# Double Constacyclic Codes over Two Finite Commutative Chain Rings 

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#### Abstract

Many kinds of codes which possess two cycle structures over two special finite commutative chain rings, such as $\mathbb{Z}_{2} \mathbb{Z}_{4}$-additive cyclic codes and quasi-cyclic codes of fractional index etc., were proved asymptotically good. In this paper we extend the study in two directions: we consider any two finite commutative chain rings with a surjective homomorphism from one to the other, and consider double constacyclic structures. We construct an extensive kind of double constacyclic codes over two finite commutative chain rings. And, developing a probabilistic method suitable for quasi-cyclic codes over fields, we prove that the double constacyclic codes over two finite commutative chain rings are asymptotically good.


Key words: Finite field; finite chain ring; constacyclic code; double constacyclic; asymptotically good.

## 1 Introduction

All rings in this paper are commutative and with identity. By $R^{\times}$we denote the multiplicative group of units (invertible elements) of a ring $R$. A ring is called a finite chain ring if it is finite and its ideals form a chain with respect to the inclusion relation. A finite ring $R$ is a chain ring if and only if there is a nilpotent element $\pi$ of $R$ such that the quotient $\bar{R}=R / R \pi=: F$ is a field, which is called the residue field of $R$; for $a \in R, \bar{a} \in F$ denotes the residue image of $a$; see Remark 2.1 below for details. Finite fields (i.e., Galois fields), residue integer rings $\mathbb{Z}_{p^{s}}$ modulo prime power $p^{s}$, Galois rings etc. are special kinds of finite chain rings.

[^0]A code sequence $C_{1}, C_{2}, \cdots$ is said to be asymptotically good if the code length of $C_{i}$ goes to infinity, and both the rates and the relative minimum distances of $C_{i}$ 's are positively bounded from below; see Definition 2.2 below for details. A class of codes is said to be asymptotically good if there is an asymptotically good code sequence in the class.

It is well-known that linear codes over a finite field are asymptotically good, see [25], [45]. More precisely, in [42] the relative minimum distance of linear codes are proved to be asymptotically distributed at the so-called GV-bound. In [5], the asymptotic distribution of the relative minimum distance of general codes (without any more algebraic structures) is characterized. On the other hand, for the quasi-abelian codes with index going to infinity, [17] proved that the relative minimum distances of such codes are also asymptotically distributed at the GV-bound.

Cyclic codes over finite fields are studied and applied extensively, e.g. see [28]. However it is still an attractive open question: are cyclic codes over a finite field asymptotically good? For example, see [34].

Quasi-cyclic codes of index 2 over a finite fields are asymptotically good, see [11], [12], [29]. Bazzi and Mitter [6] extended the result to quasi-abelian codes of index 2. Soon after, [35] proved that binary double even quasi-cyclic codes of index 2 are asymptotically good.

Self-dual quasi-cyclic codes (with index going to infinity) over a finite field are asymptotically good, see [14], [32]. Based on Artin's primitive element conjecture, Alahmadi et al. [2] proved that, if $q$ is not a square, $q$-ary self-dual quasi-cyclic codes of index 2 are asymptotically good. In both [30] and [31], the asymptotic goodness of any $q$-ary self-dual quasi-cyclic codes of index 2 is obtained.

In [19] we introduced the quasi-cyclic codes of fractional index over finite fields, and proved the asymptotically goodness of them. Mi and Cao [38] extended the result. Gao et al. [24] studied the algebraic structure of quasi-cyclic codes of index $1 \frac{1}{2}$. Aydin and Halilović [3] introduced the so-called multi-twisted codes over finite fields, which are much more extensive than the quasi-cyclic codes of fractional index.

Hammons et al. [26] initiated the study on coding over finite rings. Research on codes over finite chain rings is developed very much, e.g., [15], [21], [41].

On the other hand, $\mathbb{Z}_{2} \mathbb{Z}_{4}$-additive codes appeared in Delsarte [10]. It is extended to $\mathbb{Z}_{p^{r}} \mathbb{Z}_{p^{s}}$-additive codes in Aydogd and Siap [4]. Abualrub et al. [1] introduced $\mathbb{Z}_{2} \mathbb{Z}_{4}$-additive cyclic codes. Borges et al. generalized the results on $\mathbb{Z}_{2} \mathbb{Z}_{4}$-additive cyclic codes to $\mathbb{Z}_{p^{r}} \mathbb{Z}_{p^{s}}$-additive cyclic codes in [8]. Gao et al. [23] investigated the double cyclic codes (quasi-cyclic codes of index 2 ) over $\mathbb{Z}_{4}$. In [33] and [20], we found that $\mathbb{Z}_{2} \mathbb{Z}_{4}$-additive cyclic codes are asymptotically good. It is extended to that $\mathbb{Z}_{p^{r}} \mathbb{Z}_{p^{s}}$-additive cyclic codes are asymptotically good, see Yao et al. [46]. On the other hand, Gao et al. [22] show that $\mathbb{Z}_{4}$-double cyclic codes are asymptotically good.

Generalizing $\mathbb{Z}_{2} \mathbb{Z}_{4}$ codes, $\mathbb{Z}_{p^{r}} \mathbb{Z}_{p^{s}}$ codes etc., Borges et al. [7] introduced a general type of codes over two finite chain rings $R_{1}$ and $R_{2}$ with an epimorphism (i.e., surjective homomorphism) $R_{1} \rightarrow R_{2}$. They called these codes by $R_{1} R_{2}$ linear codes, and by $R_{1} R_{2}$-linear cyclic codes if further cyclic structures are afforded; They investigated the algebraic structures of these codes.

In this paper we introduce a more general type of codes as follows. Let $R$ and $R^{\prime}$ be finite chain rings with an epimorphism $\rho: R \rightarrow R^{\prime}$, hence they have the same residue field $\bar{R}=\overline{R^{\prime}}=: F$. Let $\lambda \in R^{\times}$, and $\lambda^{\prime}=\rho(\lambda) \in R^{\prime \times}$; hence $\bar{\lambda}=\overline{\lambda^{\prime}} \in F^{\times}$. Let $t=\operatorname{ord}_{F \times}(\bar{\lambda})$ be the order of the residue element $\bar{\lambda}$ in the unit group $F^{\times}$. Assume that $\alpha, \alpha^{\prime}$ are positive integers such that $\alpha \equiv \alpha^{\prime}(\bmod t)$ and $\operatorname{gcd}(\alpha, t)=1$. As usual, $R[X]$ denotes the polynomial ring over $R$. For any integer $n>0$ the quotient ring $R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle$ of $R[X]$ modulo the ideal $\left\langle X^{\alpha n}-\lambda\right\rangle$ generated by $X^{\alpha n}-\lambda$ is an $R[X]$-module. Through the epimorphism $\rho$, the quotient $R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle$ is also an $R[X]$-module. We define double constacyclic codes over $\left(R^{\prime}, R\right)$ as follows (please see Definition 2.5 below for more details):

- Any $R[X]$-submodule $C$ of $\left(R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle\right) \times\left(R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle\right)$ is said to be an $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes, the pair $\left(\alpha^{\prime} n, \alpha n\right)$ is called the cycle length of $C$, the fraction $\alpha^{\prime} / \alpha$ is called the ratio of $C$.

With notation as above, we will prove that such codes are asymptotically good.
Theorem 1.1. There are positive integers $n_{1}, n_{2}, \cdots$, and for each $n_{i}, i=$ $1,2, \cdots$, there is an $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic code $C_{i}$ of cycle length $\left(\alpha^{\prime} n_{i}, \alpha n_{i}\right)$ such that the code sequence $C_{1}, C_{2}, \cdots$ is asymptotically good.

Obviously, the asymptotic goodness of many kinds of codes mentioned above are straightforward consequences of the theorem. Further, we mention two special interesting cases.

If $R^{\prime}=R=F$ is a finite field (which is an important case), then $\lambda^{\prime}=\lambda$ and, following notation of [3], we also call the $(F, F)$-linear $(\lambda, \lambda)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ by double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$ over $F$. In particular:

- if $\alpha^{\prime}=\alpha$, then double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$ are just quasiconstacyclic codes of index 2 ;
- if $\lambda=1$ (hence $\alpha, \alpha^{\prime}$ are arbitrary positive integers), then $(F, F)$-linear $(1,1)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ are double cyclic codes of ratio $\alpha^{\prime} / \alpha$; and they are just quasi-cyclic codes of index $1 \frac{1}{\alpha}$ once $\alpha^{\prime}=1$.

Corollary 1.2. The double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$ over a finite field $F$ are asymptotically good.

In particular, the following two hold.
Corollary 1.3. The quasi-constacyclic codes of index 2 over any finite field are asymptotically good.

Corollary 1.4. For any positive integer $\alpha$, the quasi-cyclic codes of index $1 \frac{1}{\alpha}$ over any finite field are asymptotically good.

If $R^{\prime}=\mathbb{Z}_{p^{r}}$ and $R=\mathbb{Z}_{p^{s}}$ with $r \leq s$, then $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes are just $\left(\mathbb{Z}_{p^{r}}, \mathbb{Z}_{p^{s}}\right)$-additive $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes.

Corollary 1.5. The $\left(\mathbb{Z}_{p^{r}}, \mathbb{Z}_{p^{s}}\right)$-additive $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ are asymptotically good. In particular, $\mathbb{Z}_{p^{r}} \mathbb{Z}_{p^{s}}$-additive cyclic codes of any ratio $\alpha^{\prime} / \alpha$ are asymptotically good.

To prove Theorem 1.1, we develop a probabilistic method, which was proved effective for the quasi-cyclic codes of index 2 over finite fields, to a method suitable for the double twisted codes over finite fields. That is a special case of Theorem 1.1. Then, using the minimal ideals of $R^{\prime}$ and $R$, we make a reduction of the main result over general finite chain rings to the case over finite fields. The two skills are another contributions of the paper.

In Section 2 we describe the related notation more detailed, and sketch necessary preliminaries. In Section 3, we consider the case that $R^{\prime}=R=F$ is a finite field, and investigate the double twisted codes over $F$; by developing a probabilistic method, we prove the main result Theorem 1.1 in that case, i.e., Theorem 3.12 below. Finally, Section 4 is devoted to completing a proof of Theorem 1.1 for general case, i.e., Theorem 4.4 bellow, which is a more precise version of Theorem 1.1.

## 2 Preliminaries

In this paper any ring $R$ is commutative and with identity $1_{R}$ (or 1 for short). Subrings and ring homomorphisms are all identity-preserving. $R^{\times}$denotes the multiplicative unit group of $R$. For any set $S,|S|$ denotes the cardinality of $S$.

Remark 2.1. A finite ring $R$ is called a chain ring if all ideals of $R$ form a chain by the inclusion relation. It is easy to see that a finite $\operatorname{ring} R$ is a chain ring if and only if $R$ has a nilpotent element $\pi$ such that $\bar{R}=R / R \pi=: F$ is a field, hence $R \pi=J(R)$ is the radical of $R$; e.g., see [16, Lemma 2.4]. $F=\bar{R}$ is called the residue field of $R$; for $a \in R, \bar{a} \in F$ denotes the residue image of $a$.

In the following we assume that $R$ is a finite chain ring with radical $J(R)=$ $R \pi, \pi^{\ell}=0$ but $\pi^{\ell-1} \neq 0$, the positive integer $\ell$ is called the nilpotency index of $\pi$ (and, of the radical $R \pi$ ). Then the following hold:

- $R \supsetneq R \pi \supsetneq \cdots \supsetneq R \pi^{\ell-1} \supsetneq R \pi^{\ell}=0$ are the all ideals of $R$.
- $F:=R / R \pi=\operatorname{GF}\left(p^{r}\right)$ is a finite field (Galois field) with $|F|=q=p^{r}$, where $p$ is a prime, $F$ is called the residue field of $R$.
- $R \pi^{i} / R \pi^{i+1} \cong F$ for $i=0,1, \cdots, \ell-1$, and the cardinality $|R|=q^{\ell}$.

For more details, please see [37] or [40].

There exist several weight functions w on $R$, e.g., Hamming weigh, homogeneous weight, etc. And each weight w on $R$ is extended in a natural way to a weight function on the $R$-module $R^{n}=\left\{a=\left(a_{0}, a_{1}, \cdots, a_{n-1}\right) \mid a_{i} \in R\right\}$, denote by w again, which induces a distance on $R^{n}$ in a natural way: $\mathrm{d}(a, b)=\mathrm{w}(a-b)$, for all $a, b \in R^{n}$. Then for any code $C \subseteq R^{n}$, the minimum distance $\mathrm{d}(C)$ is defined as usual. If $C$ is linear (i.e., $C$ is an $R$-submodule of $R^{n}$ ), then $\mathrm{d}(C)=\mathrm{w}(C):=\min \{\mathrm{w}(c) \mid 0 \neq c \in C\}$, called the minimum weight of $C$.

For a code $C \subseteq R^{n}$, how to measure the code length and the information length of $C$ ? If $R=F$ is a field (i.e., $\ell=1$ ) and $|F|=q$, then $n$ is the code length, while $\log _{q}|C|$ is the information length of $C$. Because of the Grey map, for $\mathbb{Z}_{2}^{\alpha} \mathbb{Z}_{4}^{\beta}$-codes, an element of $\mathbb{Z}_{4}$ maybe viewed as an element of $\mathbb{Z}_{2} \times \mathbb{Z}_{2}$. For $C \subseteq \mathbb{Z}_{2}^{\alpha} \mathbb{Z}_{4}^{\beta}$, in some literature; e.g., see $[1,4,8,20,46]$, the information length of $C$ is defined to be $\log _{2}|C|$, and define the rate of $C$ by $\mathrm{R}(C)=\frac{\log _{2}|C|}{\alpha+2 \beta}$.

Therefore we define the code length of $C \subseteq R^{n}$ to be $n \ell$, and the information length of $C$ to be $\log _{q}|C|$. Then the relative minimum distance of $C$ is defined by $\Delta(C)=\frac{\mathrm{d}(C)}{n \ell}$, and the rate of $C$ is defined by $\mathrm{R}(C)=\frac{\log _{q}|C|}{n \ell}$.

Definition 2.2. A sequence of codes $C_{1}, C_{2}, \cdots$, where $C_{i} \subseteq R^{n_{i}}$, is said to be asymptotically good if the length $n_{i} \ell$ goes to infinity and there is a positive real number $\delta$ such that $\mathrm{R}\left(C_{i}\right) \geq \delta$ and $\Delta\left(C_{i}\right) \geq \delta$ for all $i=1,2, \cdots$.

A class of codes is said to be asymptotically good if there is an asymptotically good sequence $C_{1}, C_{2}, \cdots$, with every $C_{i}$ inside the class.

Lemma 2.3. Let $C_{1}, C_{2}, \cdots$, where $C_{i} \subseteq R^{n_{i}}$, be a sequence of codes with the positive integers $n_{i}$ going to infinity. If for a weight function w on $R$ the sequence $C_{1}, C_{2}, \cdots$ is asymptotically good, then for any weight function $\mathrm{w}^{\prime}$ on $R$ the sequence $C_{1}, C_{2}, \cdots$ is asymptotically good.

Proof. Let $\mathrm{d}(a, b)=\mathrm{w}(a-b), \mathrm{d}^{\prime}(a, b)=\mathrm{w}^{\prime}(a-b)$, for $a, b \in R$. Denote $\Delta\left(C_{i}\right)=$ $\frac{\mathrm{d}\left(C_{i}\right)}{n_{i} \ell}$ and $\Delta^{\prime}\left(C_{i}\right)=\frac{\mathrm{d}^{\prime}\left(C_{i}\right)}{n_{i} \ell}$. Since $R$ is finite, there is a real number $\omega>0$ such that $\mathrm{w}^{\prime}(a) \geq \omega \cdot \mathrm{w}(a)$ for all $a \in R$. For any $\left(a_{0}, a_{1}, \cdots, a_{n_{i}-1}\right) \in R^{n_{i}}$, we have
$\mathrm{w}^{\prime}\left(a_{0}, a_{1}, \cdots, a_{n_{i}-1}\right)=\sum_{j=0}^{n_{i}-1} \mathrm{w}^{\prime}\left(a_{j}\right) \geq \sum_{j=0}^{n_{i}-1} \omega \cdot \mathrm{w}\left(a_{j}\right)=\omega \cdot \mathrm{w}\left(a_{0}, a_{1}, \cdots, a_{n_{i}-1}\right)$.
And, for all $\mathbf{a}=\left(a_{1}, \cdots, a_{n_{i}}\right), \mathbf{b}=\left(b_{1}, \cdots, b_{n_{i}}\right) \in R^{n_{i}}$,

$$
\mathrm{d}^{\prime}(\mathbf{a}, \mathbf{b})=\sum_{j=0}^{n_{i}-1} \mathrm{w}^{\prime}\left(a_{j}-b_{j}\right) \geq \sum_{j=0}^{n_{i}-1} \omega \cdot \mathrm{w}\left(a_{j}-b_{j}\right)=\omega \cdot \mathrm{d}(\mathbf{a}, \mathbf{b})
$$

Thus $\mathrm{d}^{\prime}\left(C_{i}\right) \geq \omega \cdot \mathrm{d}\left(C_{i}\right)$, hence $\Delta^{\prime}\left(C_{i}\right) \geq \omega \cdot \Delta\left(C_{i}\right)$. So, if $\Delta\left(C_{i}\right) \geq \delta$ for all $i=1,2, \cdots$, then $\Delta^{\prime}\left(C_{i}\right) \geq \omega \delta$ for all $i=1,2, \cdots$. Take $\delta^{\prime}=\min \{\delta, \omega \delta\}$. Then $\Delta^{\prime}\left(C_{i}\right) \geq \delta^{\prime}$ for all $i=1,2, \cdots$.

Remark 2.4. Lemma 2.3 shows that the asymptotic goodness of a code sequence (or, of a class of codes) is independent of the choice of the weight functions, though in Definition 2.2 we have to specify a weight function. Similarly to Lemma 2.3, it can be also proved that the asymptotic goodness is independent
of the choice of the rates. In the following, therefore, by "w" we always denote the Hamming weight; and, the minimum distance and the rate of a code $C$ are defined as that before Definition 2.2.

Let $\lambda \in R^{\times}$. An $R$-submodule $C \subseteq R^{n}$ is called a $\lambda$-constacyclic code if

$$
\begin{equation*}
\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in C \Longrightarrow\left(\lambda c_{n-1}, c_{0}, \cdots, c_{n-2}\right) \in C \tag{2.1}
\end{equation*}
$$

Any $\lambda$-constacyclic code $C \subseteq R^{n}$ is identified with an ideal of $R[X] /\left\langle X^{n}-\lambda\right\rangle$, and vice versa; where $R[X]$ denotes the polynomial ring over $R$ and $\left\langle X^{n}-\lambda\right\rangle$ denotes the ideal generated by $X^{n}-\lambda$.

Assume that $R^{\prime}$ is also a finite chain ring with radical $R^{\prime} \pi^{\prime}$ of nilpotency index $\ell^{\prime}$. If there is an epimorphism $\rho: R \rightarrow R^{\prime}$, then the residue field $R^{\prime} / R^{\prime} \pi^{\prime} \cong$ $R / R \pi=F$, and $\ell^{\prime} \leq \ell$. But the converse is not true in general. As far as we know, once $R$ and $R^{\prime}$ are both Galois rings, or both $F$-algebras ( $F$ is the residue field), then the epimorphism $\rho: R \rightarrow R^{\prime}$ exist if and only if $R^{\prime} / R^{\prime} \pi^{\prime} \cong R / R \pi$ and $\ell^{\prime} \leq \ell$, see [16, Remark 2.5]. In general case, the equivalence no longer holds. It is still an open question how to classify the finite chain rings, see [13], [27]. Thus, the assumption in the beginning of Section 3 of [7] is invalid in general. However, the results of [7] are still valid and interesting provided such an epimorphism exists. Inspired by [7], we introduce a kind of codes which is more extensive than the kind of $R^{\prime} R$-linear cyclic codes defined in [7].

Definition 2.5. Let $R, R^{\prime}$ be finite chain rings as above. Assume that

- there is an epimorphism $\rho: R \rightarrow R^{\prime}$, hence $\bar{R} \cong \overline{R^{\prime}}$ and $\ell^{\prime} \leq \ell$;
in the following we identify $\bar{R}=\overline{R^{\prime}}:=F$, and let $|F|=q$;
- $\lambda \in R^{\times}, \operatorname{ord}_{F \times}(\bar{\lambda})=t$; and $\lambda^{\prime}=\rho(\lambda) \in R^{\prime \times}$, hence $\overline{\lambda^{\prime}}=\bar{\lambda}$;
- integers $\alpha, \alpha^{\prime}>0, \alpha^{\prime} \equiv \alpha(\bmod t)$ and $\operatorname{gcd}(\alpha, t)=1$.

The epimorphism $\rho: R \rightarrow R^{\prime}$ induces an epimorphism $\rho: R[X] \rightarrow R^{\prime}[X]$, $\sum_{i} a_{i} X^{i} \mapsto \sum_{i} \rho\left(a_{i}\right) X^{i}$. So, for any integer $n>0$, both $R[X] /\left\langle X^{n}-\lambda\right\rangle$ and $R^{\prime}[X] /\left\langle X^{n}-\lambda^{\prime}\right\rangle$ are $R[X]$-modules. Any $R[X]$-submodule $C$ of the $R[X]$-module

$$
\left(R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle\right) \times\left(R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle\right)
$$

is said to be an $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes, the pair $\left(\alpha^{\prime} n, \alpha n\right)$ is called the cycle length of $C$, the fraction $\alpha^{\prime} / \alpha$ is called the ratio of $C$. Note that, by Remark 2.4, the code length of $C$ is $\alpha^{\prime} n \ell^{\prime}+\alpha n \ell$, hence the relative minimum distance $\Delta(C)=\frac{\mathrm{w}(C)}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}$, and the rate $\mathrm{R}(C)=\frac{\log _{q}|C|}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}$.

If $R^{\prime}=R=F$ is a finite field, then $\lambda^{\prime}=\lambda$ and, following notation of [3], we also call the $(F, F)$-linear $(\lambda, \lambda)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ by double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$ over $F$.

Similarly to Eq.(2.1), any ( $R^{\prime}, R$ )-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic code $C$ defined as above is identified with an $R$-submodule $C \subseteq R^{\prime \alpha^{\prime} n} \times R^{\alpha n}$ satisfying that

$$
\begin{align*}
& \left(c_{0}^{\prime}, c_{1}^{\prime}, \cdots, c_{\alpha^{\prime} n-1}^{\prime}, c_{0}, c_{1}, \cdots, c_{\alpha n-1}\right) \in C  \tag{2.2}\\
\Longrightarrow & \left(\lambda^{\prime} c_{\alpha^{\prime} n-1}^{\prime}, c_{0}^{\prime}, \cdots, c_{\alpha^{\prime} n-2}^{\prime}, \lambda c_{\alpha n-1}, c_{0}, \cdots, c_{\alpha n-2}\right) \in C
\end{align*}
$$

and vice versa.
As mentioned in Section 1, we'll prove that $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ are asymptotically good, and the key step is to prove it for the case that $R^{\prime}=R=F$ is a finite field, i.e., Theorem 3.12 below. Thus we need a few preliminaries about the codes over finite fields.

Let $F$ be a finite field with $|F|=q=p^{r}$ as above in Definition 2.5. Let $I=\{1, \cdots, n\}$ be an index set, and $F^{I}=F^{n}$. For any subset $I^{\prime} \subseteq I, I^{\prime}=$ $\left\{i_{1}, \cdots, i_{k}\right\}, 1 \leq i_{1}<\cdots<i_{k} \leq n$, denote $F^{I^{\prime}}=\left\{\left(a_{i_{1}}, \cdots, a_{i_{k}}\right) \mid a_{i_{j}} \in F\right\} ;$ hence we have the projection $\rho_{I^{\prime}}: F^{I} \rightarrow F^{I^{\prime}}, \rho_{I^{\prime}}\left(a_{1}, \cdots, a_{n}\right)=\left(a_{i_{1}}, \cdots, a_{i_{k}}\right)$.

Definition 2.6. A code $C \subseteq F^{I}$ is said to be balanced if there are subsets (repetition allowed) $I_{1}, \cdots, I_{m} \subseteq I$ and a positive integer $t$ such that
(1) for each $I_{j}, 1 \leq j \leq m$, the projection $\rho_{I_{j}}$ induces a bijection from $C$ onto $F^{I_{j}}$ (equivalently, $q^{\left|I_{j}\right|}=|C|=\left|\rho_{I_{j}}(C)\right|$ for $\left.j=1, \cdots, m\right) ;$
(2) for each $i \in I$, there are exact $t$ indexes $1 \leq j_{1}<\cdots<j_{t} \leq m$ such that $i \in I_{j_{h}}$ for $h=1, \cdots, t$.

The following function $h_{q}(\delta)$ is called the $q$-ary entropy:

$$
\begin{equation*}
h_{q}(\delta)=\delta \log _{q}(q-1)-\delta \log _{q} \delta-(1-\delta) \log _{q}(1-\delta), \quad \delta \in\left[0,1-q^{-1}\right] \tag{2.3}
\end{equation*}
$$

where $0 \log _{q} 0=0$ as a convention. The function $h_{q}(x)$ is strictly increasing and concave in the interval $\left[0,1-q^{-1}\right]$ with $h_{q}(0)=0$ and $h_{q}\left(1-q^{-1}\right)=1$.

The following result was proved in [36], [43] and [44] for the binary case, and proved in [17, Corollary 3.4] for general case.

Lemma 2.7. Assume that $0<\delta<1-\frac{1}{q}$. Let $C \subseteq F^{n}$ be a balanced code. Set $k=\log _{q}|C|$. Denote $C^{\leq \delta}=\left\{c \in C \left\lvert\, \frac{\mathrm{w}(c)}{n} \leq \delta\right.\right\}$. Then $\left|C^{\leq \delta}\right| \leq q^{k h_{q}(\delta)}$.

The above lemma is suitable to study the asymptotic properties of group codes such as quasi-abelian codes, dihedral codes etc., e.g., [6, 9, 17, 18]. To apply it to the investigation of the constacyclic codes, we need the following lemma.

Lemma 2.8. Let $C$ be a $\lambda$-constacyclic code over $F$ of length $n$. Then $C$ is a balanced code.

Proof. Let $\operatorname{dim} C=k, I=\{0,1, \cdots, n-1\}$. Let $I_{*}=\left\{i_{1}, \cdots, i_{k}\right\} \subseteq I$ with $0 \leq i_{1}<\cdots<i_{k}<n$ such that $\left|I_{*}\right|=k=\operatorname{dim} \rho_{I_{*}}(C)$. Set

$$
\Theta_{\lambda}=\left(\begin{array}{cccc}
0 & 1 & & \\
& \ddots & \ddots & \\
& & 0 & 1 \\
\lambda & & & 0
\end{array}\right)_{n \times n}, \text { so } \mathbf{c} \Theta_{\lambda}=\left(\lambda c_{n-1}, c_{0}, \cdots, c_{n-2}\right) \in C, \forall \mathbf{c} \in C
$$

Let $\theta=(0,1, \cdots, n-1)$ be a cycle permutation, so $\theta^{-1} I_{*}=\left\{i_{1}-1, \cdots, i_{k}-1\right\}$, where $i_{1}-1$ should be replaced by $n-1$ if $i_{1}=0$. Further, set

$$
\Theta_{k}^{-1}=\left(\begin{array}{cccc}
0 & & & 1 \\
1 & \ddots & & \\
& \ddots & 0 & \\
& & 1 & 0
\end{array}\right)_{k \times k}, \quad D_{k}=\left(\begin{array}{cccc}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & \lambda^{-1}
\end{array}\right)_{k \times k}
$$

It is checked directly that:

- If $i_{1}>0$, then $\rho_{\theta^{-1} I_{*}}(\mathbf{c})=\rho_{I_{*}}\left(\mathbf{c} \Theta_{\lambda}\right) \Theta_{k}^{-1} ;$
- otherwise, $i_{1}=0$, and $\rho_{\theta^{-1} I_{*}}(\mathbf{c})=\rho_{I_{*}}\left(\mathbf{c} \Theta_{\lambda}\right) \Theta_{k}^{-1} D_{k}$.

In any case, $\rho_{\theta-1}^{I_{*}}(C)=\rho_{I_{*}}(C) \Theta_{k}^{-1}=F^{\theta^{-1} I_{*}}$. Now we take an information set $I_{1} \subseteq I$ of $C$, i.e., $\left|I_{1}\right|=k=\operatorname{dim} \rho_{I_{1}}(C)$. Let $I_{j+1}=\theta^{-j} I_{1}, j=0,1, \cdots, n-1$. Then $I_{1}, \cdots, I_{n}$ satisfy Definition $2.6(1)$. The cyclic permutation group $\langle\theta\rangle=$ $\left\langle\theta^{-1}\right\rangle$ acts on $I$ transitively (in fact, regularly). By Lemma 2.9 below, $I_{1}, \cdots, I_{n}$ satisfy Definition 2.6(2). In conclusion, $C$ is a balanced code.

Lemma 2.9. Let a finite group $G$ act transitively on a finite set $X$. Let $Y \subseteq X$, $x_{0} \in X$. Then $\left|\left\{g \in G \mid x_{0} \in g Y\right\}\right|=\frac{|G| \cdot|Y|}{|X|}$.

Proof. For any $y \in Y$, there is a $g \in G$ such that $x_{0}=g y$. And, for $g^{\prime} \in G$, $g^{\prime} y=x_{0}=g y$ if and only if $g^{-1} x_{0}=y=g^{-1} x_{0}$, if and only if $g^{\prime} g^{-1} \in G_{x_{0}}$, where $G_{x_{0}}=\left\{h \in G \mid h x_{0}=x_{0}\right\}$. Thus, $\left|\left\{g \in G \mid x_{0}=g y\right\}\right|=\left|G_{x_{0}}\right|=\frac{|G|}{|X|}$. Therefore, $\left|\left\{g \in G \mid x_{0} \in g Y\right\}\right|=|Y| \cdot \frac{|G|}{|X|}$.

## 3 Double $\lambda$-twisted codes over finite fields

Remark 3.1. In this section, we always take the following notation.

- $F$ is a finite field, the cardinality $|F|=q ; \lambda \in F^{\times}, \operatorname{ord}_{F \times}(\lambda)=t$.
- $\mathcal{R}=F[X] /\left\langle X^{n}-1\right\rangle, n$ is a positive integer with $\operatorname{gcd}(n, q t)=1$.
- $\alpha, \alpha^{\prime}>0$ are integers, $\alpha^{\prime} \equiv \alpha(\bmod t)$ and $\operatorname{gcd}(\alpha, t)=1$; set $\alpha^{\prime \prime}=\min \left\{\alpha, \alpha^{\prime}\right\}$.
- $\mathcal{R}_{\lambda, \alpha}=F[X] /\left\langle X^{\alpha n}-\lambda\right\rangle, \mathcal{R}_{\lambda, \alpha^{\prime}}=F[X] /\left\langle X^{\alpha^{\prime} n}-\lambda\right\rangle ;$ then $\mathcal{R}_{\lambda, \alpha^{\prime}} \times \mathcal{R}_{\lambda, \alpha}=\left\{\left(a^{\prime}, a\right) \mid a^{\prime} \in \mathcal{R}_{\lambda, \alpha^{\prime}}, a \in \mathcal{R}_{\lambda, \alpha}\right\}$.
- $\delta \in\left(0,1-q^{-1}\right)$.

The rings ( $F$-algebras) $\mathcal{R}, \mathcal{R}_{\lambda, \alpha}$ and $\mathcal{R}_{\lambda, \alpha^{\prime}}$ can be viewed as $F[X]$-modules. Hence $\mathcal{R}_{\lambda, \alpha^{\prime}} \times \mathcal{R}_{\lambda, \alpha}$ is an $F[X]$-module. By Definition 2.5, any $F[X]$-submodule $C \subseteq \mathcal{R}_{\lambda, \alpha^{\prime}} \times \mathcal{R}_{\lambda, \alpha}$ is called an $(F, F)$-linear $(\lambda, \lambda)$-constacyclic code of ratio $\alpha^{\prime} / \alpha$. As pointed out in Section 1, the code $C$ is also called a double $\lambda$-twisted code over $F$ of ratio $\alpha^{\prime} / \alpha$. The code length of $C$ is $\alpha^{\prime} n+\alpha n$, and the information length of $C$ is just the dimension $\operatorname{dim} C$, see Remark 2.4.

In Subsection 3.1 we relate an ideal of $\mathcal{R}$ to an ideal of $\mathcal{R}_{\lambda, \alpha}$ (and of $\mathcal{R}_{\lambda, \alpha^{\prime}}$ ). In Subsection 3.2 we construct and study a kind of random double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$. In Subsection 3.3 we prove the asymptotic goodness of the double $\lambda$-twisted codes.

### 3.1 About $\mathcal{R}, \mathcal{R}_{\lambda, \alpha}$ and $\mathcal{R}_{\lambda, \alpha^{\prime}}$

Because $\operatorname{gcd}(n, q)=1, X^{n}-1$ is a product of pairwise coprime monic irreducible $F$-polynomials $\phi_{i}(X)$ with degree $\operatorname{deg} \phi_{i}(X)=d_{i}$ as follows

$$
X^{n}-1=\phi_{0}(X) \phi_{1}(X) \cdots \phi_{m}(X)
$$

where we appoint that

$$
\begin{equation*}
\phi_{0}(X)=X-1, \quad \widehat{\phi}_{0}(X)=\phi_{1}(X) \cdots \phi_{m}(X)=X^{n-1}+\cdots+X+1 \tag{3.1}
\end{equation*}
$$

So $X^{n}-1=\phi_{0}(X) \widehat{\phi}_{0}(X)$. By Chinese Remainder Theorem,

$$
\begin{equation*}
\mathcal{R}=F_{0} \oplus F_{1} \oplus \cdots \oplus F_{m} \tag{3.2}
\end{equation*}
$$

where $F_{i} \cong F[X] /\left\langle\phi_{i}(X)\right\rangle$ are finite fields and $\operatorname{dim}_{F} F_{i}=\operatorname{deg} \phi_{i}(X)=d_{i}$ for $i=0,1, \cdots, m$. Of course, $d_{0}=1$.

Remark 3.2. Let notation be as above. We denote

$$
\begin{equation*}
\mu(n)=\min \left\{d_{1}, \cdots, d_{m}\right\} \tag{3.3}
\end{equation*}
$$

By [6, Lemma 2.6] (for binary case) and [18, Lemma 3.6] (for general case), there is a sequence $n_{1}, n_{2}, \cdots$ of positive integers $n_{i}$ coprime to $q t$ such that $\lim _{i \rightarrow \infty} \frac{\log _{q} n_{i}}{\mu\left(n_{i}\right)}=0$. Thus, in the following we further assume that $\mu(n)>\log _{q} n$.

By Eq.(3.2), any ideal $\mathcal{J}$ of $\mathcal{R}$ is a direct sum of some of $F_{0}, F_{1}, \cdots, F_{m}$, hence $\mathcal{J}=\mathcal{R} e_{\mathcal{J}}$ for an idempotent $e_{\mathcal{J}}$, and $b e_{\mathcal{J}}=b$ for all $b \in \mathcal{J}$; so $\mathcal{J}$ is a ring with identity $e_{\mathcal{J}}$. And, for all $f(X) \in F[X]$, we have $f(X) e_{\mathcal{J}} \in \mathcal{J}$ and

$$
\begin{equation*}
f(X) b(X)=f(X) e_{\mathcal{J}} b(X), \quad \forall b(X) \in \mathcal{J} \tag{3.4}
\end{equation*}
$$

that is, the $F[X]$-module structure of $\mathcal{J}$ is reduced to the $\mathcal{J}$-module structure of $\mathcal{J}$ itself.

We relate now $\mathcal{R}$ to a part of $\mathcal{R}_{\lambda, \alpha}$ (and $\mathcal{R}_{\lambda, \alpha^{\prime}}$ ). We start with a remark.
Remark 3.3. In the multiplicative group $F^{\times}$, the element $\lambda$ generates a cyclic group $\langle\lambda\rangle=\left\{\lambda^{j} \mid j \in \mathbb{Z}_{t}\right\}$; i.e., the elements of $\langle\lambda\rangle$ are 1-1 corresponding to the elements of $\mathbb{Z}_{t}$. By assumption, $\alpha, n \in \mathbb{Z}_{t}^{\times}$, so the inverses $\frac{1}{\alpha}, \frac{1}{n} \in \mathbb{Z}_{t}^{\times}$exist, and $\lambda^{\frac{1}{\alpha}}, \lambda^{\frac{1}{n}}, \lambda^{\frac{1}{\alpha n}}$, etc., make sense. Note that, since $\alpha^{\prime} \equiv \alpha(\bmod t), \frac{1}{\alpha^{\prime}}=\frac{1}{\alpha}$ in $\mathbb{Z}_{t}^{\times}$, hence $\lambda^{\frac{1}{\alpha^{\prime}}}=\lambda^{\frac{1}{\alpha}}$.

Then we have the decomposition:

$$
X^{\alpha n}-\lambda=\left(X^{n}\right)^{\alpha}-\left(\lambda^{\frac{1}{\alpha}}\right)^{\alpha}=\left(X^{n}-\lambda^{\frac{1}{\alpha}}\right) \cdot \psi_{\lambda, \alpha}(X)
$$

where

$$
\begin{equation*}
\psi_{\lambda, \alpha}(X)=\left(X^{n}\right)^{\alpha-1}+\left(X^{n}\right)^{\alpha-2} \lambda^{\frac{1}{\alpha}}+\cdots+\left(\lambda^{\frac{1}{\alpha}}\right)^{\alpha-1} \tag{3.5}
\end{equation*}
$$

Note that, in the special case " $\alpha=1 ", \psi_{\lambda, 1}=1$. By Eq.(3.1),

$$
X^{n}-\lambda^{\frac{1}{\alpha}}=\lambda^{\frac{1}{\alpha}}\left(\left(X / \lambda^{\frac{1}{\alpha n}}\right)^{n}-1\right)=\lambda^{\frac{1}{\alpha}} \phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \widehat{\phi}_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right)
$$

Setting $\psi_{\lambda, \alpha}^{+}(X)=\phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)$, we get

$$
\begin{equation*}
X^{\alpha n}-\lambda=\lambda^{\frac{1}{\alpha}} \phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \widehat{\phi}_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=\lambda^{\frac{1}{\alpha}} \widehat{\phi}_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}^{+}(X) \tag{3.6}
\end{equation*}
$$

In the special case " $\lambda=1$ and $\alpha=1 ", \psi_{1,1}^{+}(X)=\phi_{0}(X)$ since $\psi_{1,1}=1$.
Replacing $\alpha$ by $\alpha^{\prime}$, we get $\psi_{\lambda, \alpha^{\prime}}(X)$ and $\psi_{\lambda, \alpha^{\prime}}^{+}(X)$ similarly to Eq.(3.5) and Eq.(3.6), respectively.

We are concerned with the following ideals of $\mathcal{R}, \mathcal{R}_{\lambda, \alpha}$ and $\mathcal{R}_{\lambda, \alpha^{\prime}}$ :

$$
\begin{equation*}
\mathcal{I}=\mathcal{R} \phi_{0}(X), \quad \mathcal{I}_{\lambda, \alpha}=\mathcal{R}_{\lambda, \alpha} \psi_{\lambda, \alpha}^{+}(X), \quad \mathcal{I}_{\lambda, \alpha^{\prime}}=\mathcal{R}_{\lambda, \alpha^{\prime}} \psi_{\lambda, \alpha^{\prime}}^{+}(X) \tag{3.7}
\end{equation*}
$$

which are all $F[X]$-modules, as $\mathcal{R}, \mathcal{R}_{\lambda, \alpha}$ and $\mathcal{R}_{\lambda, \alpha^{\prime}}$ are $F[X]$-modules. By Eq.(3.2) and Eq.(3.3),

$$
\begin{equation*}
\mathcal{I}=F_{1} \oplus \cdots \oplus F_{m}, \quad \text { each } F_{i} \text { is a field with } d_{i}=\operatorname{dim}_{F} F_{i} \geq \mu(n) \tag{3.8}
\end{equation*}
$$

and any $F[X]$-submodule of $\mathcal{I}$ is a direct sum of some of $F_{1}, \cdots, F_{m}$. Then we relate $\mathcal{I}$ to $\mathcal{I}_{\lambda, \alpha}$ and $\mathcal{I}_{\lambda, \alpha^{\prime}}$ by the following concept.

Remark 3.4. Assume that $M, M^{\prime}$ are $F[X]$-modules, and $\sigma: F[X] \rightarrow F[X]$ is an $F$-algebra automorphism (i.e., $\sigma$ is both an $F$-linear isomorphism and a ring isomorphism). If a map $\tau: M \rightarrow M^{\prime}$ preserves additions and satisfies that: $\tau(a m)=\sigma(a) \tau(m), \forall a \in F[X], \forall m \in M$, then we say that $\tau$ is a $\sigma-F[X]-$ homomorphism. Further, if a $\sigma-F[X]$-homomorphism $\tau$ is bijective, then we say that $\tau$ is a $\sigma-F[X]$-isomorphism. Note that a $\sigma-F[X]$-isomorphism $\tau: M \rightarrow M^{\prime}$ preserves all the submodule structures, including the dimensions of submodules.

The following is clearly an $F$-algebra automorphism:

$$
\begin{equation*}
\sigma_{\lambda}: F[X] \longrightarrow F[X], \quad f(X) \longmapsto f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \tag{3.9}
\end{equation*}
$$

which is defined for both $\alpha$ and $\alpha^{\prime}$ because $\frac{1}{\alpha^{\prime}}=\frac{1}{\alpha}$ in $\mathbb{Z}_{t}^{\times}$, see Remark 3.3. In the special case " $\lambda=1$ " (i.e., cyclic case), $\sigma_{1}=\operatorname{id}_{F[X]}$ is the identity automorphism of $F[X]$.

Lemma 3.5. The following is a well-defined $\sigma_{\lambda}-F[X]$-isomorphism:

$$
\tau_{\lambda, \alpha}: \quad \mathcal{I} \longrightarrow \mathcal{I}_{\lambda, \alpha}, \quad f(X) \longmapsto f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X) .
$$

Proof. For $f(X) \in \mathcal{I}=\mathcal{R} \phi_{0}(X), f(X)=g(X) \phi_{0}(X)$ for a $g(X) \in \mathcal{R}$; then

$$
f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=g\left(X / \lambda^{\frac{1}{\alpha n}}\right) \phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=g\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}^{+}(X)
$$

so $f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X) \in \mathcal{I}_{\lambda, \alpha}$. Next, assume that both $f(X), f^{\prime}(X) \in F[X]$ represent one and the same element in $\mathcal{I}$, then $f^{\prime}(X)=f(X)+g(X)\left(X^{n}-1\right)$ for a $g(X) \in F[X]$, so

$$
f^{\prime}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)+g\left(X / \lambda^{\frac{1}{\alpha n}}\right)\left(\left(X / \lambda^{\frac{1}{\alpha n}}\right)^{n}-1\right) \psi_{\lambda, \alpha}(X) .
$$

By Eq.(3.6), in $\mathcal{I}_{\lambda, \alpha}$ we have $\left(\left(X / \lambda^{\frac{1}{\alpha n}}\right)^{n}-1\right) \psi_{\lambda, \alpha}(X)=0$. Thus

$$
f^{\prime}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X), \quad\left(\text { in } \mathcal{I}_{\lambda, \alpha} .\right)
$$

Summarizing the above, we see that the $\tau_{\lambda, \alpha}$ in the lemma is a well-defined map. Obviously, $\tau_{\lambda, \alpha}$ preserves additions. For $f(X) \in \mathcal{I}$ and $g(X) \in F[X]$,

$$
\begin{aligned}
& \tau_{\lambda, \alpha}(g(X) f(X))=g\left(X / \lambda^{\frac{1}{\alpha n}}\right) f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X) \\
& =g\left(X / \lambda^{\frac{1}{\alpha n}}\right) \tau_{\lambda, \alpha}(f(X))=\sigma_{\lambda}(g(X)) \tau_{\lambda, \alpha}(f(X))
\end{aligned}
$$

Thus, $\tau_{\lambda, \alpha}$ is a $\sigma_{\lambda}-F[X]$-homomorphism. For any $g(X) \psi_{\lambda, \alpha}^{+}(X) \in \mathcal{I}_{\lambda, \alpha}$,

$$
g(X) \psi_{\lambda, \alpha}^{+}(X)=g(X) \phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)
$$

Then $g\left(\lambda^{\frac{1}{\alpha n}} X\right) \phi_{0}(X) \in \mathcal{I}$ and

$$
\tau_{\lambda, \alpha}\left(g\left(\lambda^{\frac{1}{\alpha n}} X\right) \phi_{0}(X)\right)=g\left(\lambda^{\frac{1}{\alpha n}} X / \lambda^{\frac{1}{\alpha n}}\right) \phi_{0}\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha}(X)=g(X) \psi_{\lambda, \alpha}^{+}(X)
$$

So, $\tau_{\lambda, \alpha}$ is surjective. Finally, by Eq.(3.6) again,

$$
\begin{equation*}
\operatorname{dim}_{F} \mathcal{I}=\operatorname{dim}_{F} \mathcal{I}_{\lambda, \alpha}=n-1 \tag{3.10}
\end{equation*}
$$

Thus, $\tau_{\lambda, \alpha}$ is a bijection.
Remark 3.6. It is clear that Lemma 3.5 still holds if we replace $\alpha$ by $\alpha^{\prime}$; i.e., the following is a well-defined $\sigma_{\lambda}-F[X]$-isomorphism (recall that $\frac{1}{\alpha^{\prime}}=\frac{1}{\alpha}$ in $\mathbb{Z}_{t}^{\times}$ hence $\lambda^{\frac{1}{\alpha n}}=\lambda^{\frac{1}{\alpha^{\prime} n}}$ :

$$
\tau_{\lambda, \alpha^{\prime}}: \mathcal{I} \longrightarrow \mathcal{I}_{\lambda, \alpha^{\prime}}, \quad f(X) \longmapsto f\left(X / \lambda^{\frac{1}{\alpha n}}\right) \psi_{\lambda, \alpha^{\prime}}(X) .
$$

For $f(X) \in \mathcal{I}$, to simplify the notation, in the following we'll denote the image $\tau_{\lambda, \alpha}(f(X))$ by $f^{\tau_{\lambda, \alpha}}(X) \in \mathcal{I}_{\lambda, \alpha}$, and denote $\tau_{\lambda, \alpha^{\prime}}(f(X))$ by $f^{\tau_{\lambda, \alpha^{\prime}}}(X) \in \mathcal{I}_{\lambda, \alpha^{\prime}}$.

### 3.2 Random double twisted code $C_{a^{\prime}, a}$ over $F$

Recall that $\mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha}$ is a $2(n-1)$-dimensional $F[X]$-submodule of $\mathcal{R}_{\lambda, \alpha^{\prime}} \times \mathcal{R}_{\lambda, \alpha}$, see Eq.(3.10). In the rest of this section, we view $\mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha}$ as a probability space with equal probability for every sample.

For $\left(a^{\prime}(X), a(X)\right) \in \mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha}$, let $C_{a^{\prime}, a}=F[X]\left(a^{\prime}(X), a(X)\right)$ be the $F[X]-$ submodule of $\mathcal{R}_{\lambda, \alpha^{\prime}} \times \mathcal{R}_{\lambda, \alpha}$ generated by $\left(a^{\prime}(X), a(X)\right)$, i.e.

$$
\begin{equation*}
C_{a^{\prime}, a}=\left\{\left(g(X) a^{\prime}(X), g(X) a(X)\right) \in \mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha} \mid g(X) \in F[X]\right\} \tag{3.11}
\end{equation*}
$$

Then $C_{a^{\prime}, a}$ is a random double $\lambda$-twisted code of cycle length $\left(\alpha^{\prime} n, \alpha n\right)$ over $F$. By Lemma 3.5, Remark 3.6, we can take $b^{\prime}(X), b(X) \in \mathcal{I}$ such that $b^{\prime \prime} \tau_{\lambda, \alpha^{\prime}}(X)=$ $a^{\prime}(X)$ and $b^{\tau_{\lambda, \alpha}}(X)=a(X)$. For any $g(X) \in F[X]$, by Eq.(3.9) there is an $f(X) \in F[X]$ such that $\sigma_{\lambda}(f(X))=g(X)$. We get that

$$
\begin{aligned}
& \tau_{\lambda, \alpha}(f(X) b(X))=\sigma_{\lambda}(f(X)) a(X)=g(X) a(X) \\
& \tau_{\lambda, \alpha^{\prime}}\left(f(X) b^{\prime}(X)\right)=\sigma_{\lambda}(f(X)) a^{\prime}(X)=g(X) a^{\prime}(X)
\end{aligned}
$$

Further, by Eq.(3.4), we can take the $f(X)$ such that $f(X) \in \mathcal{I}$. So we get

$$
\begin{equation*}
C_{a^{\prime}, a}=\left\{\left(\sigma_{\lambda}(f(X)) a^{\prime}(X), \sigma_{\lambda}(f(X)) a(X)\right) \mid f(X) \in \mathcal{I}\right\} \subseteq \mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha} \tag{3.12}
\end{equation*}
$$

For each $f(X) \in \mathcal{I}$, we have a random code word

$$
c_{f, a^{\prime}, a}=\left(\sigma_{\lambda}(f(X)) a^{\prime}(X), \sigma_{\lambda}(f(X)) a(X)\right) \in C_{a^{\prime}, a}
$$

Recall that the code length equals $\alpha^{\prime} n+\alpha n$, see Remark 3.1. We get a $0-1$ random variable over the probability space $\mathcal{I}_{\lambda, \alpha^{\prime}} \times \mathcal{I}_{\lambda, \alpha}$ :

$$
Y_{f}= \begin{cases}1, & 0<\frac{\mathrm{w}\left(c_{f, a^{\prime}, a}\right)}{\alpha^{\prime} n+\alpha n} \leq \delta \\ 0, & \text { otherwise }\end{cases}
$$

Clearly, $Y_{0}=0$. We further define an non-negative integer random variable

$$
Y=\sum_{f(X) \in \mathcal{I}} Y_{f}
$$

Because of Eq.(3.12), the variable $Y$ stands for the number of the non-zero random code words whose relative weight is at most $\delta$. So

$$
\begin{equation*}
\operatorname{Pr}\left(\Delta\left(C_{a^{\prime}, a}\right) \leq \delta\right)=\operatorname{Pr}(Y \geq 1) \leq \mathrm{E}(Y) \tag{3.13}
\end{equation*}
$$

where $\mathrm{E}(Y)$ denotes the expectation of $Y$, and the inequality follows by Markov Inequality, e.g., see [39, Theorem 3.1].

In the rest of this subsection, we estimate the expectation $\mathrm{E}(Y)$. For $f(X) \in$ $\mathcal{I}$, denote

$$
\begin{equation*}
C_{f}=\mathcal{R} f(X)=\mathcal{I} f(X)=\{g(X) f(X) \mid g(X) \in \mathcal{I}\}, \quad d_{f}=\operatorname{dim}_{F} C_{f} \tag{3.14}
\end{equation*}
$$

Then $C_{f}$ is an ideal (an $F[X]$-submodule) of $\mathcal{R}$ contained in $\mathcal{I}$.

Lemma 3.7. Keep the above notation. Let $\alpha^{\prime \prime}=\min \left\{\alpha^{\prime}, \alpha\right\}$ and $\delta \in\left(0,1-q^{-1}\right)$ be as in Remark 3.1. Then the expectation

$$
\mathrm{E}\left(Y_{f}\right) \leq q^{-2 d_{f}+2 d_{f} h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)}
$$

Proof. From that $c_{f, a^{\prime}, a}=\left(\sigma_{\lambda}(f(X)) a^{\prime}(X), \sigma_{\lambda}(f(X)) a(X)\right)$, we see that

$$
\left\{c_{f, a^{\prime}, a} \mid a^{\prime}(X) \in \mathcal{I}_{\lambda, \alpha^{\prime}}, a(X) \in \mathcal{I}_{\lambda, \alpha}\right\}=\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X)) \times \mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))
$$

Denote $M=\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X)) \times \mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))$. By the notation in Lemma 2.7,

$$
\begin{equation*}
\mathrm{E}\left(Y_{f}\right)=\operatorname{Pr}\left(\mathrm{E}\left(Y_{f}\right)=1\right)=\left(\left|M^{\leq \delta}\right|-1\right) /|M| \tag{3.15}
\end{equation*}
$$

By Lemma 3.5, we have a $\sigma_{\lambda}-F[X]$-isomorphism

$$
C_{f}=\mathcal{I} f(X) \cong \mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))
$$

in particular,

$$
\operatorname{dim}_{F}\left(\mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right)=\operatorname{dim}_{F}\left(C_{f}\right)=d_{f}
$$

In the same way, we have a $\sigma_{\lambda}-F[X]$-isomorphism

$$
\begin{equation*}
C_{f}=\mathcal{I} f(X) \cong \mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X)) \quad \text { hence } \quad \operatorname{dim}_{F}\left(\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X))\right)=d_{f} \tag{3.16}
\end{equation*}
$$

So,

$$
\begin{equation*}
|M|=\left|\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X)) \times \mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right|=q^{d_{f}} q^{d_{f}}=q^{2 d_{f}} \tag{3.17}
\end{equation*}
$$

It is easy to see that

$$
\begin{aligned}
& M \leq \delta \\
&=\left(\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X)) \times \mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right)^{\leq \delta} \\
&= \bigcup_{\substack{w^{\prime}, w \geq 0 \\
w^{\prime}+w=\left\lfloor\delta\left(\alpha^{\prime}+\alpha\right) n\right\rfloor}}\left(\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w^{\prime}}{\alpha^{\prime} n}} \times\left(\mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w}{\alpha n}}
\end{aligned}
$$

Thus

$$
\left|M^{\leq \delta}\right| \leq \sum_{\substack{w^{\prime}, w \geq 0 \\ w^{\prime}+w=\left\lfloor\delta\left(\alpha^{\prime}+\alpha\right) n\right\rfloor}}\left|\left(\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w^{\prime}}{\alpha^{\prime} n}}\right| \cdot\left|\left(\mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w}{\alpha n}}\right|
$$

By Lemma 2.7, Lemma 2.8 and Eq,(3.16),

$$
\left|\left(\mathcal{I}_{\lambda, \alpha^{\prime}} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w^{\prime}}{\alpha^{\prime} n}}\right| \leq q^{d_{f} h_{q}\left(\frac{w^{\prime}}{\alpha^{\prime} n}\right)}, \quad\left|\left(\mathcal{I}_{\lambda, \alpha} \sigma_{\lambda}(f(X))\right)^{\leq \frac{w}{\alpha n}}\right| \leq q^{d_{f} h_{q}\left(\frac{w}{\alpha n}\right)}
$$

Thus

$$
\left|M^{\leq \delta}\right| \leq \sum_{\substack{w^{\prime}, w \geq 0 \\ w^{\prime}+w=\left\lfloor\delta\left(\alpha^{\prime}+\alpha\right) n\right\rfloor}} q^{d_{f}\left(h_{q}\left(\frac{w^{\prime}}{\alpha^{\prime} n}\right)+h_{q}\left(\frac{w}{\alpha n}\right)\right)}
$$

Let $\alpha^{*}=\max \left\{\alpha^{\prime}, \alpha\right\}$. Then $\alpha^{\prime \prime} \leq \alpha^{\prime}, \alpha \leq \alpha^{*}$ and $\alpha^{\prime} \alpha=\alpha^{\prime \prime} \alpha^{*}$. Recall that $h_{q}(x)$ is concave and increasing in the interval $\left[0,1-\frac{1}{q}\right]$. So

$$
\begin{aligned}
& h_{q}\left(\frac{w^{\prime}}{\alpha^{\prime} n}\right)+h_{q}\left(\frac{w}{\alpha n}\right) \leq 2 h_{q}\left(\frac{\frac{w^{\prime}}{\alpha^{\prime} n}+\frac{w}{\alpha n}}{2}\right)=2 h_{q}\left(\frac{\alpha w^{\prime}+\alpha^{\prime} w}{2 \alpha^{\prime} \alpha n}\right) \\
& \leq 2 h_{q}\left(\frac{\alpha^{*} w^{\prime}+\alpha^{*} w}{2 \alpha^{\prime \prime} \alpha^{*} n}\right)=2 h_{q}\left(\frac{w^{\prime}+w}{2 \alpha^{\prime \prime} n}\right) \leq 2 h_{q}\left(\frac{\delta\left(\alpha^{\prime}+\alpha\right) n}{2 \alpha^{\prime \prime} n}\right)=2 h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right) .
\end{aligned}
$$

Further, the number of the pairs $\left(w^{\prime}, w\right)$ satisfying that $w^{\prime}, w \geq 0$ and $w^{\prime}+w=$ $\left\lfloor\delta\left(\alpha^{\prime}+\alpha\right) n\right\rfloor$ is at most $\left(\alpha^{\prime}+\alpha\right) n$. We obtain

$$
\left|M^{\leq \delta}\right| \leq\left(\alpha^{\prime}+\alpha\right) n \cdot q^{2 d_{f} h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)}=q^{2 d_{f} h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)}
$$

Combining it with Eq.(3.15) and Eq.(3.17), we get

$$
\mathrm{E}\left(Y_{f}\right) \leq q^{2 d_{f} h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)} / q^{2 d_{f}}=q^{-2 d_{f}+2 d_{f} h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)}
$$

We are done.

Let $\mu(n)$ be as in Eq.(3.3). For $\mu(n) \leq d \leq n-1$, set

$$
\begin{equation*}
\Omega_{d}=\left\{C \mid C \text { is an } F[X] \text {-submodule of } \mathcal{I}, \operatorname{dim}_{F} C=d\right\} . \tag{3.18}
\end{equation*}
$$

For any ideal $C$ of $\mathcal{R}$, let

$$
\begin{equation*}
C^{*}=\{c \in C \mid \mathcal{R} c=C\} \quad \text { (note that } \mathcal{R} c=C c \text { for } c \in C \text { ). } \tag{3.19}
\end{equation*}
$$

Lemma 3.8. Let notation be as above. Then
(1) $\left|\Omega_{d}\right| \leq n^{\frac{d}{\mu(n)}}$.
(2) $\mathcal{I}-\{0\}=\bigcup_{d=\mu(n)}^{n-1} \bigcup_{C \in \Omega_{d}} C^{*}$.

Proof. (1). By Eq.(3.8), $\mathcal{I}=F_{1} \oplus \cdots \oplus F_{m}$ with each $F_{i}$ being a field with $d_{i}=$ $\operatorname{dim}_{F} F_{i} \geq \mu(n)$. Each $C \in \Omega_{d}$ is a direct sum of some of $F_{1}, \cdots, F_{m}$, and the number of direct summands is at most $d / \mu(n)$. Thus $\left|\Omega_{d}\right| \leq m^{d / \mu(n)} \leq n^{d / \mu(n)}$.
(2). For any $f \in \mathcal{I}-\{0\}, f \in C_{f}^{*}$ and $C_{f} \in \Omega_{d_{f}}$.

Lemma 3.9. Let $\mathrm{E}(Y)$ be as in Eq.(3.13), $\alpha^{\prime \prime}=\min \left\{\alpha^{\prime}, \alpha\right\}$ and $\delta \in\left(0,1-q^{-1}\right)$ as in Remark 3.1. If $\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{\log _{q} n}{2 \mu(n)}>0$, then

$$
\mathrm{E}(Y) \leq q^{-2 \mu(n)\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{3 \log _{q} n}{2 \mu(n)}\right)+\log _{q}\left(\alpha^{\prime}+\alpha\right)}
$$

Proof. Note that $Y_{0}=0$. By the linearity of the expectation and Lemma 3.8(2),

$$
\mathrm{E}(Y)=\sum_{f(X) \in \mathcal{I}-\{0\}} \mathrm{E}\left(Y_{f}\right)=\sum_{d=\mu(n)}^{n-1} \sum_{C \in \Omega_{d}} \sum_{f(X) \in C^{*}} \mathrm{E}\left(Y_{f}\right)
$$

By Lemma 3.7, Eq.(3.18) and Lemma 3.8(1),

$$
\begin{aligned}
\sum_{C \in \Omega_{d}} \sum_{f(X) \in C^{*}} \mathrm{E}\left(Y_{f}\right) & \leq \sum_{C \in \Omega_{d}} \sum_{f(X) \in C^{*}} q^{-2 d+2 d h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)} \\
& \leq \sum_{C \in \Omega_{d}}|C| \cdot q^{-2 d+2 d h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)} \\
& =\sum_{C \in \Omega_{d}} q^{-d+2 d h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)} \\
& \leq n^{\frac{d}{\mu(n)}} q^{-d+2 d h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)} \\
& =q^{-d+2 d h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)+\frac{d \log _{q} n}{\mu(n)}} \\
& =q^{-2 d\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{\log _{q} n}{2 \mu(n)}\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)}
\end{aligned}
$$

Since $\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{\log _{q} n}{2 \mu(n)}>0$ and $d \geq \mu(n)$,

$$
\mathrm{E}(Y) \leq \sum_{d=\mu(n)}^{n-1} q^{-2 \mu(n)\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{\log _{q} n}{2 \mu(n)}\right)+\log _{q}\left(\left(\alpha^{\prime}+\alpha\right) n\right)}
$$

The number of the indexes from $\mu(n)$ to $n-1$ is less than $n$. So

$$
\begin{aligned}
\mathrm{E}(Y) & \leq n \cdot q^{-2 \mu(n)\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{\log _{q} n}{2 \mu(n)}\right)+\log _{q} n+\log _{q}\left(\alpha^{\prime}+\alpha\right)} \\
& =q^{-2 \mu(n)\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{3 \log _{q} n}{2 \mu(n)}\right)+\log _{q}\left(\alpha^{\prime}+\alpha\right)} .
\end{aligned}
$$

We are done.
Lemma 3.10. (1) $\operatorname{dim}_{F} C_{a^{\prime}, a} \leq n-1$, i.e., $\mathrm{R}\left(C_{a^{\prime}, a}\right) \leq \frac{1}{\alpha^{\prime}+\alpha}-\frac{1}{\left(\alpha^{\prime}+\alpha\right) n}$.
(2) $\operatorname{Pr}\left(\operatorname{dim}_{F} C_{a^{\prime}, a}=n-1\right) \geq\left(\frac{1}{4}\right)^{\frac{1}{\mu(n)}}$.

Proof. (1). By Eq.(3.12), $\left|C_{a^{\prime}, a}\right| \leq|\mathcal{I}|=q^{n-1}$. That is, $\operatorname{dim}_{F} C_{a^{\prime}, a} \leq n-1$.
(2). If $a(X) \in \mathcal{I}_{\lambda, \alpha}$ satisfies that $\mathcal{I}_{\lambda, \alpha} a(X)=\mathcal{I}_{\lambda, \alpha}$, i.e., $a(X) \in \mathcal{I}_{\lambda, \alpha}^{*}$ in notation of Eq.(3.19), then $\operatorname{dim}_{F} C_{a^{\prime}, a}=n-1$ (by Eq.(3.12) again). Thus

$$
\operatorname{Pr}\left(\operatorname{dim}_{F} C_{a^{\prime}, a}=n-1\right) \geq \operatorname{Pr}\left(a(X) \in \mathcal{I}_{\lambda, \alpha}^{*}\right)=\frac{\left|\mathcal{I}_{\lambda, \alpha}^{*}\right|}{\left|\mathcal{I}_{\lambda, \alpha}\right|}
$$

By the isomorphism of Lemma 3.5, $\frac{\left|\mathcal{I}_{\lambda, \alpha}^{*}\right|}{\left|\mathcal{I}_{\lambda, \alpha}\right|}=\frac{\left|\mathcal{I}^{*}\right|}{|\mathcal{I}|}$. Thus

$$
\operatorname{Pr}\left(\operatorname{dim}_{F} C_{a, b}=n-1\right) \geq \frac{\left|\mathcal{I}^{*}\right|}{|\mathcal{I}|} .
$$

By Eq.(3.2), $|\mathcal{I}|=q^{d_{1}+\cdots+d_{m}},\left|\mathcal{I}^{*}\right|=\left(q^{d_{1}}-1\right) \cdots\left(q^{d_{m}}-1\right)$, where $d_{1}+\cdots+d_{m}=$ $n-1$ and $d_{i} \geq \mu(n)$ for $i=1, \cdots, m$ (cf. Eq.(3.3)). Hence $m \leq \frac{n}{\mu(n)}$.

$$
\begin{aligned}
\frac{\left|\mathcal{I}^{*}\right|}{|\mathcal{I}|} & =\left(1-\frac{1}{q^{d_{1}}}\right) \cdots\left(1-\frac{1}{q^{d_{m}}}\right) \geq\left(1-\frac{1}{q^{\mu(n)}}\right)^{m} \\
& \geq\left(1-\frac{1}{q^{\mu(n)}}\right)^{\frac{n}{\mu(n)}}=\left(1-\frac{1}{q^{\mu(n)}}\right)^{q^{\mu(n)}} \frac{n}{q^{\mu(n)} \mu(n)}
\end{aligned}
$$

Since the sequence $\left(1-\frac{1}{h}\right)^{h}$ for $h=2,3, \cdots$ is increasing and $\left(1-\frac{1}{2}\right)^{2}=\frac{1}{4}$, we have $\left(1-\frac{1}{q^{\mu(n)}}\right)^{q^{\mu(n)}} \geq \frac{1}{4}$. By the assumption in Remark 3.2, $\mu(n)>\log _{q} n$, hence $q^{\mu(n)}>q^{\log _{q} n}=n$. We obtain that $\frac{\left|\mathcal{I}^{*}\right|}{|\mathcal{I}|} \geq\left(\frac{1}{4}\right)^{\frac{1}{\mu(n)}}$.

### 3.3 Asymptotic property of the random code $C_{a^{\prime}, a}$ over $F$

Keep the notation in Remark 3.1. From Remark 3.2, we can assume that positive integers $n_{1}, n_{2}, \cdots$ satisfy:

$$
\begin{equation*}
\operatorname{gcd}\left(n_{i}, q t\right)=1, \forall i=1,2, \cdots, \quad \text { and } \quad \lim _{i \rightarrow \infty} \frac{\log _{q} n_{i}}{\mu\left(n_{i}\right)}=0 \tag{3.20}
\end{equation*}
$$

Note that the assumption also implies that $\mu\left(n_{i}\right)>\log _{q} n_{i}$ (for $i$ large enough) and $\mu\left(n_{i}\right) \rightarrow \infty$. In Eq.(3.12), taking $n=n_{i}$, we have the random double $\lambda$-twisted codes $C_{a^{\prime}, a}^{(i)}$ of cycle length $\left(\alpha^{\prime} n_{i}, \alpha n_{i}\right)$ over $F$. Then

$$
\begin{equation*}
C_{a^{\prime}, a}^{(1)}, C_{a^{\prime}, a}^{(2)}, \cdots, C_{a^{\prime}, a}^{(i)}, \cdots \tag{3.21}
\end{equation*}
$$

is a sequence of random double $\lambda$-twisted codes over $F$, and the length $\alpha^{\prime} n_{i}+\alpha n_{i}$ of $C_{a^{\prime}, a}^{(i)}$ goes to infinity.
Theorem 3.11. Let notation be as in Eq.(3.20) and Eq.(3.21). Assume that $\delta \in\left(0,1-q^{-1}\right)$ satisfying that $h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)<\frac{1}{2}$, where $\alpha^{\prime \prime}=\min \left\{\alpha^{\prime}, \alpha\right\}$. Then
(1) $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\Delta\left(C_{a^{\prime}, a}^{(i)}\right)>\delta\right)=1$.
(2) $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\operatorname{dim}_{F} C_{a^{\prime}, a}^{(i)}=n_{i}-1\right)=1$.

Proof. (1). By Eq.(3.13) and Lemma 3.9,

$$
\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\Delta\left(C_{a^{\prime}, a}^{(i)}\right) \leq \delta\right) \leq \lim _{i \rightarrow \infty} q^{-2 \mu\left(n_{i}\right)\left(\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{3 \log _{q} n_{i}}{2 \mu\left(n_{i}\right)}\right)+\log _{q}\left(\alpha^{\prime}+\alpha\right)} .
$$

Note that $\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)>0$. By Eq.(3.20), we have $\lim _{i \rightarrow \infty} \frac{\log _{q} n_{i}}{\mu\left(n_{i}\right)}=0$, which also implies that $\mu\left(n_{i}\right) \rightarrow \infty$. Then, there is a positive real number $\delta_{0}$ such that $\frac{1}{2}-h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)-\frac{3 \log _{q} n_{i}}{2 \mu\left(n_{i}\right)}>\delta_{0}$ for large enough $i$. So, $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\Delta\left(C_{a^{\prime}, a}^{(i)}\right) \leq \delta\right)=0$.
(2). By Lemma 3.10, $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\operatorname{dim}_{F} C_{a^{\prime}, a}^{(i)}=n_{i}-1\right) \geq \lim _{i \rightarrow \infty}\left(\frac{1}{4}\right)^{\frac{1}{\mu\left(n_{i}\right)}}=1$.

As a consequence, the double $\lambda$-twisted codes of ratio $\alpha^{\prime} / \alpha$ over finite fields are asymptotically good.

Theorem 3.12. Keep the notation in Remark 3.1. Assume that $\delta \in\left(0,1-q^{-1}\right)$ satisfying that $h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)<\frac{1}{2}$. Then there is a sequence $C_{1}, C_{2}, \cdots$ of double $\lambda$ twisted codes $C_{i}$ of ratio $\alpha^{\prime} / \alpha$ over $F$ such that the length of $C_{i}$ goes to infinity, $\lim _{i \rightarrow \infty} \mathrm{R}\left(C_{i}\right)=\frac{1}{\alpha^{\prime}+\alpha}$, and $\Delta\left(C_{i}\right)>\delta$ for all $i=1,2, \cdots$.

Proof. In Theorem 3.11, we can take $C_{i}=C_{a^{\prime}, a}^{(i)}$ for $i=1,2, \cdots$ such that:

- the length of $C_{i}$ is $\alpha^{\prime} n_{i}+\alpha n_{i}$;
- the relative minimum distance $\Delta\left(C_{i}\right)>\delta$;
- the information length of $C_{i}$ is $\operatorname{dim} C_{i}=n_{i}-1$, hence the rate
$\mathrm{R}\left(C_{i}\right)=\frac{n_{i}-1}{\alpha^{\prime} n_{i}+\alpha n_{i}}=\frac{1}{\alpha^{\prime}+\alpha}-\frac{1}{\alpha^{\prime} n_{i}+\alpha n_{i}}$.
Thus the theorem holds.


## $4\left(R^{\prime}, R\right)$-linear constacyclic codes of ratio $\alpha^{\prime} / \alpha$

Remark 4.1. In this section we turn to the general case and take the notation in Definition 2.5:

- $R, R^{\prime}$ are finite chain rings, $J(R)=R \pi$ of nilpotency index $\ell, J\left(R^{\prime}\right)=R^{\prime} \pi^{\prime}$ of nilpotency index $\ell^{\prime}$; there is an epimorphism $\rho: R \rightarrow R^{\prime}$, hence they have the same residue field $F:=\bar{R}=\overline{R^{\prime}}$, and $\ell^{\prime} \leq \ell$, set $|F|=q$.
- $\lambda \in R^{\times}, \operatorname{ord}_{F^{\times}}(\bar{\lambda})=t$; and $\lambda^{\prime}=\rho(\lambda) \in R^{\prime \times}$, hence $\bar{\lambda}=\overline{\lambda^{\prime}}$.
- Integers $\alpha^{\prime}, \alpha>0, \alpha^{\prime} \equiv \alpha(\bmod t), \operatorname{gcd}(\alpha, t)=1$; further, $\alpha^{\prime \prime}=\min \left\{\alpha^{\prime}, \alpha\right\}$.

And further,

- $n$ is a positive integer such that $\operatorname{gcd}(n, q t)=1$.
- $\delta \in\left(0,1-q^{-1}\right)$.

Any $R[X]$-submodule $C$ of $\left(R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle\right) \times\left(R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle\right)$ is an $\left(R^{\prime}, R\right)$ linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic code of ratio $\alpha^{\prime} / \alpha$, code length equals $\alpha^{\prime} n \ell^{\prime}+\alpha n \ell$, the relative minimum distance $\Delta(C)=\frac{\mathrm{w}(C)}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}$, and the rate $\mathrm{R}(C)=\frac{\log _{q}|C|}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}$.

The finite chain ring $R$ has a unique minimal ideal $R \pi^{\ell-1}$, and the following is an $R$-module epimorphism:

$$
R \longrightarrow R \pi^{\ell-1}, \quad a \longmapsto a \pi^{\ell-1} .
$$

The kernel of this $R$-module epimorphism is $R \pi$. Thus it induces an $R$-module isomorphism

$$
\begin{equation*}
\eta: F=R / R \pi \xrightarrow{\cong} R \pi^{\ell-1}, \quad \bar{a} \longmapsto a \pi^{\ell-1} . \tag{4.1}
\end{equation*}
$$

Lemma 4.2. The following is a well-defined $R[X]$-module monomorphism (i.e., injective homomorphism):

$$
\begin{align*}
\eta_{\alpha}: \quad F[X] /\left\langle X^{\alpha n}-\bar{\lambda}\right\rangle & \longrightarrow R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle \\
\sum_{j=0}^{\alpha n-1} \bar{a}_{j} X^{j} & \longmapsto \sum_{j=0}^{\alpha n-1} a_{j} \pi^{\ell-1} X^{j} \tag{4.2}
\end{align*}
$$

which preserves Hamming weights.

Proof. The above $R$-module isomorphism Eq.(4.1) induces an $R[X]$-module monomorphism, denoted by $\eta$ again:

$$
\begin{equation*}
\eta: F[X] \rightarrow R[X], \quad \sum_{i} \bar{a}_{i} X^{i} \mapsto \sum_{i} a_{i} \pi^{\ell-1} X^{i} \tag{4.3}
\end{equation*}
$$

The $R[X]$-module structure of $F[X]$ is as follows: $g(X) \cdot f(X)=\bar{g}(X) f(X)$ for $f(X) \in F[X]$ and $g(X)=\sum_{i} b_{i} X^{i} \in R[X]$, where $\bar{g}(X)=\sum_{i} \bar{b}_{i} X^{i} \in F[X]$. Because $\eta$ in Eq.(4.3) is an $R[X]$-module monomorphism, we have:

$$
\begin{equation*}
\eta(\bar{g}(X) f(X))=g(X) \eta(f(X)), \quad \forall f(X) \in F[X], g(X) \in R[X] \tag{4.4}
\end{equation*}
$$

For $f(X) \in F[X]$ we denote $\eta(f(X))=: f^{\eta}(X)$. Combining the $\eta$ in Eq.(4.3) with the quotient homomorphism $R[X] \rightarrow R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle$, we obtain the following $R[X]$-homomorphism

$$
\tilde{\eta}: F[X] \rightarrow R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle, \quad f(X) \mapsto f^{\eta}(X)\left(\bmod X^{\alpha n}-\lambda\right) .
$$

Assume that $f(X)=\sum_{i} \bar{a}_{i} X^{i} \in \operatorname{Ker}(\tilde{\eta})$, i.e., there is a $g(X)=\sum_{i} b_{i} X^{i} \in R[X]$ such that

$$
\eta(f(X))=\sum_{i} a_{i} \pi^{\ell-1} X^{i}=\left(X^{\alpha n}-\lambda\right) g(X)
$$

Since $\pi \sum_{i} a_{i} \pi^{\ell-1} X^{i}=0$, we have that $\left(X^{\alpha n}-\lambda\right) \pi g(X)=0$; further, since $X^{\alpha n}-\lambda$ is monic, we can see that $\pi g(X)=0$. Thus, for any coefficient $b_{i}$ of $g(X)$ there is a $d_{i} \in R$ such that $b_{i}=d_{i} \pi^{\ell-1}$. Then $g(X)=\eta(d(X))$ where $d(X)=\sum_{i} \bar{d}_{i} X^{i} \in F[X]$. By Eq.(4.4),

$$
\eta(f(X))=\left(X^{\alpha n}-\lambda\right) \eta(d(X))=\eta\left(\left(X^{\alpha n}-\bar{\lambda}\right) d(X)\right)
$$

Since $\eta$ is injective, $f(\underline{X})=\left(X^{\alpha n}-\bar{\lambda}\right) d(X) \in F[X]\left(X^{\alpha n}-\bar{\lambda}\right)$. We get that $\operatorname{Ker}(\tilde{\eta}) \subseteq F[X]\left(X^{\alpha n}-\bar{\lambda}\right)$. The inverse inclusion is obvious. Thus

$$
\operatorname{Ker}(\tilde{\eta})=F[X]\left(X^{\alpha n}-\bar{\lambda}\right)=\left\langle X^{\alpha n}-\bar{\lambda}\right\rangle
$$

and $\tilde{\eta}$ induces the $R[X]$-module monomorphism $\eta_{\alpha}$ in Eq.(4.2).
For $f(X)=\sum_{j=0}^{\alpha n-1} \bar{a}_{j} X^{j} \in F[X] /\left\langle X^{\alpha n}-\bar{\lambda}\right\rangle$. the image

$$
\eta_{\alpha}(f(X))=\sum_{j=0}^{\alpha n-1} a_{j} \pi^{\ell-1} X^{j} \in R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle
$$

Obviously,

$$
\bar{a}_{j} \neq 0 \quad(\text { in } F) \quad \Longleftrightarrow \quad a_{j} \pi^{\ell-1} \neq 0 \quad(\text { in } R)
$$

Thus $\mathrm{w}(f(X))=\mathrm{w}\left(\eta_{\alpha}(f(X))\right)$; i.e., $\eta_{\alpha}$ preserves the Hamming weights.
Similarly, we have the following $R^{\prime}[X]$-module monomorphism

$$
\eta^{\prime}: F[X] \rightarrow R^{\prime}[X], \quad \sum_{i} \bar{a}_{i} X^{i} \mapsto \sum_{i} a_{i} \pi^{\prime \ell^{\prime}-1} X^{i}
$$

and the following $R^{\prime}[X]$-module monomorphism:

$$
\begin{align*}
\eta_{\alpha^{\prime}}^{\prime}: F[X] /\left\langle X^{\alpha^{\prime} n}-\overline{\lambda^{\prime}}\right\rangle & \longrightarrow R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle, \\
\sum_{j=0}^{\alpha n-1} \bar{a}_{j} X^{j} & \longmapsto \sum_{j=0}^{\alpha n-1} a_{j} \pi^{\prime \ell^{\prime}-1} X^{j}, \tag{4.5}
\end{align*}
$$

which preserves Hamming weights. Note that, through the epimorphism $\rho$ : $R \rightarrow R^{\prime}$, both $\eta^{\prime}$ and $\eta_{\alpha}^{\prime}$ can be viewed as $R[X]$-module homomorphisms.

Recall the notation in Section 3 (but with assumptions in Remark 4.1 of this section, e.g., $\left.\bar{\lambda}=\overline{\lambda^{\prime}} \in F\right)$ :

- $\mathcal{I}_{\bar{\lambda}, \alpha} \subseteq \mathcal{R}_{\bar{\lambda}, \alpha}=F[X] /\left\langle X^{\alpha n}-\bar{\lambda}\right\rangle$, (Eq.(3.7))
- $\mathcal{I}_{\overline{\lambda^{\prime}}, \alpha^{\prime}} \subseteq \mathcal{R}_{\overline{\lambda^{\prime}}, \alpha^{\prime}}=F[X] /\left\langle X^{\alpha^{\prime} n}-\overline{\lambda^{\prime}}\right\rangle$, (Eq.(3.7))
- $C_{a^{\prime}, a}=F[X]\left(a^{\prime}(X), a(X)\right) \subseteq \mathcal{I}_{\overline{\lambda^{\prime}}, \alpha^{\prime}} \times \mathcal{I}_{\bar{\lambda}, \alpha}, \quad$ (Eq.(3.11))
- positive integers $n_{1}, n_{2}, \cdots$ satisfy Eq.(3.20)
- $C_{a^{\prime}, a}^{(1)}, C_{a^{\prime}, 1}^{(2)}, \cdots$, in Eq.(3.21).

By the $R[X]$-monomorphisms Eq.(4.2) and Eq.(4.5), we can embed

$$
C_{a^{\prime}, a}^{(i)}=\left\{\left(g(X) a^{\prime}(X), g(X) a(X)\right) \mid g(X) \in F[X]\right\} \quad \text { (see Eq.(3.11)) }
$$

into

$$
\left(R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n_{i}}-\lambda^{\prime}\right\rangle\right) \times\left(R[X] /\left\langle X^{\alpha n_{i}}-\lambda\right\rangle\right)
$$

as follows:

$$
\begin{equation*}
\left(g(X) a^{\prime}(X), g(X) a(X)\right) \longmapsto\left(\eta^{\prime}(g(X)) \eta_{\alpha^{\prime}}^{\prime}\left(a^{\prime}(X)\right), \eta(g(X)) \eta_{\alpha}(a(X))\right) \tag{4.6}
\end{equation*}
$$

We denote the image of $C_{a^{\prime}, a}^{(i)}$ by $\tilde{C}_{a^{\prime}, a}^{(i)}$. In this way, we obtain a sequence of $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes of ratio $\alpha^{\prime} / \alpha$ as follows:

$$
\begin{equation*}
\tilde{C}_{a^{\prime}, a}^{(1)}, \tilde{C}_{a^{\prime}, a}^{(2)}, \cdots, \tilde{C}_{a^{\prime}, a}^{(i)}, \cdots \tag{4.7}
\end{equation*}
$$

Theorem 4.3. Let notation be as in Eq.(4.7). Assume that $\delta \in\left(0,1-q^{-1}\right)$ satisfying that $h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)<\frac{1}{2}$. Then
(1) $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\Delta\left(\tilde{C}_{a^{\prime}, a}^{(i)}\right)>\frac{\alpha^{\prime}+\alpha}{\alpha^{\prime} \ell^{\prime}+\alpha \ell} \cdot \delta\right)=1$.
(2) $\lim _{i \rightarrow \infty} \operatorname{Pr}\left(\left|\tilde{C}_{a^{\prime}, a}^{(i)}\right|=q^{n_{i}-1}\right)=1$.

Proof. Because both Eq.(4.2) and Eq.(4.5) are monomorphism and preserve the Hamming weights, $\mathrm{w}\left(\tilde{C}_{a^{\prime}, a}^{(i)}\right)=\mathrm{w}\left(C_{a^{\prime}, a}^{(i)}\right)$. The code length of $C_{a^{\prime}, a}^{(i)}$ is $\alpha^{\prime} n+\alpha n$; while the code length of $\tilde{C}_{a^{\prime}, a}^{(i)}$ is $\alpha^{\prime} n \ell^{\prime}+\alpha n \ell$. So

$$
\Delta\left(\tilde{C}_{a^{\prime}, a}^{(i)}\right)=\frac{\mathrm{w}\left(C_{a^{\prime}, a}^{(i)}\right)}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}=\frac{\mathrm{w}\left(C_{a^{\prime}, a}^{(i)}\right)}{\alpha^{\prime} n+\alpha n} \cdot \frac{\alpha^{\prime} n+\alpha n}{\alpha^{\prime} n \ell^{\prime}+\alpha n \ell}=\frac{\alpha^{\prime}+\alpha}{\alpha^{\prime} \ell^{\prime}+\alpha \ell} \cdot \Delta\left(C_{a^{\prime}, a}^{(i)}\right)
$$

then

$$
\Delta\left(C_{a^{\prime}, a}^{(i)}\right) \geq \delta \quad \Longleftrightarrow \quad \Delta\left(\tilde{C}_{a^{\prime}, a}^{(i)}\right) \geq \frac{\alpha^{\prime}+\alpha}{\alpha^{\prime} \ell^{\prime}+\alpha \ell} \cdot \delta
$$

Therefore, the theorem follows from Theorem 3.11 immediately.

Finally, the following is a more precise version of Theorem 1.1.
Theorem 4.4. Assume that $\delta \in\left(0,1-q^{-1}\right)$ satisfying that $h_{q}\left(\frac{\alpha^{\prime}+\alpha}{2 \alpha^{\prime \prime}} \delta\right)<\frac{1}{2}$. Then there is a sequence $C_{1}, C_{2}, \cdots$ of $\left(R^{\prime}, R\right)$-linear $\left(\lambda^{\prime}, \lambda\right)$-constacyclic codes $C_{i}$ of ratio $\alpha^{\prime} / \alpha$ such that the length of $C_{i}$ goes to infinity, $\lim _{i \rightarrow \infty} \mathrm{R}\left(C_{i}\right)=\frac{1}{\alpha^{\prime} \ell^{\prime}+\alpha \ell}$, and $\Delta\left(C_{i}\right)>\frac{\alpha^{\prime}+\alpha}{\alpha^{\prime} \ell^{\prime}+\alpha \ell} \cdot \delta$ for all $i=1,2, \cdots$.

Proof. In Theorem 4.3, we can take $C_{i}=\tilde{C}_{a^{\prime}, a}^{(i)}$ for $i=1,2, \cdots$ such that:

- the length of $C_{i}$ is $\alpha^{\prime} n_{i} \ell^{\prime}+\alpha n_{i} \ell ;$
- the relative minimum distance $\Delta\left(C_{i}\right)>\frac{\alpha^{\prime}+\alpha}{\alpha^{\prime} \ell^{\prime}+\alpha \ell} \cdot \delta$;
- the information length of $C_{i}$ is $\log _{q}\left|C_{i}\right|=n_{i}-1$, hence the rate

$$
\mathrm{R}\left(C_{i}\right)=\frac{n_{i}-1}{\alpha^{\prime} n_{i} \ell^{\prime}+\alpha n_{i} \ell}=\frac{1}{\alpha^{\prime} \ell^{\prime}+\alpha \ell}-\frac{1}{\alpha^{\prime} n_{i} \ell^{\prime}+\alpha n_{i} \ell} .
$$

Thus the theorem holds.

## 5 Conclusion

The main contribution of this paper is that a very general type of codes is constructed and the asymptotic goodness of such codes is proved.

We introduced a type of codes: let $R$ and $R^{\prime}$ be two finite commutative chain rings with an epimorphism $\rho: R \rightarrow R^{\prime}$, let $\lambda \in R^{\times}$and $\lambda^{\prime}=\rho(\lambda)$, and $\alpha, \alpha^{\prime}, n$ be positive integers; we call any $R[X]$-submodule $C$ of the $R[X]$ module $\left(R^{\prime}[X] /\left\langle X^{\alpha^{\prime} n}-\lambda^{\prime}\right\rangle\right) \times\left(R[X] /\left\langle X^{\alpha n}-\lambda\right\rangle\right)$ by an $\left(R^{\prime}, R\right)$-linear constacyclic code. Such codes form an extensive class of codes. First, the two alphabets for the codes are finite commutative chain rings which cover many alphabets used in coding. Second, instead of cyclic structures, the more general constacyclic structures are considered. Third, the two lengths of the two constacyclic circles are not necessarily equal. Thus, $\left(R^{\prime}, R\right)$-linear constacyclic codes cover many well-known kinds of codes, e.g., quasi-cyclic codes of fractional index, $\mathbb{Z}_{2} \mathbb{Z}_{4^{-}}$ additive cyclic codes etc.

We proved in a random style that $\left(R^{\prime}, R\right)$-linear constacyclic codes are asymptotically good. The usual probabilistic method applied to quasi-cyclic codes could not applied to the constacyclic case directly. We take an algebraic skill to reform it into a developed probabilistic method effective for studying quasiconstacyclic codes. And then we reduced the proof for the asymptotic goodness of ( $R^{\prime}, R$ )-linear constacyclic codes to the quasi-constacyclic case, so that the proof of the asymptotic goodness of $\left(R^{\prime}, R\right)$-linear constacyclic codes is completed. The developed probabilistic method is another contribution of this paper.

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