Tight Exponential Analysis for Smoothing the Max-Relative Entropy and for Quantum Privacy Amplification

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Abstract

The max-relative entropy together with its smoothed version is a basic tool in quantum information theory. In this paper, we derive the exact exponent for the asymptotic decay of the small modification of the quantum state in smoothing the max-relative entropy based on purified distance. We then apply this result to the problem of privacy amplification against quantum side information, and we obtain an upper bound for the exponent of the asymptotic decreasing of the insecurity, measured using either purified distance or relative entropy. Our upper bound complements the earlier lower bound established by Hayashi, and the two bounds match when the rate of randomness extraction is above a critical value. Thus, for the case of high rate, we have determined the exact security exponent. Following this, we give examples and show that in the low-rate case, neither the upper bound nor the lower bound is tight in general. This exhibits a picture similar to that of the error exponent in channel coding. Lastly, we investigate the asymptotics of equivocation and its exponent under the security measure using the sandwiched Rényi divergence of order $s \in (1,2]$, which has not been addressed previously in the quantum setting.

Index Terms

max-relative entropy, quantum privacy amplification, exponent, sandwiched Rényi divergence, equivocation

I. Introduction

The smooth max-relative entropy is a basic tool in quantum information theory [1], [2], [3], [4], [5], [6], [7], developed in parallel with the related but different concepts of hypothesis testing relative entropy [8], [9], [10], [11], [12], [13], [14] and information spectrum relative entropy [15], [16], [8], [9], [10], [12]. In the asymptotic limit when multiple copies of underlying resource states are available, the one-shot characterizations using these quantities lead to results of the traditional information-theoretic type. Indeed, the quantum relative entropy (Kullback-Leibler divergence), arguably, finds its most direct operational interpretations in the asymptotic analysis of these quantities [17], [18], [3], [7], [15], [16], [9]. The asymptotic analysis of the quantum hypothesis testing entropy has been extended to the second-order regime [12], [19]. Moreover, large-deviation type exponential analysis for quantum hypothesis testing is well understood [20], [21], [22], [23], [24], [25], and the relation between information spectrum and quantum hypothesis testing has been well studied up to the exponential analysis [8], [10].

However, the details of the asymptotic behavior of the smoothing of the max-relative entropy is more complicated. In particular, it depends on the choice of the distance measure to define the smoothing. Originally, Renner [1] defined the smoothing of the max-relative entropy based on the trace norm distance. The paper [26] derived its type exponential behavior based on the trace norm distance in the classical case.

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Later, the references [3], [4], [27] introduced another smoothing of the max-relative entropy based on the purified distance. The reference [12] showed that this type of smoothing of the max-relative entropy has the same behavior with the quantum hypothesis testing entropy in the second-order regime. The exact large-deviation type exponential behavior in both types of smoothing of the max-relative entropy remains unclear in the general case.

In this paper, we conduct the exponential analysis for the smoothing of the max-relative entropy based on the purified distance. For two quantum states ρ and σ , consider the smoothing quantity $\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) := \min\{\ell(\rho^{\otimes n}, \widetilde{\rho^n}) \mid \widetilde{\rho^n} \leq 2^{nr}\sigma^{\otimes n}\}$, where ℓ is certain distance measure and $\widetilde{\rho^n}$ is a subnormalized quantum state. It is known that when r is larger than the relative entropy $D(\rho \| \sigma)$, the smoothing quantity can be arbitrarily small when n is big enough. We determine the precise exponent under which the smoothing quantity converges to 0 exponentially in this case for ℓ being the purified distance (cf. Theorem 1). Remarkably, this exponent is given in terms of the sandwiched Rényi divergence [28], [29]. Our result naturally covers the exponential analysis for the smoothing of a particular type of the conditional minentropy and the max-mutual information (see, e.g., Definition 3 for the conditional min-entropy).

We apply the above-mentioned result to the problem of private randomness extraction against quantum side information, a quantum information processing task also called privacy amplification [30], [31], [1], [12], [32], [33]. Asymptotic security of privacy amplification in the i.i.d. situation is known to hold when the rate of randomness extraction does not exceed the conditional entropy of the raw randomness given the quantum side information [31], [1]. We obtain an upper bound for the rate of exponential decreasing of the insecurity in the i.i.d. situation, measured either by the purified distance or by the relative entropy, in terms of a version of the sandwiched Rényi conditional entropy (cf. Theorem 2). Notice that an upper bound for the rate of exponential decreasing corresponds to a lower bound of the insecurity. This complements the previous work [32], which has established a privacy amplification theorem concerning the achievability via two-universal hash functions and obtains a corresponding lower bound for the exponent in the asymptotic case. We show that our upper bound matches the above mentioned lower bound when the rate R of randomness extraction is above a critical value R_{critical} . Thus, for the case with high rate of randomness extraction, we have determined the exact security exponent (cf. Theorem 3). For the low-rate situation, we give simple examples to show that neither the upper bound nor the lower bound is tight in general. These results exhibit a picture similar to that of the error exponent of channel coding in classical information theory [34].

In addition, we investigate the security of privacy amplification under a more general class of information measure—the sandwiched Rényi divergence of order $s \in (1,2]$. We prove tight equivocation rate for this security measure and derive the exponential rate of decay of the insecurity. This problem has been analyzed by the reference [35] in the classical case, in which they evaluate the asymptotics of equivocations and their exponents under various Rényi information measures. We generalize their results to the quantum setting here.

Our results provide operational interpretations to the sandwiched Rényi divergence and the sandwiched Rényi conditional entropy, in addition to previous operational interpretations to the sandwiched Rényi information quantities [25], [36], [37], [38], [39], [40]. However, the operational interpretations found in the present paper, as well as those in the concurrent work of [41] which addresses different problems, are in stark contrast to those of the previous ones, in the sense explained as follows. The works [25], [36], [37], [38], [39], [40] proved that the sandwiched Rényi information quantities characterize the strong converse exponents, that is, the exponential rates under which the underlying error goes to 1. Our results and the work [41], for the first time, show that the sandwiched Rényi information quantities characterize the exponents under which the underlying error goes to 0, and are therefore of greater realistic significance.

The remainder of this paper is organized as follows. In Section II, we introduce the necessary notations, definitions and some properties of quantum entropic quantities. Then in Section III, we derive the optimal exponent in smoothing the max-relative entropy. Section IV is devoted to the analysis of the asymptotic rates of exponential decreasing of the insecurity of privacy amplification. In Section V, we investigate the equivocation rate and the exponential rate of decay of the insecurity of privacy amplification measured

by the sandwiched Rényi divergence of order $s \in (1,2]$. At last, in Section VI, we conclude the paper with some discussion and open questions.

II. NOTATION AND PRELIMINARIES

A. Basic notation

Let \mathcal{H} be a finite dimensional Hilbert space. $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators on \mathcal{H} , and $\mathcal{P}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ denotes the set of positive semidefinite operators. $\mathbb{1}_{\mathcal{H}}$ is the identity operator. The set of (normalized) quantum states and subnormalized quantum states on \mathcal{H} are denoted as $\mathcal{S}(\mathcal{H})$ and $\mathcal{S}_{\leq}(\mathcal{H})$, respectively. They are given by

$$S(\mathcal{H}) = \{ \rho \in \mathcal{P}(\mathcal{H}) | \operatorname{Tr} \rho = 1 \},$$

$$S_{<}(\mathcal{H}) = \{ \rho \in \mathcal{P}(\mathcal{H}) | \operatorname{Tr} \rho \leq 1 \}$$

and also called density operators. A classical-quantum (CQ) state is a bipartite state of the form $\rho_{XA} = \sum_x p(x)|x\rangle\langle x|_X\otimes\rho_A^x$, where $\rho_A^x\in\mathcal{S}(\mathcal{H}),\ p(x)$ is a probability distribution, and $\{|x\rangle\}$ is an orthonormal basis of the underlying Hilbert space \mathcal{H}_X . If the system X is classical as in the CQ state, we also use the notation X to represent a random variable that takes the value x with probability p(x). The set of all the possible values of X is denoted by the corresponding calligraphic letter \mathcal{X} .

We write $A \ge 0$ if $A \in \mathcal{P}(\mathcal{H})$, and $A \ge B$ if $A - B \ge 0$. If $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint, we use $\{A \ge 0\}$ to denote the spectral projection of A corresponding to all non-negative eigenvalues. $\{A > 0\}$, $\{A \le 0\}$ and $\{A < 0\}$ are defined in a similar way. The positive part of A is defined as $A_+ := A\{A > 0\}$. We can easily check that, for any $D \in \mathcal{L}(\mathcal{H})$ such that $0 \le D \le 1$,

$$\operatorname{Tr} A_{+} \ge \operatorname{Tr} AD.$$
 (1)

A quantum channel (or quantum operation), which acts on quantum states, is formally described by a linear, completely positive, trace-preserving (CPTP) map $\Phi: \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_B)$. A quantum measurement is described by a set of positive semidefinite operators $\mathcal{M} = \{M_x\}_x$ such that $\sum_x M_x = \mathbb{I}$, and it converts a quantum state ρ into a probability vector \vec{p} with $\vec{p}_x = \operatorname{Tr} \rho M_x$. For each quantum measurement $\mathcal{M} = \{M_x\}_x$, there is a measurement channel $\Phi_{\mathcal{M}}: \rho \mapsto \sum_x (\operatorname{Tr} \rho M_x)|x\rangle\langle x|$, where $\{|x\rangle\}$ is an orthonormal basis.

We employ the purified distance [42], [3] to measure the closeness of two states $\rho, \sigma \in \mathcal{S}_{\leq}(\mathcal{H})$. It is defined as $P(\rho, \sigma) := \sqrt{1 - F^2(\rho, \sigma)}$, where

$$F(\rho, \sigma) := \left\| \sqrt{\rho} \sqrt{\sigma} \right\|_{1} + \sqrt{(1 - \operatorname{Tr} \rho)(1 - \operatorname{Tr} \sigma)}$$

is the fidelity function. The purified distance has some nice properties, inherited from the fidelity. *Proposition 1:* The following properties hold for the purified distance.

(i) Triangle inequality [42]: Let $\rho, \sigma, \tau \in \mathcal{S}_{<}(\mathcal{H})$. Then

$$P(\rho, \sigma) \le P(\rho, \tau) + P(\tau, \sigma);$$

(ii) Fuchs-van de Graaf inequality [43]: Let $\rho, \sigma \in \mathcal{S}_{<}(\mathcal{H})$. Then

$$d(\rho, \sigma) \le P(\rho, \sigma) \le \sqrt{2d(\rho, \sigma) - d^2(\rho, \sigma)},$$

where $d(\rho, \sigma) := \frac{1}{2}(\|\rho - \sigma\|_1 + |\operatorname{Tr}(\rho - \sigma)|)$ is the trace distance;

(iii) Data processing inequality [44]: Let $\rho, \sigma \in \mathcal{S}_{<}(\mathcal{H})$ and Φ be a CPTP map. Then

$$P(\rho, \sigma) \ge P(\Phi(\rho), \Phi(\sigma));$$

(iv) Uhlmann's theorem [45]: Let $\rho_{AB} \in \mathcal{S}_{\leq}(\mathcal{H}_{AB})$ be a bipartite state, and $\sigma_A \in \mathcal{S}_{\leq}(\mathcal{H}_A)$. Then there exists an extension σ_{AB} of σ_A such that

$$P(\rho_{AB}, \sigma_{AB}) = P(\rho_A, \sigma_A).$$

The ϵ -ball of subnormalized quantum states around $\rho \in \mathcal{S}(\mathcal{H})$ is defined using the purified distance as

$$\mathcal{B}^{\epsilon}(\rho) := \{ \tilde{\rho} \in \mathcal{S}_{<}(\mathcal{H}) | P(\tilde{\rho}, \rho) \leq \epsilon \}.$$

For an operator $A \in \mathcal{L}(\mathcal{H})$, let v(A) be the number of different eigenvalues of A. If A is self-adjoint with spectral projections $P_1, \ldots, P_{v(A)}$, then the associated pinching map $\mathcal{E}_A : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a CPTP map given by

$$\mathcal{E}_A: X \mapsto \sum_i P_i X P_i.$$

The pinching inequality [46] states that if X is positive semidefinite, we have

$$X \le v(A)\mathcal{E}_A(X). \tag{2}$$

B. Entropies and information divergences

The quantum relative entropy for $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{P}(\mathcal{H})$ is defined [47] as

$$D(\rho \| \sigma) := \begin{cases} \operatorname{Tr}(\rho(\log \rho - \log \sigma)) & \text{if } \operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma), \\ +\infty & \text{otherwise,} \end{cases}$$

where the logarithm function log is with base 2 throughout this paper. For a bipartite state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, the quantum mutual information and the conditional entropy are defined, respectively, as

$$I(A:B)_{\rho} := D(\rho_{AB} \| \rho_A \otimes \rho_B),$$

$$H(A|B)_{\rho} := -D(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B).$$

Among various inequivalent generalizations of the Rényi relative entropy to the non-commutative quantum situation, the sandwiched Rényi divergence [28], [29] is of particular interest.

Definition 1: Let $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, and $\alpha \in (0,1) \cup (1,\infty)$. If either $0 < \alpha < 1$ and $\operatorname{Tr} \rho \sigma \neq 0$ or $\alpha > 1$ and $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$, the sandwiched Rényi divergence of order α is defined as

$$D_{\alpha}(\rho\|\sigma) := \frac{1}{\alpha - 1} \log Q_{\alpha}(\rho\|\sigma), \quad \text{where} \quad Q_{\alpha}(\rho\|\sigma) := \operatorname{Tr} \left(\sigma^{\frac{1 - \alpha}{2\alpha}} \rho \sigma^{\frac{1 - \alpha}{2\alpha}}\right)^{\alpha}.$$

Otherwise, we set $D_{\alpha}(\rho \| \sigma) = +\infty$.

When α goes to infinity, $D_{\alpha}(\rho \| \sigma)$ converges to the max-relative entropy [2]

$$D_{\max}(\rho \| \sigma) := \inf\{\lambda \mid \rho \le 2^{\lambda} \sigma\}. \tag{3}$$

For $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ and $\alpha \in (0,1) \cup (1,\infty)$, we consider the sandwiched Rényi conditional entropy of order α defined as [48]

$$H_{\alpha}(A|B)_{\rho} := -D_{\alpha}(\rho_{AB} || \mathbb{1}_A \otimes \rho_B).$$

If the system B is of dimension 1, the sandwiched Rényi conditional entropy reduces to the Rényi entropy of a single system $H_{\alpha}(A)_{\rho} = -D_{\alpha}(\rho_A || \mathbb{1}_A) = \frac{1}{1-\alpha} \log \operatorname{Tr} \rho_A^{\alpha}$. We mention that these definitions can be extended to include the cases that $\alpha = 0, 1, +\infty$ by taking the limit of α . Moreover, for a CQ state ρ_{XE} , we define for s > 0,

$$\hat{R}(s) := \frac{\mathrm{d}}{\mathrm{d}s} s H_{1+s}(X|E)_{\rho} \tag{4}$$

and we set

$$R_{\text{critical}} := \hat{R}(1) = \frac{\mathrm{d}}{\mathrm{d}s} s H_{1+s}(X|E)_{\rho}\big|_{s=1}.$$
 (5)

In the next proposition, we collect a few properties of the Rényi quantities defined above.

Proposition 2: Let $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, $\xi_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, and $\omega_{XAB} = \sum_{x} p(x)|x\rangle\langle x|_X \otimes \omega_{AB}^x \in \mathcal{S}(\mathcal{H}_{XAB})$. Then the sandwiched Rényi divergence and the Rényi conditional entropy satisfy the following properties:

- (i) Monotonicity [28], [49]: If $0 < \alpha \le \beta$, then $D_{\alpha}(\rho \| \sigma) \le D_{\beta}(\rho \| \sigma)$;
- (ii) Limit of $\alpha \to 1$ [28], [29]: $\lim_{\alpha \to 1} D_{\alpha}(\rho \| \sigma) = D(\rho \| \sigma)$, and $\lim_{\alpha \to 1} H_{\alpha}(A|B)_{\rho} = H(A|B)_{\rho}$; (iii) Data processing inequality [50], [49], [28], [29]: Let $\alpha \in [\frac{1}{2}, \infty)$ and Φ be a CPTP map. Then

$$D_{\alpha}(\rho \| \sigma) \ge D_{\alpha}(\Phi(\rho) \| \Phi(\sigma));$$

- (iv) Convexity [39]: For $\alpha \in (0, +\infty)$, the function $f(\alpha) = \log Q_{\alpha}(\rho \| \sigma)$ is convex;
- (v) Invariance under isometries [28], [29]: Let $\mathcal{U}: \mathcal{H} \to \mathcal{H}'$, $\mathcal{U}_A: \mathcal{H}_A \to \mathcal{H}_{A'}$ and $\mathcal{U}_B: \mathcal{H}_B \to \mathcal{H}_{B'}$ be isometries. Then $D_{\alpha}(\mathcal{U}\rho\mathcal{U}^*||\mathcal{U}\sigma\mathcal{U}^*) = D_{\alpha}(\rho||\sigma)$ and $H_{\alpha}(A'|B')_{(\mathcal{U}_A\otimes\mathcal{U}_B)\xi_{AB}(\mathcal{U}_A^*\otimes\mathcal{U}_B^*)} = H_{\alpha}(A|B)_{\xi_{AB}}$;
- (vi) Monotonicity under discarding classical information [51]: For the state σ_{XAB} that is classical on X and for $\alpha \in (0, +\infty)$,

$$H_{\alpha}(AX|B)_{\sigma} \geq H_{\alpha}(A|B)_{\sigma}$$
.

The result of the following proposition is established by Mosonyi and Ogawa [25]. Proposition 3: For any $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, $a \in \mathbb{R}$ and t > 0, we have

$$\lim_{n \to \infty} \frac{\log \operatorname{Tr} \rho^{\otimes n} \{ \rho^{\otimes n} > t2^{na} \sigma^{\otimes n} \}}{n} = \lim_{n \to \infty} \frac{\log \operatorname{Tr} (\rho^{\otimes n} - t2^{na} \sigma^{\otimes n})_{+}}{n} = \inf_{s \ge 0} \{ s (D_{1+s}(\rho \| \sigma) - a) \}. \tag{6}$$

Remark 1: In its original statement [25], Proposition 3 appears with t being 1 and a being in the interval $(D(\rho||\sigma), D_{\max}(\rho||\sigma))$. However, it is easy to see that it holds for any t>0 and $a\in\mathbb{R}$. The reason for t is obvious, since it can be absorbed into a when $n \to \infty$. As for a, we discuss the following two cases. 1) $a > D_{\max}(\rho \| \sigma)$: it is easy to check that the three expressions in Eq. (6) are all $-\infty$. 2) $a < D(\rho \| \sigma)$: by the equalities of Eq. (6) established for $a \in (D(\rho \| \sigma), D_{\max}(\rho \| \sigma))$, we see that the two limits in Eq. (6) goes to 0 when $a \searrow D(\rho \| \sigma)$. In addition, $\operatorname{Tr} \rho^{\otimes n} \{ \rho^{\otimes n} > 2^{na} \sigma^{\otimes n} \}$ and $\operatorname{Tr} (\rho^{\otimes n} - 2^{na} \sigma^{\otimes n})_+$ are monotonically decreasing with a (cf. [9]). So, the two limits are nonnegative when $a < D(\rho \| \sigma)$. On the other hand, it is easy to see that the two limits are upper bounded by 0 because the terms in the logarithm function are upper bounded by 1. Hence, we conclude that the two limits actually equal to 0 when $a < D(\rho || \sigma)$. This coincides with the third expression of Eq. (6).

III. EXPONENT IN SMOOTHING THE MAX-RELATIVE ENTROPY

The max-relative entropy is defined in Eq. (3). The smoothed version based on the purified distance is given by the following definition [2].

Definition 2: Let $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, and $0 \le \epsilon < 1$. The smooth max-relative entropy is defined as

$$D_{\max}^{\epsilon}(\rho\|\sigma) := \min_{\tilde{\rho} \in \mathcal{B}^{\epsilon}(\rho)} D_{\max}(\tilde{\rho}\|\sigma).$$

In this section, we investigate the asymptotic behavior of the exponential decay of the small modification in smoothing the max-relative entropy. To formulate the problem in an equivalent way, we define the smoothing quantity, for any $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$ and $\lambda \in \mathbb{R}$,

$$\epsilon(\rho \| \sigma, \lambda) := \min \left\{ \epsilon \mid D_{\max}^{\epsilon}(\rho \| \sigma) \le \lambda \right\} = \min \left\{ P(\rho, \tilde{\rho}) \mid \tilde{\rho} \le 2^{\lambda} \sigma \quad \text{and} \quad \tilde{\rho} \in \mathcal{S}_{\le}(\mathcal{H}) \right\}. \tag{7}$$

We determine the precise exponential rate of decay for $\epsilon(\rho^{\otimes n} || \sigma^{\otimes n}, nr)$.

Theorem 1: For arbitrary $\rho \in \mathcal{S}(\mathcal{H})$, $\sigma \in \mathcal{P}(\mathcal{H})$, and $r \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{-1}{n} \log \epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) = \frac{1}{2} \sup_{s \ge 0} \left\{ s \left(r - D_{1+s}(\rho \| \sigma) \right) \right\}. \tag{8}$$

The above theorem can be rewritten as follows. There exists a sequence $\varepsilon_n \to 0$ such that

$$D_{\max}^{2^{-n(r_e+\varepsilon_n)}}(\rho^{\otimes n}\|\sigma^{\otimes n}) = nr$$
(9)

with $r_e = \frac{1}{2} \sup_{s \ge 0} \left\{ s \left(r - D_{1+s}(\rho \| \sigma) \right) \right\}$. When $r \le D(\rho \| \sigma)$, the right hand side of Eq. (8) is zero. Otherwise, it is strictly positive.

The quantum asymptotic equipartition property [3], [7] states that, as $n \to \infty$, $\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) \to 0$ when $r > D(\rho \| \sigma)$ and $\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) \to 1$ when $r < D(\rho \| \sigma)$. Moreover, these convergences are exponentially fast. Our result of Theorem 1 has provided the exact exponent for the decay of $\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr)$ in the case $r > D(\rho \| \sigma)$. This is in analogy to the Hoeffding bound [22], [23], [24] for the hypothesis testing relative entropy.

Proof of Theorem 1: At first, we deal with the " \geq " part. This is done by deriving a general upper bound for $\epsilon(\rho||\sigma,\lambda)$, and then we apply it to the asymptotic situation. Set

$$Q := \left\{ \mathcal{E}_{\sigma}(\rho) \le \frac{1}{v(\sigma)} 2^{\lambda} \sigma \right\},\tag{10}$$

where \mathcal{E}_{σ} is the pinching map and $v(\sigma)$ is the number of distinct eigenvalues of σ . We consider the state $\tilde{\rho} = Q\rho Q$. On the one hand, by the pinching inequality (2) and the definition of Q, we have

$$Q\rho Q \leq v(\sigma)Q\mathcal{E}_{\sigma}(\rho)Q$$

$$\leq v(\sigma)Q\left(\frac{1}{v(\sigma)}2^{\lambda}\sigma\right)Q$$

$$\leq 2^{\lambda}\sigma.$$
(11)

On the other hand, we can bound the distance between ρ and $\tilde{\rho}$ as follows. Firstly,

$$P(\rho, \tilde{\rho}) = \sqrt{1 - F(\rho, Q\rho Q)^2}$$
$$= \sqrt{1 - (\operatorname{Tr} \rho Q)^2}$$
$$\leq \sqrt{2 \operatorname{Tr} \rho (1 - Q)}.$$

Then, denoting $p = \operatorname{Tr} \rho(\mathbb{1} - Q)$ and $q = \operatorname{Tr} \sigma(\mathbb{1} - Q)$, from the definition of Q we easily see that $p \geq \frac{1}{v(\sigma)} 2^{\lambda} q$. So, for any $s \geq 0$,

$$P(\rho, \tilde{\rho}) \leq \sqrt{2p^{1+s}p^{-s}} \leq \sqrt{2\left(p^{1+s}\left(\frac{1}{v(\sigma)}2^{\lambda}q\right)^{-s}\right)}$$

$$\leq \sqrt{2\left(p^{1+s}\left(\frac{1}{v(\sigma)}2^{\lambda}q\right)^{-s} + (1-p)^{1+s}\left(\frac{1}{v(\sigma)}2^{\lambda}(\operatorname{Tr}\sigma - q)\right)^{-s}\right)}$$

$$= \sqrt{2v(\sigma)^{s} 2^{s\left(D_{1+s}((p,1-p)\|(q,\operatorname{Tr}\sigma - q)) - \lambda\right)}}$$

$$\leq \sqrt{2v(\sigma)^{s} 2^{s\left(D_{1+s}(\rho\|\sigma) - \lambda\right)}},$$

$$(12)$$

where the last line is by the data processing inequality for the sandwiched Rényi divergence under quantum measurements (Proposition 2 (iii)). Eq. (11) and Eq. (12) imply that

$$\epsilon(\rho \| \sigma, \lambda) \le \sqrt{2v(\sigma)^s 2^{s(D_{1+s}(\rho \| \sigma) - \lambda)}}.$$

This further gives

$$\liminf_{n \to \infty} \frac{-1}{n} \log \epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) \ge \frac{1}{2} \sup_{s \ge 0} \left\{ s \left(r - D_{1+s}(\rho \| \sigma) \right) \right\}. \tag{13}$$

Here we have also used the inequality $v(\sigma^{\otimes n}) \leq (n+1)^{\operatorname{rank}(\sigma)}$ (see, e.g. [52], Theorem 12.1.1).

Next, we turn to the derivation of the other direction. Let $\rho_n \in \mathcal{S}_{\leq}(\mathcal{H}^{\otimes n})$ be any subnormalized state which satisfies

$$\rho_n \le 2^{nr} \sigma^{\otimes n}. \tag{14}$$

We are to lower bound the purified distance between $\rho^{\otimes n}$ and ρ_n . Set $Q_n := \{\rho^{\otimes n} > 9 \cdot 2^{nr} \sigma^{\otimes n}\}$. Denote $p_n = \operatorname{Tr} \rho^{\otimes n} Q_n$ and $q_n = \operatorname{Tr} \rho_n Q_n$, which are the probabilities of obtaining the outcome associated with Q_n when a projective measurement $\{Q_n, \mathbb{1} - Q_n\}$ is applied to $\rho^{\otimes n}$ and ρ_n , respectively. Then, by Eq. (14) and the definition of Q_n , it is easy to see that

$$Q_n \rho^{\otimes n} Q_n \ge 9 \cdot 2^{nr} Q_n \sigma^{\otimes n} Q_n$$
$$\ge 9Q_n \rho_n Q_n,$$

which gives

$$p_n \ge 9q_n. \tag{15}$$

Now by the monotonicity of the fidelity under quantum measurements, we have

$$F(\rho^{\otimes n}, \rho_n) \le F((p_n, 1 - p_n), (q_n, \operatorname{Tr} \rho_n - q_n))$$

$$\le \sqrt{p_n} \sqrt{q_n} + \sqrt{1 - p_n}$$

$$\le \frac{p_n}{3} + \sqrt{1 - p_n},$$

where for the last line Eq. (15) is used. Thus,

$$P(\rho^{\otimes n}, \rho_n) = \sqrt{1 - F^2(\rho^{\otimes n}, \rho_n)}$$

$$\geq \sqrt{1 - \left(\frac{p_n}{3} + \sqrt{1 - p_n}\right)^2}$$

$$= \sqrt{-\frac{p_n^2}{9} + p_n - \frac{2p_n}{3}\sqrt{1 - p_n}}$$

$$\geq \sqrt{p_n}\sqrt{-\frac{p_n}{9} + 1 - \frac{2}{3}}$$

$$= \sqrt{p_n}\sqrt{\frac{1}{3} - \frac{p_n}{9}}.$$

Because ρ_n is an arbitrary subnormalized state that satisfies Eq. (14), we obtain

$$\epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) \ge \sqrt{p_n} \sqrt{\frac{1}{3} - \frac{p_n}{9}}.$$
 (16)

Proposition 3 provides the exact rate of exponential decay for p_n in (16), yielding

$$\limsup_{n \to \infty} \frac{-1}{n} \log \epsilon(\rho^{\otimes n} \| \sigma^{\otimes n}, nr) \le \frac{1}{2} \sup_{s > 0} \left\{ s \left(r - D_{1+s}(\rho \| \sigma) \right) \right\}. \tag{17}$$

Combining Eq. (13) and Eq. (17) we complete the proof.

Remark 2: For the first part (the " \geq " part) of the proof of Theorem 1, we can also employ the method introduced in [53] (cf. Lemma 7 and Lemma 8) to construct the state $\tilde{\rho}$. This method was later used and refined in [3] and [7], yielding tight upper bound for $\epsilon(\rho||\sigma,\lambda)$. Our approach here is more direct. However, the price to pay is that an additional quantity $v(\sigma)$ is involved.

IV. SECURITY EXPONENT OF PRIVACY AMPLIFICATION AGAINST QUANTUM ADVERSARIES

Assume that two parties, Alice and Bob, share some common classical randomness, represented by a random variable X which takes any value $x \in \mathcal{X}$ with probability p_x . The information of X is partially leaked to an adversary Eve, and is stored in a quantum system E whose state is correlated with X. This situation is described by the following classical-quantum (CQ) state

$$\rho_{XE} = \sum_{x} p_x |x\rangle\langle x|_X \otimes \rho_E^x. \tag{18}$$

In the procedure of privacy amplification, Alice and Bob apply a hash function $f: \mathcal{X} \to \mathcal{Z}$ to extract a random number Z, which is expected to be uniformly distributed and independent of the adversary's system E. This results in the state

$$\rho_{ZE}^f := \sum_{z} |z\rangle\langle z|_Z \otimes \sum_{x \in f^{-1}(z)} p_x \rho_E^x \tag{19}$$

on systems Z and E. The size of the extracted randomness is $|\mathcal{Z}|$ and the security is measured by the closeness of this real state to the ideal state $\frac{\mathbb{I}_Z}{|\mathcal{Z}|} \otimes \rho_E$. In this paper, we consider two security measures, the insecurity $P(\rho_{ZE}^f, \frac{\mathbb{I}_Z}{|\mathcal{Z}|} \otimes \rho_E)$ in terms of purified distance, and the insecurity $D(\rho_{ZE}^f, \frac{\mathbb{I}_Z}{|\mathcal{Z}|} \otimes \rho_E)$ in terms of relative entropy. These two measures have been extensively used in the literature for privacy amplification. See, e.g., [12], [54] for the purified distance measure, and [30], [55], [32] for the relative entropy measure. The latter is also called modified quantum mutual information and is related to the leaked information [32]. Since it can be written as

$$D(\rho_{ZE}^f \| \frac{\mathbb{1}_Z}{|\mathcal{Z}|} \otimes \rho_E) = I(Z; E)_{\rho^f} + D(\rho_Z^f \| \frac{\mathbb{1}_Z}{|\mathcal{Z}|})$$
$$= \log |\mathcal{Z}| - H(Z|E)_{\rho^f},$$

we can understand it as the leaked information plus the nonuniform of the extracted randomness, or the difference between the ideal ignorance and the real ignorance of the extracted randomness, from the viewpoint of the adversary.

The two-universal family of hash functions are commonly employed to extract private randomness. It has the advantage of being universal (irrelevant of the detailed structure of the state ρ_{XE}), as well as being efficiently realizable [56], [30], [1], [27], [32]. This is particularly useful in the cryptographic setting. Let \mathcal{F} be a set of hash functions from \mathcal{X} to \mathcal{Z} , and F represent a random choice of hash function f from (a subset of) \mathcal{F} with probability $P_F(f)$. If $\forall (x_1, x_2) \in \mathcal{X}^2$ with $x_1 \neq x_2$,

$$\Pr\left\{F(x_1) = F(x_2)\right\} \le \frac{1}{|\mathcal{Z}|},\tag{20}$$

we say that the pair (\mathcal{F}, P_F) is two-universal, and that F is a two-universal random hash function.

The preceding work [32] has derived an upper bound, in terms of the sandwiched Rényi divergence, for the insecurity of privacy amplification under the relative entropy measure. When n-multiple copies of the state (18) are available, this provides an achievable rate of the exponential decreasing of the insecurity, when the number of copies n increase. We are interested in the problem of determining the precise exponent under which the insecurity decreases.

A. Main results

At first, we derive a general upper bound for the rate of exponential decreasing of the insecurity in privacy amplification, under both the purified distance measure and the relative entropy measure.

Theorem 2: Let ρ_{XE} be a CQ state, $\mathcal{F}_n(R)$ be the set of functions from \mathcal{X}^n to $\mathcal{Z}_n = \{1, \dots, 2^{nR}\}$. Let $\rho_{Z_nE^n}^{f_n}$ denote the state resulting from applying a hash function $f_n \in \mathcal{F}_n(R)$ to $\rho_{XE}^{\otimes n}$. For any fixed randomness extraction rate $R \geq 0$, we have

$$\limsup_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n E^n}^{f_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \frac{1}{2} \sup_{s \ge 0} \left\{ s \left(H_{1+s}(X|E)_\rho - R \right) \right\}, \tag{21}$$

$$\limsup_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \sup_{s > 0} \left\{ s \left(H_{1+s}(X|E)_\rho - R \right) \right\}. \tag{22}$$

Remark 3: The work [35] have proved Eq. (22) in the classical case, where ρ_{XE} is fully classical. Theorem 2 has extended this result to the quantum setting.

By combining Theorem 2 and a lower bound derived in [32], we can get the exact exponent of the asymptotic decreasing of the insecurity when the rate of randomness extraction is above a critical value.

Theorem 3: Let ρ_{XE} be a CQ state, $\mathcal{F}_n(R)$ be the set of functions from \mathcal{X}^n to $\mathcal{Z}_n = \{1, \dots, 2^{nR}\}$, F_n be any two-universal random hash function drawn from (a subset of) $\mathcal{F}_n(R)$, and $R_{\text{critical}} := \frac{d}{ds} s H_{1+s}(X|E)_{\rho}|_{s=1}$. For the rate R of randomness extraction satisfying $R \geq R_{\text{critical}}$, we have

$$\lim_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n E^n}^{f_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = \lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_{F_n} P(\rho_{Z_n E^n}^{F_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})$$

$$= \frac{1}{2} \max_{0 \le s \le 1} \left\{ s \left(H_{1+s}(X|E)_{\rho} - R \right) \right\},$$
(23)

$$\lim_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = \lim_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_{F_n} D(\rho_{Z_n E^n}^{F_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})$$

$$= \max_{0 \le s \le 1} \left\{ s \left(H_{1+s}(X|E)_\rho - R \right) \right\}. \tag{24}$$

The proof of Theorem 2 is based on the result obtained in Section III on the exponent in smoothing the max-relative entropy. To relate privacy amplification to the smooth max-relative entropy in a proper way, we employ a version of the smooth conditional min-entropy [27], [54].

Definition 3: For a state $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$, the smooth conditional min-entropy is defined as

$$H_{\min}^{\epsilon}(A|B)_{\rho} := -D_{\max}^{\epsilon}(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B). \tag{25}$$

When $\epsilon = 0$, we recover the (non-smoothed) conditional min-entropy $H_{\min}(A|B)_{\rho} := -D_{\max}(\rho_{AB}||\mathbb{1}_A \otimes \rho_B)$.

Proposition 4: Let $\sigma_{XAB} = \sum_{x} p_x |x\rangle\langle x|_X \otimes \sigma_{AB}^x$ be a state in $\mathcal{S}(\mathcal{H}_{XAB})$. Let $f: \mathcal{X} \to \mathcal{Z}$ be a function and let Z = f(X). Then,

$$H_{\min}^{\epsilon}(XA|B)_{\sigma} \ge H_{\min}^{\epsilon}(ZA|B)_{\sigma}, \text{ where}$$

$$\sigma_{ZAB} = \sum_{z} |z\rangle\langle z|_{Z} \otimes \left(\sum_{x \in f^{-1}(z)} p_{x}\sigma_{AB}^{x}\right).$$

There is another definition of the smooth conditional min-entropy (see, e.g., [3], [4], [12]):

$$\bar{H}_{\min}^{\epsilon}(A|B)_{\rho} := -\min_{\sigma_B} D_{\max}^{\epsilon}(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B). \tag{26}$$

Since the reference [12, Proposition 3] showed the same statement as Proposition 4 under the different definition (26), the proof of Proposition 4 is analogous to the proof of Proposition 3 in [12] and is given in the Appendix. To see the relation between the smooth conditional min-entropy and the insecurity, we show the following proposition.

Proposition 5: Let ρ_{XE} be a CQ state. When $\log |\mathcal{Z}| \geq H_{\min}^{\epsilon}(X|E)_{\rho}$, any function $f: \mathcal{X} \to \mathcal{Z}$ satisfies

$$P(\rho_{ZE}^f, \frac{\mathbb{1}_Z}{|\mathcal{Z}|} \otimes \rho_E) \ge \epsilon, \tag{27}$$

where ρ_{ZE}^f is a state of the form (19) resulting from applying f to ρ_{XE} . In fact, the reference [12, Theorem 8] showed the same statement as Proposition 5 under the different definition (26). Hence, it can be shown in a similar way.

Proof of Proposition 5: For any function $f: \mathcal{X} \to \mathcal{Z}$, Proposition 4 applies, giving

$$H_{\min}^{\epsilon}(X|E)_{\rho} \ge H_{\min}^{\epsilon}(Z|E)_{\rho^f}.$$
 (28)

We choose ϵ' such that $H_{\min}^{\epsilon'}(Z|E)_{\rho^f} = \log |\mathcal{Z}|$. By the definition of the smooth conditional min-entropy, we find that

$$P(\rho_{ZE}^f, \frac{\mathbb{1}_Z}{|\mathcal{Z}|} \otimes \rho_E) \ge \epsilon'. \tag{29}$$

Also, Eq. (28) implies $\epsilon' \ge \epsilon$. Therefore, we obtain (27). Now, we are ready to prove Theorem 2 and Theorem 3.

110w, we are ready to prove Theorem 2 and Theorem 3.

Proof of Theorem 2: Eq. (21) can be shown by the combination of Theorem 1 and Proposition 5 as follows. We choose $r_e := \frac{1}{2} \sup_{s \ge 0} \left\{ s \left(H_{1+s}(X|E)_{\rho} - R \right) \right\}$. Eq. (9), i.e., Theorem 1 guarantees the existence of a sequence $\varepsilon_n \to 0$ such that $nR = H_{\min}^{2^{-n(r_e+\varepsilon_n)}}(X^n|E^n)_{\rho_{XE}^{\otimes n}}$. Hence, Proposition 5 guarantees

$$\limsup_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n E^n}^{f_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})$$

$$\leq \lim_{n \to \infty} \frac{-1}{n} \log 2^{-n(r_e + \varepsilon_n)}$$

$$= r_e,$$

which coincides with Eq. (21).

To prove Eq. (22), we make use of a relation between the relative entropy and the purified distance. By definition, we easily see that

$$D_{\frac{1}{2}}(\rho \| \sigma) = -2\log F(\rho, \sigma).$$

Meanwhile, since D_{α} is nondecreasing with α ,

$$D_{\frac{1}{2}}(\rho \| \sigma) \le D(\rho \| \sigma).$$

Thus,

$$P(\rho, \sigma) = \sqrt{1 - F^2(\rho, \sigma)} \le \sqrt{1 - 2^{-D(\rho||\sigma)}} \le \sqrt{(\ln 2)D(\rho||\sigma)}.$$
 (30)

Eq. (22) follows directly from Eq. (30) and Eq. (21), and we complete the proof.

Proof of Theorem 3: The preceding work [32, Theorem 1] has proved that under the conditions of Theorem 2 and for any two-universal hash function F_n drawn from (a subset of) $\mathcal{F}_n(R)$,

$$\liminf_{n \to \infty} \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \ge \liminf_{n \to \infty} \frac{-1}{n} \log \mathbb{E}_{F_n} D(\rho_{Z_n E^n}^{F_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \\
\ge \max_{0 \le s \le 1} \left\{ s \left(H_{1+s}(X|E)_{\rho} - R \right) \right\}.$$
(31)

Making use of Eq. (30) and the concavity of the square root function, we are able to get a similar bound for the purified distance measure from Eq. (31), under the same conditions. Namely,

$$\lim_{n \to \infty} \inf \frac{-1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{Z_n E^n}^{f_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \ge \lim_{n \to \infty} \inf \frac{-1}{n} \log \mathbb{E}_{F_n} P(\rho_{Z_n E^n}^{F_n}, \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})$$

$$\ge \lim_{n \to \infty} \inf \frac{-1}{n} \log \sqrt{(\ln 2) \mathbb{E}_{F_n} D(\rho_{Z_n E^n}^{F_n} || \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})}$$

$$\ge \frac{1}{2} \max_{0 \le s \le 1} \left\{ s \left(H_{1+s}(X|E)_{\rho} - R \right) \right\}.$$
(32)

If the lower bounds in Eq. (31) and Eq. (32) equal the upper bounds in Eq. (22) and Eq. (21), respectively, we would obtain the exact rates of exponential decay. In the following, we prove that this is indeed the case when $R \ge R_{\text{critical}}$.

Consider the optimization problem

$$E_u(R) := \sup_{s \ge 0} \left\{ s \left(H_{1+s}(X|E)_\rho - R \right) \right\}. \tag{33}$$

Since the function $s \mapsto sH_{1+s}(X|E)_{\rho}$ is concave (cf. Proposition 2 (iv)) and obviously continuously differentiable on $(0, \infty)$, $s(H_{1+s}(X|E)_{\rho} - R)$ is also concave and continuously differentiable as a function of s. So the supremum in Eq. (33) is achieved at the point with zero derivative (if it exists), given by the solution of the equation

$$R = \hat{R}(s) \equiv \frac{\mathrm{d}}{\mathrm{d}s} s H_{1+s}(X|E)_{\rho}. \tag{34}$$

Note that the critical rate is

$$R_{\text{critical}} = \hat{R}(1) \equiv \frac{\mathrm{d}}{\mathrm{d}s} s H_{1+s}(X|E)_{\rho} \Big|_{s=1}.$$
 (35)

R(s) is nonincreasing, because $s \mapsto sH_{1+s}(X|E)_{\rho}$ is concave. Also, we define

$$\hat{R}(0) := \lim_{s \to 0} \hat{R}(s) = H(X|E)_{\rho},$$

$$\hat{R}(\infty) := \lim_{s \to +\infty} \hat{R}(s) = H_{\min}(X|E)_{\rho},$$
(36)

$$\hat{R}(\infty) := \lim_{s \to +\infty} \hat{R}(s) = H_{\min}(X|E)_{\rho},\tag{37}$$

where (36) and (37) follow from (51) and (52) of [25, Lemma IV.2], respectively. There are four cases:

- (i) $R \ge H(X|E)_{\rho}$: the function $s \mapsto s(H_{1+s}(X|E)_{\rho} R)$ is monotonically dereasing. So the supremum in Eq. (33) is 0, achieved at s = 0;
- (ii) $R_{\text{critical}} \leq R < H(X|E)_{\rho}$: Eq. (34) has a solution $s^* \in (0,1]$, where Eq. (33) achieves the supremum;
- (iii) $H_{\min}(X|E)_{\rho} < R < R_{\text{critical}}$: Eq. (34) has a solution $s^* \in (1, +\infty)$, where Eq. (33) achieves the supremum;
- (iv) $R \leq H_{\min}(X|E)_{\rho}$: the function $s \mapsto s(H_{1+s}(X|E)_{\rho} R)$ is monotonically increasing. So the supremum in Eq. (33) is $+\infty$, approached when $s \to +\infty$.

In cases (i) and (ii), we have that the supremum in Eq. (33) is achieved at $s \in [0, 1]$. Therefore, the bound in Eq. (22) and that in Eq. (31) are equal, and so are the bound in Eq. (21) and that in Eq. (32). Hence we complete the proof.

Since R(s) is nonincreasing, R(s) has the inverse function ψ . The results presented in Theorem 2 and Theorem 3 can be explained by using $E_u(R)$ and $E_l(R) := \sup_{0 \le s \le 1} s(H_{1+s}(X|E)_{\rho} - R)$ as follows.

$$E_{u}(R) = \begin{cases} 0 & \text{when } R \geq H(X|E)_{\rho}, \\ \psi(R)H_{1+\psi(R)}(X|E)_{\rho} - \psi(R)R & \text{when } H(X|E)_{\rho} > R \geq H_{\min}(X|E)_{\rho}, \\ +\infty & \text{when } H_{\min}(X|E)_{\rho} > R, \end{cases}$$

$$E_{l}(R) = \begin{cases} 0 & \text{when } R \geq H(X|E)_{\rho}, \\ \psi(R)H_{1+\psi(R)}(X|E)_{\rho} - \psi(R)R & \text{when } R \geq H(X|E)_{\rho}, \\ H_{2}(X|E)_{\rho} - R & \text{when } H(X|E)_{\rho} > R > R_{\text{critical}}, \\ H_{2}(X|E)_{\rho} - R & \text{when } R_{\text{critical}} \geq R. \end{cases}$$

$$(38)$$

$$E_{l}(R) = \begin{cases} 0 & \text{when } R \ge H(X|E)_{\rho}, \\ \psi(R)H_{1+\psi(R)}(X|E)_{\rho} - \psi(R)R & \text{when } H(X|E)_{\rho} > R > R_{\text{critical}}, \\ H_{2}(X|E)_{\rho} - R & \text{when } R_{\text{critical}} \ge R. \end{cases}$$
(39)

Figure 1 illustrates the above two functions.

We make a few remarks on a related security measure. The quantity, $\min_{\sigma_E} P(\rho_{ZE}^f, \frac{\mathbb{I}_Z}{|\mathcal{Z}|} \otimes \sigma_E)$, was employed in some works to measure the insecurity of the extracted randomness Z (see, e.g., [12]). There is an additional minimization over the adversary's state, compared to $P(\rho_{ZE}^f, \frac{1}{|\mathcal{Z}|} \otimes \rho_E)$ that we use here. Denoting the minimizer in that measure as σ_E^* , we have

$$P(\rho_{ZE}^{f}, \frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \sigma_{E}^{*}) \leq P(\rho_{ZE}^{f}, \frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \rho_{E})$$

$$\leq P(\rho_{ZE}^{f}, \frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \sigma_{E}^{*}) + P(\frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \sigma_{E}^{*}, \frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \rho_{E})$$

$$\leq 2P(\rho_{ZE}^{f}, \frac{\mathbb{1}_{Z}}{|\mathcal{Z}|} \otimes \sigma_{E}^{*}).$$

So, there is no difference between these two measures regarding the rate of asymptotic exponential decreasing. However, we prefer to employ the measure $P(\rho_{ZE}^f, \frac{\mathbb{I}_Z}{|\mathcal{Z}|} \otimes \rho_E)$ because fixing ρ_E in the measure fits better the requirement of composable security (see discussions in [57] and [54]).

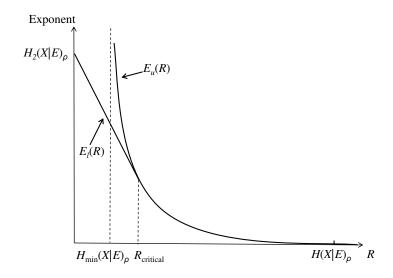


Fig. 1. Security exponent of privacy amplification. $E_u(R)$ is the upper bound derived in the present paper. $E_l(R)$ is the lower bound by the reference [32]. These two bounds are equal when $R \geq R_{\text{critical}}$, giving the exact security exponent. When $R \geq H(X|E)_{\rho}$, the security exponent is 0. Below the critical value R_{critical} , the upper bound $E_u(R)$ is larger and diverges to infinity when $R < H_{\min}(X|E)_{\rho}$, while the lower bound $E_l(R)$ becomes linear and reaches $H_2(X|E)_{\rho}$ at R=0.

B. Discussion on the low-rate case

In Theorem 3, we have obtained the exponents only when $R \geq R_{\text{critical}}$. One may guess that either the achievability bounds of Eq. (31) and Eq. (32) or the converse bounds of Theorem 2 are the exact exponents when $R < R_{\text{critical}}$. Here we give two simple examples to show that this is not true, i.e., neither of them are tight in general when $R < R_{\text{critical}}$. This indicates that R_{critical} may be indeed a critical point in the exponential analysis of privacy amplification.

Example 1 We consider the classical-quantum state $\rho_{XE} = (\frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|)\otimes\rho_E$. We have $sH_{1+s}(X|E)_{\rho} = \phi(s) := -\log((\frac{1}{3})^{1+s} + (\frac{2}{3})^{1+s})$. Then, using the binary entropy $h(x) := -x\log x - (1-x)\log(1-x)$, we have $H(X|E)_{\rho} = h(\frac{1}{3})$ and $\hat{R}(s) = \frac{\mathrm{d}}{\mathrm{d}s}sH_{1+s}(X|E)_{\rho} = \frac{(1+2^{1+s})\log 3 - 2^{1+s}\log 2}{1+2^{1+s}}$. In particular, $R_{\mathrm{critical}} = \hat{R}(1) = \frac{5\log 3 - 4\log 2}{5}$ and $\hat{R}(+\infty) = \log \frac{3}{2} = H_{\min}(X|E)_{\rho}$. In addition, $E_u(\log \frac{3}{2})$ is calculated as

$$E_u(\log \frac{3}{2}) = \lim_{s \to +\infty} \left\{ -\log((\frac{1}{3})^{1+s} + (\frac{2}{3})^{1+s}) - s\log \frac{3}{2} \right\}$$
$$= \lim_{s \to +\infty} \left\{ \log \frac{3}{2} - \log(1 + \frac{1}{2^{1+s}}) = \log \frac{3}{2} \right\}.$$

Therefore, since $H_2(X|E)_{\rho} = \log \frac{9}{5}$, $E_u(R)$, $E_l(R)$ are calculated as

$$E_{u}(R) = \begin{cases} 0 & \text{when } R \ge h(\frac{1}{3}), \\ \phi(\psi(R)) - \psi(R)R & \text{when } h(\frac{1}{3}) > R \ge \log \frac{3}{2}, \\ +\infty & \text{when } \log \frac{3}{2} > R, \end{cases}$$

$$(40)$$

$$E_{l}(R) = \begin{cases} 0 & \text{when } R \ge h(\frac{1}{3}), \\ \phi(\psi(R)) - \psi(R)R & \text{when } h(\frac{1}{3}) > R > \frac{5\log 3 - 4\log 2}{5}, \\ \log \frac{9}{5} - R & \text{when } \frac{5\log 3 - 4\log 2}{5} \ge R. \end{cases}$$
(41)

Their behaviors are plotted as Fig. 2. Notice that $\hat{R}(s)$ is strictly nonincreasing for s because $sH_{1+s}(X|E)_{\rho}$ is a strictly concave function of s. Hence, we have $\psi(R) > 1$ for $R < \frac{5\log 3 - 4\log 2}{5}$. Since $\frac{d(\phi(s) - sR)}{ds}|_{s=1} = (\phi'(s) - R)|_{s=1} > 0$, $E_u(R)$ takes a larger value than $E_l(R)$ because

$$E_u(R) = \phi(\psi(R)) - \psi(R)R > \phi(1) - R = E_l(R). \tag{42}$$

Therefore, this case has the following three possible cases. In the first case, $E_u(R)$ is the tight upper bound. In the second case, $E_l(R)$ is the tight lower bound. In the third case, neither $E_u(R)$ nor $E_l(R)$ is a tight bound. To investigate this problem, we notice that the eigenvalue of $\rho_X^{\otimes n}$ associated with the eigenvector $|0,0,\cdots,0\rangle$ is $\frac{1}{3^n}$, and all the other eigenvalues are $\frac{1}{3^n}$ multiplied by an even number. This simple fact will be crucial for our later estimation.

Let $f_n: \mathcal{X}^n \to \mathcal{Z}_n$ be an arbitrary sequence of hash function (the size $|\mathcal{Z}_n|$ is also arbitrary). Let $z_n^* = f_n(0,0,\cdots,0)$ and pick $z_n' \in \mathcal{Z}_n$ such that $z_n' \neq z_n^*$. Then $\langle z_n^* | \rho_{Z_n}^{f_n} | z_n^* \rangle$ must be $\frac{1}{3^n}$ multiplied by a odd number and $\langle z_n' | \rho_{Z_n}^{f_n} | z_n' \rangle$ be $\frac{1}{3^n}$ multiplied by an even number. So

$$d(\rho_{Z_n E^n}^{f_n}, \frac{1}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})$$

$$= \frac{1}{2} \sum_{z_n \in \mathcal{Z}_n} \left| \langle z_n | \rho_{Z_n}^{f_n} | z_n \rangle - \frac{1}{|\mathcal{Z}_n|} \right|$$

$$\geq \frac{1}{2} \left(\left| \langle z_n^* | \rho_{Z_n}^{f_n} | z_n^* \rangle - \frac{1}{|\mathcal{Z}_n|} \right| + \left| \langle z_n' | \rho_{Z_n}^{f_n} | z_n' \rangle - \frac{1}{|\mathcal{Z}_n|} \right| \right)$$

$$\geq \frac{1}{2} \left| \langle z_n^* | \rho_{Z_n}^{f_n} | z_n^* \rangle - \langle z_n' | \rho_{Z_n}^{f_n} | z_n' \rangle \right|$$

$$\geq \frac{1}{2 \times 3^n}.$$

With this in hand, the use of Pinsker's inequality and Fuchs-van de Graaf inequality [43] leads respectively to

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D(\rho_{\mathcal{Z}_n E_n}^{f_n} \| \frac{\mathbb{1}_{\mathcal{Z}_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \log 9, \tag{43}$$

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} P(\rho_{\mathcal{Z}_n E_n}^{f_n}, \frac{\mathbb{1}_{\mathcal{Z}_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \log 3, \tag{44}$$

for any randomness extraction rate R>0. Eq. (43) and Eq. (44) also provide the same bounds for the exponents in the average case where the insecurity is averaged over two-universal hash functions. On the other hand, for $R< H_{\min}(X|E)_{\rho}=\log\frac{3}{2}$, (40) shows that $E_u(R)=+\infty>\log 9=3.16993$. Hence, the upper bound $E_u(R)$ in Theorem 2 is not the tight upper bound. That is, for $R< H_{\min}(X|E)_{\rho}$, we have the second case or the third case.

Example 2 Let $\rho_{XE} = (\frac{1}{4} \sum_{i=1}^{4} |i\rangle\langle i|) \otimes \rho_{E}$, $\mathcal{Z} = \{0,1\}$. We denote by S_4 the permutation group of $\mathcal{X} = \{1,2,3,4\}$. Let Π be the random permutation over \mathcal{X} , i.e., it takes the value $\pi \in S_4$ with equal probability for all π . Define $f: \mathcal{X} \to \mathcal{Z}$ by

$$f(i) := \begin{cases} 0 & i = 1, 2, \\ 1 & i = 3, 4. \end{cases}$$

Then, we consider the random hash function $F_n := (f \circ \Pi)^{\times n}$. It is easy to see that F_n is two-universal. But on the other hand, it always holds that

$$\rho_{Z_n E^n}^{F_n} = \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n},$$

where $\mathcal{Z}_n = \mathcal{Z}^n$. Hence, $\mathbb{E}_{F_n} D(\rho_{Z_n E^n}^{F_n} \| \frac{\mathbb{I}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = \mathbb{E}_{F_n} P(\rho_{Z_n E^n}^{F_n}, \frac{\mathbb{I}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = 0$, and the corresponding exponents are $+\infty$. This is also true when the expectations are replaced by the minimization over all hash functions from \mathcal{X}^n to \mathcal{Z}_n . So, the lower bounds of Eq. (31) and Eq. (32), which are finite everywhere, are not tight in general.

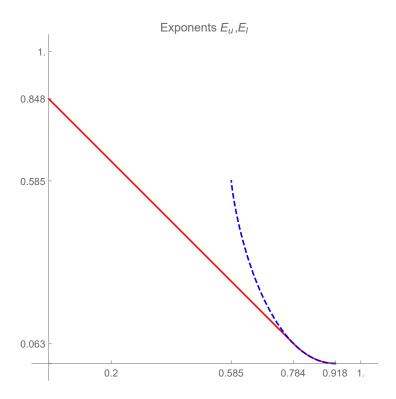


Fig. 2. Exponents in Example 1. Solid red curve expresses the lower bound $E_l(R)$ by the reference [32]. Dashed blue curve expresses the upper bound $E_u(R)$ derived in the present paper. These two bounds are equal when $R \geq \frac{5 \log 3 - 4 \log 2}{5} = 0.784963$, giving the exact security exponent. When $R \geq 0.918296$, the security exponent is 0. Below the critical value 0.784963, the upper bound $E_u(R)$ is $\log \frac{3}{2} = 0.584963$ when $R = \log \frac{3}{2}$. Since it diverges to infinity when $R < \log \frac{3}{2}$, it is not plotted in this range. The lower bound $E_l(R)$ becomes linear and reaches $\log \frac{9}{5} = 0.847997$ at R = 0.

V. Asymptotic equivocation rate and security exponent under the sandwiched Rényi divergence

The equivocation rate is the adversary's maximum ambiguity rate for a given randomness extraction rate R. Specifically, for a CQ state ρ_{XE} and a randomness extraction rate R, the equivocation rate $\mathcal{R}_s(R|\rho)$ under the sandwiched Rényi divergence of order 1+s security measure is defined as

$$\mathcal{R}_s(R|\rho) := \lim_{n \to \infty} \frac{1}{n} \max_{f_n} H_{1+s}(Z_n|E^n)_{\rho^{f_n}},$$

where the maximization is taken over all maps $f_n: \mathcal{X}^{\times n} \to \mathcal{Z}_n$ and $\rho_{Z_n E^n}^{f_n}$ is the state resulting from applying f_n to $\rho_{XE}^{\otimes n}$. In some papers, the equivocation rate is also defined as the adversary's minimum information rate for a given randomness extraction rate R, i.e.,

$$\mathcal{R}'_s(R|\rho) := \lim_{n \to \infty} \frac{1}{n} \min_{f_n} D_{1+s}(\rho_{Z_n E^n}^{f_n} || \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}).$$

These two definitions are related. Indeed, it is easy to see that

$$\mathcal{R}_s(R|\rho) = R - \mathcal{R}'_s(R|\rho).$$

In our paper, we take the second definition.

The concept of equivocation was first proposed by Wyner [58] and was studied by many researchers in the wiretap scenario. In the quantum privacy amplification scenario, the preceding work [32] derived the equivocation rate under the quantum relative entropy security measure. Later, the reference [35] derived the equivocation rate, in the classical privacy amplification scenario, under the Rényi relative entropy security measure.

In this section, we investigate the asymptotic equivocation rate and the security exponents under the sandwiched Rényi divergence security measure, with Rényi parameter in (1,2]. This generalizes the results by [35] to the quantum privacy amplification scenario. These results are presented in the following two theorems. Theorem 4 deals with the asymptotic equivocation rate, and Theorem 5 treats the security exponent. In what follows, we use $|x|^+$ to denote $\max\{x,0\}$.

Theorem 4: Let ρ_{XE} be a CQ state, and $\mathcal{F}_n(R)$ be the set of functions from \mathcal{X}^n to $\mathcal{Z}_n = \{1, \dots, 2^{nR}\}$. Let $\rho_{Z_nE^n}^{f_n}$ denote the state resulting from applying a hash function $f_n \in \mathcal{F}_n(R)$ to $\rho_{XE}^{\otimes n}$. For any randomness extraction rate $R \geq 0$ and any $s \in (0, 1]$, we have

$$\lim_{n \to \infty} \frac{1}{n} \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = |R - H_{1+s}(X|E)_{\rho}|^+. \tag{45}$$

Theorem 5: Let ρ_{XE} be a CQ state, and $\mathcal{F}_n(R)$ be the set of functions from \mathcal{X}^n to $\mathcal{Z}_n = \{1, \dots, 2^{nR}\}$. Let $\rho_{Z_nE^n}^{f_n}$ denote the state resulting from applying a hash function $f_n \in \mathcal{F}_n(R)$ to $\rho_{XE}^{\otimes n}$. For any randomness extraction rate $R \geq R_{\text{critical}}$ and any $s \in (0,1]$, we have

$$\lim_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = \Big| \max_{t \in [s,1]} \left\{ t H_{1+t}(X|E)_\rho - tR \right\} \Big|^+. \tag{46}$$

Remark 4: Actually, the results obtained in Section IV already give that for any randomness extraction rate $R \ge R_{\text{critical}}$ and any $s \in [-\frac{1}{2}, 0]$,

$$\lim_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) = \max_{0 \le t \le 1} \left\{ t \left(H_{1+t}(X|E)_{\rho} - R \right) \right\}. \tag{47}$$

To see this, we first notice that $D_{\frac{1}{2}}(\rho\|\sigma) = -\log(1 - P^2(\rho, \sigma))$. This together with Eq. (23) proves Eq. (47) for one of the endpoint $s = -\frac{1}{2}$. On the other hand, Eq. (24) confirms Eq. (47) for the other endpoint s = 0. As the function $s \mapsto D_{1+s}(\rho\|\sigma)$ is monotonically increasing in $[-\frac{1}{2},0]$, Eq. (47) for the whole interval follows.

Before the proof of Theorem 4 and Theorem 5, we first present and prove several useful lemmas. Lemma 1: Let $A_x \in \mathcal{P}(\mathcal{H})$ for $x \in \mathcal{X}$, and let λ be a positive number. Then we have

$$\operatorname{Tr}(\sum_{x \in \mathcal{X}} A_x - \lambda \mathbb{1})_+ \ge \sum_{x \in \mathcal{X}} \operatorname{Tr}(A_x - \lambda \mathbb{1})_+.$$

Proof: Let $P_x = \{A_x > \lambda \mathbb{1}\}$ and let $P = \bigvee_x P_x$ be the projection onto the subspace spanned by $\{\sup(P_x)\}_{x \in \mathcal{X}}$. Then we have

$$\operatorname{Tr}(\sum_{x \in \mathcal{X}} A_x - \lambda \mathbb{1})_+ \ge \operatorname{Tr}(\sum_{x \in \mathcal{X}} A_x - \lambda \mathbb{1}) P$$

$$= \sum_x \operatorname{Tr} A_x P - \lambda \operatorname{Tr} P$$

$$\ge \sum_x \operatorname{Tr} A_x P_x - \lambda \operatorname{Tr} P$$

$$= \sum_x \operatorname{Tr}(A_x - \lambda \mathbb{1})_+ + \sum_x \lambda \operatorname{Tr} P_x - \lambda \operatorname{Tr} P$$

$$\ge \sum_x \operatorname{Tr}(A_x - \lambda \mathbb{1})_+,$$

$$(48)$$

where the first inequality is due to (1), the second inequality results from $P \ge P_x, \forall x \in \mathcal{X}$, and the last inequality is because the sum of the dimensions of all P_x is larger than the dimension of P.

Lemma 2: Let ρ_{XE} , \mathcal{Z}_n , $\mathcal{F}_n(R)$ and $\rho_{Z_nE^n}^{f_n}$ be the same as those in Theorem 4 and Theorem 5. Then for any t > 0 and $0 < R < H(X|E)_{\rho}$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \min_{f_n \in \mathcal{F}(R)} \operatorname{Tr}(\rho_{Z_n E^n}^{f_n} - t \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})_+ \ge \inf_{s\ge 0} \left\{ s(R - H_{1+s}(X|E)_\rho) \right\}.$$

Proof: Fix $m \in \mathbb{N}$, and write n in the form n = km + r, where $k, r \in \mathbb{N}$ and $0 \le r < m$. Suppose $\rho_E^{\otimes m}$ and $\rho_E^{\otimes r}$ have spectral projections $\{E_i\}_{i \in \mathcal{I}}$ and $\{P_j\}_{j \in \mathcal{J}}$ with corresponding eigenvalues $\{\lambda_i\}_{i \in \mathcal{I}}$ and $\{\eta_j\}_{j \in \mathcal{J}}$, respectively.

Now we evaluate the left hand side. First, recalling that the trace distance decreases under the action of a channel, we have

$$\operatorname{Tr}(\rho_{Z_{n}E^{n}}^{f_{n}} - t \frac{\mathbb{1}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})_{+}$$

$$= \sum_{z_{n}} \operatorname{Tr}\left(\sum_{x_{n} \in f_{n}^{-1}(z_{n})} p(x_{n}) \rho_{E^{n}}^{x_{n}} - t \frac{\rho_{E}^{\otimes n}}{|\mathcal{Z}_{n}|}\right)_{+}$$

$$\geq \sum_{z_{n}} \operatorname{Tr}\left(\mathcal{E}_{\rho_{E}^{\otimes m}}^{\otimes k} \otimes \mathcal{E}_{\rho_{E}^{\otimes r}}\left(\sum_{x_{n} \in f_{n}^{-1}(z_{n})} p(x_{n}) \rho_{E^{n}}^{x_{n}} - t \frac{\rho_{E}^{\otimes n}}{|\mathcal{Z}_{n}|}\right)\right)_{+}$$

$$= \sum_{z_{n}} \operatorname{Tr}\left(\sum_{i_{k}, j} \sum_{x_{n} \in f_{n}^{-1}(z_{n})} E_{i_{k}} \otimes P_{j} p(x_{n}) \rho_{E^{n}}^{x_{n}} E_{i_{k}} \otimes P_{j} - t \frac{\lambda_{i_{k}} \eta_{j} E_{i_{k}} \otimes P_{j}}{|\mathcal{Z}_{n}|}\right)\right)_{+}$$

$$= \sum_{z_{n}} \sum_{i_{k}, j} \operatorname{Tr}\left(\sum_{x_{n} \in f_{n}^{-1}(z_{n})} E_{i_{k}} \otimes P_{j} p(x_{n}) \rho_{E^{n}}^{x_{n}} E_{i_{k}} \otimes P_{j} - t \frac{\lambda_{i_{k}} \eta_{j} E_{i_{k}} \otimes P_{j}}{|\mathcal{Z}_{n}|}\right)_{+}.$$

$$(49)$$

Then, with Lemma 1, we can proceed as

$$(49) \geq \sum_{z_{n}} \sum_{i_{k},j} \sum_{x_{n} \in f_{n}^{-1}(z_{n})} \operatorname{Tr}(E_{i_{k}} \otimes P_{j} p(x_{n}) \rho_{E^{n}}^{x_{n}} E_{i_{k}} \otimes P_{j} - t \frac{\lambda_{i_{k}} \eta_{j} E_{i_{k}} \otimes P_{j}}{|\mathcal{Z}_{n}|})_{+}$$

$$= \sum_{z_{n}} \sum_{x_{n} \in f_{n}^{-1}(z_{n})} \operatorname{Tr}\left(\sum_{i_{k},j} E_{i_{k}} \otimes P_{j} p(x_{n}) \rho_{E^{n}}^{x_{n}} E_{i_{k}} \otimes P_{j} - t \frac{\lambda_{i_{k}} \eta_{j} E_{i_{k}} \otimes P_{j}}{|\mathcal{Z}_{n}|}\right)_{+}$$

$$= \sum_{x_{n}} \operatorname{Tr}\left(\mathcal{E}_{\rho_{E}^{\otimes m}}^{\otimes k} \otimes \mathcal{E}_{\rho_{E}^{\otimes r}}(p(x_{n}) \rho_{E^{n}}^{x_{n}}) - t \frac{\rho_{E}^{\otimes n}}{|\mathcal{Z}_{n}|}\right)_{+}$$

$$= \operatorname{Tr}\left(\mathcal{E}_{\rho_{E}^{\otimes m}}^{\otimes k} \otimes \mathcal{E}_{\rho_{E}^{\otimes r}}(\rho_{XE}^{\otimes n}) - t \frac{1}{2^{nR}} \mathbb{1}_{X}^{\otimes n} \otimes \rho_{E}^{\otimes n}\right)_{+}$$

$$\geq \operatorname{Tr}\left(\mathcal{E}_{\rho_{E}^{\otimes m}}^{\otimes k}(\rho_{XE}^{\otimes mk}) - \frac{t|\mathcal{X}|^{r}}{2^{Rr}} \frac{\mathbb{1}_{X}^{\otimes mk} \otimes \rho_{E}^{\otimes mk}}{2^{mkR}}\right)_{+},$$

$$(50)$$

where the last inequality is because the trace distance decreases under partial trace. Since the function $A \in \mathcal{P}(\mathcal{H}) \to \text{Tr}(A)^{1+s}$ is operator monotone, Eq. (2) implies

$$v(\rho_E^{\otimes m})^{1+s} 2^{sD_{1+s}(\mathcal{E}_{\rho_E^{\otimes m}}(\rho_{XE}^{\otimes m}) \| \mathbb{1}_X^{\otimes m} \otimes \rho_E^{\otimes m})} = v(\rho_E^{\otimes m})^{1+s} Q_{1+s}(\mathcal{E}_{\rho_E^{\otimes m}}(\rho_{XE}^{\otimes m}) \| \mathbb{1}_X^{\otimes m} \otimes \rho_E^{\otimes m})$$

$$\geq Q_{1+s}(\rho_{XE}^{\otimes m} \| \mathbb{1}_X^{\otimes m} \otimes \rho_E^{\otimes m}) = 2^{sD_{1+s}(\rho_{XE}^{\otimes m} \| \mathbb{1}_X^{\otimes m} \otimes \rho_E^{\otimes m})}.$$
(51)

Therefore, we obtain

$$\lim_{n \to \infty} \inf \frac{1}{n} \log \min_{f_{n}} \operatorname{Tr} \left(\rho_{Z_{n}E^{n}}^{f_{n}} - t \frac{\mathbb{1}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n} \right)_{+}$$

$$\geq \inf_{s \geq 0} s \left(R + \frac{D_{1+s} (\mathcal{E}_{\rho_{E}^{\otimes m}} (\rho_{XE}^{\otimes m}) \| \mathbb{1}_{X}^{\otimes m} \otimes \rho_{E}^{\otimes m})}{m} \right)$$

$$\geq \inf_{s \geq 0} s \left(R + \frac{D_{1+s} (\rho_{XE}^{\otimes m} \| \mathbb{1}_{X}^{\otimes m} \otimes \rho_{E}^{\otimes m})}{m} - \frac{(s+1) \log v(\rho_{E}^{\otimes m})}{sm} \right)$$

$$= -\sup_{s \geq 0} \left\{ s \left(H_{1+s} (X|E)_{\rho} - R + \frac{\log v(\rho_{E}^{\otimes m})}{m} \right) \right\} - \frac{\log v(\rho_{E}^{\otimes m})}{m},$$
(52)

where the first inequality follows from Eq. (50) and Proposition 3, and the second inequality follows from Eq. (51).

Because the function $R \mapsto \sup_{s \ge 0} s(H_{1+s}(X|E)_{\rho} + R)$ is continuous, by letting $m \to \infty$ we conclude the proof.

Lemma 3: For a CQ state ρ_{XE} and a two-universal random hash functions $F: \mathcal{X} \to \mathcal{Z} = \{1, \dots, M\}$, we have for $s \in (0, 1]$,

$$\mathbb{E}_F Q_{1+s}(\rho_{ZE}^F || \mathbb{1}_Z \otimes \rho_E) \le v(\rho_E)^{1+s} \left(Q_{1+s}(\rho_{XE} || \mathbb{1}_X \otimes \rho_E) + \frac{1}{M^s} \right). \tag{53}$$

Proof: Let the spectral projections of ρ_E be $\{E_i\}_{i\in\mathcal{I}}$, and the corresponding eigenvalues be $\{\lambda_i\}_{i\in\mathcal{I}}$. Then, with the pinching inequality (2), we can bound $\mathbb{E}_F Q_{1+s}(\rho_{ZE}^F || \mathbb{1}_Z \otimes \rho_E)$ as follows.

$$\mathbb{E}_{F}Q_{1+s}(\rho_{ZE}^{F} \| \mathbb{1}_{Z} \otimes \rho_{E})
\leq v(\rho_{E})^{1+s} \mathbb{E}_{F}Q_{1+s}(\mathcal{E}_{\rho_{E}}(\rho_{ZE}^{F}) \| \mathbb{1}_{Z} \otimes \rho_{E})
= v(\rho_{E})^{1+s} \mathbb{E}_{F}(\sum_{z,i} Q_{1+s}(\sum_{x \in f^{-1}(z)} \Pi_{i}p(x)\rho_{E}^{x}\Pi_{i} \| \lambda_{i}\Pi_{i}))
= v(\rho_{E})^{1+s} \mathbb{E}_{F}(\sum_{z,i} \sum_{x \in f^{-1}(z)} \lambda_{i}^{-s} \operatorname{Tr} \Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}(\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i} + \sum_{x \neq x'} \Pi_{i}p(x')\rho_{E}^{x'}\Pi_{i}\mathbb{1}_{f(x')\neq f(x)})^{s}),$$
(54)

where the inequality follows from the same reason as Eq. (51). To proceed, we invoke the property that the function $f(x) = x^s$ is operator concave when $0 < s \le 1$, to see that

$$\mathbb{E}_{F}\left(\sum_{z,i}\sum_{x\in f^{-1}(z)}\lambda_{i}^{-s}\operatorname{Tr}\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}\left(\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}+\sum_{x\neq x'}\Pi_{i}p(x')\rho_{E}^{x'}\Pi_{i}1_{f(x')\neq f(x)}\right)^{s}\right)$$

$$\leq \sum_{z,i}\sum_{x\in f^{-1}(z)}\lambda_{i}^{-s}\operatorname{Tr}\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}\left(\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}+\sum_{x\neq x'}\frac{1}{M}\Pi_{i}p(x')\rho_{E}^{x'}\Pi_{i}\right)^{s}$$

$$= \sum_{z,i}\sum_{x\in f^{-1}(z)}\lambda_{i}^{-s}\operatorname{Tr}\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}\left(\frac{1}{M}\Pi_{i}\rho_{E}\Pi_{i}+\frac{M-1}{M}\Pi_{i}p(x)\rho_{E}^{x}\Pi_{i}\right)^{s}.$$
(55)

Then we use the inequality $(X + \lambda \mathbb{1})^s \le X^s + \lambda^s \mathbb{1}$ for any $X \in \mathcal{P}(\mathcal{H})$ and $\lambda \ge 0$, to bound Eq. (55) as follows.

$$\sum_{z,i} \sum_{x \in f^{-1}(z)} \lambda_{i}^{-s} \operatorname{Tr} \Pi_{i} p(x) \rho_{E}^{x} \Pi_{i} \left(\frac{1}{M} \Pi_{i} \rho_{E} \Pi_{i} + \frac{M-1}{M} \Pi_{i} p(x) \rho_{E}^{x} \Pi_{i} \right)^{s}$$

$$\leq \sum_{i} \sum_{x} \lambda_{i}^{-s} \operatorname{Tr} \Pi_{i} p(x) \rho_{E}^{x} \Pi_{i} \left(\frac{1}{M^{s}} (\Pi_{i} \rho_{E} \Pi_{i})^{s} + \frac{(M-1)^{s}}{M^{s}} (\Pi_{i} p(x) \rho_{E}^{x} \Pi_{i})^{s} \right)$$

$$\leq \sum_{x} \sum_{i} \operatorname{Tr} (\lambda_{i}^{-s} (\Pi_{i} p(x) \rho_{E}^{x} \Pi_{i})^{1+s} + \frac{1}{M^{s}} \Pi_{i} p(x) \rho_{E}^{x} \Pi_{i})$$

$$= \left(Q_{1+s} (\mathcal{E}_{\rho_{E}}(\rho_{XE}) \| \mathbb{1}_{X} \otimes \rho_{E} \right) + \frac{1}{M^{s}}$$

$$\leq Q_{1+s} (\rho_{XE} \| \mathbb{1}_{X} \otimes \rho_{E}) + \frac{1}{M^{s}},$$
(56)

where the second inequality is simply due to $\frac{M-1}{M}$ < 1. Therefore, the combination of Eqs. (54), (55), and (56) yields Eq. (53).

Lemma 4: For a CQ state ρ_{XE} and a two-universal random hash functions $F: \mathcal{X} \to \mathcal{Z} = \{1, \dots, M\}$, we have for $s \in (0, 1]$,

$$\mathbb{E}_{F} 2^{sD_{1+s}(\rho_{ZE}^{F} \| \frac{1}{|Z|} \otimes \rho_{E})} \le 1 + v(\rho_{E})^{s} 2^{s(\log M - H_{1+s}(X|E)_{\rho})}. \tag{57}$$

Proof: For $s \in (0,1]$, we have

$$\mathbb{E}_{F} 2^{sD_{1+s}(\rho_{ZE}^{F} \| \mathbb{1}_{Z} \otimes \rho_{E})} \\
= \operatorname{Tr} \mathbb{E}_{F} \sum_{m \in \mathcal{Z}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\sum_{x': F(x') = m} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{1+s} \\
= \operatorname{Tr} \mathbb{E}_{F} \sum_{m \in \mathcal{Z}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\sum_{x': F(x') = m} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right) \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\sum_{x': F(x') = m} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{s} \\
= \operatorname{Tr} \mathbb{E}_{F} \sum_{m \in \mathcal{Z}} \left(\sum_{x': F(x') = m} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\sum_{x': F(x') = m} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{s} \rho_{E}^{-\frac{s}{2(1+s)}} \\
= \operatorname{Tr} \mathbb{E}_{F} \sum_{x \in \mathcal{X}} p(x) \rho_{E}^{x} \rho_{E}^{-\frac{s}{2(1+s)}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\sum_{x': F(x') = F(x)} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{s} \rho_{E}^{-\frac{s}{2(1+s)}}. \tag{58}$$

Then we proceed as follows.

$$\mathbb{E}_{F} 2^{sD_{1+s}(\rho_{ZE}^{F} \parallel 1_{Z} \otimes \rho_{E})} \\
\stackrel{(a)}{\leq} \operatorname{Tr} \sum_{x \in \mathcal{X}} p(x) \rho_{E}^{x} \rho_{E}^{-\frac{s}{2(1+s)}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(\mathbb{E}_{F} \sum_{x': F(x') = F(x)} p(x') \rho_{E}^{x'} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{s} \rho_{E}^{-\frac{s}{2(1+s)}} \\
\stackrel{(b)}{\leq} \operatorname{Tr} \sum_{x \in \mathcal{X}} p(x) \rho_{E}^{x} \rho_{E}^{-\frac{s}{2(1+s)}} \left(\rho_{E}^{-\frac{s}{2(1+s)}} \left(v(\rho_{E}) \mathcal{E}_{\rho_{E}}(p(x) \rho_{E}^{x}) + \frac{1}{M} \rho_{E} \right) \rho_{E}^{-\frac{s}{2(1+s)}} \right)^{s} \rho_{E}^{-\frac{s}{2(1+s)}} \\
= \sum_{x \in \mathcal{X}} \operatorname{Tr} p(x) \rho_{E}^{x} \rho_{E}^{-s} \left(v(\rho_{E}) \mathcal{E}_{\rho_{E}}(p(x) \rho_{E}^{x}) + \frac{1}{M} \rho_{E} \right)^{s} \\
\leq \sum_{x \in \mathcal{X}} \operatorname{Tr} p(x) \rho_{E}^{x} \rho_{E}^{-s} \left(v(\rho_{E})^{s} (\mathcal{E}_{\rho_{E}}(p(x) \rho_{E}^{x}))^{s} + \frac{1}{M^{s}} \rho_{E}^{s} \right) \\
= v(\rho_{E})^{s} 2^{-sH_{1+s}(X|E)} \mathcal{E}_{\rho_{E}}(\rho) + \frac{1}{M^{s}} \\
\stackrel{(c)}{\leq} v(\rho_{E})^{s} 2^{-sH_{1+s}(X|E)} + \frac{1}{M^{s}}, \tag{59}$$

where (a) follows from the matrix concavity of $x \mapsto x^s$, (b) comes from the definition of the twouniversal hash functions and the pinching inequality (2) and (c) is due to the data processing inequality of the sandwiched Rényi divergence. This completes the proof.

Now, we are ready for the proofs of Theorem 4 and Theorem 5.

Proof of Theorem 4: At first, we deal with the " \leq " part. By Lemma 3, we know that for any n there exists a hash function f_n such that

$$D_{1+s}(\rho_{Z_{n}E^{n}}^{f_{n}} \| \frac{\mathbb{1}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})$$

$$= nR + \frac{1}{s} \log Q_{1+s}(\rho_{Z_{n}E^{n}}^{f_{n}} \| \mathbb{1}_{Z_{n}} \otimes \rho_{E}^{\otimes n})$$

$$\leq nR + \frac{1}{s} \log \left(Q_{1+s}^{n}(\rho_{XE} \| \mathbb{1}_{X} \otimes \rho_{E}) + \frac{1}{2^{nRs}}\right) + \frac{1+s}{s} \log v(\rho_{E}^{\otimes n})$$

$$= \frac{1}{s} \log \left(1 + 2^{nRs}Q_{1+s}^{n}(\rho_{XE} \| \mathbb{1}_{X} \otimes \rho_{E})\right) + \frac{1+s}{s} \log v(\rho_{E}^{\otimes n}).$$
(60)

This further yields

$$\limsup_{n \to \infty} \frac{1}{n} \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le |R - H_{1+s}(X|E)_{\rho}|^+.$$

Next, we turn to the derivation of the other direction. By Proposition 2 (vi), we have for any hash function f_n ,

$$D_{1+s}(\rho_{Z_{n}E^{n}}^{f_{n}} \| \frac{1}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})$$

$$= nR - H_{1+s}(Z_{n}|E^{n})_{\rho_{Z_{n}E^{n}}^{f_{n}}}$$

$$\geq nR - nH_{1+s}(X|E)_{\rho}.$$
(61)

From this and noticing that $D_{1+s}(\rho_{Z_nE^n}^{f_n}\|\frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|}\otimes\rho_E^{\otimes n})\geq 0$, it is easy to get

$$\liminf_{n\to\infty} \frac{1}{n} \min_{f_n \in \mathcal{F}(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \ge |R - H_{1+s}(X|E)_\rho|^+.$$

Proof of Theorem 5: At first, we prove the "\geq" part. The left side of Eq. (46) can be bounded as follows, thanks to the monotonicity of the sandwiched Rényi divergence (Proposition 2 (i)) and Lemma 4.

$$\min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{1}{|\mathcal{Z}_n|} \otimes \rho_E)$$

$$\leq \min_{f_n \in \mathcal{F}_n(R)} D_{1+t}(\rho_{Z_n E^n}^{f_n} \| \frac{1}{|\mathcal{Z}_n|} \otimes \rho_E)$$

$$\leq \frac{1}{t} \log \left(1 + v(\rho_E^{\otimes n})^t 2^{t(nR - nH_{1+t}(X|E)_\rho)} \right)$$

$$\leq \frac{\log e}{t} v(\rho_E^{\otimes n})^t 2^{t(nR - nH_{1+t}(X|E)_\rho)},$$
(62)

for any $t \in [s,1]$. Noticing that $D_{1+s}(\rho_{Z_nE^n}^{f_n}\|\frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|}\otimes \rho_E) \leq nR$, we see that the exponent must be non-negative. This observation and Eq. (62) implies

$$\liminf_{n\to\infty} -\frac{1}{n}\log \min_{f_n\in\mathcal{F}_n(R)} D_{1+s}(\rho_{Z_nE^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E) \ge \big| \sup_{t\in[s,1]} tH_{1+t}(X|E)_\rho - tR\big|^+.$$

Next, we prove the other direction. We will deal with the case $R \leq \hat{R}(s)$ and the case $R > \hat{R}(s)$ separately. Now we start with the former case. Let f_n be an arbitrary hash function. We choose a positive constant c such that $c^s - 2 \geq 1$. Then we construct a channel

$$\Phi(X) = (\operatorname{Tr}\{\rho_{Z_n E^n}^{f_n} > c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}\}X)|0\rangle\langle 0| + (\operatorname{Tr}\{\rho_{Z_n E^n}^{f_n} \leq c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}\}X)|1\rangle\langle 1|,$$

and denote

$$p_n = \operatorname{Tr} \rho_{Z_n E^n}^f \{ \rho_{Z_n E^n}^{f_n} > c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n} \} \quad \text{and} \quad q_n = \operatorname{Tr} (\frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \{ \rho_{Z_n E^n}^{f_n} > c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n} \}.$$

It is easy to see that

$$p_n \ge cq_n. (63)$$

Hence, by the data processing inequality for the channel Φ and Eq. (63), we have

$$D_{1+s}(\rho_{Z_{n}E^{n}}^{f_{n}} \| \frac{\mathbb{I}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})$$

$$\geq \frac{1}{s} \log \left\{ p_{n}^{s+1} q_{n}^{-s} + (1 - p_{n})^{s+1} (1 - q_{n})^{-s} \right\}$$

$$\geq \frac{1}{s} \log \left\{ c^{s} p_{n} + (1 - p_{n})^{2} \right\}$$

$$= \frac{1}{s} \log \left\{ 1 + (c^{s} - 2) p_{n} + p_{n}^{2} \right\}$$

$$\geq \frac{1}{s} \log \left\{ 1 + p_{n} \right\}$$

$$\geq \frac{1}{s} \log \left\{ 1 + \operatorname{Tr}(\rho_{Z_{n}E^{n}}^{f_{n}} - c \frac{\mathbb{I}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})_{+} \right\},$$
(64)

where the third inequality follows from $c^s - 2 > 1$.

Eq. (64) implies

$$\min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \ge \frac{1}{s} \log \left\{ 1 + \min_{f_n \in \mathcal{F}_n(R)} \operatorname{Tr}(\rho_{Z_n E^n}^{f_n} - c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})_+ \right\}
\doteq \frac{1}{s} \min_{f_n \in \mathcal{F}_n(R)} \operatorname{Tr}(\rho_{Z_n E^n}^{f_n} - c \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n})_+,$$
(65)

where $a_n \doteq b_n$ means that $\lim_{n\to\infty} \frac{1}{n} \log \frac{a_n}{b_n} = 0$. Now, we can use Eq. (65) and Lemma 2 to get

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \sup_{t > 0} (tH_{1+t}(X|E)_{\rho} - tR). \tag{66}$$

Because $\hat{R}(1) \leq R \leq \hat{R}(s)$, we have

$$\sup_{t>0} (tH_{1+t}(X|E)_{\rho} - tR) = \max_{s \le t \le 1} (tH_{1+t}(X|E)_{\rho} - tR),$$

and we complete the case $R \leq \hat{R}(s)$.

Next, we turn to the case $R > \hat{R}(s)$. We also define a channel like the above step

$$\Delta(X) = (\operatorname{Tr}\{\rho_{Z_n E^n}^{f_n} > \frac{\mathbb{1}_{Z_n}}{2^n \hat{R}(s)} \otimes \rho_E^{\otimes n}\}X)|0\rangle\langle 0| + (\operatorname{Tr}\{\rho_{Z_n E^n}^{f_n} \leq \frac{\mathbb{1}_{Z_n}}{2^n \hat{R}(s)} \otimes \rho_E^{\otimes n}\}X)|1\rangle\langle 1|,$$

and denote

$$p_n = \operatorname{Tr} \rho_{Z_n E^n}^{f_n} \{ \rho_{Z_n E^n}^{f_n} > \frac{\mathbb{1}_{Z_n}}{2^{n \hat{R}(s)}} \otimes \rho_E^{\otimes n} \} \quad \text{and} \quad q_n = \operatorname{Tr} (\frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \{ \rho_{Z_n E^n}^{f_n} > \frac{\mathbb{1}_{Z_n}}{2^{n \hat{R}(s)}} \otimes \rho_E^{\otimes n} \}.$$

We invoke a similar argument as Eq. (64) to bound $D_{1+s}(\rho_{Z_nE^n}^{f_n}\|\frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|}\otimes\rho_E^{\otimes n})$, by using the channel Δ .

$$D_{1+s}(\rho_{Z_{n}E^{n}}^{f_{n}} \| \frac{\mathbb{I}_{Z_{n}}}{|\mathcal{Z}_{n}|} \otimes \rho_{E}^{\otimes n})$$

$$\geq \frac{1}{s} \log \left\{ p_{n}^{s+1} q_{n}^{-s} + (1 - p_{n})^{s+1} (1 - q_{n})^{-s} \right\}$$

$$\geq \frac{1}{s} \log \left\{ 2^{ns(R - \hat{R}(s))} p_{n} + (1 - p_{n})^{2} \right\}$$

$$\geq \frac{1}{s} \log \left\{ 1 + (2^{sn(R - \hat{R}(s))} - 2) p_{n} \right\}$$

$$\geq \frac{1}{s} \log \left\{ 1 + (2^{sn(R - \hat{R}(s))} - 2) (\rho_{Z_{n}E^{n}}^{f} - c \frac{\mathbb{I}_{Z_{n}}}{2^{n\hat{R}(s)}} \otimes \rho_{E}^{\otimes n})_{+} \right\}.$$
(67)

Eq. (67) and Lemma 2 imply

$$\min_{f_n \in \mathcal{F}_n(R)} D_{1+s} \left(\rho_{Z_n E^n}^{f_n} \right\| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n} \right)
\dot{\geq} \frac{1}{s} \log \left\{ 1 + 2^{sn(R-\hat{R}(s))} 2^{n \inf_{t \geq 0} (t\hat{R}(s) - tH_{1+t}(X|E)_{\rho})} \right\}
= \frac{1}{s} \log \left\{ 1 + 2^{sn(R-\hat{R}(s))} 2^{n(s\hat{R}(s) - sH_{1+s}(X|E)_{\rho})} \right\},$$
(68)

where $a_n \geq b_n$ means that $\lim_{n \to \infty} \frac{\log a_n}{n} \geq \lim_{n \to \infty} \frac{\log b_n}{n}$, and the last line is because the minimum of the function $t \mapsto t(\hat{R}(s) - H_{1+t}(X|E)_{\rho})$ is achieved at t = s when $R > \hat{R}(s)$.

Eq. (68) further gives

$$\limsup_{n \to \infty} -\frac{1}{n} \log \min_{f_n \in \mathcal{F}_n(R)} D_{1+s}(\rho_{Z_n E^n}^{f_n} \| \frac{\mathbb{1}_{Z_n}}{|\mathcal{Z}_n|} \otimes \rho_E^{\otimes n}) \le \left| sH_{1+s}(X|E)_{\rho} - sR \right|_{+}$$

$$= \left| \max_{s \le t \le 1} (tH_{1+t}(X|E)_{\rho} - tR) \right|_{+}, \tag{69}$$

and this completes the proof of the case $R > \hat{R}(s)$.

VI. CONCLUSION AND DISCUSSION

Employing the sandwiched Rényi divergence, we have obtained the precise exponent in smoothing the max-relative entropy, and as an application, combining the existing result [32, Theorem 1], we have also obtained the precise exponent for quantum privacy amplification when the rate of extracted randomness is not too low. Our results, along with the concurrent work [41] which addresses different problems, clearly show that the sandwiched Rényi divergence can not only characterize the strong converse exponents [25], [36], [37], [38], [39], [40], but also accurately characterizes how the performance of certain quantum information processing tasks approach the perfect. We anticipate that more applications of the sandwiched Rényi divergence along this line will be found in the future.

Different definitions for the sandwiched Rényi conditional entropy have been proposed, among which two typical versions are [48], [28]

$$H_{\alpha}(A|B)_{\rho} = -D_{\alpha}(\rho_{AB} \| \mathbb{1}_A \otimes \rho_B), \quad \text{and}$$
 (70)

$$\bar{H}_{\alpha}(A|B)_{\rho} = -\min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D_{\alpha}(\rho_{AB} \| \mathbb{1}_A \otimes \sigma_B), \tag{71}$$

and it was not quite clear which one should be the proper formula. The version (71) has later found operational meanings in Ref. [38] and Ref. [40]. By giving an operational meaning to the version (70) in this paper, we conclude that both versions are proper expressions and the sandwiched Rényi conditional entropy is not unique.

The smoothing quantity in Theorem 1 and the insecurity in Theorem 2 as well as Theorem 3 are measured by the purified distance and/or the Kullback-Leibler divergence. Determining the respective exponents for these two problems under the trace distance is an interesting open problem. Originally, Renner [1] defined the smoothing of the max-relative entropy based on the trace norm distance to derive an upper bound of the insecurity in privacy amplification under two-universal hashing. However, the reference [26] showed that this type of entropy cannot derive the tight exponential upper bound in the classical setting of this problem while it derived the type exponential behavior based on the trace norm distance in the classical case. Instead, the references [59], [26] showed that the smoothing of the Rényi entropy of order 2 based on the trace norm distance derives the tight exponential upper bound in the classical setting of this problem. The reference [60] considered its quantum extension, but did not derive the tight exponential evaluation, while this topic has a recent progress [33] after the references [59], [60], [26].

For privacy amplification, we are only able to find out the exact exponent when the rate R of the randomness extraction is above the critical value R_{critical} . Determining the exponent for rate R less than R_{critical} is another important open question. The examples in Section IV-B indicate that this problem may be more of a combinatorial feature in the low-rate regime, at least when the rate R is such that $0 \le R \le H_{\min}(X|E)_{\varrho}$.

In addition, Section V has derived the asymptotic equivocation rate under the sandwiched Rényi divergence for any randomness extraction rate as Theorem 4. Also, this section has derived the security exponent under the sandwiched Rényi divergence in Theorem 5 when the randomness extraction rate is not smaller than the critical rate. This exponent is remained an open problem when the randomness extraction rate is larger than the critical rate R_{critical} .

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APPENDIX PROOF OF PROPOSITION 4

We need the following lemma.

Lemma 5: Let $\sigma_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ and let $U : \mathcal{H}_A \to \mathcal{H}_{A'}$ be an isometry. Then

$$H_{\min}^{\epsilon}(A|B)_{\sigma} = H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}.$$

Proof: By definition, there is a state $\tilde{\sigma}_{AB} \in \mathcal{B}^{\epsilon}(\sigma_{AB})$ satisfying

$$\tilde{\sigma}_{AB} \leq 2^{-H_{\min}^{\epsilon}(A|B)_{\sigma}} \mathbb{1}_A \otimes \sigma_B.$$

Let $\tilde{\sigma}_{A'B} := U\tilde{\sigma}_{AB}U^*$. Obviously, we have $\tilde{\sigma}_{A'B} \in \mathcal{B}^{\epsilon}(U\sigma_{AB}U^*)$, and

$$\tilde{\sigma}_{A'B} \leq 2^{-H_{\min}^{\epsilon}(A|B)_{\sigma}} U \mathbb{1}_{A} U^{*} \otimes \sigma_{B}$$
$$\leq 2^{-H_{\min}^{\epsilon}(A|B)_{\sigma}} \mathbb{1}_{A'} \otimes \sigma_{B}.$$

This verifies by definition that

$$H_{\min}^{\epsilon}(A|B)_{\sigma} \leq H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}.$$

For the opposite direction, similarly, by definition there is a state $\tilde{\sigma}_{A'B} \in \mathcal{B}^{\epsilon}(U\sigma_{AB}U^*)$ satisfying

$$\tilde{\sigma}_{A'B} \leq 2^{-H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}} \mathbb{1}_{A'} \otimes \sigma_B.$$

Then for the subnormalized state $U^*\tilde{\sigma}_{A'B}U \in \mathcal{S}_{\leq}(\mathcal{H}_{AB})$, we can check that

$$P(\sigma_{AB}, U^* \tilde{\sigma}_{A'B} U) = P(U \sigma_{AB} U^*, U U^* \tilde{\sigma}_{A'B} U U^*)$$

$$= P(U \sigma_{AB} U^*, \tilde{\sigma}_{A'B})$$

$$\leq \epsilon,$$
(72)

and

$$U^* \tilde{\sigma}_{A'B} U \leq 2^{-H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}} U^* \mathbb{1}_{A'} U \otimes \sigma_B$$
$$= 2^{-H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}} \mathbb{1}_A \otimes \sigma_B,$$

where for the second line of Eq. (72), notice that UU^* is a projection onto $U\mathcal{H}_A$, and hence we check it directly using the expression of the fidelity function. This implies by definition that

$$H_{\min}^{\epsilon}(A|B)_{\sigma} \ge H_{\min}^{\epsilon}(A'|B)_{U\sigma U^*}.$$

Proof of Proposition 4: Let $U:|x\rangle\mapsto|x\rangle\otimes|f(x)\rangle$ be the isometry from X to XZ, and write $\sigma_{XZAB}=U\sigma_{XAB}U^*$. Obviously, σ_{XZAB} is classical on X and Z, and is the extension of both σ_{XAB} and σ_{ZAB} . Since Lemma 5 gives that $H^{\epsilon}_{\min}(XA|B)_{\sigma}=H^{\epsilon}_{\min}(XZA|B)_{\sigma}$, what we need to do is to show

$$H_{\min}^{\epsilon}(XZA|B)_{\sigma} \ge H_{\min}^{\epsilon}(ZA|B)_{\sigma}.$$
 (73)

By the definition of $H^{\epsilon}_{\min}(ZA|B)_{\sigma}$, there is $\tilde{\sigma}_{ZAB} \in \mathcal{B}^{\epsilon}(\sigma_{ZAB})$ such that

$$\tilde{\sigma}_{ZAB} \le 2^{-H_{\min}^{\epsilon}(ZA|B)_{\sigma}} \mathbb{1}_{ZA} \otimes \sigma_{B}. \tag{74}$$

Now Uhlmann's theorem [45] tells us that there is $\hat{\sigma}_{XZAB} \in \mathcal{S}_{\leq}(\mathcal{H}_{XZAB})$ which extends $\tilde{\sigma}_{ZAB}$ and satisfies $P(\sigma_{XZAB}, \hat{\sigma}_{XZAB}) = P(\sigma_{ZAB}, \tilde{\sigma}_{ZAB})$. Using the measurement map $\mathcal{M}_X : L \mapsto \sum_x |x\rangle\langle x|L|x\rangle\langle x|$, we define $\tilde{\sigma}_{XZAB} := \mathcal{M}_X(\hat{\sigma}_{XZAB})$. Since $\sigma_{XZAB} = \mathcal{M}_X(\sigma_{XZAB})$,

$$P(\sigma_{XZAB}, \tilde{\sigma}_{XZAB}) \le P(\sigma_{XZAB}, \hat{\sigma}_{XZAB}) \le \epsilon. \tag{75}$$

By construction, $\tilde{\sigma}_{XZAB}$ has the form $\tilde{\sigma}_{XZAB} = \sum_{x} |x\rangle\langle x|_X \otimes \tilde{\sigma}_{ZAB}^x$ and is still an extension of $\tilde{\sigma}_{ZAB}$. So, $\tilde{\sigma}_{ZAB}^x \leq \sum_{x} \tilde{\sigma}_{ZAB}^x = \tilde{\sigma}_{ZAB}$. This, together with Eq. (74), ensures that

$$\tilde{\sigma}_{XZAB} \le 2^{-H_{\min}^{\epsilon}(ZA|B)_{\sigma}} \mathbb{1}_{XZA} \otimes \sigma_{B}. \tag{76}$$

Eq. (75) and Eq. (76) together imply Eq. (73), concluding the proof.

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