

Source Coding for Synthesizing Correlated Randomness

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Abstract

We consider a scenario wherein two parties Alice and Bob are provided X_1^n and X_2^n – samples that are IID from a PMF $P_{X_1X_2}$. Alice and Bob can communicate to Charles over (noiseless) communication links of rate R_1 and R_2 respectively. Their goal is to enable Charles generate samples Y^n such that the triple (X_1^n, X_2^n, Y^n) has a PMF that is close, in total variation, to $\prod P_{X_1X_2Y}$. In addition, the three parties may possess pairwise shared common randomness at rates C_1 and C_2 . We address the problem of characterizing the set of rate quadruples (R_1, R_2, C_1, C_2) for which the above goal can be accomplished. We provide a set of sufficient conditions, i.e. an inner bound to the achievable rate region, and necessary conditions, i.e. an outer bound to the rate region for this three party setup. We provide a joint-typicality based random coding argument involving encoding and decoding operations to perform soft covering and a pertinent relaxation of the PMF requirement for the encoders.

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I. INTRODUCTION

The task of generating correlated randomness at different terminals in a network has applications in several communication [1], [2] and computing [3]–[5] scenarios. The presence of distributed correlated randomness also serves as a primitive in several cryptographic protocols [6]. In this article, we study the problem of characterizing fundamental information-theoretic limits of generating such correlated randomness in network scenarios.

We consider the scenario depicted in Fig 1. Three distributed parties - Alice, Bob and Charles - have to generate samples that are independent and identically distributed (IID) with a target probability mass function (PMF) $P_{X_1 X_2 Y}$. Alice and Bob are provided with samples that are IID $P_{X_1 X_2}$ - the corresponding marginal of the target PMF $P_{X_1 X_2 Y}$. They have access to unlimited private randomness and share noiseless communication links of rates R_1, R_2 with Charles. In addition, the three parties share common randomness at rate C . For what rate triples (R_1, R_2, C) can Alice and Bob enable Charles to generate the required samples? In this article, we undertake a Shannon-theoretic study and characterize inner [7] and outer bounds on the aforementioned set of rate triples.

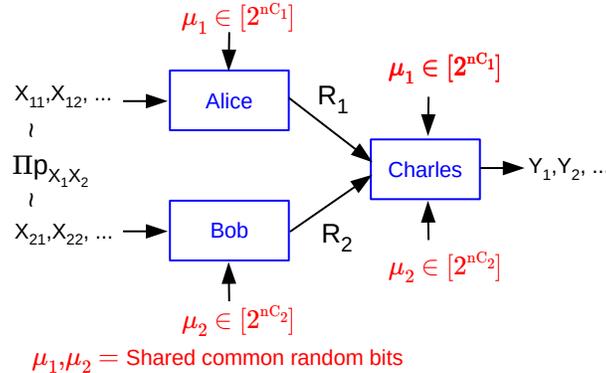


Fig. 1. Illustration of distributed agents performing source coding for synthesizing correlated randomness.

The roots of this line of study - distributed terminals generating IID copies of correlated random variables - can be traced back to the work of Wyner [8]. Wyner [8] considered the scenario of distributed parties generating IID samples distributed with PMF P_{XY} , when fed with a common information stream. In characterizing this rate, Wyner discovered a measure, commonly referred to as *Wyner's common information*, that quantifies the amount of common information between two correlated random variables. A renewed interest in this study led Cuff [9] to study the scenario depicted in Fig. 1 with just two terminals corresponding to Alice, Charles, and Bob being absent. Cuff [9] characterized the entire set of rate pairs (R, C) and showed that Wyner's common information forms one vertex of this region. Cuff's work also shares an interesting connection with an analogous problem in quantum information theory. Prior to [9], Winter [10] considered the problem of simulating quantum measurements with limited common randomness. This work was generalized in [11] where the authors characterized a complete trade-off

between communication and common randomness rates. Building on this, [12] studied a distributed scenario consisting of three distributed parties and derived inner bounds.

Motivated by applications in security [13], cryptography [14], need for co-ordinated control among distributed terminals [15], among others, this line of study has received considerable attention lately [16], [17]. The works of Wyner [8], Cuff [9] and others [18] naturally lead us to consider the scenario depicted in Fig. 1. In contrast to these works, our scenario requires two distributed terminals, observing correlated information, to co-ordinate their communication to a central decoder. This poses certain technical challenges in the design and analysis of the encoders and the decoders, thereby leaving the information-theoretic study of our scenario unresolved. As we describe in the sequel, our work overcomes these challenge via (i) a novel design of the encoders and decoder, and (ii) identification of appropriate mathematical tools for performance analysis and rate region characterization.

The key challenge here is to ensure that Bob's simulated samples Y^n are *correlated simultaneously* with X_1^n and X_2^n in a single-letter fashion. In particular, it maybe noted that the conventional side-information approach of treating one of the sources, say X_2^n , as side-information and adopting the proof of channel synthesis with side-information [18] does not work. The reason for this is the need for simultaneous correlation as mentioned above. Indeed, it maybe noted that, while the channel synthesis with side-information problem [18] has been addressed and solved several years ago, the problem of distributed channel synthesis has remained open.

We propose a novel approach to addressing this problem. We first prove an inner bound that appears smaller at first sight. Specifically, we prove achievability of one corner point of the achievable rate region wherein the lower bound on one of the rates is higher. This larger lower bound enables us simulate the generated samples to be correlated with a larger sub-collection of auxiliary random variables. We then leverage this for lowering the lower bound on the other rate components. By then using convexification, we prove that by swapping the order and performing time-sharing, we can enlarge the inner bound to what one might conjecture to be a natural inner bound via binning. The reader will find Figs. 4, 5, and 6 illustrate the new steps in our proof technique.

We also emphasize that while the stated inner bound might appear natural for a reader familiar with the problem of distributed source coding [19], the problem of distributed channel synthesis is different and involves more constraints. Indeed, in this problem, it is required that the generated random variables appear to have a single-letter distribution as specified, not just that they meet certain distortion criterion. This difference is clearly emphasized in the rate region obtained for the conventional channel synthesis problem studied by Cuff [9] for which we are aware of optimality. Observe that, as against to a single lower bound on the rate that we obtain in the Shannon's source coding problem, Cuff's problem yields two lower bounds, proving that the distributed channel synthesis problem is more involved, and perhaps hinting at the need for a new proof technique that we have developed in this article. We also note that we have the opportunity to employ a more sophisticated Chernoff-Hoeffding concentration inequality due to Ahlswede Winter [20] - a tool not regularly employed in proof of coding theorems.

Lastly, we highlight another novelty of our findings. In addressing the scenario in Fig. 1, it is natural

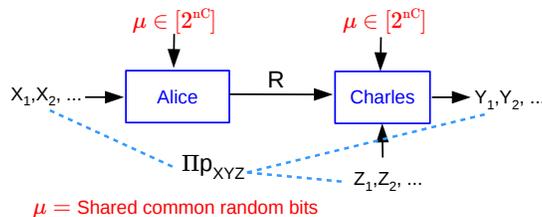


Fig. 2. Synthesizing correlated randomness with side information available at the decoder.

to try and build on Cuff's [9] findings - relying on the use of a likelihood encoder that maps the observed sequence and common random bits into a codebook of sufficient rate. Essentially, the encoder performs a MAP decoding of the observed sequence into the chosen codebook. While this choice greatly simplifies the analysis, it permits little room for generalization. Our experience in network information theory suggests that encoding and decoding via joint-typicality can be naturally generalized to diverse multi-terminal scenarios. Motivated by this, we propose joint-typicality based encoding and decoding to perform soft covering [7]. As a reader will note, the transition from a likelihood encoder to a joint-typicality based encoder results in challenges in analysis due to the hard constraints that the encoders are valid PMFs. Toward this, we develop a novel construction of random encoders, by relaxing the PMF requirement. This relaxation plays a central role in generalizing the results to the distributed case. The mathematical tools we have adopted to overcome these challenges maybe viewed as part of our technical contribution. In view of the general applicability of typicality-based coding schemes, we regard the typicality-based soft covering we propose as an important step. Furthermore, we leverage ideas from the outer bounds for the distributed source coding problem [21] to characterize an outer bound for this problem. This article therefore contains a complete suite of results for the distributed channel synthesis problem, thereby 1) filling our knowledge gaps in regards to our scenario and 2) deriving bounds for this problem that is on par with our knowledge for the distributed source coding problem. Elaborating on the last point, we note that with infinite common randomness, our rate regions reduce to those that are currently the best known for the distributed source coding problem.

A preliminary version of this work appeared in [7]. Subsequently, building on this work, the authors in [22] considered a side information and three-way common information generalization of the problem considered in [7], and derived inner and outer bounds.

The paper is organized as follows. After setting up notation and stating the problem (Sec. II), we provide our main results, the inner and outer bounds to the achievable rate-region of a distributed problem, in Sec. III. Before providing a complete proof of the above inner bound, we consider the two-terminal side-information scenario (Fig. 2) in Sec. IV, wherein the decoder is provided with side-information. This provides us with an ideal pedagogical step to present our typicality based encoder, decoder and indicate the mathematical challenges in its information-theoretic analysis. Unlike [18], we propose a joint-typicality based encoder and decoder and provide a complete proof of achievability of the full rate region. Building on the tools developed therein, we present the proof of our main results in Sec. V.

II. PRELIMINARIES AND PROBLEM STATEMENT

We supplement standard information theory notation with the following. For a PMF P_X , we let $P_X^n = \prod_{i=1}^n P_X$. Given a sequence $x^n \in \mathcal{X}^n$, let P_{x^n} denote its empirical distribution. For any distribution P_X on \mathcal{X} , define the δ -typical set $T_\delta(X)$ as

$$T_\delta(X) \triangleq \left\{ x^n \in \mathcal{X}^n : \|P_{x^n} - P_X\|_\infty \leq \frac{\delta}{|\mathcal{X}|}, P_{x^n} \ll P_X \right\}.$$

For any distribution P_{XY} on $\mathcal{X} \times \mathcal{Y}$, define the δ -jointly typical set $T_\delta(X, Y)$ as

$$T_\delta(X, Y) = T_\delta(P_{XY}) \triangleq \left\{ (x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \|P_{x^n, y^n} - P_{XY}\|_\infty \leq \frac{\epsilon}{|\mathcal{X}||\mathcal{Y}|}, P_{x^n, y^n} \ll P_{XY} \right\},$$

where P_{x^n, y^n} is the empirical joint distribution of two sequences (x^n, y^n) . Note that if $(x^n, y^n) \in T_\delta(P_{XY})$, then $x^n \in T_\delta(P_X)$, and $y^n \in T_\delta(P_Y)$. For any conditional distribution $P_{Y|X} : \mathcal{X} \rightarrow \mathcal{Y}$, and any $x^n \in \mathcal{X}^n$, define the δ -conditional typical set $T_\delta(Y|x^n)$ as

$$T_\delta(Y|x^n) \triangleq \left\{ y^n \in \mathcal{Y}^n : \|P_{x^n, y^n} - P_{Y|X} P_{x^n}\|_\infty \leq \frac{\delta}{|\mathcal{Y}|}, P_{x^n, y^n} \ll P_{x^n} P_{Y|X} \right\}.$$

For an integer $n \geq 1$, $[n] \triangleq \{1, \dots, n\}$. The total variation between PMFs P_X and Q_X defined over \mathcal{X} is denoted $\|P_X - Q_X\|_1 = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| = \sup_{\mathcal{A} \subset \mathcal{X}} |P_X(\mathcal{A}) - Q_X(\mathcal{A})|$.

Definition 1. Given a PMF P_{XYZ} on $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$, a rate pair (R, C) is said to be achievable, if $\forall \epsilon > 0$ and all sufficiently large n , there exists a collection of 2^{nC} randomized encoders $E^{(\mu)} : \mathcal{X}^n \rightarrow [\Theta]$ for $\mu \in [2^{nC}]$ and a corresponding collection of 2^{nC} randomized decoders $D^{(\mu)} : \mathcal{Z}^n \times [\Theta] \rightarrow \mathcal{Y}^n$ for $\mu \in [2^{nC}]$ such that $\|P_{X^n Y^n Z^n}^n - P_{X^n Y^n Z^n}\|_1 \leq \epsilon$, $\frac{1}{n} \log_2 \Theta \leq R + \epsilon$, where for all $x^n, y^n, z^n \in \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$

$$P_{X^n Y^n Z^n}^n(x^n, y^n, z^n) \triangleq \sum_{\mu \in [2^{nC}]} 2^{-nC} \sum_{m \in [\Theta]} P_{X^n Z^n}^n(x^n, z^n) P_{M|X^n}^{(\mu)}(m|x^n) P_{Y^n|Z^n, M}^{(\mu)}(y^n|z^n, m),$$

$P_{M|X^n}^{(\mu)}, P_{Y^n|Z^n, M}^{(\mu)}$ are the PMFs induced by encoder and decoder respectively, corresponding to shared random message μ , with M being the random variable corresponding to the message transmitted. We let $\mathcal{R}_s(P_{XYZ})$ denote the set of achievable rate pairs.

Cuff [9, Thm. II.1] provides a single-letter characterization for $\mathcal{R}_s(P_{XY})$ when $\mathcal{Z} = \phi$ is empty. A single-letter characterization of $\mathcal{R}_s(P_{XY})$ in the general case was provided in [18]. Building on this, we address the network scenario (Fig. 1) for which we state the problem below. In the following, we let $\underline{X} = (X_1, X_2)$, $\underline{x}^n = (x_1^n, x_2^n)$.

Definition 2. Given a PMF $P_{X_1 X_2 Y}$ on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}$, a rate quadruple (R_1, R_2, C_1, C_2) is said to be achievable, if $\forall \epsilon > 0$ and all sufficiently large n , there exists $2^{nC_1} \times 2^{nC_2}$ randomized encoder pairs $(E_1^{(\mu_1)}, E_2^{(\mu_2)})$, where $E_j^{(\mu_j)} : \mathcal{X}_j^n \rightarrow [\Theta_j] : \mu_j \in [2^{nC_j}], j \in [2]$, and a corresponding collection of 2^{nC} randomized decoders $D^{(\mu)} : [\Theta_1] \times [\Theta_2] \rightarrow \mathcal{Y}^n$ for $\mu \in [2^{nC}]$, where $C \triangleq C_1 + C_2$ and $\mu \triangleq (\mu_1, \mu_2)$,

such that $\|P_{\underline{X}Y}^n - P_{\underline{X}^n Y^n}\|_1 \leq \epsilon$, $\frac{1}{n} \log_2 \Theta_j \leq R_j + \epsilon : j \in [2]$, where for all $\underline{x}^n, \underline{y}^n \in \underline{\mathcal{X}}^n \times \mathcal{Y}^n$

$$P_{\underline{X}^n Y^n}(\underline{x}^n, \underline{y}^n) \triangleq \sum_{\mu \in [2^{nC}]} 2^{-nC} \sum_{\substack{(m_1, m_2) \in \\ [\Theta_1] \times [\Theta_2]}} P_{\underline{X}}^n(\underline{x}^n) P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) P_{Y^n|M_1, M_2}^{(\mu)}(\underline{y}^n|m_1, m_2),$$

$P_{M_j|X_j^n}^{(\mu_j)} : j \in [2], P_{Y^n|M_1, M_2}^{(\mu)}$ are the PMFs induced by the two randomized encoders and decoder, respectively, corresponding to common random index (μ_1, μ_2) . We let $\mathcal{R}_d(P_{\underline{X}Y})$ denote the set of achievable rate triples.

Our main results are the characterization of an inner bound and an outer bound to $\mathcal{R}_d(P_{\underline{X}Y})$ which are provided in Theorem 1 and Theorem 2, respectively.

III. DISTRIBUTED SOFT COVERING - MAIN RESULTS

In this section, we provide an inner bound and an outer bound to the achievable rate-region for the distributed setting (Fig. 1). Our first result in this regard is the following inner bound to $\mathcal{R}_d(P_{\underline{X}Y})$. In the following, we let $\underline{X} = (X_1, X_2), \underline{W} = (W_1, W_2), \underline{x} = (x_1, x_2)$ and $\underline{w} = (w_1, w_2)$.

Theorem 1. Given a PMF $P_{X_1 X_2 Y}$, let $\mathcal{P}(P_{X_1 X_2 Y})$ denote the collection of all PMFs $P_{QW_1 W_2 \underline{X}Y}$ defined on $\mathcal{Q} \times \mathcal{W}_1 \times \mathcal{W}_2 \times \underline{\mathcal{X}} \times \mathcal{Y}$ such that (i) $P_{\underline{X}Y}(\underline{x}, y) = \sum_{(q, \underline{w}) \in \mathcal{Q} \times \mathcal{W}} P_{QW \underline{X}Y}(q, \underline{w}, \underline{x}, y)$ for all (\underline{x}, y) , (ii) $\sum_{\underline{w} \in \mathcal{W}} P_{QW \underline{X}Y}(q, \underline{w}, \underline{x}, y) = P_Q(q) P_{\underline{X}Y}(\underline{x}, y)$ for all (q, \underline{x}, y) (iii) $W_1 - QX_1 - QX_2 - W_2$ and $\underline{X} - QW - Y$ are Markov chains, (iv) $|\mathcal{W}_1| \leq |\mathcal{X}_1|, |\mathcal{W}_2| \leq |\mathcal{X}_2|$, and $|\mathcal{Q}| \leq 7$. Further, let $\beta(P_{QW \underline{X}Y})$ denote the set of rates and common randomness quadruple $(R_1, R_2, C_1, C_2) \in [0, \infty)^4$ that satisfy

$$\begin{aligned} R_1 &\geq I(X_1; W_1|Q) - I(W_1; W_2|Q) \\ R_2 &\geq I(X_2; W_2|Q) - I(W_1; W_2|Q) \\ R_1 + R_2 &\geq I(X_1; W_1|Q) + I(X_2; W_2|Q) - I(W_1; W_2|Q) \\ R_1 + C_1 &\geq I(X_1 X_2 Y; W_1|Q) - I(W_1; W_2|Q), \\ R_2 + C_2 &\geq I(X_1 X_2 Y; W_2|Q) - I(W_1; W_2|Q), \\ R_1 + R_2 + C_1 &\geq I(X_1 X_2 Y; W_1|Q) + I(X_2; W_2|Q) - I(W_1; W_2|Q) \\ R_1 + R_2 + C_2 &\geq I(X_1 X_2 Y; W_2|Q) + I(X_1; W_1|Q) - I(W_1; W_2|Q) \\ R_1 + R_2 + C_1 + C_2 &\geq I(X_1 X_2 Y; W_1 W_2|Q) \end{aligned} \tag{1}$$

where the mutual information terms are evaluated with the PMF $P_{QW_1 W_2 \underline{X}Y}$. We have

$$\mathcal{R}_I(P_{\underline{X}Y}) \triangleq \text{Closure} \left(\bigcup_{P_{QW \underline{X}Y} \in \mathcal{P}(P_{X_1 X_2 Y})} \beta(P_{QW \underline{X}Y}) \right) \subseteq \mathcal{R}_d(P_{\underline{X}Y}). \tag{2}$$

In other words, (R_1, R_2, C_1, C_2) is achievable if $(R_1, R_2, C_1, C_2) \in \mathcal{R}_I(P_{\underline{X}Y})$.

Remark 1. Before providing a proof to the above theorem, we briefly discuss two corner points of the rate region with respect to the common randomness available. Firstly, consider the regime when both C_1 and C_2 are unlimited. This implies that only the first three constraints are active and hence the inner bound to the achievable rate-region reduces to the Berger-Tung inner bound [23]. Secondly, consider the case when only one of the C_1 , and C_2 , say C_2 , is unlimited. In the first glance, one may think that the rate R_1 is only constraint by the first and the sum rate ($R_1 + R_2$) constraint. However, a careful observation yields an additional constraint $R_1 + R_2 + C_1$ limiting the rate of R_1 . The insight to this is the joint distributed simulation task that the problem addresses. It suggests that if R_2 and C_2 are at their minimum then R_1 has to provide for any additional rate that is needed in simulating the joint distribution.

Proof. The proof of this theorem is provided in Section V. □

We consider an example to illustrate the significance of the inner bound.

Example 1. Consider a distributed setup as shown in Fig. 1. Let the input alphabets of the two encoders, \mathcal{X}_1 and \mathcal{X}_2 , and the output alphabet \mathcal{Y} be given by the binary set $\{0, 1\}$. Let the joint distribution $P_{X_1 X_2 Y} = P_{X_1 X_2} P_{Y|X_1 X_2}$ be defined as

$$P_{X_1 X_2}(0, 0) = P_{X_1 X_2}(1, 1) = \frac{(1-p)}{2} \quad \text{and} \quad P_{X_1 X_2}(0, 1) = P_{X_1 X_2}(1, 0) = \frac{p}{2},$$

and

$$P_{Y|X_1 X_2}(0|0, 0) = P_{Y|X_1 X_2}(0|1, 1) = 1 - \delta \quad \text{and} \quad P_{Y|X_1 X_2}(0|0, 1) = P_{Y|X_1 X_2}(0|1, 0) = \delta,$$

for $p = \delta = 0.2$. The trade-off between the achievable sum communication rate and sum common randomness rate is numerically computed and is depicted in Fig. 3. The figure demonstrates the usefulness of common randomness in decreasing the sum communication rates. However, below a certain threshold, no amount of common randomness can be used toward decreasing the communication rates further.

Our next main result for the distributed setting is the outer bound to the achievable rate region.

Theorem 2. For all $\epsilon > 0$, let $\mathcal{P}_F(\epsilon)$ denote the collection of conditional PMFs $\tilde{P}_{JQUVY|X_1 X_2}$ defined on $\mathcal{J} \times \mathcal{Q} \times \mathcal{U} \times \mathcal{V} \times \mathcal{Y}$ such that the following conditions are satisfied: (a) (Q, J) is independent of (X_1, X_2) , (b) $U - (X_1, Q, J) - (X_2, Q, J) - V$, and (c) $(X_1, X_2, Q) - (J, U, V) - Y$, and (d) $\|P_{X_1 X_2 Y} - P_{X_1 X_2} \tilde{P}_{Y|X_1 X_2}\|_1 \leq \epsilon$, where $\mathcal{J}, \mathcal{Q}, \mathcal{U}$ and \mathcal{V} are finite sets. Let \mathcal{P}_R denote the collection of conditional PMFs $P_{W|X_1 X_2}$ defined on $\mathcal{W} \times \mathcal{X}_1 \times \mathcal{X}_2$ such that the condition (e) $X_1 - W - X_2$ is satisfied where \mathcal{W} is a finite set. For a $\tilde{P}_{JQUVY|X_1 X_2} \in \mathcal{P}_F$ and a $P_{W|X_1 X_2} \in \mathcal{P}_R$, let $\lambda_\epsilon(\tilde{P}_{JQUVY|X_1 X_2}, P_{W|X_1 X_2})$ denote

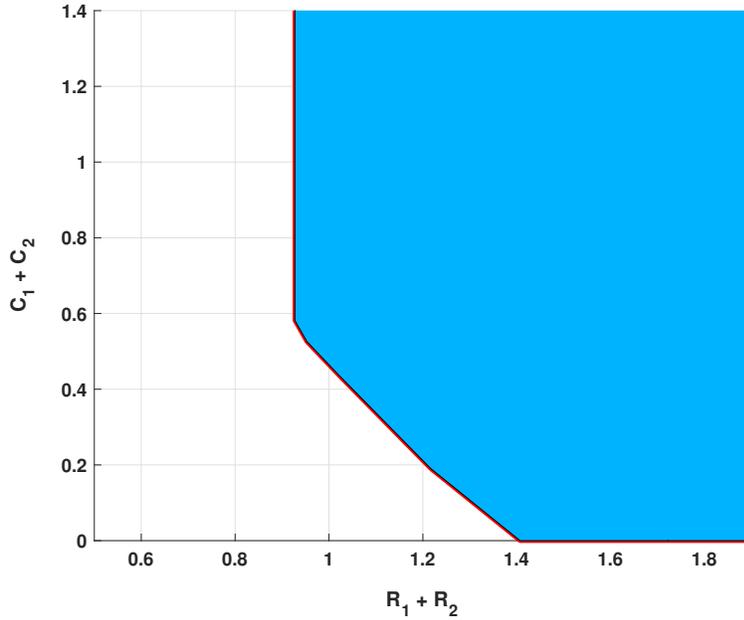


Fig. 3. Figure depicting the trade-off between the sum rate and the sum common randomness.

the set of rates and common randomness quadruple $(R_1, R_2, C_1, C_2) \in [0, \infty)^4$ that satisfy

$$\begin{aligned}
 R_1 &\geq I(W; U|V, J) + I(X_1, U|W, Q, J) - \epsilon \\
 R_2 &\geq I(W; V|U, J) + I(X_2, V|W, Q, J) - \epsilon \\
 R_1 + R_2 &\geq I(W; U, V|J) + I(X_1; U|W, Q, J) + I(X_2; V|W, Q, J) - 2\epsilon \\
 R_1 + R_2 + C_1 + C_2 &\geq I(W; U, V|J) + I(X_1, X_2, Y; U, V|Q, W, J) - g_c(\epsilon), \tag{3}
 \end{aligned}$$

under the Markov coupling between $\tilde{P}_{JQUVY|X_1X_2}$ and $P_{W|X_1X_2}$, i.e., condition (f) $W - (X_1, X_2) - (J, Q, U, V, Y)$ is satisfied, where $g_c(\epsilon) \triangleq 4\epsilon(\log(|\mathcal{X}_1||\mathcal{X}_2||\mathcal{Y}|) - \log(\epsilon))$. In other words, the joint distribution of the concerned random variables is given by $P_W P_{X_1|W} P_{X_2|W} \tilde{P}_Q \tilde{P}_J \tilde{P}_U|_{X_1QJ} \tilde{P}_V|_{X_2QJ} \tilde{P}_Y|_{UVJ}$, and with which the mutual information terms are evaluated. We have $\mathcal{R}_d(P_{\underline{XY}}) \subseteq \bigcap_{\epsilon>0} \mathcal{R}_O(P_{\underline{XY}}, \epsilon)$, where

$$\mathcal{R}_O(P_{\underline{XY}}, \epsilon) \triangleq \bigcap_{P_{W|X_1X_2} \in \mathcal{P}_R} \bigcup_{P_{JQUVY|X_1X_2} \in \mathcal{P}_F(\epsilon)} \lambda_\epsilon(P_{JQUVY|X_1X_2}, P_{W|X_1X_2}). \tag{4}$$

In other words, if $(R_1, R_2, C) \in \bigcap_{\epsilon>0} \mathcal{R}_O(P_{\underline{XY}}, \epsilon)$, then (R_1, R_2, C) is achievable.

Proof. The proof of the above theorem is provided in Appendix B. \square

Remark 2. Note that for every $P_{W|X_1X_2} \in \mathcal{P}_R$, we have an outer bound, obtained by taking the intersection over ϵ and the union over $P_{JQUVY|X_1X_2} \in \mathcal{P}_F(\epsilon)$, on $\mathcal{R}_d(P_{\underline{XY}})$. Hence we have a family of outer bounds.

Remark 3. One may question the computability of the outer bound provided in Theorem (2). The computability of this bound depends on the cardinality of the auxiliary random variables defined in the theorem. Currently, we are unable to bound the cardinality of the auxiliary random variables, but aim to provide one in our future work. As a matter of fact, the current outer bounds for the equivalent distributed rate distortion problem still suffers from the computability issue. The first outer bound to this problem was provided in [23] and a recent substantial improvement was made by authors in [21], [24]. All these bounds suffer from the absence of cardinality bounds on at least one of the variables used and hence cannot be claimed to be computable using finite resources. This problem still remains open.

Remark 4. Due to the lack of cardinality bounds, the space of probability distributions is not compact, and hence the mutual information may not be a continuous function of ϵ . Therefore, the continuity of $\mathcal{R}_O(P_{\underline{X}Y}, \epsilon)$ at $\epsilon = 0$ still remains an open question. When the cardinality bounds become available, we will have continuity at $\epsilon = 0$, and thus $\mathcal{R}_O(P_{\underline{X}Y}, 0) = \bigcap_{\epsilon > 0} \mathcal{R}_O(P_{\underline{X}Y}, \epsilon)$.

IV. SOFT COVERING WITH SIDE INFORMATION

Although our paper is mainly geared toward the distributed case (addressed in Section III), we provide a proof of the side information scenario for pedagogical reasons. We provide a new proof of achievability of $\mathcal{R}_s(P_{XYZ})$. The proof develops a new construction of random encoders by relaxing the PMF requirement, and using refined Chernoff-Hoeffding bound, which could find applications in other problems of information theory. This relaxation and the refined bound play a central role in generalizing the results to the distributed case 1. As mentioned earlier, the side-information problem was addressed in [18] using a different proof methodology.

Theorem 3. $(R, C) \in \mathcal{R}_s(P_{XYZ})$ if and only if there exists a PMF P_{WXYZ} such that (i) $P_{XYZ}(x, y, z) = \sum_{w \in \mathcal{W}} P_{WXYZ}(w, x, y, z)$ for all (x, y, z) where \mathcal{W} is the alphabet of W , (ii) $Z - X - W$ and $X - (Z, W) - Y$ are Markov chains, (iii) $|\mathcal{W}| \leq (|\mathcal{X}||\mathcal{Y}||\mathcal{Z}|)^2$, and

$$R \geq I(X; W) - I(W; Z), \quad R + C \geq I(XYZ; W) - I(W; Z). \quad (5)$$

Proof. We begin the proof by describing the encoder.

A. Encoder Description

Fix a PMF P_{WXYZ} satisfying the constraints stated in the theorem. Throughout, $\mu \in [2^{nC}]$ denotes the C bits of common randomness shared between the encoder and decoder. For each $\mu \in [2^{nC}]$, we shall design a randomized encoder $E^{(\mu)} : \mathcal{X}^n \rightarrow [\Theta]$ and a randomized decoder $D^{(\mu)} : \mathcal{Z}^n \times [\Theta] \rightarrow \mathcal{Y}^n$

that induce PMFs $P_{M|X^n}^{(\mu)}$ and $P_{Y^n|Z^n M}^{(\mu)}$ respectively, for which

$$\mathcal{Q} \triangleq \frac{1}{2} \sum_{x^n, y^n, z^n} \left| P_{XYZ}^n(x^n, y^n, z^n) - \sum_{\mu \in [2^{nC}]} \sum_{m \in [\Theta]} \frac{P_{XZ}^n(x^n, z^n)}{2^{nC}} P_{M|X^n}^{(\mu)}(m|x^n) P_{Y^n|Z^n, M}^{(\mu)}(y^n|z^n, m) \right| \leq \varepsilon. \quad (6)$$

From now on we denote $\Theta = 2^{n\tilde{R}}$. The design of these randomized encoders and decoders involves building a collection of codebooks $\mathcal{C} \triangleq (\mathcal{C}^{(\mu)} : \mu \in [2^{nC}])$ where $\mathcal{C}^{(\mu)} \triangleq (\mathbf{w}^n(l, \mu) \in \mathcal{W}^n : l \in [2^{n\tilde{R}}])$ for $\mu \in [2^{nC}]$, where \mathcal{W} is the alphabet of W in the theorem statement, and \tilde{R} will be specified shortly. On observing x^n and μ , the randomized encoder chooses an index L in $[2^{n\tilde{R}}]$ according to a PMF $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$. The chosen index is then mapped to an index in $[2^{nR}]$ which is communicated to the decoder. Before we specify the PMF $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$, let us describe how the chosen index is mapped to an index in $[2^{nR}]$. We define a binning map $b^{(\mu)} : [2^{n\tilde{R}}] \rightarrow [2^{nR}]$. On observing x^n , the encoder chooses $L \in [2^{n\tilde{R}}]$ with respect to PMF $E_{L|X^n}^{(\mu)}(\cdot|x^n)$, and communicates $b^{(\mu)}(L)$ to the decoder.

Let us relate to the above three elements that make up the encoder. The PMF $E_{L|X^n}^{(\mu)}$ is analogous to the likelihood encoder $\Gamma_{J|X^n, K}$ of Cuff [9] but with important changes to incorporate typicality-based encoding that permits the use of side-information at the decoder. The map $b^{(\mu)}$ performs standard information-theoretic binning [25] to utilize side-information. We now specify $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$. Fix $\epsilon > 0, \delta > 0, \eta > 0$, and for $x^n \in T_\delta(X)$ and $l \in [2^{n\tilde{R}}]$, let

$$E_{L|X^n}^{(\mu)}(l|x^n) \triangleq \frac{1}{2^{n\tilde{R}}} \frac{1 - \epsilon}{1 + \eta} \sum_{w^n \in T_\delta(W|x^n)} \mathbb{1}_{\{\mathbf{w}^n(l, \mu) = w^n\}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)}.$$

In specifying $E_{L|X^n}^{(\mu)}$, we have relaxed the requirement that $E_{L|X^n}^{(\mu)}(\cdot|x^n)$ be a PMF. This relaxation - a novelty of our work - yields analytical tractability of a random coding ensemble to be described in the sequel. However, note that these maps depend on the choice of the codebook \mathcal{C} . We prove in Appendix A-A that with high probability, $E_{L|X^n}^{(\mu)}(\cdot|x^n) : [2^{n\tilde{R}}] \rightarrow \mathbb{R}$ is a PMF for every $x^n \in \notin T_\delta(X)$. This will form a part of our random codebook analysis and in fact, as we see in Lemma 1, one of the rate constraints is a consequence of the conditions necessary for the above definition of $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$ to be a PMF. We also note that $E_{L|X^n}^{(\mu)}$ being a PMF guarantees $P_{M|X^n}$ is a PMF.

Having specified $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$, we now characterize $P_{M|X^n}$ for $m \in [2^{nR}] \cup \{0\}$ as

$$P_{M|X^n}^{(\mu)}(m|x^n) \triangleq \begin{cases} \mathbb{1}_{\{m=0\}} & \text{if } s^{(\mu)}(x^n) > 1, \\ 1 - s^{(\mu)}(x^n) & \text{if } m = 0 \text{ and } s^{(\mu)}(x^n) \in [0, 1], \\ \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \mathbb{1}_{\{b^{(\mu)}(l)=m\}} & \text{if } m \neq 0 \text{ and } s^{(\mu)}(x^n) \in [0, 1] \end{cases} \quad (7)$$

for all $x^n \in T_\delta(X)$, and $s^{(\mu)}(x^n)$ defined as $s^{(\mu)}(x^n) \triangleq \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n)$. For $x^n \notin T_\delta(X)$, we let $P_{M|X^n}^{(\mu)}(m|x^n) = \mathbb{1}_{\{m=0\}}$. It can be verified that $P_{M|X^n}$ is a valid PMF. We have thus described the

encoder and $P_{M|X^n}$.

B. Decoder Description

We now describe the decoder. On observing $z^n \in \mathcal{Z}^n, \mu$ and the index $m \in [2^{nR}] \cup \{0\}$ communicated by the encoder, for $m \neq 0$, the decoder populates

$$\mathcal{D}^{(\mu)}(z^n, m) \triangleq \{l \in [2^{n\tilde{R}}] : b^{(\mu)}(l) = m, (\mathbf{w}^n(l, \mu), z^n) \in T_\delta(W, Z)\}$$

Let

$$f^{(\mu)}(m, z^n) \triangleq \begin{cases} \mathbf{w}^n(l, \mu) & \text{if } \mathcal{D}^{(\mu)}(z^n, m) = \{l\} \\ w_0 & \text{otherwise, i.e., } |\mathcal{D}^{(\mu)}(z^n, m)| \neq 1 \text{ or } m = 0. \end{cases}$$

The decoder chooses y^n according to PMF $P_{Y^n|WZ}^n(y^n | f^{(\mu)}(m, z^n), z^n)$. This implies the PMF $P_{Y^n|Z^n M}^{(\mu)}(\cdot | \cdot)$ is given by

$$P_{Y^n|Z^n M}^{(\mu)}(\cdot | z^n, m) = P_{Y^n|WZ}^n(\cdot | f^{(\mu)}(m, z^n), z^n). \quad (8)$$

C. Distribution of Codebook

To prove of existence of a codebook for which the above terms are arbitrarily small, we employ random coding. Specifically, we let the codewords of \mathcal{C} to be IID with distribution

$$\tilde{P}_{W^n}(w^n) = \begin{cases} \frac{P_W^n(w^n)}{1-\epsilon} & \text{if } w^n \in T_{\bar{\delta}}(W) \\ 0 & \text{otherwise,} \end{cases} \quad (9)$$

where $\bar{\delta} \triangleq \delta|\mathcal{X} + \mathcal{W}|$, and $\epsilon(\delta, n) \triangleq \sum_{w^n \notin T_{\bar{\delta}}(W)} P_W^n(w^n)$. Note that $\epsilon(\delta, n) \searrow 0$ as $n \rightarrow \infty$ for every $\delta > 0$ sufficiently small. The binning of the codewords is performed independently, where each $b^{(\mu)}(\cdot)$ is chosen randomly, uniformly and independently from $[2^{nR}]$.

D. Analysis of Total Variation

We begin by splitting \mathcal{Q} into two terms using an indicator function $\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}}$ as

$$\mathcal{Q} = \mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} + \mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}}^c \quad (10)$$

where $\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}}$ is defined as

$$\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = \begin{cases} 1 & \text{if } s^{(\mu)}(x^n) \in [0, 1] \text{ for all } x^n \in T_\delta(X), \mu \in [2^{nC}], \\ 0 & \text{otherwise,} \end{cases}$$

and recalling $s^{(\mu)}(x^n) = \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n)$. Taking expectation over the codebooks and bounding \mathcal{Q} in the right hand side of (10) by 1¹ gives

$$\mathbb{E}[\mathcal{Q}] \leq \mathbb{E}[\mathcal{Q}\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}}] + \mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = 0\}. \quad (11)$$

We now show using the lemma below, that by appropriately constraining \tilde{R} , $\mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = 0\}$ can be made arbitrarily small. In other words, with high probability, we will have $E_{L|X^n}^{(\mu)}$ such that $0 \leq \sum_{l_1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)} \leq 1$ for all $\mu \in [2^{nC}]$ and $x^n \in T_\delta(X)$.

Lemma 1. *For any $\delta, \eta \in (0, 1/2)$, if $\tilde{R} > I(X : W) + 4\delta_1$ then*

$$\mathbb{P}\left(\bigcap_{\mu=1}^{2^{nC}} \bigcap_{x^n \in T_\delta(X)} \left(\sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq 1\right)\right) \rightarrow 1 \text{ as } n \rightarrow \infty, \quad (12)$$

where $\delta_1(\delta), \delta_2(\delta) \searrow 0$ as $\delta \searrow 0$,

Proof. The proof is provided in Appendix A-A. □

Since, we have

$$\mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = 0\} = 1 - \mathbb{P}\left(\bigcap_{\mu_1=1}^{2^{nC_1}} \bigcap_{\substack{x_1^n \in \\ T_\delta(X_1)}} \left(E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \leq 1\right)\right),$$

from Lemma 1, for any $\delta \in (0, 1)$, we have $\mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = 0\} \leq \epsilon_p$ for all sufficiently large n , where $\epsilon_p(\delta) \searrow 0$ as $\delta \searrow 0$.

We now look at the first term in (10), i.e., $\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}}$. This can be expanded as

$$\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = \left[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \mathcal{Q}_{x^n} + \sum_{x^n \notin T_\delta(X)} P_X^n(x^n) \mathcal{Q}_{x^n} \right] \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}},$$

where \mathcal{Q}_{x^n} is defined as

$$\mathcal{Q}_{x^n} \triangleq \frac{1}{2} \sum_{y^n, z^n} \left| P_{YZ|X}^n(y^n, z^n|x^n) - \sum_{\mu \in [2^{nC}]} \sum_{m \in [2^{nR}] \cup \{0\}} \frac{P_{Z|X}^n(z^n|x^n)}{2^{nC}} P_{M|X^n}^{(\mu)}(m|x^n) P_{Y^n|Z^n, M}^{(\mu)}(y^n|z^n, m) \right|.$$

Using the standard typicality arguments², we obtain, for all sufficiently large n ,

$$\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} = \sum_{x^n \in T_\delta(X)} P_X^n(x^n) \mathcal{Q}_{x^n} \mathbb{1}_{\{\text{PMF}(\mathcal{C})\}} + \epsilon_t(\delta), \quad (13)$$

¹Total Variation is bounded from above by 1

²Note that \mathcal{Q}_{x^n} is a total variational distance between two conditional PMFs, conditioned on X , for each x^n , and hence it is bounded from above by one.

where $\epsilon_t(\delta) \searrow 0$ as $\delta \searrow 0$. Now, what remains is the first term in (13). A major portion of our analysis from here on deals with arguing that this term can be made arbitrarily small. Further, since this term contains the indicator $\mathbb{1}_{\{\text{PMF}(C)\}}$, we can restrict our analysis to only the set of random collection of codebook C that satisfy $0 \leq \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq 1$ for all $x^n \in T_\delta(X)$ and $\mu \in [2^{nC}]$.

Step 1: Isolating the error induced by not covering

We begin our analysis by isolating the error induced by not covering the product distribution P_{XYZ}^n . Note that under the condition that $\mathbb{1}_{\{\text{PMF}(C)\}} = 1$, we have $P_{M|X^n}^{(\mu)}(m|x^n) = \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n)$ when $m \neq 0$, and $P_{M|X^n}^{(\mu)}(0|x^n) = 1 - \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n)$. Using this, we substitute the definition of randomized encoder (7) and the decoder (8) in the second term within the modulus of \mathcal{Q}_{x^n} . This gives

$$\frac{1}{2^{nC}} \sum_{\mu \in [2^{nC}]} \sum_{m \in [2^{nR}] \cup \{0\}} P_{Z|X}^n(z^n|x^n) P_{M|X^n}^{(\mu)}(m|x^n) P_{Y^n|Z^n M}^{(\mu)}(y^n|z^n, m) = T_1 + T_2,$$

where³,

$$\begin{aligned} T_1 &\triangleq \sum_{\mu \in [2^{nC}]} \sum_{m \in [2^{nR}]} \sum_{l=1}^{2^{n\tilde{R}}} \sum_{w^n \in T_\delta(W|x^n)} \frac{(1-\epsilon)}{(1+\eta)} \frac{1}{2^{n(\tilde{R}+C)}} \frac{P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \\ &\quad \mathbb{1}_{\{w^n = w^n(l, \mu), b^{(\mu)}(l) = m\}} P_{Y|WZ}^n(y^n | f^{(\mu)}(b^{(\mu)}(l), z^n), z^n) \\ &= \sum_{\mu \in [2^{nC}]} \sum_{l=1}^{2^{n\tilde{R}}} \sum_{w^n \in T_\delta(W|x^n)} \frac{(1-\epsilon)}{(1+\eta)} \frac{1}{2^{n(\tilde{R}+C)}} \frac{P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = w^n(l, \mu)\}} \\ &\quad P_{Y|WZ}^n(y^n | f^{(\mu)}(b^{(\mu)}(l), z^n), z^n), \\ T_2 &\triangleq \frac{1}{2^{nC}} \sum_{\mu \in [2^{nC}]} P_{Z|X}^n(z^n|x^n) \left[1 - \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \right] P_{Y|WZ}^n(y^n|w_0, z^n). \end{aligned}$$

Substituting T_1, T_2 for the second term within the modulus of \mathcal{Q}_{x^n} , and applying triangle inequality, we obtain $\mathcal{Q}_{x^n} \mathbb{1}_{\{\text{PMF}(C)\}} \leq \left[\frac{1}{2} \sum_{y^n, z^n} (S + \tilde{S}) \right] \mathbb{1}_{\{\text{PMF}(C)\}} \leq \frac{1}{2} \sum_{y^n, z^n} (S + \tilde{S} \mathbb{1}_{\{\text{PMF}(C)\}})$, where

$$\begin{aligned} S &\triangleq \left| P_{Y|Z|X}^n(y^n, z^n|x^n) - \right. \\ &\quad \left. \frac{(1-\epsilon)}{(1+\eta)} \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} \sum_{w^n \in T_\delta(W|x^n)} \frac{P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} P_{Y|WZ}^n(y^n | f^{(\mu)}(b^{(\mu)}(l), z^n), z^n) \mathbb{1}_{\{w^n = w^n(l, \mu)\}} \right|, \\ \tilde{S} &\triangleq \left| \frac{1}{2^{nC}} \sum_{\mu \in [2^{nC}]} P_{Z|X}^n(z^n|x^n) \left(1 - \sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \right) P_{Y|WZ}^n(y^n|w_0, z^n) \right|. \end{aligned}$$

Note that the term corresponding to \tilde{S} captures the error induced by not covering the product distribution $P_{XYZ}^n(\cdot)$ and we bound this term employing the following proposition.

³For the ease of notation, we do not show the dependency of T_1 , and T_2 on x^n, y^n and z^n .

Proposition 1. *There exist functions $\epsilon_{\tilde{S}}(\delta)$, and $\delta_{\tilde{S}}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} \tilde{S} \mathbb{1}_{\text{PMF}(C)}] \leq \epsilon_{\tilde{S}}(\delta)$, if $\tilde{R} > I(X; W) + \delta_{\tilde{S}}$, where $\epsilon_{\tilde{S}}, \delta_{\tilde{S}} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-A □

Now we move on to isolating the error component of S caused by binning the randomized encoders.

Step 2: Error caused by binning

We now consider the term corresponding to S . By adding and subtracting an appropriate term within the modulus of S and using triangle inequality, S can be bounded as $S \leq S_1 + S_2$, where

$$S_1 \triangleq \left| P_{YZ|X}^n(y^n, z^n | x^n) - \sum_{\mu, l} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{X|W}^n(x^n | w^n) P_{Z|X}^n(z^n | x^n) P_{Y|WZ}^n(y^n | w^n, z^n)}{2^{n(\tilde{R}+C)} (1+\eta) P_X^n(x^n)} \mathbb{1}_{\{w^n = \tilde{w}^n(l, \mu)\}} \right|,$$

$$S_2 \triangleq \left| \frac{(1-\epsilon)}{(1+\eta)} \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{P_{Z|X}^n(z^n | x^n) P_{X|W}^n(x^n | w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \tilde{w}^n(l, \mu)\}} \right. \\ \left. \left(P_{Y|WZ}^n(y^n | w^n, z^n) - P_{Y|WZ}^n(y^n | f^{(\mu)}(b^{(\mu)}(l), z^n), z^n) \right) \right|.$$

Note that the term S_2 captures the error introduced due to the binning operation. To bound this term, we provide the following proposition.

Proposition 2 (Mutual Packing). *There exist $\epsilon_{S_2}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E} \left[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} S_2 \right] \leq \epsilon_{S_2}(\delta)$, if $(\tilde{R}_1 - R_1) \leq I(W; Z) + \delta_{S_2}$, where $\epsilon_{S_2}, \delta_{S_2}(\delta) \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-B. □

Now we are left with the analysis of the term S_1 .

Step 3: Bounding the approximation/covering error

In this last step, we analyze the term S_1 which captures the action of the encoder in approximating the product distribution $P_{XYZ}^n(\cdot)$. For that, we split S_1 as $S_1 \leq S_{11} + S_{12}$, where

$$S_{11} = \left| P_{YZ|X}^n(y^n, z^n | x^n) - 2^{-n(\tilde{R}+C)} \sum_{\mu, l} P_{Z|X}^n(z^n | x^n) \sum_{w^n \in T_\delta(W)} \frac{P_{X|W}^n(x^n | w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \tilde{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n | w^n, z^n) \right|$$

$$S_{12} = 2^{-n(\tilde{R}+C)} \left| \sum_{\mu, l} P_{Z|X}^n(z^n | x^n) \sum_{w^n \in T_\delta(W)} \frac{P_{X|W}^n(x^n | w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \tilde{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n | w^n, z^n) - \right. \\ \left. \left(\frac{1-\epsilon}{1+\eta} \right) \sum_{\mu, l} P_{Z|X}^n(z^n | x^n) \sum_{w^n \in T_\delta(W|x^n)} \frac{P_{X|W}^n(x^n | w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \tilde{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n | w^n, z^n) \right|.$$

(14)

Using the Markov chains $Z - X - W$ and $X - (Z, W) - Y$ which P_{WXYZ} satisfies, and the fact that $\sum_{w^n \in \mathcal{T}_\delta(W)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} = 1$, we can simplify the second term in S_{11} as

$$\begin{aligned} & 2^{-n(\tilde{R}+C)} \sum_{\mu, l} P_{Z|X}^n(z^n|x^n) \sum_{w^n \in \mathcal{T}_\delta(W)} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n|w^n, z^n) \\ &= \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} \sum_{w^n \in \mathcal{T}_\delta(W)} \frac{P_{X|W}^n(x^n|\mathbf{w}^n(l, \mu))}{P_X^n(x^n)} P_{Z|XW}^n(z^n|x^n, \mathbf{w}^n(l, \mu)) P_{Y|WXZ}^n(y^n|\mathbf{w}^n(l, \mu), x^n, z^n) \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} \\ &= \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} \frac{P_{XYZ|W}^n(x^n, y^n, z^n|\mathbf{w}^n(l, \mu))}{P_X^n(x^n)}. \end{aligned}$$

Substituting the above simplification into the expression for S_{11} gives

$$S_{11} = \left| P_{YZ|X}^n(y^n, z^n|x^n) - \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} \frac{P_{XYZ|W}^n(x^n, y^n, z^n|\mathbf{w}^n(l, \mu))}{P_X^n(x^n)} \right|. \quad (15)$$

Substituting this simplification in $\mathbb{E}[\sum_{x^n \in \mathcal{T}_\delta(X)} P_X^n(x^n) (\sum_{y^n, z^n} S_{11})]$, we obtain

$$\begin{aligned} & \mathbb{E} \left[\sum_{x^n \in \mathcal{T}_\delta(X)} P_X^n(x^n) \left(\sum_{y^n, z^n} S_{11} \right) \right] \\ &= \mathbb{E} \left[\sum_{x^n \in \mathcal{T}_\delta(X)} \sum_{y^n, z^n} \left| P_{XYZ}^n(x^n, y^n, z^n) - \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} P_{XYZ|W}^n(x^n, y^n, z^n|\mathbf{w}^n(l, \mu)) \right| \right] \\ &\leq \mathbb{E} \left[\sum_{x^n, y^n, z^n} \left| \tilde{P}_{XYZ}^n(x^n, y^n, z^n) - \frac{1}{2^{n(\tilde{R}+C)}} \sum_{\mu, l} P_{XYZ|W}^n(x^n, y^n, z^n|\mathbf{w}^n(l, \mu)) \right| \right] + 2\epsilon, \quad (16) \end{aligned}$$

where the last inequality follows by defining $\tilde{P}_{XYZ}^n(\cdot)$ as $\tilde{P}_{XYZ}^n(x^n, y^n, z^n) \triangleq \sum_{w^n \in \mathcal{T}_\delta(W)} P_{XYZ|W}^n(x^n, y^n, z^n|w^n) \tilde{P}_{W^n}(w^n)$.

Lemma 2 (One-shot Soft Covering). *Let P_{AB} be a joint PMF defined on $\mathcal{A} \times \mathcal{B}$ with \mathcal{A} and \mathcal{B} being finite sets. Further, suppose we are given a subset $\mathcal{T} \subset \mathcal{A}$ and a collection of subsets $\mathcal{T}_b \subset \mathcal{A}$ for all $b \in \mathcal{B}$ which satisfy the following hypotheses for all $b \in \mathcal{B}$:*

$$P_A(\mathcal{T}) \geq 1 - \epsilon, \quad (17a)$$

$$P_{A|B}(\mathcal{T}_b|b) \geq 1 - \epsilon, \quad (17b)$$

$$\left(\sum_{a \in \mathcal{T}} \sqrt{P_A(a)} \right)^2 \leq D, \quad \text{and} \quad (17c)$$

$$P_{A|B}(a|b) \leq \frac{1}{d}, \forall a \in \mathcal{T}_b, \quad (17d)$$

for some $\epsilon \in (0, 1)$ and $d < D$. Let M be a finite non-negative integer and let a random covering code $\mathbb{C} \triangleq \{C_m\}_{m \in [1, M]}$ be defined as a collection of codewords C_m that are chosen pairwise independently

according to the distribution P_B from \mathcal{B} . Then we have

$$\mathbb{E}_{\mathbb{C}} \left[\sum_{a \in \mathcal{A}} \left| P_A(a) - \frac{1}{M} \sum_{m=1}^M P_{A|B}(a|C_m) \right| \right] \leq \sqrt{\frac{D}{Md}} + 4\delta(\epsilon). \quad (18)$$

Proof. The proof is provided in Appendix A-B □

Now we prove the above term is small by using the following identification. Identify \mathcal{A} by $(\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n)$, \mathcal{B} by $T_{\delta}(W)$, \mathcal{T} by $T_{\delta}(XYZ)$, \mathcal{T}_b by $T_{\delta}(XYZ|w^n)$ for all $w^n \in T_{\delta}(W)$, and P_{AB} by $P_{XYZ|W}^n \tilde{P}_{W^n}$ (with $\tilde{P}_{W^n}(\cdot)$ as defined in (9)). Using this identification we first compute D and d that satisfy the hypothesis of the lemma. This gives $D = 2^{n(H(X,Y,Z)+\delta_{XYZ})}$ and $d = 2^{n(H(XYZ|W)-\delta'_{XYZ})}$. To satisfy (17a), we use the fact that if $\|P_A - Q_A\| \leq \epsilon_A$, for P_A and Q_A defined as two distributions on \mathcal{A} then for any subset $\bar{\mathcal{A}} \subset \mathcal{A}$, we have $P_A(\bar{\mathcal{A}}) \geq Q_A(\bar{\mathcal{A}}) - \epsilon_A$. Since $\|P_{XYZ}^n - \tilde{P}_{X^n Y^n Z^n}\| \leq 2\epsilon$, we have $\tilde{P}_{X^n Y^n Z^n}(T_{\delta}(XYZ)) \geq 1 - 3\epsilon(\delta)$ which can be made arbitrarily close to 1 for a sufficiently large n . The hypotheses (17b) and (17d) can be shown to be true using the basic typicality arguments. For the hypothesis (17c) we use the bound $\tilde{P}_{X^n Y^n Z^n}(\cdot) \leq \frac{1}{(1-\epsilon)} P_{XYZ}^n(\cdot)$, which gives the D mentioned above.

Using this identification and applying Lemma (2) on (16) we obtain $\mathbb{E}[\sum_{x^n \in T_{\delta}(X)} P_X^n(x^n) (\sum_{y^n, z^n} S_{11})] \leq \epsilon_{S_{11}}$, if $\tilde{R} + C \geq I(XYZ; W) + \delta_{S_{11}}$ for sufficiently n , where $\delta_{S_{11}}(\delta), \epsilon_{S_{11}}(\delta) \searrow 0$ as $\delta \searrow 0$.

Now consider S_{12} . This term can be split into the S'_{12} and S''_{12} such that $S_{12} = S'_{12} + S''_{12}$, where

$$S'_{12} \triangleq 2^{-n(\tilde{R}+C)} \left| \left(1 - \frac{1-\epsilon}{1+\eta} \right) \sum_{\mu, l} P_{Z|X}^n(z^n|x^n) \sum_{\substack{w^n \in \\ T_{\delta}(W|x^n) \cap T_{\delta}(W)}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n|w^n, z^n) \right|,$$

$$S''_{12} \triangleq 2^{-n(\tilde{R}+C)} \left| \sum_{\mu, l} P_{Z|X}^n(z^n|x^n) \sum_{\substack{w^n \notin T_{\delta}(W|x^n) \\ w^n \in T_{\delta}(W)}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} P_{Y|WZ}^n(y^n|w^n, z^n) \right|.$$

Now, we apply expectation over each of the following to obtain,

$$\begin{aligned} \mathbb{E} \left[\sum_{y^n, z^n} S'_{12} \right] &= 2^{-n(\tilde{R}+C)} \frac{\eta + \epsilon}{1 + \eta} \sum_{y^n, z^n} \sum_{\mu, l} P_{Z|X}^n(z^n|x^n) \sum_{\substack{w^n \in \\ T_{\delta}(W|x^n) \cap T_{\delta}(W)}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \frac{P_W^n(w^n)}{1 - \epsilon} P_{Y|WZ}^n(y^n|w^n, z^n) \\ &\leq \left(\frac{\eta + \epsilon}{1 + \eta} \right) \frac{1}{(1 - \epsilon)} \sum_{y^n, z^n} \sum_{w^n \in T_{\delta}(W|x^n)} P_{WYZ|X}^n(w^n, y^n, z^n|x^n) \leq \frac{(\eta + \epsilon)}{(1 + \eta)(1 - \epsilon)}. \end{aligned} \quad (19)$$

And similarly, we have

$$\begin{aligned} \mathbb{E} \left[\sum_{y^n, z^n} S''_{12} \right] &= \frac{1}{(1 - \epsilon)} \sum_{y^n, z^n} \sum_{\substack{w^n \notin T_{\delta}(W|x^n) \\ w^n \in T_{\delta}(W)}} P_{WYZ|X}^n(w^n, y^n, z^n|x^n) \\ &\leq \frac{1}{(1 - \epsilon)} \sum_{w^n \notin T_{\delta}(W|x^n)} P_{W|X}^n(w^n|x^n) \leq \frac{\epsilon''}{1 - \epsilon}, \end{aligned} \quad (20)$$

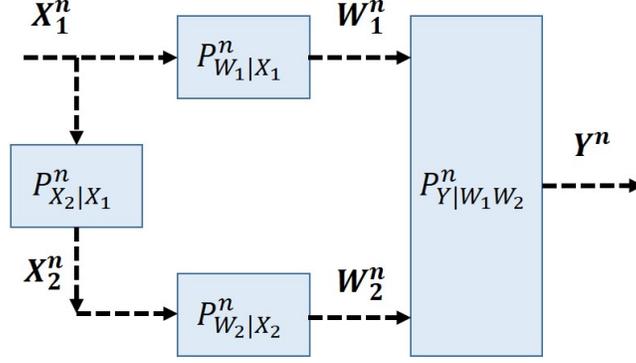


Fig. 4. Figure demonstrating the generation of random variables X_1, X_2, W_1, W_2, Y from the joint PMF $P_{\underline{XWY}}$ while incorporating the Markov chains specified in the theorem statement.

where $\epsilon''(\delta) \searrow 0$ as $\delta \searrow 0$. We have argued that terms S_1, S_2 and \tilde{S} are small in expectation for sufficiently large n , which implies $\mathbb{E}[\mathcal{Q}_{x^n} \mathbb{1}_{\{\text{PMF}(C)\}}] \leq \epsilon_Q$ for sufficiently large n , where $\epsilon_Q(\delta) \searrow 0$ as $\delta \searrow 0$. Using this in (13), and subsequently in (11), and eventually in (10) gives $\mathbb{E}[\mathcal{Q}] \leq \epsilon$, for sufficiently large n if

$$\tilde{R} \geq R \geq 0, \quad C \geq 0, \quad \tilde{R} \geq I(X; W), \quad \tilde{R} - R \leq I(W; Z) \quad \text{and} \quad \tilde{R} + C \geq I(XYZ; W). \quad (21)$$

Lastly, the proof is completed using the Fourier-Motzkin Elimination [26]. \square

V. PROOF OF THEOREM 1

Having designed a randomized encoding scheme based on typicality for the side-information case, we are in a position to employ the same encoder for the distributed scenario. In contrast to the side-information scenario, both randomized encoders choose codewords resulting in the need to prove that the individual codewords chosen by the distributed randomized encoders are with high probability jointly typical with the observed source sequences. This involves new elements in the context of soft covering. Fix a PMF $P_{QW_1W_2X_1X_2Y}$ satisfying the constraints stated in the theorem. Since Q , the time sharing random variable, is employed in the standard way, for ease of exposition, we provide the proof of the special case of $Q = 0$. Its generalization can be obtained in a straightforward way. Let $\mu \in [2^{nC}]$ denote the common randomness shared amidst all terminals. The first encoder uses a part of the entire common randomness available to it, say C_1 bits out of the C bits, which is denoted by $\mu_1 \in [2^{nC_1}]$. Similarly, let $\mu_2 \in [2^{nC_2}]$ denote the common randomness used by the second encoder. Note that $C = C_1 \times C_2$ and $\mu \triangleq (\mu_1, \mu_2)$. Our goal is to prove the existence of PMFs $P_{M_1|X_1}^{(\mu_1)}(m_1|x_1^n) : x_1^n \in \mathcal{X}_1^n, m_1 \in [\Theta_1], \mu_1 \in [2^{nC_1}]$, $P_{M_2|X_2}^{(\mu_2)}(m_2|x_2^n) : x_2^n \in \mathcal{X}_2^n, m_2 \in [\Theta_2], \mu_2 \in [2^{nC_2}]$, $P_{Y^n|M_1, M_2}^{(\mu)}(y^n|m_1, m_2) : y^n \in \mathcal{Y}^n, (m_1, m_2) \in [\Theta_1] \times [\Theta_2]$ such that

$$\mathcal{Q} \triangleq \frac{1}{2} \sum_{x_1^n, x_2^n, y^n} \left| P_{X_1X_2Y}^n(x_1^n, x_2^n, y^n) - \sum_{\mu \in [2^{nC}]} \sum_{m_1 \in [\Theta_1]} \sum_{m_2 \in [\Theta_2]} \frac{P_{X_1X_2}^n(x_1^n, x_2^n)}{2^{nC}} P_{M_1|X_1}^{(\mu_1)}(m_1|x_1^n) P_{M_2|X_2}^{(\mu_2)}(m_2|x_2^n) P_{Y^n|M_1, M_2}^{(\mu)}(y^n|m_1, m_2) \right| \leq \epsilon, \quad (22)$$

for $\frac{\log \Theta_j}{n} = R_j : j \in [2]$ and for all sufficiently large n . Consider the collections $\mathcal{C}_1 \triangleq (\mathcal{C}_1^{(\mu_1)} : 1 \leq \mu_1 \leq 2^{nC_1})$ where $\mathcal{C}_1^{(\mu_1)} \triangleq (\mathbf{w}_1(l_1, \mu_1) : 1 \leq l_1 \leq 2^{n\tilde{R}_1})$ and $\mathcal{C}_2 \triangleq (\mathcal{C}_2^{(\mu_2)} : 1 \leq \mu_2 \leq 2^{nC_2})$ where $\mathcal{C}_2^{(\mu_2)} \triangleq (\mathbf{w}_2(l_2, \mu_2) : 1 \leq l_2 \leq 2^{n\tilde{R}_2})$. For this collection, we let

$$E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \triangleq \frac{1}{2^{n\tilde{R}_1}} \frac{1 - \epsilon_1}{1 + \eta} \sum_{w_1^n \in T_{\delta}(W_1|x_1^n)} \mathbb{1}_{\{\mathbf{w}^n(l_1, \mu_1) = w_1^n\}} \frac{P_{X_1|W_1}^n(x_1^n|w_1^n)}{P_{X_1}^n(x_1^n)}$$

$$E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n) \triangleq \frac{1}{2^{n\tilde{R}_2}} \frac{1 - \epsilon_2}{1 + \eta} \sum_{w_2^n \in T_{\delta}(W_2|x_2^n)} \mathbb{1}_{\{\mathbf{w}^n(l_2, \mu_2) = w_2^n\}} \frac{P_{X_2|W_2}^n(x_2^n|w_2^n)}{P_{X_2}^n(x_2^n)}$$

where $\bar{\delta}_i \triangleq \delta|\mathcal{X}_i + \mathcal{W}_i|$ and $\epsilon_i = 1 - P_W^n(T_{\bar{\delta}_i}(W_i))$; $i = 1, 2$. The definition of $E_{L_1|X_1^n}^{(\mu_1)}$ and $E_{L_2|X_2^n}^{(\mu_2)}$ can be thought of as encoding rules that do not exploit the additional rebate obtained by using binning techniques.

A. Binning of Random Encoders

Further, we define maps $b_1^{(\mu_1)} : [2^{n\tilde{R}_1}] \rightarrow [2^{nR_1}]$ and $b_2^{(\mu_2)} : [2^{n\tilde{R}_2}] \rightarrow [2^{nR_2}]$ performing standard information-theoretic binning, with $0 < R_1 \leq \tilde{R}_1$ and $0 < R_2 \leq \tilde{R}_2$. Using these maps, we induce the PMF $P_{M_1|X_1^n}^{(\mu_1)}$ on the message to be transmitted by the first encoder as

$$P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = \begin{cases} \mathbb{1}_{\{m_1=0\}} & \text{if } s_1^{(\mu_1)}(x_1^n) > 1, \\ 1 - s_1^{(\mu_1)}(x_1^n) & \text{if } m_1 = 0 \text{ and } s_1^{(\mu_1)}(x_1^n) \in [0, 1], \\ \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \mathbb{1}_{\{b_1^{(\mu_1)}(l_1) = m_1\}} & \text{if } m_1 \neq 0 \text{ and } s_1^{(\mu_1)}(x_1^n) \in [0, 1] \end{cases} \quad (23)$$

for all $x_1^n \in T_{\delta}(X_1)$ and $s_1^{(\mu_1)}(x_1^n)$ defined as $s_1^{(\mu_1)}(x_1^n) = \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n)$. For $x_1^n \notin T_{\delta}(X_1)$, we let $P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = \mathbb{1}_{\{m_1=0\}}$.

We similarly define the PMF $P_{M_2|X_2^n}^{(\mu_2)}$ for the second encoder as

$$P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) = \begin{cases} \mathbb{1}_{\{m_2=0\}} & \text{if } s_2^{(\mu_2)}(x_2^n) > 1, \\ 1 - s_2^{(\mu_2)}(x_2^n) & \text{if } m_2 = 0 \text{ and } s_2^{(\mu_2)}(x_2^n) \in [0, 1], \\ \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n) \mathbb{1}_{\{b_2^{(\mu_2)}(l_2) = m_2\}} & \text{if } m_2 \neq 0 \text{ and } s_2^{(\mu_2)}(x_2^n) \in [0, 1] \end{cases} \quad (24)$$

for all $x_2^n \in T_{\delta}(X_2)$ and $s_2^{(\mu_2)}(x_2^n)$ defined as $s_2^{(\mu_2)}(x_2^n) = \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n)$. For $x_2^n \notin T_{\delta}(X_2)$, we let $P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) = \mathbb{1}_{\{m_2=0\}}$.

With this definition note that, $\sum_{m_1=0}^{2^{nR_1}} P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) = 1$ for all $\mu_1 \in [2^{nC_1}]$ and $x_1^n \in \mathcal{X}_1^n$ and similarly,

$$\sum_{m_2=0}^{2^{nR_2}} P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) = 1 \text{ for all } \mu_2 \in [2^{nC_2}] \text{ and } x_2^n \in \mathcal{X}_2^n.$$

B. Decoder Mapping

We now describe the decoder. On observing μ and the indices $m_1, m_2 \in [2^{nR_1}] \times [2^{nR_2}]$ communicated by the encoders, the decoder first deduces (μ_1, μ_2) from μ and then populates

$$\mathcal{D}^{(\mu_1, \mu_2)}(m_1, m_2) = \left\{ \begin{array}{l} (l_1, l_2) \in [2^{n\tilde{R}_1}] \times [2^{n\tilde{R}_2}] : b_1^{(\mu_1)}(l_1) = m_1, b_2^{(\mu_2)}(l_2) = m_2, \\ (\mathbf{w}_1^n(l_1, \mu_1), \mathbf{w}_2^n(l_2, \mu_2)) \in T_\delta(W_1, W_2) \end{array} \right\}. \quad (25)$$

Let

$$f^{(\mu)}(m_1, m_2) = \begin{cases} (\mathbf{w}_1^n(l_1, \mu_1), \mathbf{w}_2^n(l_2, \mu_2)) & \text{if } \mathcal{D}^{(\mu_1, \mu_2)}(m_1, m_2) = \{(l_1, l_2)\} \\ (\tilde{w}_1^n, \tilde{w}_2^n) & \text{otherwise, i.e., } |\mathcal{D}^{(\mu_1, \mu_2)}(m_1, m_2)| \neq 1 \end{cases}.$$

The decoder chooses y^n according to PMF $P_{Y|W_1W_2}^n(y^n | f^{(\mu)}(m_1, m_2))$. This implies the PMF $P_{Y^n|M_1M_2}^{(\mu)}(\cdot | \cdot)$ is given by

$$P_{Y^n|M_1M_2}^{(\mu)}(\cdot | m_1, m_2) = P_{Y|W_1W_2}^n(y^n | f^{(\mu)}(m_1, m_2)). \quad (26)$$

C. Distribution of Codebooks

The PMF defined on the ensemble of codebooks is as specified below. The codewords of the random codebook $\mathcal{C}_1^{(\mu_1)} = (\mathbf{w}_1(l_1, \mu_1) : 1 \leq l_1 \leq 2^{n\tilde{R}_1})$ for each $\mu_1 \in 2^{nC_1}$ are mutually independent and distributed with PMF

$$\mathbb{P}(W_1(l_1, \mu_1) = w_1^n) = \frac{P_{W_1}^n(w_1^n)}{(1 - \epsilon_1)} \mathbb{1}_{\{w_1^n \in T_\delta^n(W_1)\}}$$

Similarly, $\mathcal{C}_2^{(\mu_2)} = (\mathbf{w}_2(l_2, \mu_2) : 1 \leq l_2 \leq 2^{n\tilde{R}_2})$ for each $\mu_2 \in [2^{nC_2}]$ are mutually independent and distributed with PMF

$$\mathbb{P}(W_2(l_2, \mu_2) = w_2^n) = \frac{P_{W_2}^n(w_2^n)}{(1 - \epsilon_2)} \mathbb{1}_{\{w_2^n \in T_\delta^n(W_2)\}}$$

where, recall $\epsilon_i = 1 - P_{W_i}^n(T_\delta(W_i))$; $i = 1, 2$. Finally, the binning functions $b_1^{(\mu_1)}(\cdot)$ and $b_2^{(\mu_2)}(\cdot)$ are chosen random, uniformly and independently from the sets $[2^{nR_1}]$ and $[2^{nR_2}]$, respectively.

We now begin our analysis of (22). Our goal is to prove the existence of a collections $\mathcal{C}_1, \mathcal{C}_2$ for which (22) holds. We do this via random coding. Specifically, we prove that $\mathbb{E}\mathcal{Q} \leq \epsilon$ where the expectation is over the ensemble of codebooks.

D. Analysis of Total Variation

We begin by splitting \mathcal{Q} into two terms using an indicator function $\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}$ as

$$\mathbb{E}\mathcal{Q} = \mathbb{E}[\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}] + \mathbb{E}[\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}^c}] \leq \mathbb{E}[\mathcal{Q} \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}] + \mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = 0\} \quad (27)$$

where $\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}$ is defined as

$$\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = \begin{cases} 1 & \text{if } s_1^{(\mu_1)}(x_1^n) \in [0, 1] \text{ and } s_2^{(\mu_2)}(x_2^n) \in [0, 1] \\ & \text{for all } x_1^n \in T_\delta(X_1), x_2^n \in T_\delta(X_2), \mu_1 \in [2^{n\mathcal{C}_1}], \mu_2 \in [2^{n\mathcal{C}_2}] \\ 0 & \text{otherwise,} \end{cases}$$

and (27) follows from the upper bound of 1 over the total variation. We now show using the lemma below, that by appropriately constraining \tilde{R}_1 and \tilde{R}_2 , $\mathbb{P}\{\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = 0\}$ can be made arbitrarily small.

In other words, with high probability, we will have $E_{L_1|X_1^n}^{(\mu_1)}$ and $E_{L_2|X_2^n}^{\mu_2}$ such that $0 \leq \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)} \leq 1$ for all $\mu_1 \in [2^{n\mathcal{C}_1}]$ and $x_1^n \in T_\delta(X_1)$, and $0 \leq \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)} \leq 1$ for all $\mu_2 \in [2^{n\mathcal{C}_2}]$ and $x_2^n \in T_\delta(X_2)$.

Proposition 3. For any $\delta, \eta \in (0, 1/2)$, if $\tilde{R}_1 > I(X_1 : W_1) + 4\delta_1$ and $\tilde{R}_2 > I(X_2 : W_2) + 4\delta_2$, where $\delta_1(\delta), \delta_2(\delta) \searrow 0$ as $\delta \searrow 0$, then

$$\mathbb{P} \left[\left(\bigcap_{\mu=1}^{2^{n\mathcal{C}_1}} \bigcap_{x^n \in T_\delta(X_1)} \left(\sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \leq 1 \right) \right) \cap \left(\bigcap_{\mu_2=1}^{2^{n\mathcal{C}_2}} \bigcap_{x_2^n \in T_\delta(X_2)} \left(\sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n) \leq 1 \right) \right) \right] \rightarrow 1 \text{ as } n \rightarrow \infty. \quad (28)$$

Proof. The proof follows from Lemma 1. \square

We now look at the first term in (27), i.e., $\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}$. This can be expanded as

$$\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = \left[\sum_{\underline{x}^n \in T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \mathcal{Q}_{\underline{x}^n} + \sum_{\underline{x}^n \notin T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \mathcal{Q}_{\underline{x}^n} \right] \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}, \quad (29)$$

where $T_\delta(\underline{X})$ is defined as $T_\delta(\underline{X}) \triangleq \{\underline{x}^n : (x_1^n, x_2^n) \in T_\delta(X_1, X_2)\}$ and $\mathcal{Q}_{\underline{x}^n}$ is defined as

$$\mathcal{Q}_{\underline{x}^n} \triangleq \frac{1}{2} \sum_{y^n} \left| P_{Y^n|\underline{X}}^n(y^n|\underline{x}^n) - \sum_{\substack{\mu_1, \mu_2 \\ m_1 \in [2^{n\mathcal{R}_1}] \cup \{0\} \\ m_2 \in [2^{n\mathcal{R}_2}] \cup \{0\}}} \frac{1}{2^{n(\mathcal{C}_1 + \mathcal{C}_2)}} \sum_{\substack{m_1 \in [2^{n\mathcal{R}_1}] \cup \{0\} \\ m_2 \in [2^{n\mathcal{R}_2}] \cup \{0\}}} P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) P_{Y^n}^{(\mu)}(y^n|m_1, m_2) \right|.$$

Since, using the standard typicality arguments one can argue $\sum_{\underline{x}^n \notin T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \leq \epsilon_t$, where $\epsilon_t(\delta) \searrow 0$ as $\delta \searrow 0$. We bound $\mathcal{Q}_{\underline{x}^n}$ within the second summation in the right hand side of the above equation⁴ to

⁴Note that $\mathcal{Q}_{x_1^n, x_2^n}$ is a total variational distance between two conditional PMFs, conditioned on (X_1, X_2) , for each (x_1^n, x_2^n) and hence it is bounded from above by one.

obtain,

$$\mathcal{Q} \cdot \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = \sum_{\underline{x}^n \in T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \mathcal{Q}_{\underline{x}^n} \mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} + \epsilon_t(\delta). \quad (30)$$

Now, what remains is the first term in (30). A major portion of our analysis from here on deals with arguing that this term can be made arbitrarily small. Further, since this term contains the indicator $\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}}$, we can restrict our analysis to only the set of random codebooks $(\mathcal{C}_1, \mathcal{C}_2)$ that satisfy $0 \leq \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \leq 1$ and $0 \leq \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n) \leq 1$ for all $\underline{x}^n \in T_\delta(\underline{X})$ and $\mu_1 \in [2^{nC_1}]$, $\mu_2 \in [2^{nC_2}]$.

Step 1: Isolating the error induced by not covering

As a first step, we separate the error induced by not covering the product distribution $P_{X_1 X_2 Y}^n(\cdot)$ through the randomized encoders and provide a bound to it. Note that under the condition that $\mathbb{1}_{\{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)\}} = 1$, we have $P_{M_i|X_i^n}^{(\mu_i)}(m_i|x_i^n) = \sum_{l_i=1}^{2^{n\tilde{R}_i}} E_{L_i|X_i^n}^{(\mu_i)}(l_i|x_i^n)$ when $m_i \neq 0$, and $P_{M_i|X_i^n}^{(\mu_i)}(0|x_i^n) = 1 - \sum_{l_i=1}^{2^{n\tilde{R}_i}} E_{L_i|X_i^n}^{(\mu_i)}(l_i|x_i^n)$, for $i \in \{1, 2\}$. Using this, we substitute the definition of the randomized encoders (23), (24) and the decoder (26) in the second term within the modulus of $\mathcal{Q}_{\underline{x}^n}$ for $\underline{x}^n \in T_\delta(\underline{X})$, which gives,

$$\sum_{\substack{\mu_1 \in [2^{nC_1}], \mu_2 \in [2^{nC_2}] \\ m_1 \in [2^{n\tilde{R}_1}] \cup \{0\} \\ m_2 \in [2^{n\tilde{R}_2}] \cup \{0\}}} \frac{P_{M_1|X_1^n}^{(\mu_1)}(m_1|x_1^n) P_{M_2|X_2^n}^{(\mu_2)}(m_2|x_2^n) P_{Y^n|M_1 M_2}^{(\mu)}(y^n|m_1, m_2)}{2^{n(C_1+C_2)}} = T_1 + T_2 + T_3 + T_4,$$

where⁵,

$$\begin{aligned} T_1 &\triangleq \sum_{\substack{\mu_1, \mu_2 \\ m_1 \in [2^{n\tilde{R}_1}] \\ m_2 \in [2^{n\tilde{R}_2}]} \sum_{l_1, l_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C)} (1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \\ &\quad \mathbb{1}_{\{w_1^n = \mathbf{w}_1^n(l_1, \mu_1), b_1^{(\mu_1)}(l_1) = m_1\}} \mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2), b_2^{(\mu_2)}(l_2) = m_2\}} P_{Y^n|W_1 W_2}^n(y^n | f^{(\mu)}(b_1^{(\mu_1)}(l_1), b_2^{(\mu_2)}(l_2))) \\ &= \sum_{\mu_1, \mu_2} \sum_{l_1, l_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \\ &\quad \mathbb{1}_{\{w_1^n = \mathbf{w}_1^n(l_1, \mu_1)\}} \mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} P_{Y^n|W_1 W_2}^n(y^n | f^{(\mu)}(b_1^{(\mu_1)}(l_1), b_2^{(\mu_2)}(l_2))), \\ T_2 &\triangleq \sum_{\mu_1, \mu_2, l_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{\left[1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n)\right] (1 - \epsilon_1) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(C_1+C_2)}} \frac{1}{2^{n\tilde{R}_2} (1 + \eta) P_{X_2}^n(x_2^n)} \mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} P_{Y^n|W_1 W_2}^n(y^n | \tilde{w}_1^n, \tilde{w}_2^n), \end{aligned}$$

⁵For the ease of notation, we do not show the dependency of T_1, T_2, T_3 and T_4 on \underline{x}^n , however, in principle they depend on \underline{x}^n and in fact, are only defined for $\underline{x}^n \in T_\delta(\underline{X})$

$$T_3 \triangleq \sum_{\mu_1, \mu_2, l_1} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \frac{\left[1 - \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n)\right] (1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n)}{2^{n(C_1+C_2)}} \frac{1}{2^{n\tilde{R}_1}(1 + \eta) P_{X_1}^n(x_1^n)} \mathbb{1}_{\{w_1^n = \tilde{w}_1^n(l_1, \mu_1)\}} P_{Y|W_1W_2}^n(y^n|\tilde{w}_1^n, \tilde{w}_2^n),$$

$$T_4 \triangleq \sum_{\mu_1, \mu_2} \frac{\left[1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n)\right] \left[1 - \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n)\right]}{2^{n(C_1+C_2)}} P_{Y|W_1W_2}^n(y^n|\tilde{w}_1^n, \tilde{w}_2^n)$$

The above simplification in the expression for T_1 is obtained by using $\sum_{m_1 \in [2^{nR_1}]} \mathbb{1}_{\{w_1^n = \tilde{w}_1^n(l_1, \mu_1)\}} = 1$ and $\sum_{m_2 \in [2^{nR_2}]} \mathbb{1}_{\{w_2^n = \tilde{w}_2^n(l_2, \mu_2)\}} = 1$, which follows from the definition of the maps $b_1^{(\mu_1)}$ and $b_2^{(\mu_2)}$. A similar simplification for the expressions T_2 and T_3 is used while substituting $P_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n) = 1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n)$ and $P_{M_2|X_2^n}^{(\mu_2)}(0|x_2^n) = 1 - \sum_{l_2=1}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n)$, respectively. Finally, T_4 uses the substitution for both $P_{M_1|X_1^n}^{(\mu_1)}(0|x_1^n)$ and $P_{M_2|X_2^n}^{(\mu_2)}(0|x_2^n)$. Substituting T_1, T_2, T_3 and T_4 for the second term within the modulus of (22), we obtain $\mathcal{Q}_{\underline{x}^n} \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} \leq \frac{1}{2} \sum_{y^n} (S + \tilde{S}) \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} \leq \frac{1}{2} \sum_{y^n} (S + \tilde{S} \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}})$, where

$$S \triangleq \left| P_{Y|X_1X_2}^n(y^n|x_1^n, x_2^n) - \sum_{\mu_1, \mu_2} \sum_{l_1, l_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \right. \\ \left. \mathbb{1}_{\{w_1^n = \tilde{w}_1^n(l_1, \mu_1)\}} \mathbb{1}_{\{w_2^n = \tilde{w}_2^n(l_2, \mu_2)\}} P_{Y|W_1W_2}^n(y^n|f^{(\mu)}(b_1^{(\mu_1)}(l_1), b_2^{(\mu_2)}(l_2))) \right|,$$

and $\tilde{S} \triangleq |T_2| + |T_3| + |T_4|$. Note that the term corresponding to \tilde{S} captures the error induced by not covering the product distribution $P_{X_1X_2Y}^n(\cdot)$ and we bound this term employing the following proposition.

Proposition 4. *There exist functions $\epsilon_{\tilde{S}}(\delta)$, and $\delta_{\tilde{S}}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E} \left[\frac{1}{2} \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n} P_{\underline{X}}^n(\underline{x}^n) \tilde{S} \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} \right] \leq \epsilon_{\tilde{S}}(\delta)$, if $\tilde{R}_1 > I(X_1; W_1) + \delta_{\tilde{S}}$ and $\tilde{R}_2 > I(X_2; W_2) + \delta_{\tilde{S}}$, where $\epsilon_{\tilde{S}}, \delta_{\tilde{S}} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-C. \square

Now we move on to isolating the error component of S caused by binning the randomized encoders.

Step 2: Error caused by binning

By adding and subtracting an appropriate term within the modulus of S and using triangle inequality, S can be bounded as $S \leq S_1 + S_2$, where

$$S_1 \triangleq \left| P_{Y|X_1X_2}^n(y^n|x_1^n, x_2^n) - \sum_{\mu_1, \mu_2} \sum_{l_1, l_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \right. \\ \left. \mathbb{1}_{\{w_1^n = \tilde{w}_1^n(l_1, \mu_1)\}} \mathbb{1}_{\{w_2^n = \tilde{w}_2^n(l_2, \mu_2)\}} P_{Y|W_1W_2}^n(y^n|w_1^n, w_2^n) \right|$$

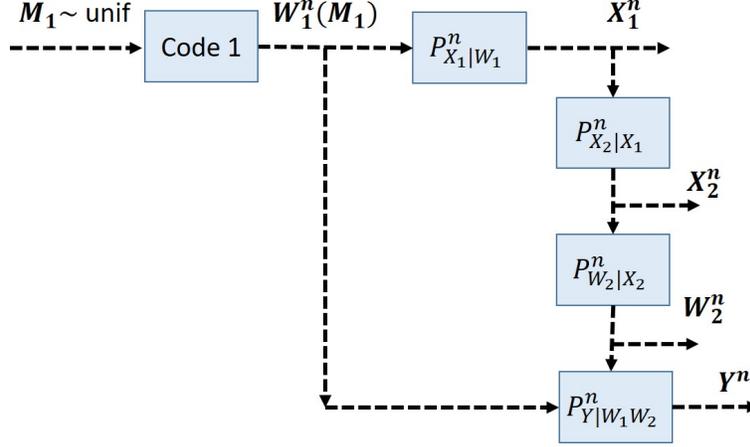


Fig. 5. Depiction of the approximation performed by Alice (encoder 1) while assuming a product distribution on Bob's (encoder 2) end.

$$S_2 \triangleq \sum_{\mu_1, \mu_2} \sum_{l_1, l_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \left| P_{Y|W_1W_2}^n(y^n|w_1^n, w_2^n) - P_{Y|W_1W_2}^n(y^n|f^{(\mu)}(b_1^{(\mu_1)}(l_1), b_2^{(\mu_2)}(l_2))) \right|.$$

Note that the term S_2 captures the error introduced due to the binning operation. To bound this term, we provide the following proposition.

Proposition 5 (Mutual Packing). *There exist $\epsilon_{S_2}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E} \left[\sum_{x^n \in T_\delta(\underline{X})} P_{\underline{X}}^n(x^n) S_2 \right] \leq \epsilon_{S_2}(\delta)$, if $(\tilde{R}_1 - R_1) + (\tilde{R}_2 - R_2) \leq I(W_1; W_2) + \delta_{S_2}$, where $\epsilon_{S_2}, \delta_{S_2}(\delta) \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-D. □

Now, we are left with the analysis of the term S_1 . For this, we segregate the effect of two encoders within the term S_1 , and separately analyze each of them, starting with the Alice's encoder.

Step 3: Term concerning Alice's encoding

For notational convenience, we first define $E_{W_1^n|X_1^n}^{(\mu_1)}$ and $E_{W_2^n|X_2^n}^{(\mu_2)}$, as

$$E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \triangleq \frac{1}{2^{n\tilde{R}_1}} \frac{(1 - \epsilon_1)}{(1 + \eta)} \sum_{l_1=1}^{2^{n\tilde{R}_1}} \frac{P_{X_1|W_1}^n(x_1^n|w_1^n)}{P_{X_1}^n(x_1^n)} \mathbb{1}_{\{w_1^n = \tilde{w}_1^n(l_1, \mu_1), w_1^n \in T_\delta(W_1|x_1^n)\}},$$

$$E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) \triangleq \frac{1}{2^{n\tilde{R}_2}} \frac{(1 - \epsilon_2)}{(1 + \eta)} \sum_{l_2=1}^{2^{n\tilde{R}_2}} \frac{P_{X_2|W_2}^n(x_2^n|w_2^n)}{P_{X_2}^n(x_2^n)} \mathbb{1}_{\{w_2^n = \tilde{w}_2^n(l_2, \mu_2), w_2^n \in T_\delta(W_2|x_2^n)\}}.$$

Note that when $\mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} = 1$, we also have $0 \leq \sum_{w_1^n \in \mathcal{W}_1^n} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) \leq 1$ and $0 \leq$

$\sum_{w_2^n \in \mathcal{W}_2^n} E_{W_2^n | X_2^n}^{(\mu_2)}(w_2^n | x_2^n) \leq 1$. This simplifies S_1 as

$$S_1 = \left| P_{Y^n | X_1 X_2}^n(y^n | x_1^n, x_2^n) - \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) E_{W_2^n | X_2^n}^{(\mu_2)}(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) \right|.$$

Now we add and subtract a term that separates the action of first encoder from that of second encoder allowing us to separately bound the error introduced by each of these encoders. This term essentially assumes that the second encoder is simply a conditional product PMF $P_{W_2^n | X_2^n}^n$ as opposed to the n-letter PMF, while keeping the first encoder the same. Figure 5 illustrates the dynamics of this term. The term is given as

$$\frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n, w_2^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2^n | X_2^n}^n(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n).$$

By adding and subtracting this term and using triangle inequality we obtain $S_1 \leq Q_1 + Q_2$, where

$$Q_1 \triangleq \left| P_{Y^n | X_1 X_2}^n(y^n | x_1^n, x_2^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n, w_2^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2^n | X_2^n}^n(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) \right|,$$

$$Q_2 \triangleq \left| \frac{1}{2^{nC_1}} \sum_{\mu_1} \sum_{w_1^n, w_2^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2^n | X_2^n}^n(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) - \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) E_{W_2^n | X_2^n}^{(\mu_2)}(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) \right|.$$

Our objective here is to show $1/2 \sum_{\substack{\underline{x}^n \in T_\delta(\underline{X}), y^n \\ \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}}}} P_{\underline{X}^n}(\underline{x}^n) S_1 \cdot \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} \leq 1/2 \sum_{\substack{\underline{x}^n \in T_\delta(\underline{X}), y^n \\ \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}}}} P_{\underline{X}^n}(\underline{x}^n) [Q_1 + Q_2] \cdot \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}}$ is small, which eventually leads to (while also showing other terms corresponding to S_2 and \tilde{S} , are small), establishing $\sum_{\underline{x}^n \in T_\delta(\underline{X})} P_{\underline{X}^n}(\underline{x}^n) \mathcal{Q}_{\underline{x}^n} \cdot \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}}$ vanishes in expectation. With this partition, the terms within the modulus of Q_1 differ only in the action of Alice's encoding/approximation, and similarly, the terms within Q_2 differ only in the action of Bob's encoding/approximation. Showing that these two terms are small forms a major portion of the achievability proof. To begin with, let us consider Q_1 .

Analysis of Q_1 : To prove $\left[\frac{1}{2} \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}^n}(\underline{x}^n) Q_1 \cdot \mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} \right]$ is small, we characterize the rate constraints which ensure that an upper bound to Q_1 can be made to vanish in an expected sense. In addition, this upper bound becomes useful in obtaining a single-letter characterization for the rate needed to make the term corresponding to Q_2 vanish. For this, we define J for each $\underline{x}^n \in T_\delta(\underline{X})$ as,

$$J = \left| P_{Y^n | W_2^n | \underline{X}}^n(y^n, w_2^n | \underline{x}^n) - \frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2^n | X_2^n}^n(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n), \right| \quad (31)$$

where $E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \in [0, 1]$ for all $x_1^n \in T_\delta(X_1)$. By defining J we have added the random variable

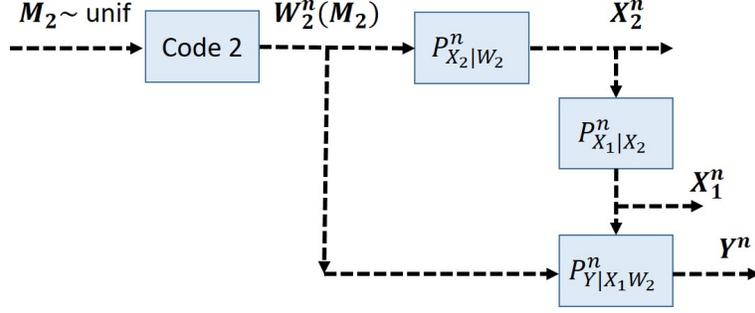


Fig. 6. Depiction of the approximation performed by Bob (encoder 2).

W_2 into the collection of random variables which first encoder is trying to approximate. Hence, this encoder now approximates the joint product PMF $P_{\underline{X}W_2Y}$. To make J small, we expect the sum of the encoding rate of first encoder and common randomness i.e., $\tilde{R}_1 + C_1$ to be larger than $I(W_1; X_1 X_2 W_2 Y)$. We prove this by bounding $\sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J$ using the following proposition.

Proposition 6. *There exist $\epsilon_J(\delta), \delta_J(\delta)$ such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J \right] \leq \epsilon_J$ if $S_1 + C_1 \geq I(W_1; X_1 X_2 Y W_2) + \delta_J$, where $\epsilon_J, \delta_J \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-E. □

Now in regards to Q_1 , applying triangle inequality on the summation over w_2 gives

$$\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n) Q_1 \leq \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J \quad (32)$$

Using the above proposition concludes the proof for the term corresponding to Q_1 . Now, we move on to bounding the term Q_2 .

Step 4: Analysis of Bob's encoding Using the term J in Step 3, we ensured that the random variables $X_1 X_2 Y W_2$ are close to a product PMF in total variation. In this step, we approximate the PMF of random variables $X_1 X_2 Y$ using the Bob's encoding rule and bound the term corresponding to Q_2 (as illustrated in Figure 6). We proceed with the following proposition.

Proposition 7. *There exist functions $\epsilon_{Q_2}(\delta)$ and $\delta_{Q_2}(\delta)$, such that for all sufficiently small δ and sufficiently large n , we have $\mathbb{E}[Q_2] \leq \epsilon_{Q_2}$, if $S_1 + C_1 \geq I(W_1; X_1 X_2 Y W_2) + \delta_{Q_2}$ and $S_2 + C_2 \geq I(W_2; X_1 X_2 Y) + \delta_{Q_2}$, where $\epsilon_{Q_2}, \delta_{Q_2} \searrow 0$ as $\delta \searrow 0$.*

Proof. The proof is provided in Appendix C-F. □

Hence, in bounding the terms corresponding to Q_1 and Q_2 , we have obtained the following constraints:

$$\tilde{R}_1 + C_1 \geq I(W_1; X_1 X_2 Y W_2), \quad \tilde{R}_2 + C_2 \geq I(W_2; X_1 X_2 Y). \quad (33)$$

E. Rate Constraints

To sum-up, we showed that (22) holds for sufficiently large n and with probability sufficiently close to 1, if the following bounds holds while incorporating the time sharing random variable Q taking values over the finite set \mathcal{Q} :

$$\begin{aligned} \tilde{R}_1 &\geq I(X_1; W_1|Q), & \tilde{R}_2 &\geq I(X_2; W_2|Q), \\ \tilde{R}_1 + C_1 &\geq I(X_1 X_2 Y W_2; W_1|Q), & \tilde{R}_2 + C_2 &\geq I(X_1 X_2 Y; W_2|Q), \\ \tilde{R}_1 + \tilde{R}_2 - (R_1 + R_2) &\leq I(W_1; W_2|Q), & 0 \leq R_1 \leq \tilde{R}_1, & 0 \leq R_2 \leq \tilde{R}_2, & C_1 \geq 0, C_2 \geq 0. \end{aligned} \quad (34)$$

Let us denote the above achievable rate-region by \mathcal{R}_1 . By doing an exact symmetric analysis, but by replacing the first encoder by a product distribution instead of the second encoder in S_1 , all the constraints remain the same, except that the constraints on $\tilde{R}_1 + C_1$ and $\tilde{R}_2 + C_2$ change as follows

$$\tilde{R}_1 + C_1 \geq I(W_1; X_1 X_2 Y|Q), \quad \tilde{R}_2 + C_2 \geq I(W_2; X_1 X_2 Y W_1|Q). \quad (35)$$

Let us denote the above region by \mathcal{R}_2 . By time sharing between the any two points of \mathcal{R}_1 and \mathcal{R}_2 one can achieve any point in the convex hull of $(\mathcal{R}_1 \cup \mathcal{R}_2)$. The following lemma gives a symmetric characterization of the convex hull of the union of the above achievable rate-regions.

Lemma 3. *For the above defined rate regions \mathcal{R}_1 and \mathcal{R}_2 , we have $\mathcal{R}_3 = \text{Convex Hull}(\mathcal{R}_1 \cup \mathcal{R}_2)$, where \mathcal{R}_3 is given by the set of all the sextuples $(\tilde{R}_1, \tilde{R}_2, R_1, R_2, C_1, C_2)$ satisfying the following constraints:*

$$\begin{aligned} \tilde{R}_1 &\geq I(X_1; W_1|Q), & \tilde{R}_2 &\geq I(X_2; W_2|Q), \\ \tilde{R}_1 + C_1 &\geq I(X_1 X_2 Y; W_1|Q), & \tilde{R}_2 + C_2 &\geq I(X_1 X_2 Y; W_2|Q), \\ \tilde{R}_1 + \tilde{R}_2 + C_1 + C_2 &\geq I(X_1 X_2 Y; W_1 W_2|Q) + I(W_1; W_2|Q), \\ \tilde{R}_1 + \tilde{R}_2 - (R_1 + R_2) &\leq I(W_1; W_2|Q) & 0 \leq R_1 \leq \tilde{R}_1 & 0 \leq R_2 \leq \tilde{R}_2 & C_1 \geq 0, C_2 \geq 0 \end{aligned} \quad (36)$$

Proof. The proof follows from elementary set-theoretic analysis, and hence is omitted. \square

Lemma 4. *Let $\bar{\mathcal{R}}_3$ denote the set of all quadruples (R_1, R_2, C_1, C_2) for which there exists $(\tilde{R}_1, \tilde{R}_2)$ such that the sextuple $(R_1, R_2, C_1, C_2, \tilde{R}_1, \tilde{R}_2)$ satisfies the inequalities in (36). Let \mathcal{R}_F denote the set of all quadruples (R_1, R_2, C_1, C_2) that satisfy the inequalities in (1) given in the statement of the theorem. Then, $\bar{\mathcal{R}}_3 = \mathcal{R}_F$.*

Proof. This follows by Fourier-Motzkin elimination [26]. \square

The cardinality bounds on the auxilliary random variables follows from an argument using supporting hyperplanes of convex sets [27], and Caratheodary theorem [28].

APPENDIX A

PROOF OF LEMMAS

A. *Proof of Lemma 1: $E_{L|X^n}^{(\mu)}(\cdot|\cdot)$ is a PMF with high probability*

From the definition of $E_{L|X^n}^{(\mu)}(l|x^n)$, we have for $x^n \in T_\delta(X)$,

$$\sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n) = \frac{1}{2^{n\bar{R}}} \left(\frac{1-\epsilon}{1+\eta} \right) \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \sum_{l=1}^{2^{n\bar{R}}} \mathbb{1}_{\{w^n(l,\mu)=w^n\}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)}.$$

Let us define $Z_l^{(\mu)}(x^n)$, for $x^n \in T_\delta(X)$ as

$$Z_l^{(\mu)}(x^n) = \sum_{w^n \in T_\delta(W|x^n)} \mathbb{1}_{\{w^n(l,\mu)=w^n\}} P_{X|W}^n(x^n|w^n) (1-\epsilon) \quad (37)$$

and let $D = 2^{n(H(X|W)-\delta_1)}$, where $\delta_1(\delta) \searrow 0$ as $\delta \searrow 0$. This gives us the following bound on the expectation of the empirical average of $\{Z_l^{(\mu)}(x^n)\}_{l \in [2^{n\bar{R}}]}$ as

$$\begin{aligned} \mathbb{E} \left[\frac{1}{N} \sum_{l=1}^N D Z_l^{(\mu)}(x^n) \right] &= 2^{n(H(X|W)-\delta_1)} \sum_{w^n \in T_\delta(W|x^n) \cap T_\delta(W)} P_W^n(w^n) P_{X|W}^n(x^n|w^n) \\ &= 2^{n(H(X|W)-\delta_1)} \sum_{w^n \in T_\delta(W|x^n)} P_W^n(w^n) P_{X|W}^n(x^n|w^n) \\ &\geq 2^{n(H(X|W)-\delta_1)} 2^{-n(H(X,W)+2\delta_1)} 2^{n(H(W|X)-\delta_1)} \geq 2^{-n(I(X;W)+4\delta_1)}, \end{aligned} \quad (38)$$

for all sufficiently large n , where in the above equations we use the fact that $\mathbb{E}[\mathbb{1}_{\{W^n(l,\mu)=w^n\}}] = \frac{P_W^n(w^n)}{1-\epsilon}$ for $w^n \in T_\delta(W)$, and the fact that if $x^n \in T_\delta(X)$ and $w^n \in T_\delta(W|x^n)$, then $(x^n, w^n) \in T_\delta(X, W)$, and consequently $w^n \in T_\delta(W)$. Furthermore, for sufficiently large n , we also have

$$D Z_l^{(\mu)}(x^n) \leq 2^{n(H(X|W)-\delta_1)} 2^{-n(H(X|W)-\delta_1)} (1-\epsilon) \sum_{w^n \in T_\delta(W|x^n)} \mathbb{1}_{\{W^n(l,\mu)=w^n\}} \leq 1 \quad (39)$$

where we have bounded $\sum_{w^n \in T_\delta(W|x^n)} \mathbb{1}_{\{W^n(l,\mu)=w^n\}}$ by 1.

Since $\{Z_l^{(\mu)}(x^n)\}_l$ is a sequence of IID Random variables, we can approximate its empirical average, for $x^n \in T_\delta(X)$, using a refined Chernoff-Hoeffding bound given by

Lemma 5. *Let $\{Z_n\}_{n=1}^N$ be a sequence of N IID random variables bounded between zero and one, i.e., $Z_n \in [0, 1] \quad \forall n \in [N]$, and suppose $\mathbb{E} \left[\frac{1}{N} \sum_{n=1}^N Z_n \right] = \mu$ be bounded below by a positive constant θ*

as $\mu \geq \theta$ where $\theta \in (0, 1)$, then for every $\eta \in (0, 1/2)$ and $(1 + \eta)\theta < 1$, we can bound the probability that the ensemble average of the sequence $\{Z_n\}_{n=1}^N$ lies in $(1 \pm \eta)\mu$ as

$$\mathbb{P}\left(\frac{1}{N} \sum_{n=1}^N Z_n \in [(1 - \eta)\mu, (1 + \eta)\mu]\right) \geq 1 - 2 \exp\left(-\frac{N\eta^2\theta}{4 \ln 2}\right) \quad (40)$$

Proof. Follows from Operator Chernoff Bound [20]. \square

Note that $\{DZ_l^{(\mu)}(x^n)\}_l$ satisfies the constraints of the above lemma from Eqns. (38) and (39). Thus applying Lemma (5) to $\{DZ_l^{(\mu)}(x^n)\}_l$ for every $x^n \in T_\delta(X)$ gives

$$\mathbb{P}\left(\frac{1}{2^{n\bar{R}}} \sum_{l=1}^{2^{n\bar{R}}} Z_l^{(\mu)}(x^n) \in [(1 - \eta)\mathbb{E}[Z^{(\mu)}(x^n)], (1 + \eta)\mathbb{E}[Z^{(\mu)}(x^n)]]\right) \geq 1 - 2 \exp\left(-\frac{\eta^2 2^{n(\bar{R} - I(X;W) - 4\delta_1)}}{4 \ln 2}\right), \quad (41)$$

where $Z^{(\mu)}(x^n) \triangleq \frac{1}{2^{n\bar{R}}} \sum_{l=1}^{2^{n\bar{R}}} Z_l^{(\mu)}(x^n)$, the ensemble mean of the IID sequence $\{Z_l^{(\mu)}(x^n)\}_l$. Substituting the following simplification

$$\frac{1}{2^{n\bar{R}}} \sum_{l=1}^{2^{n\bar{R}}} Z_l^{(\mu)}(x^n) = (1 + \eta)P_X^n(x^n) \sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n), \quad (42)$$

which follows from the definition of $Z_l^n(x^n)$ in (41) gives

$$\mathbb{P}\left((1 + \eta)P_X^n(x^n) \sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq (1 + \eta)\mathbb{E}[Z^{(\mu)}(x^n)]\right) \geq 1 - 2 \exp\left(-\frac{\eta^2 2^{n(\bar{R} - I(X;W) - 4\delta_1)}}{4 \ln 2}\right). \quad (43)$$

Further we can bound $\mathbb{E}[Z(x^n)]$ as

$$\frac{\mathbb{E}[Z^{(\mu)}(x^n)]}{P_X^n(x^n)} \leq \frac{(1 - \epsilon)}{P_X^n(x^n)} \sum_{w^n \in T_\delta(W|x^n)} \tilde{P}_W^n(w^n) P_{X|W}^n(x^n|w^n) \leq \frac{1}{P_X^n(x^n)} \sum_{w^n} P_{X,W}^n(x^n, w^n) = 1.$$

This simplifies the above probability term as

$$\mathbb{P}\left(\sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq 1\right) \geq 1 - 2 \exp\left(-\frac{\eta^2 2^{n(\bar{R} - I(X;W) - 4\delta_1)}}{4 \ln 2}\right).$$

Using the union bound, we extend the above probability to the intersection of all $\mu \in [2^{nC}]$ and $x^n \in T_\delta(X)$ as

$$\mathbb{P}\left[\bigcap_{\mu=1}^{2^{nC}} \bigcap_{x^n \in T_\delta(X)} \left(\sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq 1\right)\right] \geq 1 - \sum_{\mu=1}^{2^{nC}} \sum_{x^n \in T_\delta(X)} \mathbb{P}\left(\sum_{l=1}^{2^{n\bar{R}}} E_{L|X^n}^{(\mu)}(l|x^n) \leq 1\right)$$

$$\geq 1 - 2^{nC} |T_\delta(X)| 2 \exp\left(-\frac{\eta^2 2^{n(\tilde{R}-I(X,W)-4\delta_1)}}{4 \ln 2}\right). \quad (44)$$

Therefore, if $\tilde{R} > I(X;W) + 4\delta_1$, the second term in the right hand side of (44) decays exponentially to zero and as a result the probability of the above intersections goes to 1. This completes the proof of the lemma.

B. Proof of Lemma 2

Be begin by defining K as

$$\begin{aligned} K &\triangleq \sum_{a \in \mathcal{A}} \left| P_A(a) - \frac{1}{M} \sum_{m=1}^M P_{A|B}(a|C_m) \right| \\ &= \sum_{a \in \mathcal{A}} \left| \sum_{b \in \mathcal{B}} P_{AB}(a,b) - \frac{1}{M} \sum_{m=1}^M \sum_{b \in \mathcal{B}} P_{A|B}(a|b) \mathbb{1}_{\{C_m=b\}} \right|, \end{aligned}$$

where in the above equality we have used $\sum_{b \in \mathcal{B}} \mathbb{1}_{\{C_m=b\}} = 1$. Using triangle inequality, we obtain $K \leq K_1 + K_2 + K_3 + K_4$, where

$$\begin{aligned} K_1 &\triangleq \sum_{a \in \mathcal{T}} \left| \sum_{b \in \mathcal{B}} P_{AB}(a,b) \mathbb{1}_{\{a \in \mathcal{T}_b\}} - \frac{1}{M} \sum_{m=1}^M \sum_{b \in \mathcal{B}} P_{A|B}(a|b) \mathbb{1}_{\{C_m=b\}} \mathbb{1}_{\{a \in \mathcal{T}_b\}} \right|, \\ K_2 &\triangleq \sum_{a \in \mathcal{T}} \sum_{b \in \mathcal{B}} P_{AB}(a,b) \mathbb{1}_{\{a \notin \mathcal{T}_b\}}, \quad K_3 \triangleq \sum_{a \in \mathcal{T}} \frac{1}{M} \sum_{m=1}^M \sum_{b \in \mathcal{B}} P_{A|B}(a|b) \mathbb{1}_{\{C_m=b\}} \mathbb{1}_{\{a \notin \mathcal{T}_b\}}, \quad \text{and} \\ K_4 &\triangleq \sum_{a \in \mathcal{A} \setminus \mathcal{T}} \left| P_A(a) - \frac{1}{M} \sum_{m=1}^M P_{A|B}(a|C_m) \right|. \end{aligned}$$

We begin by first bounding the terms corresponding to K_2, K_3 and K_4 , finally and delve into bounding the main term corresponding to K_1 . Note that K_2 can be written as

$$K_2 \leq \sum_{b \in \mathcal{B}} P_B(b) \sum_{a \notin \mathcal{T}_b} P_{A|B}(a|b) = \sum_{b \in \mathcal{B}} P_B(b) (1 - P_{A|B}(\mathcal{T}_b|b)) \leq \epsilon,$$

where the last inequality uses the hypothesis (17b) from the statement of the lemma. Considering the term K_3 , applying expectation yields

$$\mathbb{E}[K_3] \leq \sum_{b \in \mathcal{B}} P_B(b) \sum_{a \notin \mathcal{T}_b} P_{A|B}(a|b) = \sum_{b \in \mathcal{B}} P_B(b) (1 - P_{A|B}(\mathcal{T}_b|b)) \leq \epsilon,$$

where the last inequality again uses the hypothesis (17b) from the statement of the lemma. Considering the term K_4 , we use the fact that $\mathbb{E}[\frac{1}{M} \sum_{m=1}^M P_{A|B}(a|C_m)] = P_A(a)$, and bound K_4 as

$$\mathbb{E}[K_4] \leq 2 \sum_{a \in \mathcal{A} \setminus \mathcal{T}} \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M P_{A|B}(a|C_m) \right] = 2 \sum_{a \in \mathcal{A} \setminus \mathcal{T}} P_A(a) = 2(1 - P_A(\mathcal{T})) \leq 2\epsilon.$$

Finally, we consider the term K_1 . Using the concavity of the square-root function, we have

$$\mathbb{E}[K_1] \leq \sum_{a \in \mathcal{T}} \sqrt{\text{Var} \left(\frac{1}{M} \sum_{m=1}^M \sum_{b \in \mathcal{B}} P_{A|B}(a|b) \mathbb{1}_{\{C_m=b\}} \mathbb{1}_{\{a \in \mathcal{T}_b\}} \right)}. \quad (45)$$

Further, the term within the variance can be simplified as

$$\begin{aligned} \text{Var} \left(\frac{1}{M} \sum_{m=1}^M \sum_{b \in \mathcal{B}} P_{A|B}(a|b) \mathbb{1}_{\{C_m=b\}} \mathbb{1}_{\{a \in \mathcal{T}_b\}} \right) &\leq \frac{1}{M} \mathbb{E} \left[\sum_{b \in \mathcal{B}} P_{A|B}^2(a|b) \mathbb{1}_{\{a \in \mathcal{T}_b\}} \mathbb{1}_{\{b=C_1\}} \right] \\ &= \frac{1}{M} \sum_{b \in \mathcal{B}} P_{A|B}^2(a|b) P_B(b) \mathbb{1}_{\{a \in \mathcal{T}_b\}} \\ &\leq \frac{1}{M} \frac{1}{d} \sum_{b \in \mathcal{B}} P_{A|B}(a|b) P_B(b) = \frac{P_A(a)}{Md}, \end{aligned}$$

where in the first inequality we use (i) the fact that codewords are generated pairwise independently from P_B , and (ii) $\text{Var}(\cdot) \leq \mathbb{E}[(\cdot)^2]$, in the first equality we have used $\mathbb{E}[\mathbb{1}_{\{C_m=b\}}] = P_B(b)$, and in the second inequality we have used the hypothesis (17d) from the statement of the lemma. Finally, substituting the above bounds in (45), and using the hypothesis (17c), we obtain

$$\mathbb{E}[K_1] \leq \sqrt{\frac{D}{Md}}.$$

Combining all the bounds on K_1, K_2, K_3 and K_4 completes the proof.

APPENDIX B

PROOF OF THEOREM 2

Let (R_1, R_2, C_1, C_2) be an achievable quadruple. Fix an arbitrary $\epsilon > 0$ and a sufficiently large n . From Definition 2 it follows that there exists $2^{nC_1} \times 2^{nC_2}$ randomized encoder pairs $(E_1^{(\mu_1)}, E_2^{(\mu_2)})$, $j \in [2]$, and a corresponding collection of 2^{nC} randomized decoders $D^{(\mu)}$ that satisfy the following constraints: $\frac{1}{n} \log \Theta_j \leq R_j + \epsilon$, and $\|P_{\underline{X}Y}^n - P_{\underline{X}Y^n}\|_1 \leq \epsilon$. Let M_1 and M_2 be the messages communicated by the first and second encoders, respectively, and let $K_1 \in [2^{nC_1}]$ and $K_2 \in [2^{nC_2}]$ denote the common randomness shared among the first encoder and the decoder, and the second encoder and the decoder, respectively. The source sequence pair (X_1^n, X_2^n) and K_1 and K_2 are mutually independent. Let $P_{M_j|X_j^n K_j} : j \in \{0, 1\}$ denote the two distributed stochastic encoders, respectively. $Y^n \sim P_{Y^n|M_1 M_2, K_1, K_2}$ be the samples generated by the decoder using the messages received and the common randomness available. Lastly, let R_1 and C_1 denote communication rate and the common randomness rate of the first encoder, respectively, and similarly, let R_2 and C_2 denote communication rate and the common randomness rate of the second encoder, respectively. Additionally, let $P_{W|X_1 X_2}$ be an arbitrary distribution in \mathcal{P}_R . Recall that $P_{X_1 X_2} P_{W|X_1 X_2}$ satisfies the Markov chain $X_1 - W - X_2$. We generate n copies of the auxiliary random variable W denoted as W^n , from (X_1^n, X_2^n) in a memoryless fashion using $P_{W|X_1, X_2}$ to yield (i) $X_1^n - W^n - X_2^n$. Enforce a Markov coupling of this with the n -letter encoders and decoder

to result in the following n -letter Markov chains: (ii) $M_1 - (X_1^n, K_1) - (W^n, X_2^n, K_2, M_2)$, and (iii) $M_2 - (X_2^n, K_2) - (W^n, X_1^n, K_1, M_1)$. This simplifies the joint distribution as

$$\begin{aligned}
& P_{K_1 K_2 X_1^n X_2^n W^n M_1 M_2 Y^n}(k_1, k_2, x_1^n, x_2^n, w^n, m_1, m_2, y^n) \\
&= P_{K_1}(k_1) P_{K_2}(k_2) P_W^n(w^n) P_{X_1^n|W^n}(x_1^n|w^n) P_{X_2^n|W^n}(x_2^n|w^n) \\
&\quad P_{M_1|X_1^n K_1}(m_1|x_1^n, k_1) P_{M_2|X_2^n K_2}(m_2|x_2^n, k_2) P_{Y^n|M_1 K_1 M_2 K_2}(y^n|m_1, k_1, m_2, k_2). \quad (46)
\end{aligned}$$

Further, define for $i \in [n]$, U_i and V_i as $U_i \triangleq (M_1, K_1, W^{i-1})$ and $V_i \triangleq (M_2, K_2, W^{i-1})$.

Step 1: Rate Constraints: Using this, we have

$$\begin{aligned}
n(R_1 + \epsilon) &\geq H(M_1) \\
&\geq H(M_1|M_2, K_1, K_2) \\
&\geq I(X_1^n; M_1|M_2, K_1, K_2) \\
&\stackrel{(a)}{=} I(X_1^n; M_1|M_2, K_1, K_2) + I(W^n; M_1|M_2, K_1, K_2, X_1^n) \\
&\stackrel{(b)}{=} I(X_1^n, W^n; M_1|M_2, K_1, K_2) + I(X_1^n, W^n; K_1|M_2, K_2) \\
&= I(X_1^n, W^n; M_1, K_1|M_2, K_2) \\
&= I(W^n; M_1, K_1|M_2, K_2) + I(X_1^n; M_1, K_1|M_2, K_2, W^n) \\
&\stackrel{(c)}{=} I(W^n; M_1, K_1|M_2, K_2) + I(X_1^n; M_1, K_1|W^n) \\
&= \sum_{i=1}^n \left[I(W_i; M_1, K_1|M_2, K_2, W^{i-1}) + I(X_{1i}; M_1, K_1|W^n, X_1^{i-1}) \right] \\
&\stackrel{(d)}{=} \sum_{i=1}^n \left[I(W_i; U_i|V_i) + H(X_{1i}|W^n) - H(X_{1i}|M_1, K_1, W^n, X_1^{i-1}) \right] \\
&\stackrel{(e)}{\geq} \sum_{i=1}^n \left[I(W_i; U_i|V_i) + H(X_{1i}|W_i, Q_i) - H(X_{1i}|M_1, K_1, W_i, Q_i) \right] \\
&= \sum_{i=1}^n \left[I(W_i; U_i|V_i) + I(X_{1i}, U_i|W_i, Q_i) \right] \\
&\stackrel{(f)}{=} n \left(I(W; U|V, J) + I(X_1, U|W, Q, J) \right), \quad (47)
\end{aligned}$$

where (a) follows from the fact that

$$\begin{aligned}
I(W^n; M_1|M_2, K_1, K_2, X_1^n) &= I(M_2 K_2 W^n; M_1|X_1^n, K_1) - I(M_2 K_2; M_1|X_1^n, K_1) \\
&= -I(M_2 K_2; M_1|X_1^n, K_1) \leq 0,
\end{aligned}$$

which is only true if $I(W^n; M_1 | M_2, K_1, K_2, X_1^n) = 0$, (b) follows from $I(X_1^n, W^n; K_1 | M_2, K_2) = 0$ which is true given the decomposition in (46), (c) uses the fact that for a Markov Chain $(A, B) - C - D$, we have $I(A; B | C, D) = I(A; B | C)$, (d) is obtained using the definitions of U_i and V_i , and the memoryless nature of the source X_1^n , (e) follows from defining for all $i \in [n]$, $Q_i \triangleq W^n \setminus i$ and using the result - conditioning reduces entropy, and finally (f) follows by (i) defining a time-sharing random variable J which is uniformly distributed in $[1, n]$ and independent of $(W^n, U^n, V^n, Q^n, X_1^n, X_2^n, Y^n)$, and (ii) defining W, U, V, Q, X_1 and X_2 as $W_J, U_J, V_J, Q_J, X_{1J}$ and X_{2J} , respectively. Using identical steps for the bound R_2 , we get the following bound for R_2

$$n(R_2 + \epsilon) \geq n(I(W; V | U) + I(X_2, V | W, Q)).$$

We now provide a bound on the sum rate $R_1 + R_2$ as

$$\begin{aligned}
n(R_1 + R_2 + \epsilon) &\geq H(M_1, M_2) \\
&\geq I(X_1^n, X_2^n, K_1, K_2; M_1, M_2) \\
&\stackrel{(a)}{=} I(W^n, X_1^n, X_2^n, K_1, K_2; M_1, M_2) \\
&= I(W^n; M_1, M_2) + I(K_1, K_2; M_1, M_2 | W^n) + I(X_1^n, X_2^n; M_1, M_2 | W^n, K_1, K_2) \\
&\stackrel{(b)}{\geq} I(W^n; M_1, M_2) + I(K_1, K_2; W^n | M_1, M_2) + I(X_1^n, X_2^n; M_1, M_2, K_1, K_2 | W^n) \\
&\stackrel{(c)}{=} I(W^n; M_1, M_2, K_1, K_2) + I(X_1^n; M_1, K_1 | W^n) + I(X_2^n; M_2, K_2 | W^n) \\
&= \sum_i^n \left[I(W_i; M_1, M_2, K_1, K_2 | W^{i-1}) + I(X_{1i}; M_1, K_1 | W^n, X_1^{i-1}) \right. \\
&\qquad \qquad \left. + I(X_{2i}; M_2, K_2 | W^n, X_2^{i-1}) \right] \\
&= \sum_i^n \left[I(W_i; M_1, M_2, K_1, K_2 | W^{i-1}) + I(X_{1i}; U_i | W_i, Q_i) + I(X_{2i}; V_i | W_i, Q_i) \right] \\
&= \sum_i^n \left[I(W_i; M_1, M_2, K_1, K_2 | W^{i-1}) + I(W_i; W^{i-1}) \right. \\
&\qquad \qquad \left. + I(X_{1i}; U_i | W_i, Q_i) + I(X_{2i}; V_i | W_i, Q_i) \right] \\
&\stackrel{(d)}{=} \sum_i^n \left[I(W_i; U_i, V_i) + I(X_{1i}; U_i | W_i, Q_i) + I(X_{2i}; V_i | W_i, Q_i) \right] \\
&= n(I(W; U, V | J) + I(X_1; U | W, Q, J) + I(X_2; V | W, Q, J)), \tag{48}
\end{aligned}$$

where (a) follows from the Markov Chain $W^n - (X_1^n, X_2^n, K_1, K_2) - (M_1, M_2)$ which makes

$I(W^n; M_1, M_2 | X_1^n, X_2^n, K_1, K_2) = 0$, (b) follows from

$$\begin{aligned} (i) \quad I(K_1, K_2; M_1, M_2 | W^n) &= H(K_1, K_2 | W^n) - H(K_1, K_2 | W^n, M_1, M_2) \\ &= H(K_1, K_2) - H(K_1, K_2 | W^n, M_1, M_2) \\ &\geq I(K_1, K_2; W^n | M_1, M_2), \end{aligned}$$

$$\text{and } (ii) \quad I(K_1, K_2; X_1^n, X_2^n | W^n) = 0.$$

(c) follows from the Markov Chain $M_1 - (X_1^n, K_1) - W^n - (X_2^n, K_2) - M_2$, and (d) follows from similar arguments as in (47-d).

We now provide the bound for $R_1 + R_2 + C_1 + C_2$ as follows.

$$\begin{aligned} n(R_1 + R_2 + C_1 + C_2 + \epsilon) &\geq H(M_1, M_2, K_1, K_2) \\ &\geq I(M_1, M_2, K_1, K_2; X_1^n, X_2^n, Y^n) \\ &\stackrel{(a)}{=} I(X_1^n, X_2^n, Y^n, W^n; M_1, M_2, K_1, K_2) \\ &= I(W^n; M_1, M_2, K_1, K_2) + I(X_1^n, X_2^n, Y^n; M_1, M_2, K_1, K_2 | W^n), \quad (49) \end{aligned}$$

where (a) follows from using the Markov chain $W^n - (X_1^n, X_2^n) - (M_1, M_2, K_1, K_2) - Y^n$ which implies $I(W^n; M_1, M_2, K_1, K_2 | X_1^n, X_2^n, Y^n) = 0$. Again the first term in the right hand side of (49) can be simplified following the approach in (48) as $I(W^n; M_1, M_2, K_1, K_2) = \sum_{i=1}^n I(W_i; U_i, V_i)$. For the second term, we have

$$\begin{aligned} &I(X_1^n, X_2^n, Y^n; M_1, M_2, K_1, K_2 | W^n) \\ &= \sum_{i=1}^n \left[I(X_{1i}, X_{2i}, Y_i; U_i, V_i, X_1^{i-1}, X_2^{i-1}, Y^{i-1} | Q_i, W_i) - I(X_{1i}, X_{2i}, Y_i; X_1^{i-1}, X_2^{i-1}, Y^{i-1} | Q_i, W_i) \right] \\ &\geq \sum_{i=1}^n I(X_{1i}, X_{2i}, Y_i; U_i, V_i | Q_i, W_i) - ng_c(\epsilon), \end{aligned}$$

where in the last inequality above we use $\|P_{X_1^n X_2^n Y^n}^n - P_{X_1^n X_2^n Y^n}\|_1 \leq \epsilon$, implies $I(X_{1i}, X_{2i}, Y_i; X_1^{i-1}, X_2^{i-1}, Y^{i-1} | Q_i, W_i) \leq ng_c(\epsilon)$, and define $g_c(\epsilon)$ as in the statement of the theorem using Lemma VI.3 from [9] obtaining $g_c(\epsilon) \searrow 0$ as $\epsilon \searrow 0$ which follows from the memoryless nature of $P_{W^n | X_1^n X_2^n}$. Substituting the above simplification in (49), we obtain

$$\begin{aligned} n(R_1 + R_2 + C_1 + C_2 + \epsilon) &\geq \sum_{i=1}^n \left[I(W_i; U_i, V_i) + I(X_{1i}, X_{2i}, Y_i; U_i, V_i | Q_i, W_i) - g_c(\epsilon) \right] \\ &= n \left(I(W; U, V | J) + I(X_1, X_2, Y; U, V | Q, W, J) - g_c(\epsilon) \right), \quad (50) \end{aligned}$$

where the equality above follows by defining J as an averaging random variable which is uniformly

distributed in $[1, n]$ and Y as Y_J .

Step 2: Single-letter l_1 constraint (d): Since the encoders and decoder satisfy the l_1 distance constraint $\|P_{X_1 X_2 Y}^n - P_{X_1^n X_2^n Y^n}\|_1 \leq \epsilon$ (as in Definition 2), using the Lemma VI.2 from [9] we have

$$\|P_{X_1 X_2 Y} - P_{X_{1,J} X_{2,J} Y_J}\|_1 \leq \|P_{X_1 X_2 Y}^n - P_{X_1^n X_2^n Y^n}\|_1 \leq \epsilon. \quad (51)$$

Step 3: Markov Chains: We now argue that the Markov Chains (a), (b), (c), (e) and (f) stated in the theorem statement hold. The Markov Chain (a) follows from the standard information-theoretic arguments with time-sharing random variables [29] and using the fact that (i) J is independent of (X_1^n, X_2^n, Q^n) and (ii) the stationary and memoryless nature of the sources (X_1^n, X_2^n, W^n) which makes (X_1, X_2) independent of Q . Moving on to the next, the Markov chain (b) $U - (X_1, Q, J) - (X_2, Q, J) - V$ holds true from the following arguments. Since J is uniform and is independent of the sources X_1^n and X_2^n , this is equivalent to showing $U_i - (X_{1i}, Q_i) - (X_{2i}, Q_i) - V_i$ for $i \in [n]$. This is equivalent to $(M_1, K_1, W^{i-1}) - (X_{1i}, W^{i-1}, W_{i+1}^n) - (X_{2i}, W^{i-1}, W_{i+1}^n) - (M_2, K_2, W^{i-1})$. Hence we need to show $(M_1, K_1) - (X_{1i}, W^{i-1}, W_{i+1}^n) - (X_{2i}, W^{i-1}, W_{i+1}^n) - (M_2, K_2)$. We show this in the following. For an arbitrary $i \in [n]$ and for $m_j \in [2^{nR_j}], k_j \in [2^{nC_j}] : j = 1, 2, x_{1i} \in \mathcal{X}_1, x_{2i} \in \mathcal{X}_2, w^{[i]} \triangleq w^{n \setminus i} \in \mathcal{W}^{n-1}$, we have

$$\begin{aligned} & P[M_1 = m_1, K_1 = k_1 | X_{1i} = x_{1i}, X_{2i} = x_{2i}, W^{[i]} = w^{[i]}, M_2 = m_2, K_2 = k_2] \\ &= \frac{P[M_1 = m_1, K_1 = k_1, X_{1i} = x_{1i}, X_{2i} = x_{2i}, W^{[i]} = w^{[i]}, M_2 = m_2, K_2 = k_2]}{P[X_{1i} = x_{1i}, X_{2i} = x_{2i}, W^{[i]} = w^{[i]}, M_2 = m_2, K_2 = k_2]} \\ &= \frac{P(K_1 = k_1)P(K_2 = k_2)P(W^{[i]} = w^{[i]}, X_{1i} = x_{1i}, X_{2i} = x_{2i})}{P(K_2 = k_2)P(W^{[i]} = w^{[i]}, X_{1i} = x_{1i}, X_{2i} = x_{2i})} \\ &\quad \times \frac{\sum_{x_1^{[i]}} P(X_1^{[i]} = x_1^{[i]} | W^{[i]} = w^{[i]}) P(M_1 = m_1 | X_1^n = x_1^n, K_1 = k_1)}{\sum_{x_2^{[i]}} P(X_2^{[i]} = x_2^{[i]} | W^{[i]} = w^{[i]}) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)} \\ &\quad \times \sum_{x_2^{[i]}} P(X_2^{[i]} = x_2^{[i]} | W^{[i]} = w^{[i]}) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2) \\ &= P(K_1 = k_1) \left(\sum_{x_1^{[i]}} P(X_1^{[i]} = x_1^{[i]} | W^{[i]} = w^{[i]}) P(M_1 = m_1 | X_1^n = x_1^n, K_1 = k_1) \right). \end{aligned}$$

Note that the right hand side in the above simplification does not depend on (x_{2i}, m_2, k_2) . Hence we have shown $(M_1, K_1, W^{i-1}) - (X_{1i}, W^{n \setminus i}) - (X_{2i}, W^{n \setminus i}, M_2, K_2)$. Similarly, using identical arguments, we can show $(M_2, K_2, W^{i-1}) - (X_{2i}, W^{n \setminus i}) - (X_{1i}, W^{n \setminus i}, M_1, K_1)$. These imply that $U_i - (X_{1i}, Q_i) - (X_{2i}, Q_i) - V_i$ for all $i = 1, 2, \dots, n$.

To prove the next Markov Chain (c) given by $(X_1, X_2, Q) - (J, U, V) - Y$, consider the following arguments: Since J is uniform and independent of the sources, the Markov chain (c) is equivalent to $(X_{1i}, X_{2i}, W^{n \setminus i}) - (W^{i-1}, M_1, K_1, M_2, K_2) - Y_i$ for all $i \in [n]$. We prove this using the following. For an arbitrary $i \in [n]$ and for $m_j \in [2^{nR_j}], k_j \in [2^{nC_j}] : j = 1, 2, x_{1i} \in \mathcal{X}_1, x_{2i} \in \mathcal{X}_2, y_i \in \mathcal{Y}$,

$w^{[i]} \triangleq w^{n \setminus i} \in \mathcal{W}^{n-1}$, we have

$$\begin{aligned}
& P(X_{1i} = x_{1i}, X_{2i} = x_{2i}, W^{[i]} = w^{[i]} | M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^{i-1} = w^{i-1}, Y_i = y_i) \\
&= \frac{P(M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^{[i]} = w^{[i]}, X_{1i} = x_{1i}, X_{2i} = x_{2i}, Y_i = y_i)}{P(M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^{i-1} = w^{i-1}, Y_i = y_i)} \\
&= \frac{\left[\sum_{x_1^{[i]}, x_2^{[i]}, w_i} P(W^n = w^n, X_1^n = x^n, X_2^n = x_2^n) \right]}{\left[\sum_{x_1^n, x_2^n, w_i^n} P(W^n = w^n, X_1^n = x^n, X_2^n = x_2^n) \right]} \\
&\quad \times \frac{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)}{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)} \\
&\quad \times \frac{\sum_{y^{[i]}} P(Y^n = y^n | M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2)}{\sum_{y^{[i]}} P(Y^n = y^n | M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2)} \\
&= \frac{P(X_{1i} = x_{1i}, X_{2i} = x_{2i}, W^{[i]} = w^{[i]}, M_1 = m_1, M_2 = m_2 | K_1 = k_1, K_2 = k_2)}{P(W^{i-1} = w^{i-1}, M_1 = m_1, M_2 = m_2 | K_1 = k_1, K_2 = k_2)}.
\end{aligned}$$

Since the right hand side of the above simplification is independent of y_i , we therefore have the Markov chain (c) to be satisfied. Progressing ahead, we have the Markov chain (e) given by $X_{1J} - W_J - X_{2J}$ which is satisfied from the choice of $P_{W|X_1, X_2}$ similar to the arguments made in showing the Markov chain (a). Finally, toward showing the Markov chain (f) given by $W - (X_1, X_2) - (J, Q, U, V, Y)$ consider the following set of arguments: For an arbitrary $i \in [n]$ and for $m_j \in [2^{nR_j}], k_j \in [2^{nC_j}] : j = 1, 2, x_{1i} \in \mathcal{X}_1, x_{2i} \in \mathcal{X}_2, y_i \in \mathcal{Y}, w^{[i]} \triangleq w^{n \setminus i} \in \mathcal{W}^{n-1}$, we have

$$\begin{aligned}
& P(W_i = w_i | M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^{[i]} = w^{[i]}, Y_i = y_i, X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i) \\
&= \frac{P(M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^n = w^n, X_{1i} = x_{1i}, X_{2i} = x_{2i}, Y_i = y_i, J = i)}{P(M_1 = m_1, M_2 = m_2, K_1 = k_1, K_2 = k_2, W^{[i]} = w^{[i]}, Y_i = y_i, X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i)} \\
&= \frac{\left[\sum_{x_1^{[i]}, x_2^{[i]}} P(W^n = w^n, X_1^n = x^n, X_2^n = x_2^n, J = i) \right]}{\left[\sum_{x_1^{[i]}, x_2^{[i]}} P(W^{[i]} = w^{[i]}, X_1^n = x^n, X_2^n = x_2^n, J = i) \right]} \\
&\quad \times \frac{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)}{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)}
\end{aligned}$$

$$\begin{aligned}
& P(W_i = w_i, X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i) \left[\sum_{x_1^{[i]}, x_2^{[i]}} P(X_1^{[i]} = x^{[i]}, X_2^{[i]} = x_2^{[i]} | W^{[i]} = w^{[i]}) \right] \\
= & \frac{P(W_i = w_i, X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i) \left[\sum_{x_1^{[i]}, x_2^{[i]}} P(X_1^{[i]} = x^{[i]}, X_2^{[i]} = x_2^{[i]} | W^{[i]} = w^{[i]}) \right]}{P(X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i) \left[\sum_{x_1^{[i]}, x_2^{[i]}} P(X_1^{[i]} = x^{[i]}, X_2^{[i]} = x_2^{[i]} | W^{[i]} = w^{[i]}) \right]} \\
& \times \frac{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)}{P(M_1 = m_1 | X_1^n = x^n, K_1 = k_1) P(M_2 = m_2 | X_2^n = x_2^n, K_2 = k_2)} \\
= & \frac{P(W_i = w_i, X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i)}{P(X_{1i} = x_{1i}, X_{2i} = x_{2i}, J = i)} = P_{W|X_1 X_2}(w_i | x_{1i}, x_{2i}).
\end{aligned}$$

We now have the right hand side of the above simplification independent of $j, w^{[i]}, m_1, m_2, k_1, k_2$, and y_i , which proves that the Markov Chain (f) is satisfied.

We have shown that (R_1, R_2, C_1, C_2) belongs to $\mathcal{R}_O(P_{XY}, \epsilon)$ for all $\epsilon > 0$, which is the desired proof of the outer bound.

APPENDIX C

PROOF OF PROPOSITIONS

A. Proof of Proposition 1

We begin by using the lower bound from (41) given in Appendix A-A. If $\tilde{R} > I(X; W) + 4\delta_1$, we have

$$\begin{aligned}
\sum_{l=1}^{2^{n\tilde{R}}} E_{L|X^n}^{(\mu_1)}(l|x^n) &= \frac{1}{2^{n\tilde{R}}} \left(\frac{1-\epsilon}{1+\eta} \right) \sum_{w^n \in T_\delta(W|x^n)} \sum_{l=1}^{2^{n\tilde{R}}} \mathbb{1}_{\{w^n(l, \mu_1) = w^n\}} \frac{P_{X|W}^n(x^n|w^n)}{P_X^n(x^n)} \\
&= \left(\frac{1}{1+\eta} \right) \frac{1}{P_X^n(x^n)} \frac{1}{2^{n\tilde{R}}} \sum_{l=1}^{2^{n\tilde{R}}} Z_l^{(\mu)}(x^n) \\
&\stackrel{w.h.p.}{\geq} \left(\frac{1}{1+\eta} \right) \frac{1}{P_X^n(x^n)} (1-\eta) \mathbb{E}[Z^{(\mu)}(x^n)] \\
&\geq \left(\frac{1-\eta}{1+\eta} \right) (1-\epsilon_c), \tag{52}
\end{aligned}$$

where the second equality follows from the definition of $Z_l^{(\mu)}(x^n)$ as defined in (37), the first inequality uses the lower bound from (41) which is true with probability greater than $1 - \delta_\tau$, where $\delta_\tau \triangleq 2 \exp\left(-\frac{\eta^2 2^{n(\tilde{R}-I(X,W)-4\delta_1)}}{4 \ln 2}\right)$, and the second inequality uses the fact that $\mathbb{E}[Z^{(\mu)}(x^n)] = P_X^n(x^n) \sum_{w^n \in T_\delta(W|x^n)} P_{W|X}^n(w^n|x^n) \geq P_X^n(x^n)(1-\epsilon_c)$, for sufficiently large n and $\epsilon_c(\delta) \searrow 0$ as

$\delta \searrow 0$. Using this we get, with high probability,

$$\begin{aligned} \sum_{y^n, z^n} \tilde{S} \cdot \mathbb{1}_{\text{PMF}(\mathcal{C})} &\leq \frac{2\eta + \epsilon_c(1-\eta)}{1+\eta} \sum_{y^n, z^n} P_{Z|X}^n(z^n|x^n) P_{Y|WZ}^n(y^n|w_0, z^n) \\ &\leq \frac{2\eta + \epsilon_c(1-\eta)}{1+\eta} \sum_{y^n, z^n} P_{YZ|WX}^n(y^n, z^n|w_0, x^n) = \frac{2\eta + \epsilon_c(1-\eta)}{1+\eta} \end{aligned}$$

where the first equality follows by using the Markov Chains $Z - X - W$ and $X - (W, Z) - Y$. Finally, since $\tilde{S} \cdot \mathbb{1}_{\text{PMF}(\mathcal{C})} \leq 1$, using the above result, we have for sufficiently large n ,

$$\mathbb{E} \left[\sum_{y^n, z^n} \tilde{S} \cdot \mathbb{1}_{\text{PMF}(\mathcal{C})} \right] \leq \frac{2\eta + \epsilon_c(1-\eta)}{1+\eta} (1 - \delta_\tau) + \delta_\tau. \quad (53)$$

Therefore $\mathbb{E} \left[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} \tilde{S} \cdot \mathbb{1}_{\text{PMF}(\mathcal{C})} \right]$ can be made arbitrarily small for all sufficiently large n .

B. Proof of Proposition 2

The term $\mathbb{E} \left[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} S_2 \right]$ captures the binning error in terms of total variation. If we let $\tilde{w}^n = f^{(\mu)}(b^{(\mu)}(l), z^n)$, we have

$$\sum_{y^n} \left| \left(P_{Y|WZ}^n(y^n|w^n, z^n) - P_{Y|WZ}^n(y^n|\tilde{w}^n, z^n) \right) \right| \leq 2 \cdot \mathbb{1}_{\{w^n \neq \tilde{w}^n\}}. \quad (54)$$

Substituting (54) in $\mathbb{E}[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} S_2]$, and using union bound, we obtain $\mathbb{E}[\sum_{x^n \in T_\delta(X)} P_X^n(x^n) \sum_{y^n, z^n} S_2] \leq J_1 + J_2$, where

$$\begin{aligned} J_1 &\triangleq 2 \cdot \mathbb{E} \left[\sum_{x^n \in T_\delta(X)} \sum_{z^n} \sum_{\mu, l} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n|w^n)}{2^{n(\tilde{R}+C)} (1+\eta)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} \mathbb{1}_{(w^n, z^n) \notin T_\delta(W, Z)} \right] \\ J_2 &\triangleq 2 \cdot \mathbb{E} \left[\sum_{x^n \in T_\delta(X)} \sum_{z^n} \sum_{\mu, l} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n|w^n)}{2^{n(\tilde{R}+C)} (1+\eta)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} \right. \\ &\quad \left. \sum_{m, l'} \sum_{\substack{\tilde{w}^n: \\ w^n \neq \tilde{w}^n}} \mathbb{1}_{\{(\tilde{w}^n, z^n) \in T_\delta(W, Z)\}} \mathbb{1}_{\{b^{(\mu)}(l) = m\}} \mathbb{1}_{\{b^{(\mu)}(l') = m\}} \mathbb{1}_{\{\mathbf{w}(l', \mu) = \tilde{w}^n\}} \right] \end{aligned}$$

We begin by showing J_1 can be made arbitrarily small for sufficiently large n . Using the fact that $\mathbb{E}[\mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}}] = \frac{P_W^n(w^n)}{(1-\epsilon)}$, for $w^n \in T_\delta(W)$, we have

$$J_1 = 2 \sum_{x^n \in T_\delta(X)} \sum_{z^n} \sum_{\substack{w^n: w^n \in \\ T_\delta(W|x^n) \cap T_\delta(W), \\ (w^n, z^n) \notin T_\delta(W, Z)}} \frac{P_{Z|X}^n(z^n|x^n) P_{X|W}^n(x^n, w^n)}{(1+\eta)}$$

$$\begin{aligned}
&\leq \frac{2}{(1+\eta)} \sum_{x^n \in T_\delta(X)} \sum_{z^n} \sum_{\substack{w^n: w^n \in \\ T_\delta(W|x^n) \cap T_\delta(W), \\ (w^n, z^n) \notin T_\delta(W, Z)}} P_{XWZ}^n(x^n, w^n, z^n) \\
&\leq \frac{2}{(1+\eta)} \sum_{(w^n, z^n) \notin T_\delta(W, Z)} P_{WZ}^n(w^n, z^n) \leq \frac{2\epsilon_{J_1}}{(1+\eta)},
\end{aligned}$$

where $\epsilon_{J_1}(\delta) \searrow 0$ as $\delta \searrow 0$. Proceeding with J_2 , we have

$$\begin{aligned}
J_2 &\leq 2 \cdot \mathbb{E} \left[\sum_{x^n \in T_\delta(X)} \sum_{z^n} \sum_{\mu, l} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{XZ}^n(x^n, z^n) P_{X|W}^n(x^n|w^n)}{2^{n(\tilde{R}+C)}(1+\eta)} \mathbb{1}_{\{w^n = \mathbf{w}^n(l, \mu)\}} \right. \\
&\quad \left. \sum_{m, l'} \sum_{\substack{\tilde{w}^n: \\ w^n \neq \tilde{w}^n}} \mathbb{1}_{\{(\tilde{w}^n, z^n) \in T_\delta(W, Z)\}} \mathbb{1}_{\{b^{(\mu)}(l) = m\}} \mathbb{1}_{\{b^{(\mu)}(l') = m\}} \mathbb{1}_{\{\mathbf{w}(l', \mu) = \tilde{w}^n\}} \right] \\
&= 2 \sum_{x^n \in T_\delta(X)} \sum_{z^n \in T_\delta(Z)} \sum_{m=1}^{2^{nR}} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{XZ}^n(x^n, z^n) P_{X|W}^n(x^n|w^n)}{2^{n(\tilde{R}+C)}(1+\eta)} \\
&\quad \sum_{\substack{\tilde{w}^n: (\tilde{w}^n, z^n) \\ \in T_\delta(W, Z), \\ w^n \neq \tilde{w}^n}} \sum_{\mu, l, l'} \mathbb{E} \left[\mathbb{1}_{\{\mathbf{w}^n(l, \mu) = w^n\}} \mathbb{1}_{\{\mathbf{w}(l', \mu) = \tilde{w}^n\}} \mathbb{1}_{\{b^{(\mu)}(l) = m\}} \mathbb{1}_{\{b^{(\mu)}(l') = m\}} \right] \\
&= 2 \sum_{x^n \in T_\delta(X)} \sum_{z^n \in T_\delta(Z)} \sum_{m=1}^{2^{nR}} \sum_{\substack{w^n \in \\ T_\delta(W|x^n)}} \frac{(1-\epsilon) P_{XZ}^n(x^n, z^n) P_{X|W}^n(x^n|w^n)}{2^{n(\tilde{R}+C)}(1+\eta)} \\
&\quad \sum_{\substack{\tilde{w}^n: (\tilde{w}^n, z^n) \in T_\delta(W, Z), \\ w^n \neq \tilde{w}^n}} \sum_{\mu, l, l'} \left[\frac{P_W^n(w^n)}{(1-\epsilon)} \frac{P_W^n(\tilde{w}^n)}{(1-\epsilon)} 2^{-2nR} \right] \\
&= 2 \cdot 2^{n(\tilde{R}-R)} \sum_{x^n \in T_\delta(X)} \sum_{z^n \in T_\delta(Z)} \sum_{w^n \in T_\delta(W|x^n)} \frac{P_{XWZ}^n(x^n, w^n, z^n)}{(1+\eta)} \sum_{\substack{\tilde{w}^n: (\tilde{w}^n, z^n) \\ \in T_\delta(W, Z), \\ w^n \neq \tilde{w}^n}} \frac{P_W^n(\tilde{w}^n)}{(1-\epsilon)} \\
&\leq 2 \cdot 2^{n(\tilde{R}-R)} \sum_{x^n \in T_\delta(X)} \sum_{z^n \in T_\delta(Z)} \sum_{w^n \in T_\delta(W|x^n)} \frac{P_{WZ|X}^n(w^n, z^n|x^n)}{(1+\eta)(1-\epsilon)} 2^{-n(I(W;Z)-\delta_I)} \\
&\leq 2^{n(\tilde{R}-R-I(W;Z)+\delta_I+\delta')}
\end{aligned}$$

where the second equality follows by using $\mathbb{E}[\mathbb{1}_{\{b^{(\mu)}(l)=m\}}] = 2^{-nR}$, the third equality follows from the Markov Chain $Z - X - W$, the second inequality follows from the properties of δ -typical sets where $\delta_I(\delta) \searrow 0$ as $\delta \searrow 0$. Therefore, from above $\mathbb{E}[\sum_{y^n, z^n} S_2]$ can be made arbitrarily small, for sufficiently large n , if $\tilde{R} - R \leq I(W; Z) + \epsilon_1$, where $\epsilon_1 = \delta_I + \delta'$.

C. Proof of Proposition 4

We begin by defining $\tilde{S}_i \triangleq |T_i|$ for $i \in \{2, 3, 4\}$. Firstly, consider the following simplification of \tilde{S}_2 .

$$\begin{aligned}
& \sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_2 \\
&= \sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} \left| \sum_{\mu_1, \mu_2, l_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{P_{X_1 X_2}^n(x_1^n, x_2^n) \left[1 - \sum_{m_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right]}{2^{n(C_1+C_2)}} \right. \\
&\quad \left. \frac{(1 - \epsilon_2) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n\tilde{R}_2}(1 + \eta) P_{X_2}^n(x_2^n)} \mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} P_{Y|W_1 W_2}^n(y^n | \tilde{w}_1^n, \tilde{w}_2^n) \right| \\
&= \sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} \sum_{\mu_1, \mu_2, l_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{P_{X_1 X_2}^n(x_1^n, x_2^n) \left| 1 - \sum_{m_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right|}{2^{n(C_1+C_2)}} \\
&\quad \frac{(1 - \epsilon_2) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n\tilde{R}_2}(1 + \eta) P_{X_2}^n(x_2^n)} \mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} P_{Y|W_1 W_2}^n(y^n | \tilde{w}_1^n, \tilde{w}_2^n)
\end{aligned}$$

Taking expectation over the second encoder's codebook, we obtain

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}_2} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_2 \right] \\
&\leq \sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} \sum_{\mu_1, \mu_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{\left| 1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right|}{2^{n(C_1+C_2)}(1 + \eta)} P_{X_1, X_2, W_2}^n(x_1^n, x_2^n, w_2^n) P_{Y|W_1 W_2}^n(y^n | \tilde{w}_1^n, \tilde{w}_2^n) \\
&\leq \sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{w_2^n} \sum_{\mu_1, \mu_2} \frac{\left| 1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right|}{2^{n(C_1+C_2)}(1 + \eta)} P_{X_1, X_2, W_2}^n(x_1^n, x_2^n, w_2^n) \\
&\leq \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{x_1^n \in T_\delta(X_1)} P_{X_1}^n(x_1^n) \frac{\left| 1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right|}{(1 + \eta)}. \tag{55}
\end{aligned}$$

Further, taking expectation over the first encoder's codebook and introducing the indicator $\mathbb{1}_{\text{PMF}(C_1, C_2)}$, we get

$$\begin{aligned}
\mathbb{E} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_2 \cdot \mathbb{1}_{\text{PMF}(C_1, C_2)} \right] &\leq \mathbb{E}_{\mathcal{C}_1} \left[\mathbb{E}_{\mathcal{C}_2} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_2 \right] \mathbb{1}_{\text{PMF}(C_1)} \right] \\
&\leq \frac{2\eta + \epsilon_c(1 - \eta)}{1 + \eta} (1 - \delta_\tau) + \delta_\tau.
\end{aligned}$$

where the last inequality follows using the result from (53) provided in Appendix C-A, and $\delta_\tau(\delta) \searrow 0$ as $\delta \searrow 0$ if $\tilde{R}_1 \geq I(X_1; W_1) + 4\delta_1$, for all sufficiently large n , where $\delta_1 \searrow 0$, $\epsilon_c(\delta) \searrow 0$ as $\delta \searrow 0$.

Using very similar arguments as above, it can also be shown that if $\tilde{R}_2 \geq I(X_2; W_2) + 4\delta_1$, then

$\mathbb{E} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_3 \cdot \mathbb{1}_{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)} \right]$ can be made arbitrarily small for all sufficiently large n .

Similarly consider the final term corresponding to \tilde{S}_4 . For \tilde{R}_1 and \tilde{R}_2 satisfying the above constraints, i.e., $\tilde{R}_1 \geq I(X_1; W_1) + 4\delta$ and $\tilde{R}_2 \geq I(X_2; W_2) + 4\delta$, we will have

$$\begin{aligned} & \mathbb{E} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{y^n} P_{X_1 X_2}^n(x_1^n, x_2^n) \tilde{S}_4 \cdot \mathbb{1}_{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)} \right] \\ & \leq \mathbb{E} \left[\sum_{x_1^n, x_2^n \in T_\delta(\underline{X})} \sum_{\mu_1, \mu_2} \frac{P_{X_1 X_2}^n(x_1^n, x_2^n) \left| 1 - \sum_{l_1=1}^{2^{n\tilde{R}_1}} E_{L_1|X_1^n}^{(\mu_1)}(l_1|x_1^n) \right| \left| 1 - \sum_{l_2=2}^{2^{n\tilde{R}_2}} E_{L_2|X_2^n}^{(\mu_2)}(l_2|x_2^n) \right|}{2^{nC}} \mathbb{1}_{\text{PMF}(\mathcal{C}_1, \mathcal{C}_2)} \right] \\ & \leq \left[\left(\frac{2\eta + \epsilon_c(1-\eta)}{1+\eta} \right)^2 (1 - 2\delta_\tau) + 2\delta_\tau \right], \end{aligned}$$

where the second inequality again uses the result from Appendix C-A. This completes the proof.

D. Proof of Proposition 5

Define $(\hat{w}_1^n, \hat{w}_2^n) = f^{(\mu)}(b^{(\mu_1)}(l_1), b^{(\mu_2)}(l_2))$. Consider,

$$\left| P_{Y|W_1 W_2}^n(y^n | w_1^n, w_2^n) - P_{Y|W_1 W_2}^n(y^n | f^{(\mu)}(b_1^{(\mu_1)}(l_1), b_2^{(\mu_2)}(l_2))) \right| \leq 2 \cdot \mathbb{1}_{\{(w_1^n, w_2^n) \neq (\hat{w}_1^n, \hat{w}_2^n)\}}.$$

Substituting the above bound in the S_{12} term and using the union bound, we obtain

$\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \sum_{y^n} S_2 \right] \leq J_1 + J_2$, where

$$\begin{aligned} J_1 & \triangleq 2 \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{\mu_1, \mu_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1-\epsilon_1)(1-\epsilon_2) P_{X_1 X_2}^n(x_1^n, x_2^n) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1+\eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \\ & \quad \sum_{l_1, l_2} \sum_{m_1, m_2} \mathbb{E} \left[\mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} \mathbb{1}_{\{w_1^n = \mathbf{w}_1^n(l_1, \mu_1)\}} \mathbb{1}_{\{(w_1^n, w_2^n) \notin T_\delta(W_1, W_2)\}} \mathbb{1}_{\{b_1^{(\mu_1)}(l_1) = m_1\}} \mathbb{1}_{\{b_2^{(\mu_1)}(l_2) = m_2\}} \right], \\ J_2 & \triangleq 2 \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{\mu_1, \mu_2} \sum_{\substack{w_1^n \in \\ T_\delta(W_1|x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2|x_2^n)}} \frac{(1-\epsilon_1)(1-\epsilon_2) P_{X_1 X_2}^n(x_1^n, x_2^n) P_{X_1|W_1}^n(x_1^n|w_1^n) P_{X_2|W_2}^n(x_2^n|w_2^n)}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)} (1+\eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \\ & \quad \sum_{l_1, l_2} \sum_{\substack{\hat{w}_1^n, \hat{w}_2^n: \\ (\hat{w}_1^n, \hat{w}_2^n) \neq (w_1^n, w_2^n)}} \sum_{m_1, m_2} \sum_{l'_1, l'_2} \mathbb{E} \left[\mathbb{1}_{\{w_2^n = \mathbf{w}_2^n(l_2, \mu_2)\}} \mathbb{1}_{\{w_1^n = \mathbf{w}_1^n(l_1, \mu_1)\}} \mathbb{1}_{\{(\hat{w}_1^n, \hat{w}_2^n) \in T_\delta(W_1, W_2)\}} \right. \\ & \quad \left. \mathbb{1}_{\{b_1^{(\mu_1)}(l_1) = m_1\}} \mathbb{1}_{\{b_1^{(\mu_1)}(l'_1) = m_1\}} \mathbb{1}_{\{b_2^{(\mu_1)}(l_2) = m_2\}} \mathbb{1}_{\{b_2^{(\mu_1)}(l'_2) = m_2\}} \mathbb{1}_{\{w_1(l'_1, \mu_1) = w'_1\}} \mathbb{1}_{\{w_2(l'_2) = w'_2\}} \right]. \end{aligned}$$

Consider the term J_1 . This can be bounded as

$$J_1 \leq 2 \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{(w_1^n, w_2^n) \notin T_\delta(W_1, W_2)} \frac{P_{X_1 X_2 W_1 W_2}^n(x_1^n, x_2^n, w_1^n, w_2^n)}{(1 + \eta)^2} \leq 2 \sum_{\substack{(w_1^n, w_2^n) \notin \\ T_\delta(W_1, W_2)}} \frac{P_{W_1 W_2}^n(w_1^n, w_2^n)}{(1 + \eta)^2} \leq \epsilon_{J_1},$$

where $\epsilon_{J_1}(\delta) \searrow 0$ as $\delta \searrow 0$. Now, consider the term corresponding to J_2 .

$$\begin{aligned} J_2 &\leq 2 \cdot 2^{n(\tilde{R}_1 + \tilde{R}_2 - R_1 - R_2)} \sum_{x_1^n, x_2^n} \sum_{\substack{w_1^n \in \\ T_\delta(W_1 | x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2 | x_2^n)}} \frac{(1 - \epsilon_1)(1 - \epsilon_2) P_{X_1 X_2}^n(x_1^n, x_2^n)}{(1 + \eta)^2 P_{X_1}^n(x_1^n) P_{X_2}^n(x_2^n)} \\ &\quad P_{X_1 | W_1}^n(x_1^n | w_1^n) P_{X_2 | W_2}^n(x_2^n | w_2^n) \sum_{\substack{\hat{w}_1^n, \hat{w}_2^n: (\hat{w}_1^n, \hat{w}_2^n) \neq (w_1^n, w_2^n) \\ (\hat{w}_1^n, \hat{w}_2^n) \in T_\delta(W_1, W_2)}} \frac{P_{W_1}^n(w_1^n)}{(1 - \epsilon)} \frac{P_{W_2}^n(w_2^n)}{(1 - \epsilon)} \frac{P_{W_1}^n(\hat{w}_1)}{(1 - \epsilon)} \frac{P_{W_2}^n(\hat{w}_2)}{(1 - \epsilon)} \\ &\leq \frac{2 \cdot 2^{n(\tilde{R}_1 + \tilde{R}_2 - R_1 - R_2)}}{(1 - \epsilon_1)(1 - \epsilon_2)(1 + \eta)^2} \sum_{x_1^n, x_2^n} \sum_{\substack{w_1^n \in \\ T_\delta(W_1 | x_1^n)}} \sum_{\substack{w_2^n \in \\ T_\delta(W_2 | x_2^n)}} P_{X_1 X_2}^n(x_1^n, x_2^n) P_{W_1 | X_1}^n(w_1^n | x_1^n) P_{W_2 | X_2}^n(w_2^n | x_2^n) \\ &\quad \sum_{\substack{\hat{w}_1^n, \hat{w}_2^n: (\hat{w}_1^n, \hat{w}_2^n) \neq (w_1^n, w_2^n) \\ (\hat{w}_1^n, \hat{w}_2^n) \in T_\delta(W_1, W_2)}} P_{W_1}^n(\hat{w}_1) P_{W_2}^n(\hat{w}_2) \\ &\leq \frac{2 \cdot 2^{n(\tilde{R}_1 + \tilde{R}_2 - R_1 - R_2)}}{(1 - \epsilon_1)(1 - \epsilon_2)(1 + \eta)^2} \sum_{w_1^n, w_2^n} P_{W_1 W_2}^n(w_1^n, w_2^n) \sum_{\substack{\hat{w}_1^n, \hat{w}_2^n: (\hat{w}_1^n, \hat{w}_2^n) \neq (w_1^n, w_2^n) \\ (\hat{w}_1^n, \hat{w}_2^n) \in T_\delta(W_1, W_2)}} P_{W_1}^n(\hat{w}_1) P_{W_2}^n(\hat{w}_2) \\ &\leq \frac{2 \cdot 2^{n(\tilde{R}_1 + \tilde{R}_2 - R_1 - R_2)}}{(1 - \epsilon_1)(1 - \epsilon_2)(1 + \eta)^2} 2^{-n(I(W_1; W_2) + \delta'_j)}. \end{aligned} \tag{56}$$

Hence, from above if $\tilde{R}_1 + \tilde{R}_2 - R_1 - R_2 \leq I(W_1; W_2) + \delta'_j$, then the term $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X})} P_{\underline{X}}^n(\underline{x}^n) \sum_{y_n} S_2 \right]$ goes to zero exponentially, where $\delta'_j(\delta), \delta''_j(\delta) \searrow 0$ as $\delta \searrow 0$.

E. Proof of Proposition 6

We begin by considering the second term within the modulus of $P_{\underline{X}}^n(\underline{x}^n) J$, for $\underline{x}^n \in T_\delta(\underline{X})$, i.e.,

$$\begin{aligned} &\frac{1}{2^{nC}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n} P_{X_1 X_2}^n(x_1^n, x_2^n) E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2 | X_2}^n(w_2^n | x_2^n) P_{Y | W_1 W_2}^n(y^n | w_1^n, w_2^n), \\ &= \frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n} P_{X_1}^n(x_1^n) E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \\ &\quad \left(P_{X_2 | X_1 W_1}^n(x_2^n | x_1^n, w_1^n) P_{W_2 | X_1 X_2 W_1}^n(w_2^n | x_1^n, x_2^n, w_1^n) P_{Y | X_1 X_2 W_1 W_2}^n(y^n | x_1^n, x_2^n, w_1^n, w_2^n) \right) \\ &= \frac{1}{2^{n(\tilde{R}_1 + C_1)}} \frac{(1 - \epsilon_1)}{(1 + \eta)} \sum_{\mu_1, l_1} \sum_{\substack{w_1 \in \\ T_\delta(W_1 | x_1^n)}} P_{X_1 | W_1}^n(x_1^n | w_1^n) P_{X_2 W_2 Y | X_1 W_1}^n(x_2^n, w_2^n, y^n | x_1^n, w_1^n) \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \end{aligned} \tag{57}$$

$$= \frac{1}{2^{n(\tilde{R}_1 + C_1)}} \frac{(1 - \epsilon_1)}{(1 + \eta)} \sum_{\mu_1, l_1} \sum_{\substack{w_1 \in \\ T_\delta(W_1|x_1^n)}} P_{X_1 X_2 W_2 Y|W_1}^n(x_1^n, x_2^n, w_2^n, y^n | w_1^n) \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}}. \quad (58)$$

We use the simplification from above and again using triangle inequality bound $\sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J$

by the following:

$$\begin{aligned} & \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J \\ & \leq \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} \left| P_{X_1 X_2 W_2 Y}^n(x_1^n, x_2^n, w_2^n, y^n) \right. \\ & \quad \left. - \frac{1}{2^{n(\tilde{R}_1 + C_1)}} \sum_{\mu_1, l} \sum_{w_1} P_{X_1 X_2 W_2 Y|W_1}^n(x_1^n, x_2^n, w_2^n, y^n | w_1^n) \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \right| \\ & \quad + \sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} \left| \frac{1}{2^{n(\tilde{R}_1 + C_1)}} \sum_{\mu_1, l} \sum_{w_1} P_{X_1 X_2 W_2 Y|W_1}^n(x_1^n, x_2^n, w_2^n, y^n | w_1^n) \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \right. \\ & \quad \left. - \frac{1}{2^{n(\tilde{R}_1 + C_1)}} \frac{(1 - \epsilon_1)}{(1 + \eta)} \sum_{\mu_1, l} \sum_{\substack{w_1 \in \\ T_\delta(W_1|x_1^n)}} P_{X_1 X_2 W_2 Y|W_1}^n(x_1^n, x_2^n, w_2^n, y^n | w_1^n) \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \right| \quad (59) \end{aligned}$$

The first term in (59) can be shown to be small in the expected sense using the Lemma 2 given the constraint $\tilde{R}_1 + C_1 \geq I(X_1, X_2, W_2, Y; W_1)$. Further, the second term in (59) can be bounded by first taking the expectation over the codebook of W_1 and then using a technique similar to that of bounding (14). We therefore have $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X})} \sum_{y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n) J \right] \leq \epsilon_J$ for $\tilde{R}_1 + C \geq I(X_1, X_2, W_2, Y; W_1)$ and sufficiently large n .

F. Proof of Proposition 7

Analysis of $\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n) Q_2$: We recall $Q_2, E_{W_1^n|X_1^n}^{(\mu_1)}(\cdot|\cdot), E_{W_2^n|X_2^n}^{(\mu_2)}(\cdot|\cdot)$.

$$\begin{aligned} Q_2 &= \left| \frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n, w_2^n} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) P_{W_2|X_2}^n(w_2^n|x_2^n) P_{Y|W_1 W_2}^n(y^n|w_1^n, w_2^n) \right. \\ & \quad \left. - \frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} \sum_{w_1^n, w_2^n} E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) P_{Y|W_1 W_2}^n(y^n|w_1^n, w_2^n) \right| \quad (60) \end{aligned}$$

$$E_{W_1^n|X_1^n}^{(\mu_1)}(w_1^n|x_1^n) = \frac{1}{2^{n\tilde{R}_1}} \frac{1 - \epsilon_1}{1 + \eta} \sum_{l_1=1}^{2^{n\tilde{R}_1}} \frac{P_{X_1|W_1}^n(x_1^n|w_1^n)}{P_{X_1}^n(x_1^n)} \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{w_1^n \in T_\delta(W_1|x_1^n)\}}$$

$$E_{W_2^n|X_2^n}^{(\mu_2)}(w_2^n|x_2^n) = \frac{1}{2^{n\tilde{R}_2}} \frac{1 - \epsilon_2}{1 + \eta} \sum_{l_2=1}^{2^{n\tilde{R}_2}} \frac{P_{X_2|W_2}^n(x_2^n|w_2^n)}{P_{X_2}^n(x_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \mathbb{1}_{\{w_2^n \in T_\delta(W_2|x_2^n)\}}.$$

Let us define the $\tilde{P}_{\underline{X}^n \underline{W}^n \underline{Y}^n}(\underline{x}^n, \underline{w}^n, y^n)$ on $\mathcal{X}_1^n \times \mathcal{X}_2^n \times \mathcal{W}_1^n \times \mathcal{W}_2^n \times \mathcal{Y}^n$ as

$$\tilde{P}_{\underline{X}^n \underline{W}^n \underline{Y}^n}(\underline{x}^n, \underline{w}^n, y^n) \triangleq \frac{1}{2^{nC_1}} \sum_{\mu_1 \in [2^{nC_1}]} P_{\underline{X}}^n(\underline{x}^n) E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) P_{W_2^n | X_2^n}^n(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n). \quad (61)$$

We remind the reader that $0 \leq \sum_{w_1^n \in \mathcal{W}_1^n} E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \leq 1$, since we only need to consider the case $\mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} = 1$. Refer to Lemma 3 for an upper bound on $\mathbb{P}(\mathbb{1}_{\{\text{PMF}(C_1, C_2)\}} = 0)$.

From the definition (61), the first term in $P_{\underline{X}}^n(\underline{x}^n)Q_2$ is simply $\sum_{\underline{w}^n} \tilde{P}_{\underline{X}^n \underline{W}^n \underline{Y}^n}(\underline{x}^n, \underline{w}^n, y^n)$. Let us denote this expression by $\tilde{P}_{\underline{X}^n \underline{Y}^n}(\underline{x}^n, y^n)$. Further, its second term can be simplified as

$$\begin{aligned} & \frac{1}{2^{n(C_1+C_2)}} \sum_{\mu_1, \mu_2} \sum_{w_1^n, w_2^n} P_{X_1 X_2}^n(x_1^n, x_2^n) E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) E_{W_2^n | X_2^n}^{(\mu_2)}(w_2^n | x_2^n) P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) \\ &= \frac{1}{2^{n(\tilde{R}_2+C_1+C_2)}} \frac{(1-\epsilon_1)}{(1-\eta)} \sum_{\mu_1, \mu_2, l_2} \sum_{\substack{w_1^n, \\ w_2^n \in T_\delta(W_2 | x_2^n)}} P_{X_1 X_2}^n(x_1^n, x_2^n) E_{W_1^n | X_1^n}^{(\mu_1)}(w_1^n | x_1^n) \\ & \quad \frac{P_{W_2^n | X_2^n}^n(w_2^n | x_2^n)}{P_{W_2^n}^n(w_2^n)} P_{Y^n | W_1 W_2}^n(y^n | w_1^n, w_2^n) \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \\ &= \frac{1}{2^{n(\tilde{R}_2+C_2)}} \frac{(1-\epsilon_1)}{(1+\eta)} \sum_{\mu_2, l_2} \sum_{\substack{w_1^n, \\ w_2^n \in T_\delta(W_2 | x_2^n)}} \frac{\tilde{P}_{\underline{X}^n \underline{W}^n \underline{Y}^n}(\underline{x}^n, \underline{w}^n, y^n) \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}}}{P_{W_2^n}^n(w_2^n)} \end{aligned}$$

where the last equality follows by the definition from (61). We therefore have

$$\begin{aligned} & \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_2 = \\ & \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \left| \tilde{P}_{\underline{X}^n \underline{Y}^n}(\underline{x}^n, y^n) - \frac{(1-\epsilon_1)}{(1+\eta)2^{n(\tilde{R}_2+C_2)}} \sum_{l_2, \mu_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2 | x_2^n)}} \frac{\tilde{P}_{\underline{X}^n \underline{Y}^n \underline{W}_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2^n}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right|. \end{aligned}$$

To bound the above term, we add and subtract the following three terms within the modulus

$$\begin{aligned} (i) & P_{\underline{X} \underline{Y}}^n(\underline{x}^n, y^n) \\ (ii) & \frac{1}{2^{n(\tilde{R}_2+C_2)}} \sum_{\mu_2, l_2} \sum_{w_2^n \in T_\delta(W_2)} P_{\underline{X} \underline{Y} | W_2}^n(\underline{x}^n, y^n | w_2^n) \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \\ (iii) & \frac{1}{2^{n(\tilde{R}_2+C_2)}} \sum_{\mu_2, l_2} \sum_{w_2^n \in T_\delta(W_2)} \frac{\tilde{P}_{\underline{X}^n \underline{Y}^n \underline{W}_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2^n}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \end{aligned}$$

Using triangle inequality on each pair of terms within the modulus, we obtain

$$\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_2 \leq \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)[Q_{21} + Q_{22} + Q_{23} + Q_{24}] \quad (62)$$

where, for all $\underline{x}^n \in T_\delta(\underline{X})$, we define

$$\begin{aligned}
P_{\underline{X}}^n(\underline{x}^n)Q_{21} &= \left| \tilde{P}_{\underline{X}^n Y^n}(\underline{x}^n, y^n) - P_{\underline{X} Y}^n(\underline{x}^n, y^n) \right|, \\
P_{\underline{X}}^n(\underline{x}^n)Q_{22} &= \left| P_{\underline{X} Y}^n(\underline{x}^n, y^n) - \frac{1}{2^{n(\tilde{R}_2 + C_2)}} \sum_{\mu_2, l_2} \sum_{w_2^n \in T_\delta(W_2)} P_{\underline{X} Y | W_2}^n(\underline{x}^n, y^n | w_2^n) \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2\}} \right| \\
P_{\underline{X}}^n(\underline{x}^n)Q_{23} &= \frac{1}{2^{n(\tilde{R}_2 + C_2)}} \sum_{\mu_2, l_2} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2\}} \left| \sum_{\substack{w_2^n \in \\ T_\delta(W_2)}} P_{\underline{X} Y | W_2}^n(\underline{x}^n, y^n | w_2^n) - \sum_{\substack{w_2^n \in \\ T_\delta(W_2)}} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \right| \\
P_{\underline{X}}^n(\underline{x}^n)Q_{24} &= \left| \frac{1}{2^{n(\tilde{R}_2 + C_2)}} \sum_{\mu_2, l_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2)}} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right. \\
&\quad \left. - \frac{(1 - \epsilon_1)}{(1 + \eta)2^{n(\tilde{R}_2 + C_2)}} \sum_{\mu_2, l_2} \sum_{\substack{w_2^n \in \\ T_\delta(W_2 | x_2^n)}} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right| \quad (63)
\end{aligned}$$

Now we look at bounding each of these four terms, starting with the term corresponding to Q_{21} . Since $\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{21} \leq \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n, w_2^n} P_{\underline{X}}^n(\underline{x}^n)J$, the result from Proposition 6 implies if $\tilde{R}_1 + C_1 \geq I(X_1, X_2, W_2, Y; W_1)$ then, for sufficiently large n , the term corresponding to Q_{21} can be made arbitrarily small in expected sense.

Secondly, we look at $\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{22}$. Using Lemma 2, we get, if $\tilde{R}_2 + C_2 \geq I(X_1 X_2 Y; W_2) + \delta_{Q_{22}}$, then for sufficiently large n , $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{22} \right] \leq \epsilon_{Q_{22}}$, where $\epsilon_{Q_{22}}, \delta_{Q_{22}} \searrow 0$ as $\delta \searrow 0$. Thirdly, consider $\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{23}$. Applying expectation over the second codebook followed by the first gives

$$\begin{aligned}
&\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{23} \right] \\
&= \mathbb{E}_{\mathcal{C}_1} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \frac{P_{W_2}^n(w_2^n)}{(1 - \epsilon_2)} \left| \sum_{w_2^n \in T_\delta(W_2)} P_{\underline{X} Y | W_2}^n(\underline{x}^n, y^n | w_2^n) - \sum_{w_2^n \in T_\delta(W_2)} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \right| \right] \\
&\leq \mathbb{E}_{\mathcal{C}_1} \left[\frac{1}{(1 - \epsilon_2)} \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \sum_{w_2^n} P_{\underline{X}}^n(\underline{x}^n)J \right] \quad (64)
\end{aligned}$$

where the first equality follows by expectation of the indicator function over the second codebook, and the subsequent inequality follows from using the triangle inequality and using the definition of J (31).

Again using Proposition 6 proves $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{23} \right]$ can be made arbitrarily small.

Finally, we remain with $\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q_{24}$. This term can be split into two terms such that

$Q_{24} = Q'_{24} + Q''_{24}$ where

$$P_{\underline{X}}^n(\underline{x}^n)Q'_{24} = 2^{-n(\tilde{R}_2+C_2)} \left| \left(1 - \frac{1-\epsilon_1}{1+\eta} \right) \sum_{\mu_2, l_2} \sum_{w_2^n \in T_\delta(W_2|x_2^n)} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right|$$

$$P_{\underline{X}}^n(\underline{x}^n)Q''_{24} = 2^{-n(\tilde{R}_2+C_2)} \left(\frac{1-\epsilon_1}{1+\eta} \right) \left| \sum_{\mu_2, l_2} \sum_{\substack{w_2^n \notin T_\delta(W_2|x_2^n) \\ w_2^n \in T_\delta(W_2)}} \frac{\tilde{P}_{\underline{X}^n Y^n W_2^n}(\underline{x}^n, y^n, w_2^n)}{P_{W_2}^n(w_2^n)} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right|$$

Consider $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q'_{24} \right]$,

$$= \mathbb{E} \left[\frac{(\eta - \epsilon_1)(1 - \epsilon_1)}{(1 + \eta)^2} \frac{1}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)}} \right. \\ \left. \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \sum_{\mu_1, \mu_2, l_1, l_2} \sum_{\substack{w_1^n \in T_\delta(W_1|x_1^n) \\ w_2^n \in T_\delta(W_2|x_2^n)}} \frac{P_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n)}{P_{W_1}^n(w_1^n)P_{W_2}^n(w_2^n)} \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right] \\ = \frac{(\eta - \epsilon_1)}{(1 + \eta)^2(1 - \epsilon_1)} \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \sum_{\substack{w_1^n \in T_\delta(W_1|x_1^n) \\ w_2^n \in T_\delta(W_2|x_2^n)}} P_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) \\ \leq \frac{(\eta - \epsilon_1)}{(1 + \eta)^2(1 - \epsilon_1)} \sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} \sum_{w_1^n, w_2^n} P_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n) = \frac{(\eta - \epsilon)}{(1 + \eta)^2(1 - \epsilon)}$$

where the first equality above is obtained by substituting the definition of $\tilde{P}_{\underline{X}^n Y^n W^n}(\underline{x}^n, y^n, \underline{w}^n)$ followed by using the simplification from (58), and the second equality is followed by using the fact that

$$\mathbb{E} \left[\mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right] = \frac{P_{W_1}^n(w_1^n)P_{W_2}^n(w_2^n)}{(1 - \epsilon_1)(1 - \epsilon_2)} \quad (65)$$

Similarly, consider $\mathbb{E} \left[\sum_{\underline{x}^n \in T_\delta(\underline{X}), y^n} P_{\underline{X}}^n(\underline{x}^n)Q''_{24} \right]$,

$$= \mathbb{E} \left[\frac{1 - \epsilon_1}{1 + \eta} \frac{1}{2^{n(\tilde{R}_1 + \tilde{R}_2 + C_1 + C_2)}} \left| \sum_{\substack{\mu_1, \mu_2, \\ l_1, l_2}} \sum_{y^n} \sum_{\substack{w_1^n \in T_\delta(W_1|x_1^n) \\ \{w_2^n \notin T_\delta(W_2|x_2^n) \\ w_2^n \in T_\delta(W_2)\}}} \frac{P_{\underline{X}WY}^n(\underline{x}^n, \underline{w}^n, y^n)}{P_{W_1}^n(w_1^n)P_{W_2}^n(w_2^n)} \right. \right. \\ \left. \left. \mathbb{1}_{\{w_1^n(l_1, \mu_1) = w_1^n\}} \mathbb{1}_{\{w_2^n(l_2, \mu_2) = w_2^n\}} \right| \right] \\ \leq \frac{1}{(1 + \eta)(1 - \epsilon_2)} \sum_{\substack{x_2 \in T_\delta(X_2) \\ w_2 \notin T_\delta(W_2|x_2^n)}} P_{X_2 W_2}^n(x_2^n, w_2^n) \sum_{x_1, w_1} P_{X_1 W_1 | X_2}(x_1^n, w_1^n | x_2^n) \sum_{y^n} P_{Y | W_1 W_2}(y^n | w_1^n, w_2^n) \\ \leq \frac{\epsilon'}{(1 + \eta)(1 - \epsilon_2)} \quad (66)$$

This completes the analysis of all the terms corresponding to Q_2 .

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