# SOME NOTES ON VARIATIONAL PRINCIPLE FOR METRIC MEAN DIMENSION

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ABSTRACT. Firstly, we answer the problem 1 asked by Gutman and Śpiewak in [GS20], then we establish a double variational principle for mean dimension in terms of Rēnyi information dimension and show the order of sup and lim sup (or lim inf) of the variational principle for the metric mean dimension in terms of Rēnyi information dimension obtained by Gutman and Śpiewak can be changed under the marker property. Finally, we attempt to introduce the notion of maximal metric mean dimension measure, which is an analogue of the concept called classical maximal entropy measure related to the topological entropy.

### 1. INTRODUCTION

By a pair  $(\mathcal{X}, T)$  we mean a topological dynamical system (TDS for short), where  $\mathcal{X}$  is a compact metrizable topological space and  $T : \mathcal{X} \to \mathcal{X}$  is a continuous self-map. The set of metrics on  $\mathcal{X}$  compatible with the topology is denoted by  $\mathscr{D}(\mathcal{X})$ . By  $M(\mathcal{X}), M(\mathcal{X}, T), E(\mathcal{X}, T)$  we denote the sets of all Borel probability measures on  $\mathcal{X}$ , all *T*-invariant Borel probability measures on  $\mathcal{X}$ , all ergodic measures on  $\mathcal{X}$ , respectively.

Mean topological dimension introduced by Gromov [Gro99] is a new topological invariant in topological dynamical systems, which was systematically studied by Lindenstrauss and Weiss [LW00]. They also introduced a notion called metric mean dimension to capture the topological complexity of infinite topological entropy systems and revealed a well-known result that metric mean dimension is an upper bound of mean topological dimension.

One says that a compact metric space  $(\mathcal{X}, d)$  admits tame growth of covering numbers if for each  $\theta > 0$ ,

$$\lim_{\epsilon \to 0} \epsilon^{\theta} \log r_1(T, d, \epsilon, \mathcal{X}) = 0,$$

where  $r_1(T, d, \epsilon, \mathcal{X})$  denotes the minimal cardinality of  $\mathcal{X}$  covered by the balls  $B(x, \epsilon) := \{y \in \mathcal{X} : d(x, y) < \epsilon\}.$ 

<sup>2020</sup> Mathematics Subject Classification: 37A05, 37A35, 37B40, 94A34.

Key words and phrases: Variational principle; Metric mean dimension; Mean dimension; Rate distortion dimension;  $R\bar{e}nyi$  information dimension.

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In 2018, to inject ergodic theoretic ideas into mean dimension theory, Lindenstrauss and Tsukamoto [LT18] established a variational principle for metric mean dimension in terms of rate distortion dimensions. Namely,

**Theorem A.** Let  $(\mathcal{X}, T)$  be a TDS with a metric d. Then

$$\overline{mdim}_M(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in M(\mathcal{X},T)} R_{\mu,L^{\infty}}(\epsilon)}{\log \frac{1}{\epsilon}}.$$

Additionally, if  $(\mathcal{X}, d)$  has tame growth of covering numbers, then for  $p \in [1, \infty)$ ,

$$\overline{mdim}_M(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in M(\mathcal{X},T)} R_{\mu,p}(\epsilon)}{\log \frac{1}{\epsilon}}$$

where  $\overline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$  denotes upper metric mean dimension of  $\mathcal{X}$ ,  $R_{\mu,p}(\epsilon), R_{\mu,L^{\infty}}(\epsilon)$  are called the  $L^{p}$  and  $L^{\infty}$  rate distortion function, respectively. Moreover, the above two results are valid for lower metric mean dimension  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$  by changing lim sup into lim inf.

The amenable version of Theorem A was proved by Chen et al. [CDZ22] by using abundant non-trivial quasi-tiling methods. Besides, Gutman and Śpiewak [GS20] showed that the second variational principle in Theorem A can only range over all ergodic measures and posed a problem [GS20, Problem 1] if the first variational principle in Theorem A can only range over all ergodic measures. In this paper, we give a positive answer to this problem.

**Theorem 1.1.** Let  $(\mathcal{X}, T)$  be a TDS with a metric d. Then

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} R_{\mu,L^{\infty}}(\epsilon)}{\log \frac{1}{\epsilon}},$$
  
$$\underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \liminf_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} R_{\mu,L^{\infty}}(\epsilon)}{\log \frac{1}{\epsilon}}.$$

Given  $\mu \in M(\mathcal{X}, T)$ , recall that lower and upper R $\bar{e}$ nyi information dimensions of  $\mu$  are respectively given by

$$\underline{MRID}(\mathcal{X}, T, \mu, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \le \epsilon} h_{\mu}(T, P),$$
$$\overline{MRID}(\mathcal{X}, T, \mu, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \le \epsilon} h_{\mu}(T, P),$$

where the infimums range over all finite partitions of  $\mathcal{X}$  with diameter at most  $\epsilon$ , and  $h_{\mu}(T, P)$  denotes the measure-theoretic entropy of  $\mu$ with respect to T and P, its precise definition is given in section 2.

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One says that a topological dynamical system  $(\mathcal{X}, T)$  admits *marker* property if for any N > 0 there exists an open set  $U \subset \mathcal{X}$  with property that

$$U \cap T^n U = \emptyset, 1 \le n \le N$$
, and  $\mathcal{X} = \bigcup_{n \in \mathbb{Z}} T^n U$ .

After establishing variational principle for metric mean dimension, Lindenstrauss and Tsukamoto [LT19] further established a double variational principle for mean dimension in terms of  $L^1$ -rate distortion dimension under marker property, which can be considered an important link between mean dimension theory and ergodic theory. Later, Tsukamoto [T20] extended this result to mean dimension with potential. In this paper, replacing  $L^1$ -rate distortion dimension by Rēnyi information dimension, we also establish a double variational principle for mean dimension in terms of Rēnyi information dimension and show the order of sup and lim sup (or lim inf) of the variational principle for the metric mean dimension in terms of Rēnyi information dimension obtained by Gutman and Spiewak can be changed under marker property.

**Theorem 1.2.** Let  $(\mathcal{X}, T)$  be a TDS admitting marker property. Then

$$\operatorname{mdim}(\mathcal{X}, T) = \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in M(\mathcal{X}, T)} \underline{MRID}(\mathcal{X}, T, \mu, d),$$
$$= \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in M(\mathcal{X}, T)} \overline{MRID}(\mathcal{X}, T, \mu, d),$$

and for any  $d \in \mathscr{D}'(\mathcal{X})$ ,

$$\underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \mathrm{mdim}_{M}(T,\mathcal{X},d)$$

$$= \sup_{\mu \in M(\mathcal{X},T)} \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)$$

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 $\begin{array}{l} \operatorname{mdim}(\mathcal{X},T) \ denotes \ the \ mean \ dimension, \ see \ [{\bf T20}, \ Subsection \ 1.2] \\ \underline{for \ its \ explicit \ definition, \ and \ \mathscr{D}'(\mathcal{X}) = \ \{d \in \mathscr{D}(\mathcal{X}) : \ mdim(\mathcal{X},T) = \\ \overline{\mathrm{mdim}}_M(T,\mathcal{X},d) \}. \end{array}$ 

We would like to remark that if a TDS admits marker property, Tsukamoto [T20, Theorem 1.8] showed there exists  $d \in \mathscr{D}(\mathcal{X})$  such that  $mdim(\mathcal{X}, T) = \overline{\mathrm{mdim}}_M(T, \mathcal{X}, d)$ , this implies  $\mathscr{D}'(\mathcal{X})$  is not empty.

#### 2. Preliminary

In this section, we recall the definition of metric mean dimension and collect several types of measure-theoretic entropies for forthcoming proofs.

2.1. Metric mean dimension. Let  $n \in \mathbb{N}$ , for  $x, y \in \mathcal{X}$ , we define the *n*-th Bowen metric  $d_n$  on  $\mathcal{X}$  as

$$d_n(x,y) := \max_{0 \le j \le n-1} d(T^j(x), T^j(y)).$$

For a non-empty subset  $Z \subset \mathcal{X}$ . A set  $E \subset \mathcal{X}$  is an  $(n, \epsilon)$ -spanning set of Z if for any  $x \in Z$ , there exists  $y \in E$  such that  $d_n(x, y) < \epsilon$ . The smallest cardinality of  $(n, \epsilon)$ -spanning set of Z is denoted by  $r_n(T, d, \epsilon, Z)$ . A set  $F \subset Z$  is an  $(n, \epsilon)$ -separated set of Z if  $d_n(x, y) \ge \epsilon$ for any  $x, y \in F$  with  $x \ne y$ . The largest cardinality of  $(n, \epsilon)$ -separated set of Z is denoted by  $s_n(T, d, \epsilon, Z)$ .

Put

$$r(T, \mathcal{X}, d, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(T, d, \epsilon, \mathcal{X})$$

and

$$s(T, \mathcal{X}, d, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, \epsilon, \mathcal{X}).$$

By a standard method used in [W82], we have  $r(T, \mathcal{X}, d, \epsilon) \leq s(T, \mathcal{X}, d, \epsilon) \leq r(T, \mathcal{X}, d, \frac{\epsilon}{2})$ .

Let  $(\mathcal{X}, T)$  be a TDS, we define the upper metric mean dimension of T on  $\mathcal{X}$  as

$$\overline{mdim}_M(T,\mathcal{X},d) := \limsup_{\epsilon \to 0} \frac{r(T,\mathcal{X},d,\epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{s(T,\mathcal{X},d,\epsilon)}{\log \frac{1}{\epsilon}}$$

Similarly, one can define the lower metric mean dimension  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$ by replacing lim sup with lim inf. If  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d) = \overline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$ , we call the common value denoted by  $mdim_{M}(T, \mathcal{X}, d)$  metric mean dimension.

2.2. Measure-theoretical entropy. Let P be a partition of  $\mathcal{X}$  and  $\mu \in M(\mathcal{X}, T)$ , then the partition entropy of P is given by

$$H_{\mu}(P) = \sum_{A \in P} -\mu(A) \log \mu(A),$$

here we use the convention that  $\log = \log_e$  and  $0 \cdot \infty = 0$ .

Let P, Q be two finite partitions of  $\mathcal{X}$ , the join of P and Q is defined by  $P \lor Q := \{A \cap B : A \in P, B \in Q\}$ . Set  $P^n := \lor_{j=0}^{n-1} T^{-j} P$ , and we define the measure-theoretic entropy of  $\mu$  with respect to T and Pas

$$h_{\mu}(T,P) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(P^n).$$

The measure-theoretic entropy of  $\mu$  is defined by

$$h_{\mu}(T) = \sup_{P} h_{\mu}(T, P),$$

where P ranges over all finite partitions of  $\mathcal{X}$ .

2.3. Rate distortion theory. Let  $(\Omega, \mathbb{P})$  be a probability space and  $\mathcal{X}$  and  $\mathcal{Y}$  be measurable spaces, and let  $\xi : \Omega \to \mathcal{X}$  and  $\eta : \Omega \to \mathcal{Y}$  be measurable maps. We define the *mutual information*  $I(\xi; \eta)$  as the supremum of

$$\sum_{\substack{1 \le m \le M, \\ 1 \le n \le N}} \mathbb{P}(\xi \in P_m, \eta \in Q_n) \log \frac{\mathbb{P}(\xi \in P_m, \eta \in Q_n)}{\mathbb{P}(\xi \in P_m)\mathbb{P}(\eta \in Q_n)},$$

where  $\{P_1, ..., P_M\}$  and  $\{Q_1, ..., Q_N\}$  are partitions of  $\mathcal{X}$  and  $\mathcal{Y}$  respectively, here we use the convention that  $0 \log \frac{0}{a} = 0$  for all  $a \ge 0$ .

A measurable map  $\xi : \Omega \to \mathcal{X}$  with finite image naturally associates a finite partition on  $\Omega$ , the *preimage partition* of  $\Omega$ . In this case we denote the entropy of  $\xi$  by  $H(\xi)$ .

If  $\mathcal{X}$  and  $\mathcal{Y}$  are finite sets, then  $I(\xi; \eta)$  is given by

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \mathbb{P}(\xi = x, \eta = y) \log \frac{\mathbb{P}(\xi = x, \eta = y)}{\mathbb{P}(\xi = x)\mathbb{P}(\eta = y)},$$
$$= H(\xi) - H(\xi|\eta) = H(\xi) + H(\eta) - H(\xi \lor \eta),$$

where  $H(\xi|\eta)$  is the conditional entropy of  $\xi$  given  $\eta$ , the value shows the amount of information which the random variables  $\xi$  and  $\eta$  share.

Let  $(\mathcal{X}, T)$  be a TDS with a metric d and  $\mu \in M(\mathcal{X}, T)$ . Given  $1 \leq p < \infty$  and  $\epsilon > 0$ , we define the  $L^p$ -rate distortion function  $R_{\mu,p}(\epsilon)$  as the infimum of

$$\frac{I(\xi;\eta)}{n}$$

where n runs over all natural numbers, and  $\xi$  and  $\eta = (\eta_0, ..., \eta_{n-1})$  are random variables defined on some probability space  $(\Omega, \mathbb{P})$  such that

(1)  $\xi$  takes values in  $\mathcal{X}$ , and its law is given by  $\mu$ .

(2) Each  $\eta_k$  takes values in  $\mathcal{X}$ , and

$$\mathbb{E}\left(\frac{1}{n}\sum_{k=0}^{n-1}d(T^k\xi,\eta_k)^p\right) < \epsilon^p,$$

where  $\mathbb{E}(\cdot)$  is the expectation with respect to the probability measure  $\mathbb{P}$ .

Let s > 0, we define the  $L^{\infty}$ -rate distortion function  $R_{\mu,L^{\infty}}(\epsilon, s)$  as the infimum of

$$\frac{I(\xi;\eta)}{r}$$

where *n* runs over all natural numbers, and  $\xi$  and  $\eta = (\eta_0, ..., \eta_{n-1})$  are random variables defined on some probability space  $(\Omega, \mathbb{P})$  such that

- (1)  $\xi$  takes values in  $\mathcal{X}$ , and its law is given by  $\mu$ .
- (2) Each  $\eta_k$  takes values in  $\mathcal{X}$ , and

 $\mathbb{E}$  (the number of  $0 \le k \le n-1$  with  $d(T^k\xi, \eta_k) \ge \epsilon$ ) < sn, and we set  $R_{\mu,L^{\infty}}(\epsilon) = \lim_{s \to 0} R_{\mu,L^{\infty}}(\epsilon, s).$ 

In above two definitions, Lindenstrauss and Tsukamoto [LT18, Section IV, reamrk 14] showed that the infimums can only consider random variable  $\eta$  taking finitely many values, and they also showed [LT18, III,B] that for  $1 \leq p < \infty$ ,  $R_{\mu,p}(\epsilon) \leq R_{\mu,L^{\infty}}(\epsilon')$  holds for  $0 < \epsilon' \leq \epsilon$ . We define the lower and upper  $L^P$  rate distortion dimensions as

$$\frac{rdim_{L^p}(\mathcal{X}, T, d, \mu)}{rdim_{L^p}(\mathcal{X}, T, d, \mu)} = \liminf_{\epsilon \to 0} \frac{R_{\mu, p}(\epsilon)}{\log \frac{1}{\epsilon}}$$
$$\overline{rdim}_{L^p}(\mathcal{X}, T, d, \mu) = \limsup_{\epsilon \to 0} \frac{R_{\mu, p}(\epsilon)}{\log \frac{1}{\epsilon}}$$

Replacing  $R_{\mu,p}(\epsilon)$  with  $R_{\mu,L^{\infty}}(\epsilon)$ , one can similarly define lower and upper  $L^{\infty}$  rate distortion dimensions.

2.4. Brin-Katok local entropy. Measure-theoretic entropy is given from the viewpoint of the local perspective. Let  $\mu \in \mathcal{M}(X)$ . Define

$$\overline{h}^{BK}_{\mu}(T,\epsilon) := \int \overline{h}^{BK}_{\mu}(T,x,\epsilon) d\mu$$
$$\underline{h}^{BK}_{\mu}(T,\epsilon) := \int \underline{h}^{BK}_{\mu}(T,x,\epsilon) d\mu,$$

where

$$\overline{h}_{\mu}^{BK}(T, x, \epsilon) = \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}$$
$$\underline{h}_{\mu}^{BK}(T, x, \epsilon) = \liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}.$$

If  $\mu \in M(\mathcal{X}, T)$ , Brin and Katok [K80] showed

$$\lim_{\epsilon \to 0} \underline{h}_{\mu}^{BK}(T, \epsilon) = \lim_{\epsilon \to 0} \overline{h}_{\mu}^{BK}(T, \epsilon) = h_{\mu}(T).$$

2.5. Katok's entropy. Measure-theoretic entropy defined by spanning set.

Let  $\mu \in \mathcal{M}(\mathcal{X}), \epsilon > 0$  and  $n \in \mathbb{N}$ . Given  $\delta \in (0, 1)$  and put

$$R^{\delta}_{\mu}(T, n, \epsilon) := \min\{\#E : \mu(\bigcup_{x \in E} B_n(x, \epsilon)) > 1 - \delta\}.$$

Define

$$\overline{h}_{\mu}^{K}(T,\epsilon,\delta) = \limsup_{n \to \infty} \frac{1}{n} \log R_{\mu}^{\delta}(T,n,\epsilon)$$
$$\underline{h}_{\mu}^{K}(T,\epsilon,\delta) = \liminf_{n \to \infty} \frac{1}{n} \log R_{\mu}^{\delta}(T,n,\epsilon).$$

If  $\mu \in E(\mathcal{X}, T)$ , Katok [K80] showed

$$\lim_{\epsilon \to 0} \overline{h}_{\mu}^{K}(T, \epsilon, \delta) = \lim_{\epsilon \to 0} \underline{h}_{\mu}^{K}(T, \epsilon, \delta) = h_{\mu}(T).$$

We can define two quantities related to Katok's entropy by an alternative way.

Let  $\mu \in M(\mathcal{X})$ . Note that the quantities  $\overline{h}_{\mu}^{K}(T, \epsilon, \delta), \underline{h}_{\mu}^{K}(T, \epsilon, \delta)$  are non-decreasing when  $\delta$  decreases. Therefore, we define

$$\overline{h}_{\mu}^{K}(T,\epsilon) := \lim_{\delta \to 0} \overline{h}_{\mu}^{K}(T,\epsilon,\delta), \ \underline{h}_{\mu}^{K}(T,\epsilon) := \lim_{\delta \to 0} \underline{h}_{\mu}^{K}(T,\epsilon,\delta).$$

2.6. **Pfister and Sullivan's entropy.** Measure-theoretic entropy defined by separated set. Let  $\mu \in E(\mathcal{X}, T)$  and  $\epsilon > 0$ . Define

$$PS_{\mu}(T, d, \epsilon) := \inf_{F \ni \mu} \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, \epsilon, \mathcal{X}_{n,F}),$$

where the infimum ranges over all neighborhoods of  $\mu$  in  $M(\mathcal{X})$  and  $\mathcal{X}_{n,F} := \{x \in \mathcal{X} : \mathcal{E}_n(x) = \frac{1}{n} \sum_{j=1}^{n-1} \delta_{T^j(x)} \in F\}$ . In fact, the infimum can only range over any base of open neighborhoods of  $\mu$ .

Pfister and Sullivan [PS07] proved that  $h_{\mu}(T) = \lim_{\epsilon \to 0} PS_{\mu}(T, d, \epsilon).$ 

2.7. Generic points. Let  $\mu \in M(\mathcal{X}, T)$ , by

$$G_{\mu} := \{ x \in \mathcal{X} : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^{j}(x)} \to \mu, n \to \infty \}$$

we denote the set of generic points of  $\mu$ . Note that if  $\mu \in E(\mathcal{X}, T)$ , then we know  $\mu(G_{\mu}) = 1$  by Birkhoff's ergodic theorem.

### 3. Proofs of main results

In this section, we prove Theorem 1.1 and Theorem 1.2.

Firstly, we give the proof of Theorem 1.1.

The following lemma slightly modifies the statement in [W21, Lemma 4.1,(2)], which plays a key role in the proof of Theorem 1.1.

**Lemma 3.1.** Let  $(\mathcal{X}, T)$  be a TDS with a metric d and  $\{E_n\}_{n\geq 1}$  be a sequence non-empty subsets of  $\mathcal{X}$ . Let  $\epsilon > 0$ , and let  $F_n$  be an  $(n, 6\epsilon)$ -separated set of  $E_n$  with maximal cardinality  $s_n(T, d, 6\epsilon, E_n)$ . Set

$$\mu_n = \frac{1}{n \# F_n} \sum_{x \in F_n} \sum_{j=0}^{n-1} \delta_{T^j(x)}.$$

Choose a subsequence  $n_j$  such that  $\mu_{n_j}$  convergences to  $\mu \in M(\mathcal{X}, T)$ in the weak<sup>\*</sup> topology, then

$$\limsup_{j \to \infty} \frac{1}{n_j} \log s_{n_j}(T, d, 6\epsilon, E_{n_j}) \le R_{\mu, L^{\infty}}(\epsilon).$$

*Proof.* This lemma can be proved by repeating the proof of [LT18, Proposition 35].  $\hfill \Box$ 

For sake of readers, we slightly modify the statement in [W21, Proposition 4.3] and repeat the proof.

**Lemma 3.2.** Let  $(\mathcal{X}, T)$  be a TDS with a metric d. Then for any  $\epsilon > 0$  and  $\mu \in E(\mathcal{X}, T)$ , we have

$$\overline{h}_{\mu}^{K}(T,\epsilon) \leq PS_{\mu}(T,d,\epsilon) \leq R_{\mu,L^{\infty}}(\frac{1}{6}\epsilon).$$

Proof. Given  $\epsilon > 0$  and let  $\mu \in E(\mathcal{X}, T)$ . Fix a base of open neighborhoods  $\mathcal{F}_{\mu}$  of  $\mu$ , if  $F \in \mathcal{F}_{\mu}$ , then  $G_{\mu} \subset \bigcup_{N \ge 1} \bigcap_{n \ge N} \mathcal{X}_{n,F}$ . Let  $\delta \in (0, 1)$  and note that  $\mu(G_{\mu}) = 1$ , we can find  $N_0$  such that for any  $n \ge N_0$ ,  $\mu(\mathcal{X}_{n,F}) > 1 - \delta$ . Hence,  $R^{\delta}_{\mu}(T, n, \epsilon) \le s_n(T, d, \epsilon, \mathcal{X}_{n,F})$  for any  $n \ge N_0$ . This implies that  $\overline{h}^K_{\mu}(T, \epsilon, \delta) \le \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, \epsilon, \mathcal{X}_{n,F})$  holds for any  $\delta \in (0, 1)$ . Letting  $\delta \to 0$  and by the arbitrariness of F, we get  $\overline{h}^K_{\mu}(T, \epsilon) \le PS_{\mu}(T, d, \epsilon)$ .

Next, given  $\epsilon > 0$  and let  $\mu \in E(\mathcal{X}, T)$  again, we show  $PS_{\mu}(T, d, 6\epsilon) \leq R_{\mu,L^{\infty}}(\epsilon)$ . Without loss of generality, we may assume that  $R_{\mu,L^{\infty}}(\epsilon) < \infty$ . If

$$PS_{\mu}(T, d, 6\epsilon) = \inf_{F \ni \mu} \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, 6\epsilon, \mathcal{X}_{n,F}) > R_{\mu,L^{\infty}}(\epsilon),$$

then we can choose  $\gamma_0 > 0$  and a decreasing sequence of closed convex neighborhood  $\{C_n\}$  of  $\mu$  such that

(3.1) 
$$\limsup_{n \to \infty} \frac{1}{n} \log s_n(T, d, 6\epsilon, \mathcal{X}_{n, C_n}) > R_{\mu, L^{\infty}}(\epsilon) + \gamma_0$$

and  $\cap_{n\geq 1}C_n = \{\mu\}.$ 

Let  $F_n \subset \mathcal{X}_{n,C_n}$  be an  $(n, 6\epsilon)$ -separated set of  $\mathcal{X}_{n,C_n}$  with the maximal cardinality  $s_n(T, d, 6\epsilon, \mathcal{X}_{n,C_n})$ . Set

$$\mu_n = \frac{1}{n \# F_n} \sum_{x \in F_n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$$

Then  $\mu_n \in C_n$  and  $\lim_{n \to \infty} \mu_n = \mu$ .

By Lemma 3.1, we know that  $\limsup_{n\to\infty} \frac{1}{n} \log s_n(T, d, 6\epsilon, \mathcal{X}_{n,C_n}) \leq R_{\mu,L^{\infty}}(\epsilon)$ , which contradicts with the inequality (3.1).

**Theorem 3.3.** Let  $(\mathcal{X}, T)$  be a TDS with a metric d. Then

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} \underline{h}_{\mu}^{K}(T,\epsilon)}{\log \frac{1}{\epsilon}}$$
$$= \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} \overline{h}_{\mu}^{K}(T,\epsilon)}{\log \frac{1}{\epsilon}}.$$

The two variational principles are valid for  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$  by changing lim sup into lim inf and the supremum can range over all invariant measures.

*Proof.* Fix  $\delta_0 \in (0, 1)$ , then we have

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} \underline{h}_{\mu}^{K}(T,\epsilon,\delta_{0})}{\log \frac{1}{\epsilon}}, \text{ by [S21, Proposition 7.3]}$$
$$\leq \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} \underline{h}_{\mu}^{K}(T,\epsilon)}{\log \frac{1}{\epsilon}}$$
$$\leq \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} \overline{h}_{\mu}^{K}(T,\epsilon)}{\log \frac{1}{\epsilon}}.$$

On the other hand, fix  $\epsilon > 0$  and  $\mu \in M(\mathcal{X})$ . Let  $\delta \in (0,1)$ , then  $R^{\delta}_{\mu}(T,n,\epsilon) \leq r_n(T,d,\epsilon,\mathcal{X})$  for every  $n \in \mathbb{N}$ , which yields that  $\overline{h}^K_{\mu}(T,\epsilon,\delta) \leq r(T,\mathcal{X},d,\epsilon)$  holds for every  $\delta \in (0,1)$ . Letting  $\delta \to 0$  gives  $\overline{h}^K_{\mu}(T,\epsilon) \leq r(T,\mathcal{X},d,\epsilon)$ . Hence, we finally obtain that

$$\limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X}, T)} \overline{h}_{\mu}^{K}(T, \epsilon)}{\log \frac{1}{\epsilon}} \le \overline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d).$$

This completes the proof.

Now, we are ready to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* It suffices to show show the first equality, and the second one can be obtained in a similar manner. By Theorem A, it is clear that

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) \geq \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in E(\mathcal{X},T)} R_{\mu,L^{\infty}}(\epsilon)}{\log \frac{1}{\epsilon}}$$

On the other hand, by Lemma 3.2 and Theorem 3.3, we get the converse inequality.  $\hfill \Box$ 

Next, we proceed to give the proof of Theorem 1.2.

**Proposition 3.4.** Let  $(\mathcal{X}, T)$  be a TDS and  $\mu \in M(\mathcal{X}, T)$ . Then for every  $\epsilon > 0$  and  $\mu \in M(\mathcal{X}, T)$ , we have

$$R_{\mu,L^{\infty}}(2\epsilon) \leq \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P),$$

where the infimum ranges over all finite partitions of  $\mathcal{X}$ .

Proof. Fix  $\epsilon > 0$  and  $\mu \in M(\mathcal{X}, T)$ , we can choose a finite partition Q of  $\mathcal{X}$  so that  $\inf_{\text{diam}P \leq \epsilon} h_{\mu}(T, P) \leq \log \# Q < \infty$ . Let P be a finite partition of  $\mathcal{X}$  with diameter at most  $\epsilon$ , and let  $\xi$  be a random variable taking values in  $\mathcal{X}$  and obeying  $\mu$ . For every  $n \in \mathbb{N}$  and every  $A \in P^n$ ,

we choose  $x_A \in A$  and define a map  $f : \mathcal{X} \to \mathcal{X}$  by setting  $f(x) = x_A$ if  $x \in A$ . Put  $\eta = (f(\xi), Tf(\xi), ..., T^{n-1}f(\xi))$ , then

 $\mathbb{E}\left(\text{the number of } k \in [0, n-1] \text{ with } d(T^k \xi, T^k f(\xi)) \geq 2\epsilon\right) = 0 < sn,$ 

for any s > 0. Hence,  $R_{\mu,L^{\infty}}(2\epsilon, s) \leq \frac{I(\xi;\eta)}{n} \leq \frac{H(\eta)}{n} = \frac{H_{\mu}(P^n)}{n}$ , this implies that  $R_{\mu,L^{\infty}}(2\epsilon, s) \leq h_{\mu}(T, P)$ . Letting  $s \to 0$  gives the desired result.

Proof of Theorem 1.2. By [GS20, Theorem 3.1], we have

(3.2) 
$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{\sup_{\mu \in M(\mathcal{X},T)} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)}{\log \frac{1}{\epsilon}}$$
$$\geq \sup_{\mu \in M(\mathcal{X},T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)$$

By the fact obtained in [LT18, III,B] that for  $1 \leq p < \infty$ ,  $R_{\mu,p}(\epsilon) \leq R_{\mu,L^{\infty}}(\epsilon')$  holds for  $0 < \epsilon' \leq \epsilon$ , we have  $R_{\mu,1}(3\epsilon) \leq R_{\mu,L^{\infty}}(2\epsilon)$ . Together with the Proposition 3.4, we obtain that for any  $\mu \in M(\mathcal{X}, T)$ ,

$$\underline{rdim}_{L^1}(\mathcal{X}, T, d, \mu) \le \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam} P \le \epsilon} h_{\mu}(T, P).$$

Therefore, for any  $d \in \mathscr{D}(\mathcal{X})$ ,

$$\begin{split} mdim(\mathcal{X},T) &\leq \sup_{\mu \in M(\mathcal{X},T)} \underline{rdim}_{L^{1}}(\mathcal{X},T,d,\mu) \text{ by Corollary [T20, Corollary 1.7]} \\ &\leq \sup_{\mu \in M(\mathcal{X},T)} \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \leq \epsilon} h_{\mu}(T,P) \\ &\leq \sup_{\mu \in M(\mathcal{X},T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \leq \epsilon} h_{\mu}(T,P) \\ &\leq \overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) \text{ by } (3\cdot 2). \end{split}$$

Using the fact obtained in [T20, Theorem 1.8], if  $(\mathcal{X}, T)$  admits marker property, then there exists  $d \in \mathscr{D}(\mathcal{X})$  such that  $mdim(\mathcal{X}, T) = \overline{\mathrm{mdim}}_M(T, \mathcal{X}, d)$ . This implies that

$$\operatorname{mdim}(\mathcal{X}, T) = \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in M(\mathcal{X}, T)} \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \le \epsilon} h_{\mu}(T, P),$$
$$= \min_{d \in \mathscr{D}(\mathcal{X})} \sup_{\mu \in M(\mathcal{X}, T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\operatorname{diam} P \le \epsilon} h_{\mu}(T, P).$$

By the Lindenstrauss and Weiss's classical inequality [LW00],  $\operatorname{mdim}(\mathcal{X}, T) \leq \underline{\operatorname{mdim}}_M(T, \mathcal{X}, d) \leq \overline{\operatorname{mdim}}_M(T, \mathcal{X}, d)$  for any  $d \in \mathscr{D}(\mathcal{X})$ , then for any  $d \in \mathscr{D}'(\mathcal{X})$ , we have  $\operatorname{mdim}(\mathcal{X}, T) = \underline{\operatorname{mdim}}_M(T, \mathcal{X}, d) = \overline{\operatorname{mdim}}_M(T, \mathcal{X}, d)$ .

$$\underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d)$$

$$= \sup_{\mu \in M(\mathcal{X},T)} \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)$$

$$= \sup_{\mu \in M(\mathcal{X},T)} \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)$$

$$= \liminf_{\epsilon \to 0} \sup_{\mu \in M(\mathcal{X},T)} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P)$$

$$= \limsup_{\epsilon \to 0} \sup_{\mu \in M(\mathcal{X},T)} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\mathrm{diam}P \leq \epsilon} h_{\mu}(T,P),$$

where the last two equalities hold by [GS20, Theorem 3.1].

For a TDS admits marker property, Theorem 1.2 shows that for some "nice" metrics, the variational principles are still valid if we change the order of sup and lim sup (or lim inf). It is not clear that whether we can drop the marker property imposed on the topological dynamical system or not, and removing the condition requires us to answer a central problem that if for every topological dynamical system, there exists a metric d such that  $mdim(\mathcal{X}, T) = mdim_M(T, \mathcal{X}, d)$ . This open problem was also mentioned in [GLT16, LT19, T20].

Finally, we attempt to introduce the notion of maximal metric mean dimension measure analogous to the classical notion of maximal entropy measure related to topological entropy. We begin this new concept with the following example.

**Example 3.5.** Let  $\sigma : [0,1]^{\mathbb{Z}} \to [0,1]^{\mathbb{Z}}$  be the (left) shift on alphabet [0,1], where [0,1] is the unit interval with the standard metric. Equipped  $[0,1]^{\mathbb{Z}}$  with a metric given by

$$d(x,y) = \sum_{n \in \mathbb{Z}} 2^{-|n|} |x_n - y_n|.$$

Let  $\mu = \mathcal{L}^{\otimes \mathbb{Z}}$ , where  $\mathcal{L}$  is the Lebesgue measure on [0, 1].

Let  $\mathcal{X} = [0, 1]^{\mathbb{Z}}$  and  $T = \sigma$ . It is well-known that  $mdim_M(T, \mathcal{X}, d) = 1$ , see [LT18, Section II, E. Example] for more details. Shi [S21, Exmaple 6.1] showed that  $\lim_{\epsilon \to 0} \frac{\overline{h}_{\mu}^{BK}(T,\epsilon)}{\log \frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \frac{\underline{h}_{\mu}^{BK}(T,\epsilon)}{\log \frac{1}{\epsilon}} = 1$ . By [YCZ22, Proposition 3.1], we know  $\lim_{\epsilon \to 0} \frac{\dim_{\epsilon \to 0} h_{\mu}(T,P)}{\log \frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \frac{\overline{h}_{\mu}^{BK}(T,\epsilon)}{\log \frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \frac{\overline{h}_{\mu}^{BK$ 

1. This shows that

$$mdim_M(T, \mathcal{X}, d) = \lim_{\epsilon \to 0} \frac{\inf_{\dim P \le \epsilon} h_\mu(T, P)}{\log \frac{1}{\epsilon}}$$
$$= \lim_{\epsilon \to 0} \frac{\overline{h}_\mu^{BK}(T, \epsilon)}{\log \frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \frac{\underline{h}_\mu^{BK}(T, \epsilon)}{\log \frac{1}{\epsilon}} = 1.$$

Unlike the measure-theoretical entropy used to establish variational principle for topological entropy, there are abundant choices that can be considered as a object to establish a variational principle for metric mean dimension [S21], for example Brin-Katok local entropy, Katok entropy,... Therefore, to inject ergodic theoretic ideas into mean dimension theory, a reasonable quantity related to measure-theoretical entropy is crucial. This leads to the following

**Definition 3.6.** A non-negative real-valued function  $F(\mu, \epsilon)$  defined on  $M(\mathcal{X}, T) \times \mathbb{R}_+$  (or  $E(\mathcal{X}, T) \times \mathbb{R}_+$ ) is said to be a candidate if for any fixed  $\mu$ ,  $F(\mu, \epsilon)$  is non-decreasing as  $\epsilon$  decreases and  $h_{\mu}(T) = \lim_{\epsilon \to 0} F(\mu, \epsilon)$ , and we define upper measure-theoretical metric mean dimension of  $\mu$  as

$$\overline{mdim}_M(T, \mathcal{X}, \mu) = \limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}}.$$

Similarly, one can define lower measure-theoretical metric mean dimension  $\underline{mdim}_{M}(T, \mathcal{X}, \mu)$  by replacing  $\limsup_{\epsilon \to 0}$  by  $\liminf_{\epsilon \to 0}$ . If  $\underline{mdim}_{M}(T, \mathcal{X}, \mu)$  $= \overline{mdim}M(T, \mathcal{X}, \mu)$ , we call the common value measure-theoretical metric mean dimension of  $\mu$ . Given  $\mu \in E(\mathcal{X}, T)$ , such candidates can be  $\overline{h}_{\mu}^{K}(T, \epsilon, \delta)$ ,  $\underline{h}_{\mu}^{K}(T, \epsilon, \delta)$ ,  $\overline{h}_{\mu}^{BK}(T, \epsilon)$ ,  $\underline{h}_{\mu}^{BK}(T, \epsilon)$   $R_{\mu,L^{\infty}}(\epsilon)$ , readers can turn to [YCZ22, Subsection 2.2] for more candidates.

The quantity (or we refer to "speed")  $\overline{mdim}(T, \mathcal{X}, \mu)$  can be interpreted as how fast the candidate  $F(\mu, \epsilon)$  approximate the (infinite) measure-theoretical entropy  $h_{\mu}(T)$  as  $\epsilon \to 0$ . Namely, when  $\epsilon > 0$  is sufficiently small, we may approximate  $F(\mu, \epsilon)$  as

$$F(\mu, \epsilon) \approx \overline{mdim}_M(T, \mathcal{X}, \mu) \log \frac{1}{\epsilon}.$$

**Definition 3.7.** Let  $(\mathcal{X}, T)$  be a TDS and  $d \in \mathscr{D}(\mathcal{X})$ . Given a candidate  $F(\mu, \epsilon)$  satisfying

$$\overline{\mathrm{mdim}}_M(T,\mathcal{X},d) = \sup_{\mu \in M(\mathcal{X},T)} \overline{mdim}_M(T,\mathcal{X},\mu),$$

and we call  $\mu$  a maximal upper (resp. lower) metric mean dimension measure if  $\overline{\mathrm{mdim}}_M(T, \mathcal{X}, d) = \overline{\mathrm{mdim}}_M(T, \mathcal{X}, \mu)$  (resp.  $\underline{\mathrm{mdim}}_M(T, \mathcal{X}, d) = \overline{\mathrm{mdim}}_M(T, \mu, \mathcal{X}, \mu)$ ). The set of all maximal upper (resp. lower) metric mean dimension measures is denoted by  $\overline{M}_{max}(T, \mathcal{X}, d)$  (resp.  $\underline{M}_{max}(T, \mathcal{X}, d)$ ).

Obviously,  $\overline{M}_{max}(T, \mathcal{X}, d)$  and  $\underline{M}_{max}(T, \mathcal{X}, d)$  depend on the metric and the candidate that we choose.

**Proposition 3.8.** Let  $(\mathcal{X}, T)$  be a TDS and  $d \in \mathcal{D}(\mathcal{X})$ . Given a candidate  $F(\mu, \epsilon)$  satisfying

$$\overline{\mathrm{mdim}}_M(T,\mathcal{X},d) = \sup_{\mu \in M(\mathcal{X},T)} \limsup_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}}$$

and

$$\underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \sup_{\mu \in M(\mathcal{X},T)} \liminf_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}}.$$

Then the following statements hold

- (1) If  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d) = \overline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d)$ , then  $\underline{M}_{max}(T, \mathcal{X}, d) \subset$  $\overline{M}_{max}(T, \mathcal{X}, d).$
- (2) If for any  $\mu \in \overline{M}_{max}(T, \mathcal{X}, d)$  satisfies  $\limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} = \liminf_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}}$ , then  $\underline{M}_{max}(T, \mathcal{X}, d) \supset \overline{M}_{max}(T, \mathcal{X}, d).$
- (3) If for every fixed  $\epsilon > 0$ ,  $F(\mu, \epsilon)$  is a concave function on  $M(\mathcal{X}, T)$ and  $\underline{M}_{max}(T, \mathcal{X}, d) \neq \emptyset$ , then the set  $\underline{M}_{max}(T, \mathcal{X}, d)$  is a convex subset of  $M(\mathcal{X}, T)$ . Additionally, if  $\underline{\mathrm{mdim}}_M(T, \mathcal{X}, d) = \infty$ , then  $\underline{M}_{max}(T, \mathcal{X}, d) \neq \emptyset.$

*Proof.* (1) Without loss of generality, we assume that  $\underline{M}_{max}(T, \mathcal{X}, d) \neq d$  $\emptyset$ . Let  $\mu \in \underline{M}_{max}(T, \mathcal{X}, d)$ , then

$$\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d) = \liminf_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} \\ \leq \limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} \leq \overline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d).$$

This implies that  $\underline{M}_{max}(T, \mathcal{X}, d) \subset \overline{M}_{max}(T, \mathcal{X}, d)$ . (2)Let  $\mu \in \overline{M}_{max}(T, \mathcal{X}, d)$  with  $\limsup_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}} = \liminf_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}}$ , then

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}} = \liminf_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}} \le \underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) \le \overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d),$$

which yields that  $\underline{M}_{max}(T, \mathcal{X}, d) \supset \overline{M}_{max}(T, \mathcal{X}, d)$ . (3) Let  $\underline{mdim}_{M}(T, \mathcal{X}, \mu) = \liminf_{\epsilon \to 0} \frac{F(\mu, \epsilon)}{\log \frac{1}{\epsilon}}$ . Since  $F(\mu, \epsilon)$  is a concave function on  $M(\mathcal{X}, T)$  for any fixed  $\epsilon > 0$ , then  $\underline{mdim}_{M}(T, \mathcal{X}, \mu)$  is also concave. Let  $\mu_1, \mu_2 \in \underline{M}_{max}(T, \mathcal{X}, d)$  and  $p \in [0, 1]$ , this yields that  $\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, d) = p\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, \mu_{1}) + (1-p)\underline{\mathrm{mdim}}_{M}(T, \mathcal{X}, \mu_{2})$  $< mdim_M(T, \mathcal{X}, p\mu_1 + (1-p)\mu_2) < mdim_M(T, \mathcal{X}, d).$ 

It follows that  $p\mu_1 + (1-p)\mu_2 \in \underline{M}_{max}(T, \mathcal{X}, d)$ , which shows  $\underline{M}_{max}(T, \mathcal{X}, d)$ is convex.

If  $\overline{\mathrm{mdim}}_M(T,\mathcal{X},d) = \infty$ , then for each  $n \in \mathbb{N}$ , we can choose  $\mu_n \in M(\mathcal{X},T)$  such that  $\underline{mdim}_M(T,\mathcal{X},\mu_n) > 2^n$ . Set  $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_n = \sum_{n=1}^{N} \frac{1}{2^n} \mu_n + \frac{1}{2^N} \nu_N$  for every  $N \in \mathbb{N}$ , where  $\nu_N \in M(\mathcal{X},T)$ . Using the concavity of  $\underline{mdim}_M(T,\mathcal{X},\mu)$  with respect to  $\mu$ , we have

$$\underline{mdim}_{M}(T, \mathcal{X}, \mu) \geq \sum_{n=1}^{N} \frac{1}{2^{n}} \underline{mdim}_{M}(T, \mathcal{X}, \mu_{n}) > N$$

Letting  $N \to \infty$  gives  $\underline{mdim}_M(T, \mathcal{X}, \mu) = \infty$ , which shows that  $\underline{M}_{max}(T, \mathcal{X}, d) \neq \emptyset$ .

We finally end up this paper with a question as follows.

Question 1 For every topological dynamical system  $(\mathcal{X}, T)$ , can we choose proper metric d and proper candidate  $F(\mu, \epsilon)$  such that there exists  $\mu \in M(\mathcal{X}, T)$  (or  $E(\mathcal{X}, T)$ ) satisfying

$$\overline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \limsup_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}}$$
$$\underline{\mathrm{mdim}}_{M}(T,\mathcal{X},d) = \liminf_{\epsilon \to 0} \frac{F(\mu,\epsilon)}{\log \frac{1}{\epsilon}}?$$

In other words, such a metric d and  $\mu$  have the same speed that respectively approximate infinite topological entropy and infinite measuretheoretical entropy.

#### Acknowledgement

The work was supported by the National Natural Science Foundation of China (Nos.12071222 and 11971236), China Postdoctoral Science Foundation (No.2016M591873), and China Postdoctoral Science Special Foundation (No.2017T100384). The work was also funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. We would like to express our gratitude to Tianyuan Mathematical Center in Southwest China(11826102), Sichuan University and Southwest Jiaotong University for their support and hospitality.

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