Analytical algorithm for capacities of classical and classical-quantum channels

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Abstract—We derive an analytical algorithm for the channel capacity of a classical channel without any iteration, while its existing algorithms require iterations and the number of iterations depends on the required precision level. Hence, our algorithm is its first analytical algorithm for this task without any iteration, while this algorithm needs several conditions for the channel. We apply the obtained algorithm to examples, and see how the obtained algorithm works in these examples. Then, we extend it to the channel capacity of a classical-quantum (cq-) channel. Many existing studies proposed algorithms for a cq-channel and all of them require iterations. Our extended analytical algorithm has also no iteration, and outputs the exactly optimum value.

Index Terms—mutual information, maximization, channel capacity, classical-quantum channel, analytical algorithm

I. INTRODUCTION

One of the key problems in classical and quantum information theory is the maximization of information quantities. However, it is not so easy to perform such a maximization analytically because all of existing methods require certain iterations, whose number depends on the required precision level. The most common maximization problem is the channel capacity, which is given as the maximization of mutual information [6], and its calculation has been studied by Arimoto [2], Blahut [3], and their related studies [7], [8], [9]. However, these are iterative approximation algorithms to calculate the maximum of the mutual information. In addition, the reference [4] calculated only its upper bound and the references [24], [10] developed other type of method to approximately calculate it. Hence, they cannot calculate the exact value for the channel capacity. As variants, the references [14], [25] extended the above method to the wire-tap capacity [12], [13] when the wire-tap channel is degraded. Also, the references [15], [16], [17], [18], [19] extended it to the quantum setting, so called the capacity of classical-quantum channel. However, these results are also iterative approximation algorithms.

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This paper proposes an algorithm to analytically calculate the channel capacity of the classical channel without iteration. The proposed algorithm is composed of solving simultaneous linear equations and calculation of logarithm and exponential because it employs an information-geometrical structure. However, the proposed method works under certain conditions. Since our method is analytical, we can derive several analytical formulas for the capacity when these conditions are satisfied. Then, to see this possibility, we apply our algorithm to several examples, and derive analytical expressions of the capacities in these examples. Further, we extend our analytical algorithm to the calculations of the capacity of classical-quantum channel.

The remaining part of this paper is organized as follows. First, Section II derives our algorithm for the capacity of a classical channel. Section III applies the obtained result to several examples. Next, Section IV extends this method to the capacity of classical-quantum channel. Finally, Section V discusses the merit and the demerit of our method over existing methods.

II. CAPACITY OF CLASSICAL CHANNEL

We consider the input and output alphabets $\mathcal{X}:=\{1,\ldots,n_1\}$ and $\mathcal{Y}:=\{1,\ldots,n_2\}$ that are finite sets. We denote the sets of probability distributions on \mathcal{X} and \mathcal{Y} by $\mathcal{P}_{\mathcal{X}}$ and $\mathcal{P}_{\mathcal{Y}}$, respectively. For distributions $P,Q\in\mathcal{P}_{\mathcal{X}}$, the entropy H(P) and the divergence $D(P\|Q)$ are defined as

$$H(P) := -\sum_{x \in \mathcal{X}} P(x) \log P(x), \tag{1}$$

$$D(P||Q) := \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.$$
 (2)

Throughout this paper, the base of the logarithm is chosen to be the natural logarithm.

A channel from \mathcal{X} to \mathcal{Y} is given as conditional distribution on \mathcal{Y} conditioned with \mathcal{X} . That is, using the notation $W_x(y) := W(y|x)$, it can be considered as a map $W: \mathcal{X} \to \mathcal{P}_{\mathcal{Y}}$. For $Q_X \in \mathcal{P}_{\mathcal{X}}$ and $Q_Y \in \mathcal{P}_{\mathcal{Y}}, W \cdot Q_X \in \mathcal{P}_{\mathcal{Y}}, W \times Q_X \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$, and $Q_X \times Q_Y \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ are defined by $(W \cdot Q_X)(x,y) := \sum_{x \in \mathcal{X}} W(y|x)Q_X(x), \ (W \times Q_X)(x,y) := W(y|x)Q_X(x)$, and $(Q_Y \times Q_X)(x,y) := Q_X(x)Q_Y(y)$, respectively.

The channel capacity of a channel W is given by [6], [27, p.124]

$$C(W) := \max_{Q_X \in \mathcal{P}_{\mathcal{X}}} \sum_{x \in \mathcal{X}} Q_X(x) D(W_x || W \cdot Q_X)$$

$$= \min_{Q_Y \in \mathcal{P}_{\mathcal{Y}}} \max_{x \in \mathcal{X}} D(W_x || Q_Y)$$

$$= \min_{Q_X \in \mathcal{P}_{\mathcal{X}}} \max_{x \in \mathcal{X}} D(W_x || W \cdot Q_X). \tag{3}$$

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To discuss C(W), we assume the following conditions.

(A) W_1, \ldots, W_{n_1} are linearly independent.

Then, we have the following:

Lemma 1: When a distribution $Q_Y = W \cdot Q_X$ realizes the minimum in (3), it satisfies the following condition: $D(W_x||Q_Y)$ does not depend on $x \in \text{supp}(Q_X)$.

Lemma 1 is shown in Appendix B. We define the set \mathcal{M}_0 as

$$\mathcal{M}_0 := \Big\{ Q_Y \in \mathcal{P}_{\mathcal{Y}} \Big| Q_Y = \sum_{x \in \mathcal{X}} c(x) W_x, \quad \sum_{x \in \mathcal{X}} c(x) = 1 \Big\}.$$
(4)

Here, the condition $c(x) \geq 0$ is not imposed. Hence, \mathcal{M}_0 is characterized by as linear constraints, which will be explained in Appendix C. Then, we introduce another condition for the distribution Q_Y :

(B) $D(W_x||Q_Y)$ does not depend on $x \in \mathcal{X}$.

Lemma 2: When Condition (A) holds, only one distribution $Q_Y \in \mathcal{M}_0$ satisfies Condition (B).

Lemma 2 is shown in Appendix C.

In the following, we denote the element of \mathcal{M}_0 to satisfy the condition (B) by $Q_{Y,*}$. Since $Q_{Y,*}$ belongs to \mathcal{M}_0 , there exists a function $\widehat{Q}_{X,*}$ on \mathcal{X} as the solution of the following equation:

$$\sum_{x \in \mathcal{X}} W(y|x) \widehat{Q}_{X,*}(x) = Q_{Y,*}(y).$$
 (5)

Condition (A) guarantees the uniqueness of $\widehat{Q}_{X,*}$. We have the following theorem:

Theorem 1: Assume Condition (A). The following conditions are equivalent

- (i) The relation $D(W_x||Q_{Y,*}) = C(W)$ holds.
- (ii) The function $Q_{X,*}$ satisfies the condition

$$\widehat{Q}_{X,*}(x) \ge 0 \text{ for } x \in \mathcal{X}.$$
 (6)

Theorem 1 is shown in Appendix D.

When the function $\widehat{Q}_{X,*}$ does not satisfy (6), it is not a distribution on \mathcal{X} . Due to Theorem 1, the condition (i) does not hold. That is, there exists an element $x \in \mathcal{X}$ such that $D(W_x || Q_{Y,*}) > C(W)$.

Hence, under the condition (ii), the capacity C(W) is given by $D(W_x\|Q_{Y,*})$. To consider the case that the condition (ii) does not hold, we prepare the following theorem. For any function f on \mathcal{X} , we define $\mathcal{N}(f) := \{x \in \mathcal{X} | f(x) < 0\}$ and $\mathcal{N}^c(f) := \{x \in \mathcal{X} | f(x) \geq 0\}$.

Theorem 2: Assume Condition (A). Then, we have

$$C(W) = \max_{Q_X \in \mathcal{P}_{\mathcal{N}^c(\widehat{Q}_{X,*})}} \sum_{x \in \mathcal{N}^c(\widehat{Q}_{X,*})} Q_X(x) D(W_x || W \cdot Q_X).$$
(7)

Theorem 2 is shown in Appendix E. Therefore, the capacity C(W) is obtained only with the input set $\mathcal{N}^c(\widehat{Q}_{X,*})$. That is, the function $\widehat{Q}_{X,*}$ gives an important information for computing C(W).

We choose $n_2 - 1$ linearly independent functions f_1, \ldots, f_{n_2-1} on \mathcal{Y} such that they are not constant function and

$$\sum_{y \in \mathcal{Y}} W_{n_2}(y) f_j(y) = 0 \tag{8}$$

for $j = 1, ..., n_2 - 1$. We define the matrix $(h_{i,j})$

$$h_{i,j} := \sum_{y \in \mathcal{Y}} W_i(y) f_j(y). \tag{9}$$

Given an $n_2 - 1$ -dimensional parameter $\theta = (\theta^1, \dots, \theta^{n_2 - 1})$, we define the distribution $P_{\theta, Y}$ as

$$P_{\theta,Y}(y) = e^{\sum_{j=1}^{n_2-1} f_j(y)\theta^j - \phi(\theta)},$$
(10)

where

$$\phi(\theta) := \log \left(\sum_{y \in \mathcal{Y}} e^{\sum_{j=1}^{n_2 - 1} f_j(y)\theta^j} \right). \tag{11}$$

The parameterization (10) is called the natural parameter [5]. We have the following theorem.

Theorem 3: Assume that the parameters $\theta^1, \dots, \theta^{n_2-1}$ satisfy the condition

$$\sum_{j=1}^{n_2-1} h_{i,j}\theta^j = -H(W_i) + H(W_{n_1}). \tag{12}$$

for $i = 1, \ldots, n_1 - 1$. Then, we have

$$D(W_x || P_{\theta, Y}) = \phi(\theta) - H(W_{n_1}). \tag{13}$$

for $x \in \mathcal{X}$.

Proof: The condition (12) implies that

$$\sum_{y \in \mathcal{Y}} W_i(y) \sum_{j=1}^{n_2 - 1} f_j(y) \theta^j = \sum_{j=1}^{n_2 - 1} h_{i,j} \theta^j$$
$$= -H(W_i) + H(W_{n_2}). \tag{14}$$

For $x(\neq n_2) \in \mathcal{X}$, we have

$$D(W_x || P_{\theta, Y}) = \sum_{y \in \mathcal{V}} W_x(y) \left(\log W_x(y) - \log P_{\theta, Y}(y) \right)$$

$$= -H(W_x) - \sum_{y \in \mathcal{Y}} W_x(y) \left(\sum_{j=1}^{n_2 - 1} f_j(y) \theta^j - \phi(\theta) \right)$$

$$= -H(W_x) - \left(-H(W_x) + H(W_{n_2}) - \phi(\theta) \right)$$

$$= \phi(\theta) - H(W_{n_2}). \tag{15}$$

Also, we have

$$D(W_n || P_{\theta, Y}) = \sum_{y \in \mathcal{Y}} W_{n_2}(y) (\log W_{n_2}(y) - \log P_{\theta, Y}(y))$$

$$= -H(W_{n_2}) - \sum_{y \in \mathcal{Y}} W_x(y) \left(\sum_{j=1}^{n_2-1} f_j(y) \theta^j - \phi(\theta) \right)$$

= $-H(W_{n_2}) - \left(-\phi(\theta) \right) = \phi(\theta) - H(W_{n_2}).$ (16)

We define the set \mathcal{E}_0 as

$$\mathcal{E}_0 := \{ P_{\theta, Y} | \text{ The condition (12) holds.} \}$$
 (17)

Lemma 3: The set $\mathcal{M}_0 \cap \mathcal{E}_0$ is composed of one element

Lemma 3 is shown in Appendix F. Therefore, $P_{\theta_*,Y}$ equals $Q_{Y,*}$.

Now, as a stronger assumption than Condition (A), we assume the following condition (Condition (C)).

(C) $n_1=n_2$ and W_1,\ldots,W_{n_1} are linearly independent. Since $\mathcal{M}_0=\mathcal{P}_{\mathcal{X}}$, due to Lemma 2, only one set of parameters $\theta^1,\ldots,\theta^{n_2-1}$ satisfies the condition (12). Due to Theorem 3, solving the equation (12), we find $Q_{Y,*}$ as $P_{\theta,Y}$. To construct our algorithm, we add the n_2 -th function f_{n_2} on $\mathcal Y$ and define $h_{i,j}$ by (9) for $i,j=1,\ldots,n_2$. We rewrite the equation (5) as

$$\sum_{x \in \mathcal{X}} \widehat{Q}_{X,*}(x) h_{x,j} = \sum_{x \in \mathcal{X}} \widehat{Q}_{X,*}(x) \sum_{y \in Y} W_x(y) f_j(y)$$
$$= \sum_{y \in \mathcal{Y}} P_{\theta,Y}(y) f_j(y). \tag{18}$$

We obtain the function $\widehat{Q}_{X,*}$ on \mathcal{X} as the solution of (18), and $W \cdot \widehat{Q}_{X,*} = Q_{Y,*}$ satisfies the condition (B). When the function $\widehat{Q}_{X,*}$ satisfies the condition (6), the value $D(W_x || P_{\theta,Y})$ is the capacity of the channel W due to Theorem 3. Therefore, we have Algorithm 1 to compute C(W) under Condition (C).

In fact, $(W_i(j))_{i,j}$ and $(f_j(i))_{i,j}$ form $n_2 \times n_2$ matrices. When $(f_j(i))_{i,j}$ is the inverse matrix of $(W_i(j))_{i,j}$, $(h_{i,j})_{i,j}$ is the identity matrix. Due to Theorem 1, Theorem 3 does not necessarily work for calculating C(W). Hence, based on Theorems 1 and 3, we propose Algorithm 1 to check the condition in Theorem 1, and compute C(W) under this condition.

In Algorithm 1, Step 1 has calculation complexity $O(n_2^3)$. Steps 2 and 3 have calculation complexity $O(n_2^2)$ because $h_{i,j}$ is an upper triangle matrix. Step 5 has calculation complexity $O(n_2^2)$. Hence, the total calculation complexity is $O(n_2^3)$.

Algorithm 1 Exact algorithm for classical channel capacity

Step 1: Choose f_1, \ldots, f_{n_2} such that $(f_j(i))_{i,j}$ is the inverse matrix of $(W_i(j))_{i,j}$. Hence, $h_{i,j} = \delta_{i,j}$.

Step 2: Set the parameter $\theta^i = -H(W_i) + H(W_{n_2})$ for $i = 1, \dots, n_2 - 1$, which is the solution of (12).

Step 3: Calculate $\phi(\theta)$ by using (11).

Step 4: Calculate $\widehat{Q}_{X,*}(x) := \sum_{y \in \mathcal{Y}} P_{\theta,Y}(y) f_x(y)$, where $P_{\theta,Y}(y)$ is calculated by (10). This step follows from (18).

Step 5: If the condition (6) holds, we consider that the condition in Theorem 1 holds and output $\phi(\theta) - H(W_n)$ as the capacity. Otherwise, we consider that the condition in Theorem 1 does not hold and output "the capacity cannot be computed."

Next, instead of Condition (C), we consider the following condition.

(C') The relation $n_1 \ge n_2$ holds. Any n_2 elements among W_1, \ldots, W_{n_1} are linearly independent.

Under this condition, we can apply Algorithm 1 to any n_2 elements x_1, \ldots, x_{n_2} in \mathcal{X} . If the capacity is calculated

under this choice, it is denoted by $C(W; x_1, \ldots, x_{n_2})$. When the capacity is calculated under all choices of x_1, \ldots, x_{n_2} , the maximum of $C(W; x_1, \ldots, x_{n_2})$ is the capacity of the channel.

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In this case, we need to try $\binom{n_1}{n_2}$ combinations, which requires too large calculation amount. However, it is possible to avoid such repetition as follows. First, we apply the conventional iterative algorithm by [2], [3] or the improved iterative algorithm by [9]. Then, we obtain an approximately optimal input distribution. If the distribution has the majority of the probability in n_2 elements of \mathcal{X} , we can consider the support of the optimal input distribution is composed of these n_2 elements of \mathcal{X} . Hence, we apply Algorithm 1 to the case when \mathcal{X} is the above n_2 elements. That is, it is sufficient to check whether Algorithm 1 outputs the capacity only in this case. When we employ this method, we do not need $\binom{n_1}{n_2}$ repetitions. That is, the above hybrid method works for analytical calculation.

However, if $C(W; x_1, ..., x_{n_2})$ depends on the choice of n_2 elements $x_1, ..., x_{n_2}$, and the minimum difference

$$\min_{\substack{(x_1,\dots,x_{n_2})\\ \neq (x_1',\dots,x_{n_2}')}} \left| C(W;x_1,\dots,x_{n_2}) - C(W;x_1',\dots,x_{n_2}') \right|$$

is very small, this idea does not work. In this case, it is expected that the approximately optimal input distribution the majority of the probability in more than n_2 elements of \mathcal{X} . Hence, the above method does not work.

Also, even under Condition (C'), there is the case that the support of the optimal input distribution is composed of a smaller element than n_2 . In this case, even when we apply Algorithm 1 for $\binom{n_1}{n_2}$ combinations, we cannot obtain the capacity.

When only Condition (A) holds, $Q_{Y,*}$ can be characterized as follows.

Theorem 4: Assume that $h_{i,j}=0$ for $j=n_1+,\ldots,n_2-1$ and the parameters $\theta^1,\ldots,\theta^{n_1-1}$ satisfies the condition (9). When the parameters $\theta^{n_1},\ldots,\theta^{n_2-1}$ are given as

$$(\theta^{n_1}, \dots, \theta^{n_2-1}) = \underset{\eta^{n_1}, \dots, \eta^{n_2-1}}{\operatorname{argmin}} \phi(\theta^1, \dots, \theta^{n_1-1}, \eta^{n_1}, \dots, \eta^{n_2-1}),$$
(19)

we have $P_{\theta,Y} = Q_{Y,*}$.

Proof: Since the objective function in (19), the parameters $\theta^{n_1}, \ldots, \theta^{n_2-1}$ achieves the minimum (19) if and only if

$$\frac{\partial \phi(\theta^1, \dots, \theta^{n_2 - 1})}{\partial \theta^j} = 0 \text{ for } j = n_1, \dots, n_2 - 1.$$
 (20)

This is because the concavity of ϕ guarantees that there are no local minima. The above condition is equivalent to

$$\sum_{y \in \mathcal{V}} f_j(y) P_{\theta, Y}(y) = 0 \text{ for } j = n_1, \dots, n_2 - 1.$$
 (21)

Due to (131), when $\theta^{n_1}, \dots, \theta^{n_2-1}$ are given by (19), $P_{\theta,Y}$ belongs to \mathcal{M}_0 . Due to the uniqueness by Lemma 3, we obtain the desired statement.

Theorem 4 guarantees that $Q_{Y,*}$ is given as the solution of the minimization (19), which is a convex minimization. While the analytical solution of (19) is difficult in general, it

is possible in the following case. We impose the following condition for the functions $f_{n_1}, \ldots, f_{n_2-1}$: (I) $f_j(y)$ takes non-zero value only with two elements $y_j, y'_i \in \mathcal{Y}$ for j = n_1+,\ldots,n_2-1 . (II) The sets $\{y_j,y_j'\}$ for $j=n_1+,\ldots,n_2-1$

In this case, the relation (19), i.e., (20), can be simplified as

$$0 = f_j(y_j)e^{f_j(y_j)\theta^j + \sum_{i=1}^{n_1-1} f_i(y_j)\theta^i} + f_j(y_j')e^{f_j(y_j')\theta^j + \sum_{i=1}^{n_1-1} f_i(y_j')\theta^i}$$
(22)

for $j = n_1 + \dots, n_2 - 1$. The equation (22) is solved as

$$\theta^{j} = \frac{1}{f_{j}(y_{j}) - f_{j}(y'_{j})} \left(\sum_{i=1}^{n_{1}-1} (f_{i}(y'_{j}) - f_{i}(y_{j})) \theta^{i} + \log \frac{-f_{j}(y'_{j})}{f_{j}(y_{j})} \right)$$
(23)

for $j = n_1 + \dots, n_2 - 1$. Therefore, we can analytically calculate $P_{\theta,Y} = Q_{Y,*}$ under the conditions (I) and (II).

III. EXAMPLE

A. Output system with two elements

are disjoint with each other.

First, we consider the case with $Y = \{1, 2\}$. When \mathcal{X} and the channel W satisfies Condition (C') in this case, the method described after Condition (C') works well as follows. For any two elements $x_1 \neq x_2 \in \mathcal{X}$, the channel only with two inputs x_1, x_2 always satisfies the condition (6) because $Q_{Y,*}$ is located between W_{x_1} and W_{x_2} . Hence, the condition in Theorem 1 holds. Therefore, it is sufficient to derive a general formula for the capacity when two elements in \mathcal{X} are fixed.

Therefore, in the following, we consider the case with $\mathcal{X} =$ $\{1,2\}$ and $Y=\{1,2\}$. We define the distributions W_x for $x \in \mathcal{X}$ by the following vector form:

$$W_1 := \begin{pmatrix} 1-p \\ p \end{pmatrix}, \quad W_2 := \begin{pmatrix} 1-q \\ q \end{pmatrix}.$$
 (24)

For simplicity, we assume that q > p. We define the 2×2 matrix V as $V := (W_1, W_1)$. The inverse matrix is

$$V^{-1} = \frac{1}{q - p} \begin{pmatrix} q & q - 1 \\ -p & 1 - p \end{pmatrix}.$$
 (25)

In this case, the parameter θ is one-dimensional and is solved to h(q) - h(p), where h(p) is the binary entropy. Then, $\phi(\theta)$ is calculated as

$$\phi(\theta) = \log\left(e^{\frac{q(h(q) - h(p))}{q - p}} + e^{\frac{-p(h(q) - h(p))}{q - p}}\right)$$

$$= \log\left(e^{\frac{q(h(q) - h(p))}{q - p}}\left(1 + e^{\frac{-(q + p)(h(q) - h(p))}{q - p}}\right)\right)$$

$$= \frac{q(h(q) - h(p))}{q - p} + \log\left(1 + e^{\frac{-(q + p)(h(q) - h(p))}{q - p}}\right). (26)$$

The capacity is calculated as

$$C(W) = \phi(\theta) - h(p)$$

$$= \frac{ph(q) - qh(p)}{q - p} + \log\left(1 + e^{\frac{-(q+p)(h(q) - h(p))}{q - p}}\right), (27)$$

which is a general capacity formula with $\mathcal{X} = \{1, 2\}$ and $Y = \{1, 2\}$. Then,

$$P_{\theta,Y} = \begin{pmatrix} e^{\frac{q(h(q) - h(p))}{q - p} - \phi(\theta)} \\ e^{\frac{-p(h(q) - h(p))}{q - p} - \phi(\theta)} \end{pmatrix}.$$
(28)

Hence, the optimal input distribution is

$$\widehat{Q}_{X,*} = \begin{pmatrix} \frac{1}{q-p} \left(q - e^{\frac{-p(h(q)-h(p))}{q-p} - \phi(\theta)} \right) \\ \frac{1}{q-p} \left(-p + e^{\frac{-p(h(q)-h(p))}{q-p} - \phi(\theta)} \right) \end{pmatrix}.$$
 (29)

B. Output system with three elements

1) General problem description: Next, we consider the case with $Y = \{1, 2, 3\}$. In this case, Algorithm 1 does not necessarily work even under the condition (C). Moreover, the method described after Condition (C') does not necessarily work even under the condition (C'). To see such a case, we consider the following example with $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{Y} = \{1, 2, 3\}$ with $\epsilon \in [0, 1/2]$. We define the distributions W_x for $x \in \mathcal{X}$ by the following vector form:

$$W_1 := \begin{pmatrix} 1 - \epsilon \\ 0 \\ \epsilon \end{pmatrix}, \quad W_2 := \begin{pmatrix} 0 \\ 1 - \epsilon \\ \epsilon \end{pmatrix}$$
 (30)

$$W_3 := \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad W_4 := \begin{pmatrix} \frac{1}{2} - \epsilon \\ \frac{1}{2} - \epsilon \\ 2\epsilon \end{pmatrix}. \tag{31}$$

We define 3×3 matrix V_j for $j \in \mathcal{X}$ as $V_1 :=$ $(W_2, W_3, W_4), V_2 := (W_1, W_3, W_4), V_3 := (W_1, W_2, W_4),$ $V_4 := (W_1, W_2, W_3)$. Their inverse matrices are

$$V_1^{-1} = \begin{pmatrix} -\frac{1}{1-\epsilon} & \frac{1}{1-\epsilon} & 0\\ \frac{3-2\epsilon}{2(1-\epsilon)} & \frac{1-2\epsilon}{2(1-\epsilon)} & \frac{-1+2\epsilon}{2\epsilon}\\ \frac{1}{2(1-\epsilon)} & -\frac{1}{2(1-\epsilon)} & \frac{1}{2\epsilon} \end{pmatrix}$$
(32)

$$V_2^{-1} = \begin{pmatrix} \frac{1}{1-\epsilon} & -\frac{1}{1-\epsilon} & 0\\ \frac{1-2\epsilon}{2(1-\epsilon)} & \frac{3-2\epsilon}{2(1-\epsilon)} & \frac{-1+2\epsilon}{2\epsilon}\\ -\frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2\epsilon} \end{pmatrix}$$
(33)

$$V_3^{-1} = \begin{pmatrix} \frac{3-2\epsilon}{2(1-\epsilon)} & \frac{1-2\epsilon}{2(1-\epsilon)} & -\frac{1-2\epsilon}{2\epsilon} \\ \frac{1-2\epsilon}{2(1-\epsilon)} & \frac{3-2\epsilon}{2(1-\epsilon)} & -\frac{1-2\epsilon}{2\epsilon} \\ -1 & -1 & \frac{1-\epsilon}{\epsilon} \end{pmatrix}$$
(34)

$$V_4^{-1} = \begin{pmatrix} \frac{1}{2(1-\epsilon)} & -\frac{1}{2(1-\epsilon)} & \frac{1}{2\epsilon} \\ -\frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2\epsilon} \\ 1 & 1 & \frac{-1+\epsilon}{2(1-\epsilon)} \end{pmatrix}.$$
(35)

Also, we have

$$H(W_1) = H(W_2) = h(\epsilon) \tag{36}$$

$$H(W_3) = \log 2$$
, $H(W_4) = h(2\epsilon) + (1 - 2\epsilon) \log 2$. (37)

When we apply Algorithm 1 to the three components in V_i , we denote θ , $\phi(\theta)$, $P_{\theta,Y}$, $Q_{X,*}$, and $\phi(\theta) - H(W_n)$ by θ_j , $= \frac{ph(q) - qh(p)}{q - p} + \log\left(1 + e^{\frac{-(q+p)(h(q) - h(p))}{q - p}}\right), (27) \quad \phi_j(\theta_j), P_{j,Y} \text{ and } \widehat{Q}_{j,X}, \text{ and } C_j, \text{ respectively. In the following,}$ we discuss our model dependently of the value of j.

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2) Case that j=4: First, we consider the case that j=4, i.e., the channel is composed of three inputs $\{1,2,3\}$. Then, we have

$$\theta_4 = \begin{pmatrix} h_{4,\epsilon} \\ h_{4,\epsilon} \end{pmatrix} \tag{38}$$

with $h_{4,\epsilon} := \log 2 - h(\epsilon)$ and

$$\phi_4(\theta_4) = \log\left(2 + e^{\frac{h_{4,\epsilon}}{\epsilon}}\right). \tag{39}$$

Thus,

$$C_4 = \phi_4(\theta_4) - \log 2 = \log\left(1 + \frac{e^{\frac{h_{4,\epsilon}}{\epsilon}}}{2}\right) \tag{40}$$

Hence,

$$P_{4,Y} = \begin{pmatrix} e^{-\phi_4(\theta_4)} \\ e^{-\phi_4(\theta_4)} \\ e^{\frac{h_{4,\epsilon}}{\epsilon} - \phi_4(\theta_4)} \end{pmatrix}. \tag{41}$$

Therefore,

$$\widehat{Q}_{4,X} = V_4^{-1} P_{4,Y}
= \begin{pmatrix} \frac{1}{2\epsilon} e^{\frac{h_{4,\epsilon}}{\epsilon} - \phi_4(\theta_4)} \\ \frac{1}{2\epsilon} e^{\frac{h_{4,\epsilon}}{\epsilon} - \phi_4(\theta_4)} \\ 2e^{-\phi_4(\theta_4)} - \frac{1-\epsilon}{\epsilon} e^{\frac{h_{4,\epsilon}}{\epsilon} - \phi_4(\theta_4)} \end{pmatrix}.$$
(42)

While the first and second components of $\widehat{Q}_{4,X}$ are always positive value, the third component has a possibility to have a negative value. The non-negativity of the first component is equivalent to the following condition:

$$1 \ge g_1(\epsilon),\tag{43}$$

where $g_1(\epsilon):=\frac{1-\epsilon}{2\epsilon}e^{\frac{h_{4,\epsilon}}{\epsilon}}$. In (43), the first inequality corresponds to the non-negativity of the third component and the second inequality corresponds to the non-negativity of the first component. Fig. 1 numerically plots the function $g_1(\epsilon)$. It shows that $\widehat{Q}_{4,X}$ is a probability distribution when $0.3588 \le \epsilon$. That is, C_4 is achievable for $0.3588 \le \epsilon$. When $\epsilon < 0.3588$, the third component of $\widehat{Q}_{4,X}$ is negative. Hence, C_4 is not achievable. Due to Theorem 2, the optimal input distribution in this case has the support in $\{1,2\}$. In this case, due to the symmetry, the uniform distribution on $\{1,2\}$ is optimal. That is, the capacity with the input set $\{1,2,3\}$ is

$$C_* := -(1 - \epsilon) \log \frac{1 - \epsilon}{2} - \epsilon \log \epsilon - h(\epsilon) = (1 - \epsilon) \log 2. \tag{44}$$

3) Case that j=3: We consider the case that j=3, i.e., the channel is composed of three inputs $\{1,2,4\}$. Then, we have

$$\theta_3 = \begin{pmatrix} h_{3,\epsilon} \\ h_{3,\epsilon} \end{pmatrix} \tag{45}$$

with $h_{3,\epsilon} := h(2\epsilon) + (1 - 2\epsilon) \log 2 - h(\epsilon)$ and

$$\phi_3(\theta_3) = \log\left(2e^{2h_{3,\epsilon}} + e^{-\frac{(1-2\epsilon)h_{3,\epsilon}}{\epsilon}}\right)$$
$$= \log\left(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}\right) + 2h_{3,\epsilon}. \tag{46}$$

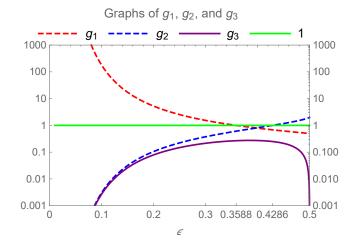


Fig. 1. Graphs of functions g_1, g_2 , and g_3 with logarithmic scale. Red dashed curve expresses g_1 . Blue dashed curve expresses g_2 . Purple solid curve expresses g_3 . Green solid line expresses 1. Red dashed curve g_1 and Blue dashed curve g_2 across Green solid line at 0.3588 and 0.4286, respectively.

Thus,

$$C_3 = \phi_3(\theta_3) - h(2\epsilon) - (1 - 2\epsilon) \log 2$$

$$= \log\left(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}\right) + 2h_{3,\epsilon} - h(2\epsilon) - (1 - 2\epsilon) \log 2$$

$$= \log\left(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}\right) + h(2\epsilon) + (1 - 2\epsilon) \log 2 - 2h(\epsilon). \tag{47}$$

Hence,

$$P_{3,Y} = \frac{1}{2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}} \begin{pmatrix} 1\\1\\e^{-\frac{h_{3,\epsilon}}{\epsilon}} \end{pmatrix}. \tag{48}$$

Therefore,

$$\widehat{Q}_{3,X} = V_3^{-1} P_{3,Y}
= \frac{1}{2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}} \begin{pmatrix} 1 - \frac{1-2\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \\ 1 - \frac{1-2\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \\ -2 + \frac{1-\epsilon}{\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \end{pmatrix}.$$
(49)

While the first and second components of $\widehat{Q}_{3,X}$ always have positive values, the third component has a possibility to have a negative value. The non-negativity of the first component is equivalent to the following condition:

$$g_2(\epsilon) \ge 1 \ge g_3(\epsilon),$$
 (50)

where $g_2(\epsilon):=\frac{1-\epsilon}{2\epsilon}e^{-\frac{h_{3,\epsilon}}{\epsilon}}$ and $g_3(\epsilon):=\frac{1-2\epsilon}{2\epsilon}e^{-\frac{h_{3,\epsilon}}{\epsilon}}$. In (50), the first inequality corresponds to the non-negativity of the third component and the second inequality corresponds to the non-negativity of the first component. However, as numerically plotted in Fig. 1, $g_3(\epsilon) \leq 1$ for $\epsilon < \frac{1}{2}$ and $g_2(\epsilon) < 1$ for $\epsilon < 0.4286$. Hence, when $\epsilon \geq 0.4286$, C_3 is achievable, i.e., it gives the capacity under the case j=3.

When $\epsilon < 0.4286$, the third component of $\widehat{Q}_{3,X}$ is negative. In this case, due to Theorem 2, the optimal distribution has support $\{1,2\}$. Hence, the capacity with the input set $\{1,2,4\}$ is C_* defined in (44).

4) Case that j=1: Next, we consider the case that j=1, i.e., the channel is composed of three inputs $\{2,3,4\}$. Then, we have

$$\theta_1 = \begin{pmatrix} h_{3,\epsilon} \\ h_{1,\epsilon} \end{pmatrix} \tag{51}$$

with $h_{1,\epsilon} := h(2\epsilon) - 2\epsilon \log 2$ and

$$\phi_{1}(\theta_{1}) = \log \left(e^{-\frac{1}{1-\epsilon}h_{3,\epsilon} + (\frac{1}{2(1-\epsilon)} + 1)h_{1,\epsilon}} + e^{\frac{1}{1-\epsilon}h_{3,\epsilon} + (-\frac{1}{2(1-\epsilon)} + 1)h_{1,\epsilon}} + e^{(-\frac{1}{2\epsilon} + 1)h_{1,\epsilon}} \right)
= \log \left(\frac{1}{1-\epsilon} \left(\frac{1-2\epsilon}{4} \right)^{\frac{1-2\epsilon}{2-2\epsilon}} + (1-\epsilon) \left(\frac{1-2\epsilon}{4} \right)^{-\frac{1-2\epsilon}{2-2\epsilon}} + 4\epsilon (1-2\epsilon)^{\frac{1-2\epsilon}{2\epsilon}} \right) + h_{1,\epsilon}. (52)$$

Thus,

$$C_{1} = \phi_{1}(\theta_{1}) - h(2\epsilon) - (1 - 2\epsilon) \log 2$$

$$= \log \left(\frac{1}{1 - \epsilon} \left(\frac{1 - 2\epsilon}{4}\right)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}} + (1 - \epsilon) \left(\frac{1 - 2\epsilon}{4}\right)^{-\frac{1 - 2\epsilon}{2 - 2\epsilon}} + 4\epsilon (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}}\right) - \log 2.$$
(53)

Since

$$e^{-\frac{1}{2\epsilon}h_{1,\epsilon}} = 4\epsilon (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}},\tag{54}$$

we have

$$P_{1,Y} = \begin{pmatrix} e^{-\frac{1}{1-\epsilon}h_{3,\epsilon} + (\frac{1}{2(1-\epsilon)} + 1)h_{1,\epsilon} - \phi_{1}(\theta_{1})} \\ e^{\frac{1}{1-\epsilon}h_{3,\epsilon} + (-\frac{1}{2(1-\epsilon)} + 1)h_{1,\epsilon} - \phi_{1}(\theta_{1})} \\ e^{(-\frac{1}{2\epsilon} + 1)h_{1,\epsilon} - \phi_{1}(\theta_{1})} \end{pmatrix}$$

$$= e^{h_{1,\epsilon} - \phi_{1}(\theta_{1})} \begin{pmatrix} \frac{1}{1-\epsilon} (\frac{1-2\epsilon}{4})^{\frac{1-2\epsilon}{2-2\epsilon}} \\ (1-\epsilon)(\frac{1-2\epsilon}{4})^{-\frac{1-2\epsilon}{2-2\epsilon}} \\ 4\epsilon(1-2\epsilon)^{\frac{1-2\epsilon}{2\epsilon}} \end{pmatrix}.$$

$$4\epsilon(1-2\epsilon)^{\frac{1-2\epsilon}{2\epsilon}}$$

Therefore,

$$\widehat{Q}_{1,X} = V_1^{-1} P_{1,Y} = e^{h_{1,\epsilon} - \phi_1(\theta_1)} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{pmatrix}, \quad (56)$$

where

$$\begin{split} \kappa_1 &:= -\frac{1}{(1-\epsilon)^2} (\frac{1-2\epsilon}{4})^{\frac{1-2\epsilon}{2-2\epsilon}} + (\frac{1-2\epsilon}{4})^{-\frac{1-2\epsilon}{2-2\epsilon}} \\ \kappa_2 &:= \frac{3-2\epsilon}{2(1-\epsilon)^2} (\frac{1-2\epsilon}{4})^{\frac{1-2\epsilon}{2-2\epsilon}} + \frac{1-2\epsilon}{2} (\frac{1-2\epsilon}{4})^{-\frac{1-2\epsilon}{2-2\epsilon}} \\ &\quad + (-2+4\epsilon)(1-2\epsilon)^{\frac{1-2\epsilon}{2-2\epsilon}} \\ \kappa_3 &:= \frac{1}{2(1-\epsilon)^2} (\frac{1-2\epsilon}{4})^{\frac{1-2\epsilon}{2-2\epsilon}} - \frac{1}{2} (\frac{1-2\epsilon}{4})^{-\frac{1-2\epsilon}{2-2\epsilon}} \\ &\quad + 2(1-2\epsilon)^{\frac{1-2\epsilon}{2\epsilon}}. \end{split}$$

Taking the limit $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \widehat{Q}_{1,X} = \frac{2}{5} \begin{pmatrix} \frac{3}{2} \\ \frac{7}{4} - \frac{2}{\epsilon} \\ \frac{2}{2} - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{7}{10} - \frac{4}{5\epsilon} \\ \frac{4}{5} - \frac{3}{10} \end{pmatrix}.$$
 (57)

Fig. 2 shows numerical plots of $\widehat{Q}_{1,X}(2)$, $\widehat{Q}_{1,X}(3)$, and $\widehat{Q}_{1,X}(4)$. Although $\widehat{Q}_{1,X}(3)$ and $\widehat{Q}_{1,X}(4)$ are always positive,

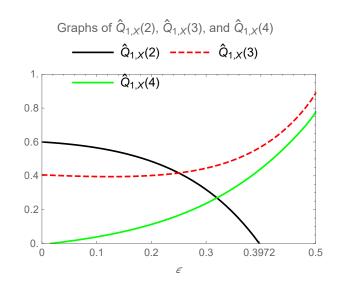


Fig. 2. Graph of the function $\widehat{Q}_{1,X}$. Black solid curve expresses $\widehat{Q}_{1,X}(2)$. Red dashed curve expresses $\widehat{Q}_{1,X}(3)$. Green solid line expresses $\widehat{Q}_{1,X}(4)$. The values $\widehat{Q}_{1,X}(3)$ and $\widehat{Q}_{1,X}(4)$ are always positive. The value $\widehat{Q}_{1,X}(2)$ is positive only when $\epsilon \leq 0.3972$.

 $\widehat{Q}_{1,X}(2)$ is positive only for $\epsilon \geq 0.3972$. Hence, when $\epsilon < 0.3972$, due to Theorem 2, the capacity of case j=1 equals the capacity of the channel with inputs 3 and 4. In this case, we cannot use Algorithm 1 because the size of the input system is smaller than the size of the output system. Assume that $P_X(3) = 1 - p$ and $P_X(4) = p$. The mutual information between X and Y is

$$h(2\epsilon p) - ph(2\epsilon)$$

$$= (1 - 2\epsilon p)\log 2 + h(2\epsilon p)$$

$$- (1 - p)\log 2 - p(h(2\epsilon) + (1 - 2\epsilon)\log 2).$$
 (58)

Then, the maximum mutual information is achieved when

$$p = \frac{1}{2\epsilon \left(1 + e^{\frac{h(2\epsilon)}{2\epsilon}}\right)} = \frac{1}{2\epsilon \left(1 + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}} (2\epsilon)^{-1}\right)}$$
$$= \frac{1}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}.$$

The capacity of this case is

$$C_{**} := h\left(\frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}\right) - \frac{h(2\epsilon)}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}$$

$$= \frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}} \log\left(2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}\right)$$

$$-\left(\frac{(1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}\right) \log\left(\frac{(1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}}\right)$$

$$-\frac{1 - 2\epsilon}{2\epsilon + (1 - 2\epsilon)^{-\frac{1 - 2\epsilon}{2\epsilon}}} \log(1 - 2\epsilon). \tag{59}$$

Due to the symmetry, we can discuss the case with j=2. 5) Derivation of C(W): Based on the above discussion, we discuss the capacity of the channel W with input system $\mathcal{X} = \{1, 2, 3, 4\}$. When $0 \le \epsilon \le 0.3588$, C_1 is the capacity for the case j=1, and C_* is the capacity for the case j=3,4. Since $C_* \geq C_1$ in this case, C_* is the capacity of the channel W.

When $0.3588 < \epsilon \le 0.3972$, C_1 is the capacity for the case j=1, C_* is the capacity for the cases j=3, and C_4 is the capacity for the cases j=4. Since $C_4 \ge C_*$, C_1 in this case, C_4 is the capacity of the channel W.

In fact, as seen in Fig. 4, 4 curves C_1 , C_3 , C_4 , and C_{**} intersect at 0.3972. For $0.3972 < \epsilon \le \frac{1}{2}$, C_{**} is the capacity for the case j=1, C_3 or C_* is the capacity for the cases j=3, and C_4 is the capacity for the cases j=4. Since $C_{**} \ge C_*, C_3, C_4$ in this case, C_{**} is the capacity of the channel W. Overall, the capacity C(W) of the channel W is calculated as follows.

$$C(W) = \begin{cases} C_* & \text{when } 0 \le \epsilon \le 0.3588 \\ C_4 & \text{when } 0.3588 < \epsilon \le 0.3972 \\ C_{**} & \text{when } 0.3972 < \epsilon \le 1/2. \end{cases}$$
 (60)

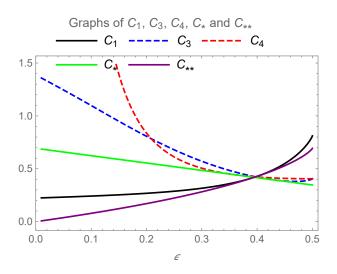


Fig. 3. Graphs of functions C_1, C_3, C_4, C_* and C_{**} . Black solid curve expresses C_1 . Blue dashed curve expresses C_3 . Red dashed curve expresses C_4 . Green solid line expresses C_* . Purple solid curve expresses C_{**} . Its enlarged view is given as Fig. 4.

IV. CAPACITY OF CLASSICAL-QUANTUM CHANNEL

Next, we discuss a classical-quantum channel from the classical system $\mathcal{X}:=\{1,\ldots,n_1\}$ to the quantum system \mathcal{H} with dimension n_2 , which is given as a set of density matrices $\{W_j\}_{j=1}^{n_1}$. We denote the set of density matrices on \mathcal{H} by $\mathcal{S}(\mathcal{H})$. For density matrices $\rho,\sigma\in\mathcal{S}(\mathcal{H})$, the entropy $H(\rho)$ and the divergence $D(\rho\|\sigma)$ are defined as

$$H(\rho) := -\operatorname{Tr} \rho \log \rho, \quad D(\rho \| \sigma) := \operatorname{Tr} \rho (\log \rho - \log \sigma).$$
 (61)

Under this classical-quantum channel, the capacity of classical-quantum channel $W = \{W_j\}_{j=1}^{n_1}$ is defined as [28], [29], [30], [31]

$$C_q(W) := \max_{P \in \mathcal{P}_{\mathcal{X}}} \sum_{x \in \mathcal{X}} P(x) D\left(W_x \bigg\| \sum_{x' \in \mathcal{X}} P(x') W_{x'}\right), \quad (62)$$

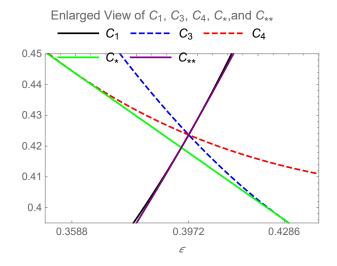


Fig. 4. Enlarged view of graphs of functions C_1, C_3, C_4, C_* and C_{**} . The explanations for 5 curves are the same as Fig. 3. 4 curves C_1, C_3, C_4 , and C_{**} intersect at 0.3972. In particular, C_1 touches C_{**} at 0.3972 C_3 and C_4 touch C_* at 0.3588 and 0.4286, respectively. That is, the inequalities $C_1 \geq C_{**}$ and $C_3, C_4 \geq C_*$ hold always.

The capacity of classical-quantum channel has the following form [32], [33]

$$C_q(W) = \min_{\sigma \in \mathcal{S}(\mathcal{H})} \max_{x \in \mathcal{X}} D(W_x || \sigma), \tag{63}$$

Statements similar to statements in Section II can be shown in this case of cq-channel by using quantum information geometry based on Kubo-Mori-Bogoliubov Fisher information, which is directly linked to quantum relative entropy (61) [5], [39, Chapter 7]. Here, for the calculation of $C_q(W)$, we consider only the algorithm corresponding to Algorithm 1. Hence, we consider the case under the following condition similar to Condition (C).

(D) $n_1=n_2^2$ and $W_1,\ldots,W_{n_2^2}$ are linearly independent. We choose n^2-1 linearly independent Hermitian matrices $A_1,\ldots,A_{n_2^2-1}$ on $\mathcal H$ such that

$$\operatorname{Tr} W_{n_2^2} A_j = 0 \tag{64}$$

for $j = 1, ..., n_2^2 - 1$. We define the matrix $(h_{i,j})$

$$h_{i,j} := \operatorname{Tr} W_i A_j. \tag{65}$$

Due to Condition (D), the n_2^2-1 vectors $\{(h_{x,j})_{j=1}^{n_2^2-1}\}_{x=1}^{n_2^2-1}$ are linearly independent.

Given an $n_2^2 - 1$ -dimensional parameter $\theta = (\theta^1, \dots, \theta^{n_2^2 - 1})$, we define the density matrix ρ_θ as

$$\rho_{\theta} = \exp\left(\sum_{j=1}^{n_2^2 - 1} A_j \theta^j - \phi(\theta)\right),\tag{66}$$

where

$$\phi(\theta) := \log \operatorname{Tr} \exp\left(\sum_{j=1}^{n_2^2 - 1} A_j \theta^j\right). \tag{67}$$

We have the following theorem.

8

Theorem 5: Assume that the parameters $\theta^1, \dots, \theta^{n_2^2-1}$ satisfies the condition

$$\sum_{j=1}^{n_2^2-1} h_{i,j} \theta^j = -H(W_i) + H(W_{n_2^2})$$
 (68)

for $i = 1, \dots, n_2^2 - 1$ Then, we have

$$D(W_x \| \rho_\theta) = \phi(\theta) - H(W_{n_2^2})$$
 (69)

for any element $x \in \mathcal{A}$.

Proof: The condition (68) implies that

$$\operatorname{Tr} W_i \sum_{j=1}^{n_2^2 - 1} A_j \theta^j = \sum_{j=1}^{n_2^2 - 1} h_{i,j} \theta^j = -H(W_i) + H(W_{n_2^2}).$$
(70)

For any element $x(\neq n_2^2) \in \mathcal{X}$, we have

$$D(W_x \| \rho_\theta) = \text{Tr } W_x (\log W_x - \log \rho_\theta)$$

$$= -H(W_x) - \text{Tr } W_x \left(\sum_{j=1}^{n_2^2 - 1} A_j \theta^j - \phi(\theta) \right)$$

$$= -H(W_x) - \left(-H(W_x) + H(W_{n_2^2}) - \phi(\theta) \right)$$

$$= \phi(\theta) - H(W_{n_2^2}). \tag{71}$$

Also, we have

$$D(W_{n_2^2} \| \rho_{\theta}) = \text{Tr } W_{n_2^2} (\log W_{n_2^2} - \log \rho_{\theta})$$

$$= -H(W_{n_2^2}) - \text{Tr } W_{n_2^2} \left(\sum_{j=1}^{n_2^2 - 1} A_j \theta^j - \phi(\theta) \right)$$

$$= -H(W_{n_2^2}) - (-\phi(\theta)) = \phi(\theta) - H(W_{n_2^2}). \tag{72}$$

The combination of (71) and (72) implies the desired statement.

Then, we consider the following condition:

(E) $D(W_x || \sigma)$ does not depend on $x \in \mathcal{X}$.

Lemma 4: When Condition (D) holds, only one density matrix σ on \mathcal{H} satisfies the condition (E). We denote such a density matrix by σ_* .

Proof:

We have

$$D(W_x \| \rho_\theta) = -H(W_x) - \sum_{i=1}^{n_2^2 - 1} \theta^j h_{x,j} - \phi(\theta).$$
 (73)

Condition (E) with $\sigma = \rho_{\theta}$ is rewritten as

$$-H(W_x) - \sum_{j=1}^{n_2^2 - 1} \theta^j h_{x,j} - \phi(\theta)$$

$$= -H(W_{n_2^2}) - \sum_{j=1}^{n_2^2 - 1} \theta^j h_{n_2^2} - \phi(\theta) = -H(W_{n_2^2}) - \phi(\theta)$$
(74)

for $x = 1, ..., n_2^2 - 1$, where the final equation follows from (64). This condition is rewritten as

$$-H(W_x) + H(W_{n_2^2}) = -\sum_{j=1}^{n_2^2 - 1} h_{x,j} \theta^j$$
 (75)

for $x = 1, ..., n_2^2 - 1$. Since the matrix $h_{i,j}$ is invertible, only one vector $\theta = (\theta^1, ..., \theta^{n_2^2 - 1})$ satisfies (107), i.e., Condition (E).

The relation (63) guarantees that

$$C_q(W) \le D(W_x \| \sigma_*) \tag{76}$$

for any element $x \in \mathcal{X}$.

Due to Theorem 5, when θ satisfies the condition (68), the density matrix ρ_{θ} equals σ_{*} . To construct our algorithm, we add the n_{2}^{2} -th Hermitian matrix $A_{n_{2}^{2}}$ and define $h_{i,j}$ by (65) for $i,j=1,\ldots,n_{2}^{2}$. To find the input distribution $\widehat{Q}_{X,*}$ to achieve the maximum (63), we consider the equation $\sum_{x}W(y|x)\widehat{Q}_{X,*}(x)=\rho_{\theta}$, which can be rewritten as

$$\sum_{x \in \mathcal{X}} \widehat{Q}_{X,*}(x) h_{x,j} \left(= \sum_{x \in \mathcal{X}} \widehat{Q}_{X,*}(x) \operatorname{Tr} W_x A_j \right) = \operatorname{Tr} \rho_{\theta} A_j.$$
(77)

If we have

$$\widehat{Q}_{X,*}(x) \ge 0 \text{ for } x \in \mathcal{X},\tag{78}$$

since Lemma 4 guarantees that $\rho_{\theta} = \sigma_*$, (76) guarantees that

$$D(W_x \| \rho_\theta) = C_q(W), \tag{79}$$

i.e., the solution gives the capacity.

Therefore, in the same way as Algorithm 1, we propose Algorithm 2 based on Theorem 5 and Lemma 4.

Now, we describe two Hermitian matrices X, Y on \mathcal{H} by two n^2 -dimensional vectors $x = (x_j)_{j=1}^{n_2^2}$ and $y = (y_j)_{j=1}^{n_2^2}$ as follows

$$X = \sum_{j=1}^{n_2} x_j |j\rangle\langle j|$$

$$+ \sum_{j=1}^{n_2-1} \sum_{j'=1}^{j-1} \frac{x_{n_2+j(j-1)/2+j'}}{\sqrt{2}} (|j\rangle\langle j'| + |j'\rangle\langle j|)$$

$$+ \sum_{j=1}^{n_2-1} \sum_{j'=1}^{j-1} \frac{x_{n_2(n_2+1)/2+j(j-1)/2+j'}}{\sqrt{2}} (i|j\rangle\langle j'| - i|j'\rangle\langle j|).$$
(80)

Here, the matrix Y is defined in the same way by using $y = (y_j)_{j=1}^{n_2^2}$. Then, we have

$$\operatorname{Tr} XY = \sum_{j=1}^{n_2^2} x_j y_j. \tag{81}$$

In this sense, $(W_1,\ldots,W_{n_2^2})$ and $(A_1,\ldots,A_{n_2^2})$ can be considered as $n_2^2 \times n_2^2$ matrices. Then, Step 1 of Algorithm 2 can be done by calculating the inverse matrix of the matrix corresponding to $(W_1,\ldots,W_{n_2^2})$. Hence, Step 1 has calculation complexity $O(n_2^6)$. In Step 2, the calculation of all of $H(W_i)$ needs calculation complexity $O(n_1n_2^3) = O(n_2^5)$. Hence, Step 2 has calculation complexity $O(n_2^5)$ in total. In Step 3, the calculation of $\sum_{j=1}^{n_2^2-1} A_j \theta^j$ has calculation complexity $O(n_2^4)$. and the calculation of $\exp\left(\sum_{j=1}^{n_2^2-1} A_j \theta^j\right)$ and its

trace has calculation complexity $O(n_2^3)$. Step 3 has calculation complexity $O(n_2^4)$ in total. In Step 5, the calculation of all of $\operatorname{Tr} \rho_\theta A_j$ has calculation complexity $O(n_1 n_2^2) = O(n_2^4)$ since $\exp\left(\sum_{j=1}^{n_2^2-1} A_j \theta^j\right)$ and its trace are already calculated. Hence, the total calculation complexity is $O(n_2^6)$.

Algorithm 2 Exact algorithm for classical channel capacity

Step 1: Choose $A_1, \ldots, A_{n_2^2}$ such that $h_{i,j}$ is the identity matrix.

Step 2: Set the parameter $\theta^i = -H(W_i) + H(W_{n_2^2})$ for $i = 1, \ldots, n_2^2 - 1$, which is the solution of (68).

Step 3: Calculate $\phi(\theta)$ by using (67).

Step 4: Calculate $Q_{X,*}(x) := \operatorname{Tr} \rho_{\theta} A_j$, where ρ_{θ} is calculated by (66).

Step 5: If the condition (78) holds, we consider that (79) holds and output $\phi(\theta) - H(W_n)$ as the capacity. Otherwise, we output "the capacity cannot be computed."

V. COMPARISON

In the calculation of the capacity of classical channel, when an error ϵ is allowed, the conventional method [2], [3] has calculation amount $O(\frac{n_1 n_2 \log n_1}{\epsilon})$ because each iteration has calculation amount $n_1 n_2$ and the number of iterations is $O(\frac{\log n_1}{\epsilon})$. While it is smaller than our method (Algorithm 1) when $n_1 = n_2$, our method derives the exact value of the maximum without iteration.

When only Condition (A) holds, we can consider to solve the minimization (19) due to Theorem 4. However, it is difficult to analytically solve (19) in general. Since this method needs larger calculation amount to obtain $\theta^1, \ldots, \theta^{n_1-1}$, the algorithm based on Theorem 4 does not have advantage over the conventional method [2], [3] except for the case that the minimization (19) is analytically solved.

Next, we compare Algorithm 2 with existing algorithms for the capacity of a classical-quantum channel. The algorithm by [15], [19] has calculation complexity $O(\frac{(n_1n_2^2+n_2^3)\log n_1}{\epsilon}+n_1n_2^3)$ The algorithm by [17] has calculation complexity $O(\frac{\max(n_1,n_2)n_2^3\sqrt{\log n_1}}{\epsilon})$. Unfortunately, these existing algorithms are smaller than our method, Algorithm2 when $n_1=n_3(n_2-1)+1$ or $n_1=n_2^2$. However, our method derives the exact value of the maximum without iteration when we calculate the inverse matrix exactly. This point is an advantage over existing methods.

Indeed, in practice, to evaluate the precision of our algorithm, we need to evaluate the precision for each step including the calculations of the inverse matrix, logarithm, and exponential. Such an analysis is left for a future study.

VI. CONCLUSION AND FUTURE STUDY

We have proposed an exact algorithm to calculate the channel capacities of classical and classical-quantum channels. However, we have various conditions to apply our algorithm. Therefore, it is a future problem to remove conditions. Indeed, Toyota [11] studied information geometrical structure [5] for

Arimoto-Blahut algorithm for the capacity of a classical channel. Hence, it is an interesting topic to derive an information theoretical characterization of our method.

Further, it is a challenging problem to extend our method to the maximization of Gallager's function, i.e., Rényi mutual information, including classical-quantum setting, which is related to the exponential decreasing rate [34], [35] of the decoding error probability and the strong converse exponent [36], [37], [38]. As another future study, we can consider an extension of our algorithm to wire-tap channel capacity [12], [13], [14], [25].

VII. ADDITIONAL DISCUSSION

After completing the review process of this paper, the author found the reference [40] that has already derived an analytical calculation method for the channel capacity under a certain condition. The reference [41] derived the same method as [40]. The method by [40], [41] is the following; First, assume $n_1 = n_2$ and the existence of the inverse matrix $(g_{x,x'})_{x,x'}$ of the transition matrix $(W(y|x))_{x,y}$, i.e., $\sum_{x'} g_{x,x'} W(y|x') = \delta_{y,x}$. Then, the capacity is calculated as

$$C = \log \left(\sum_{x'=1}^{n_1} G_{x'} \right), \tag{82}$$

where

$$G_{x'} := \sum_{x,y} g_{s,x'} W(y|x) \log W(y|x). \tag{83}$$

In addition, the input distribution P_* realizing the capacity is given as

$$P_*(x) = \exp(-C) \sum_{x'=1}^{n_1} g_{x,x'} \exp(G_{x'}).$$
 (84)

Our method has the following advantage over the above method. First, our method works even with classical-quantum channel while their method works only with classical channel. Second, their method needs to assume the existence of the inverse matrix of the transition matrix $(W(y|x))_{x,y}$. Although Algorithm 1 requires the existence of the inverse matrix of the transition matrix $(W(y|x))_{x,y}$ in Step 1, our method can relax this condition in the following way because it is sufficient to find functions f_1, \ldots, f_{n_1-1} satisfying the conditions (8) and (9) with $h_{i,j} = \delta_{i,j}$ and $n_2 = n_1$.

Now, instead of the existence of the inverse matrix of the transition matrix $(W(y|x))_{x,y}$, we assume the existence of the inverse matrix of the matrix $(W(y|x) - \frac{W(n_1|x)W(y|n_1)}{W(n_1|n_1)})_{x,y=1,\dots,n_1-1}$ by $c_{j,y}$, i.e., $\sum_{y=1}^{n_1-1} c_{j,y}(W(y|x) - \frac{W(n_1|x)W(y|n_1)}{W(n_1|n_1)}) = \delta_{x,j}$. Then, we set $f_1,\dots f_{n_1-1}$ as $f_j(y) = c_{j,y}$ for $y=1,\dots,n_1-1$, $f_j(n_1) = -\sum_{y=1}^{n_1-1} c_{j,y} \frac{W(y|n_1)}{W(n_1|n_1)}$, and $f_j(y) = 0$ for $y=n_1,\dots,n_2$. We find that the functions f_1,\dots,f_{n_1-1} satisfy the conditions (8) and (9) with $h_{i,j} = \delta_{i,j}$. Since the existence of the inverse matrix of the matrix $(W(y|x) - \frac{W(n_1|x)W(y|n_1)}{W(n_1|n_1)})_{x,y=1,\dots,n_1-1}$ is a weaker condition than the existence of the inverse matrix of the transition matrix $(W(y|x))_{x,y}$, our method is better than the method by [40] even for the classical channel.

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APPENDIX A

SUMMARY FOR INFORMATION GEOMETRY

To show Theorems 1 and 2, we summarize basic knowledge for information geometry, which was established in the reference [5]. The following contents are used in Appendices D and E. Given a finite probability space \mathcal{X} , we define an exponential family as follows. Consider l linearly independent random variables f_1, \ldots, f_l on \mathcal{X} . We define the distribution $P_{\theta, X}$ as

$$P_{\theta,X}(x) := e^{\sum_{j=1}^{l} \theta^{j} f_{j}(x) - \phi(\theta)},$$
 (85)

where $\phi(\theta) := \log \sum_{x \in \mathcal{X}} e^{\sum_{j=1}^{l} \theta^{j} f_{j}(x)}$. The set $\mathcal{E} := \{P_{\theta,X} | \theta \in \mathbb{R}^{l}\} \subset \mathcal{P}_{\mathcal{X}}$ is called an exponential family generated by random variables f_{1}, \ldots, f_{l} . Also, the set

$$\mathcal{M} := \{ Q_X \in \mathcal{P}_{\mathcal{X}} | Q_X \text{ satisfies (87).} \}$$
 (86)

is called the mixture family generated by the constraint

$$\sum_{x \in \mathcal{X}} f_j(x) Q_X(x) = a_j. \tag{87}$$

The following is a typical example of a mixture family. For a subset $\mathcal{X}_0 \subset \mathcal{X}$, as a generalization of \mathcal{M}_0 , we define the mixture family $\mathcal{M}_{\mathcal{X}_0}$ as

$$\mathcal{M}_{\mathcal{X}_0} := \Big\{ Q_Y \in \mathcal{P}_{\mathcal{Y}} \Big| Q_Y = \sum_{x \in \mathcal{X} \setminus \mathcal{X}_0} c(x) W_x, \sum_{x \in \mathcal{X} \setminus \mathcal{X}_0} c(x) = 1 \Big\}.$$
(88)

When \mathcal{X}_0 is the empty set, $\mathcal{M}_{\mathcal{X}_0}$ coincides with \mathcal{M}_0 . Also, we simplify $\mathcal{M}_{\{x\}}$ to \mathcal{M}_x .

The following is known as Pythagorean theorem [5].

Theorem 6: There uniquely exists an element $P_{X,*} \in \mathcal{E} \cap \mathcal{M}$. Any elements $P_{X,1} \in \mathcal{M}$ and $P_{X,2} \in \mathcal{E}$ satisfy

$$D(P_{X,1}||P_{X,2}) = D(P_{X,1}||P_{X,*}) + D(P_{X,*}||P_{X,2}).$$
(89)

Using this theorem, we can show the following corollaries. Corollary 1: Given a distribution Q_X on \mathcal{X} , there uniquely exists an element $Q_{X,*} \in \mathcal{M}$ such that

$$D(P_{X,1}||Q_X) = D(P_{X,1}||Q_{X,*}) + D(Q_{X,*}||Q_X)$$
 (90)

for any element $P_{X,1} \in \mathcal{M}$. $Q_{X,*}$ is called the projection of Q_X to \mathcal{M} , and is denoted by $\Gamma_{\mathcal{M}}^{(m)}(Q_X)$.

Corollary 2: Given a distribution Q_X on \mathcal{X} , there uniquely exists an element $Q_{X,*} \in \mathcal{E}$ such that

$$D(Q_X || P_{X,2}) = D(Q_X || Q_{X,*}) + D(Q_{X,*} || P_{X,2})$$
(91)

for any element $P_{X,2} \in \mathcal{E}$. $Q_{X,*}$ is called the projection of Q_X to \mathcal{E} , and is denoted by $\Gamma_{\mathcal{E}}^{(e)}(Q_X)$.

Now, we consider a one-parameter exponential family $\{P_t\}$. Lemma 5: For $t_1 \le t_2 \le t_3$, we have

$$D(P_{t_1}||P_{t_2}) + D(P_{t_2}||P_{t_3}) \le D(P_{t_1}||P_{t_3}). \tag{92}$$

Proof: Let J_t be the Fisher information in the one-parameter exponential family $\{P_t\}$. Then, we have

$$D(P_t || P_{t'}) = \int_{t'}^{s} J_s(s - t') ds.$$
 (93)

The expression (93) implies (92).

APPENDIX B PROOF OF LEMMA 1

We show this lemma by contradiction. We assume that $D(W_x\|Q_Y)$ depends on $x \in \operatorname{supp}(Q_X)$. Then, the set $\mathcal{X}_0 := \{x_0 \in \mathcal{X} | D(W_{x_0}\|Q_Y) < \max_{x \in \mathcal{X}} D(W_x\|Q_Y) \}$ is not empty. With a small $\epsilon > 0$, we choose $Q_{X,\epsilon}$ as

$$Q_{X,\epsilon}(x_0) := Q_X(x_0) - \frac{\epsilon}{|\mathcal{X}_0|}$$
(94)

$$Q_{X,\epsilon}(x') := Q_X(x') + \frac{\epsilon}{|\mathcal{X} \setminus \mathcal{X}_0|}$$
 (95)

for $x_0 \in \mathcal{X}_0$ and $x' \in \mathcal{X} \setminus \mathcal{X}_0$. The above choices of $Q_{X,\epsilon}$ guarantee that $W \cdot Q_{X,\epsilon}$ is closer to $W_{x'}$ than $W \cdot Q_X$ for $x' \in \mathcal{X} \setminus \mathcal{X}_0$, which implies the relation

$$D(W_{x'}||W \cdot Q_{X,\epsilon}) < D(W_{x'}||W \cdot Q_X).$$
 (96)

Since $W \cdot Q_X$ is closer to W_{x_0} than $W \cdot Q_{X,\epsilon}$ for $x_0 \in \mathcal{X}_0$, we have

$$D(W_{x_0} || W \cdot Q_X) < D(W_{x_0} || W \cdot Q_{X,\epsilon}). \tag{97}$$

Since $D(W_{x_0}||W \cdot Q_X) < D(W_{x'}||W \cdot Q_X)$, we can choose a sufficiently small $\epsilon > 0$ such that

$$D(W_{x_0} || W \cdot Q_X) < D(W_{x_0} || W \cdot Q_{X,\epsilon})$$

$$< D(W_{x'} || W \cdot Q_X). \tag{98}$$

The relations (96) and (98) imply

$$\max_{x \in \mathcal{X}} D(W_x || W \cdot Q_{X,\epsilon})$$

$$< D(W_{x'} || W \cdot Q_X) = \max_{x \in \mathcal{X}} D(W_x || W \cdot Q_X). \tag{99}$$

However, $W \cdot Q_X$ is the minimizer of (3), which contradicts (99).

APPENDIX C PROOF OF LEMMA 2

We choose n_2-1 linearly independent functions f_1,\ldots,f_{n_2-1} on $\mathcal Y$ such that they are not constant function and

$$\sum_{y \in \mathcal{V}} W_x(y) f_j(y) = 0, \quad \sum_{y \in \mathcal{V}} W_{n_1}(y) f_{j'}(y) = 0 \quad (100)$$

for $j=n_1,\ldots,n_2-1,$ $j'=1,\ldots,n_2-1,$ and $x=1,\ldots,n_1.$ In fact, the set \mathcal{M}_0 is rewritten as

$$\mathcal{M}_{0} = \left\{ Q_{Y} \in \mathcal{P}_{\mathcal{Y}} \middle| \sum_{y \in \mathcal{Y}} Q_{Y}(y) f_{j}(y) = 0 \text{ for } j = n_{1}, \dots, n_{2} - 1 \right\}.$$

$$(101)$$

Then, any distribution on \mathcal{Y} is parameterized as

$$P_{\theta,Y}(y) := e^{\sum_{j=1}^{n_2-1} \theta^j f_j(y) - \phi(\theta)}, \tag{102}$$

where $\phi(\theta):=\log\sum_{y\in\mathcal{Y}}e^{\sum_{j=1}^{n_2-1}\theta^jf_j(y)}$. For any vector $\theta_1=(\theta^1,\ldots,\theta^{n_1-1})$, there exist parameters $\theta_2(\theta_1)=(\theta^{n_1}(\theta_1),\ldots,\theta^{n_2-1}(\theta_1))$ such that $P_{(\theta_1,\theta_2(\theta_1)),Y}\in\mathcal{M}_0$. This fact can be shown as follows. Given a vector θ_1 , we define the set

$$\mathcal{G}(\theta_1) := \left\{ \left(\sum_{y \in \mathcal{Y}} P_{(\theta_1, \theta_2), Y}(y) f_j(y) \right)_{j=n_1}^{n_2 - 1} \middle| \theta_2 \in \mathbb{R}^{n_2 - n_1} \right\}.$$
(103)

This set equals the inner of the convex hull of $\{(f_j(y))_{j=n_1}^{n_2-1}\}_{y\in\mathcal{Y}}$. That is, the set $\mathcal{G}(\theta_1)$ does not depend on $\theta_1\in\mathbb{R}^{n_1-1}$. The first equation shows that the origin $(0,\ldots,0)$ belongs to $\bigcup_{\theta_1\in\mathbb{R}^{n_1-1}}\mathcal{G}(\theta_1)$. Hence, the origin $(0,\ldots,0)$ belongs to $\mathcal{G}(\theta_1)$ for an element $\theta_1\in\mathbb{R}^{n_1-1}$. Therefore, there exist parameters $\theta_2(\theta_1)=(\theta^{n_1}(\theta_1),\ldots,\theta^{n_2-1}(\theta_1))$ such that $P(\theta_1,\theta_2(\theta_1)),Y\in\mathcal{M}_0$.

Then, we choose the parameters $h_{x,j}$ as

$$h_{x,j} := \sum_{y \in \mathcal{Y}} W_x(y) f_j(y) \tag{104}$$

for $j=1,\ldots,n_1-1$. Since functions f_1,\ldots,f_{n_1-1} are linearly independent, due to Condition (A), the vectors $\{(h_{x,j})_{j=1}^{n_1-1}\}_{x=1}^{n_1-1}$ are linearly independent.

Then, we have

$$D(W_x || P_{\theta,Y}) = -H(W_x) - \sum_{j=1}^{n_1 - 1} \theta^j h_{x,j} - \phi(\theta_1, \theta_2(\theta_1)).$$
(105)

Condition (B) with $Q_Y = P_{\theta,Y}$ is rewritten as

$$-H(W_x) - \sum_{j=1}^{n_1-1} \theta^j h_{x,j} - \phi(\theta_1, \theta_2(\theta_1))$$

$$= -H(W_{n_1}) - \phi(\theta_1, \theta_2(\theta_1))$$
(106)

for $x = 1, ..., n_1$. This condition is rewritten as

$$-H(W_x) + H(W_{n_1}) = \sum_{j=1}^{n_1-1} h_{x,j} \theta^j$$
 (107)

for $x=1,\ldots,n_1-1$. Since the matrix $h_{x,j}$ is invertible, only one vector $\theta_1=(\theta^1,\ldots,\theta^{n_1-1})$ satisfies (107), i.e., Condition (B).

APPENDIX D PROOF OF THEOREM 1

Assume the condition (ii). For $Q_X(\neq \widehat{Q}_{X,*}) \in \mathcal{P}_{\mathcal{X}}$, we have $\max_{x \in \mathcal{X}} D(W_x || W \cdot Q_X) > \max_{x \in \mathcal{X}} D(W_x || W \cdot \widehat{Q}_{X,*})$ because $W \cdot Q_X$ and belongs to \mathcal{M}_0 and only one element of \mathcal{M}_0 satisfy Condition (B) due to Lemma 2. Hence, $Q_{Y,*}$ achieves C(W), which implies Condition (i).

Assume the condition (i). There exists $Q_X \in P_X$ such that $D(W_x || Q_{Y,*}) = \sum_{x \in \mathcal{X}} Q_X(x) D(W_x || W \cdot Q_X)$. For any element $x \in \operatorname{supp}(Q_X)$, we have

$$D(W_x || W \cdot Q_X) = D(W_x || Q_{Y_x}). \tag{108}$$

Then, the distribution $\overline{Q}_{Y,*}:=\Gamma^{(m)}_{\mathcal{M}_0}(Q_{Y,*})$ satisfies $D(W_x\|Q_{Y,*})=D(W_x\|\overline{Q}_{Y,*})+D(\overline{Q}_{Y,*}\|Q_{Y,*}),$ where the projection $\Gamma^{(m)}_{\mathcal{M}_0}$ is defined in Appendix A. Hence, $D(W_x\|Q_{Y,*})\geq D(W_x\|\overline{Q}_{Y,*}).$

Since $\min_{Q_Y \in \mathcal{P}_{\mathcal{Y}}} \max_{x \in \mathcal{X}} D(W_x \| Q_Y) = \max_{x \in \mathcal{X}} D(W_x \| W \cdot Q_X)$, we have $D(W_x \| W \cdot Q_X) = D(W_x \| Q_{Y,*}) = D(W_x \| \overline{Q}_{Y,*})$ for $x \in \operatorname{supp}(Q_X)$. Hence, $D(\overline{Q}_{Y,*} \| Q_{Y,*}) = 0$, i.e., $\overline{Q}_{Y,*} = Q_{Y,*}$. That is, $Q_{Y,*}$ belongs to \mathcal{M}_0 . Due to Condition (A) and Lemma 2, the condition (108) uniquely determines $Q_{Y,*}$. Hence, $W \cdot \widehat{Q}_{X,*} = Q_{Y,*} = W \cdot Q_X$. Condition (A) guarantees the relation $\widehat{Q}_{X,*} = Q_X$, which implies the condition (ii).

APPENDIX E PROOF OF THEOREM 2

Due to Condition (A), there uniquely exists a distribution $Q_{X,*} \in \mathcal{P}_{\mathcal{X}}$ to achieve the capacity C(W). It is sufficient to show that $Q_{X,*}(x_0) = 0$ for any element $x_0 \in \mathcal{N}(\widehat{Q}_{X,*})$. For this aim, we fix an arbitrary element $x_0 \in \mathcal{N}(\widehat{Q}_{X,*})$.

Step 1: We show that there exists a distribution $Q_{Y,0} \in \mathcal{M}_{x_0}$ such that

$$\max_{x \in \mathcal{X}} D(W_x || Q_{Y,*}) \ge \max_{x \in \mathcal{X}} D(W_x || Q_{Y,0}) = D(W_{x'} || Q_{Y,0})$$
(109)

for any element $x' \in \mathcal{X} \setminus \{x_0\}$.

Since $Q_{Y,*}$ is the unique element of \mathcal{M}_0 to satisfy Condition B, any element $x' \in \mathcal{X} \setminus \{x_0\}$ satisfies

$$D(W_{x_0}||Q_{Y,*}) = D(W_{x'}||Q_{Y,*}).$$
(110)

We choose a function f_{x_0} on \mathcal{X} such that

$$\sum_{y \in \mathcal{Y}} f_{x_0}(y) W_{x_0}(y) = 1, \tag{111}$$

$$\sum_{y \in \mathcal{Y}} f_{x_0}(y) W_x(y) = 0 \tag{112}$$

for any element $x(\neq x_0) \in \mathcal{X}$. We denote $-\widehat{Q}_{X,*}(x_0) > 0$ by a. Then, we have

$$\frac{1}{1+a}Q_{Y,*} + \frac{a}{1+a}W_{x_0} \in \mathcal{M}_{x_0}.$$
 (113)

The combination of (112) and (113) implies that

$$\sum_{y \in \mathcal{Y}} f_{x_0}(y) \left(\frac{1}{1+a} Q_{Y,*}(y) + \frac{a}{1+a} W_{x_0}(y) \right) = 0.$$
 (114)

Then, the combination of (111) and (114) yields that

$$\sum_{y \in \mathcal{Y}} f_{x_0}(y) Q_{Y,*}(y) = -a. \tag{115}$$

The distribution $Q_{Y,0} := \Gamma^{(m)}_{\mathcal{M}_{x_0}}(Q_{Y,*}) \in \mathcal{M}_{x_0}$ satisfies

$$D(Q_Y || Q_{Y,*}) = D(Q_Y || Q_{Y,0}) + D(Q_{Y,0} || Q_{Y,*})$$
(116)

for any distribution $Q_Y \in \mathcal{M}_{x_0}$. We define the exponential family $\mathcal{E}_1 := \{Q_{Y,t}\}_{t \in \mathbb{R}}$ as

$$Q_{Y,t}(y) := Q_{Y,0}(y)e^{tf_{x_0}(y) - \varphi(y)}, \tag{117}$$

where

$$\varphi(y) := \log \sum_{y \in \mathcal{Y}} Q_{Y,0}(y) e^{t f_{x_0}(y)}.$$
(118)

Hence, $Q_{Y,0}$ coincides with the case with t=0. We choose t_* such that $Q_{Y,t_*}=Q_{Y,*}$. The relation (115) guarantees that $t_*<0$. Also, we choose t_0 as $Q_{Y,t_0}=\Gamma^{(e)}_{\mathcal{E}_1}(W_{x_0})$. Then, we have

$$D(W_{x_0}||Q_{Y,t}) = D(W_{x_0}||Q_{Y,t_0}) + D(Q_{Y,t_0}||Q_{Y,t}).$$
(119)

for any $t_0 \in \mathbb{R}$. The relation (115) guarantees that $t_0 > 0$. Since $t_* < 0$ and $t_0 > 0$, Lemma 5 yields that

$$D(Q_{Y,t_0}||Q_{Y,0}) \le D(Q_{Y,t_0}||Q_{Y,t_*}) - D(Q_{Y,0}||Q_{Y,t_*}).$$
(120)

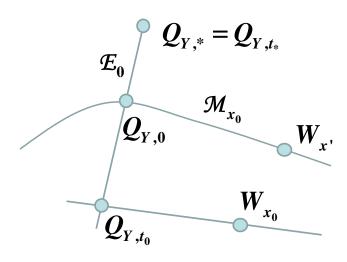


Fig. 5. Relation among various distributions appearing in Step 1 of the proof of Theorem 2. This figure shows the topological relation among the distributions $Q_{Y,*} = Q_{Y,t_*}, Q_{Y,0}, \ Q_{Y,t_0}$, the exponential family \mathcal{E}_0 , and the mixture family \mathcal{M}_{x_0} .

The combination of (119) and (120) guarantees that

$$D(W_{x_0} \| Q_{Y,0}) \stackrel{(a)}{=} D(W_{x_0} \| Q_{Y,t_0}) + D(Q_{Y,t_0} \| Q_{Y,0})$$

$$\stackrel{(b)}{\leq} D(W_{x_0} \| Q_{Y,t_0}) + D(Q_{Y,t_0} \| Q_{Y,t_*}) - D(Q_{Y,0} \| Q_{Y,t_*})$$

$$\stackrel{(c)}{=} D(W_{x_0} \| Q_{Y,t_*}) - D(Q_{Y,0} \| Q_{Y,t_*})$$

$$\stackrel{(d)}{=} D(W_{x_0} \| Q_{Y,*}) - D(Q_{Y,0} \| Q_{Y,*})$$

$$\stackrel{(e)}{=} D(W_{x'} \| Q_{Y,*}) - D(Q_{Y,0} \| Q_{Y,*}) \stackrel{(f)}{=} D(W_{x'} \| Q_{Y,0}) \quad (121)$$

for any element $x' \in \mathcal{X} \setminus \{x_0\}$. Each step is shown in the following way. Steps (a) and (c) follow from (119). Step (b) follows from (120). Step (d) follows from $Q_{Y,t_*} = Q_{Y,*}$. Step (e) follows from (110). Step (f) follows from (116). (121) shows the following two facts. One is $D(W_{x'}\|Q_{Y,0})$ does not depend on $x' \in \mathcal{X} \setminus \{x_0\}$. The other is $D(W_{x_0}\|Q_{Y,0}) \leq D(W_{x'}\|Q_{Y,0})$. The combination of these two facts implies

$$\max_{x \in \mathcal{X}} D(W_x || Q_{Y,0}) = D(W_{x'} || Q_{Y,0})$$

$$\stackrel{(a)}{\leq} D(W_{x'} || Q_{Y,*}) \leq \max_{x \in \mathcal{X}} D(W_x || Q_{Y,*}), \qquad (122)$$

where Step (a) follows from (116). Hence, we obtain (109). **Step 2:** We choose a function $\widehat{Q}_{X,1}$ on $\mathcal{X} \setminus \{x_0\}$ such that

$$\sum_{x \in \mathcal{X} \setminus \{x_0\}} \widehat{Q}_{X,1}(x) W_x = Q_{Y,1}, \tag{123}$$

where $\widehat{Q}_{X,1}$ uniquely exists because $Q_{Y,1} \in \mathcal{M}_{x_0}$. We show the desired statement $Q_{X,*}(x_0) = 0$ when $\widehat{Q}_{X,1}(x) \leq 0$ for $x \in \mathcal{X} \setminus \{x_0\}$.

In this case, it is sufficient to show that $Q_{X,*}(x_0) = \widehat{Q}_{X,1}$, i.e., $\widehat{Q}_{X,1}$ achieves the capacity C(W). We have

$$\sum_{x \in \mathcal{X} \setminus \{x_0\}} Q_{X,1}(x) D(W_x || Q_{Y,1}) \stackrel{(a)}{=} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x || Q_{Y,1})$$

$$\stackrel{(b)}{=} \max_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \sum_{x \in \mathcal{X} \setminus \{x_0\}} Q_X(x) D(W_x || W \cdot Q_X)$$

$$\stackrel{(c)}{=} \min_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x || W \cdot Q_X)$$

$$\stackrel{(d)}{=} \min_{Q_Y \in \mathcal{M}_{x_0}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x || Q_Y)$$

$$\stackrel{(e)}{\leq} \min_{Q_X \in \mathcal{P}_{\mathcal{X}}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x || W \cdot Q_X)$$

$$\leq \min_{Q_X \in \mathcal{P}_X} \max_{x \in \mathcal{X}} D(W_x || W \cdot Q_X)
\leq \max_{x \in \mathcal{X}} D(W_x || W \cdot Q_{X,1}).$$
(124)

In the above relations, $\mathcal{P}_{\mathcal{X}\setminus\{x_0\}}$ means the set of probability distributions on the set $\mathcal{X}\setminus\{x_0\}$. Each step is shown in the following way. Step (a) follows from the second equation in (109). Step (b) follows from Theorem 1. Step (c) follows from (3). Step (d) is shown as follows. Since $Q_Y\mapsto D(W_x\|Q_Y)$ is convex, $Q_Y\mapsto\max_{x\in\mathcal{X}\setminus\{x_0\}}D(W_x\|Q_Y)$ is also convex. Since $Q_{Y,1}$ achieves a local minimum, it also achieve the global minimum in \mathcal{M}_{x_0} .

Step (e) is shown as follows. For $Q_X \in \mathcal{P}_{\mathcal{X}}$, the distribution $Q'_Y := \Gamma^{(m)}_{\mathcal{M}_{\pi^n}}(W \cdot Q_X)$ satisfies

$$D(W_x || W \cdot Q_X) = D(W_x || Q'_Y) + D(Q'_Y || W \cdot Q_X)$$

 $\geq D(W_x || Q'_Y) \text{ for } x \in \mathcal{X} \setminus \{x_0\},$ (125)

which shows (e).

Hence, we have

$$C(W) = \sum_{x \in \mathcal{X} \setminus \{x_0\}} Q_{X,1}(x) D(W_x || W \cdot Q_{X,1}).$$
 (126)

Step 3: We show the desired statement $Q_{X,*}(x_0) = 0$ when there exists $x_1 \in \mathcal{X} \setminus \{x_0\}$ such that $Q_{X,1}(x_1) < 0$. Applying the same discussion as Step 1 with replacing $Q_{Y,*}$ and $Q_{Y,0}$ by $Q_{Y,1}$ and $Q_{Y,2}$, respectively, we find that there exists a distribution $Q_{Y,2} \in \mathcal{M}_{\{x_0,x_1\}}$ such that

$$\max_{x \in \mathcal{X}} D(W_x \| Q_{Y,1}) \ge \max_{x \in \mathcal{X}} D(W_x \| Q_{Y,2}) = D(W_{x'} \| Q_{Y,2})$$
(127)

for any element $x' \in \mathcal{X} \setminus \{x_0, x_1\}$. Then, we choose $\widehat{Q}_{X,2}$ in the same way as (123). If $\widehat{Q}_{X,2}(x) \geq 0$ for $x \in \mathcal{X} \setminus \{x_0, x_1\}$, we find that $Q_{X,*}(x_0) = 0$ in the same way as Step 2. Otherwise, we repeat the above procedure up to i times

until we have $\widehat{Q}_{X,i}(x) \geq 0$ for $x \in \mathcal{X} \setminus \{x_0, x_1, \dots, x_{i-1}\}$. Once we obtain the above condition, we find $Q_{X,*}(x_0) = 0$ in the same way as Step 2.

APPENDIX F PROOF OF LEMMA 3

To show Lemma 3, we prepare functions $\overline{f}_1,\ldots,\overline{f}_{n_2-1}$ to satisfy the condition in Theorem 4. We denote the distribution defined in (10) based on these functions $\overline{f}_1,\ldots,\overline{f}_{n_2-1}$ by $\overline{P}_{\theta,Y}$. Such functions are given as linear combination of the original functions f_1,\ldots,f_{n_2-1} by using coefficient $a^j_{i'}$ as

$$\sum_{j} f_j a_{j'}^j = \overline{f}_{j'}. \tag{128}$$

Hence, we have

$$\sum_{j'=1}^{n_2-1} \overline{f}_{j'}(y)\theta^{j'} = \sum_{j=1}^{n_2-1} f_j(y) \left(\sum_{j'=1}^{n_2-1} a_{j'}^j \theta^{j'}\right). \tag{129}$$

Using this relation, we find that $\overline{P}_{\theta,Y} = P_{\overline{\theta},Y}$, where $\overline{\theta}^j = \sum_{j'=1}^{n_2-1} a^j_{j'} \theta^{j'}$. Thus, the set \mathcal{E}_0 can be characterized with the new functions $\overline{f}_1, \ldots, \overline{f}_{n_2-1}$. Therefore, without loss of generality, we can assume that the functions f_1, \ldots, f_{n_2-1} satisfies the condition in Theorem 4.

We choose $\theta^1, \dots, \theta^{n_1-1}$ satisfies the condition (9). Then, we have

$$\mathcal{E}_{0} = \{ P_{\theta^{1},\dots,\theta^{n_{1}-1},\eta^{n_{1}},\dots,\eta^{n_{2}-1},Y} | (\eta^{n_{1}},\dots,\eta^{n_{2}-1}) \in \mathbb{R}^{n_{2}-n_{1}} \}.$$
(130)

Hence, \mathcal{E}_0 is an exponential family generated by $f_{n_1}, \ldots, f_{n_2-1}$.

Since $f_{i,j} = 0$ for $j = n_1 + \dots, n_2 - 1$, \mathcal{M}_0 can be written

$$\mathcal{M}_{0}$$
= $\{Q_{Y} \in \mathcal{P}_{\mathcal{Y}} | \sum_{y \in \mathcal{Y}} f_{j}(y)Q_{Y}(y) = 0 \text{ for } j = n_{1}, \dots, n_{2} - 1\}.$
(131)

Hence, \mathcal{M}_0 is a mixture family generated by the same functions $f_{n_1}, \ldots, f_{n_2-1}$. Therefore, Theorem 6 implies Lemma 3

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