Rate-Exponent Region for a Class of Distributed Hypothesis Testing Against Conditional Independence Problems

1

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Abstract

We study a class of K-encoder hypothesis testing against conditional independence problems. Under the criterion that stipulates minimization of the Type II error subject to a (constant) upper bound ϵ on the Type I error, we characterize the set of encoding rates and exponent for both discrete memoryless and memoryless vector Gaussian settings. For the DM setting, we provide a converse proof and show that it is achieved using the Quantize-Bin-Test scheme of Rahman and Wagner. For the memoryless vector Gaussian setting, we develop a tight outer bound by means of a technique that relies on the de Bruijn identity and the properties of Fisher information. In particular, the result shows that for memoryless vector Gaussian sources the rate-exponent region is exhausted using the Quantize-Bin-Test scheme with Gaussian test channels; and there is no loss in performance caused by restricting the sensors' encoders not to employ time sharing. Furthermore, we also study a variant of the problem in which the source, not necessarily Gaussian, has finite differential entropy and the sensors' observations noises under the null hypothesis are Gaussian. For this model, our main result is an upper bound on the exponent-rate function. The bound is shown to mirror a corresponding explicit lower bound, except that the lower bound involves the source power (variance) whereas the upper bound has the source entropy power. Part of the utility of the established bound is for investigating asymptotic exponent/rates and losses incurred by distributed detection as function of the number of sensors.

I. Introduction

Consider the multiterminal detection system shown in Figure 1. In this problem, a memoryless vector source $(X, Y_0, Y_1, ..., Y_K)$, $K \ge 1$, has a joint distribution that depends on two hypotheses, a null hypothesis H_0 and an alternate hypothesis H_1 . A detector that observes directly the pair (X, Y_0) but only receives summary information of the sensors' observations $(Y_1, ..., Y_K)$ seeks to determine which of the two hypotheses is true. Specifically, Encoder k, $1 \le k \le K$, which observes an independent and identically distributed (i.i.d.) string Y_k^n , sends a message M_k to the detector at finite rate of R_k bits per observation over a noise-free channel; and the detector makes its decision between the two hypotheses on the basis of the received messages $(M_1, ..., M_K)$ as well as the available pair (X^n, Y_0^n) .

In doing so, the detector can make two types of error: Type I error (guessing H_1 while H_0 is true) and Type II error (guessing H_0 while H_1 is true). The type II error probability can decrease exponentially fast with the size n of the i.i.d. strings, say with an exponent E; and, classically, one is interested is characterizing the set of achievable rate-exponent tuples (R_1, \ldots, R_K, E) in the regime in which the probability of the Type I error is kept below a prescribed small value ϵ . This problem, which was first introduced by Berger [1] and then studied further in [2]–[4], arises naturally in many applications. Recent developments include analysis of the tradeoff between the two types of error exponents [5] or from the perspective of information spectrum [6], and extensions to networks with multiple sensors [7]–[11], multiple detectors [12], [13], interactive terminals [14], [15], multi-hop networks [8], [16]–[19], noisy channels [20], [21] and scenarios with privacy constraints [22]–[25]. Its theoretical understanding, however, is far from complete, even from seemingly simple instances of it.

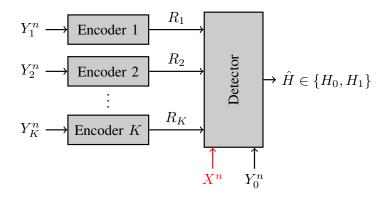


Fig. 1: Distributed hypothesis testing against conditional independence.

One important such instances was studied by Rahman and Wagner in [7]. In [7], the two hypotheses are such that X and $(Y_1, ..., Y_K)$ are correlated conditionally given Y_0 under the null hypothesis H_0 ; and they are independent conditionally given Y_0 under the alternate hypothesis H_1 , i.e., ¹

$$H_0: P_{X,Y_0,Y_1,\dots,Y_K} = P_{Y_0} P_{X,Y_1,\dots,Y_K|Y_0}$$
(1a)

$$H_1: Q_{X,Y_0,Y_1,\dots,Y_K} = P_{Y_0} P_{X|Y_0} P_{Y_1,\dots,Y_K|Y_0}.$$
(1b)

Note that $(Y_0, Y_1, ..., Y_K)$ and (Y_0, X) have the same distributions under both hypotheses; and the multiterminal problem (1) is a multi-encoder version of the single-encoder test against independence studied by Ahlswede and Csiszár in [2, Theorem 2]. For the problem (1) Rahman and Wagner provided inner and outer bounds on the rate-exponent region which do *not* match in general (see [7, Theorem 1] for the inner bound and [7, Theorem 2] for the outer bound). The inner bound of [7, Theorem 1] is similar to a generalized Berger-Tung inner bound for distributed source coding [26], [27]; and is based on a scheme, named Quantize-Bin-Test (QBT) therein, in which like in the Shimokawa–Han–Amari scheme [28] the encoders quantize and then bin their observations but the detector performs the test directly using the bins.

¹In fact, the model of [7] also involves a random variable Y_{K+1} , which is chosen here to be deterministic as it is not relevant for the analysis and discussion that will follow in this paper.

In this paper, we study a class of the distributed hypothesis testing problem (1) obtained by restricting the joint distribution of the variables under the null hypothesis H_0 to satisfy the Markov chain

$$Y_{\mathcal{S}} \leftrightarrow (X, Y_0) \leftrightarrow Y_{\mathcal{S}^c} \quad \forall \ \mathcal{S} \subseteq \mathcal{K} := \{1, \dots, K\}$$
 (2)

i.e., the encoders' observations $\{Y_k\}_{k\in\mathcal{K}}$ are independent conditionally given (X,Y_0) . We investigate both discrete memoryless (DM) and memoryless vector Gaussian models. For the DM setting, we provide a converse proof and show that it is achieved using the Quantize-Bin-Test scheme of [7, Theorem 1]. Our converse proof is strongly inspired by that of the rate-distortion region of the Chief-Executive Officer (CEO) problem under logarithmic loss of Courtade and Weissman [29, Theorem 10]. In fact, with an easy entropy characterization of the rate-exponent region that we develop here the problem is shown equivalent operationally to an CEO problem in which the remote source is X_i , agent k observes Y_k , the decoder observes side information (SI) Y_0 and wants to reconstruct the remote source X to within average distortion level $(H(X|Y_0)-E)$, and where the distortion is measured under logarithmic loss. It appears that the result of our converse can be implied by Rahman-Wagner outer bound of [7, Theorem 2] when in the problem (1) one imposes the Markov condition (2) on the distribution under the null hypothesis. This, moreover, also means that for the multiterminal CEO problem under logarithmic loss of [29] the outer bound of Wagner-Anantharam of [30, Theorem 1] implies the converse part of their Theorem 10 therein. Finally, we note that, for general distributions under the null hypothesis, i.e., without the Markov chain (2), prior to this work the optimality of the Quantize-Bin-Test scheme of [7] for the problem of testing against conditional independence was known only for the special case of a single encoder, i.e., K = 1 [7, Theorem 3], a result which can also be recovered from our result in this paper.

For the vector Gaussian setting we provide an explicit characterization of the rate-exponent region. For the proof of the converse part of this result, essentially we develop an outer bound by means of a technique that relies on the de Bruijn identity and the properties of Fisher information; and we show that it is tight. Past application of these techniques was shown recently to yield the optimal region for the related vector Gaussian CEO problem under logarithmic loss in [11], while previously found generally non-tight for the classic squared error distortion measure [31]. In particular, our result here shows that for memoryless vector Gaussian sources the rate-exponent region is exhausted using the Quantize-Bin-Test Scheme of [7, Theorem 1] with *Gaussian* test channels. Furthermore, it also shows that there is no loss in performance caused by restricting the sensors' encoders *not* to employ time sharing. This provides what appears to be the first optimality result for the Gaussian hypothesis testing against conditional independence problem in the vector sources case.

Furthermore, we broaden our view to also study a generalization of the *K*-encoder scalar Gaussian hypothesis testing against independence problem in which the sensors' observations under the null hypothesis are independent noisy versions of *X*, with Gaussian noises, but *X* itself is an arbitrary continuous memoryless source. For instance, the distribution of *X*, not necessarily Gaussian, is arbitrary and has non-zero finite entropy power. We recall that the entropy power of a continuous random

variable *X* which has density $p_X(x)$ is defined as

$$N(X) = \frac{e^{2h(X)}}{2\pi e} \tag{3}$$

where h(X) denotes the differential entropy of X. In this case, we establish an upper bound on the exponent rate function. It is shown that the bound exactly mirrors a corresponding *explicit* lower bound, except that the lower bound has the source power (variance) whereas the upper bound has the source entropy power. The bounds do not depend on auxiliaries; and, while they hold generally for arbitrary distributions of source X with finite differential entropy, their utility is mostly in that they reflect the right behavior as a function of the number of sensors.

A. Outline and Notation

The rest of this paper is organized as follows. Section II provides a formal description of the hypothesis testing problem that we study in this paper, as well as some definitions that are related to it. Sections III and IV contain the main results of this paper. Section III provides a single-letter characterization of the rate-exponent region in the DM setting, as well as an explicit characterization of the region for the case of memoryless vector Gaussian sources. Section IV provides an upper bound on the exponent-rate function for the case in which the sensors' noises are Gaussian but the source itself is memoryless continuous with arbitrary density that has finite differential entropy. This section also contains application to the study of asymptotics of the exponent-rate function for a large number of sensors. The proofs are deferred to the appendices section.

Throughout this paper, we use the following notation. Upper case letters are used to denote random variables, e.g., X; lower case letters are used to denote realizations of random variables, e.g., x; and calligraphic letters denote sets, e.g., X. The cardinality of a set X is denoted by |X|. The closure of a set \mathcal{A} is denoted by \mathcal{A} . The length-*n* sequence (X_1, \ldots, X_n) is denoted as X^n ; and, when confusion is not possible, for integers j and k such that $1 \le k \le j \le n$ the sub-sequence $(X_k, X_{k+1}, \dots, X_j)$ is denoted as X_{ν}^{I} . Probability mass functions (pmfs) are denoted by $P_{X}(x) = \Pr\{X = x\}$; and, sometimes, for short, as p(x). We use $\mathcal{P}(X)$ to denote the set of discrete probability distributions on X. Boldface upper case letters denote vectors or matrices, e.g., X, where context should make the distinction clear. For an integer $K \ge 1$, we denote the set of integers smaller or equal K as $\mathcal{K} = \{k \in \mathbb{N} : 1 \le k \le K\}$. For a set of integers $S \subseteq \mathcal{K}$, the complementary set of S is denoted by S^c , i.e., $S^c = \{k \in \mathbb{N} : k \in \mathcal{K} \setminus S\}$. Sometimes, for convenience we will need to define \bar{S} as $\bar{S} = \{0\} \cup S^c$. For a set of integers $S \subseteq K$; the notation X_S designates the set of random variables $\{X_k\}$ with indices in the set S, i.e., $X_S = \{X_k\}_{k \in S}$. We denote the covariance of a zero mean, complex-valued, vector **X** by $\Sigma_{\mathbf{x}} = \mathbb{E}[\mathbf{X}\mathbf{X}^{\dagger}]$, where $(\cdot)^{\dagger}$ indicates conjugate transpose. Similarly, we denote the cross-correlation of two zero-mean vectors X and Y as $\Sigma_{x,y} = \mathbb{E}[XY^{\dagger}]$, and the conditional correlation matrix of **X** given **Y** as $\Sigma_{x|y} = \mathbb{E}[(X - \mathbb{E}[X|Y])(X - \mathbb{E}[X|Y])^{\dagger}]$ i.e., $\Sigma_{x|y} = \Sigma_x - \Sigma_{x,y} \Sigma_y^{-1} \Sigma_{y,x}$. For matrices **A** and **B**, the notation diag(**A**, **B**) denotes the block diagonal matrix whose diagonal elements are the matrices A and B and its off-diagonal elements are the all zero matrices. Also, for a set of integers $\mathcal{J} \subset \mathbb{N}$ and a family of matrices $\{A_i\}_{i \in \mathcal{J}}$ of the same size, the notation $A_{\mathcal{J}}$ is used to denote the (super) matrix obtained by concatenating vertically the matrices $\{\mathbf{A}_i\}_{i\in\mathcal{I}}$, where the indices are sorted in the ascending order, e.g., $\mathbf{A}_{[0,2]} = [\mathbf{A}_0^{\dagger}, \mathbf{A}_2^{\dagger}]^{\dagger}$.

II. PROBLEM FORMULATION

Consider a (K+2)-dimensional memoryless source (X, Y_0 , Y_1 ,..., Y_K) with finite alphabet $X \times \mathcal{Y}_0 \times \mathcal{Y}_1 \times ... \times \mathcal{Y}_K$. The joint probability mass function (pmf) of (X, Y_0 , Y_1 ,..., Y_K) is assumed to be determined by a hypothesis H that takes one of two values, a null hypothesis H_0 and an alternate hypothesis H_1 . Under the null hypothesis H_0 , it is assumed that X and (Y_0 , Y_1 ,..., Y_K) are correlated and the joint distribution of (X, Y_0 , Y_1 ,..., Y_K) satisfies the following Markov chain

$$Y_S \leftrightarrow (X, Y_0) \leftrightarrow Y_{S^c} \quad \forall S \subseteq \mathcal{K} := \{1, \dots, K\}.$$
 (4)

Under the alternate hypothesis H_1 , it is assumed that X and $(Y_1, ..., Y_K)$ are independent conditionally given Y_0 . That is,

$$H_0: P_{X,Y_0,Y_1...,Y_K} = P_{X,Y_0} \prod_{k=1}^K P_{Y_k|X,Y_0}$$
 (5a)

$$H_1: Q_{X_1Y_0,Y_1,...,Y_K} = Q_{Y_0}Q_{X_1Y_0}Q_{Y_1,...,Y_K|Y_0}.$$
 (5b)

Throughout we make the assumption that the distributions P and Q have same (X, Y_0) - and (Y_0, Y_1, \dots, Y_K) marginals, i.e.,

$$P_{X,Y_0} = Q_{X,Y_0}$$
 and $P_{Y_0,Y_1,...,Y_K} = Q_{Y_0,Y_1,...,Y_K}$. (6)

Let now $\{(X_i, Y_{0,i}, Y_{1,i}, \dots, Y_{K,i})\}_{i=1}^n$ be a sequence of n independent copies of $(X, Y_0, Y_1, \dots, Y_K)$; and consider the detection system shown in Figure 1. Here, there are K sensors and one detector. Sensor $k \in \mathcal{K}$ observes the memoryless source component Y_k^n and sends a message $M_k = \phi_k^{(n)}(Y_k^n)$ to the detector, where the mapping

$$\phi_k^{(n)}: \mathcal{Y}_k^n \to \{1, \dots, M_k^{(n)}\}$$
 (7)

designates the encoding operation at this sensor. The detector observes the pair (X^n, Y_0^n) and uses them, as well as the messages $\{M_1, \ldots, M_K\}$ gotten from the sensors, to make a decision between the two hypotheses, based on a decision rule

$$\psi^{(n)}: \{1, \dots, M_1^{(n)}\} \times \dots \times \{1, \dots, M_K^{(n)}\} \times \mathcal{X}^n \times \mathcal{Y}_0^n \to \{H_0, H_1\}. \tag{8}$$

The mapping (8) is such that $\psi^{(n)}(m_1, \dots, m_K, x^n, y_0^n) = H_0$ if $(m_1, \dots, m_K, x^n, y_0^n) \in \mathcal{A}_n$ and H_1 otherwise, with

$$\mathcal{A}_n \subseteq \prod_{k=1}^n \{1,\ldots,M_k^{(n)}\} \times \mathcal{X}^n \times \mathcal{Y}_0^n$$

designating the acceptance region for H_0 . The encoders $\{\phi_k^{(n)}\}_{k=1}^K$ and the detector $\psi^{(n)}$ are such that the Type I error probability does not exceed a prescribed level $\epsilon \in [0,1]$, i.e.,

$$P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X^{n},Y_{0}^{n}}(\mathcal{A}_{n}^{c}) \le \epsilon \tag{9}$$

and the Type II error probability does not exceed β , i.e.,

$$Q_{\phi_1^{(n)}(Y_1^n),\dots,\phi_K^{(n)}(Y_K^n),X^n,Y_0^n}(\mathcal{A}_n) \le \beta.$$
(10)

Definition 1. A rate-exponent tuple $(R_1, ..., R_K, E)$ is achievable for a fixed $\epsilon \in [0, 1]$ if for any positive δ and sufficiently large n there exist encoders $\{\phi_k^{(n)}\}_{k=1}^K$ and a detector $\psi^{(n)}$ such that

$$\frac{1}{n}\log M_k^{(n)} \le R_k + \delta \text{ for all } k \in \mathcal{K}, \text{ and}$$
 (11a)

$$-\frac{1}{n}\log\beta \ge E - \delta. \tag{11b}$$

The rate-exponent region \mathcal{R}_{HT} is defined as

$$\mathcal{R}_{HT} := \bigcap_{\epsilon > 0} \mathcal{R}_{HT,\epsilon},\tag{12}$$

where $\mathcal{R}_{HT,\epsilon}$ is the set of all achievable rate-exponent vectors for a fixed $\epsilon \in [0,1]$.

III. RATE-EXPONENT RESULTS

A. Discrete Memoryless Case

We start with an entropy characterization of the rate-exponent region \mathcal{R}_{HT} as defined by (12). Let

$$\mathcal{R}^{\star} = \bigcup_{n} \bigcup_{\{\phi_{k}^{(n)}\}_{k \in \mathcal{K}}} \mathcal{R}^{\star} \left(n, \{\phi_{k}^{(n)}\}_{k \in \mathcal{K}} \right)$$

$$\tag{13}$$

where

$$\mathcal{R}^{\star} \left(n, \{ \phi_k^{(n)} \}_{k \in \mathcal{K}} \right) = \left\{ (R_1, \dots, R_K, E) \text{ s.t.} \right.$$

$$R_k \ge \frac{1}{n} \log |\phi_k^{(n)}(Y_k^n)| \text{ for all } k \in \mathcal{K}, \text{ and}$$

$$E \le \frac{1}{n} I(\{ \phi_k^{(n)}(Y_k^n) \}_{k \in \mathcal{K}}; X^n | Y_0^n) \right\}.$$
(14a)

We have the following proposition the proof of which is essentially similar to that of [2, Theorem 1] and appears in Appendix A.

Proposition 1. $\mathcal{R}_{HT} = \overline{\mathcal{R}^{\star}}$.

The result of Proposition 1 essentially means that the studied hypothesis testing problem is operationally equivalent to a chief executive officer (CEO) source coding problem where the distortion is measured under logarithmic loss. Specifically, this equivalent CEO problem is one in which the remote source is X; there are K agents observing noisy versions of it, with agent k observing Y_k ; and the decoder observes side information (SI) Y_0 and wants to reconstruct the remote source X to within average logarithmic loss distortion ($H(X|Y_0) - E$). The latter problem was solved in [29, Theorem 10] in the case of no decoder SI (i.e., $Y_0 = \emptyset$) but its proof carries over with minimal changes to the case in which the decoder is equipped with SI Y_0 . Thus, with the result of Proposition 1 and a rather straightforward generalization of [29, Theorem 10] we have the following theorem which provides a single-letter characterization of the rate-exponent region \mathcal{R}_{HT} .

Theorem 1. The rate-exponent region \mathcal{R}_{HT} is given by the union of all non-negative tuples (R_1, \ldots, R_K, E) that satisfy, for all subsets $S \subseteq \mathcal{K}$,

$$E \le I(U_{S^c}; X | Y_0, Q) + \sum_{k \in S} \left(R_k - I(Y_k; U_k | X, Y_0, Q) \right)$$
 (15)

for some auxiliary random variables (U_1, \ldots, U_K, Q) with distribution $P_{U_K,Q}$ such that

$$P_{X,Y_0,Y_K,U_K,Q} = P_Q P_{X,Y_0} \prod_{k=1}^K P_{Y_k|X,Y_0} \prod_{k=1}^K P_{U_k|Y_k,Q}.$$
 (16)

A direct proof of the achievability part of Theorem 1 follows by an easy application of the Quantize-Bin-Test scheme of Rahman and Wagner [7, Theorem 1]. The interested reader may also find an alternate, direct, proof of its converse part in Appendix B.

Comparatively, the hypothesis testing model of [7] is one in which under the null hypothesis $(Y_1, ..., Y_K)$ are arbitrarily correlated among them and with the pair (X, Y_0) ; and under the alternate hypothesis Y_0 induces conditional independence between $(Y_1, ..., Y_K)$ and X. More precisely, the joint distributions of $(X, Y_0, Y_1, ..., Y_K)$ under the null and alternate hypotheses as considered in [7] are

$$H_0: \tilde{P}_{X,Y_0,Y_1...,Y_K} = P_{Y_0} P_{X|Y_0} P_{Y_1,...,Y_K|X,Y_0}$$
(17a)

$$H_1: \tilde{Q}_{X,Y_0,Y_1...,Y_K} = P_{Y_0} P_{X|Y_0} P_{Y_1,...,Y_K|Y_0}. \tag{17b}$$

For this more general model, they provide inner and outer bounds on the rate-exponent region which do *not* match in general (see [7, Theorem 1] for the inner bound and [7, Theorem 2] for the outer bound). Our Theorem 1 shows that if, in addition, the joint distribution of the variables under the null hypothesis H_0 is restricted to satisfy the Markov chain condition (4), then the Quantize-Bin-Test scheme of [7, Theorem 1] is optimal. Accordingly, the reader may wonder whether the converse of Theorem 1 could be implied by Rahman-Wagner outer bound of [7, Theorem 2] when specialized to the test setting studied here. The answer to this question, brought to the attention of the author during the revision of this paper, appears to be affirmative. To see this, recall that the outer bound of [7, Theorem 2], denoted hereafter as \mathcal{R}_{RW}^{out} , is given by

$$\mathcal{R}_{\text{RW}}^{\text{out}} = \bigcap_{A \in \mathcal{A}} \bigcup_{\lambda_0 \in \Lambda_0} \mathcal{R}_{\text{RW}}^{\text{out}}(A, \lambda_0)$$
(18)

where:

- i) \mathcal{A} is the set of finite-alphabet random variable A such that Y_1, \ldots, Y_K, X are conditionally independent given (A, Y_0) ;
- ii) Λ_0 is the set of finite-alphabet random variables $\lambda_0 = (U_1, \dots, U_K, W, Q)$ such that:
 - (a) (W,Q) is independent of $(Y_1, \ldots, Y_K, X, Y_0)$
 - (b) $U_k \rightarrow (Y_k, W, Q) \rightarrow (U_{k^c}, Y_{k^c}, X, Y_0)$ for all $k \in \mathcal{K}$;
- iii) for given $A \in \mathcal{A}$ and $\lambda_0 \in \Lambda_0$ for which the joint distribution of A, $(X, Y_0, Y_1, ..., Y_K)$ and λ_0 satisfies the Markov chain condition

$$A \leftrightarrow (Y_1, \dots, Y_K, X, Y_0) \leftrightarrow \lambda_0 \tag{19}$$

and $\mathcal{R}^{\text{out}}_{\text{RW}}(A, \lambda_0)$ is defined as the set of all non-negative (R_1, \dots, R_K, E) for which

$$\sum_{k \in \mathcal{S}} R_k \ge I(\mathbf{U}_{\mathcal{S}}; A | \mathbf{U}_{\mathcal{S}^c}, Y_0, Q) + \sum_{k \in \mathcal{S}} I(U_k; Y_k | A, W, Y_0, Q), \quad \forall \mathcal{S} \subseteq \mathcal{K}$$
(20a)

$$E \le I(U_1, \dots, U_K; X|Y_0, Q).$$
 (20b)

Let $(U_1, ..., U_K, W, Q) \in \Lambda_0$. Noticing that $X \in \mathcal{A}$ and setting A = X, the inequality (20a) can be weakened as

$$\sum_{k \in S} R_k \ge I(\mathbf{U}_S; X | \mathbf{U}_{S^c}, Y_0, Q) + \sum_{k \in S} I(U_k; Y_k | X, W, Y_0, Q)$$
(21)

$$= I(\mathbf{U}_{\mathcal{K}}; X | Y_0, Q) - I(\mathbf{U}_{\mathcal{S}^c}; X | Y_0, Q) + \sum_{k \in \mathcal{S}} I(U_k; Y_k | X, W, Y_0, Q)$$
 (22)

$$\geq E - I(\mathbf{U}_{S^c}; X | Y_0, Q) + \sum_{k \in S} I(U_k; Y_k | X, W, Y_0, Q)$$
 (23)

where the last inequality follows by using (20b). Also, we have

$$I(\mathbf{U}_{S^c}; X|Y_0, Q) = H(X|Y_0, Q) - H(X|\mathbf{U}_{S^c}, Y_0, Q)$$
(24)

$$\stackrel{(a)}{=} H(X|Y_0, W, Q) - H(X|\mathbf{U}_{S^c}, Y_0, Q)$$
 (25)

$$\stackrel{(b)}{\leq} H(X|Y_0, W, Q) - H(X|\mathbf{U}_{S^c}, Y_0, W, Q) \tag{26}$$

$$= I(\mathbf{U}_{S^c}; X | Y_0, W, Q) \tag{27}$$

where (*a*) holds since (*W*, *Q*) is independent of (*X*, Y_0) and (*b*) holds since conditioning reduces entropy. Combining (23) and (27), we get that for all $S \subseteq K$ we have Thus, we have the bound

$$\sum_{k \in S} R_k \ge E - I(\mathbf{U}_{S^c}; X | Y_0, W, Q) + \sum_{k \in S} I(U_k; Y_k | X, W, Y_0, Q). \tag{28}$$

Thus, the variable W can be absorbed into the time sharing random variable Q, and one gets the expression of the above Theorem 1.

Remark 1. For reasons that are essentially similar to the above it is not difficult to see that, for the related K-encoder CEO problem under logarithmic loss, $k \ge 2$, the outer bound of Wagner-Anantharam of [30, Theorem 1] implies the converse part of Courtade-Weissman [29, Theorem 10].

Remark 2. Prior to this work, the optimality of the QBT scheme of [7] for the problem of testing against conditional independence was known only for the special case of a single encoder, i.e., K = 1 [7, Theorem 3], a result which can also be recovered from Theorem 1.

B. Memoryless Vector Gaussian Case

We now turn to a continuous example of the hypothesis testing problem studied in this paper. Here, $(X, Y_0, Y_1, ..., Y_K)$ is a zero-mean circularly-symmetric complex-valued Gaussian random vector. Without loss of generality, let

$$\mathbf{Y}_0 = \mathbf{H}_0 \mathbf{X} + \mathbf{Z}_0 \tag{29}$$

where $\mathbf{H}_0 \in \mathbb{C}^{n_0 \times n_x}$, $\mathbf{X} \in \mathbb{C}^{n_x}$ and $\mathbf{Z}_0 \in \mathbb{C}^{n_0}$ are independent Gaussian vectors with zero-mean and covariance matrices $\mathbf{\Sigma}_{\mathbf{X}} > \mathbf{0}$ and $\mathbf{\Sigma}_0 > \mathbf{0}$, respectively. The vectors $(\mathbf{Y}_1, \dots, \mathbf{Y}_K)$ and \mathbf{X} are correlated under the null hypothesis H_0 and are independent under the alternate hypothesis H_1 . Specifically, under the null hypothesis

$$H_0: \mathbf{Y}_k = \mathbf{H}_k \mathbf{X} + \mathbf{Z}_k, \quad \text{for all } k \in \mathcal{K}$$
 (30)

where the noise vectors $(\mathbf{Z}_1, \dots, \mathbf{Z}_K)$ are jointly Gaussian with zero mean and covariance matrix $\Sigma_{\mathbf{n}_K} > \mathbf{0}$, and assumed to be independent from \mathbf{X} but correlated among them and with \mathbf{Z}_0 , such that for every $S \subseteq \mathcal{K}$,

$$\mathbf{Z}_{\mathcal{S}} \to \mathbf{Z}_0 \to \mathbf{Z}_{\mathcal{S}^c}. \tag{31}$$

For every $k \in \mathcal{K}$ we denote by Σ_k the *conditional* covariance matrix of noise \mathbf{Z}_k conditionally given \mathbf{Z}_0 . Under the alternate hypothesis H_1 , the joint distribution of $(\mathbf{X}, \mathbf{Y}_0, \mathbf{Y}_1, \dots, \mathbf{Y}_K)$, denoted as $Q_{\mathbf{X}, \mathbf{Y}_0, \mathbf{Y}_1, \dots, \mathbf{Y}_K}$, factorizes as

$$H_1: Q_{X,Y_0,Y_1,\dots,Y_K} = Q_{Y_0}Q_{X|Y_0}Q_{Y_1,\dots,Y_K|Y_0}.$$
(32)

Here $Q_{\mathbf{X},\mathbf{Y}_0} = P_{\mathbf{X},\mathbf{Y}_0}$ where $P_{\mathbf{X},\mathbf{Y}_0}$ is the joint distribution of the vector $(\mathbf{X},\mathbf{Y}_0)$ under H_0 as induced by (29) and $Q_{\mathbf{Y}_0,\mathbf{Y}_1,...,\mathbf{Y}_K} = P_{\mathbf{Y}_0,\mathbf{Y}_1,...,\mathbf{Y}_K}$ where $P_{\mathbf{Y}_0,\mathbf{Y}_1,...,\mathbf{Y}_K}$ is the joint distribution of the vector $(\mathbf{Y}_0,\mathbf{Y}_1,...,\mathbf{Y}_K)$ under H_0 as induced by (29), (30) and (31).

Let $\mathcal{R}_{VG\text{-HT}}$ denote the rate-exponent region of this vector Gaussian hypothesis testing against conditional independence problem.

For convenience, we now introduce the following notation which will be instrumental in what follows. Let, for every set $S \subseteq \mathcal{K}$, the set $\bar{S} = \{0\} \cup S^c$. Also, for $S \subseteq \mathcal{K}$ and given matrices $\{\Omega_k\}_{k=1}^K$ such that $\mathbf{0} \leq \Omega_k \leq \Sigma_k^{-1}$, let $\Lambda_{\bar{S}}$ designate the block-diagonal matrix given by

$$\Lambda_{\bar{S}} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}\left(\{\Sigma_k - \Sigma_k \Omega_k \Sigma_k\}_{k \in S^c}\right) \end{bmatrix}$$
(33)

where **0** in the principal diagonal elements is the $n_0 \times n_0$ -all zero matrix.

The following theorem provides an explicit characterization of \mathcal{R}_{VG-HT} .

Theorem 2. The rate-exponent region $\mathcal{R}_{VG\text{-HT}}$ of the vector Gaussian hypothesis testing against conditional independence problem is given by the set of all non-negative tuples (R_1, \ldots, R_K, E) that satisfy, for all subsets $S \subseteq \mathcal{K}$,

$$E \leq \sum_{k \in \mathcal{S}} \left(R_k + \log |\mathbf{I} - \mathbf{\Omega}_k \mathbf{\Sigma}_k| \right) - \log \left| \mathbf{I} + \mathbf{\Sigma}_{\mathbf{X}} \mathbf{H}_0^{\dagger} \mathbf{\Sigma}_0^{-1} \mathbf{H}_0 \right|$$

$$+ \log \left| \mathbf{I} + \mathbf{\Sigma}_{\mathbf{X}} \mathbf{H}_{\tilde{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\tilde{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\tilde{S}} \mathbf{\Sigma}_{\mathbf{n}_{\tilde{S}}}^{-1} \right) \mathbf{H}_{\tilde{S}} \right|$$
(34)

for matrices $\{\Omega_k\}_{k=1}^K$ such that $\mathbf{0} \leq \Omega_k \leq \Sigma_k^{-1}$, where $\bar{S} = \{0\} \cup S^c$ and $\Lambda_{\bar{S}}$ is given by (33).

Proof. The proof of Theorem 2 appears in Appendix C.

The direct part of Theorem 2 is obtained by evaluating the region of Theorem 1, which can be shown easily to extend to the continuous alphabet case through standard discretization arguments,

using Gaussian test channels and no-time sharing. Specifically, we let $Q = \emptyset$ and $P_{U_k|Y_k,Q}(u_k|\mathbf{y}_k,q) = C\mathcal{N}(\mathbf{y}_k,[(\mathbf{I} - \mathbf{\Omega}_k\mathbf{\Sigma}_k)^{-1} - \mathbf{I}]^{-1}\mathbf{\Sigma}_k)$. The main contribution of Theorem 2 is its converse part, the proof of which uses a technique that relies on the de Bruijn identity and the properties of Fisher information. The bound is similar to the outer bound for the vector Gaussian CEO problem under logarithmic loss given by Ugur *et al.* in [11]. In particular, the result of Theorem 2 shows that there is no loss in performance if one restricts the auxiliaries (test channels) of the Quantize-Test-Bin scheme to be *Gaussian*. Furthermore, there is *no* loss in performance caused by restricting the encoders not to employ time sharing.

In the rest of this section, we elaborate on two special cases of Theorem 2, the one-encoder vector Gaussian testing against conditional independence problem (i.e., K = 1) and the K-encoder scalar Gaussian testing against independence problem.

1) The one-encoder vector Gaussian HT problem against conditional independence: Set K = 1 in (30), (31) and (39). In this case the Markov chain (31) is non-restrictive as it is trivially satisfied for all arbitrarily correlated noise at the sensor and side information \mathbf{Y}_0 at the detector. Theorem 2 then provides a complete solution of the (general) one-encoder vector Gaussian testing against conditional independence problem. The result is stated in the following Corollary.

Corollary 1. For the one-encoder vector Gaussian HT against conditional independence problem, the rate-exponent region is given by the set of all non-negative pairs (R_1, E) that satisfy

$$E \le R_1 + \log|\mathbf{I} - \mathbf{\Omega}_1 \mathbf{\Sigma}_1| \tag{35a}$$

$$E \le \log \left| \mathbf{I} + \Sigma_{\mathbf{x}} \mathbf{H}_{\{0,1\}}^{\dagger} \Sigma_{\mathbf{n}_{\{0,1\}}}^{-1} \left(\mathbf{I} - \Lambda_{\{0,1\}} \Sigma_{\mathbf{n}_{\{0,1\}}}^{-1} \right) \mathbf{H}_{\{0,1\}} \right| - \log \left| \mathbf{I} + \Sigma_{\mathbf{x}} \mathbf{H}_{0}^{\dagger} \Sigma_{0}^{-1} \mathbf{H}_{0} \right|, \tag{35b}$$

for some $n_1 \times n_1$ matrix Ω_1 such that $0 \le \Omega_1 \le \Sigma_1^{-1}$, where $\mathbf{H}_{\{0,1\}} = [\mathbf{H}_0^{\dagger}, \mathbf{H}_1^{\dagger}]^{\dagger}$, $\Sigma_{\mathbf{n}_{\{0,1\}}}$ is the covariance matrix of noise $(\mathbf{Z}_0, \mathbf{Z}_1)$ and

$$\Lambda_{\{0,1\}} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_1 - \Sigma_1 \mathbf{\Omega}_1 \Sigma_1 \end{bmatrix} \tag{36}$$

with the **0** in its principal diagonal denoting the $n_0 \times n_0$ -all zero matrix.

In particular, for the setting of testing against independence, i.e., $\mathbf{Y}_0 = \emptyset$ and the detector's task reduced to guessing whether \mathbf{Y}_1 and \mathbf{X} are independent or not, the optimal trade-off expressed by (35) reduces to the set of (R_1, E) pairs that satisfy, for some $n_1 \times n_1$ matrix $\mathbf{\Omega}_1$ such that $\mathbf{0} \leq \mathbf{\Omega}_1 \leq \mathbf{\Sigma}_1^{-1}$,

$$E \le \min \left\{ R_1 + \log |\mathbf{I} - \mathbf{\Omega}_1 \mathbf{\Sigma}_1|, \log |\mathbf{I} + \mathbf{\Sigma}_{\mathbf{x}} \mathbf{H}_1^{\dagger} \mathbf{\Omega}_1 \mathbf{H}_1 | \right\}. \tag{37}$$

Observe that (35) is the counter-part, to the vector Gaussian setting, of the result of [7, Theorem 3] which provides a single-letter formula for the Type II error exponent for the one-encoder DM testing against conditional independence problem. Similarly, (37) is the solution of the vector Gaussian version of the one-encoder DM testing against independence problem which is studied, and solved, by Ahlswede and Csiszár in [2, Theorem 2].

2) The K-encoder scalar Gaussian HT problem against independence: Consider now the special case of the setup of Theorem 2 in which $K \ge 2$, $Y_0 = \emptyset$, and the sources and noises are all scalar complex-valued Gaussian, i.e., $n_x = 1$ and $n_k = 1$ for all $k \in \mathcal{K}$. The vector (Y_1, \ldots, Y_K) and X are correlated under the null hypothesis H_0 with

$$H_0: Y_k = X + Z_k, \quad \text{for all } k \in \mathcal{K}$$
 (38)

The noises $Z_1, ..., Z_K$ are zero-mean jointly Gaussian, mutually independent and independent from X. Also, we assume that the variances σ_k^2 of noise Z_k , $k \in \mathcal{K}$, and σ_X^2 of X are all non-negative. Under the alternate hypothesis H_1 , the joint distribution of $(X, Y_1, ..., Y_K)$, denoted as $Q_{X,Y_1,...,Y_K}$, factorizes as

$$H_1: Q_{X,Y_1,...,Y_K} = Q_X Q_{Y_1,...,Y_K}$$
(39)

where $Q_X = P_X$, i.e., complex Gaussian with zero-mean and variance σ_X^2 and $Q_{Y_1,...,Y_K} = P_{Y_1,...,Y_K}$ where $P_{Y_1,...,Y_K}$ is the joint distribution of the vector $(Y_1,...,Y_K)$ under H_0 as induced by (38). In this case, the result of Theorem 2 reduces as stated in the following corollary.

Corollary 2. For the K-encoder scalar Gaussian HT against independence problem described by (38) and (39), the rate-exponent region is given by the set of all non-negative tuples $(R_1, ..., R_K, E)$ that satisfy

$$\mathcal{R}_{SG-HT} = \left\{ (R_1, \dots, R_K, E) : \exists (\gamma_1, \dots, \gamma_K) \in \mathbb{R}_+^K \text{ s.t.} \right.$$

$$\gamma_k \le \frac{1}{\sigma_k^2}, \ \forall k \in \mathcal{K}, \text{ and } \forall S \subseteq \mathcal{K}$$

$$\sum_{k \in S} R_k \ge E + \log \left[\left(\left(1 + \sigma_X^2 \sum_{k \in S^c} \gamma_k \right) \prod_{k \in S} (1 - \gamma_k \sigma_k^2) \right)^{-1} \right] \right\}. \tag{40}$$

The region \mathcal{R}_{SG-HT} as given by (40) can be used to, e.g., characterize the centralized rate region, i.e., the set of rate vectors (R_1, \ldots, R_K) that achieve the centralized Type II error exponent

$$I(Y_1, ..., Y_K; X) = \sum_{k=1}^K \log \frac{\sigma_X^2}{\sigma_k^2}.$$
 (41)

We close this section by mentioning that, as it can be seen from the proof of Theorem 2, the Quantize-Bin-Test scheme of [7, Theorem 1] evaluated with Gaussian test channels and no time-sharing is optimal for the vector Gaussian K-encoder hypothesis testing against conditional independence problem described by (30) and (39). Furthermore, we note that Rahman and Wagner also characterized the optimal rate-exponent region of a different² Gaussian hypothesis testing against independence problem, called the Gaussian many-help-one hypothesis testing against independence problem therein, in the case of scalar valued sources [7, Theorem 7]. Specialized to the case K = 1, the result of Theorem 2 recovers that of [7, Theorem 7] in the case of no helpers; and extends it to vector-valued sources and testing against conditional independence in that case.

 2 This problem is related to the Gaussian many-help-one problem [32]–[34]. Here, different from the setup of Figure 1, the source X is observed directly by a main encoder who communicates with a detector that observes Y in the aim of making a decision on whether X and Y are independent or not. Also, there are helpers that observe independent noisy versions of X and communicate with the detector in the aim of facilitating that test.

IV. TESTING UNDER GAUSSIAN NOISE: DUAL ROLES OF POWER AND ENTROPY POWER

In this section, we broaden our view to study a generalization of the K-encoder scalar Gaussian HT against independence problem described by (38) and (39) in which the sensors' observations (Y_1, \ldots, Y_K) under the null hypothesis are still independent noisy versions of X, with Gaussian noises, but X itself is an arbitrary continuous memoryless source. In particular, X is not necessarily Gaussian. Throughout this section we assume that X has density $P_X(x)$ (not necessarily Gaussian) which has finite differential entropy h(X), variance σ_X^2 and non-zero finite entropy power

$$N(X) = \frac{e^{2h(X)}}{2\pi e}. (42)$$

Specifically, X and $(Y_1, ..., Y_K)$ are correlated under the null hypothesis H_0 with

$$H_0: Y_k = X + Z_k, \text{ for } k = 1, ..., K$$
 (43)

where the noise Z_k is zero-mean Gaussian with variance σ_k^2 and is independent from all other noises and from X; and they are independent under H_1 with their joint distribution given by

$$H_1: Q_{X,Y_1,...,Y_K} = P_X P_{Y_1,...,Y_K}$$
(44)

where P_X is the distribution of X under H_0 (not necessarily Gaussian!) and $P_{Y_1,...,Y_K}$ is the joint distribution of $(Y_1,...,Y_K)$ under H_0 as induced by (43).

In this section sometimes we will be interested in the sum-rate exponent function, which is defined as

$$R_{\text{sum}}(E) = \min_{(R_1, \dots, R_K, E) \in \mathcal{R}_{\text{HT}, \varepsilon}} \sum_{k=1}^K R_k.$$
 (45)

Throughout it will be convenient to use the following shorthand notation. For any non-empty subset $S \subseteq K$ the sufficient statistic for X given $\{Y_k\}_{k \in S}$ is given by

$$Y(S) = \frac{1}{|S|} \sum_{k \in S} \frac{\sigma_S^2}{\sigma_k^2} Y_k \tag{46a}$$

$$= X + Z(S) \tag{46b}$$

where

$$Z(S) = \frac{1}{|S|} \sum_{k \in S} \frac{\sigma_S^2}{\sigma_k^2} Z_k \tag{47}$$

is a zero-mean Gaussian random variable of variance $\sigma_S^2/|S|$, and σ_S^2 denotes the harmonic mean of the noise variances in the set S, given by

$$\sigma_{\mathcal{S}}^2 = \left(\frac{1}{|\mathcal{S}|} \sum_{k \in \mathcal{S}} \frac{1}{\sigma_k^2}\right)^{-1}.$$
 (48)

For the special case of empty set $S = \emptyset$, we set $Y(\emptyset) = Z(\emptyset) = \text{constant}$.

In the rest of this section, we will develop bounds on the rate-exponent region of this model which exhibit a pleasant duality between power and entropy power. Bounds of the same kind of duality were already observed in the context of source coding under the classic squared error distortion measure for point-to-point [35, p. 338] and multiterminal CEO [36] settings. The recent work [37] is somewhat related, but to a lesser extent.

A. Special Case K = 1

Set K = 1 in (43) and (44). For notational convenience, we use the substitutions $Y = Y_1$, $Z = Z_1$ and $\sigma_Z^2 = \sigma_1^2$. First let us recall that for a given non-negative rate R the optimal rate exponent is given by [2, Theorem 2]

$$E(R) = \max_{P_{U|Y}: I(U;Y) \le R} I(U;X). \tag{49}$$

It is rather easy to see that a simple lower bound on the exponent-rate function is given by

$$E(R) \ge \frac{1}{2} \log^+ \left(\frac{\sigma_{\gamma}^2}{\sigma_X^2 e^{-2R} + \sigma_Z^2} \right). \tag{50}$$

This can be obtained by evaluating the right hand side (RHS) of (49) using the choice of auxiliary

$$U = Y + V \tag{51}$$

where V is zero-mean Gaussian with variance

$$\sigma_V^2 = \frac{\sigma_X^2 + \sigma_Z^2}{e^{2R} - 1} \tag{52}$$

and is independent from (X, Z).

Also, it can be shown (see Appendix D) that

$$E(R) \le \frac{1}{2} \log^+ \left(\frac{N(Y)}{N(X)e^{-2R} + \sigma_7^2} \right).$$
 (53)

Part of the appeal of these bounds is the interesting duality that is played by the source power and its entropy power. Also, this directly implies their tightness in the special case in which the source X is Gaussian since power and entropy power are equal in that case. Moreover, the above also implies that among all sources with the same variance (power) the Gaussian is the worst (i.e., has the smallest Type-II error exponent for given R). Conversely, among all sources with the same entropy power the Gaussian is the best (i.e., has the largest Type-II error exponent for given R).

B. Upper Bound

We now turn to the *K*-encoder test described by (43) and (44). The main result of this section is an upper bound on the exponent-rate function for an arbitrary continuous source *X* with finite differential entropy. Its strength is in that a direct consequence of it (Corollary 4 below) is shown to reflect the right behavior as a function of the number of observations/sensors.

Recall the definition of the sufficient statistic Y(S) for X given $Y_S = \{Y_k\}_{k \in S}$ as given by (46) for given $S \subseteq K$.

Theorem 3. If a rate-exponent tuple $(R_1, ..., R_K, E)$ is achievable, i.e., $(R_1, ..., R_K, E) \in \mathcal{R}_{HT}$, then there must exist non-negative real numbers $(\gamma_1, ..., \gamma_K)$ with $\gamma_k \leq 1/\sigma_k^2$ for all $k \in \mathcal{K}$ such that for all (strict) subsets $S \subset \mathcal{K}$, we have

$$E \leq \frac{1}{2} \log \left(|S^{c}| \frac{N(Y(S^{c}))}{\sigma_{S^{c}}^{2}} - N(X) \sum_{k \in S^{c}} \left(\frac{1}{\sigma_{k}^{2}} - \gamma_{k} \right) \right) + \sum_{k \in S} \left(R_{k} - \frac{1}{2} \log \frac{1}{1 - \gamma_{k} \sigma_{k}^{2}} \right);$$

$$(54)$$

and for the full set S = K we have

$$E \le \sum_{k=1}^{K} \left(R_k - \frac{1}{2} \log \frac{1}{1 - \gamma_k \sigma_k^2} \right), \tag{55}$$

where $Y(S^c)$ and $\sigma_{S^c}^2$ are defined using (46) and (48) respectively.

Proof. The proof of Theorem 3 appears in Appendix E.

Remark 3. A simple entropy power inequality argument can be used to show that the term inside the logarithm in the RHS of (54) is guaranteed to be larger than 1 for all non-negative choices of $(\gamma_1, ..., \gamma_K)$ that satisfy $0 \le \gamma_k \le 1/\sigma_k^2$ for all $k \in \mathcal{K}$, making the expression well defined.

We now state the next corollary which provides a lower bound on the exponent-rate function for an arbitrary continuous source X with finite differential entropy, and whose proof, omitted here for brevity, can be obtained easily from that of the direct part of Theorem 2 (for instance, see Eq. (40) of Corollary 2). In fact, while the result of Theorem 2 pertains to the case of jointly Gaussian (X, Y_1, \ldots, Y_K) , the key argument of its direct part is the Markov lemma [35] whose proof only uses the fact that conditioned on the source sequence X^n the noisy observations $\{Y_k\}_{k=1}^K$ and the auxiliaries $\{U_k\}_{k=1}^K$ are Gaussian. Clearly, this still holds here even though X is not necessarily Gaussian.

Corollary 3. If there exist non-negative real numbers $(\gamma_1, \ldots, \gamma_K)$ with $\gamma_k \leq 1/\sigma_k^2$ for all $k \in \mathcal{K}$ such that for all (strict) subsets $S \subset \mathcal{K}$, we have

$$E \ge \frac{1}{2} \log \left(|S^c| \frac{\sigma_{Y(S^c)}^2}{\sigma_{S^c}^2} - \sigma_X^2 \sum_{k \in S^c} \left(\frac{1}{\sigma_k^2} - \gamma_k \right) \right) + \sum_{k \in S} \left(R_k - \frac{1}{2} \log \frac{1}{1 - \gamma_k \sigma_k^2} \right)$$

$$(56)$$

and for the full set S = K we have

$$E \ge \sum_{k=1}^{K} \left(R_k - \frac{1}{2} \log \frac{1}{1 - \gamma_k \sigma_k^2} \right),$$
 (57)

then the tuple $(R_1, ..., R_K, E)$ is achievable, i.e., $(R_1, ..., R_K, E) \in \mathcal{R}_{HT}$, where $Y(S^c)$ and $\sigma^2_{S^c}$ are defined using (46) and (48) respectively.

Investigating the above bounds of Theorem 3 and Corollary 3, it is interesting to observe a pleasant duality, in the sense that the power (variance) terms of the lower bound are replaced by entropy power terms (note that $\sigma_{Y(S^c)}^2 = \sigma_X^2 + \sigma_{Z(S^c)}^2$ but we prefer to use the form given in the theorem so as to emphasize such duality). Among other aspects, this directly implies tightness of the bounds in the special case in which the source X is Gaussian; thus providing an alternative proof of Theorem 2 in the special case of testing against independence and scalar Gaussian sources. Furthermore, similar to the single-sensor setting of Section IV-A, the bounds also imply that for given entropy power the Gaussian is the best distribution and for given power the Gaussian is the worst distribution.

C. Sum-Rate Exponent Function

For simplicity, we set all the noise variances to be equal, i.e., $\sigma_k^2 = \sigma_Z^2$ for all $k \in \mathcal{K}$. Note that in this case, the harmonic mean of the noise variances as defined by (48) is $\sigma_S^2 = \sigma_Z^2$ for all $S \subseteq \mathcal{K}$. Using Theorem 3, we have

$$\sum_{k=1}^{K} R_k \stackrel{(a)}{\ge} E + \frac{1}{2} \log \prod_{k=1}^{K} \frac{1}{1 - \gamma_k \sigma_Z^2}$$
 (58)

$$\stackrel{(b)}{\geq} E - \frac{K}{2} \log \frac{1}{K} \sum_{k=1}^{K} \left(1 - \gamma_k \sigma_Z^2 \right) \tag{59}$$

$$\stackrel{(c)}{\geq} E + \frac{K}{2} \log \frac{KN(X)}{\sigma_X^2} \left(\frac{KN(Y(\mathcal{K}))}{\sigma_{\mathcal{K}}^2} - e^{2E} \right)^{-1}$$
 (60)

where: (a) follows by (55), (b) follows by using Jensen's inequality, and (c) follows by applying (54) for $S = \emptyset$.

The result of the next corollary follows directly from (60).

Corollary 4. If a sum-rate exponent pair (R_{sum}, E) is achievable, i.e., $(R_1, ..., R_K, E) \in \mathcal{R}_{HT}$ with $(R_1 + ... + R_K) = R_{sum}$, then the following holds,

$$R_{sum} \ge E + \frac{K}{2} \log^{+} \left(\frac{KN(X)}{KN(Y(\mathcal{K})) - \sigma_{Z}^{2} e^{2E}} \right)$$
 (61)

for E for which $\sigma_7^2 e^{2E} < KN(Y(\mathcal{K}))$, where $Y(\mathcal{K})$ is defined using (46).

Using Corollary 3, it is easy to see that for given exponent E for which $\sigma_Z^2 e^{2E} < K \sigma_{Y(K)}^2$ the sum-rate exponent is upper bounded as

$$R_{\text{sum}} \le E + \frac{K}{2} \log \left(\frac{K \sigma_X^2}{K \sigma_{Y(K)}^2 - \sigma_Z^2 e^{2E}} \right). \tag{62}$$

D. Application

Part of the utility of the results of Theorem 3 and Corollary 4 is, e.g., for investigating asymptotic exponent/rates and losses incurred by distributed detection as function of the number of observations. The gap between the bounds (61) and (62) is upper-bounded by

$$\Delta(K) := \frac{K}{2} \log^{+} \left(\frac{\sigma_X^2}{N(X)} \left(\frac{KN(Y(K)) - \sigma_Z^2 e^{2E}}{K\sigma_{Y(K)}^2 - \sigma_Z^2 e^{2E}} \right) \right). \tag{63}$$

Recalling that $Y(K) = X + (\sigma_Z/\sqrt{K})G$ where $G \sim \mathcal{N}(0,1)$ the behavior of $\Delta(K)$ for large K can be obtained easily using de Bruijn identity type bound for entropy power [38, Eq. (15)]

$$N(Y(\mathcal{K})) \le N(X) + \frac{\sigma_Z^2}{K} \left(\frac{d}{dt} N(X + \sqrt{t}G)|_{t=0} \right). \tag{64}$$

More precisely, we obtain

$$0 \le \lim_{K \to \infty} \Delta(K) \le \frac{\sigma_Z^2}{2} \left[\left(\frac{\kappa_X}{N(X)} - \frac{1}{\sigma_X^2} \right) - e^{2E} \left(\frac{1}{N(X)} - \frac{1}{\sigma_X^2} \right) \right]^+ \tag{65a}$$

$$\leq \frac{\sigma_Z^2}{2} \left(\frac{\kappa_X}{N(X)} - \frac{1}{\sigma_Y^2} \right) \tag{65b}$$

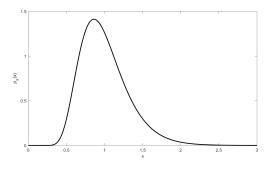
where the scalar coefficient κ_X is defined as

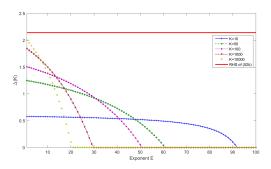
$$\kappa_X := \frac{d}{dt} N(X + \sqrt{t}G)|_{t=0}. \tag{66}$$

(Note that if X itself is Gaussian, $\kappa_X = 1$ and $N(X) = \sigma_X^2$). Figure 2 depicts the evolution of the RHS of (63) as a function of the exponent E for an example non-Gaussian distribution, the Wald distribution given by

 $p_X(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\frac{-\lambda(x-\mu)^2}{2\mu^2 x}, \quad x \in]0, +\infty[$ (67)

with the scale parameter μ set to 1 and the shape parameter λ set to 10. Also shown for comparison, the upper bound on the limit of $\Delta(K)$ at large K as given by (65b). Observe that $\Delta(K)$, and so the gap between our bounds (61) and (62), are relatively small for this example. Also, the gap is larger for larger values of K and smaller values of the exponent (intuitively, this is because the Gaussian part of Y(K), which is $(\sum_{i=1}^{K} Z_i)/K$, is weaker for increasing values of K).





- (a) Wald distribution (67) with $\mu = 1$ and $\lambda = 10$
- (b) Evolution of $\Delta(K)$ v.s. the exponent

Fig. 2: Evolution of $\Delta(K)$ as given by Eq. (50) as a function of the exponent E for the Wald distribution with scale parameter $\mu = 1$ and shape parameter $\lambda = 10$.

Consider now a setup with a single sensor that observes the vector $(Y_1, ..., Y_K)$. Given that Y(K) is a sufficient statistic for X given $(Y_1, ..., Y_K)$, this is equivalent to a point-to-point detection system with a sensor that has Y(K) and a detector that has X^n . Let K denote the rate needed to achieve exponent K for this setting. Using (61) and (50) we get that the cost of distributed processing (rate redundancy) is lower-bounded as

$$(R_{sum} - R) \ge \frac{1}{2} \log^{+} \left(\left(\frac{N(X)}{N(Y(\mathcal{K})) - \sigma_{Z(\mathcal{K})}^{2}} e^{2E} \right)^{K} \frac{\sigma_{Y(\mathcal{K})}^{2} - \sigma_{Z(\mathcal{K})}^{2}}{\sigma_{X}^{2}} \right)$$

$$= \frac{1}{2} \log^{+} \left(\left(\frac{N(X)}{N(Y(\mathcal{K})) - \frac{\sigma_{Z}^{2}}{K} e^{2E}} \right)^{K} \left(1 + \frac{\sigma_{Z}^{2}}{K \sigma_{X}^{2}} (1 - e^{2E}) \right) \right). \tag{68}$$

Appendix

Throughout this section we denote the set of strongly jointly ϵ -typical sequences [39, Chapter 14.2] with respect to the distribution $P_{X,Y}$ as $\mathcal{T}_{\epsilon}^{n}(P_{X,Y})$.

A. Proof of Proposition 1

i) Assume that $(R_1, ..., R_K, E) \in \mathcal{R}_{HT}$. Fix $\epsilon > 0$ and $\delta > 0$ and let a hypothesis test $(\phi_1^{(n)}, ..., \phi_K^{(n)}, \psi^{(n)}, \mathcal{A}_n)$ with Type-I probability of error $(1 - \alpha_n) \in [0, 1]$ and Type-II probability of error $\beta_n \in [0, 1]$ such that

$$\log \|\phi_k^{(n)}\| \le n(R_k + \delta), \qquad k = 1, \dots, K$$
 (A-1a)

$$\alpha_n \ge (1 - \epsilon)$$
 (A-1b)

$$-\frac{1}{n}\log\beta_n \ge E - \delta. \tag{A-1c}$$

First note that we have

$$\begin{split} D\Big(P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X^{n},Y_{0}^{n}} \|Q_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X^{n},Y_{0}^{n}}\Big) &\stackrel{(a)}{=} D\Big(P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}}\Big) \\ &+ \mathbb{E}_{P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}}} \Big[D\Big(P_{X^{n}|\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{X^{n}|\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}}}\Big) \Big] \\ &\stackrel{(b)}{=} D\Big(P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{X^{n}|Y_{0}^{n}}\Big) \\ &+ \mathbb{E}_{P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}}} \Big[D\Big(P_{X^{n}|\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{X^{n}|Y_{0}^{n}}\Big) \Big] \\ &\stackrel{(c)}{=} D\Big(P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{X^{n}|Y_{0}^{n}}\Big) \Big] \\ &\stackrel{(c)}{=} D\Big(P_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \|Q_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),Y_{0}^{n}} \Big) \\ &+ I(\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X_{0}^{n}} \|Q_{Y_{1}^{n},\dots,Y_{K}^{n},Y_{0}^{n}}\Big) \\ &\stackrel{(d)}{\leq} D\Big(P_{Y_{1},\dots,Y_{K}^{n},Y_{0}^{n}} \|Q_{Y_{1},\dots,Y_{K}^{n},Y_{0}^{n}}\Big) \\ &+ I(\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X_{0}^{n}} \|Q_{Y_{1},\dots,Y_{K}^{n},Y_{0}^{n}}\Big) \\ &\stackrel{(d)}{=} D\Big(P_{Y_{1},\dots,Y_{K},Y_{0}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &+ I(\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X_{0}^{n}) + n\mathbb{E}_{P_{Y_{0}}} \Big[D\Big(P_{X^{n}|Y_{0}^{n}} \|Q_{X^{n}|Y_{0}^{n}}\Big)\Big] \\ &\stackrel{(d)}{=} D\Big(P_{Y_{1},\dots,Y_{K},Y_{0}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &+ I(\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X_{0}^{n}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &+ I(\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X_{0}^{n}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &\stackrel{(d)}{=} D\Big(P_{Y_{1},\dots,Y_{K},Y_{0}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &\stackrel{(d)}{=} D\Big(P_{Y_{1},\dots,Y_{K},Y_{0}} \|Q_{Y_{1},\dots,Y_{K},Y_{0}}\Big) \\ &\stackrel{(d)$$

$$\stackrel{(f)}{=} I(\phi_1^{(n)}(Y_1^n), \dots, \phi_K^{(n)}(Y_K^n); X^n | Y_0^n)$$
(A-7)

where: (a) holds by the chain rule for KL divergence; (b) holds since X^n is independent of (Y_1^n, \ldots, Y_K^n) conditionally given Y_0^n under H_1 ; (c) follows by straightforward algebra; (d) holds by the data processing inequality; (e) holds since the vector $(X^n, Y_0^n, Y_1^n, \ldots, Y_K^n)$ is i.i.d. under both H_0 and H_1 ; and (f) holds since as per the condition (6) the distributions P and Q have same (X, Y_0) - and (Y_0, Y_1, \ldots, Y_K) -marginals.

Thus, for any $\epsilon' > 0$, provided that ϵ is small enough and n is large enough we get

$$I(\phi_1^{(n)}(Y_1^n), \dots, \phi_K^{(n)}(Y_K^n); X^n | Y_0^n) \stackrel{(a)}{=} D\Big(P_{\phi_1^{(n)}(Y_1^n), \dots, \phi_K^{(n)}(Y_K^n), X^n, Y_0^n} | | Q_{\phi_1^{(n)}(Y_1^n), \dots, \phi_K^{(n)}(Y_K^n), X^n, Y_0^n} \Big)$$
(A-8)

$$\stackrel{(b)}{\geq} \alpha_n \log \frac{\alpha_n}{\beta_n} + (1 - \alpha_n) \log \frac{1 - \alpha_n}{1 - \beta_n} \tag{A-9}$$

$$\stackrel{(c)}{=} -h(\alpha_n) - \alpha_n \log \beta_n - (1 - \alpha_n) \log(1 - \beta_n) \tag{A-10}$$

$$\stackrel{(d)}{\geq} (1 - \epsilon)n(E - \delta) - h(\alpha_n) - (1 - \alpha_n)\log(1 - \beta_n) \tag{A-11}$$

$$= n(E - \epsilon') \tag{A-12}$$

where (a) follows from (A-7); (b) holds by application of the log-sum inequality [35, Theorem 2.7.1]; in (c), for $u \in (0,1)$, h(u) denotes the entropy of a Bernoulli-(u) random variable, i.e.,

$$h(u) = -u \log u - (1 - u) \log(1 - u); \tag{A-13}$$

and (d) holds by using (A-1b) and (A-1c).

The inequalities (A-1a) and (A-12) together show that the tuple $(R_1 + \delta, ..., R_K + \delta, E - \epsilon') \in \mathcal{R}^*$; and, hence, $(R_1, ..., R_K, E) \in \overline{\mathcal{R}^*}$. Thus, $\mathcal{R}_{HT} \subseteq \overline{\mathcal{R}^*}$.

ii) Assume now that $(R_1, \ldots, R_K, E) \in \overline{\mathbb{R}^*}$. For any $\epsilon > 0$ and $\delta > 0$, since $(R_1 + \delta, \ldots, R_K + \delta, E - \delta) \in \mathbb{R}^*$ there must exist $p \in \mathbb{N}$ and functions $(f_1^{(p)}, \ldots, f_K^{(p)})$ such that

$$\log ||f_k^{(p)}|| \le p(R_k + \delta), \qquad k = 1, \dots, K$$
 (A-14)

$$E - \delta \le \frac{1}{p} I(f_1^{(p)}(Y_{1,1}^p), \dots, f_K^{(p)}(Y_{K,1}^p); X^p | Y_{0,1}^p). \tag{A-15}$$

By application of Stein's lemma to

$$H_{0}: \tilde{P}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}} \qquad H_{1}: \tilde{Q}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}}$$
(A-16)

where

$$\tilde{P}_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),X^p,Y_0^p} = P_{X^p,Y_0^p} \prod_{k=1}^K P_{f_k^{(p)}(Y_k^p)|X^p,Y_0^p}$$
(A-17a)

$$\tilde{Q}_{f_{i}^{(p)}(Y_{i}^{p}),...,f_{k}^{(p)}(Y_{k}^{p}),X_{i}^{p},Y_{0}^{p}} = Q_{Y_{0}^{p}}Q_{X_{i}^{p}|Y_{0}^{p}}Q_{f_{i}^{(p)}(Y_{i}^{p}),...,f_{k}^{(p)}(Y_{k}^{p})|Y_{0}^{p}},$$
(A-17b)

we get for every $\epsilon \in [0,1]$

$$\lim_{l \to \infty} \sup -\frac{1}{l} \log \beta_{\mathbf{R}}(lp, \epsilon) \ge D\left(\tilde{P}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}} || \tilde{Q}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}}\right)$$
(A-18)

where $\mathbf{R} \triangleq (R_1, \dots, R_K)$ and

$$\beta_{\mathbf{R}}(lp,\epsilon) \triangleq \min_{(g_1,\dots,g_K) : \log ||g_1|| \le lp(R_1+\delta),\dots,\log ||g_K|| \le lp(R_K+\delta)} \min_{\mathcal{A}} \left\{ \tilde{Q}_{g_1(Y_1^{lp}),\dots,g_K(Y_K^{lp}),X^{lp},Y_0^{lp}}(\mathcal{A}) \text{ s.t. } : \right.$$

$$\mathcal{A} \subset g_1(\mathcal{Y}_1^{lp}) \times \dots \times g_K(\mathcal{Y}_K^{lp}) \times \mathcal{X}^{lp} \times \mathcal{Y}_0^{lp}, \ \tilde{P}_{g_1(Y_1^{lp}),\dots,g_K(Y_K^{lp}),X^{lp},Y_0^{lp}}(\mathcal{A}) \ge 1 - \epsilon \right\}. \tag{A-19}$$

Let, for large n, functions $\phi_1^{(n)}, \ldots, \phi_K^{(n)}$ such that for $1 \le k \le K$ the function $\phi_k^{(n)}$ is defined over $\mathcal{Y}_{k,1}^n$ by concatenation from $f_k^{(p)}$ defined over $\mathcal{Y}_{k,1}^p$ as

$$\phi_k^{(n)}(y_{k,1},\ldots,y_{k,n}) \triangleq \left(f_k^{(p)}(y_{k,1},\ldots,y_{k,p}),\ldots,f_k^{(p)}(y_{k,(l-1)p+1},\ldots,y_{k,lp})\right), \quad lp \leq n \leq (l+1)p.$$
 (A-20)

Using (A-14) and (A-20), it is easy to see that

$$\log \|\phi_k^{(n)}\| \le n(R_k + \delta), \qquad k = 1, \dots, K.$$
 (A-21)

Also, noting that for $lp \le n \le (l+1)p$ we have

$$\beta_{\mathbf{R}}((l+1)p,\epsilon) \le \beta_{\mathbf{R}}(n,\epsilon) \le \beta_{\mathbf{R}}(lp,\epsilon)$$
 (A-22)

and using (A-18) it follows that

$$\lim_{n \to \infty} \sup -\frac{1}{n} \log \beta_{\mathbf{R}}(n, \epsilon) \ge D\left(\tilde{P}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}} || \tilde{Q}_{f_{1}^{(p)}(Y_{1}^{p}), \dots, f_{K}^{(p)}(Y_{K}^{p}), X^{p}, Y_{0}^{p}}\right)$$
(A-23)

$$\stackrel{(a)}{=} I(f_1^{(p)}(Y_1^p), \dots, f_K^{(p)}(Y_{K,1}^p); X^p | Y_{0,1}^p)$$
(A-24)

$$\stackrel{(b)}{\geq} E - \delta \tag{A-25}$$

where (a) follows by noticing that by the condition (6) the joint distributions $\tilde{P}_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),X^p,Y_0^p}$ and $\tilde{Q}_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),X^p,Y_0^p}$ as defined by (A-17) have same (X^p,Y_0^p) - and $(f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),Y_0^p)$ -marginals, i.e., $\tilde{P}_{X^p,Y_0^p} = \tilde{Q}_{X^p,Y_0^p} = P_{X^p,Y_0^p}$ and $\tilde{P}_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),Y_0^p} = \tilde{Q}_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),Y_0^p} = P_{f_1^{(p)}(Y_1^p),\dots,f_K^{(p)}(Y_K^p),Y_0^p}$ and then applying the steps leading to (A-7); and (b) holds by using (A-15).

Now, for convenience let us denote by $\phi^{(n)} = (\phi_1^{(n)}, \dots, \phi_K^{(n)})$ and

$$\beta(n,\epsilon,\boldsymbol{\phi}^{(n)}) \triangleq \min_{\mathcal{A}} \left\{ \tilde{\mathcal{Q}}_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X^{n},Y_{0}^{n}}(\mathcal{A}) \text{ s.t.} : \right.$$

$$\mathcal{A} \subset \phi_{1}^{(n)}(\mathcal{Y}_{1}^{n}) \times \dots \times \phi_{K}^{(n)}(\mathcal{Y}_{K}^{n}) \times \mathcal{X}^{n} \times \mathcal{Y}_{0}^{n}, \ \tilde{P}_{\phi_{1}^{(n)}(Y_{1}^{n}),\dots,\phi_{K}^{(n)}(Y_{K}^{n}),X^{n},Y_{0}^{n}}(\mathcal{A}) \geq 1 - \epsilon \right\}. \tag{A-26}$$

Noticing that as per (A-19) the term $\beta_{\mathbb{R}}(n,\epsilon)$ of the LHS of (A-25) involves a minimization over all functions (g_1,\ldots,g_K) for which $\|g_k\| \leq n(R_k+\delta)$, $k=1,\ldots,K$, and recalling that the functions $(\phi_1^{(n)},\ldots,\phi_K^{(n)})$ as defined by (A-20) satisfy (A-21), then by (A-25) we get

$$\lim_{n \to \infty} \sup -\frac{1}{n} \log \beta(n, \epsilon, \phi^{(n)}) \ge E - \delta. \tag{A-27}$$

Finally, using (A-21) and (A-27) it follows that $(R_1, \ldots, R_K, E) \in \mathcal{R}_{HT}$. Thus, $\overline{\mathcal{R}^{\star}} \subseteq \mathcal{R}_{HT}$.

B. Proof of Converse of Theorem 1

Let a non-negative tuple $(R_1, ..., R_K, E) \in \mathcal{R}_{HT}$ be given. Since $\mathcal{R}_{HT} = \overline{\mathcal{R}^*}$, then there must exist a series of non-negative tuples $\{(R_1^{(m)}, ..., R_K^{(m)}, E^{(m)})\}_{m \in \mathbb{N}}$ such that

$$(R_1^{(m)}, \dots, R_K^{(m)}, E^{(m)}) \in \mathcal{R}^*$$
 for all $m \in \mathbb{N}$, and (B-1a)

$$\lim_{m \to \infty} (R_1^{(m)}, \dots, R_K^{(m)}, E^{(m)}) = (R_1, \dots, R_K, E).$$
 (B-1b)

Fix $\delta' > 0$. Then, $\exists m_0 \in \mathbb{N}$ such that for all $m \ge m_0$, we have

$$R_k \ge R_k^{(m)} - \delta'$$
 for all $k \in \mathcal{K}$, and (B-2a)

$$E \le E^{(m)} + \delta'. \tag{B-2b}$$

For $m \ge m_0$, there exist a series $\{n_m\}_{m \in \mathbb{N}}$ and functions $\{\phi_k^{(n_m)}\}_{k \in \mathcal{K}}$ such that

$$R_k^{(m)} \ge \frac{1}{n_m} \log |\phi_k^{(n_m)}| \text{ for all } k \in \mathcal{K}, \text{ and}$$
 (B-3a)

$$E^{(m)} \le \frac{1}{n_m} I(\{\phi_k^{(n_m)}(Y_k^{n_m})\}_{k \in \mathcal{K}}; X^{n_m} | Y_0^{n_m}).$$
 (B-3b)

Combining (B-2) and (B-3) we get that for all $m \ge m_0$,

$$R_k \ge \frac{1}{n_m} \log |\phi_k^{(n_m)}(Y_k^{n_m})| - \delta' \text{ for all } k \in \mathcal{K}, \text{ and}$$
 (B-4a)

$$E \le \frac{1}{n_m} I(\{\phi_k^{(n_m)}(Y_k^{n_m})\}_{k \in \mathcal{K}}; X^{n_m} | Y_0^{n_m}) + \delta'.$$
(B-4b)

The second inequality of (B-4) implies that

$$H(X^{n_m}|\{\phi_k^{(n_m)}(Y_k^{n_m})\}_{k\in\mathcal{K}}, Y_0^{n_m}) \le n_m(H(X|Y_0) - E) + n_m\delta'.$$
(B-5)

Let $S \subseteq \mathcal{K}$ a given subset of \mathcal{K} and $J_k := \phi_k^{(n_m)}(Y_k^{n_m})$. Also, define, for $i = 1, ..., n_m$, the following auxiliary random variables

$$U_{k,i} := (J_k, Y_k^{i-1}), \quad Q_i := (X^{i-1}, X_{i+1}^{n_m}, Y_0^{i-1}, Y_{0,i+1}^{n_m}).$$
 (B-6)

Note that, for all $k \in \mathcal{K}$, it holds that $U_{k,i} - Y_{k,i} - (X_i, Y_{0,i}) - Y_{\mathcal{K} \setminus k,i} - U_{\mathcal{K} \setminus k,i}$ is a Markov chain in this order.

We have

$$n_{m} \sum_{k \in S} R_{k} \geq \sum_{k \in S} H(J_{k})$$

$$\geq H(J_{S}|J_{S^{c}}, Y_{0}^{n_{m}})$$

$$\geq I(J_{S}; X^{n_{m}}, Y_{S}^{n_{m}}|J_{S^{c}}, Y_{0}^{n_{m}})$$

$$= I(J_{S}; X^{n_{m}}|J_{S^{c}}, Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}})$$

$$= H(X^{n_{m}}|J_{S^{c}}, Y_{0}^{n_{m}}) - H(X^{n_{m}}|J_{K}, Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}})$$

$$\stackrel{(a)}{\geq} H(X^{n_{m}}|J_{S^{c}}, Y_{0}^{n_{m}}) - H(X^{n_{m}}|Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) + n_{m}E - n_{m}\delta'$$

$$= \sum_{i=1}^{n_{m}} H(X_{i}|J_{S^{c}}, X^{i-1}, Y_{0}^{n_{m}}) - H(X^{n_{m}}|Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) + n_{m}E - n_{m}\delta'$$

$$\stackrel{(b)}{\geq} \sum_{i=1}^{n_{m}} H(X_{i}|J_{S^{c}}, X^{i-1}, X_{i+1}^{n_{m}}, Y_{S^{c}}^{i-1}, Y_{0}^{n_{m}}) - H(X^{n_{m}}|Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) + n_{m}E - n_{m}\delta'$$

$$\stackrel{(c)}{=} \sum_{i=1}^{n_{m}} H(X_{i}|U_{S^{c},i}, Y_{0,i}, Q_{i}) - H(X^{n_{m}}|Y_{0}^{n_{m}}) + I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) + n_{m}E - n_{m}\delta'$$

$$\stackrel{(d)}{=} I(J_{S}; Y_{S}^{n_{m}}|X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) - \sum_{i=1}^{n_{m}} I(U_{S^{c},i}, X_{i}|Y_{0,i}, Q_{i}) + n_{m}E - n_{m}\delta'$$
(B-7)

where (a) follows by using (B-5); (b) holds since conditioning reduces entropy; and (c) follows by substituting using (B-6); and (d) holds since $(X^{n_m}, Y_0^{n_m})$ is memoryless and Q_i is independent of $(X_i, Y_{0,i})$ for all $i = 1, ..., n_m$.

The term $I(J_S; Y_S^{n_m} | X^{n_m}, J_{S^c}, Y_0^{n_m})$ on the RHS of (B-7) can be lower bounded as

$$I(J_{S}; Y_{S}^{n_{m}} | X^{n_{m}}, J_{S^{c}}, Y_{0}^{n_{m}}) \stackrel{(a)}{\geq} \sum_{k \in S} I(J_{k}; Y_{k}^{n_{m}} | X^{n_{m}}, Y_{0}^{n_{m}})$$

$$= \sum_{k \in S} \sum_{i=1}^{n_{m}} I(J_{k}; Y_{k,i} | Y_{k}^{i-1}, X^{n_{m}}, Y_{0}^{n_{m}})$$

$$\stackrel{(b)}{=} \sum_{k \in S} \sum_{i=1}^{n_{m}} I(J_{k}, Y_{k}^{i-1}; Y_{k,i} | X^{n_{m}}, Y_{0}^{n_{m}})$$

$$\stackrel{(c)}{=} \sum_{k \in S} \sum_{i=1}^{n_{m}} I(U_{k,i}; Y_{k,i} | X_{i}, Y_{0,i}, Q_{i})$$
(B-8)

where (a) follows due to the Markov chain $J_k \leftrightarrow Y_k^{n_m} \leftrightarrow (X^{n_m}, Y_0^{n_m}) \leftrightarrow Y_{S \setminus k}^{n_m} \leftrightarrow J_{S \setminus k}$; (b) follows due to the Markov chain $Y_{k,i} \leftrightarrow (X^{n_m}, Y_0^{n_m}) \leftrightarrow Y_k^{i-1}$; and (c) follows by substituting using (B-6).

Then, combining (B-7) and (B-8), we get

$$n_m E \le \sum_{i=1}^{n_m} I(U_{S^c,i}, X_i | Y_{0,i}, Q_i) + n_m \sum_{k \in S} R_k - \sum_{k \in S} \sum_{i=1}^{n_m} I(U_{k,i}; Y_{k,i} | X_i, Y_{0,i}, Q_i) + n_m \delta'.$$
 (B-9)

Noticing that δ' in (B-9) can be chosen arbitrarily small, a standard time-sharing argument completes the proof of the converse part.

C. Proof of Theorem 2

First note that a characterization (in terms of auxiliaries) of the rate-exponent region of the memoryless vector Gaussian hypothesis testing against conditional independence problem of Section IV, obtained by an easy extension of the result of Theorem 1 to the continuous alphabet case through standard discretization arguments), is given by the union of all non-negative tuples $(R_1, ..., R_K, E)$ that satisfy for all $S \subseteq K$,

$$E - \sum_{k \in S} R_k \le I(U_{S^c}; \mathbf{X} | \mathbf{Y}_0, Q) - \sum_{k \in S} I(\mathbf{Y}_k; U_k | \mathbf{X}, \mathbf{Y}_0, Q)$$
(C-1)

for some joint distribution of the form that factorizes as

$$P_{\mathbf{X},\mathbf{Y}_{0},\mathbf{Y}_{K},U_{K},Q}(\mathbf{x},\mathbf{y}_{0},\mathbf{y}_{K},u_{K},q) = P_{Q}(q)P_{\mathbf{X},\mathbf{Y}_{0}}(\mathbf{x},\mathbf{y}_{0})$$

$$\times \prod_{k=1}^{K} P_{\mathbf{Y}_{k}|\mathbf{X},\mathbf{Y}_{0}}(\mathbf{y}_{k}|\mathbf{x},\mathbf{y}_{0}) \prod_{k=1}^{K} P_{U_{k}|\mathbf{Y}_{k},Q}(u_{k}|\mathbf{y}_{k},q). \tag{C-2}$$

1) Converse part: Let an achievable tuple $(R_1, ..., R_K, E)$ be given. Using the above there must exist auxiliary random variables $(U_1, ..., U_K, Q)$ with distribution that factorizes as (C-2) such that (C-1) holds for any subset $S \subseteq \mathcal{K}$. The converse proof of Theorem 2 relies on deriving an upper bound on the RHS of (C-1). In doing so, we use the technique of [31, Theorem 8] which relies on the de Bruijn identity and the properties of Fisher information; and extend the argument to account for the time-sharing variable Q and side information \mathbf{Y}_0 .

For convenience, we first state the following lemma.

Lemma 1. [31], [40] Let (X, Y) be a pair of random vectors with pmf p(x, y). We have

$$\log |(\pi e)\mathbf{J}^{-1}(\mathbf{X}|\mathbf{Y})| \le h(\mathbf{X}|\mathbf{Y}) \le \log |(\pi e) \text{mmse}(\mathbf{X}|\mathbf{Y})|$$

where the conditional Fisher information matrix is defined as

$$\mathbf{J}(\mathbf{X}|\mathbf{Y}) := \mathbb{E}\left[\nabla \log p(\mathbf{X}|\mathbf{Y})\nabla \log p(\mathbf{X}|\mathbf{Y})^{\dagger}\right]$$

and the minimum mean squared error (MMSE) matrix is

$$\text{mmse}(X|Y) := \mathbb{E}\left[\left(X - \mathbb{E}[X|Y]\right)\left(X - \mathbb{E}[X|Y]\right)^{\dagger}\right]. \quad \Box$$

Fix $q \in Q$ and $S \subseteq Q$. There must exists a matrix $\Omega_{k,q}$ such that $0 \le \Omega_{k,q} \le \Sigma_k^{-1}$ and

$$\operatorname{mmse}\left(\mathbf{Y}_{k}|\mathbf{X},U_{k,q},\mathbf{Y}_{0},q\right)=\Sigma_{k}-\Sigma_{k}\Omega_{k,q}\Sigma_{k}.\tag{C-3}$$

It is easy to see that such $\Omega_{k,q}$ always exists since

$$\mathbf{0} \le \operatorname{mmse} \left(\mathbf{Y}_{k} | \mathbf{X}, U_{k,q}, \mathbf{Y}_{0}, q \right) \le \mathbf{\Sigma}_{\mathbf{y}_{k} | (\mathbf{x}, \mathbf{y}_{0})} = \mathbf{\Sigma}_{k}. \tag{C-4}$$

Then, we have

$$I(\mathbf{Y}_{k}; U_{k}|\mathbf{X}, \mathbf{Y}_{0}, Q = q) = \log |(\pi e)\mathbf{\Sigma}_{k}| - h(\mathbf{Y}_{k}|\mathbf{X}, U_{k,q}, \mathbf{Y}_{0}, Q = q)$$

$$\stackrel{(a)}{\geq} \log |\mathbf{\Sigma}_{k}| - \log \left| \text{mmse}(\mathbf{Y}_{k}|\mathbf{X}, U_{k,q}, \mathbf{Y}_{0}, Q = q) \right|$$

$$\stackrel{(b)}{=} - \log \left| \mathbf{I} - \mathbf{\Omega}_{k,q} \mathbf{\Sigma}_{k} \right|$$
(C-5)

where (a) is due to Lemma 1; and (b) is due to (C-3).

Now, let the matrix $\Lambda_{\bar{\mathcal{S}},q}$ be defined as

$$\Lambda_{\bar{S},q} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\{\Sigma_k - \Sigma_k \mathbf{\Omega}_{k,q} \Sigma_k\}_{k \in S^c}) \end{bmatrix}. \tag{C-6}$$

Then, we have

$$I(U_{S^{c}}; \mathbf{X}|\mathbf{Y}_{0}, Q = q) = h(\mathbf{X}|\mathbf{Y}_{0}) - h\left(\mathbf{X}|U_{S^{c},q}, \mathbf{Y}_{0}, Q = q\right)$$

$$\stackrel{(a)}{\leq} h(\mathbf{X}|\mathbf{Y}_{0}) - \log\left|(\pi e)\mathbf{J}^{-1}\left(\mathbf{X}|\mathbf{U}_{S^{c},q}, \mathbf{Y}_{0}, q\right)\right|$$

$$\stackrel{(b)}{=} h(\mathbf{X}|\mathbf{Y}_{0}) - \log\left|(\pi e)\left(\boldsymbol{\Sigma}_{\mathbf{x}}^{-1} + \mathbf{H}_{\tilde{\mathbf{S}}}^{+}\boldsymbol{\Sigma}_{\mathbf{n}_{\tilde{\mathbf{S}}}}^{-1}\left(\mathbf{I} - \boldsymbol{\Lambda}_{\tilde{\mathbf{S}},q}\boldsymbol{\Sigma}_{\mathbf{n}_{\tilde{\mathbf{S}}}}^{-1}\right)\mathbf{H}_{\tilde{\mathbf{S}}}\right)^{-1}\right|$$
(C-7)

where (a) follows by using Lemma 1; and for (b) holds by using the equality

$$\mathbf{J}(\mathbf{X}|U_{S^{c},q},\mathbf{Y}_{0},q) = \mathbf{\Sigma}_{\mathbf{x}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S},q} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}}. \tag{C-8}$$

the proof of which uses a connection between MMSE and Fisher information as shown next. More precisely, for the proof of (C-8) first recall de Brujin identity which relates Fisher information and MMSE.

Lemma 2. [31] Let (V_1, V_2) be a random vector with finite second moments and $\mathbf{Z} \sim \mathcal{CN}(\mathbf{0}, \Sigma_z)$ independent of (V_1, V_2) . Then

mmse
$$(\mathbf{V}_2|\mathbf{V}_1,\mathbf{V}_2+\mathbf{Z}) = \Sigma_{\mathbf{z}} - \Sigma_{\mathbf{z}} \mathbf{J} (\mathbf{V}_2+\mathbf{Z}|\mathbf{V}_1) \Sigma_{\mathbf{z}}$$
.

From MMSE estimation of Gaussian random vectors, we have

$$X = \mathbb{E}[X|Y_{\bar{S}}] + W_{\bar{S}}$$

$$= G_{\bar{S}}Y_{\bar{S}} + W_{\bar{S}}$$
(C-9)

where $G_{\bar{S}} := \Sigma_{w_{\bar{S}}} H^{\dagger}_{\bar{S}} \Sigma_{n_{\bar{S}}'}^{-1}$ and $W_{\bar{S}} \sim \mathcal{CN}(0, \Sigma_{w_{\bar{S}}})$ is a Gaussian vector that is independent of $Y_{\bar{S}}$ and

$$\Sigma_{\mathbf{w}_{\tilde{\mathbf{S}}}}^{-1} := \Sigma_{\mathbf{x}}^{-1} + \mathbf{H}_{\tilde{\mathbf{S}}}^{\dagger} \Sigma_{\mathbf{n}_{\tilde{\mathbf{S}}}}^{-1} \mathbf{H}_{\tilde{\mathbf{S}}}. \tag{C-10}$$

Next, we show that the cross-terms of mmse $(\mathbf{Y}_{S^c}|\mathbf{X},U_{S^c,q},\mathbf{Y}_0,q)$ are zero. For $i \in S^c$ and $j \neq i$, we have

$$\mathbb{E}\left[\left(Y_{i} - \mathbb{E}[Y_{i}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q]\right)\left(Y_{j} - \mathbb{E}[Y_{j}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q]\right)^{\dagger}\right]$$

$$\stackrel{(a)}{=} \mathbb{E}\left[\mathbb{E}\left[\left(Y_{i} - \mathbb{E}[Y_{i}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q]\right)\left(Y_{j} - \mathbb{E}[Y_{j}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q]\right)^{\dagger}|\mathbf{X}, \mathbf{Y}_{0}\right]\right]$$

$$\stackrel{(b)}{=} \mathbb{E}\left[\mathbb{E}\left[\left(Y_{i} - \mathbb{E}\left[Y_{i}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q\right]\right)|\mathbf{X}, \mathbf{Y}_{0}\right] \times \mathbb{E}\left[\left(Y_{j} - \mathbb{E}[Y_{j}|\mathbf{X}, U_{\mathcal{S}^{c},q}, \mathbf{Y}_{0}, q]\right)^{\dagger}|\mathbf{X}, \mathbf{Y}_{0}\right]\right]$$

$$= \mathbf{0}, \tag{C-11}$$

where (a) is due to the law of total expectation; (b) is due to the Markov chain $\mathbf{Y}_k \leftrightarrow (\mathbf{X}, \mathbf{Y}_0) \leftrightarrow \mathbf{Y}_{K \setminus k}$. Then, we have

$$\operatorname{mmse}(\mathbf{G}_{\bar{S}}\mathbf{Y}_{\bar{S}}|\mathbf{X}, U_{S^{c},q}, \mathbf{Y}_{0}, q) = \mathbf{G}_{\bar{S}} \operatorname{mmse}(\mathbf{Y}_{\bar{S}}|\mathbf{X}, U_{S^{c},q}, \mathbf{Y}_{0}, q) \mathbf{G}_{\bar{S}}^{\dagger}$$

$$\stackrel{(a)}{=} \mathbf{G}_{\bar{S}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\{\operatorname{mmse}(\mathbf{Y}_{k}|\mathbf{X}, U_{S^{c},q}, \mathbf{Y}_{0}, q)\}_{k \in S^{c}}) \end{bmatrix} \mathbf{G}_{\bar{S}}^{\dagger}$$

$$\stackrel{(b)}{=} \mathbf{G}_{\bar{S}} \mathbf{\Lambda}_{\bar{S},q} \mathbf{G}_{\bar{S}}^{\dagger} \qquad (C-12)$$

where (a) follows since the cross-terms are zero as shown in (C-11); and (b) follows due to (C-3) and the definition of $\Lambda_{\bar{S},q}$ given in (C-6).

We note that $\mathbf{W}_{\tilde{S}}$ is independent of $\mathbf{Y}_{\tilde{S}} = (\mathbf{Y}_0, \mathbf{Y}_{S^c})$; and, with the Markov chain $U_{S^c} \leftrightarrow \mathbf{Y}_{S^c} \leftrightarrow (\mathbf{X}, \mathbf{Y}_0)$, which itself implies $U_{S^c} \leftrightarrow \mathbf{Y}_{S^c} \leftrightarrow (\mathbf{X}, \mathbf{Y}_0, \mathbf{W}_{\tilde{S}})$, this yields that $\mathbf{W}_{\tilde{S}}$ is independent of U_{S^c} . Thus, $\mathbf{W}_{\tilde{S}}$ is independent of $(\mathbf{G}_{\tilde{S}}\mathbf{Y}_{\tilde{S}}, U_{S^c}, \mathbf{Y}_0, Q)$. Applying Lemma 2 with

$$\mathbf{V}_1 := (U_{\mathcal{S}^c}, \mathbf{Y}_0, Q) \tag{C-13a}$$

$$V2 := G_{\bar{S}}Y_{\bar{S}} \tag{C-13b}$$

$$\mathbf{Z} := \mathbf{W}_{\tilde{\mathbf{S}}},\tag{C-13c}$$

we get

$$\begin{split} \mathbf{J}(\mathbf{X}|U_{S^{c},q},\mathbf{Y}_{0},q) &= \boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1} - \boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1} \text{ mmse} \big(\mathbf{G}_{\bar{S}}\mathbf{Y}_{\bar{S}}^{-1}\big|\mathbf{X},U_{S^{c},q},\mathbf{Y}_{0},q\big)\boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1} \\ &\stackrel{(a)}{=} \boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1} - \boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1}\mathbf{G}_{\bar{S}}\boldsymbol{\Lambda}_{\bar{S},q}\mathbf{G}_{\bar{S}}^{\dagger}\boldsymbol{\Sigma}_{\mathbf{w}_{S}}^{-1} \\ &\stackrel{(b)}{=} \boldsymbol{\Sigma}_{\mathbf{x}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger}\boldsymbol{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1}\mathbf{H}_{\bar{S}} - \mathbf{H}_{\bar{S}}^{\dagger}\boldsymbol{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1}\boldsymbol{\Lambda}_{\bar{S},q}\boldsymbol{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1}\mathbf{H}_{\bar{S}} \\ &= \boldsymbol{\Sigma}_{\mathbf{x}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger}\boldsymbol{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \big(\mathbf{I} - \boldsymbol{\Lambda}_{\bar{S},q}\boldsymbol{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1}\big)\mathbf{H}_{\bar{S}}, \end{split}$$

where (a) is due to (C-12); and (b) follows due to the definitions of $\Sigma_{\mathbf{w}_{S}}^{-1}$ and $\mathbf{G}_{\bar{S}}$. This proves (C-8). Next, averaging over the time sharing random variable Q both sides of the inequalities (C-5) and (C-7) and letting $\Omega_{k} := \sum_{q \in Q} p(q) \Omega_{k,q}$ we get

$$I(\mathbf{Y}_{k}; \mathbf{U}_{k} | \mathbf{X}, \mathbf{Y}_{0}, Q) = \sum_{q \in Q} p(q)I(\mathbf{Y}_{k}; \mathbf{U}_{k} | \mathbf{X}, \mathbf{Y}_{0}, Q = q)$$

$$\stackrel{(a)}{\geq} - \sum_{q \in Q} p(q) \log |\mathbf{I} - \mathbf{\Omega}_{k,q} \mathbf{\Sigma}_{k}|$$

$$\stackrel{(b)}{\geq} - \log |\mathbf{I} - \sum_{q \in Q} p(q)\mathbf{\Omega}_{k,q} \mathbf{\Sigma}_{k}|$$

$$= -\log |\mathbf{I} - \mathbf{\Omega}_{k} \mathbf{\Sigma}_{k}|$$
(C-14)

where (*a*) follows from (C-5); and (*b*) follows from the concavity of the log-det function and Jensen's Inequality.

Besides, we have

$$I(U_{\mathcal{S}^c}; \mathbf{X} | \mathbf{Y}_0, Q) = h(\mathbf{X} | \mathbf{Y}_0) - \sum_{q \in Q} p(q) h(\mathbf{X} | U_{\mathcal{S}^c, q}, \mathbf{Y}_0, Q = q)$$

$$\stackrel{(a)}{\leq} h(\mathbf{X}|\mathbf{Y}_{0}) - \sum_{q \in Q} p(q) \log \left| (\pi e) \left(\mathbf{\Sigma}_{\mathbf{X}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S}, q} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \right)^{-1} \right| \\
\stackrel{(b)}{\leq} h(\mathbf{X}|\mathbf{Y}_{0}) - \log \left| (\pi e) \left(\mathbf{\Sigma}_{\mathbf{X}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S}} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \right)^{-1} \right|, \tag{C-15}$$

where (*a*) is due to (C-7); and (*b*) is due to the concavity of the log-det function and Jensen's inequality and the definition of $\Lambda_{\bar{S}}$ given in (33).

Using (C-15), we get

$$I(U_{S^{c}}; \mathbf{X}|\mathbf{Y}_{0}, Q) = \mathbb{E}_{Q} \Big[I(U_{S^{c}}; \mathbf{X}|\mathbf{Y}_{0}, Q = q) \Big]$$

$$\leq h(\mathbf{X}|\mathbf{Y}_{0}) - \log \Big| (\pi e) \left(\mathbf{\Sigma}_{\mathbf{X}}^{-1} + \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S}} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \right)^{-1} \Big|$$

$$= \log \Big| \mathbf{I} + \mathbf{\Sigma}_{\mathbf{X}} \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S}} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \Big| - I(\mathbf{X}; \mathbf{Y}_{0})$$

$$= \log \Big| \mathbf{I} + \mathbf{\Sigma}_{\mathbf{X}} \mathbf{H}_{\bar{S}}^{\dagger} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \mathbf{\Lambda}_{\bar{S}} \mathbf{\Sigma}_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \Big| - \log \Big| \mathbf{I} + \mathbf{\Sigma}_{\mathbf{X}} \mathbf{H}_{0}^{\dagger} \mathbf{\Sigma}_{0}^{-1} \mathbf{H}_{0} \Big|.$$
 (C-16)

Finally, substituting in (C-1) using (C-14) and (C-16) we get (34). The proof of the converse terminates by taking the union over all matrices Ω_k and observing that those satisfy $\Omega_k = \sum_{q \in Q} p(q) \Omega_{k,q} \leq \Sigma_k^{-1}$ since $\mathbf{0} \leq \Omega_{k,q} \leq \Sigma_k^{-1}$ for all $k \in \mathcal{K}$.

2) Direct part: The proof of the direct part follows by evaluating the region described by (C-1) and (C-2) using Gaussian test channels and no time-sharing. Specifically, we set $Q = \emptyset$ and

$$\mathbf{U}_k = \mathbf{Y}_k + \mathbf{V}_k \tag{C-17a}$$

$$= \mathbf{H}_k \mathbf{X} + \mathbf{Z}_k + \mathbf{V}_k \tag{C-17b}$$

where the noise V_k is zero-mean Gaussian with covariance matrix

$$\Gamma_k = \left[\left(\mathbf{I} - \mathbf{\Omega}_k \mathbf{\Sigma}_k \right)^{-1} - \mathbf{I} \right]^{-1} \mathbf{\Sigma}_k \tag{C-18}$$

for some matrix Ω_k such that $0 \le \Omega_k \le \Sigma_k^{-1}$; and is independent of Y_k and of other noises. Specifically, using such choice $(U_1, \ldots, U_K, X, Y_0, Y_1, \ldots, Y_K)$ is jointly Gaussian and we get

$$I(\mathbf{Y}_{k}; \mathbf{U}_{k} | \mathbf{X}, \mathbf{Y}_{0}) \stackrel{(a)}{=} h(\mathbf{Z}_{k} + \mathbf{V}_{k} | \mathbf{X}, \mathbf{Y}_{0}) - h(\mathbf{V}_{k} | \mathbf{X}, \mathbf{Y}_{0}, \mathbf{Y}_{k})$$
(C-19)

$$\stackrel{(b)}{=} h(\mathbf{Z}_k + \mathbf{V}_k | \mathbf{X}, \mathbf{Y}_0, \mathbf{Z}_0) - h(\mathbf{V}_k | \mathbf{X}, \mathbf{Y}_0, \mathbf{Y}_k, \mathbf{Z}_0, \mathbf{Z}_k)$$
(C-20)

$$\stackrel{(c)}{=} h(\mathbf{Z}_k + \mathbf{V}_k | \mathbf{X}, \mathbf{Z}_0) - h(\mathbf{V}_k | \mathbf{X}, \mathbf{Z}_0, \mathbf{Z}_k)$$
 (C-21)

$$\stackrel{(d)}{=} h(\mathbf{Z}_k + \mathbf{V}_k | \mathbf{Z}_0) - h(\mathbf{V}_k | \mathbf{Z}_0, \mathbf{Z}_k)$$
 (C-22)

$$\stackrel{(e)}{=} h(\mathbf{Z}_k + \mathbf{V}_k | \mathbf{Z}_0) - h(\mathbf{V}_k) \tag{C-23}$$

$$= \log |(\pi e) \text{mmse}(\mathbf{Z}_k + \mathbf{V}_k | \mathbf{Z}_0)| - \log |(\pi e) \mathbf{\Gamma}_k|$$
 (C-24)

$$= \log |\Sigma_k + \Gamma_k| - \log |\Gamma_k| \tag{C-25}$$

$$\stackrel{(f)}{=} -\log|\mathbf{I} - \mathbf{\Omega}_k \mathbf{\Sigma}_k| \tag{C-26}$$

where: (a) follows by substituting using (C-17); (b) and (c) hold using (29) and (30); (d) holds since $(\mathbf{Z}_0, \mathbf{Z}_k, \mathbf{V}_k)$ is independent of $(\mathbf{X}_0, \mathbf{Z}_k)$; (f) follows by substituting using (C-18).

Similarly, for given $S \subseteq \mathcal{K}$ we have

$$I(\mathbf{U}_{\mathcal{S}^c}; \mathbf{X}|\mathbf{Y}_0) \stackrel{(a)}{=} h(\mathbf{U}_{\mathcal{S}^c}|\mathbf{Y}_0) - h(\mathbf{Z}_{\mathcal{S}^c} + \mathbf{V}_{\mathcal{S}^c}|\mathbf{X}, \mathbf{Y}_0)$$
(C-27)

$$\stackrel{(b)}{=} h(\mathbf{U}_{S^c}|\mathbf{Y}_0) - h(\mathbf{Z}_{S^c} + \mathbf{V}_{S^c}|\mathbf{X}, \mathbf{Y}_0, \mathbf{Z}_0)$$
 (C-28)

$$\stackrel{(c)}{=} h(\mathbf{U}_{\mathcal{S}^c}|\mathbf{Y}_0) - h(\mathbf{Z}_{\mathcal{S}^c} + \mathbf{V}_{\mathcal{S}^c}|\mathbf{Z}_0) \tag{C-29}$$

$$\stackrel{(b)}{=} h(\mathbf{U}_{\mathcal{S}^c}|\mathbf{Y}_0) - \log|(\pi e)(\mathbf{\Sigma}_{\mathcal{S}^c} + \mathbf{\Gamma}_{\mathcal{S}^c})| \tag{C-30}$$

$$\stackrel{(d)}{=} h(\mathbf{Y}_{\mathcal{S}^c} + \mathbf{V}_{\mathcal{S}^c} | \mathbf{Y}_0) - \log |(\pi e)(\mathbf{\Sigma}_{\mathcal{S}^c} + \mathbf{\Gamma}_{\mathcal{S}^c})| \tag{C-31}$$

$$\stackrel{(e)}{=} \log \left| \mathbf{I} + \Sigma_{\mathbf{x}} \mathbf{H}_{\bar{S}}^{\dagger} \Sigma_{\mathbf{n}_{\bar{S}}}^{-1} \left(\mathbf{I} - \Lambda_{\bar{S}} \Sigma_{\mathbf{n}_{\bar{S}}}^{-1} \right) \mathbf{H}_{\bar{S}} \right| - \log \left| \mathbf{I} + \Sigma_{\mathbf{x}} \mathbf{H}_{0}^{\dagger} \Sigma_{0}^{-1} \mathbf{H}_{0} \right|$$
 (C-32)

here: (a) follows by substituting using (C-17); (b) holds using (29); (c) holds using \mathbf{Y}_0 is a deterministic function of $(\mathbf{X}, \mathbf{Z}_0)$ and $(\mathbf{Z}_{S^c}, \mathbf{V}_{S^c}, \mathbf{Z}_0)$ is independent of \mathbf{X} ; (d) follows by substituting using (C-17); and (e) holds using (C-18) and straightforward algebra which is omitted here for brevity.

Finally, substituting in (C-1) using (C-26) and (C-32) we get (34); and this completes the proof of the direct part of Theorem 2.

D. Proof of the Inequality (53)

Since Y = X + Z with Z Gaussian and independent from X, invoking the strong entropy power inequality of Courtade [38, Theorem 1] gives

$$e^{2\left[h(Y)-I(X;U)\right]} \ge e^{2\left[h(X)-I(U;Y)\right]} + e^{2h(Z)}.$$
 (D-1)

Thus, we get

$$I(U;X) \le h(Y) - \frac{1}{2} \log \left(e^{2\left[h(X) - I(U;Y)\right]} + e^{2h(Z)} \right).$$
 (D-2)

Using (49), we have

$$E(R) = \max_{P_{U|Y}: I(U;Y) \le R} I(U;X)$$
 (D-3)

$$\stackrel{(a)}{\leq} \max_{P_{U|Y}: I(U;Y) \leq R} \ h(Y) - \frac{1}{2} \log \left(e^{2\left[h(X) - I(U;Y)\right]} + e^{2h(Z)} \right) \tag{D-4}$$

$$\stackrel{(b)}{\leq} \max_{P_{U|Y}: I(U;Y) \leq R} \ h(Y) - \frac{1}{2} \log \left(e^{2\left[h(X) - R\right]} + e^{2h(Z)} \right) \tag{D-5}$$

$$= h(Y) - \frac{1}{2} \log \left(e^{2\left[h(X) - R\right]} + e^{2h(Z)} \right)$$
 (D-6)

$$\stackrel{(c)}{=} \frac{1}{2} \log \left(\frac{N(Y)}{N(X)e^{-2R} + \sigma_Z^2} \right) \tag{D-7}$$

where (a) holds by using (D-2), (b) holds by using that $I(U;Y) \le R$ and (c) holds by substituting using (42).

E. Proof of Theorem 3

Recall the result of Theorem 1. Specializing it to the model described by (43) and (44), we get that the region \mathcal{R}_{HT} is given by the union of all non-negative tuples (R_1, \ldots, R_K, E) that satisfy, for all subsets $S \subseteq \mathcal{K}$,

$$E \le I(U_{S^c}; X|Q) + \sum_{k \in S} (R_k - I(Y_k; U_k|X, Q))$$
 (E-1)

for some auxiliary random variables (U_1, \ldots, U_K, Q) with distribution $P_{U_K,Q}(u_K,q)$ such that

$$P_{X,Y_K,U_K,Q} = P_Q P_X \prod_{k=1}^K P_{Y_k|X} \prod_{k=1}^K P_{U_k|Y_k,Q}.$$
 (E-2)

Let $S \subseteq \mathcal{K}$ be given, and for $k \in \mathcal{K}$ define

$$\gamma_k = \frac{1}{\sigma_k^2} \left(1 - e^{-2I(U_k; Y_k | X, Q)} \right).$$
 (E-3)

Note that for all $k \in \mathcal{K}$, we have $0 \le \gamma_k \le 1/\sigma_k^2$.

If S = K, it is easy to see that (55) follows directly from (E-1) using the substitution (E-3). In the rest of the proof, we therefore suppose that S is a strict subset of S, i.e., $S \subset K$.

Using (E-2), it is easy to see that $X \to Y(S^c) \to (Y_{S^c}, U_{S^c})$ forms a Markov chain conditionally given Q. That is,

$$X \to Y(S^c) \to (Y_{S^c}, U_{S^c}) \mid Q. \tag{E-4}$$

Since $Y(S^c) = X + Z(S^c)$ with X and $Z(S^c)$ being independent conditionally given Q and $Z(Z^c)$ conditionally Gaussian given Q, invoking the conditional strong entropy power inequality of Courtade [38, Corollary 2] yields

$$e^{2\left[h(Y(S^c)|Q) - I(X;U_{S^c}|Q)\right]} \ge e^{2\left[h(X|Q) - I(U_{S^c};Y(S^c)|Q)\right]} + e^{2h(Z(S^c)|Q)}.$$
 (E-5)

Continuing from (E-5) using that Q is independent from $(X, Y(S^c), Z(S^c))$, we get

$$e^{2\left[h(Y(S^c))-I(X;U_{S^c}|Q)\right]} \ge \frac{e^{2h(X)}}{e^{2h(Y(S^c))}} e^{2h(Y(S^c)|U_{S^c},Q)} + e^{2h(Z(S^c))}. \tag{E-6}$$

The conditional entropy term $h(Y(S^c)|U_{S^c},Q)$ is given by

$$h(Y(S^c)|U_{S^c},Q) = h(Y(S^c)|U_{S^c},X,Q) + I(X;Y(S^c),U_{S^c}|Q) - I(X;U_{S^c}|Q)$$
(E-7)

$$= h(Y(S^c)|U_{S^c}, X, Q) + I(X; Y(S^c)) - I(X; U_{S^c}|Q)$$
 (E-8)

where the last equality follows using (E-4).

Also, recalling (46) we have

$$e^{2h(Y(S^c)|U_{S^c},X,Q)} \stackrel{(a)}{=} e^{2h\left(\frac{1}{|S^c|}\sum_{k\in S^c} \frac{\sigma_{S^c}^2}{\sigma_k^2}Y_k|U_{S^c},X,Q\right)}$$
 (E-9)

$$\stackrel{(b)}{\geq} \frac{1}{|\mathcal{S}^c|^2} \sum_{k \in \mathcal{S}^c} \left(\frac{\sigma_{\mathcal{S}^c}^2}{\sigma_k^2}\right)^2 e^{2h(Y_k|U_k, X, Q)} \tag{E-10}$$

where: (a) follows by substituting using (46), and (b) follows using the entropy power inequality noticing that for all \mathcal{A} such that $k \notin \mathcal{A}$ we have $U_k \multimap Y_k \multimap (X, U_{\mathcal{A}}) \mid Q$ and the random variables $\{Y_k \mid U_k, X, Q\}$ are independent.

Furthermore, we have

$$h(Y_k|U_k, X, Q) = h(Y_k|X, Q) + I(U_k; Y_k|X, Q)$$
 (E-11)

$$= h(Z_k) + I(U_k; Y_k | X, Q)$$
 (E-12)

where the last equality follows by substituting using $Y_k = X + Z_k$ and the noise Z_k is independent from (X,Q).

Now, substituting in (E-6) using (E-8), (E-10) and (E-12), we get

$$e^{2\left[h(Y(S^{c}))-I(X;U_{S^{c}}|Q)\right]} \ge e^{2h(Z(S^{c}))} + \frac{e^{2h(X)}}{e^{2h(Y(S^{c}))}} e^{2\left[I(X;Y(S^{c}))-I(X;U_{S^{c}}|Q)\right]}$$

$$\times \frac{1}{|S^{c}|^{2}} \sum_{k \in S^{c}} \left(\frac{\sigma_{S^{c}}^{2}}{\sigma_{k}^{2}}\right)^{2} e^{2h(Z_{k})} e^{-2I(U_{k};Y_{k}|X,Q)}.$$
(E-13)

Using (E-13), we have

$$I(U_{S^c}; X|Q) \le \frac{1}{2} \log \left(e^{-2h(Z(S^c))} \left[e^{2h(Y(S^c))} - \frac{e^{2h(X)}e^{2I(X;Y(S^c))}}{e^{2h(Y(S^c))}} \frac{1}{|S^c|^2} \sum_{k \in S^c} \left(\frac{\sigma_{S^c}^2}{\sigma_k^2} \right)^2 e^{2h(Z_k)} e^{-2I(U_k;Y_k|X,Q)} \right] \right)$$
(E-14)

$$= \frac{1}{2} \log \left(\frac{e^{2h(Y(S^c))}}{e^{2h(Z(S^c))}} - \frac{e^{2h(X)}}{e^{4h(Z(S^c))}} \frac{1}{|S^c|^2} \sum_{k \in S^c} \left(\frac{\sigma_{S^c}^2}{\sigma_k^2} \right)^2 e^{2h(Z_k)} e^{-2I(U_k; Y_k | X, Q)} \right)$$
 (E-15)

$$\stackrel{(a)}{=} \frac{1}{2} \log \left(\frac{N(Y(\mathcal{S}^c))}{N(Z(\mathcal{S}^c))} - \frac{N(X)}{(N(Z(\mathcal{S}^c)))^2} \sum_{k \in \mathcal{S}^c} \left(\frac{\sigma_{\mathcal{S}^c}^2}{|\mathcal{S}^c|} \right)^2 \left(\frac{1}{\sigma_k^2} - \gamma_k \right) \right)$$
(E-16)

$$\stackrel{(b)}{=} \frac{1}{2} \log \left(|\mathcal{S}^c| \frac{N(Y(\mathcal{S}^c))}{\sigma_{\mathcal{S}^c}^2} - N(X) \sum_{k \in \mathcal{S}^c} (\frac{1}{\sigma_k^2} - \gamma_k) \right) \tag{E-17}$$

where (a) holds by substituting that for a continuous random variable A we have $e^{2h(A)} = 2\pi e N(A)$ and Z_k is Gaussian with variance σ_k^2 , and (b) holds by noticing that $N(Z(S^c)) = \sigma_{S^c}^2/|S^c|$ and using (E-3). Finally, combining (E-1) and (E-17) and substituting using (E-3), we get (54); and this completes the proof of the theorem.

Acknowledgment

The author would like to thank Aaron Wagner for fruitful discussions about the relation of Theorem 1 to the outer bound of [7, Theorem 2]. In particular the steps (21)- (28), as well as the note of Remark 1, are due to him. The author also thanks the anonymous reviewers for various useful comments and suggestions which improved the quality of this paper.

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