Numerical results to demonstrate the performance of Vectorized Hankel Lift is presented at the end of Section 2. The proofs of the main result are provided from Section 3 to Section 6. Finally, we conclude this paper with a few future directions in Section 7.

2 Vectorized Hankel Lift and Frequency Retrieval

2.1 Vectorized Hankel Lift and recovery guarantee

Under the assumption that $\mathcal{H}(X^{\natural})$ is a low rank matrix, it is natural to reconstruct X^{\natural} by solving the affine rank minimization problem

$$\min \operatorname{rank}(\mathcal{H}(\boldsymbol{X})) \text{ s.t. } \boldsymbol{y} = \mathcal{A}(\boldsymbol{X}).$$
(2.1)

However, the problem (2.1) is computational intractable due to the rank objective. Since the nuclear norm of a matrix is the tightest convex envelope of the matrix rank, seeking a solution with a small nuclear norm is also able to enforce the low rank structure. Therefore, instead of solving (2.1) directly, we consider the following nuclear norm minimization problem for the recovery of X^{\natural} :

$$\min_{\mathbf{X} \in \mathbb{C}^{s \times n}} \left\| \mathcal{H}(\mathbf{X}) \right\|_{*} \text{ s.t. } \mathcal{A}(\mathbf{X}) = \mathbf{y}.$$
(2.2)

In this paper, we refer to (2.2) as Vectorized Hankel Lift. There are many existing software packages that can be used to solve this problem. Thus we restrict our attention on the theoretical recovery guarantee of Vectorized Hankel Lift and investigate when the solution of (2.2) coincides with X^{\natural} .

We need to reformulate (2.2) in order to facilitate the analysis. Let Z be an $sn_1 \times n_2$ matrix which can be expressed as

$$oldsymbol{Z} = egin{bmatrix} oldsymbol{z}_{0,0} & \cdots & oldsymbol{z}_{0,n_2-1} \ dots & \ddots & dots \ oldsymbol{z}_{n_1-1,0} & \cdots & oldsymbol{z}_{n_1-1,n_2-1} \end{bmatrix} \in \mathbb{C}^{sn_1 imes n_2},$$

where $\mathbf{z}_{j,k} = \mathbf{Z}(js : (j+1)s - 1, k)$ for $j = 0, \dots, n_1 - 1$ and $k = 0, \dots, n_2 - 1$. Recall that \mathcal{H} is the vectorized Hankel lift operator defined in (1.13). The adjoint of \mathcal{H} , denoted \mathcal{H}^* , is a linear mapping from $sn_1 \times n_2$ matrices to matrices of size $s \times n$. In particular, for any matrix $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, the *i*th column of $\mathcal{H}^*(\mathbf{Z})$ is given by

$$\mathcal{H}^{*}(\mathbf{Z})\mathbf{e}_{i} = \sum_{\substack{j+k=i\\0 \le j \le n_{1}-1\\0 \le k \le n_{2}-1}} \mathbf{z}_{j,k}, \text{ for } i = 0, \cdots, n-1.$$

Letting $\mathcal{D}^2 = \mathcal{H}^* \mathcal{H}$, we have

$$\mathcal{D}^2(\boldsymbol{X}) = \begin{bmatrix} w_0 \boldsymbol{x}_0 & \cdots & w_{n-1} \boldsymbol{x}_{n-1} \end{bmatrix}, \text{ for any } \boldsymbol{X} \in \mathbb{C}^{s \times n},$$

where the scalar w_i is defined as

$$w_i = \#\{(j,k)|j+k=i, 0 \le j \le n_1 - 1, 0 \le k \le n_2 - 1\}$$
 for $i = 0, \dots, n-1$

Moreover, we define $\mathcal{G} = \mathcal{H}\mathcal{D}^{-1}$. Then

$$\mathcal{G}(\boldsymbol{X}) = \sum_{i=0}^{n-1} \mathcal{G}\left(\boldsymbol{x}_i \boldsymbol{e}_i^{\mathsf{T}}\right) = \sum_{i=0}^{n-1} \boldsymbol{G}_i \otimes \boldsymbol{x}_i, \qquad (2.3)$$

where the set of matrices $\{G_i\}_{i=0}^{n-1}$ defined in (1.18) forms an orthonormal basis of the space of $n_1 \times n_2$ Hankel matrices. The adjoint of \mathcal{G} , denoted \mathcal{G}^* , is given by $\mathcal{G}^* = \mathcal{D}^{-1}\mathcal{H}^*$. Additionally, \mathcal{G} and \mathcal{G}^* satisfy

$$\mathcal{G}^*\mathcal{G} = \mathcal{I}$$
 $\|\mathcal{G}\| = 1$, and $\|\mathcal{G}^*\| \le 1$.

Letting $\mathbf{Z} = \mathcal{H}(\mathbf{X}) = \mathcal{GD}(\mathbf{X})$, it can be readily verified that

$$\mathcal{D}(\boldsymbol{X}) = \mathcal{G}^*(\boldsymbol{Z}) \quad ext{and} \quad (\mathcal{I} - \mathcal{G}\mathcal{G}^*)(\boldsymbol{Z}) = \boldsymbol{0}.$$

Furthermore, define $D = \text{diag}(\sqrt{w_0}, \cdots, \sqrt{w_{n-1}})$. We have $\mathcal{AD}(X) = D\mathcal{A}(X)$ for any matrix X. Therefore, the optimization problem (2.2) can be reformulated as

$$\min_{\boldsymbol{Z} \in \mathbb{C}^{sn_1 \times n_2}} \|\boldsymbol{Z}\|_* \text{ s.t. } \boldsymbol{D}\boldsymbol{y} = \mathcal{A}\mathcal{G}^*(\boldsymbol{Z}) \text{ and } (\mathcal{I} - \mathcal{G}\mathcal{G}^*)(\boldsymbol{Z}) = \boldsymbol{0}.$$
(2.4)

Due to the equivalence between (2.2) and (2.4), it suffices to investigate the recovery guarantee of (2.4). To this end, we make two assumptions. **Assumption 2.1.** The column vectors $\{b_j\}_{j=0}^{n-1}$ of the subspace matrix B^* are independently and identically sampled from a distribution F which obeys the following properties:

• Isotropy property. A distribution F obeys the isotropy property if for $b \sim F$,

$$\mathbb{E}\left[\boldsymbol{b}\boldsymbol{b}^*\right] = \boldsymbol{I}_s. \tag{2.5}$$

• Incoherence property. A distribution F satisfies the incoherence property with parameter μ_0 if for $\mathbf{b} \sim F$,

$$\max_{0 \le \ell \le s-1} |\boldsymbol{b}[\ell]|^2 \le \mu_0 \tag{2.6}$$

holds, where $b[\ell]$ denotes the ℓ th entry of b.

• For $\boldsymbol{b} \sim F$, the sampled column vectors $\{\boldsymbol{b}_j\}_{j=0}^{n-1}$ satisfy

$$\min_{0 \le j \le n-1} \| \boldsymbol{b}_j \|_2^2 \ge 1.$$
(2.7)

The first two conditions (2.5) and (2.6) in Assumption 2.1 are first introduced in [10] in the context of compressed sensing and these two properties are also made in [18, 60, 33] for the blind super-resolution problem. If F has mean zero, the isotropy condition states that the entries of \boldsymbol{b} have unit variance and are uncorrelated, which implies $\mu_0 \geq 1$ in the incoherence property. The lower bound $\mu_0 = 1$ is achievable by several examples, for instance, when the components of \boldsymbol{b} are Rademacher random variables taking the values ± 1 with equal probability or \boldsymbol{b} is uniformly sampled from the rows of a Discrete Fourier Transform (DFT) matrix. In addition to (2.5) and (2.6), we also need (2.7) to establish our main result. However, we would like to point out that (2.7) is not a stringent condition, but holds (either trivially or with high probability) by many common random ensembles.

- If the components of **b** are Rademacher random variables or **b** is uniformly sampled from the rows of a DFT matrix, it is trivial that for any fixed $j \in [n]$, $\|\mathbf{b}_j\|_2^2 = s \ge 1$.
- Suppose the components of **b** are independently and identically sampled from a distribution with mean zero and unit variance, such as the uniform distribution on the interval $\left[-\sqrt{3}, \sqrt{3}\right]$. In such case, we can apply the bounded difference inequality to show that (2.7) holds with high probability, see Lemma 3.1.

Assumption 2.2. There exists a constant $\mu_1 > 0$ such that

$$\sigma_{\min}(\boldsymbol{E}_L^*\boldsymbol{E}_L) \ge \frac{n_1}{\mu_1} \quad and \quad \sigma_{\min}(\boldsymbol{E}_R^*\boldsymbol{E}_R) \ge \frac{n_2}{\mu_1},\tag{2.8}$$

where E_L and E_R are given in (1.20) and (1.16) and $\sigma_{\min}(\cdot)$ denotes the smallest singular value of a matrix.

Assumption 2.2 is the same as the one made in [15, 6, 7] for spectrally sparse signal recovery. Later, we will show that $\sigma_{\min}(\boldsymbol{E}_{h,L}^*\boldsymbol{E}_{h,L}) \geq \frac{n_1}{\mu_1}$ also holds when $\sigma_{\min}(\boldsymbol{E}_L^*\boldsymbol{E}_L) \geq \frac{n_1}{\mu_1}$, see Lemma 3.3. Recalling the definition of \boldsymbol{E}_L and \boldsymbol{E}_R , this assumption is essentially about the conditioning property of the Vandermonde matrix. This property is studied in [41] through the discrete Ingham inequality [30] and in [44] through the discrete large sieve inequality [56]. In particular, it follows from [44] that Assumption 2.2 holds when the minimum wrap-around distance between the frequencies, denoted Δ , satisfies

$$\Delta \ge \frac{2\mu_1/(\mu_1 - 1)}{n}.$$
(2.9)

We are in position to present the main result of this paper.

Theorem 2.1 (Exact recovery guarantee of Vectorized Hankel Lift). Under Assumptions 2.1 and 2.2, $Z^{\natural} = \mathcal{H}(X^{\natural})$ is the unique optimal solution to (2.4) with probability exceeding $1 - c_0(sn)^{-c_1} - ns^{-c_2}$, provided that $n \gtrsim \mu_0 \mu_1 \cdot sr \log^4(sn)$, where c_0, c_1, c_2 are absolute constants.

Remark 2.1. The sampling complexity established in [60] for the atomic norm minimization method is $n \gtrsim \mu_0 \cdot \operatorname{sr} \log^3(\operatorname{sn})$. While this is slightly better than the sampling complexity for Vectorized Hankel Lift, our analysis is based on less stringent assumptions. In our analysis, the coefficients $\{\mathbf{h}_k\}_{k=1}^r$ are not required to be i.i.d. samples from the uniform distribution on the complex unit sphere, but can be any unit norm vectors. In addition, noting that the right-hand side of (2.9) is about 2/n for moderately large μ_1 , which is smaller than 4/n, the separation required in the main result of [60]. It is worth noting that the robust analysis of the atomic norm minimization method has been studied in [33] and we will leave the robust analysis of Vectorized Hankel Lift for future work.

The proof of Theorem 2.1 follows a well established route that has been widely used for compressed sensing and low rank matrix recovery. In a nutshell, a dual variable needs to be constructed to verify the optimality of Z^{\natural} . That being said, the details of the proof itself are nevertheless quite involved and technical, and cannot be covered by the results from existing works. In particular, we need to show that there exists a partition of the measurements satisfying a list of desirable properties in order to construct the dual certificate.

2.2 Variants of MUSIC for frequency retrieval

In this section, we discuss the subspace method, particularly the MUltiple SIgnal Classification (MUSIC) algorithm [50], for computing the frequency parameters $\{\tau_k\}_{k=1}^r$ from the matrix \mathbf{X}^{\natural} . Note that once $\{\tau_k\}_{k=1}^r$ are obtained, the weights $\{d_k, \mathbf{h}_k\}$ can be computed by solving an overdetermined linear system. As can be seen later, applying the idea of the single snapshot MUSIC to $\mathcal{H}(\mathbf{X}^{\natural})$ yields a variant which is equivalent to the existing spatial smoothing technique proposed to improve the performance of the Multiple Measurement Vector (MMV) MUSIC.

The careful reader may notice that every single row of \mathbf{X}^{\natural} is a spectrally sparse signal of the form (1.11), and moreover, all the rows share the same frequency parameters $\{\tau_k\}_{k=1}^r$. Thus we can apply the single snapshot MUSIC algorithm to a row of \mathbf{X}^{\natural} for frequency retrieval. Let $\mathbf{x}_{\ell} = \sum_{k=1}^r d_k \mathbf{h}_k[\ell] \mathbf{a}_{\tau_k}^{\mathsf{T}}, 1 \leq \ell \leq s$. Recall that $\mathcal{H}(\mathbf{x}_{\ell})$ is the Hankel matrix of rank r and it admits the Vandermonde decomposition

$$\mathcal{H}(\boldsymbol{x}_{\ell}) = \boldsymbol{E}_L \operatorname{diag}(d_1 \boldsymbol{h}_1[\ell], \cdots, d_r \boldsymbol{h}_r[\ell]) \boldsymbol{E}_R^{\mathsf{T}}.$$
(2.10)

Moreover, letting

$$\mathcal{H}(\boldsymbol{x}_{\ell})^{\mathsf{T}} = \begin{bmatrix} \boldsymbol{U} & \boldsymbol{U}_{\perp} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma} & \\ & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}^{*} \\ \boldsymbol{V}_{\perp}^{*} \end{bmatrix}$$
(2.11)

be the SVD of $\mathcal{H}(\boldsymbol{x}_{\ell})^{\mathsf{T}}$, where $\boldsymbol{U} \in \mathbb{C}^{n_2 \times r}, \boldsymbol{U}_{\perp} \in \mathbb{C}^{n_2 \times (n_2 - r)}, \boldsymbol{\Sigma} \in \mathbb{R}^{r \times r}, \boldsymbol{V} \in \mathbb{C}^{n_1 \times r}$ and $\boldsymbol{V}_{\perp} \in \mathbb{C}^{n_1 \times (n_1 - r)}$, it is evident that \boldsymbol{U} and \boldsymbol{E}_R span the same column space. Note that $\boldsymbol{E}_R = [\boldsymbol{a}_{\tau_1}, \cdots, \boldsymbol{a}_{\tau_r}]$, where $\boldsymbol{a}_{\tau_k} = [1, \cdots, e^{-2\pi i \tau_k \cdot (n_2 - 1)}]^{\mathsf{T}}$. It follows from the property of the Vandermonde matrix that

$$\boldsymbol{a}_{\tau} \in \operatorname{Range}(\boldsymbol{E}_R)$$
 if and only if $\tau \in \{\tau_1, \cdots, \tau_r\}$.

Therefore we conclude that $\tau \in {\tau_1, \dots, \tau_r}$ if and only if $1/\|\boldsymbol{U}_{\perp}^*\boldsymbol{a}_{\tau}\|_2^2 = \infty$. The single snapshot MUSIC algorithm utilizes this idea to identify the frequencies, and it consists of the following two steps:

- 1. Compute the SVD of $\mathcal{H}(\boldsymbol{x}_{\ell})^{\mathsf{T}}$ as in (2.11);
- 2. Identify $\{\tau_k\}_{k=1}^r$ as the *r* largest local maxima of the pseudospectrum: $f(\tau) = 1/\|U_{\perp}^* a_{\tau}\|_2^2$.

Here we present the single snapshot MUSIC algorithm directly based on the Hankel matrix $\mathcal{H}(\boldsymbol{x}_{\ell})$. Equivalently, it can be interpreted from the autocorrelation matrix model for signals, see for example [32] and references therein. In the noiseless setting, it is easy to see that the single snapshot MUSIC algorithm is able to compute $\{\tau_k\}_{k=1}^r$ exactly. When noise exists in \boldsymbol{x}_{ℓ} , the procedure of the algorithm remains unchanged, but with the SVD of $\mathcal{H}(\boldsymbol{x}_{\ell})^{\mathsf{T}}$ being replaced by the SVD of the noisy Hankel matrix and with \boldsymbol{U}_{\perp} being the left singular vectors corresponding to the $n_2 - r$ smallest singular values. The stability analysis of the single snapshot algorithm is discussed in [41].

To motivate the new variant of the MUSIC algorithm for estimating the frequencies from X^{\natural} , we note that E_R appears as a separate component both in the Vandermonde decomposition of $\mathcal{H}(x_{\ell})$ and that of $\mathcal{H}(X^{\natural})$, see (1.14) and (2.10). Therefore, we can replace the SVD of $\mathcal{H}(x_{\ell})^{\mathsf{T}}$ with the SVD of $\mathcal{H}(X^{\natural})^{\mathsf{T}}$ in the first step of the single snapshot MUSIC algorithm. This gives the following variant:

- 1. Compute the SVD of $\mathcal{H}(\mathbf{X}^{\natural})^{\mathsf{T}}$: $\mathcal{H}(\mathbf{X}^{\natural})^{\mathsf{T}} = \begin{bmatrix} \mathbf{U} & \mathbf{U}_{\perp} \end{bmatrix} \mathbf{\Sigma} \mathbf{V}^*$, where $\mathbf{U} \in \mathbb{C}^{n_2 \times r}$ and $\mathbf{U}_{\perp} \in \mathbb{C}^{n_2 \times (n_2 r)}$;
- 2. Identify $\{\tau_k\}_{k=1}^r$ as the *r* largest local maxima of the pseudospectrum: $f(\tau) = 1/\|\boldsymbol{U}_{\perp}^*\boldsymbol{a}_{\tau}\|_2^2$.

The following lemma establishes a connection between this variant and the single snapshot MUSIC, showing that the former one actually utilizes the SVD of the matrix formed by stacking all $\mathcal{H}(\boldsymbol{x}_{\ell})$ ($\ell = 1, \dots, s$) together.

Lemma 2.2. Let $\widetilde{\mathcal{H}}(X^{\natural})$ be a matrix constructed by stacking all $\mathcal{H}(x_{\ell})$ on top of one another:

$$\widetilde{\mathcal{H}}(oldsymbol{X}^{\natural}) = egin{bmatrix} \mathcal{H}(oldsymbol{x}_1) \ dots \ \mathcal{H}(oldsymbol{x}_s) \end{bmatrix} \in \mathbb{C}^{sn_1 imes n_2}.$$

There exists a permutation matrix $P \in \mathbb{R}^{sn_1 \times sn_1}$ such that $\widetilde{\mathcal{H}}(X^{\natural}) = P\mathcal{H}(X^{\natural})$.

Proof. Following the Vandermonde decomposition, the ℓ th block of $\mathcal{H}(X^{\natural})$ can be rewritten as

$$\mathcal{H}(\boldsymbol{e}_{\ell}^{\mathsf{T}}\boldsymbol{X}^{\natural}) = \boldsymbol{E}_{L} \begin{bmatrix} d_{1} \cdot \boldsymbol{h}_{1}[\ell] & & \\ & \ddots & \\ & & d_{r} \cdot \boldsymbol{h}_{r}[\ell] \end{bmatrix} \boldsymbol{E}_{R}^{\mathsf{T}}$$
$$= (\boldsymbol{E}_{L} \odot \boldsymbol{e}_{\ell}^{\mathsf{T}}\boldsymbol{H}) \begin{bmatrix} d_{1} & & \\ & \ddots & \\ & & d_{r} \end{bmatrix} \boldsymbol{E}_{R}^{\mathsf{T}}$$

where h_i is the *i*th column of H and $h_i[\ell]$ is the ℓ th entry of h_i . Thus $\widetilde{\mathcal{H}}(X^{\natural})$ has the following decomposition

$$\widetilde{\mathcal{H}}(\boldsymbol{X}^{\natural}) = (\boldsymbol{H} \odot \boldsymbol{E}_L) \boldsymbol{D} \boldsymbol{E}_R^{\mathsf{T}}.$$

According to the commutative law in [66, Section 1.10.3], there exists a permutation matrix P such that $H \odot E_L = P(E_L \odot H)$.

Based on Lemma 2.2, we will see that the variant obtained by applying the single snapshot MUSIC idea to $\mathcal{H}(\mathbf{X}^{\natural})$ corresponds to the spatial smoothing technique (more precisely the forward only spatial smoothing technique). First, treating the rows of \mathbf{X}^{\natural} as i.i.d samples of a random signal whose covariance matrix can be used to compute the signal space U as in (2.11), MMV MUSIC [52] uses the principal eigenspace of the empirical covariance matrix (up to a scaling factor 1/s)

$$oldsymbol{R} = \sum_{i=1}^s oldsymbol{x}_i oldsymbol{x}_i^st$$

to compute U. However, when the signal comes from coherence sources, the performance of MMV MUSIC will degrade. To deal with this difficulty, the forward only spatial smooth technique proposes to increase the number of samples by partitioning each x_i into n_2 overlapped short samples (with each short sample being of length n_1 , where $n_1 + n_2 = n + 1$), and then construct the empirical covariance matrix from all the $s \cdot n_2$ short samples. A simple algebra yields that the new empirical covariance matrix is indeed given by (up to a scaling factor $1/(sn_2)$)

$$\widehat{oldsymbol{R}} = \sum_{i=1}^s \mathcal{H}(oldsymbol{x}_i)\mathcal{H}(oldsymbol{x}_i)^*.$$

It is not hard to see that the principal eigenspace of \hat{R} is the same as the principal singular vector space of $\tilde{\mathcal{H}}(X^{\natural})$. Thus, by Lemma 2.2, we know that the variant obtained by applying the single snapshot MUSIC idea to $\mathcal{H}(X^{\natural})$ is equivalent to the spatial smoothing MUSIC. For more details about spatial smoothing, see [25, 24, 62].

2.3 Extension to higher dimension

Vectorized Hankel Lift and the analysis are easily extended to higher dimensional array recovery problem. For ease of exposition, we give a brief discussion of the two-dimensional (2D) case but emphasize that the situation in higher dimensions is similar. For the 2D blind super-resolution problem, the data matrix can be expressed as

$$\mathbf{Y}_{j,\ell} = \sum_{k=1}^{r} d_k e^{-2\pi i (j \cdot \tau_{1k} + \ell \cdot \tau_{2k})} \mathbf{G}_k[j,\ell].$$

where d_k is the amplitude, $\tau_k := (\tau_{1k}, \tau_{2k})$ is the 2D frequency and G_k corresponds to the Fourier samples of the unknown 2D point spread function. Letting $a_{\tau_{sk}} = \begin{bmatrix} 1 & e^{-2\pi i \tau_{sk} \cdot 1} & \cdots & e^{-2\pi i \tau_{sk} \cdot (n-1)} \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^n$ for s = 1, 2, the 2D data array can be rewritten in a more compact form:

$$\boldsymbol{Y} = \sum_{k=1}^{r} d_k \left(\boldsymbol{a}_{\tau_{1k}} \boldsymbol{a}_{\tau_{2k}}^{\mathsf{T}} \right) \circ \boldsymbol{G}_k,$$

Likewise, we assume that there exists a subspace matrix $B \in \mathbb{C}^{n^2 \times s}$ such that $\operatorname{vec}(G_k) = Bh_k$ for any $k = 1, \dots, r$. Then

$$\boldsymbol{y} := \operatorname{vec}(\boldsymbol{Y}) = \sum_{k=1}^{r} d_k \operatorname{vec}(\boldsymbol{a}_{\tau_{1k}} \boldsymbol{a}_{\tau_{2k}}^{\mathsf{T}}) \circ \operatorname{vec}(\boldsymbol{G}_k) = \sum_{k=1}^{r} d_k \left(\boldsymbol{a}_{\tau_{2k}} \otimes \boldsymbol{a}_{\tau_{1k}}\right) \circ \left(\boldsymbol{B}\boldsymbol{h}_k\right).$$

For any $0 \le j, \ell \le n-1$, the $(jn+\ell)$ th entry of \boldsymbol{y} is given by

$$\begin{aligned} \boldsymbol{y}_{jn+\ell} &= \sum_{k=1}^{r} d_k \left(\boldsymbol{a}_{\tau_{2k}} \otimes \boldsymbol{a}_{\tau_{1k}} \right)^{\mathsf{T}} \boldsymbol{e}_{jn+\ell} \left(\boldsymbol{b}_{jn+\ell}^* \boldsymbol{h}_k \right) \\ &= \sum_{k=1}^{r} \operatorname{trace} \left(d_k \left(\boldsymbol{a}_{\tau_{2k}} \otimes \boldsymbol{a}_{\tau_{1k}} \right)^{\mathsf{T}} \boldsymbol{e}_{jn+\ell} \left(\boldsymbol{b}_{jn+\ell}^* \boldsymbol{h}_k \right) \right) \\ &= \sum_{k=1}^{r} \operatorname{trace} \left(\boldsymbol{e}_{jn+\ell} \boldsymbol{b}_{jn+\ell}^* d_k \boldsymbol{h}_k \left(\boldsymbol{a}_{\tau_{2k}} \otimes \boldsymbol{a}_{\tau_{1k}} \right)^{\mathsf{T}} \right) \\ &= \left\langle \boldsymbol{b}_{jn+\ell} \boldsymbol{e}_{jn+\ell}^{\mathsf{T}}, \sum_{k=1}^{r} d_k \boldsymbol{h}_k \left(\boldsymbol{a}_{\tau_{2k}} \otimes \boldsymbol{a}_{\tau_{1k}} \right)^{\mathsf{T}} \right\rangle, \end{aligned}$$

where $b_{jn+\ell}$ is the $(jn+\ell)$ th column of B^* . Therefore, we have $\boldsymbol{y} = \mathcal{A}(\boldsymbol{X}^{\natural})$, where $\boldsymbol{X}^{\natural} = \sum_{k=1}^r d_k \boldsymbol{a}_{\tau_{2k}}^{\mathsf{T}} \otimes (\boldsymbol{h}_k \boldsymbol{a}_{\tau_{1k}}^{\mathsf{T}})$, and $\mathcal{A} : \mathbb{C}^{s \times n^2} \to \mathbb{C}^{n^2}$ is a linear operator given by

$$\left[\mathcal{A}(oldsymbol{X})
ight]_{jn+\ell} = \left\langle oldsymbol{b}_{jn+\ell}oldsymbol{e}_{jn+\ell}^{\mathsf{T}},oldsymbol{X}
ight
angle$$

As in the 1D case, the blind super-resolution problem is essentially about recovering the target matrix X^{\natural} from the observation vector y.

Note that the target matrix X^{\natural} can be rewritten as the following block form:

$$\boldsymbol{X}^{\natural} = \begin{bmatrix} \sum_{k=1}^{r} d_k \left(\boldsymbol{h}_k \boldsymbol{a}_{\tau_{1k}}^{\mathsf{T}} \right) & \sum_{k=1}^{r} d_k e^{-2\pi i \tau_{2k}} \left(\boldsymbol{h}_k \boldsymbol{a}_{\tau_{1k}}^{\mathsf{T}} \right) & \cdots & \sum_{k=1}^{r} d_k e^{-2\pi i \tau_{2k} \cdot (n-1)} \left(\boldsymbol{h}_k \boldsymbol{a}_{\tau_{1k}}^{\mathsf{T}} \right) \end{bmatrix}$$

Letting $X_{\ell}^{\natural} := \sum_{k=1}^{r} d_{k} e^{-2\pi i \tau_{2k} \cdot \ell} \left(h_{k} a_{\tau_{1k}}^{\mathsf{T}} \right)$, we define the two-fold vectorized Hankel lift of X^{\natural} as follows:

$$\mathcal{H}(\boldsymbol{X}^{\natural}) = \begin{bmatrix} \mathcal{H}(\boldsymbol{X}_{0}^{\natural}) & \mathcal{H}(\boldsymbol{X}_{1}^{\natural}) & \cdots & \mathcal{H}(\boldsymbol{X}_{n_{2}-1}^{\natural}) \\ \mathcal{H}(\boldsymbol{X}_{1}^{\natural}) & \mathcal{H}(\boldsymbol{X}_{2}^{\natural}) & \cdots & \mathcal{H}(\boldsymbol{X}_{n_{2}}^{\natural}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}(\boldsymbol{X}_{n_{1}-1}^{\natural}) & \mathcal{H}(\boldsymbol{X}_{n_{1}}^{\natural}) & \cdots & \mathcal{H}(\boldsymbol{X}_{n-1}^{\natural}) \end{bmatrix}$$

where $\mathcal{H}(\mathbf{X}_i^{\natural})$ is the vectorized Hankel matrix defined in (1.13). It can be readily shown that $\mathcal{H}(\mathbf{X}^{\natural})$ has the following decomposition

$$\mathcal{H}(\mathbf{X}^{\natural}) = \begin{bmatrix} (\mathbf{E}_{L} \odot \mathbf{H}) \mathbf{Y}^{0} \\ (\mathbf{E}_{L} \odot \mathbf{H}) \mathbf{Y}^{1} \\ \vdots \\ (\mathbf{E}_{L} \odot \mathbf{H}) \mathbf{Y}^{n_{1}-1} \end{bmatrix} \mathbf{D} \begin{bmatrix} \mathbf{Y}^{0} \mathbf{E}_{R}^{\mathsf{T}} & \mathbf{Y}^{1} \mathbf{E}_{R}^{\mathsf{T}} & \cdots & \mathbf{Y}^{n_{2}-1} \mathbf{E}_{R}^{\mathsf{T}} \end{bmatrix} := \mathbf{L} \mathbf{D} \mathbf{R}^{\mathsf{T}}, \quad (2.12)$$

where E_L, E_R are two matrices defined in (1.20) and (1.16) but with the frequencies $\{\tau_{1k}\}_{k=1}^r$, $H = [h_1 \cdots h_r] \in \mathbb{C}^{s \times r}$, $D = \operatorname{diag}(d_1, \cdots, d_r)$ and $Y = \operatorname{diag}(e^{-2\pi i \tau_{21}}, \cdots, e^{-2\pi i \tau_{2r}})$.

If all frequencies τ_{1k}, τ_{2k} are distinct and all d_k are non-zeros, it is not hard to see that $\mathcal{H}(\mathbf{X}^{\natural})$ is a low rank matrix. Therefore, we can recover \mathbf{X}^{\natural} by solving the following convex programming

$$\min_{\boldsymbol{X} \in \mathbb{C}^{s \times n^2}} \|\mathcal{H}(\boldsymbol{X})\|_* \text{ s.t. } \mathcal{A}(\boldsymbol{X}) = \boldsymbol{y}.$$
(2.13)

The recovery guarantee of (2.13) can be similarly established in the following theorem. The proof details are overall similar to that for Theorem 2.1, and thus are omitted.

Theorem 2.3. Under Assumption II.1 and suppose $\sigma_{\min}(\mathbf{L}^*\mathbf{L}) \geq \frac{n_1^2}{\mu_1}$ and $\sigma_{\min}(\mathbf{R}^*\mathbf{R}) \geq \frac{n_2^2}{\mu_1}$, the data matrix $\mathbf{X}^{\ddagger} \in \mathbb{C}^{s \times n^2}$ is the unique optimal solution to (2.13) with probability at least $1 - c_0(sn)^{-c_1} - n^2 s^{-c_2}$ for absolute constants c_0, c_1, c_2 , provided that $n^2 \gtrsim \mu_0 \mu_1 \cdot sr \log^5(sn)$.

After the matrix \mathbf{X}^{\natural} is recovered, the frequency $\{\boldsymbol{\tau}_{k} = (\tau_{1k}, \tau_{2k})\}_{k=1}^{r}$ can be estimated by a 2D-MUSIC algorithm [3, 40, 67] based on the two-fold vectorized Hankel matrix $\mathcal{H}(\mathbf{X}^{\natural})$ in (2.12), followed by the recovery of $\{d_k \boldsymbol{h}_k\}_{k=1}^{r}$ through least-squares.

2.4 Numerical Experiments

In this section, we empirically evaluate the performance of Vectorized Hankel Lift for the recovery of X^{\natural} in the blind super-resolution problem. Vectorized Hankel Lift is solved by SDPT3 [54] based on CVX [27]. The recovery ability of Vectorized Hankel Lift will be evaluated via the framework of empirical phase transition and we compare it with the atomic norm minimization method [60]. The locations $\{\tau_k\}_{k=1}^r$ of the point source signals are generated randomly from [0, 1), while the amplitudes $\{d_k\}_{k=1}^r$ are generated via $d_k = (1 + 10^{c_k})e^{-i\psi_k}$ with ψ_k being uniformly sampled from $[0, 2\pi)$ and c_k being uniformly sampled from [0, 1]. The subspace matrix B are sampled from two random ensembles which all satisfy the conditions in Assumption 2.1. The first one is the random submatrix sampled from the DFT matrix, and the other one is the random matrix whose entries satisfy the uniform distribution over $[-\sqrt{3}, \sqrt{3}]$. The coefficients $\{h_k\}_{k=1}^r$ are i.i.d. standard Gaussian random vectors followed by normalization. In our tests, 20 Monte Carlo trails are repeated for each problem instance and we report the probability of successful recovery out of those trials. A trail is declared to be successful if the relative reconstruction error of X^{\natural} in terms of the Frobenius norm is less than 10^{-3} .

We first fix n = 64 and vary the values of r and s. Figure 1(a) and Figure 1(b) show the phase transitions of Vectorized Hankel Lift and atomic norm minimization method when the subspace matrix B is randomly sampled from the DFT matrix and the locations of point sources are randomly generated without imposing the separation condition, and Figure 1(c) illustrates the phase transition of the atomic minimization method when the separation condition $\Delta := \min_{k\neq j} |\tau_k - \tau_j| \geq \frac{1}{n}$ is imposed. Here we omit the phase transition plot of Vectorized Hankel Lift for the frequency separation case because the plot is similar to Figure 1(a). It can be observed that the atomic norm minimization method has a higher phase transition curve when the separation condition is satisfied. However, in contrast to Vectorized Hankel Lift, its performance degrades severely when there is no frequency separation requirement. That is, Vectorized Hankel Lift is less sensitive to the separation condition. We also conduct the phase transition tests when the entries of B are i.i.d. sampled from the uniform distribution over $[-\sqrt{3}, \sqrt{3}]$. The phase transition diagrams are presented in Figure 2, and similar observations can be made. Note that the phase transition plot of Vectorized Hankel Lift for the frequency separation case is still omitted due to the high similarity with Figure 2(a).



Figure 1: The phase transitions of Vectorized Hankel Lift and the atomic norm minimization method when the subspace matrix \boldsymbol{B} is randomly sampled from the DFT matrix. (a) Vectorized Hankel Lift for randomly generated frequencies, (b) atomic norm minimization for randomly generated frequencies, and (c) atomic norm minimization for frequencies obeying the separation condition $\Delta := \min_{k \neq j} |\tau_k - \tau_j| \geq \frac{1}{n}$. The number of measurements is fixed to be n = 64. The red curve plots the hyperbola curve rs = 20.

In the above phase transition tests, the coefficients $\{\mathbf{h}_k\}_{k=1}^r$ are sampled from random Gaussian with normalization. In order to test whether the choice of $\{\mathbf{h}_k\}_{k=1}^r$ matters, we also test another two cases for the coefficients. One is the Identical Gaussian, where $\{\mathbf{h}_k\}_{k=1}^r$ are the same across r (sampled from random Gaussian with normalization). The other one is QR where $\{\mathbf{h}_k\}_{k=1}^r$ are obtained from the Q matrix in the QR decomposition of an $s \times r$ random Gaussian matrix. Tests are conducted for fixed s = 4 and n = 64, and the plots of successful recovery probability against the number of spikes r are presented in Figure 3. It can be clearly seen that no significant differences over different types of $\{\mathbf{h}_k\}_{k=1}^r$ are observed from the plots. Therefore, the numerical results validate that our main result can hold without any conditions of $\{\mathbf{h}_k\}_{k=1}^r$.

In order to examine the effect of the separation condition more carefully, we further conduct tests for fixed s = 3, r = 3, and vary the number of samples n. In the tests, we impose that there are at least two spikes with separation equal to 1.0/n and 0.5/n, respectively. For each problem instance, we repeat



Figure 2: The phase transitions of Vectorized Hankel Lift and the atomic norm minimization method when the entries of \boldsymbol{B} are i.i.d. sampled from the uniform distribution over $[-\sqrt{3},\sqrt{3}]$. (a) Vectorized Hankel Lift for randomly generated frequencies, (b) atomic norm minimization for randomly generated frequencies, and (c) atomic norm minimization for frequencies obeying the separation condition $\Delta := \min_{k\neq j} |\tau_k - \tau_j| \geq \frac{1}{n}$. The number of measurements is fixed to be n = 64. The red curve plots the hyperbola curve rs = 20.



Figure 3: The probability of successful recovery of Vectorized Hankel Lift against r with three different subspace coefficients $\{\mathbf{h}_k\}_{k=1}^r$ (s = 4, n = 64). (a): The subspace matrix \mathbf{B} are randomly sampled from the DFT matrix. (b): The entries of \mathbf{B} are i.i.d. sampled from the uniform distribution over $[-\sqrt{3}, \sqrt{3}]$.

50 Monte Carlo trails and report the probability of successful recovery out of those trials. The numerical results are presented in Figure 4. It is evident that Vectorized Hankel Lift presents a better performance when the minimum separation is 0.5/n. When the spikes are well separated (i.e., the minimum separation is $\Delta = 1.0/n$), the atomic norm minimization method performs better. In addition, the results confirm that Vectorized Hankel Lift is overall not affected by the separation condition.

We also plot the locations of the point sources $\{\tau_k\}_{k=1}^r$ and the unknown point spread function samples $\{g_k\}_{k=1}^r$ computed from X^{\natural} for a random instance corresponding to n = 64, s = 3 and r = 4. We apply the MUSIC variant introduced in Section 2.2 (i.e., the spatial smoothing MUSIC) to localize the $\{\tau_k\}_{k=1}^r$. Figure 5(a) shows the pseudospectrum $f(\tau)$ on a set of points on [0,1] with equal distance 10^{-4} . As can be seen from this figure, the function $f(\tau)$ peaks at the locations of true frequencies. After the $\{\tau_k\}_{k=1}^r$ are identified, the coefficients $\{h_k\}_{k=1}^r$ are computed by solving a least squares problem and $\{g_k\}_{k=1}^r$ are



Figure 4: The probability of successful recovery of Vectorized Hankel Lift and the atomic norm minimization method under two separation conditions, $\Delta = \frac{0.5}{n}$ and $\Delta = \frac{1.0}{n}$. The dimension of subspace and the number of spikes are both fixed to be s = 3 and r = 3. The number of samples n is varied. (a): The subspace matrix **B** are randomly sampled from the DFT matrix. (b): The entries of **B** are i.i.d. sampled from the uniform distribution over $[-\sqrt{3}, \sqrt{3}]$.

estimated as Bh_k . Figure 5(b) includes the plots of the estimates of $\{|g_k|\}_{k=1}^r$ against the true values which clearly show that $\{g_k\}_{k=1}^r$ can be recovered.



Figure 5: (a) Plots of pseudospectrum $f(\tau)$ when n = 64, s = 3, r = 4 and locations of the true frequencies when the subspace **B** is generated randomly from the standard Gaussian distribution. (b) The magnitudes of Fourier samples of the point spread functions g_1, g_2, g_3, g_4 and their estimates from least squares.

3 Proof Architecture of Main Result

3.1 Preliminaries

We first apply the bounded difference inequality to show that for the column vectors $\{b_j\}_{j=0}^{n-1}$ with independent entries, the condition (2.7) in Assumption 2.1 holds with high probability given (2.5) and (2.6).

Lemma 3.1. The column vectors $\{b_j\}_{j=0}^{n-1}$ of the subspace matrix B^* are independently and identically sampled from a distribution F which obeys the conditions (2.5) and (2.6) in Assumption 2.1. Assume the components of **b** are independent, the event

$$\min_{0 \le j \le n-1} \|\boldsymbol{b}_j\|_2^2 \ge 1 \tag{3.1}$$

occurs with probability at least $1 - n \exp\left(-\frac{s}{16\mu_0^2}\right)$.

Proof. Since \boldsymbol{b}_j satisfies (2.5), we first have

$$\mathbb{E}\left[\left\|\boldsymbol{b}_{j}\right\|_{2}^{2}\right] = \mathbb{E}\left[\operatorname{trace}(\boldsymbol{b}_{j}^{*}\boldsymbol{b}_{j})\right] = \mathbb{E}\left[\operatorname{trace}(\boldsymbol{b}_{j}\boldsymbol{b}_{j}^{*})\right] = s$$

Define $f(x_1, \dots, x_s) = \sum_{i=1}^s |x_i|^2$. It is evident that

$$|f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_s) - f(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_s)| \le |x_i|^2 + |x'_i|^2 \le 2\mu_0$$

when $|x_i|^2 \leq \mu_0$ and $|x'_i|^2 \leq \mu_0$. Because b_j also satisfies (2.6), the application of the bounded difference inequality yields that

$$\mathbb{P}\left[\left|\left\|\boldsymbol{b}_{j}\right\|_{2}^{2}-s\right|\geq t\right]\leq 2\exp\left(-\frac{t^{2}}{4s\mu_{0}^{2}}\right).$$

Consequently, we can take $t = \frac{s}{2}$ to obtain

$$\mathbb{P}\left[\left\|\boldsymbol{b}_{j}\right\|_{2}^{2} \geq \frac{s}{2}\right] \geq 1 - \exp\left(-\frac{s}{16\mu_{0}^{2}}\right).$$

Taking the uniform bound yields that for all $j \in [n]$, with probability at least $1 - n \exp\left(-\frac{s}{16\mu_0^2}\right)$, $\|\boldsymbol{b}_j\|_2^2 \ge \frac{s}{2} \ge 1$ when $s \ge 2$.

Next, we present a lemma about the basic properties of the linear operator \mathcal{A} .

Lemma 3.2. Under Assumption 2.1, the following properties hold:

$$\langle \boldsymbol{y}, \mathcal{A}\mathcal{A}^*(\boldsymbol{y}) \rangle \ge \|\boldsymbol{y}\|_2^2 \quad \text{for any fixed vector } \boldsymbol{y} \in \mathbb{C}^n,$$
(3.2)

$$\|\mathcal{A}\mathcal{A}^* - \mathcal{I}\| \le s\mu_0 \text{ and } \|\mathcal{A}\| \le \sqrt{s\mu_0}.$$
(3.3)

Proof. Since

$$\mathcal{AA}^{*}(\boldsymbol{y}) = \mathcal{A}\left(\sum_{i=0}^{n-1} \boldsymbol{y}[i]\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)$$
$$= \begin{bmatrix} \left\langle \boldsymbol{b}_{0}\boldsymbol{e}_{0}^{\mathsf{T}}, \sum_{i=0}^{n-1} \boldsymbol{y}[i]\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right\rangle \\ \vdots \\ \left\langle \boldsymbol{b}_{n-1}\boldsymbol{e}_{n-1}^{\mathsf{T}}, \sum_{i=0}^{n-1} \boldsymbol{y}[i]\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right\rangle \end{bmatrix} = \begin{bmatrix} \|\boldsymbol{b}_{0}\|_{2}^{2} \cdot \boldsymbol{y}[0] \\ \vdots \\ \|\boldsymbol{b}_{n-1}\|_{2}^{2} \cdot \boldsymbol{y}[n-1] \end{bmatrix} \in \mathbb{C}^{n},$$

(3.2) follows immediately from (3.1).

The properties in (3.3) follows directly from the definition of \mathcal{A} . For the left inequality, we have

$$egin{aligned} |\mathcal{A}\mathcal{A}^* - \mathcal{I}|| &= \sup_{oldsymbol{y} \in \mathbb{C}^n: \|oldsymbol{y}\|_2 = 1} \|\mathcal{A}\mathcal{A}^*(oldsymbol{y}) - oldsymbol{y}\|_2 \ &= \sup_{oldsymbol{y} \in \mathbb{C}^n: \|oldsymbol{y}\|_2 = 1} \sqrt{\sum_{i=0}^{n-1} \left(\|oldsymbol{b}_i\|_2^2 - 1
ight)^2 \cdot |oldsymbol{y}[i]|^2} \ &\leq \max_{0 \leq i \leq n-1} \left\|oldsymbol{b}_i\|_2^2 - 1
ight| \ &\leq s \mu_0. \end{aligned}$$

The right one can be proved as follows

$$\begin{split} |\mathcal{A}\| &= \sup_{\boldsymbol{X} \in \mathbb{C}^{s \times n} : \|\boldsymbol{X}\|_{\mathsf{F}} = 1} \|\mathcal{A}(\boldsymbol{X})\|_{2} \\ &= \sup_{\boldsymbol{X} \in \mathbb{C}^{s \times n} : \|\boldsymbol{X}\|_{\mathsf{F}} = 1} \sqrt{\sum_{i=0}^{n-1} |\boldsymbol{b}_{i}^{*} \boldsymbol{X} \boldsymbol{e}_{i}|^{2}} \\ &\leq \sup_{\boldsymbol{X} \in \mathbb{C}^{s \times n} : \|\boldsymbol{X}\|_{\mathsf{F}} = 1} \sqrt{\sum_{i=0}^{n-1} \|\boldsymbol{b}_{i}\|_{2}^{2} \cdot \|\boldsymbol{X} \boldsymbol{e}_{i}\|_{2}^{2}} \\ &\leq \max_{0 \leq i \leq n-1} \|\boldsymbol{b}_{i}\|_{2} \cdot \sup_{\boldsymbol{X} \in \mathbb{C}^{s \times n} : \|\boldsymbol{X}\|_{\mathsf{F}} = 1} \sqrt{\sum_{i=0}^{n-1} \|\boldsymbol{X} \boldsymbol{e}_{i}\|_{2}^{2}} \\ &\leq \sqrt{s\mu_{0}}. \end{split}$$

The proof is now complete.

The following lemma suggests that the smallest singular value of $E_{h,L}$ can be lower bounded by the smallest singular value of E_L .

Lemma 3.3. Recall that $H = \begin{bmatrix} h_1 & \cdots & h_r \end{bmatrix} \in \mathbb{C}^{s \times r}$ and suppose all columns of H are of unit norm. Under the incoherence condition (2.8), we have

$$\sigma_{\min}(\boldsymbol{E}_{\boldsymbol{h},L}^*\boldsymbol{E}_{\boldsymbol{h},L}) \geq \frac{n_1}{\mu_1},$$

where $E_{h,L}$ is the matrix defined in (1.15).

Proof. Let $\boldsymbol{a}_{\tau_{\ell}} = \begin{bmatrix} 1 & e^{-2\pi i \tau_{\ell}} & \cdots & e^{-2\pi i \tau_{\ell} \cdot (n_1 - 1)} \end{bmatrix}^{\mathsf{T}} \in \mathbb{C}^{n_1}$ be the ℓ th column of \boldsymbol{E}_L . Since $\boldsymbol{E}_{\boldsymbol{h},L} = \boldsymbol{E}_L \odot \boldsymbol{H}$, it can be easily seen that

$$egin{aligned} E^*_{m{h},L}E_{m{h},L}&=egin{bmatrix} a^*_{ au_1}\otimes h^*_1\dots\ a^*_{ au_r}\otimes h^*_r \end{bmatrix}egin{aligned} a_{ au_1}\otimes h_1&\cdots&a_{ au_r}\otimes h_r \end{bmatrix}\ &=egin{bmatrix} (a^*_{ au_1}\otimes h^*_1)(a_{ au_1}\otimes h_1)&\cdots&(a^*_{ au_1}\otimes h^*_1)(a_{ au_r}\otimes h_r)\dots\ dots\ dots\$$

$$=egin{bmatrix} egin{aligned} a_{ au_1}a_{ au_1}&\cdots&a_{ au_1}^*a_{ au_r}\dots&do$$

Recall that a selection matrix $\boldsymbol{P} \in \mathbb{R}^{n^2 \times n}$ is the unique matrix such that

$$Pz = \operatorname{vec} (\operatorname{diag}(z)) \text{ for all } z \in \mathbb{C}^n$$

and it has the remarkable property that $P^{\mathsf{T}}(A \otimes B)P = A \circ B$ [57, Corollary 2]. Thus we have

$$\begin{split} \sigma_{\min}(\boldsymbol{E}_{\boldsymbol{h},L}^{*}\boldsymbol{E}_{\boldsymbol{h},L}) &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} |\boldsymbol{\beta}^{*}\left((\boldsymbol{E}_{L}^{*}\boldsymbol{E}_{L})\circ(\boldsymbol{H}^{*}\boldsymbol{H})\right)\boldsymbol{\beta}| \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} |\boldsymbol{\beta}^{*}\boldsymbol{P}^{\mathsf{T}}\left((\boldsymbol{E}_{L}^{*}\boldsymbol{E}_{L})\otimes(\boldsymbol{H}^{*}\boldsymbol{H})\right)\boldsymbol{P}\boldsymbol{\beta}| \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} |\boldsymbol{\beta}^{*}\boldsymbol{P}^{\mathsf{T}}(\boldsymbol{E}_{L}^{*}\otimes\boldsymbol{H}^{*})(\boldsymbol{E}_{L}\otimes\boldsymbol{H})\boldsymbol{P}\boldsymbol{\beta}| \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} \|(\boldsymbol{E}_{L}\otimes\boldsymbol{H})\boldsymbol{P}\boldsymbol{\beta}\|_{2}^{2} \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} \|(\boldsymbol{E}_{L}\otimes\boldsymbol{H})\operatorname{vec}\left(\operatorname{diag}(\boldsymbol{\beta})\right)\|_{2}^{2} \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} \|\operatorname{vec}\left(\boldsymbol{H}\operatorname{diag}(\boldsymbol{\beta})\boldsymbol{E}_{L}^{\mathsf{T}}\right)\|_{2}^{2} \\ &= \inf_{\|\boldsymbol{\beta}\|_{2}=1} \|\boldsymbol{H}\operatorname{diag}(\boldsymbol{\beta})\boldsymbol{E}_{L}^{\mathsf{T}}\right\|_{\mathsf{F}}^{2} \\ &\geq \sigma_{\min}^{2}(\boldsymbol{E}_{L})\cdot\inf_{\|\boldsymbol{\beta}\|_{2}=1} \|\boldsymbol{H}\operatorname{diag}(\boldsymbol{\beta})\|_{\mathsf{F}}^{2} \\ &= \sigma_{\min}^{2}(\boldsymbol{E}_{L})\cdot\inf_{\|\boldsymbol{\beta}\|_{2}=1}\sum_{k=1}^{r} \|\boldsymbol{\beta}[k]\cdot\boldsymbol{h}_{k}\|_{2}^{2} \\ &= \sigma_{\min}^{2}(\boldsymbol{E}_{L})\cdot\inf_{\|\boldsymbol{\beta}\|_{2}=1}\sum_{k=1}^{r} |\boldsymbol{\beta}[k]|^{2} \\ &\geq \frac{n_{1}}{\mu_{1}}, \end{split}$$

which completes the proof.

A straightforward application of Lemma 3.3 yields the following result, which can be regarded as a variant of [7, Lemma 1].

Lemma 3.4. Suppose $\mathcal{H}(\mathbf{X}^{\natural})$ obeys the incoherence condition (2.8) with parameter μ_1 . Let $\mathcal{H}(\mathbf{X}^{\natural}) = \mathbf{U}\mathbf{S}\mathbf{V}^*$ be the singular value decomposition of $\mathcal{H}(\mathbf{X}^{\natural})$, where $\mathbf{U} \in \mathbb{C}^{sn_1 \times r}$, $\mathbf{S} \in \mathbb{R}^{r \times r}$ and $\mathbf{V} \in \mathbb{C}^{n_2 \times r}$. If we rewrite \mathbf{U} as

$$oldsymbol{U} = egin{bmatrix} oldsymbol{U}_0 \ dots \ oldsymbol{U}_{n_1-1} \end{bmatrix},$$

where the ℓ th block is $U_{\ell} = U(\ell s : (\ell + 1)s - 1, :)$ for $\ell = 0, \dots, n_1 - 1$, then

$$\max_{0 \le \ell \le n_1 - 1} \| \boldsymbol{U}_{\ell} \|_{\mathsf{F}}^2 \le \frac{\mu_1 r}{n} \text{ and } \max_{0 \le j \le n_2 - 1} \| \boldsymbol{e}_j^{\mathsf{T}} \boldsymbol{V} \|_2^2 \le \frac{\mu_1 r}{n},$$
(3.4)

Proof. We only need to prove the left inequality in (3.4) as the right one can be similarly established. Recall that $\mathcal{H}(\mathbf{X}^{\natural}) = \mathbf{E}_{\mathbf{h},L} \operatorname{diag}(d_1, \cdots, d_r) \mathbf{E}_R^{\mathsf{T}}$. Since $\mathbf{U} \in \mathbb{C}^{sn_1 \times r}$ and $\mathbf{E}_{\mathbf{h},L}$ span the same subspace and \mathbf{U} is orthogonal, there exists an orthonormal matrix $\mathbf{Q} \in \mathbb{C}^{r \times r}$ such that $\mathbf{U} = \mathbf{E}_{\mathbf{h},L} (\mathbf{E}_{\mathbf{h},L}^* \mathbf{E}_{\mathbf{h},L})^{-1/2} \mathbf{Q}$. Therefore,

$$\begin{split} \|\boldsymbol{U}_{\ell}\|_{\mathsf{F}}^{2} &= \sum_{j=\ell s}^{(\ell+1)s-1} \left\|\boldsymbol{e}_{j}^{\mathsf{T}}\boldsymbol{E}_{\boldsymbol{h},L}(\boldsymbol{E}_{\boldsymbol{h},L}^{*}\boldsymbol{E}_{\boldsymbol{h},L})^{-1/2}\right\|_{2}^{2} \\ &\leq \sum_{j=\ell s}^{(\ell+1)s-1} \left\|\boldsymbol{e}_{j}^{\mathsf{T}}\boldsymbol{E}_{\boldsymbol{h},L}\right\|_{2}^{2} \cdot \left\|(\boldsymbol{E}_{\boldsymbol{h},L}^{*}\boldsymbol{E}_{\boldsymbol{h},L})^{-1/2}\right\|^{2} \\ &\leq \frac{\mu_{1}}{n} \cdot \sum_{j=\ell s}^{(\ell+1)s-1} \left\|\boldsymbol{e}_{j}^{\mathsf{T}}\boldsymbol{E}_{\boldsymbol{h},L}\right\|_{2}^{2} \\ &= \frac{\mu_{1}}{n} \cdot \sum_{k=1}^{r} \left\|\boldsymbol{e}^{-2\pi i \tau_{k} \cdot \ell} \boldsymbol{h}_{k}\right\|_{2}^{2} \\ &= \frac{\mu_{1}r}{n}, \end{split}$$

where the second inequality is due to Lemma 3.3.

The following corollary is a direct consequence of Lemma 3.4 and will be frequently used in the sequel. Corollary 3.5. Suppose $\mathcal{H}(X^{\natural})$ obeys the incoherence condition (2.8) with parameter μ_1 . Then,

$$\max_{\substack{0 \le i \le n-1 \\ 0 \le i \le n-1 \\ 0 \le j \le n_2 - 1 }} \frac{1}{w_i} \sum_{\substack{\ell+j=i \\ 0 \le \ell \le n_1 - 1 \\ 0 \le j \le n_2 - 1 }} \|U_\ell\|_{\mathsf{F}}^2 \le \frac{\mu_1 r}{n} \text{ and } \max_{\substack{0 \le i \le n-1 \\ 0 \le i \le n-1 \\ 0 \le j \le n_2 - 1 }} \frac{1}{w_i} \sum_{\substack{\ell+j=i \\ 0 \le \ell \le n_1 - 1 \\ 0 \le j \le n_2 - 1 }} \|e_j^\mathsf{T} V\|_2^2 \le \frac{\mu_1 r}{n}.$$
(3.5)

The matrix Bernstein inequality, stated below, will be used frequently in our analysis.

Lemma 3.6 ([55, 15]). Let $\{X_{\ell}\}_{\ell=1}^{n}$ be a set independent random matrices of dimension $n_1 \times n_2$, which satisfy $\mathbb{E} [X_{\ell}] = 0$ and $||X_{\ell}|| \leq B$. Define

$$\sigma^{2} := \max\left\{ \left\| \mathbb{E}\left[\sum_{\ell=1}^{n} \boldsymbol{X}_{\ell} \boldsymbol{X}_{\ell}^{*} \right] \right\|, \left\| \mathbb{E}\left[\sum_{\ell=1}^{n} \boldsymbol{X}_{\ell}^{*} \boldsymbol{X}_{\ell} \right] \right\| \right\}.$$

Then the event

$$\left\|\sum_{\ell=1}^{n} \boldsymbol{X}_{\ell}\right\| \le c \left(\sqrt{\sigma^2 \log(n_1 + n_2)} + B \log(n_1 + n_2)\right)$$
(3.6)

holds with probability at least $1 - (n_1 + n_2)^{-c_1}$, where $c, c_1 > 0$ are absolute constants.

3.2 Deterministic optimality condition

As is typical in the analysis of low rank matrix recovery, in order to show that Z^{\natural} is the unique optimal solution to the convex program (2.4), we need to construct a dual certificate which satisfies a set of sufficient conditions. These conditions can be viewed as a variant of the KKT condition for the optimality of Z^{\natural} . Recall that the singular value decomposition (SVD) of $\mathcal{H}(X^{\natural})$ is $\mathcal{H}(X^{\natural}) = USV^*$. The tangent space T of the nuclear norm at $\mathcal{H}(X^{\natural})$ can be defined as

$$T = \left\{ \boldsymbol{U}\boldsymbol{A}^* + \boldsymbol{B}\boldsymbol{V}^* : \boldsymbol{A} \in \mathbb{C}^{n_2 \times r}, \boldsymbol{B} \in \mathbb{C}^{sn_1 \times r} \right\}.$$

The projections $\mathcal{P}_T(\mathbf{Z})$ onto the tangent space can be defined as

$$\mathcal{P}_T(\boldsymbol{Z}) := \boldsymbol{U}\boldsymbol{U}^*\boldsymbol{Z} + \boldsymbol{Z}\boldsymbol{V}\boldsymbol{V}^* - \boldsymbol{U}\boldsymbol{U}^*\boldsymbol{Z}\boldsymbol{V}\boldsymbol{V}^*.$$
(3.7)

and the corresponding projector onto the orthogonal complement of T is given by $\mathcal{P}_{T^{\perp}}(\mathbf{Z}) = \mathbf{Z} - \mathcal{P}_{T}(\mathbf{Z}).$

Theorem 3.7. Suppose $\|AA^*\| \ge 1$ and

$$\|\mathcal{P}_T \mathcal{G} \mathcal{A}^* \mathcal{A} \mathcal{G}^* \mathcal{P}_T - \mathcal{P}_T \mathcal{G} \mathcal{G}^* \mathcal{P}_T\| \le \frac{1}{2}.$$
(3.8)

If there exists a dual certificate $\Lambda \in \mathbb{C}^{sn_1 imes n_2}$ such that

$$\|\mathcal{P}_T(\boldsymbol{U}\boldsymbol{V}^* - \boldsymbol{\Lambda})\|_{\mathsf{F}} \le \frac{1}{16s\mu_0},\tag{3.9}$$

$$\|\mathcal{P}_{T^{\perp}}(\mathbf{\Lambda})\| \le \frac{1}{2},\tag{3.10}$$

$$\mathcal{G}^*(\mathbf{\Lambda}) \in \operatorname{Range}(\mathcal{A}^*),$$
(3.11)

then Z^{\natural} is the unique solution to (2.4).

Proof. The structure of the proof is overall similar to those in [15, 16, 14]. Consider any feasible solution $Z^{\natural} + M$, where the perturbation $M \in \mathbb{C}^{sn_1 \times n_2}$ satisfies

$$\mathcal{AG}^*(M) = 0, \tag{3.12}$$

$$(\mathcal{I} - \mathcal{G}\mathcal{G}^*)(\boldsymbol{M}) = 0. \tag{3.13}$$

The first condition (3.12) implies that $\mathcal{G}^*(M)$ is in the null space of \mathcal{A} , while the second condition (3.13) guarantees that M has the vectorized Hankel structure. Note that for any matrix M, there exists an $sn_1 \times n_2$ matrix $S \in T^{\perp}$ such that

$$\langle \boldsymbol{M}, \boldsymbol{S} \rangle = \| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{*} \text{ and } \| \boldsymbol{S} \| \leq 1.$$

In the meantime, we have $UV^* + S \in \partial \|Z^{\natural}\|_*$. Thus,

$$\begin{aligned} \Delta &:= \left\| \boldsymbol{Z}^{\natural} + \boldsymbol{M} \right\|_{*} - \left\| \boldsymbol{Z}^{\natural} \right\|_{*} \\ &\geq \langle \boldsymbol{U}\boldsymbol{V}^{*} + \boldsymbol{S}, \boldsymbol{M} \rangle \\ &= \langle \boldsymbol{U}\boldsymbol{V}^{*}, \boldsymbol{M} \rangle + \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{*} \\ &\geq \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{*} - \left| \langle \boldsymbol{U}\boldsymbol{V}^{*} - \boldsymbol{\Lambda}, \boldsymbol{M} \rangle \right| - \left| \langle \boldsymbol{\Lambda}, \boldsymbol{M} \rangle \right|. \end{aligned}$$
(3.14)

The condition (3.11) directly implies that there exists a vector $\boldsymbol{p} \in \mathbb{C}^n$ such that

$$\mathcal{G}^*(\boldsymbol{\Lambda}) = \mathcal{A}^*(\boldsymbol{p}).$$

Therefore, combining (3.11) and (3.13), we obtain

$$|\langle \mathbf{\Lambda}, \mathbf{M} \rangle| = |\langle \mathbf{\Lambda}, \mathcal{GG}^*(\mathbf{M}) \rangle| = |\langle \mathcal{G}^*(\mathbf{\Lambda}), \mathcal{G}^*(\mathbf{M}) \rangle| = |\langle \mathcal{A}^*(\mathbf{p}), \mathcal{G}^*(\mathbf{M}) \rangle| = \langle \mathbf{p}, \mathcal{AG}^*(\mathbf{M}) \rangle = 0.$$

Moreover, the second term of (3.14) can be upper bounded as follows:

$$\begin{split} |\langle \boldsymbol{U}\boldsymbol{V}^* - \boldsymbol{\Lambda}, \boldsymbol{M} \rangle| &\leq |\langle \mathcal{P}_T(\boldsymbol{U}\boldsymbol{V}^* - \boldsymbol{\Lambda}), \boldsymbol{M} \rangle| + |\langle \mathcal{P}_{T^{\perp}}(\boldsymbol{U}\boldsymbol{V}^* - \boldsymbol{\Lambda}), \boldsymbol{M} \rangle| \\ &\leq \|\mathcal{P}_T(\boldsymbol{U}\boldsymbol{V}^* - \boldsymbol{\Lambda})\|_{\mathsf{F}} \cdot \|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}} + \|\mathcal{P}_{T^{\perp}}(\boldsymbol{\Lambda})\| \cdot \|\mathcal{P}_{T^{\perp}}(\boldsymbol{M})\|_{*} \\ &\leq \frac{1}{16s\mu_0} \cdot \|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}} + \frac{1}{2} \cdot \|\mathcal{P}_{T^{\perp}}(\boldsymbol{M})\|_{*} \,, \end{split}$$

where the last step is due to (3.9) and (3.10). Consequently,

$$\begin{split} \Delta &\geq \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{*} - \left| \langle \boldsymbol{U}\boldsymbol{V}^{*} - \boldsymbol{\Lambda}, \boldsymbol{M} \rangle \right| - \left| \langle \boldsymbol{\Lambda}, \boldsymbol{M} \rangle \right| \\ &\geq \frac{1}{2} \cdot \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{*} - \frac{1}{16s\mu_{0}} \cdot \left\| \mathcal{P}_{T}(\boldsymbol{M}) \right\|_{\mathsf{F}} \\ &\geq \frac{1}{2} \cdot \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{\mathsf{F}} - \frac{1}{16s\mu_{0}} \cdot \left\| \mathcal{P}_{T}(\boldsymbol{M}) \right\|_{\mathsf{F}} \\ &\geq \left(\frac{1}{2} - \frac{1}{16s\mu_{0}} \cdot 4s\mu_{0} \right) \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{\mathsf{F}} \\ &= \frac{1}{4} \left\| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \right\|_{\mathsf{F}}, \end{split}$$

where the fourth line is due to Lemma 6.1 in Section 6. It follows that $\Delta > 0$ unless $\|\mathcal{P}_{T^{\perp}}(M)\|_{\mathsf{F}} = 0$. Note that $\Delta = 0$ requires $\mathcal{P}_{T^{\perp}}(M) = 0$, which in turn requires $M = \mathcal{P}_T(M)$. In this case, we have

$$\begin{split} \|\mathcal{P}_{T}(\boldsymbol{M})\|_{\mathsf{F}}^{2} &= \langle \mathcal{P}_{T}(\boldsymbol{M}), \mathcal{M} \rangle \\ &= \langle \mathcal{P}_{T}(\boldsymbol{M}), \mathcal{G}\mathcal{G}^{*}(\boldsymbol{M}) \rangle \\ &= \langle \boldsymbol{M}, \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) - \mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) \rangle + \langle \boldsymbol{M}, \mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) \rangle \\ &= \langle \boldsymbol{M}, \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) - \mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) \rangle + \langle \boldsymbol{M}, \mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}(\boldsymbol{M}) \rangle \\ &= \langle \boldsymbol{M}, \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) - \mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{M}) \rangle \\ &\leq \|\mathcal{P}_{T}\mathcal{G}\mathcal{A}^{*}\mathcal{A}\mathcal{G}^{*}\mathcal{P}_{T} - \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T} \| \cdot \|\mathcal{P}_{T}(\boldsymbol{M})\|_{\mathsf{F}}^{2} \\ &\leq \frac{1}{2} \|\mathcal{P}_{T}(\boldsymbol{M})\|_{\mathsf{F}}^{2}, \end{split}$$

which implies that $\mathcal{P}_T(M) = 0$. Thus Z^{\natural} is the unique minimizer.

3.3 Constructing the dual certificate

It is intuitively clear that we may construct a dual certificate $\Lambda \in \mathbb{C}^{sn_1 \times n_2}$ obeying the conditions (3.9), (3.10) and (3.11) by solving the following constrained least squares problem:

$$\min_{\boldsymbol{\Lambda}} \|\mathcal{P}_T(\boldsymbol{U}\boldsymbol{V}^*-\boldsymbol{\Lambda})\|_{\mathsf{F}}^2 \text{ s.t. } \mathcal{G}^*(\boldsymbol{\Lambda}) \in \operatorname{Range}(\mathcal{A}^*).$$

Here only the conditions (3.9) and (3.11) are taken into account because once $\|\mathcal{P}_T(UV^* - \Lambda)\|_{\mathsf{F}}$ is small, the projection of Λ onto T^{\perp} can be simultaneously small.

Applying the projected gradient method to solve the above optimization problem, we obtain the following update rule:

$$\mathbf{Y}^k = \mathbf{Y}^{k-1} + (\mathcal{GA}^* \mathcal{AG}^* + \mathcal{I} - \mathcal{GG}^*) \mathcal{P}_T(\mathbf{UV}^* - \mathbf{Y}^{k-1}).$$

However, due to the statistical dependence among the iterations, the convergence analysis of the vanilla gradient iteration is difficult. Therefore, the golfing scheme [28] proposes to break the statistical independence by dividing all the linear measurements into a few disjoint partitions and use a fresh partition in each iteration.

Assume we divide the linear measurements in (1.8) into k_0 partitions, denoted $\{\Omega_k\}_{k=1}^{k_0}$, and let $m = \frac{n}{k_0}$. Define

$$\mathcal{A}_{k}(\boldsymbol{X}) = \left\{ \left\langle \boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}, \boldsymbol{X} \right\rangle \right\}_{i \in \Omega_{k}} \in \mathbb{C}^{|\Omega_{k}|}$$

$$(3.15)$$

and

$$\mathcal{A}_{k}^{*}\mathcal{A}_{k}(\boldsymbol{X}) = \sum_{i \in \Omega_{k}} \left\langle \boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}, \boldsymbol{X} \right\rangle \boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} = \sum_{i \in \Omega_{k}} \boldsymbol{b}_{i} \boldsymbol{b}_{i}^{\mathsf{T}} \boldsymbol{X} \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \in \mathbb{C}^{s \times n}.$$
(3.16)

Then the golfing scheme for the construction of Λ satisfying the conditions in Theorem 3.7 can be formally expressed as

$$\mathbf{Y}^{0} = \mathbf{0} \in \mathbb{C}^{sn_{1} \times n_{2}},$$

$$\mathbf{Y}^{k} = \mathbf{Y}^{k-1} + \left(\frac{n}{m}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*} + \mathcal{I} - \mathcal{G}\mathcal{G}^{*}\right)\mathcal{P}_{T}(\mathbf{U}\mathbf{V}^{*} - \mathbf{Y}^{k-1}), \quad \text{for } k = 1, \cdots, k_{0},$$

$$\mathbf{\Lambda} := \mathbf{Y}^{k_{0}}.$$
(3.17)

Evidently the property of Λ relies on the partitions $\{\Omega_k\}_{k=1}^{k_0}$. In order to construct the desirable Λ , we require $\{\Omega_k\}_{k=1}^{k_0}$ to satisfy a set of conditions list in the following lemma, in which we have

$$\|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} = \sqrt{\sum_{i=0}^{n-1} \frac{\|\mathcal{G}^*(\boldsymbol{Z})e_i\|_2^2}{w_i}} \quad \text{and} \quad \|\boldsymbol{Z}\|_{\mathcal{G},\infty} = \max_{0 \le i \le n-1} \frac{\|\mathcal{G}^*(\boldsymbol{Z})e_i\|_2}{\sqrt{w_i}} \quad \text{for any } \boldsymbol{Z} \in \mathbb{C}^{sn_1 \times n_2}.$$
(3.18)

The proof of this lemma will be presented in Section 4.

Lemma 3.8. Let $k_0 \in \{1, \dots, n\}$ and set $m = \frac{n}{k_0}$. If $n \gtrsim k_0 \cdot \max\{\mu_1 r \log(sn), \log(k_0)\}$, then there exists a partition $\{\Omega_k\}_{k=1}^{k_0}$ such that the following properties hold :

$$\frac{m}{2} \le |\Omega_k| \le \frac{3m}{2}, \quad k = 1, \cdots, k_0,$$
(3.19)

$$\max_{1 \le k \le k_0} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* \mathcal{P}_T \right\| \le \frac{1}{4},$$
(3.20)

$$\max_{1 \le k \le k_0} \left\| \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* (\mathbf{Z}) \right\| \lesssim \left(\sqrt{\frac{n \log(sn)}{m}} \, \| \mathbf{Z} \|_{\mathcal{G}, \mathsf{F}} + \frac{n \log(sn)}{m} \, \| \mathbf{Z} \|_{\mathcal{G}, \infty} \right), \tag{3.21}$$

$$\max_{1 \le k \le k_0} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* (\mathbf{Z}) \right\|_{\mathcal{G}, \mathsf{F}} \lesssim \sqrt{\frac{\mu_1 r \log(sn)}{n}} \left(\sqrt{\frac{n \log(sn)}{m}} \, \|\mathbf{Z}\|_{\mathcal{G}, \mathsf{F}} + \frac{n \log(sn)}{m} \, \|\mathbf{Z}\|_{\mathcal{G}, \mathsf{F}} \right), \tag{3.22}$$

$$\max_{1 \le k \le k_0} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* (\mathbf{Z}) \right\|_{\mathcal{G},\infty} \lesssim \frac{\mu_1 r}{n} \left(\sqrt{\frac{n \log(sn)}{m}} \left\| \mathbf{Z} \right\|_{\mathcal{G},\mathsf{F}} + \frac{n \log(sn)}{m} \left\| \mathbf{Z} \right\|_{\mathcal{G},\infty} \right).$$
(3.23)

Here $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$ is fixed. Recalling the definition of the operator $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16), the expectation is taken with respect to $\{\mathbf{b}_i\}_{i \in \Omega_k}$.

3.4 Validating the dual certificate and completing the proof

In this section we show that the dual certificate Λ constructed from the iteration (3.17) satisfies the conditions in Theorem 3.7. The result follows from several lemmas that will be proved in Section 5. In these lemmas, $\{\Omega_k\}_{k=1}^{k_0}$ is a partition of $\{1, \dots, n\}$ satisfying the conditions in Lemma 3.8, and $\{\mathcal{A}_k\}_{k=1}^{k_0}$ are the associated linear operators defined in (3.15). Note that we assume (3.4) holds in the remainder of this paper, which follows from Assumption 2.2 and Lemma 3.4.

Lemma 3.9. Assume $n \gtrsim k_0 s \mu_0 \cdot \mu_1 r \log(sn)$. Under the condition (III.18) of Lemma 3.8, the event

$$\max_{1 \le k \le k_0} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_k^* \mathcal{A}_k \right) \mathcal{G}^* \mathcal{P}_T \right\| \le \frac{1}{2}$$
(3.24)

occurs with probability at least $1 - (sn)^{-c_1}$ for a universal constant $c_1 > 0$.

The following corollary is the special case of Lemma 3.9 when $k_0 = 1$ and n = m.

Corollary 3.10. Assume $n \gtrsim s\mu_0 \cdot \mu_1 r \log(sn)$. The event

$$\|\mathcal{P}_T \mathcal{G} \mathcal{A}^* \mathcal{A} \mathcal{G}^* \mathcal{P}_T - \mathcal{P}_T \mathcal{G} \mathcal{G}^* \mathcal{P}_T \| \le \frac{1}{2}$$
(3.25)

occurs with probability at least $1 - (sn)^{-c_1}$ for a universal constant $c_1 > 0$.

Lemma 3.11. Under the condition (III.19) of Lemma 3.8, for any $1 \le k \le k_0$ and fixed $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, the event

$$\left\| \left(\frac{n}{m} \mathcal{G} \mathcal{A}_k^* \mathcal{A}_k \mathcal{G}^* - \mathcal{G} \mathcal{G}^* \right) (\mathbf{Z}) \right\| \lesssim \sqrt{\frac{4nk_0 s\mu_0 \log(sn)}{m}} \, \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_0 \log(sn)}{m} \, \|\mathbf{Z}\|_{\mathcal{G},\infty} \tag{3.26}$$

occurs with probability at least $1 - (sn)^{-c_1}$ for a universal constant $c_1 > 0$.

Lemma 3.12. Under the condition (III.20) of Lemma 3.8, for any $1 \le k \le k_0$ and fixed $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, the event

$$\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right)\mathcal{G}^{*}(\boldsymbol{Z})\right\|_{\mathcal{G},\mathsf{F}} \lesssim \sqrt{\frac{\mu_{1}r\log(sn)}{n}} \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\boldsymbol{Z}\|_{\mathcal{G},\infty}\right)$$
(3.27)

occurs with probability at least $1 - (sn)^{-c_1}$ for a universal constant $c_1 > 0$.

Lemma 3.13. Under the condition (III.21) of Lemma 3.8, for any $1 \le k \le k_0$ and fixed $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, the event

$$\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \right) \mathcal{G}^{*}(\boldsymbol{Z}) \right\|_{\mathcal{G},\infty} \lesssim \frac{\mu_{1} r}{n} \left(\sqrt{\frac{4nk_{0} s\mu_{0} \log(sn)}{m}} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0} \log(sn)}{m} \|\boldsymbol{Z}\|_{\mathcal{G},\infty} \right)$$
(3.28)

occurs with probability at least $1 - ns^{-c_2}$ for a numerical constant $c_2 > 2$.

Lemma 3.14. Recalling that U and V satisfy (3.4), we have

$$\|\boldsymbol{U}\boldsymbol{V}^*\|_{\mathcal{G},\mathsf{F}}^2 \lesssim \frac{\mu_1 r \log(sn)}{n} \quad and \quad \|\boldsymbol{U}\boldsymbol{V}^*\|_{\mathcal{G},\infty} \le \frac{\mu_1 r}{n}.$$
(3.29)

Equipped with these lemmas, we are in position to validate the conditions in Theorem 3.7. Note that $\|\mathcal{AA}^*\| \geq 1$ holds due to (3.2) in Lemma 3.2, and (3.8) is proved in Corollary 3.10. As for (3.11), it follows immediately from the construction of Λ . Thus, it remains to validate (3.9) and (3.10).

Validating (3.9) A simple calculation yields that

$$\mathbf{E}^{k} := \mathcal{P}_{T} \left(\mathbf{U} \mathbf{V}^{*} - \mathbf{Y}^{k} \right)
= \mathcal{P}_{T} \left(\mathbf{U} \mathbf{V}^{*} - \mathbf{Y}^{k-1} - \left(\frac{n}{m} \mathcal{G} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \mathcal{G}^{*} + \mathcal{I} - \mathcal{G} \mathcal{G}^{*} \right) \mathcal{P}_{T} (\mathbf{E}^{k-1}) \right)
= \mathcal{P}_{T} (\mathbf{E}^{k-1}) - \mathcal{P}_{T} \left(\frac{n}{m} \mathcal{G} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \mathcal{G}^{*} + \mathcal{I} - \mathcal{G} \mathcal{G}^{*} \right) \mathcal{P}_{T} (\mathbf{E}^{k-1})
= \mathcal{P}_{T} \left(\mathcal{G} \mathcal{G}^{*} - \frac{n}{m} \mathcal{G} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \mathcal{G}^{*} \right) \mathcal{P}_{T} (\mathbf{E}^{k-1}),$$
(3.30)

where the second line is due to (3.17). By the construction of Λ , we can obtain

$$\begin{aligned} \left\| \mathcal{P}_{T} \left(\boldsymbol{U} \boldsymbol{V}^{*} - \boldsymbol{\Lambda} \right) \right\|_{\mathsf{F}} &= \left\| \boldsymbol{E}^{k_{0}} \right\|_{\mathsf{F}} \\ &= \left\| \mathcal{P}_{T} \left(\mathcal{G} \mathcal{G}^{*} - \frac{n}{m} \mathcal{G} \mathcal{A}_{k_{0}}^{*} \mathcal{A}_{k_{0}} \mathcal{G}^{*} \right) \mathcal{P}_{T} (\boldsymbol{E}^{k_{0}-1}) \right\|_{\mathsf{F}} \end{aligned}$$

$$\begin{split} &\leq \left\| \mathcal{P}_T \left(\mathcal{G}\mathcal{G}^* - \frac{n}{m} \mathcal{G}\mathcal{A}_{k_0}^* \mathcal{A}_{k_0} \mathcal{G}^* \right) \mathcal{P}_T \right\| \cdot \left\| \boldsymbol{E}^{k_0 - 1} \right\|_{\mathsf{F}} \\ &\leq \frac{1}{2} \left\| \boldsymbol{E}^{k_0 - 1} \right\|_{\mathsf{F}} \leq \frac{1}{2^{k_0}} \left\| \boldsymbol{E}^0 \right\|_{\mathsf{F}} \\ &= \frac{1}{2^{k_0}} \left\| \boldsymbol{U} \boldsymbol{V}^* \right\|_{\mathsf{F}} \leq \frac{r}{2^{k_0}} \\ &\leq \frac{1}{16s\mu_0}, \end{split}$$

where step (a) is due to Lemma 3.9 and the last inequality holds when $k_0 = \lceil \log_2(16rs\mu_0) \rceil$.

Validating (3.10) First recall that $\boldsymbol{E}^{k} := \mathcal{P}_{T} \left(\boldsymbol{U} \boldsymbol{V}^{*} - \boldsymbol{Y}^{k} \right)$. According to (3.17), we have

$$\mathbf{\Lambda} = \sum_{k=1}^{k_0} \left(\frac{n}{m} \mathcal{G} \mathcal{A}_k^* \mathcal{A}_k \mathcal{G}^* + \mathcal{I} - \mathcal{G} \mathcal{G}^* \right) (\mathbf{E}^{k-1}).$$

Then it follows that

$$\|\mathcal{P}_{T^{\perp}}(\mathbf{\Lambda})\| = \left\| \mathcal{P}_{T^{\perp}}\left(\sum_{k=1}^{k_{0}} \left(\frac{n}{m}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*} + \mathcal{I} - \mathcal{G}\mathcal{G}^{*}\right)(\mathbf{E}^{k-1})\right) \right\|$$
$$= \left\| \mathcal{P}_{T^{\perp}}\left(\sum_{k=1}^{k_{0}} \left(\frac{n}{m}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*} - \mathcal{G}\mathcal{G}^{*}\right)(\mathbf{E}^{k-1})\right) \right\|$$
$$\leq \sum_{k=1}^{k_{0}} \left\| \left(\frac{n}{m}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*} - \mathcal{G}\mathcal{G}^{*}\right)(\mathbf{E}^{k-1}) \right\|, \qquad (3.31)$$

where the second line follows from the fact that $E^{k-1} \in T$.

For any $1 \le k \le k_0$, Lemma 3.11 implies that

$$\left\| \left(\frac{n}{m} \mathcal{G} \mathcal{A}_k^* \mathcal{A}_k \mathcal{G}^* - \mathcal{G} \mathcal{G}^* \right) (\mathbf{E}^{k-1}) \right\| \lesssim \sqrt{\frac{4nk_0 s\mu_0 \log(sn)}{m}} \left\| \mathbf{E}^{k-1} \right\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_0 \log(sn)}{m} \left\| \mathbf{E}^{k-1} \right\|_{\mathcal{G},\infty}.$$
 (3.32)

Recalling from the equality (3.30), we have

$$\boldsymbol{E}^{k-1} = \mathcal{P}_T \left(\mathcal{G}\mathcal{G}^* - \frac{n}{m} \mathcal{G}\mathcal{A}_{k-1}^* \mathcal{A}_{k-1} \mathcal{G}^* \right) \mathcal{P}_T(\boldsymbol{E}^{k-2}).$$

Applying Lemma 3.12 and Lemma 3.13 yields that

$$\begin{aligned} \left\| \boldsymbol{E}^{k-1} \right\|_{\mathcal{G},\mathsf{F}} &= \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k-1}^{*} \mathcal{A}_{k-1} \right) \mathcal{G}^{*} \mathcal{P}_{T} (\boldsymbol{E}^{k-2}) \right\|_{\mathcal{G},\mathsf{F}} \\ &= \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k-1}^{*} \mathcal{A}_{k-1} \right) \mathcal{G}^{*} (\boldsymbol{E}^{k-2}) \right\|_{\mathcal{G},\mathsf{F}} \\ &\lesssim \sqrt{\frac{\mu_{1} r \log(sn)}{n}} \left(\sqrt{\frac{4nk_{0} s \mu_{0} \log(sn)}{m}} \left\| \boldsymbol{E}^{k-2} \right\|_{\mathcal{G},\mathsf{F}} + \frac{2ns \mu_{0} \log(sn)}{m} \left\| \boldsymbol{E}^{k-2} \right\|_{\mathcal{G},\infty} \right) \end{aligned}$$
(3.33)

and

$$\begin{aligned} \left\| \boldsymbol{E}^{k-1} \right\|_{\mathcal{G},\infty} &= \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k-1}^* \mathcal{A}_{k-1} \right) \mathcal{G}^* \mathcal{P}_T (\boldsymbol{E}^{k-2}) \right\|_{\mathcal{G},\infty} \\ &= \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k-1}^* \mathcal{A}_{k-1} \right) \mathcal{G}^* (\boldsymbol{E}^{k-2}) \right\|_{\mathcal{G},\infty} \end{aligned}$$

$$\lesssim \frac{\mu_1 r}{n} \left(\sqrt{\frac{4nk_0 s\mu_0 \log(sn)}{m}} \left\| \boldsymbol{E}^{k-2} \right\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_0 \log(sn)}{m} \left\| \boldsymbol{E}^{k-2} \right\|_{\mathcal{G},\infty} \right).$$
(3.34)

After substituting (3.33) and (3.34) into (3.32), we have

$$\begin{split} \left\| \left(\frac{n}{m} \mathcal{G} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \mathcal{G}^{*} - \mathcal{G} \mathcal{G}^{*}\right) (\mathbf{E}^{k-1}) \right\| &\lesssim \sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{k-1}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{k-1}\|_{\mathcal{G},\infty} \\ &\lesssim \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \cdot \sqrt{\frac{\mu_{1}r\log(sn)}{n}} + \frac{2ns\mu_{0}\log(sn)}{m} \cdot \frac{\mu_{1}r}{n} \right) \\ &\cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\infty} \right) \\ &= \left(\sqrt{\frac{4k_{0}s\mu_{0}\mu_{1}r\log^{2}(sn)}{m}} + \frac{2s\mu_{0}\mu_{1}r\log(sn)}{m} \right) \\ &\cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\infty} \right) \\ &\leq \frac{1}{2} \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{k-2}\|_{\mathcal{G},\infty} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{E}^{0}\|_{\mathcal{G},\mathsf{F}} \right) \\ &\leq \left(\frac{1}{2}\right)^{k-1} \cdot \left($$

where step (a) holds provided $m \gtrsim k_0 s \mu_0 \mu_1 r \log^2(sn)$. Finally, noting that $E^0 = UV^*$, the application of Lemma 3.14 gives

$$\begin{aligned} \|\mathcal{P}_{T^{\perp}}(\mathbf{\Lambda})\| &\leq \sum_{k=1}^{k_0} \left\| \left(\frac{n}{m} \mathcal{G}\mathcal{A}_k^* \mathcal{A}_k \mathcal{G}^* - \mathcal{G}\mathcal{G}^* \right) (\mathbf{E}^{k-1}) \right\| \\ &\leq \sum_{k=1}^{k_0} \left(\frac{1}{2} \right)^{k-1} \cdot \left(\sqrt{\frac{4nk_0 s\mu_0 \log(sn)}{m}} \, \|\mathbf{E}^0\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_0 \log(sn)}{m} \, \|\mathbf{E}^0\|_{\mathcal{G},\infty} \right) \\ &\lesssim \frac{1}{2} \left(\sqrt{\frac{4k_0 s\mu_0 \mu_1 r \log^2(sn)}{m}} + \frac{2s\mu_0 \mu_1 r \log(sn)}{m} \right) \\ &\leq \frac{1}{2} \end{aligned}$$

when $m \gtrsim k_0 s \mu_0 \mu_1 r \log^2(sn)$, where the first inequality follows from (3.31).

Thus we have shown that the dual certificate Λ constructed from the iteration (3.17) satisfies the conditions in Theorem 3.7 with probability at least $1-c_0(sn)^{-c_1}-ns^{-c_2}$ provided that $n = mk_0 \gtrsim \mu_0 \mu_1 \cdot sr \log^4(sn)$. Corollary 3.10 implies (3.8) holds with probability at least $1 - (sn)^{-c_3}$ if $n \gtrsim \mu_0 \mu_1 \cdot sr \log(sn)$. Taking an upper bound on the number of measurements completes the proof of Theorem 2.1.

Proof of Lemma 3.8 4

In this section, we will use probabilistic argument to show that the events (3.19) - (3.23) occur with high probability if we construct $\{\Omega_k\}_{k=1}^{k_0}$ in a random manner and thus conclude that there at least exists a partition satisfying (3.19) - (3.23).

Let $\{\epsilon_i\}_{i=0}^{n-1}$ be *n* independent random variables, each of which takes value in $\{1, \dots, k_0\}$ uniformly at random. For any $k \in \{1, \dots, k_0\}$, we construct $\{\Omega_k\}_{k=1}^{k_0}$ as follows:

$$\Omega_k = \{ i \in [n] : \epsilon_i = k \}.$$

Clearly, $\{\Omega_k\}_{k=1}^{k_0}$ form a partition of [n]. For any fixed $k \in \{1, \dots, k_0\}$, we also have

$$\mathbb{P}\left\{i \in \Omega_k\right\} = \mathbb{P}\left\{\epsilon_i = k\right\} = \frac{1}{k_0} \quad \text{for all } i = 0, \cdots, n-1.$$

Therefore $|\Omega_k|$ can be viewed as the sum of Bernoulli random variables, i.e.,

$$|\Omega_k| = \sum_{i=0}^{n-1} \mathbf{1}\{i \in \Omega_k\} =: \sum_{i=0}^{n-1} \delta_i,$$
(4.1)

where $\{\delta_i\}_{i=0}^{n-1}$ are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{k_0} = \frac{m}{n}$. The application of the Hoeffding inequality yields that $\frac{m}{2} \leq |\Omega_k| \leq \frac{3m}{2}$ holds with probability at least $1-2\exp(-cm)$ for a universal constant c > 0. Then we can take the uniform bound to obtain

$$\mathbb{P}\left\{\frac{m}{2} \le |\Omega_k| \le \frac{3m}{2} \text{ for all } k\right\} \ge 1 - 2k_0 \exp(-cm) \ge \frac{1}{2},$$

where the last inequality is due to $m = \frac{n}{k_0} \gtrsim \log(k_0)$. Our next goal is to show that the events (3.20) - (3.23) occur with high probability. We will first apply the matrix Bernstein inequality (3.6) to obtain the desired upper bounds for fixed k, and then take the uniform bound analysis to complete the proof.

4.1 Proof of (3.20)

For any $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, by the definition of $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16), we have

$$\mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k
ight] \mathcal{G}^* \mathcal{P}_T(oldsymbol{Z}) = \mathbb{E} \left[\sum_{i \in \Omega_k} \left\langle oldsymbol{b}_i oldsymbol{e}_i^\mathsf{T}, \mathcal{G}^* \mathcal{P}_T(oldsymbol{Z})
ight
angle oldsymbol{b}_i oldsymbol{e}_i^\mathsf{T}
ight]
onumber \ = \sum_{i \in \Omega_k} \mathbb{E} \left[oldsymbol{b}_i oldsymbol{b}_i^*
ight] \mathcal{G}^* \mathcal{P}_T(oldsymbol{Z}) oldsymbol{e}_i oldsymbol{e}_i^\mathsf{T}
onumber \ = \mathcal{G}^* \mathcal{P}_T(oldsymbol{Z}) \sum_{i \in \Omega_k} oldsymbol{e}_i oldsymbol{e}_i^\mathsf{T},$$

where the third line follows from the isotropy property (2.5) of $\{b_i\}$.

As a result, one has the following equality

$$\begin{aligned} \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{1}{p}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}\mathcal{P}_{T} \right\| &= \sup_{\left\|\mathbf{W}\right\|_{\mathsf{F}}=1} \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{1}{p}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}\mathcal{P}_{T}(\mathbf{W}) \right\|_{\mathsf{F}} \\ &= \sup_{\left\|\mathbf{W}\right\|_{\mathsf{F}}=1} \left\| \frac{1}{p}\mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T}(\mathbf{W})\sum_{i\in\Omega_{k}}\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}} - \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*}\mathcal{P}_{T}(\mathbf{W}) \right\|_{\mathsf{F}} \\ &= \sup_{\left\|\mathbf{W}\right\|_{\mathsf{F}}=1} \left\| \sum_{i=0}^{n-1}\left(\frac{\delta_{i}}{p} - 1\right)\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}\mathcal{P}_{T}(\mathbf{W})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \right\|_{\mathsf{F}} \\ &=: \left\| \sum_{i=0}^{n-1}\left(\frac{\delta_{i}}{p} - 1\right)\mathcal{X}_{i} \right\|, \end{aligned}$$

where δ_i is the Bernoulli random variable defined in (4.1) and \mathcal{X}_i is the operator defined as

$$\mathcal{X}_i(\boldsymbol{W}) = \mathcal{P}_T \mathcal{G} \left(\mathcal{G}^* \mathcal{P}_T(\boldsymbol{W}) \boldsymbol{e}_i \boldsymbol{e}_i^\mathsf{T} \right)$$

for any $W \in \mathbb{C}^{sn_1 \times n_2}$. It is easy to verify that \mathcal{X}_i is self-adjoint and positive semi-definite.

In order to apply the matrix Bernstein inequality (3.6) to bound $\left\|\sum_{i=0}^{n-1} \left(\frac{\delta_i}{p} - 1\right) \mathcal{X}_i\right\|$, one needs to

bound $\left\| \left(\frac{\delta_i}{p} - 1 \right) \mathcal{X}_i \right\|$ and $\left\| \mathbb{E} \left[\sum_{i=0}^{n-1} \left(\frac{\delta_i}{p} - 1 \right)^2 \mathcal{X}_i^2 \right] \right\|$. For the upper bound of $\left\| \left(\frac{\delta_i}{p} - 1 \right) \mathcal{X}_i \right\|$, a simple calculation yields that $\left\| \left(\frac{\delta_i}{p} - 1 \right) \mathcal{X}_i \right\| \leq \frac{1}{p} \|\mathcal{X}_i\|$ $= \frac{1}{p} \sup_{\|\mathbf{W}\|_{\mathsf{F}}=1} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{G}^* \mathcal{P}_T (\mathbf{W}) \mathbf{e}_i \mathbf{e}_i^\mathsf{T} \right) \right\|_{\mathsf{F}}$ $\leq \frac{1}{p} \sup_{\|\mathbf{W}\|_{\mathsf{F}}=1} \left\| \mathbf{W} \right\|_{\mathsf{F}} \cdot \frac{2\mu_1 r}{n}$ $= \frac{2\mu_1 r}{np}, \qquad (4.2)$

where the third line follows from Corollary 6.5.

To bound
$$\left\| \mathbb{E} \left[\sum_{i=0}^{n-1} \left(\frac{\delta_i}{p} - 1 \right)^2 \mathcal{X}_i^2 \right] \right\|$$
, we have
 $\left\| \sum_{i=0}^{n-1} \mathbb{E} \left[\left(\frac{\delta_i}{p} - 1 \right)^2 \mathcal{X}_i^2 \right] \right\| \leq \frac{1}{p} \left\| \sum_{i=0}^{n-1} \mathcal{X}_i^2 \right\|$
 $\leq \frac{1}{p} \max_{0 \leq i \leq n-1} \left\| \mathcal{X}_i \right\| \cdot \left\| \sum_{i=0}^{n-1} \mathcal{X}_i \right\|$
 $\leq \frac{2\mu_1 r}{np} \sup_{\|\mathbf{W}\|_{\mathbb{F}}=1} \left\| \sum_{i=0}^{n-1} \mathcal{X}_i (\mathbf{W}) \right\|$
 $= \frac{2\mu_1 r}{np} \sup_{\|\mathbf{W}\|_{\mathbb{F}}=1} \left\| \sum_{i=0}^{n-1} \mathcal{P}_T \mathcal{G} \left(\mathcal{G}^* \mathcal{P}_T (\mathbf{W}) \mathbf{e}_i \mathbf{e}_i^\mathsf{T} \right) \right\|_{\mathbb{F}}$
 $= \frac{2\mu_1 r}{np} \sup_{\|\mathbf{W}\|_{\mathbb{F}}=1} \left\| \mathcal{P}_T \mathcal{G} \mathcal{G}^* \mathcal{P}_T (\mathbf{W}) \right\|_{\mathbb{F}}$
 $= \frac{2\mu_1 r}{np} \left\| \mathcal{P}_T \mathcal{G} \mathcal{G}^* \mathcal{P}_T \right\|$
 $\leq \frac{2\mu_1 r}{np},$

where the second line is due to the positive semi-definite property of \mathcal{X}_i , the third line follows from (4.2), and the last line follows from the fact that $\|\mathcal{G}\| = 1$, $\|\mathcal{G}^*\| \leq 1$ and \mathcal{P}_T is the projection operator.

The application of the matrix Bernstein inequality implies that

$$\left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{1}{p} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* \mathcal{P}_T \right\| \lesssim \sqrt{\frac{\mu_1 r \log(sn)}{np}} + \frac{\mu_1 r \log(sn)}{np}$$
$$\lesssim \sqrt{\frac{\mu_1 r \log(sn)}{np}}$$

holds with probability at least $1 - (sn)^{-c}$ for a universal constant c > 0, where the second and third lines are due to $p \gtrsim \frac{\mu_1 r \log(sn)}{n}$. Finally, we take the uniform bound to obtain that

$$\mathbb{P}\left\{\max_{1\leq k\leq k_0} \left\| \mathcal{P}_T \mathcal{G}\left(\mathcal{I} - \frac{n}{m} \mathbb{E}\left[\mathcal{A}_k^* \mathcal{A}_k\right]\right) \mathcal{G}^* \mathcal{P}_T \right\| \leq \frac{1}{4} \right\} \geq 1 - k_0(sn) - c \geq 1 - (sn) - (c-1),$$

where the last inequality follows from the fact that $k_0 \ll sn$.

4.2 Proof of (3.21)

Following the definition of $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16) and the isotropy property of $\{b_i\}$ in (2.5), we have

$$\begin{aligned} \left\| \mathcal{G}\left(\mathcal{I} - \frac{1}{p} \mathbb{E}\left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\| &= \left\| \frac{1}{p} \mathcal{G} \mathcal{G}^{*}(\mathbf{Z}) \sum_{i \in \Omega_{k}} e_{i} e_{i}^{\mathsf{T}} - \mathcal{G} \mathcal{G}^{*}(\mathbf{Z}) \right\| \\ &= \left\| \sum_{i=0}^{n-1} \left(\frac{\delta_{i}}{p} - 1 \right) \mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z}) e_{i} e_{i}^{\mathsf{T}} \right) \right\| \\ &=: \left\| \sum_{i=0}^{n-1} \mathbf{X}_{i} \right\|, \end{aligned}$$

where δ_i is defined in (4.1) and $\mathbf{X}_i := \left(\frac{\delta_i}{p} - 1\right) \mathcal{G}\left(\mathcal{G}^*(\mathbf{Z}) \mathbf{e}_i \mathbf{e}_i^{\mathsf{T}}\right) \in \mathbb{C}^{sn_1 \times n_2}$ are independent random matrices with zero mean.

Firstly, $||X_i||$ can be bounded as follows:

$$\begin{split} \|\boldsymbol{X}_{i}\| &\leq \frac{1}{p} \left\| \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\| \\ &= \frac{1}{p} \left\| \boldsymbol{G}_{i} \otimes \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \right) \right\| \\ &\leq \frac{1}{p} \left\| \boldsymbol{G}_{i} \right\| \cdot \left\| \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \right\|_{2} \\ &\leq \frac{1}{p} \frac{1}{\sqrt{w_{i}}} \left\| \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \right\|_{2} \\ &\leq \frac{1}{p} \left\| \boldsymbol{Z} \right\|_{\mathcal{G}, \infty}, \end{split}$$

where the second line is due to (2.3), the third line follows from the fact that $\|\mathbf{A} \otimes \mathbf{B}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$, and the last line directly follows from the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18).

Secondly, we have

$$\begin{aligned} \left\| \mathbb{E}\left[\sum_{i=0}^{n-1} \boldsymbol{X}_{i} \boldsymbol{X}_{i}^{*}\right] \right\| &= \left\| \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\frac{\delta_{i}}{p} - 1\right)^{2}\right] \left(\mathcal{G}\left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right) \left(\mathcal{G}\left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right)^{*} \right\| \\ &= \left\| \sum_{i=0}^{n-1} \mathbb{E}\left[\left(\frac{\delta_{i}}{p} - 1\right)^{2}\right] \left(\boldsymbol{G}_{i} \otimes \left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right)\right) \left(\boldsymbol{G}_{i} \otimes \left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right)\right)^{*} \right\| \\ &\leq \frac{1}{p} \sum_{i=0}^{n-1} \left\| \left(\boldsymbol{G}_{i} \otimes \left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right)\right) \left(\boldsymbol{G}_{i} \otimes \left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right)\right)^{*} \right\| \end{aligned}$$

$$\begin{split} &\leq \frac{1}{p} \sum_{i=0}^{n-1} \left\| \left(G_i G_i^* \right) \otimes \left(\left(\mathcal{G}^* (Z) e_i \right) \left(\mathcal{G}^* (Z) e_i \right)^* \right) \right\| \\ &\leq \frac{1}{p} \sum_{i=0}^{n-1} \left\| G_i G_i^* \right\| \cdot \left\| \mathcal{G}^* (Z) e_i \right\|_2^2 \\ &\leq \frac{1}{p} \sum_{i=0}^{n-1} \frac{1}{w_i} \left\| \mathcal{G}^* (Z) e_i \right\|_2^2 \\ &= \frac{1}{p} \left\| Z \right\|_{\mathcal{G},\mathsf{F}}^2 \,. \end{split}$$

Since $\left\|\mathbb{E}\left[\sum_{i=0}^{n-1} X_i^* X_i\right]\right\|$ can be bounded by the same quantity, the application of the matrix Bernstein inequality (3.6) implies that

$$\left\| \mathcal{G}\left(\mathcal{I} - \frac{1}{p} \mathbb{E}\left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^*(\mathbf{Z}) \right\| = \left\| \sum_{i=0}^{n-1} \mathbf{X}_i \right\| \lesssim \left(\sqrt{\frac{\log(sn)}{p}} \, \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{\log(sn)}{p} \, \|\mathbf{Z}\|_{\mathcal{G},\infty} \right)$$

holds with probability at least $1 - (sn)^{-c}$ for a numerical constant c > 0.

By the uniform bound we conclude that the event (3.21) occurs with probability at least $1 - (sn)^{-(c-1)}$.

4.3 Proof of (3.22)

By the definition of $\|\cdot\|_{\mathcal{G},\mathsf{F}}$ in (3.18) and the isotropy property of $\{b_i\}$ in (2.5), it follows that

$$\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{1}{p}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\mathsf{F}}^{2} = \left\| \sum_{i=0}^{n-1} \left(\frac{\delta_{i}}{p} - 1\right)\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right\|_{\mathcal{G},\mathsf{F}}^{2}$$
$$= \sum_{j=0}^{n-1} \frac{1}{w_{j}} \left\| \mathcal{G}^{*}\left(\sum_{i=0}^{n-1} \left(\frac{\delta_{i}}{p} - 1\right)\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right)\boldsymbol{e}_{j}\right\|_{2}^{2}$$
$$= \sum_{j=0}^{n-1} \frac{1}{w_{j}} \left\| \left(\sum_{i=0}^{n-1} \left(\frac{\delta_{i}}{p} - 1\right)\mathcal{G}^{*}\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right)\boldsymbol{e}_{j}\right\|_{2}^{2}$$

If we construct a new vector $\boldsymbol{z}_i \in \mathbb{C}^{sn \times 1}$ as

$$\boldsymbol{z}_{i} := \left(\frac{\delta_{i}}{p} - 1\right) \begin{bmatrix} \frac{1}{\sqrt{w_{0}}} \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{e}_{0} \\ \vdots \\ \frac{1}{\sqrt{w_{\ell}}} \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{e}_{\ell} \\ \vdots \\ \frac{1}{\sqrt{w_{n-1}}} \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{e}_{n-1} \end{bmatrix},$$

then it can be easily seen that

$$\left\| \mathcal{P}_T \mathcal{G}\left(\mathcal{I} - \frac{1}{p} \mathbb{E}\left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^*(\mathbf{Z}) \right\|_{\mathcal{G}, \mathsf{F}}^2 =: \left\| \sum_{i=0}^{n-1} \mathbf{z}_i \right\|_2^2.$$

For the upper bound of $\|\boldsymbol{z}_i\|_2,$ a direct calculation yields that

$$\left\|\boldsymbol{z}_{i}\right\|_{2} \leq rac{1}{p} \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}
ight) \right\|_{\mathcal{G},\mathsf{F}}$$

$$= \frac{1}{p} \frac{1}{\sqrt{w_i}} \left\| \mathcal{P}_T \mathcal{G} \left(\sqrt{w_i} \mathcal{G}^*(\boldsymbol{Z}) \boldsymbol{e}_i \boldsymbol{e}_i^\mathsf{T} \right) \right\|_{\mathcal{G},\mathsf{F}}$$
$$\lesssim \frac{1}{p} \sqrt{\frac{\mu_1 r \log(sn)}{n}} \cdot \frac{\left\| \mathcal{G}^*(\boldsymbol{Z}) \boldsymbol{e}_i \right\|_2}{\sqrt{w_i}}$$
$$\lesssim \frac{1}{p} \sqrt{\frac{\mu_1 r \log(sn)}{n}} \left\| \boldsymbol{Z} \right\|_{\mathcal{G},\infty},$$

where the third line follows from Lemma 6.9 and the last line is due to the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18). In addition,

$$\begin{split} \left\| \mathbb{E}\left[\sum_{i=0}^{n-1} \boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*}\right] \right\| &\leq \sum_{i=0}^{n-1} \mathbb{E}\left[\|\boldsymbol{z}_{i}\|_{2}^{2} \right] \\ &\leq \frac{1}{p} \sum_{i=0}^{n-1} \left\| \mathcal{P}_{T} \mathcal{G}\left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathcal{G},\mathsf{F}}^{2} \\ &\lesssim \frac{1}{p} \frac{\mu_{1} r \log(sn)}{n} \sum_{i=0}^{n-1} \frac{\left\| \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \right\|_{2}^{2}}{w_{i}} \\ &= \frac{1}{p} \frac{\mu_{1} r \log(sn)}{n} \left\| \boldsymbol{Z} \right\|_{\mathcal{G},\mathsf{F}}^{2}, \end{split}$$

where the third inequality is due to Lemma 6.9, and the same bound can be obtained for $\left\|\mathbb{E}\left[\sum_{i=0}^{n-1} \boldsymbol{z}_{i}^{*}\boldsymbol{z}_{i}\right]\right\|$.

Therefore, by the matrix Bernstein inequality (3.6), we can show that

$$\left\|\sum_{i=0}^{n-1} \mathbf{z}_i\right\|_2 \lesssim \sqrt{\frac{\mu_1 r \log(sn)}{n}} \left(\sqrt{\frac{\log(sn)}{p}} \, \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{\log(sn)}{p} \, \|\mathbf{Z}\|_{\mathcal{G},\infty}\right)$$

holds with probability at least $1-(sn)^{-c}$ for a universal constant c > 0. Taking the uniform bound completes the proof.

4.4 Proof of (3.23)

The definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18) allows us to express $\left\|\mathcal{P}_T\mathcal{G}\left(\mathcal{I}-\frac{1}{p}\mathbb{E}\left[\mathcal{A}_k^*\mathcal{A}_k\right]\mathcal{G}^*(Z)\right)\right\|_{\mathcal{G},\infty}$ as

$$\begin{aligned} \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{1}{p}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\mathcal{G}^{*}(\mathbf{Z})\right)\right\|_{\mathcal{G},\infty} &= \left\|\sum_{i=0}^{n-1}\left(\frac{\delta_{i}}{p} - 1\right)\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right\|_{\mathcal{G},\infty} \\ &= \max_{0 \leq j \leq n-1} \frac{1}{\sqrt{w_{j}}} \left\|\mathcal{G}^{*}\left(\sum_{i=0}^{n-1}\left(\frac{\delta_{i}}{p} - 1\right)\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right)\boldsymbol{e}_{j}\right\|_{2} \\ &= \max_{0 \leq j \leq n-1} \left\|\left(\sum_{i=0}^{n-1}\left(\frac{\delta_{i}}{p} - 1\right)\frac{1}{\sqrt{w_{j}}}\mathcal{G}^{*}\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\mathbf{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right)\boldsymbol{e}_{j}\right\|_{2}.\end{aligned}$$

Define \boldsymbol{z}_i^j to be the *s*-dimensional vector

$$\boldsymbol{z}_{i}^{j} := \left(\frac{\delta_{i}}{p} - 1\right) \frac{1}{\sqrt{w_{j}}} \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{e}_{j}, \quad (i, j) \in [n] \times [n]$$

Then one can easily see that

$$\left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{1}{p} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \mathcal{G}^* (\mathbf{Z}) \right) \right\|_{\mathcal{G}, \infty} =: \max_{0 \le j \le n-1} \left\| \sum_{i=0}^{n-1} \mathbf{z}_i^j \right\|_2.$$

For any fixed $j \in [n], \left\| \boldsymbol{z}_{i}^{j} \right\|_{2}$ can be bounded as follows:

$$\begin{aligned} z_{i}^{j} \Big\|_{2} &\leq \frac{1}{p} \frac{1}{\sqrt{w_{i}}} \frac{\sqrt{w_{i}}}{\sqrt{w_{j}}} \left\| \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*} (\mathbf{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \boldsymbol{e}_{j} \right\|_{2} \\ &= \frac{1}{p} \frac{1}{\sqrt{w_{i}}} \frac{\sqrt{w_{i}}}{\sqrt{w_{j}}} \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left| \left\langle \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*} (\mathbf{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \boldsymbol{e}_{j}, \boldsymbol{\beta} \right\rangle \right| \\ &= \frac{1}{p} \frac{1}{\sqrt{w_{i}}} \sup_{\|\boldsymbol{\beta}\|_{2}=1} \frac{\sqrt{w_{i}}}{\sqrt{w_{j}}} \left| \left\langle \mathcal{P}_{T} \mathcal{G} \left(\mathcal{G}^{*} (\mathbf{Z}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right), \mathcal{G} (\boldsymbol{\beta} \boldsymbol{e}_{j}^{\mathsf{T}}) \right\rangle \right| \\ &\leq \frac{1}{p} \frac{1}{\sqrt{w_{i}}} \frac{3\mu_{1}r}{n} \left\| \mathcal{G}^{*} (\mathbf{Z}) \boldsymbol{e}_{i} \right\|_{2} \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left\| \boldsymbol{\beta} \right\|_{2} \\ &= \frac{3\mu_{1}r}{np} \cdot \frac{\left\| \mathcal{G}^{*} (\mathbf{Z}) \boldsymbol{e}_{i} \right\|_{2}}{\sqrt{w_{i}}} \\ &\leq \frac{3\mu_{1}r}{np} \left\| \mathbf{Z} \right\|_{\mathcal{G},\infty}, \end{aligned}$$

$$(4.3)$$

where the fourth line follows from Lemma 6.6 and the last line is due to the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18). Moreover, we have

$$\begin{split} \left\|\sum_{i=0}^{n-1} \mathbb{E}\left[\boldsymbol{z}_{i}^{j}(\boldsymbol{z}_{i}^{j})^{*}\right]\right\| &\leq \sum_{i=0}^{n-1} \mathbb{E}\left[\left\|\boldsymbol{z}_{i}^{j}\right\|_{2}^{2}\right] \\ &\leq \frac{1}{p}\sum_{i=0}^{n-1}\left\|\frac{1}{\sqrt{w_{j}}}\mathcal{G}^{*}\mathcal{P}_{T}\mathcal{G}\left(\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\boldsymbol{e}_{j}\right\|_{2}^{2} \\ &\leq \frac{1}{p}\left(\frac{3\mu_{1}r}{n}\right)^{2} \cdot \sum_{i=0}^{n-1}\left(\frac{\left\|\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right\|_{2}}{\sqrt{w_{i}}}\right)^{2} \\ &= \frac{1}{p}\left(\frac{3\mu_{1}r}{n}\right)^{2} \cdot \left\|\boldsymbol{Z}\right\|_{\mathcal{G},\mathsf{F}}^{2}, \end{split}$$

where the third inequality follows from (4.3). The same bound can be obtained for $\left\|\sum_{i=0}^{n-1} \mathbb{E}\left[(z_i^j)^* z_i^j\right]\right\|$ as well.

The matrix Bernstein inequality (3.6) taken collectively with the uniform bound yields that

$$\begin{split} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{1}{p} \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \mathcal{G}^* (\mathbf{Z}) \right) \right\|_{\mathcal{G}, \infty} &= \max_{0 \le j \le n-1} \left\| \sum_{i=0}^{n-1} \mathbf{z}_i^j \right\|_2 \\ &\lesssim \frac{\mu_1 r}{n} \left(\sqrt{\frac{\log(sn)}{p}} \, \| \mathbf{Z} \|_{\mathcal{G}, \mathsf{F}} + \frac{\log(sn)}{p} \, \| \mathbf{Z} \|_{\mathcal{G}, \infty} \right) \end{split}$$

holds with probability at least $1 - ns^{-c_2}$ for a universal constant $c_2 > 2$.

Finally, we take the uniform bound over all $k \in \{1, \dots, k_0\}$ again to complete the proof.

5 Proofs of Lemmas 3.9 to 3.14

This section presents the proofs of Lemmas 3.9 to 3.14, which have been used to verify (3.9) and (3.10).

5.1 Proof of Lemma 3.9

Note that

$$\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right)\mathcal{G}^{*}\mathcal{P}_{T} \right\| \leq \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}\mathcal{P}_{T} \right\| + \frac{n}{m}\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}\mathcal{P}_{T} \right\|$$

According to (3.20) in Lemma 3.8, the first term is upper bounded by $\frac{1}{4}$. We will bound the second term via the matrix Bernstein inequality (3.6).

For any $\mathbf{Z} \in \mathbb{C}^{sn_1 \times n_2}$, by the definition of $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16), we have

$$\begin{split} \mathcal{P}_{T}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{Z}) &= \mathcal{P}_{T}\mathcal{G}\left(\sum_{i\in\Omega_{k}}\left\langle\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}},\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{Z})\right\rangle\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \\ &= \sum_{i\in\Omega_{k}}\left\langle\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}},\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{Z})\right\rangle\mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \\ &= \sum_{i\in\Omega_{k}}\left\langle\mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right),\boldsymbol{Z}\right\rangle\mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right). \end{split}$$

If we define $\boldsymbol{z}_i := \operatorname{vec}\left(\mathcal{P}_T \mathcal{G}\left(\boldsymbol{b}_i \boldsymbol{e}_i^\mathsf{T}\right)\right) \in \mathbb{C}^{sn_1n_2 \times 1}$, then it follows that

$$\begin{split} \|\mathcal{P}_{T}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*}\mathcal{P}_{T}\| &= \sup_{\|\boldsymbol{W}\|_{\mathsf{F}}=1} \|\mathcal{P}_{T}\mathcal{G}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{W})\|_{\mathsf{F}} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathsf{F}}=1} \left\|\sum_{i\in\Omega_{k}} \left\langle \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right), \boldsymbol{W}\right\rangle \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{b}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right)\right\|_{\mathsf{F}} \\ &= \sup_{\|\operatorname{vec}(\boldsymbol{W})\|_{2}=1} \left\|\sum_{i\in\Omega_{k}}\boldsymbol{z}_{i}^{*}\operatorname{vec}(\boldsymbol{W})\boldsymbol{z}_{i}\right\|_{2} \\ &= \sup_{\|\operatorname{vec}(\boldsymbol{W})\|_{2}=1} \left\|\sum_{i\in\Omega_{k}}\boldsymbol{z}_{i}\boldsymbol{z}_{i}^{*}\operatorname{vec}(\boldsymbol{W})\right\|_{2} \\ &= \left\|\sum_{i\in\Omega_{k}}\boldsymbol{z}_{i}\boldsymbol{z}_{i}^{*}\right\|, \end{split}$$

where it is obvious that $z_i z_i^*$ are independent and positive semi-definite random matrices. Hence,

$$\left\|\mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}
ight]
ight)\mathcal{G}^{*}\mathcal{P}_{T}
ight\|=\left\|\sum_{i\in\Omega_{k}}\left(oldsymbol{z}_{i}oldsymbol{z}_{i}^{*}-\mathbb{E}\left[oldsymbol{z}_{i}oldsymbol{z}_{i}^{*}
ight]
ight)
ight\|.$$

Firstly, $\left\| \boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*} - \mathbb{E}\left[\boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*}
ight] \right\|$ can be bounded as follows:

$$egin{aligned} ig\|oldsymbol{z}_ioldsymbol{z}_i^* &- \mathbb{E}\left[oldsymbol{z}_ioldsymbol{z}_i^*
ight]ig\| &\leq \max\left\{\|oldsymbol{z}_ioldsymbol{z}_i^*\|, \mathbb{E}\left[\|oldsymbol{z}_ioldsymbol{z}_i^*\|
ight]
ight\} &\leq \max\left\{\|oldsymbol{z}_i\|_2^2, \mathbb{E}\left[\|oldsymbol{z}_i\|_2^2
ight]
ight\}, \end{aligned}$$

where the second line is due to the Jensen inequality. By the definition of \boldsymbol{z}_i , we have $\|\boldsymbol{z}_i\|_2^2 = \|\mathcal{P}_T \mathcal{G}(\boldsymbol{b}_i \boldsymbol{e}_i^T)\|_{\mathsf{F}}^2$. Then applying (6.6) in Corollary 6.3 implies that

$$\left\|\boldsymbol{z}_{i}\boldsymbol{z}_{i}^{*}-\mathbb{E}\left[\boldsymbol{z}_{i}\boldsymbol{z}_{i}^{*}
ight]
ight\|\leq\max\left\{\left\|\boldsymbol{z}_{i}
ight\|_{2}^{2},\mathbb{E}\left[\left\|\boldsymbol{z}_{i}
ight\|_{2}^{2}
ight]
ight\}\leqrac{2\mu_{1}rs\mu_{0}}{n}.$$

Secondly,

$$\begin{split} \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[\left(z_i z_i^* - \mathbb{E} \left[z_i z_i^* \right] \right)^2 \right] \right\| &= \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[(z_i z_i^*)^2 \right] - \left(\mathbb{E} \left[z_i z_i^* \right] \right)^2 \right\| \\ &\leq \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[(z_i z_i^*)^2 \right] \right\| \\ &\leq \max_{i \in \Omega_k} \left\| z_i z_i^* \right\| \cdot \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[(z_i z_i^*) \right] \right\| \\ &\leq \frac{2\mu_1 r s \mu_0}{n} \cdot \frac{5m}{4n}, \end{split}$$

Here the last line follows from a direct calculation:

$$\begin{split} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[(\boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*}) \right] \right\| &= \sup_{\|\boldsymbol{v} \in (\boldsymbol{W})\|_{2} = 1} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[\operatorname{vec}\left(\mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}})\right) \operatorname{vec}\left(\mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}})\right)^{*} \operatorname{vec}(\boldsymbol{W}) \right] \right\|_{2} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[\left\langle \mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}), \boldsymbol{W} \right\rangle \operatorname{vec}\left(\mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}})\right) \right] \right\|_{\mathrm{F}} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[\left\langle \mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}), \boldsymbol{W} \right\rangle \mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}) \right] \right\|_{\mathrm{F}} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[\left(\boldsymbol{b}_{i}^{*} \mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{W}) \boldsymbol{e}_{i} \right) \mathcal{P}_{T} \mathcal{G}(\boldsymbol{b}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}) \right] \right\|_{\mathrm{F}} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathbb{E}\left[\mathcal{P}_{T} \mathcal{G}\left(\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{*} \mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{W}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right] \right\|_{\mathrm{F}} \\ &= \sup_{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G}\left(\mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{W}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathrm{F}} \\ &= \frac{5m}{\|\boldsymbol{W}\|_{\mathrm{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G}\left(\mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{W}) \boldsymbol{e}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathrm{F}} \end{aligned}$$

where in the last inequality we have utilized (3.20) in the following way,

$$\frac{1}{4} \geq \left\| \mathcal{P}_{T}\mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathbb{E} \left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*} \mathcal{P}_{T} \right\| \\ \geq \frac{n}{m} \left\| \mathcal{P}_{T}\mathcal{G}\mathbb{E} \left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \mathcal{G}^{*} \mathcal{P}_{T} \right\| - \left\| \mathcal{P}_{T}\mathcal{G}\mathcal{G}^{*} \mathcal{P}_{T} \right\| \\ \geq \frac{n}{m} \sup_{\left\| \mathbf{W} \right\|_{\mathsf{F}} = 1} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T}\mathcal{G} \left(\mathcal{G}^{*} \mathcal{P}_{T}(\mathbf{W}) \mathbf{e}_{i} \mathbf{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathsf{F}} - 1.$$

Since we can obtain the same bound for $\left\|\sum_{i\in\Omega_k}\mathbb{E}\left[\left(\boldsymbol{z}_i^*\boldsymbol{z}_i - \mathbb{E}\left[\boldsymbol{z}_i^*\boldsymbol{z}_i\right]\right)^2\right]\right\|$, applying the matrix Bernstein inequality (3.6) implies that with probability at least $1 - (sn)^{-c}$,

$$\frac{n}{m} \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{A}_{k}^{*} \mathcal{A}_{k} - \mathbb{E} \left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*} \mathcal{P}_{T} \right\| = \frac{n}{m} \left\| \sum_{i \in \Omega_{k}} \left(\boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*} - \mathbb{E} \left[\boldsymbol{z}_{i} \boldsymbol{z}_{i}^{*} \right] \right) \right\|$$
$$\lesssim \frac{n}{m} \cdot \left(\sqrt{\frac{5m}{4n} \cdot \frac{2\mu_{1} r s \mu_{0}}{n} \log(sn)} + \frac{2\mu_{1} r s \mu_{0} \log(sn)}{n} \right)$$

$$= \frac{1}{m} \cdot \left(\sqrt{\frac{5m\mu_1 rs\mu_0 \log(sn)}{2}} + 2\mu_1 rs\mu_0 \log(sn) \right)$$
$$\lesssim \frac{1}{m} \cdot \sqrt{\frac{5m\mu_1 rs\mu_0 \log(sn)}{2}}$$
$$\leq \frac{1}{4},$$

where the fourth line and the last line hold when $m \gtrsim \mu_1 r s \mu_0 \log(sn)$.

Finally, combining the two terms together completes the proof.

5.2 Proof of Lemma 3.11

Notice that

$$\left\| \mathcal{G}\left(\mathcal{I} - \frac{n}{m} \mathcal{A}_{k}^{*} \mathcal{A}_{k} \right) \mathcal{G}^{*}(\mathbf{Z}) \right\| \leq \left\| \mathcal{G}\left(\mathcal{I} - \frac{n}{m} \mathbb{E}\left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\| + \frac{n}{m} \left\| \mathcal{G}\left(\mathcal{A}_{k}^{*} \mathcal{A}_{k} - \mathbb{E}\left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\|$$

$$\lesssim \sqrt{\frac{n \log(sn)}{m}} \left\| \mathbf{Z} \right\|_{\mathcal{G},\mathsf{F}} + \frac{n \log(sn)}{m} \left\| \mathbf{Z} \right\|_{\mathcal{G},\infty} + \frac{n}{m} \left\| \mathcal{G}\left(\mathcal{A}_{k}^{*} \mathcal{A}_{k} - \mathbb{E}\left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\|,$$
(5.1)

where the second line follows from (3.21). In order to prove (3.26), it suffices to bound the last term.

Recalling the definition of $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16) and using the isotropy property of $\{b_i\}$ in (2.5), we can rewrite the last term as

$$\frac{n}{m} \left\| \mathcal{G} \left(\mathcal{A}_k^* \mathcal{A}_k - \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^*(\mathbf{Z}) \right\| = \frac{n}{m} \left\| \sum_{i \in \Omega_k} \mathcal{G} \left((\mathbf{b}_i \mathbf{b}_i^* - \mathbf{I}) \mathcal{G}^*(\mathbf{Z}) \mathbf{e}_i \mathbf{e}_i^\mathsf{T} \right) \right\|$$
$$=: \frac{n}{m} \left\| \sum_{i \in \Omega_k} \mathbf{X}_i \right\|,$$

where $X_i = \mathcal{G}\left((b_i b_i^* - I)\mathcal{G}^*(Z)e_i e_i^{\mathsf{T}}\right) \in \mathbb{C}^{sn_1 \times n_2}$. It can be easily seen that X_i are independent random matrices with zero mean.

The upper bound of $||X_i||$ can be established as follows:

$$egin{aligned} \|m{X}_i\| &= \left\|m{\mathcal{G}}\left((m{b}_im{b}_i^* - m{I})m{\mathcal{G}}^*(m{Z})m{e}_im{e}_i^{\mathsf{T}}
ight)
ight\| \ &= \|m{G}_i\otimes((m{b}_im{b}_i^* - m{I})m{\mathcal{G}}^*(m{Z})m{e}_i)\| \ &\leq \|m{G}_i\|\cdot\|(m{b}_im{b}_i^* - m{I})m{\mathcal{G}}^*(m{Z})m{e}_i\| \ &\leq rac{1}{\sqrt{w_i}}\max\left\{\|m{b}_i\|_2^2,1
ight\}\cdot\|m{\mathcal{G}}^*(m{Z})m{e}_i\|_2 \ &\leq s\mu_0\,\|m{Z}\|_{m{G}}\,\infty\,, \end{aligned}$$

where the second line follows from (2.3), the third line is due to $\|\mathbf{A} \otimes \mathbf{B}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$, and the last line follows from the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18).

To bound $\|\mathbb{E}\left[\sum_{i\in\Omega_k} X_i^*X_i\right]\|$, we first define $\mathbf{z}_i = (\mathbf{b}_i\mathbf{b}_i^* - \mathbf{I})\mathcal{G}^*(\mathbf{Z})\mathbf{e}_i \in \mathbb{C}^s$. Then a simple calculation yields that

$$\begin{split} \mathbb{E}\left[\left\|\boldsymbol{z}_{i}\right\|_{2}^{2}\right] &= \mathbb{E}\left[\boldsymbol{e}_{i}^{\mathsf{T}}\left(\mathcal{G}^{*}(\boldsymbol{Z})\right)^{*}\left(\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}-\boldsymbol{I}\right)^{2}\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right] \\ &= \boldsymbol{e}_{i}^{\mathsf{T}}\left(\mathcal{G}^{*}(\boldsymbol{Z})\right)^{*}\mathbb{E}\left[\left(\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}-\boldsymbol{I}\right)^{2}\right]\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i} \\ &= \boldsymbol{e}_{i}^{\mathsf{T}}\left(\mathcal{G}^{*}(\boldsymbol{Z})\right)^{*}\left(\mathbb{E}\left[\left\|\boldsymbol{b}_{i}\right\|_{2}^{2}\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}-2\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}+\boldsymbol{I}\right]\right)\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i} \end{split}$$

$$= \boldsymbol{e}_{i}^{\mathsf{T}} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \right)^{*} \left(\mathbb{E} \left[\|\boldsymbol{b}_{i}\|_{2}^{2} \boldsymbol{b}_{i} \boldsymbol{b}_{i}^{*} - \boldsymbol{I} \right] \right) \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i}$$

$$\leq \boldsymbol{e}_{i}^{\mathsf{T}} \left(\mathcal{G}^{*}(\boldsymbol{Z}) \right)^{*} \left(\boldsymbol{s} \mu_{0} \mathbb{E} \left[\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{*} \right] - \boldsymbol{I} \right) \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i}$$

$$\leq \boldsymbol{s} \mu_{0} \cdot \| \mathcal{G}^{*}(\boldsymbol{Z}) \boldsymbol{e}_{i} \|_{2}^{2}, \qquad (5.2)$$

where the last two inequalities follow from the incoherence property (2.6) and the isotropy property (2.5) of $\{b_i\}$. Furthermore, it follows that

$$\begin{split} \left\| \mathbb{E} \left[\sum_{i \in \Omega_k} \mathbf{X}_i^* \mathbf{X}_i \right] \right\| &= \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[(\mathbf{G}_i \otimes \mathbf{z}_i)^* (\mathbf{G}_i \otimes \mathbf{z}_i) \right] \right\| \\ &= \left\| \sum_{i \in \Omega_k} \mathbb{E} \left[(\mathbf{G}_i^\mathsf{T} \mathbf{G}_i) \otimes (\mathbf{z}_i^* \mathbf{z}_i) \right] \right\| \\ &= \left\| \sum_{i \in \Omega_k} (\mathbf{G}_i^\mathsf{T} \mathbf{G}_i) \mathbb{E} \left[\| \mathbf{z}_i \|_2^2 \right] \right\| \\ &\leq s \mu_0 \cdot \left\| \sum_{i \in \Omega_k} \| \mathcal{G}^* (\mathbf{Z}) \mathbf{e}_i \|_2^2 (\mathbf{G}_i^\mathsf{T} \mathbf{G}_i) \right\| \\ &\leq s \mu_0 \cdot \sum_{i \in \Omega_k} \left\| \mathbf{G}_i^\mathsf{T} \mathbf{G}_i \right\| \| \mathcal{G}^* (\mathbf{Z}) \mathbf{e}_i \|_2^2 \\ &\leq s \mu_0 \cdot \sum_{i \in \Omega_k} \frac{\| \mathcal{G}^* (\mathbf{Z}) \mathbf{e}_i \|_2^2}{w_i} \\ &\leq s \mu_0 \cdot \sum_{i = 1}^n \frac{\| \mathcal{G}^* (\mathbf{Z}) \mathbf{e}_i \|_2^2}{w_i} \\ &= s \mu_0 \cdot \| \mathbf{Z} \|_{\mathcal{G},\mathsf{F}}^2, \end{split}$$

where the fourth line follows from (5.2), and $\|\mathbb{E}\left[\sum_{i\in\Omega_k} X_i X_i^*\right]\|$ can be similarly bounded. Therefore, by the matrix Bernstein inequality (3.6),

$$\frac{n}{m} \left\| \mathcal{G} \left(\mathcal{A}_k^* \mathcal{A}_k - \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^*(\mathbf{Z}) \right\| = \frac{n}{m} \left\| \sum_{i \in \Omega_k} \mathbf{X}_i \right\|$$
$$\lesssim \frac{n}{m} \left(\sqrt{s\mu_0 \log(sn)} \| \mathbf{Z} \|_{\mathcal{G},\mathsf{F}} + s\mu_0 \log(sn) \| \mathbf{Z} \|_{\mathcal{G},\infty} \right)$$
$$= \sqrt{\frac{nk_0 s\mu_0 \log(sn)}{m}} \| \mathbf{Z} \|_{\mathcal{G},\mathsf{F}} + \frac{ns\mu_0 \log(sn)}{m} \| \mathbf{Z} \|_{\mathcal{G},\infty}$$

holds with probability at least $1 - (sn)^{-c}$ for a universal constant c > 0. Inserting this bound into (5.1) we conclude that

$$\begin{split} \left\| \mathcal{G}\left(\mathcal{I} - \frac{n}{m} \mathcal{A}_k^* \mathcal{A}_k \right) \mathcal{G}^*(\mathbf{Z}) \right\| &\lesssim \left(\sqrt{\frac{nk_0 s \mu_0 \log(sn)}{m}} + \sqrt{\frac{n\log(sn)}{m}} \right) \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \left(\frac{ns \mu_0 \log(sn)}{m} + \frac{n\log(sn)}{m} \right) \|\mathbf{Z}\|_{\mathcal{G},\infty} \\ &\lesssim \sqrt{\frac{4nk_0 s \mu_0 \log(sn)}{m}} \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns \mu_0 \log(sn)}{m} \|\mathbf{Z}\|_{\mathcal{G},\infty} \end{split}$$

holds with probability exceeding $1 - (sn)^{-c}$.

5.3 Proof of Lemma 3.12

Notice that

$$\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\mathsf{F}} \leq \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\mathsf{F}} + \frac{n}{m}\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\mathsf{F}} \\ \lesssim \sqrt{\frac{\mu_{1}r\log(sn)}{n}}\left(\sqrt{\frac{n\log(sn)}{m}}\left\|\mathbf{Z}\right\|_{\mathcal{G},\mathsf{F}} + \frac{n\log(sn)}{m}\left\|\mathbf{Z}\right\|_{\mathcal{G},\mathsf{F}}\right) \\ + \frac{n}{m}\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\mathsf{F}},$$
(5.3)

where the second line follows from (3.22). We will adopt the matrix Bernstein inequality (3.6) to bound the second term.

Recalling the definition of $\mathcal{A}_k^* \mathcal{A}_k$ in (3.16) and letting $\mathbf{z}_i := (\mathbf{b}_i \mathbf{b}_i^* - \mathbf{I}) \mathcal{G}^*(\mathbf{Z}) \mathbf{e}_i \in \mathbb{C}^s$, we have

$$\frac{n}{m} \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{A}_{k}^{*} \mathcal{A}_{k} - \mathbb{E} \left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\|_{\mathcal{G},\mathsf{F}} = \frac{n}{m} \left\| \mathcal{P}_{T} \mathcal{G} \left(\sum_{i \in \Omega_{k}} \left((b_{i} b_{i}^{*} - \mathbb{E} \left[b_{i} b_{i}^{*} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) e_{i} e_{i}^{\mathsf{T}} \right) \right\|_{\mathcal{G},\mathsf{F}}$$

$$= \frac{n}{m} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G} \left((b_{i} b_{i}^{*} - \mathbf{I}) \mathcal{G}^{*}(\mathbf{Z}) e_{i} e_{i}^{\mathsf{T}} \right) \right\|_{\mathcal{G},\mathsf{F}}$$

$$= \frac{n}{m} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G}(\mathbf{z}_{i} e_{i}^{\mathsf{T}}) \right\|_{\mathcal{G},\mathsf{F}}$$

$$= \frac{n}{m} \sqrt{\sum_{j=0}^{n-1} \frac{1}{w_{j}}} \left\| \mathcal{G}^{*} \left(\sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G}(\mathbf{z}_{i} e_{i}^{\mathsf{T}}) \right) e_{j} \right\|_{2}^{2}$$

$$= \frac{n}{m} \sqrt{\sum_{j=0}^{n-1} \frac{1}{w_{j}}} \left\| \sum_{i \in \Omega_{k}} \mathcal{G}^{*} \mathcal{P}_{T} \mathcal{G}(\mathbf{z}_{i} e_{i}^{\mathsf{T}}) e_{j} \right\|_{2}^{2},$$

where the second equality is due to the isotropy property of $\{b_i\}$ in (2.5). Furthermore, denoting by $y_i \in \mathbb{C}^{sn \times 1}$ the vector

$$oldsymbol{y}_i := egin{bmatrix} rac{1}{\sqrt{w_0}}\mathcal{G}^*\mathcal{P}_T\mathcal{G}(oldsymbol{z}_ioldsymbol{e}_i^{\mathsf{T}})oldsymbol{e}_0\ dots\ rac{1}{\sqrt{w_\ell}}\mathcal{G}^*\mathcal{P}_T\mathcal{G}(oldsymbol{z}_ioldsymbol{e}_i^{\mathsf{T}})oldsymbol{e}_\ell\ dots\ rac{1}{\sqrt{w_{n-1}}}\mathcal{G}^*\mathcal{P}_T\mathcal{G}(oldsymbol{z}_ioldsymbol{e}_i^{\mathsf{T}})oldsymbol{e}_{n-1} \end{bmatrix},$$

the second term can be expressed as

$$\frac{n}{m} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{A}_k^* \mathcal{A}_k - \mathbb{E} \left[\mathcal{A}_k^* \mathcal{A}_k \right] \right) \mathcal{G}^* (\mathbf{Z}) \right\|_{\mathcal{G}, \mathsf{F}} =: \frac{n}{m} \left\| \sum_{i \in \Omega_k} \mathbf{y}_i \right\|_2.$$
(5.4)

Clearly, \boldsymbol{y}_i are independent random vectors with zero mean.

A direct calculation yields that

$$\left\|\boldsymbol{y}_{i}\right\|_{2} = \sqrt{\sum_{j=0}^{n-1} \frac{1}{w_{j}} \left\|\mathcal{G}^{*}\mathcal{P}_{T}\mathcal{G}(\boldsymbol{z}_{i}\boldsymbol{e}_{i}^{\mathsf{T}})\boldsymbol{e}_{j}\right\|_{2}^{2}}$$

$$= \left\| \mathcal{P}_{T}\mathcal{G}(\boldsymbol{z}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}) \right\|_{\mathcal{G},\mathsf{F}}$$

$$= \frac{1}{\sqrt{w_{i}}} \left\| \mathcal{P}_{T}\mathcal{G}\left(\sqrt{w_{i}}\boldsymbol{z}_{i}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \right\|_{\mathcal{G},\mathsf{F}}$$

$$\lesssim \frac{1}{\sqrt{w_{i}}} \left\| \boldsymbol{z}_{i} \right\|_{2} \sqrt{\frac{\mu_{1}r\log(sn)}{n}}$$

$$= \frac{1}{\sqrt{w_{i}}} \sqrt{\frac{\mu_{1}r\log(sn)}{n}} \cdot \left\| (\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*} - \boldsymbol{I}) \mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i} \right\|_{2}$$

$$\leq \frac{1}{\sqrt{w_{i}}} \sqrt{\frac{\mu_{1}r\log(sn)}{n}} \cdot \left\| \boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*} - \boldsymbol{I} \right\| \cdot \left\| \mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i} \right\|_{2}$$

$$\leq \sqrt{\frac{\mu_{1}r\log(sn)}{n}} \cdot s\mu_{0} \cdot \left\| \boldsymbol{Z} \right\|_{\mathcal{G},\infty},$$

where the fourth line follows from Lemma 6.9 and the last line is due to the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18). Additionally, we have

$$\begin{aligned} \left\| \mathbb{E} \left[\sum_{i \in \Omega_k} \boldsymbol{y}_i \boldsymbol{y}_i^* \right] \right\| &\leq \sum_{i \in \Omega_k} \mathbb{E} \left[\| \boldsymbol{y}_i \|_2^2 \right] \\ &= \sum_{i \in \Omega_k} \mathbb{E} \left[\| \mathcal{P}_T \mathcal{G}(\boldsymbol{z}_i \boldsymbol{e}_i^\mathsf{T}) \|_{\mathcal{G},\mathsf{F}}^2 \right] \\ &\lesssim \sum_{i \in \Omega_k} \frac{1}{w_i} \frac{\mu_1 r \log(sn)}{n} \cdot \mathbb{E} \left[\| \boldsymbol{z}_i \|_2^2 \right] \\ &\lesssim s \mu_0 \frac{\mu_1 r \log(sn)}{n} \cdot \sum_{i \in \Omega_k} \frac{1}{w_i} \| \mathcal{G}^*(\boldsymbol{Z}) \boldsymbol{e}_i \|_2^2 \\ &\lesssim \frac{s \mu_0 \cdot \mu_1 r \log(sn)}{n} \cdot \| \boldsymbol{Z} \|_{\mathcal{G},\mathsf{F}}^2 \,, \end{aligned}$$

where the third line is due to Lemma 6.9 and the fourth line follows from

$$\mathbb{E}\left[\left\|\boldsymbol{z}_{i}\right\|_{2}^{2}\right] = \mathbb{E}\left[\left\|\left(\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}-\boldsymbol{I}\right)\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right\|_{2}^{2}\right] \\
= \mathbb{E}\left[\boldsymbol{e}_{i}^{T}\left(\mathcal{G}^{*}(\boldsymbol{Z})\right)^{*}\left(\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}-\boldsymbol{I}\right)^{2}\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right] \\
= \boldsymbol{e}_{i}^{T}\left(\mathcal{G}^{*}(\boldsymbol{Z})\right)^{*}\left(\mathbb{E}\left[\left(\left\|\boldsymbol{b}_{i}\right\|_{2}^{2}\boldsymbol{b}_{i}\boldsymbol{b}_{i}^{*}\right)\right]-\boldsymbol{I}\right)\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i} \\
\leq s\mu_{0}\left\|\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right\|_{2}^{2}.$$
(5.5)

The same upper bound can be obtained for $\|\mathbb{E}\left[\sum_{i\in\Omega_k} y_i^* y_i\right]\|$. Applying the matrix Bernstein inequality yields that

$$\frac{n}{m} \left\| \sum_{i \in \Omega_k} \boldsymbol{y}_i \right\|_2 \lesssim \frac{n}{m} \left(\sqrt{\frac{s\mu_0\mu_1 r \log^2(sn)}{n}} \, \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \sqrt{\frac{\mu_1 r \log(sn)}{n}} \cdot s\mu_0 \log(sn) \cdot \|\boldsymbol{Z}\|_{\mathcal{G},\infty} \right)$$
$$= \sqrt{\frac{\mu_1 r \log(sn)}{n}} \left(\sqrt{\frac{nk_0 s\mu_0 \log(sn)}{m}} \, \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{ns\mu_0 \log(sn)}{m} \, \|\boldsymbol{Z}\|_{\mathcal{G},\infty} \right)$$

holds with probability at least $1 - (sn)^{-c}$ for a universal constant c > 0. Noting (5.3) and (5.4), it follows immediately that

$$\left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{I} - \frac{n}{m} \mathcal{A}_k^* \mathcal{A}_k \right) \mathcal{G}^*(\mathbf{Z}) \right\|_{\mathcal{G}, \mathsf{F}} \lesssim \sqrt{\frac{\mu_1 r \log(sn)}{n}} \left(\sqrt{\frac{n \log(sn)}{m}} \, \|\mathbf{Z}\|_{\mathcal{G}, \mathsf{F}} + \frac{n \log(sn)}{m} \, \|\mathbf{Z}\|_{\mathcal{G}, \mathsf{K}} \right)$$

$$+\sqrt{\frac{\mu_{1}r\log(sn)}{n}}\left(\sqrt{\frac{nk_{0}s\mu_{0}\log(sn)}{m}} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{ns\mu_{0}\log(sn)}{m}\|\boldsymbol{Z}\|_{\mathcal{G},\infty}\right)$$
$$\lesssim \sqrt{\frac{\mu_{1}r\log(sn)}{n}}\left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m}\|\boldsymbol{Z}\|_{\mathcal{G},\infty}\right)$$

holds with probability greater than $1 - (sn)^{-c}$.

Proof of Lemma 3.13 **5.4**

By the triangle inequality, we have

$$\begin{aligned} \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\infty} &\leq \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I}-\frac{n}{m}\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\infty} + \frac{n}{m}\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\infty} \\ &\lesssim \frac{\mu_{1}r}{n}\left(\sqrt{\frac{n\log(sn)}{m}}\left\|\mathbf{Z}\right\|_{\mathcal{G},\mathsf{F}} + \frac{n\log(sn)}{m}\left\|\mathbf{Z}\right\|_{\mathcal{G},\infty}\right) \\ &+ \frac{n}{m}\left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{A}_{k}^{*}\mathcal{A}_{k}-\mathbb{E}\left[\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right]\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\infty}, \end{aligned}$$
(5.6)

where the second line is due to (3.23). In the following proof, we will upper bound the second term by the

matrix Bernstein inequality (3.6) and the uniform bound argument. If we define $\mathbf{z}_i = (\mathbf{b}_i \mathbf{b}_i^* - \mathbf{I}) \mathcal{G}^*(\mathbf{Z}) \mathbf{e}_i \in \mathbb{C}^s$ and $\mathbf{y}_i^j = \frac{1}{\sqrt{w_j}} \mathcal{G}^* \mathcal{P}_T \mathcal{G}\left(\mathbf{z}_i \mathbf{e}_i^\mathsf{T}\right) \mathbf{e}_j \in \mathbb{C}^s$, the second term can be rewritten as

$$\frac{n}{m} \left\| \mathcal{P}_{T} \mathcal{G} \left(\mathcal{A}_{k}^{*} \mathcal{A}_{k} - \mathbb{E} \left[\mathcal{A}_{k}^{*} \mathcal{A}_{k} \right] \right) \mathcal{G}^{*}(\mathbf{Z}) \right\|_{\mathcal{G},\infty} = \frac{n}{m} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G} \left((\mathbf{b}_{i} \mathbf{b}_{i}^{*} - \mathbf{I}) \mathcal{G}^{*}(\mathbf{Z}) \mathbf{e}_{i} \mathbf{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathcal{G},\infty} \\ = \frac{n}{m} \left\| \sum_{i \in \Omega_{k}} \mathcal{P}_{T} \mathcal{G} (\mathbf{z}_{i} \mathbf{e}_{i}^{\mathsf{T}}) \right\|_{\mathcal{G},\infty} \\ = \frac{n}{m} \sup_{0 \leq j \leq n-1} \frac{1}{\sqrt{w_{j}}} \left\| \sum_{i \in \Omega_{k}} \mathcal{G}^{*} \left(\mathcal{P}_{T} \mathcal{G} (\mathbf{z}_{i} \mathbf{e}_{i}^{\mathsf{T}}) \right) \mathbf{e}_{j} \right\|_{2} \\ =: \frac{n}{m} \sup_{0 \leq j \leq n-1} \left\| \sum_{i \in \Omega_{k}} \mathbf{y}_{i}^{j} \right\|_{2},$$
(5.7)

where the first equation follows from (3.16) and the isotropy property of $\{b_i\}$ in (2.5).

For any fixed $j \in [n]$, $\left\| \boldsymbol{y}_{i}^{j} \right\|_{2}$ can be bounded as follows:

$$\begin{aligned} \left\| \boldsymbol{y}_{i}^{j} \right\|_{2} &= \frac{1}{\sqrt{w_{j}}} \left\| \boldsymbol{\mathcal{G}}^{*} \boldsymbol{\mathcal{P}}_{T} \boldsymbol{\mathcal{G}} \left(\boldsymbol{z}_{i} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \boldsymbol{e}_{j} \right\|_{2} \\ &= \frac{1}{\sqrt{w_{j}}} \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left| \left\langle \boldsymbol{\mathcal{G}}^{*} \boldsymbol{\mathcal{P}}_{T} \boldsymbol{\mathcal{G}} (\boldsymbol{z}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}) \boldsymbol{e}_{j}, \boldsymbol{\beta} \right\rangle \right| \\ &= \frac{1}{\sqrt{w_{i}}} \sup_{\|\boldsymbol{\beta}\|_{2}=1} \frac{\sqrt{w_{i}}}{\sqrt{w_{j}}} \left| \left\langle \boldsymbol{\mathcal{P}}_{T} \boldsymbol{\mathcal{G}} (\boldsymbol{z}_{i} \boldsymbol{e}_{i}^{\mathsf{T}}), \boldsymbol{\mathcal{G}} (\boldsymbol{\beta} \boldsymbol{e}_{j}^{\mathsf{T}}) \right\rangle \right| \\ &\leq \frac{1}{\sqrt{w_{i}}} \frac{3\mu_{1}r}{n} \left\| \boldsymbol{z}_{i} \right\|_{2} \\ &= \frac{1}{\sqrt{w_{i}}} \frac{3\mu_{1}r}{n} \left\| (\boldsymbol{b}_{i} \boldsymbol{b}_{i}^{*} - \boldsymbol{I}) \boldsymbol{\mathcal{G}}^{*} (\boldsymbol{Z}) \boldsymbol{e}_{i} \right\|_{2} \end{aligned}$$
(5.8)

$$egin{aligned} &\leq rac{1}{\sqrt{w_i}}rac{3\mu_1r}{n} \left\|oldsymbol{b}_i^* - oldsymbol{I}
ight\| \cdot \left\|\mathcal{G}^*(oldsymbol{Z})oldsymbol{e}_i
ight\|_2 \ &\leq s\mu_0\cdotrac{3\mu_1r}{n} \left\|oldsymbol{Z}
ight\|_{\mathcal{G},\infty}, \end{aligned}$$

where the fourth line follows from Lemma 6.6 and the last line is due to the incoherence property of $\{b_i\}$ in (2.6) and the definition of $\|\cdot\|_{\mathcal{G},\infty}$ in (3.18).

Moreover,

$$\begin{split} \mathbb{E}\left[\sum_{i\in\Omega_{k}}\boldsymbol{y}_{i}^{j}(\boldsymbol{y}_{i}^{j})^{*}\right] &\leq \mathbb{E}\left[\sum_{i\in\Omega_{k}}\left\|\boldsymbol{y}_{i}^{j}\right\|_{2}^{2}\right] \\ &\lesssim \sum_{i\in\Omega_{k}}\frac{1}{w_{i}}\left(\frac{\mu_{1}r}{n}\right)^{2}\mathbb{E}\left[\left\|\boldsymbol{z}_{i}\right\|_{2}^{2}\right] \\ &\lesssim \sum_{i\in\Omega_{k}}\frac{1}{w_{i}}\left(\frac{\mu_{1}r}{n}\right)^{2}s\mu_{0}\cdot\left\|\mathcal{G}^{*}(\boldsymbol{Z})\boldsymbol{e}_{i}\right\|_{2}^{2} \\ &\lesssim \left(\frac{\mu_{1}r}{n}\right)^{2}s\mu_{0}\cdot\left\|\boldsymbol{Z}\right\|_{\mathcal{G},\mathsf{F}}^{2}, \end{split}$$

where the second line is due to (5.8) and the third line follows from (5.5). It also holds that $\mathbb{E}\left[\sum_{i\in\Omega_k} (\boldsymbol{y}_i^j)^* \boldsymbol{y}_i^j\right] \leq 1$ $\left(\frac{\mu_1 r}{n}\right)^2 s\mu_0 \cdot \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}}^2.$ Applying the matrix Bernstein inequality and taking the uniform bound implies that

$$\frac{n}{m} \sup_{0 \le j \le n-1} \left\| \sum_{i \in \Omega_k} \boldsymbol{y}_i^j \right\|_2 \lesssim \frac{n}{m} \left(\frac{\mu_1 r}{n} \sqrt{s\mu_0 \log(sn)} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + s\mu_0 \log(sn) \frac{\mu_1 r}{n} \|\boldsymbol{Z}\|_{\mathcal{G},\infty} \right)$$
$$= \frac{\mu_1 r}{n} \left(\sqrt{\frac{nk_0 s\mu_0 \log(sn)}{m}} \|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{ns\mu_0 \log(sn)}{m} \|\boldsymbol{Z}\|_{\mathcal{G},\infty} \right)$$

holds with probability at least $1 - ns^{-c_2}$ for a numerical constant $c_2 > 2$. Noting (5.6) and (5.7) we can conclude that

$$\begin{aligned} \left\| \mathcal{P}_{T}\mathcal{G}\left(\mathcal{I} - \frac{n}{m}\mathcal{A}_{k}^{*}\mathcal{A}_{k}\right)\mathcal{G}^{*}(\mathbf{Z})\right\|_{\mathcal{G},\infty} &\lesssim \frac{\mu_{1}r}{n}\left(\sqrt{\frac{n\log(sn)}{m}} \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{n\log(sn)}{m} \|\mathbf{Z}\|_{\mathcal{G},\infty}\right) \\ &+ \frac{\mu_{1}r}{n}\left(\sqrt{\frac{nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{ns\mu_{0}\log(sn)}{m} \|\mathbf{Z}\|_{\mathcal{G},\infty}\right) \\ &\lesssim \frac{\mu_{1}r}{n}\left(\sqrt{\frac{4nk_{0}s\mu_{0}\log(sn)}{m}} \|\mathbf{Z}\|_{\mathcal{G},\mathsf{F}} + \frac{2ns\mu_{0}\log(sn)}{m} \|\mathbf{Z}\|_{\mathcal{G},\infty}\right) \end{aligned}$$

holds with probability exceeding $1 - ns^{-c_2}$.

5.5Proof of Lemma 3.14

According to (3.4), a simple algebra yields that

$$\max_{0 \le i \le n_1 - 1} \| \boldsymbol{U}_i \boldsymbol{V}^* \|_{\mathsf{F}}^2 \le \max_{0 \le i \le n_1 - 1} \| \boldsymbol{U}_i \|_{\mathsf{F}}^2 \le \frac{\mu_1 r}{n}.$$

Then the application of Corollary 6.8 implies that

$$\|\boldsymbol{U}\boldsymbol{V}^*\|_{\mathcal{G},\mathsf{F}}^2 \lesssim rac{\mu_1 r \log(sn)}{n}.$$

The upper bound of $\| \boldsymbol{U} \boldsymbol{V}^* \|_{\mathcal{G},\infty}$ can be established as follows. Note that

$$\|\boldsymbol{U}\boldsymbol{V}^*\|_{\mathcal{G},\infty} = \max_{0 \le i \le n-1} \frac{\|\mathcal{G}^*(\boldsymbol{U}\boldsymbol{V}^*)\boldsymbol{e}_i\|_2}{\sqrt{w_i}}.$$

For any fixed $i \in [n]$, we have

$$\begin{split} \frac{\|\mathcal{G}^{*}(UV^{*})e_{i}\|_{2}}{\sqrt{w_{i}}} &= \frac{1}{\sqrt{w_{i}}} \sup_{\|\beta\|_{2}=1} |\langle \mathcal{G}^{*}(UV^{*})e_{i},\beta\rangle| \\ &= \frac{1}{\sqrt{w_{i}}} \sup_{\|\beta\|_{2}=1} |\langle UV^{*},\mathcal{G}(\beta e_{i}^{\mathsf{T}})\rangle| \\ &= \frac{1}{\sqrt{w_{i}}} \sup_{\|\beta\|_{2}=1} |\langle UV^{*},\mathcal{G}_{i}\otimes\beta\rangle| \\ &= \frac{1}{w_{i}} \sup_{\|\beta\|_{2}=1} |\langle UV^{*}, \left(\sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} e_{j}e_{k}^{\mathsf{T}}\right) \otimes \beta \rangle \\ &= \frac{1}{w_{i}} \sup_{\|\beta\|_{2}=1} |\langle UV^{*}, \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} (e_{j}\otimes\beta)e_{k}^{\mathsf{T}}\rangle \\ &= \frac{1}{w_{i}} \sup_{\|\beta\|_{2}=1} \left|\sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \langle (e_{j}\otimes\beta)^{*}U, e_{k}^{\mathsf{T}}V\rangle \right| \\ &\leq \frac{1}{w_{i}} \sup_{\|\beta\|_{2}=1} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|(e_{j}\otimes\beta)^{*}U\|_{2} \|e_{k}^{\mathsf{T}}V\|_{2} \\ &\leq \sup_{\|\beta\|_{2}=1} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|(e_{j}\otimes\beta)^{*}U\|_{2}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|e_{k}^{\mathsf{T}}V\|_{2}^{2} \\ &\leq \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|\beta^{*}U_{j}\|_{2}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|e_{k}^{\mathsf{T}}V\|_{2}^{2} \\ &\leq \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|U_{j}\|_{\mathsf{F}}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|e_{k}^{\mathsf{T}}V\|_{2}^{2} \\ &\leq \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|U_{j}\|_{\mathsf{F}}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}} \|e_{k}^{\mathsf{T}}V\|_{2}^{2} \\ &\leq \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{\mathsf{F}}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|e_{k}^{\mathsf{T}}V\|_{2}^{2} \\ &\leq \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{\mathsf{F}}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{\mathsf{F}}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{1}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{i}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{1}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{2}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{1}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq j\leq n_{2}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{1}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq k\leq n_{2}-1\\0\leq k\leq n_{2}-1}}} \|U_{j}\|_{1}^{2} \sqrt{\frac{1}{w_{i}}} \sum_{\substack{j+k=i\\0\leq k\leq n_{2}-1$$

where the fourth line is due to the definition of G_i in (1.18) and the last line follows from (3.5). Therefore, $\|UV^*\|_{\mathcal{G},\infty} \leq \frac{\mu_1 r}{n}$.

6 Auxiliary Results

In this section, we present some necessary results which have been used in the previous proofs. The following lemma is used in the proof of Theorem 3.7.

Lemma 6.1. Suppose $\|\mathcal{A}\mathcal{A}^*\| \geq 1$ and $\|\mathcal{P}_T\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*\mathcal{P}_T - \mathcal{P}_T\mathcal{G}\mathcal{G}^*\mathcal{P}_T\| \leq \frac{1}{2}$. For any $M \in \mathbb{C}^{sn_1 \times n_2}$ which obeys

$$\mathcal{AG}^*(M) = 0$$
 and $(\mathcal{I} - \mathcal{GG}^*)(M) = 0$,

we have

$$\left\|\mathcal{P}_{T}(\boldsymbol{M})\right\|_{\mathsf{F}} \leq 4s\mu_{0}\left\|\mathcal{P}_{T^{\perp}}(\boldsymbol{M})\right\|_{\mathsf{F}}$$

Proof. It follows (3.12) and (3.13) that

$$\begin{split} 0 &= \left\| \left(\mathcal{G}\mathcal{A}^* \mathcal{A}\mathcal{G}^* + (\mathcal{I} - \mathcal{G}\mathcal{G}^*) \right)(M) \right\|_{\mathsf{F}} \\ &\geq \left\| \left(\mathcal{G}\mathcal{A}^* \mathcal{A}\mathcal{G}^* + (\mathcal{I} - \mathcal{G}\mathcal{G}^*) \right) \mathcal{P}_T(M) \right\|_{\mathsf{F}} - \left\| \left(\mathcal{G}\mathcal{A}^* \mathcal{A}\mathcal{G}^* + (\mathcal{I} - \mathcal{G}\mathcal{G}^*) \right) \mathcal{P}_{T^{\perp}}(M) \right\|_{\mathsf{F}}. \end{split}$$

For the first term,

$$\begin{split} \|(\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^* + (\mathcal{I} - \mathcal{G}\mathcal{G}^*))\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 &= \|\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 + \|(\mathcal{I} - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 \\ &= \langle \mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M}), \mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M}) \rangle + \langle \mathcal{P}_T(\boldsymbol{M}), (\mathcal{I} - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T(\boldsymbol{M}) \rangle \\ &= \langle \mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M}), (\mathcal{A}\mathcal{A}^*)\mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M}) \rangle + \langle \mathcal{P}_T(\boldsymbol{M}), (\mathcal{I} - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T(\boldsymbol{M}) \rangle \\ &\geq \langle \mathcal{P}_T(\boldsymbol{M}), \mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*\mathcal{P}_T(\boldsymbol{M}) \rangle + \langle \mathcal{P}_T(\boldsymbol{M}), (\mathcal{I} - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T(\boldsymbol{M}) \rangle \\ &= \|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 + \langle \mathcal{P}_T(\boldsymbol{M}), \mathcal{P}_T(\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^* - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T(\boldsymbol{M}) \rangle \\ &\geq \|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 - \|\mathcal{P}_T(\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^* - \mathcal{G}\mathcal{G}^*)\mathcal{P}_T\| \cdot \|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2 \\ &\geq \frac{1}{2}\|\mathcal{P}_T(\boldsymbol{M})\|_{\mathsf{F}}^2. \end{split}$$

where the fourth step is due to (3.2) in Lemma 3.2.

For the second term,

$$\begin{split} \| (\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^* + (\mathcal{I} - \mathcal{G}\mathcal{G}^*))\mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} &\leq \| (\mathcal{G}\mathcal{A}^*\mathcal{A}\mathcal{G}^*)\mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} + \| (\mathcal{I} - \mathcal{G}\mathcal{G}^*)\mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} \\ &\leq \| \mathcal{G} \| \cdot \| \mathcal{A}^*\mathcal{A} \| \cdot \| \mathcal{G}^* \| \cdot \| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} + \| \mathcal{I} - \mathcal{G}\mathcal{G}^* \| \cdot \| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} \\ &\leq (1 + s\mu_0) \| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} \\ &\leq 2s\mu_0 \| \mathcal{P}_{T^{\perp}}(\boldsymbol{M}) \|_{\mathsf{F}} \end{split}$$

where the third line is due to $\|\mathcal{G}\| = 1$, $\|\mathcal{G}^*\| \le 1$ and (3.3) in Lemma 3.2.

Combining these two terms together completes the proof.

The following lemmas play an important role in the proofs of Lemmas 3.8 to 3.14.

Lemma 6.2. Recall that U and V obey (III.4). For any fixed $z \in \mathbb{C}^s$, there holds

$$\max_{0 \le i \le n-1} \left\| \boldsymbol{U}^* \mathcal{G}(\boldsymbol{z} \boldsymbol{e}_i^\mathsf{T}) \right\|_\mathsf{F}^2 \le \left\| \boldsymbol{z} \right\|_2^2 \cdot \frac{\mu_1 r}{n},\tag{6.1}$$

$$\max_{0 \le i \le n-1} \left\| \mathcal{G}(\boldsymbol{z}\boldsymbol{e}_i^{\mathsf{T}}) \boldsymbol{V} \right\|_{\mathsf{F}}^2 \le \|\boldsymbol{z}\|_2^2 \cdot \frac{\mu_1 r}{n},\tag{6.2}$$

$$\max_{0 \le i \le n-1} \left\| \mathcal{P}_T \mathcal{G}(\boldsymbol{z} \boldsymbol{e}_i^{\mathsf{T}}) \right\|_{\mathsf{F}}^2 \le 2 \left\| \boldsymbol{z} \right\|_2^2 \cdot \frac{\mu_1 r}{n}.$$
(6.3)

Proof. To show (6.1), note that for any $0 \le i \le n-1$,

$$egin{aligned} \mathcal{G}(oldsymbol{z}oldsymbol{e}_i^{\mathsf{T}}) &= oldsymbol{G}_i\otimesoldsymbol{z} \ &= \left(\sum_{\substack{j+k=i\ 0\leq j\leq n_1-1\ 0\leq k\leq n_2-1}} rac{1}{\sqrt{w_i}}oldsymbol{e}_joldsymbol{e}_k^{\mathsf{T}}
ight)\otimesoldsymbol{z} \ &= \sum_{\substack{j+k=i\ 0\leq j\leq n_1-1\ 0\leq k\leq n_2-1}} rac{1}{\sqrt{w_i}}\left(oldsymbol{e}_j\otimesoldsymbol{z}
ight)oldsymbol{e}_k^{\mathsf{T}}, \end{aligned}$$

where the second equality is due to the definition of ${m G}_i$ in (1.18). It follows that

$$\begin{split} \left\| \boldsymbol{U}^{*} \mathcal{G}(\boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}}) \right\|_{\mathsf{F}}^{2} &= \left\langle \boldsymbol{U}^{*} \mathcal{G}(\boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}}), \boldsymbol{U}^{*} \mathcal{G}(\boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}}) \right\rangle \\ &= \frac{1}{w_{i}} \left\langle \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \boldsymbol{U}^{*}\left(\boldsymbol{e}_{j} \otimes \boldsymbol{z}\right) \boldsymbol{e}_{k}^{\mathsf{T}}, \sum_{\substack{p+q=i\\0 \leq p \geq n_{1}-1\\0 \leq q \leq n_{2}-1}} \boldsymbol{U}^{*}\left(\boldsymbol{e}_{p} \otimes \boldsymbol{z}\right) \boldsymbol{e}_{q}^{\mathsf{T}} \right\rangle \\ &= \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \left\| \boldsymbol{U}^{*}\left(\boldsymbol{e}_{j} \otimes \boldsymbol{z}\right) \right\|_{2}^{2} \\ &= \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0 \leq k \leq n_{2}-1\\0 \leq k \leq n_{2}-1}} \left\| \boldsymbol{U}^{*}_{j} \boldsymbol{z} \right\|_{2}^{2} \\ &= \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0 \leq k \leq n_{2}-1\\0 \leq k \leq n_{2}-1}} \left\| \boldsymbol{U}_{j}^{*} \boldsymbol{z} \right\|_{F}^{2} \\ &\leq \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0 \leq k \leq n_{2}-1\\0 \leq k \leq n_{2}-1}} \left\| \boldsymbol{z} \right\|_{2}^{2} \cdot \left\| \boldsymbol{U}_{j} \right\|_{F}^{2} \\ &\leq \| \boldsymbol{z} \|_{2}^{2} \cdot \frac{\mu_{1}r}{n}, \end{split}$$

where the last step follows from (3.5).

As for (6.2), note that

$$\begin{split} \left\| \mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}})\boldsymbol{V} \right\|_{\mathsf{F}}^{2} &= \left\langle \mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}})\boldsymbol{V}, \mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}})\boldsymbol{V} \right\rangle \\ &= \frac{1}{w_{i}} \left\langle \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} (\boldsymbol{e}_{j} \otimes \boldsymbol{z})\boldsymbol{e}_{k}^{\mathsf{T}}\boldsymbol{V}, \sum_{\substack{p+q=i\\0 \leq q \leq n_{1}-1\\0 \leq q \leq n_{2}-1}} (\boldsymbol{e}_{p} \otimes \boldsymbol{z})\boldsymbol{e}_{q}^{\mathsf{T}}\boldsymbol{V} \right\rangle \\ &= \frac{1}{w_{i}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \sum_{\substack{p+q=i\\0 \leq p \leq n_{1}-1\\0 \leq q \leq n_{2}-1}} \left\langle (\boldsymbol{e}_{j} \otimes \boldsymbol{z})\boldsymbol{e}_{k}^{\mathsf{T}}\boldsymbol{V}, (\boldsymbol{e}_{p} \otimes \boldsymbol{z})\boldsymbol{e}_{q}^{\mathsf{T}}\boldsymbol{V} \right\rangle \end{split}$$

$$\begin{split} &= \frac{1}{w_i} \sum_{\substack{j+k=i\\0 \leq j \leq n_1-1 \ 0 \leq p \leq n_1-1\\0 \leq k \leq n_2-1 \ 0 \leq q \leq n_2-1}} \sum_{\substack{p+q=i\\0 \leq j \leq n_1-1 \ 0 \leq q \leq n_2-1}} \left\langle \left(\boldsymbol{e}_p^{\mathsf{T}} \otimes \boldsymbol{z}^*\right) \left(\boldsymbol{e}_j \otimes \boldsymbol{z}\right) \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V}, \boldsymbol{e}_q^{\mathsf{T}} \boldsymbol{V} \right\rangle \\ &= \frac{1}{w_i} \sum_{\substack{j+k=i\\0 \leq j \leq n_1-1 \ 0 \leq p \leq n_1-1\\0 \leq k \leq n_2-1}} \sum_{\substack{p+q=i\\0 \leq q \leq n_2-1}} \left\langle \left(\boldsymbol{e}_p^{\mathsf{T}} \boldsymbol{e}_j\right) \otimes \left(\boldsymbol{z}^* \boldsymbol{z}\right) \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V}, \boldsymbol{e}_q^{\mathsf{T}} \boldsymbol{V} \right\rangle \\ &= \frac{1}{w_i} \sum_{\substack{j+k=i\\0 \leq j \leq n_1-1\\0 \leq k \leq n_2-1}} \left\langle \boldsymbol{z}^* \boldsymbol{z} \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V}, \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V} \right\rangle \\ &= \frac{\|\boldsymbol{z}\|_2^2}{w_i} \sum_{\substack{j+k=i\\0 \leq j \leq n_1-1\\0 \leq k \leq n_2-1}} \left\langle \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V}, \boldsymbol{e}_k^{\mathsf{T}} \boldsymbol{V} \right\rangle \\ &\leq \|\boldsymbol{z}\|_2^2 \frac{\mu_1 r}{n}, \end{split}$$

where the last step is also due to (3.5).

For the inequality (6.3), by the definition of \mathcal{P}_T in (3.7), we have

$$\begin{split} \left\| \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \right\|_{\mathsf{F}}^{2} &= \left\langle \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right), \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \right\rangle \\ &= \left\langle \mathcal{P}_{T}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right), \mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}) \right\rangle \\ &= \left\langle \boldsymbol{U}\boldsymbol{U}^{*}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) + \mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{V}\boldsymbol{V}^{*} - \boldsymbol{U}\boldsymbol{U}^{*}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{V}\boldsymbol{V}^{*}, \mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \right\rangle \\ &= \left\| \boldsymbol{U}^{*}\mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}) \right\|_{\mathsf{F}}^{2} + \left\| \mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{V} \right\|_{\mathsf{F}}^{2} - \left\| \boldsymbol{U}^{*}\mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{V} \right\|_{\mathsf{F}}^{2} \\ &\leq \left\| \boldsymbol{U}^{*}\mathcal{G}(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}) \right\|_{\mathsf{F}}^{2} + \left\| \mathcal{G}\left(\boldsymbol{z}\boldsymbol{e}_{i}^{\mathsf{T}}\right) \boldsymbol{V} \right\|_{\mathsf{F}}^{2} \\ &\leq 2 \left\| \boldsymbol{z} \right\|_{2}^{2} \frac{\mu_{1}r}{n}, \end{split}$$

which completes the proof.

After replacing z with b_i in Lemma 6.2, we obtain the following corollary based on the incoherence property (2.6) of b_i , where b_i is the *i*th column of B^* .

Corollary 6.3. Under the condition (3.4), there holds

$$\max_{0 \le i \le n-1} \left\| \boldsymbol{U}^* \mathcal{G}(\boldsymbol{b}_i \boldsymbol{e}_i^\mathsf{T}) \right\|_\mathsf{F}^2 \le \frac{\mu_0 \mu_1 s r}{n},\tag{6.4}$$

$$\max_{0 \le i \le n-1} \left\| \mathcal{G}(\boldsymbol{b}_i \boldsymbol{e}_i^{\mathsf{T}}) \boldsymbol{V} \right\|_{\mathsf{F}}^2 \le \frac{\mu_0 \mu_1 s r}{n},\tag{6.5}$$

$$\max_{0 \le i \le n-1} \left\| \mathcal{P}_T \mathcal{G}(\boldsymbol{b}_i \boldsymbol{e}_i^{\mathsf{T}}) \right\|_{\mathsf{F}}^2 \le \frac{2\mu_0 \mu_1 sr}{n}.$$
(6.6)

Lemma 6.4. Under the condition (3.4), for any fixed matrix $\boldsymbol{W} \in \mathbb{C}^{sn_1 \times n_2}$,

$$\|\mathcal{G}^*\mathcal{P}_T(\boldsymbol{W})\boldsymbol{e}_i\|_2 \le \|\boldsymbol{W}\|_{\mathsf{F}} \cdot \sqrt{\frac{2\mu_1 r}{n}}.$$
(6.7)

Proof. The result follows from a direct calculation:

$$\left\|\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{W})\boldsymbol{e}_{i}
ight\|_{2}=\sup_{\left\|\boldsymbol{eta}
ight\|_{2}=1}\left|\left\langle\mathcal{G}^{*}\mathcal{P}_{T}(\boldsymbol{W})\boldsymbol{e}_{i},\boldsymbol{eta}
ight
angle
ight|$$

$$= \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left| \left\langle \mathcal{G}^{*} \mathcal{P}_{T}(\boldsymbol{W}), \boldsymbol{\beta} \boldsymbol{e}_{i}^{\mathsf{T}} \right\rangle \right|$$

$$= \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left| \left\langle \boldsymbol{W}, \mathcal{P}_{T} \mathcal{G}(\boldsymbol{\beta} \boldsymbol{e}_{i}^{\mathsf{T}}) \right\rangle \right|$$

$$\leq \|\boldsymbol{W}\|_{\mathsf{F}} \cdot \sup_{\|\boldsymbol{\beta}\|_{2}=1} \left\| \mathcal{P}_{T} \mathcal{G}(\boldsymbol{\beta} \boldsymbol{e}_{i}^{\mathsf{T}}) \right\|_{\mathsf{F}}$$

$$\leq \|\boldsymbol{W}\|_{\mathsf{F}} \cdot \sqrt{\frac{2\mu_{1}r}{n}},$$

where the last line follows from (6.3) in Lemma 6.2.

By combining Lemmas 6.2 and 6.4, the following corollary can be established, which is used in the proof of (3.20).

Corollary 6.5. For any fixed matrix $W \in \mathbb{C}^{sn_1 \times n_2}$, under the condition (3.4), there holds

$$\max_{0 \le i \le n-1} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{G}^* \mathcal{P}_T (\boldsymbol{W}) \boldsymbol{e}_i \boldsymbol{e}_i^\mathsf{T} \right) \right\|_\mathsf{F}^2 \le \|\boldsymbol{W}\|_\mathsf{F}^2 \cdot \left(\frac{2\mu_1 r}{n} \right)^2,$$

Proof. Applying Lemma 6.2 yields that

$$\max_{0 \le i \le n-1} \left\| \mathcal{P}_T \mathcal{G} \left(\mathcal{G}^* \mathcal{P}_T (\boldsymbol{W}) \boldsymbol{e}_i \boldsymbol{e}_i^\mathsf{T} \right) \right\|_\mathsf{F}^2 \le \left\| \mathcal{G}^* \mathcal{P}_T (\boldsymbol{W}) \boldsymbol{e}_i \right\|_2^2 \cdot \frac{2\mu_1 n}{n} \\ \le \left\| \boldsymbol{W} \right\|_\mathsf{F}^2 \cdot \left(\frac{2\mu_1 r}{n} \right)^2,$$

where the last line is due to Lemma 6.4.

Lemma 6.6. For any two fixed vectors $\beta, \gamma \in \mathbb{C}^s$,

$$\sqrt{\frac{w_i}{w_j}} \left| \left\langle \mathcal{P}_T \mathcal{G}(\boldsymbol{\beta} \boldsymbol{e}_i^{\mathsf{T}}), \mathcal{G}(\boldsymbol{\gamma} \boldsymbol{e}_j^{\mathsf{T}}) \right\rangle \right| \le \frac{3\mu_1 r}{n} \cdot \|\boldsymbol{\beta}\|_2 \|\boldsymbol{\gamma}\|_2$$

holds for any $(i, j) \in [n] \times [n]$. Proof. Recall that

$$\mathcal{G}(\boldsymbol{\beta}\boldsymbol{e}_{i}^{\mathsf{T}}) = \boldsymbol{G}_{i} \otimes \boldsymbol{\beta} = \left(\frac{1}{\sqrt{w_{i}}} \sum_{\substack{k+t=i\\ 0 \leq k \leq n_{1}-1\\ 0 \leq t \leq n_{2}-1}} \boldsymbol{e}_{k}\boldsymbol{e}_{t}^{\mathsf{T}}\right) \otimes \boldsymbol{\beta} \quad \text{and} \quad \mathcal{G}(\boldsymbol{\gamma}\boldsymbol{e}_{j}^{\mathsf{T}}) = \boldsymbol{G}_{j} \otimes \boldsymbol{\gamma} = \left(\frac{1}{\sqrt{w_{j}}} \sum_{\substack{p+q=j\\ 0 \leq p \leq n_{1}-1\\ 0 \leq q \leq n_{2}-1}} \boldsymbol{e}_{p}\boldsymbol{e}_{q}^{\mathsf{T}}\right) \otimes \boldsymbol{\gamma}.$$

By the definition of \mathcal{P}_T in (3.7), we have

$$egin{aligned} &\sqrt{rac{w_i}{w_j}} \left| \left\langle \mathcal{P}_T \mathcal{G}(oldsymbol{eta} oldsymbol{e}_i^{\mathsf{T}}), \mathcal{G}(oldsymbol{\gamma} oldsymbol{e}_j^{\mathsf{T}})
ight
angle
ight| &\leq \sqrt{rac{w_i}{w_j}} \left| \left\langle oldsymbol{U} oldsymbol{U}^* \left(oldsymbol{G}_i \otimes oldsymbol{eta}
ight), oldsymbol{G}_j \otimes oldsymbol{\gamma}
ight
angle
ight| + \sqrt{rac{w_i}{w_j}} \left| \left\langle oldsymbol{U} oldsymbol{U}^* \left(oldsymbol{G}_i \otimes oldsymbol{eta}
ight), oldsymbol{V} oldsymbol{V}^*, oldsymbol{G}_j \otimes oldsymbol{\gamma}
ight
angle
ight| + \sqrt{rac{w_i}{w_j}} \left| \left\langle oldsymbol{U} oldsymbol{U}^* \left(oldsymbol{G}_i \otimes oldsymbol{eta}
ight) oldsymbol{V} oldsymbol{V}^*, oldsymbol{G}_j \otimes oldsymbol{\gamma}
ight
angle
ight|. \end{aligned}$$

It suffices to bound each of the three terms separately. For the first term, we have

$$\sqrt{\frac{w_i}{w_j}} \left| \left\langle \boldsymbol{U}\boldsymbol{U}^* \left(\boldsymbol{G}_i \otimes \boldsymbol{\beta}\right), \boldsymbol{G}_j \otimes \boldsymbol{\gamma} \right\rangle \right| = \sqrt{\frac{w_i}{w_j}} \left| \left\langle \boldsymbol{U}\boldsymbol{U}^* \left(\left(\sum_{\substack{k+t=i\\0 \leq k \leq n_1-1\\0 \leq t \leq n_2-1}} \frac{1}{\sqrt{w_i}} \boldsymbol{e}_k \boldsymbol{e}_t^\mathsf{T} \right) \otimes \boldsymbol{\beta} \right), \left(\sum_{\substack{p+q=j\\0 \leq p \leq n_1-1\\0 \leq q \leq n_2-1}} \frac{1}{\sqrt{w_j}} \boldsymbol{e}_p \boldsymbol{e}_q^\mathsf{T} \right) \otimes \boldsymbol{\gamma} \right\rangle$$

$$\begin{split} &= \sqrt{\frac{w_i}{w_j}} \left| \frac{1}{\sqrt{w_i w_j}} \left\langle \sum_{\substack{k+t=i\\0 \le k \le n_1 - 1\\0 \le t \le n_2 - 1}} UU^*(e_k \otimes \beta) e_t^{\mathsf{T}}, \sum_{\substack{p+q=j\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} (e_p \otimes \gamma) e_q^{\mathsf{T}} \right\rangle \right| \\ &= \frac{1}{w_j} \left| \sum_{\substack{k+t=i\\0 \le k \le n_1 - 1\\0 \le t \le n_2 - 1}} \sum_{\substack{p+q=j\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \langle UU^*(e_k \otimes \beta) e_t^{\mathsf{T}}, (e_p \otimes \gamma) e_q^{\mathsf{T}} \right\rangle \right| \\ &= \frac{1}{w_j} \left| \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \langle U^*(e_{i-q} \otimes \beta), U^*(e_p \otimes \gamma) \rangle \right| \\ &\leq \frac{1}{w_j} \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \|U^*(e_{i-q} \otimes \beta)\|_2 \cdot \|U^*(e_p \otimes \gamma)\|_2 \\ &= \frac{1}{w_j} \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \|U_{i-q}\|_{\mathsf{F}} \cdot \|U_p\|_{\mathsf{F}} \cdot \|\beta\|_2 \cdot \|\gamma\|_2 \\ &\leq \frac{1}{w_j} \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \|U_{i-q}\|_{\mathsf{F}}^2 \cdot \sqrt{\frac{1}{w_j} \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \|U_{i-q}\|_{\mathsf{F}}^2 \cdot \sqrt{\frac{1}{w_j} \sum_{\substack{p+q=j,q \le i\\0 \le p \le n_1 - 1\\0 \le q \le n_2 - 1}} \|U_{p}\|_2 \cdot \|\gamma\|_2 \\ &\leq \frac{\mu_1 r}{n} \cdot \|\beta\|_2 \cdot \|\gamma\|_2 . \end{split}$$

The second term can be bounded in a similar way. For the last term, we have

$$\begin{split} \sqrt{\frac{w_i}{w_j}} \left| \langle \boldsymbol{U}\boldsymbol{U}^* \left(\boldsymbol{G}_i \otimes \boldsymbol{\beta}\right) \boldsymbol{V}\boldsymbol{V}^*, \boldsymbol{G}_j \otimes \boldsymbol{\gamma} \rangle \right| &= \sqrt{\frac{w_i}{w_j}} \left| \langle \boldsymbol{U}\boldsymbol{U}^* \left(\boldsymbol{G}_i \otimes \boldsymbol{\beta}\right), \left(\boldsymbol{G}_j \otimes \boldsymbol{\gamma}\right) \boldsymbol{V}\boldsymbol{V}^* \rangle \right| \\ &= \frac{1}{w_j} \left| \sum_{\substack{k+t=i \\ 0 \leq k \leq n_1 - 1}} \sum_{\substack{p+q=j \\ 0 \leq t \leq n_2 - 1}} \langle \boldsymbol{U}\boldsymbol{U}^* \left(\boldsymbol{e}_k \otimes \boldsymbol{\beta}\right) \boldsymbol{e}_t^\mathsf{T}, \left(\boldsymbol{e}_p \otimes \boldsymbol{\gamma}\right) \boldsymbol{e}_q^\mathsf{T} \boldsymbol{V}\boldsymbol{V}^* \rangle \right| \\ &= \frac{1}{w_j} \left| \sum_{\substack{k+t=i \\ 0 \leq t \leq n_2 - 1}} \sum_{\substack{p+q=j \\ 0 \leq k \leq n_1 - 1}} \langle \left(\boldsymbol{e}_p^\mathsf{T} \otimes \boldsymbol{\gamma}^*\right) \boldsymbol{U}\boldsymbol{U}^* \left(\boldsymbol{e}_k \otimes \boldsymbol{\beta}\right), \boldsymbol{e}_q^\mathsf{T} \boldsymbol{V} \boldsymbol{V}^* \boldsymbol{e}_t \rangle \end{split} \right| \end{split}$$

$$\begin{split} &= \frac{1}{w_j} \left| \sum_{\substack{k+t=i \\ 0 \leq k \leq n_1 - 1 \\ 0 \leq k \leq n_1 - 1 \\ 0 \leq k \leq n_1 - 1 \\ 0 \leq t \leq n_2 - 1$$

where the last step is due to (3.5).

Combining the three bounds together completes the proof.

The following lemma is established in [15] and the proof will be omitted here.

Lemma 6.7. Suppose a matrix $F \in \mathbb{C}^{n_1 \times n_2}$ satisfies

$$\max_{0 \le i \le n_1 - 1} \left\| \boldsymbol{e}_i^\mathsf{T} \boldsymbol{F} \right\|_2^2 \le B.$$
(6.8)

We have

$$\sum_{i=0}^{n-1} \frac{1}{w_i} \left| \langle \boldsymbol{F}, \boldsymbol{G}_i \rangle \right|^2 \lesssim B \log(n).$$
(6.9)

We will apply this lemma to upper bound $\|Z\|_{\mathcal{G},\mathsf{F}}$ for $Z \in \mathbb{C}^{sn_1 \times n_2}$. Note that Z can be written as

where $\boldsymbol{z}_{i,j} \in \mathbb{C}^s$ is the (i,j)th block of \boldsymbol{Z} .

Corollary 6.8. For any matrix $Z \in \mathbb{C}^{sn_1 \times n_2}$ satisfying

$$\max_{0 \le i \le n_1 - 1} \sum_{j=0}^{n_2 - 1} \|\boldsymbol{z}_{i,j}\|_2^2 \le B,$$
(6.10)

we have

$$\|\boldsymbol{Z}\|_{\mathcal{G},\mathsf{F}}^2 \lesssim B\log(n). \tag{6.11}$$

Proof. Define the matrix

$$\widetilde{Z} = egin{bmatrix} \|m{z}_{0,0}\|_2 & \cdots & \|m{z}_{0,n_2-1}\|_2 \ dots & \ddots & dots \ \|m{z}_{n_1-1,0}\|_2 & \cdots & \|m{z}_{n_1-1,n_2-1}\|_2 \end{bmatrix} \in \mathbb{R}^{n_1 imes n_2}.$$

The definition of \mathcal{G}^* implies that the *i*th column of $\mathcal{G}^*(Z)$ is given by

$$\mathcal{G}^*(\boldsymbol{Z}) \boldsymbol{e}_i = rac{1}{\sqrt{w_i}} \sum_{\substack{j+k=i \ 0 \leq j \leq n_1-1 \ 0 \leq k \leq n_2-1}} \boldsymbol{z}_{j,k},$$

It follows that

$$\begin{split} \|Z\|_{\mathcal{G},\mathsf{F}}^{2} &= \sum_{i=0}^{n-1} \frac{1}{w_{i}} \|\mathcal{G}^{*}(Z)e_{i}\|_{2}^{2} \\ &= \sum_{i=0}^{n-1} \frac{1}{w_{i}} \left\| \frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} z_{j,k} \right\|_{2}^{2} \\ &\leq \sum_{i=0}^{n-1} \frac{1}{w_{i}} \left(\frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \|z_{j,k}\|_{2} \right)^{2} \\ &= \sum_{i=0}^{n-1} \frac{1}{w_{i}} \left(\frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \left\langle \widetilde{Z}e_{k}, e_{j} \right\rangle \right)^{2} \\ &= \sum_{i=0}^{n-1} \frac{1}{w_{i}} \left(\frac{1}{\sqrt{w_{i}}} \sum_{\substack{j+k=i\\0 \leq j \leq n_{1}-1\\0 \leq k \leq n_{2}-1}} \left\langle \widetilde{Z}, e_{j}e_{k}^{\mathsf{T}} \right\rangle \right)^{2} \\ &= \sum_{i=0}^{n-1} \frac{1}{w_{i}} \left(\left\langle \widetilde{Z}, G_{i} \right\rangle \right)^{2}, \end{split}$$

where the last line follows from the definition of G_i in (1.18). Since the condition (6.10) implies that $\max_{0 \le i \le n_1 - 1} \left\| \boldsymbol{e}_i^{\mathsf{T}} \widetilde{\boldsymbol{Z}} \right\|_2^2 \le B$, applying Lemma 6.7 completes the proof.

The following lemma can be established based on Corollary 6.8. It has been used in the proofs of (3.22)and (3.27).

Lemma 6.9. For any fixed $z \in \mathbb{C}^s$,

$$\left\| \mathcal{P}_T \mathcal{G}(\sqrt{w_i} \boldsymbol{z} \boldsymbol{e}_i^{\mathsf{T}}) \right\|_{\mathcal{G},\mathsf{F}}^2 \lesssim \|\boldsymbol{z}\|_2^2 \cdot \frac{\mu_1 r \log(sn)}{n}.$$

Proof. Recalling the definition of \mathcal{P}_T in (3.7), we have

$$\mathcal{P}_{T}\mathcal{G}(\sqrt{w_{i}}ze_{i}^{\mathsf{T}}) = UU^{*}\mathcal{G}\left(\sqrt{w_{i}}ze_{i}^{\mathsf{T}}\right) + \mathcal{G}\left(\sqrt{w_{i}}ze_{i}^{\mathsf{T}}\right)VV^{*} - UU^{*}\mathcal{G}\left(\sqrt{w_{i}}ze_{i}^{\mathsf{T}}\right)VV^{*}.$$

It suffices to bound the three terms separately. For the first term, recall that $U \in \mathbb{C}^{sn_1 \times r}$ can be rewritten as

$$oldsymbol{U} = egin{bmatrix} oldsymbol{U}_0 \ dots \ oldsymbol{U}_{n_1-1} \end{bmatrix},$$

where $U_{\ell} \in \mathbb{C}^{s \times r}$ is the ℓ -th block. Since

$$\begin{split} \left\| \boldsymbol{U}_{\ell} \boldsymbol{U}^{*} \mathcal{G} \left(\sqrt{w_{i}} \boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\|_{\mathsf{F}}^{2} &= w_{i} \left\| \boldsymbol{U}_{\ell} \boldsymbol{U}^{*} \left(\boldsymbol{G}_{i} \otimes \boldsymbol{z} \right) \right\|_{\mathsf{F}}^{2} \\ &\leq w_{i} \left\| \boldsymbol{U}_{\ell} \right\|_{\mathsf{F}}^{2} \cdot \left\| \boldsymbol{U} \right\|^{2} \cdot \left\| \boldsymbol{G}_{i} \otimes \boldsymbol{z} \right\|^{2} \\ &\leq w_{i} \frac{\mu_{1} r}{n} \cdot \left\| \boldsymbol{G}_{i} \right\|^{2} \cdot \left\| \boldsymbol{z} \right\|_{2}^{2} \\ &\leq \frac{\mu_{1} r}{n} \cdot \left\| \boldsymbol{z} \right\|_{2}^{2}, \end{split}$$

where the third line follows from (3.4), then the application of Corollary 6.8 yields that

$$\left\| \boldsymbol{U} \boldsymbol{U}^{*} \boldsymbol{\mathcal{G}} \left(\sqrt{w_{i}} \boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \right\|_{\boldsymbol{\mathcal{G}},\mathsf{F}}^{2} \lesssim rac{\mu_{1} r \log(sn)}{n} \cdot \left\| \boldsymbol{z} \right\|_{2}^{2}.$$

The same bound can be obtained for $\mathcal{G}\left(\sqrt{w_i}ze_i^{\mathsf{T}}\right)VV^*$.

For the last term, we have

$$\begin{split} \left\| \boldsymbol{U}_{\ell} \boldsymbol{U}^{*} \mathcal{G} \left(\sqrt{w_{i}} \boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}} \right) \boldsymbol{V} \boldsymbol{V}^{*} \right\|_{\mathsf{F}}^{2} &\leq w_{i} \left\| \boldsymbol{U}_{\ell} \right\|_{\mathsf{F}}^{2} \cdot \left\| \boldsymbol{U} \right\|^{2} \cdot \left\| \boldsymbol{\mathcal{G}} (\boldsymbol{z} \boldsymbol{e}_{i}^{\mathsf{T}}) \right\|^{2} \cdot \left\| \boldsymbol{V} \boldsymbol{V}^{*} \right\|^{2} \\ &\leq w_{i} \frac{\mu_{1} r}{n} \left\| \boldsymbol{G}_{i} \right\|^{2} \cdot \left\| \boldsymbol{z} \right\|_{2}^{2} \\ &\leq \frac{\mu_{1} r}{n} \cdot \left\| \boldsymbol{z} \right\|_{2}^{2}, \end{split}$$

where the second line is due to (3.4). Applying Corollary 6.8 again yields that

$$\| \boldsymbol{U} \boldsymbol{U}^* \mathcal{G} \left(\sqrt{w_i} \boldsymbol{z} \boldsymbol{e}_i^\mathsf{T} \right) \boldsymbol{V} \boldsymbol{V}^* \|_{\mathcal{G},\mathsf{F}}^2 \lesssim \frac{\mu_1 r \log(sn)}{n} \cdot \| \boldsymbol{z} \|_2^2.$$

The proof is completed after combining the three bounds together.

7 Conclusion

A convex approach called Vectorized Hankel Lift is proposed for blind super-resolution. It is based on the observation that the corresponding vectorized Hankel matrix is low rank if the Fourier samples of the unknown PSFs lie in a low dimensional subspace. Theoretical guarantee has been established for Vectorized Hankel Lift, showing that exact resolution can be achieved provided the number of samples is nearly optimal. We leave the robust analysis of the method to the future work. In particular, we would like to see whether the technique that bridges convex and nonconvex programs in [17] may yield an optimal error bound for the blind super-resolution problem.

For low rank matrix recovery and spectrally sparse signal recovery, many simple yet efficient nonconvex iterative algorithms have been developed and analysed based on inherent low rank structures of the problems [59, 58, 5, 7, 6]. Thus, it is also interesting to develop nonconvex optimization methods for blind super-resolution based on the low rank structure of the vectorized Hankel matrix. In fact, preliminary numerical results suggest that a variant of the gradient method in [6] is also able to reconstruct the target matrix arsing in the blind super-resolution problem from a few number of the spectrum samples. A detailed discussion towards this line of research will be reported separately.

For the single snapshot MUSIC and the MMV MUSIC, the super-resolution effect has been studied in [41, 34, 36]. Since the spatial smoothing MUSIC is designed to improve the performance of the MMV MUSIC, it is also interesting to investigate the super-resolution effect of this variant. The equivalence between it and MUSIC through Vectorized Hankle Lift (i.e., Lemma 2.2) may provide a new perspective to approach this problem.

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