# Group Properties of Polar Codes for Automorphism Ensemble Decoding

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Abstract-In this paper, we propose an analysis of the automorphism group of polar codes, with the scope of designing codes tailored for automorphism ensemble (AE) decoding. We prove the equivalence between the notion of decreasing monomial codes and the universal partial order (UPO) framework for the description of polar codes. Then, we analyze the algebraic properties of the affine automorphisms group of polar codes, providing a novel description of its structure and proposing a classification of automorphisms providing the same results under permutation decoding. Finally, we propose a method to list all the automorphisms that may lead to different candidates under AE decoding; by introducing the concept of redundant automorphisms, we find the maximum number of permutations providing possibly different codeword candidates under AE-SC, proposing a method to list all of them. A numerical analysis of the error correction performance of AE algorithm for the decoding of polar codes concludes the paper.

*Index Terms*—Polar codes, monomial codes, permutation decoding, AE decoding, automorphisms groups.

#### I. INTRODUCTION

Polar codes [1] are a class of linear block codes relying on the phenomenon of channel polarization. Under successive cancellation (SC) decoding, they can provably achieve capacity of binary memoryless symmetric channels for infinite block length. However, in the short length regime, the performance of polar codes under SC decoding is far from state-of-the-art channel codes. To improve their error correction capabilities, the use of a list decoder based on SC scheduling, termed as SC list (SCL) decoding, has been proposed in [2]. The concatenation of a cyclic redundancy check (CRC) code to the polar code permits to greatly improve its error correction performance [3], making the resulting CRC-aided SCL (CA-SCL) decoding algorithm the de-facto standard decoder for polar codes adopted in 5G standard [4]. In order to avoid the extra decoding delay due to information exchange among parallel SC decoders in hardware implementation of CA-SCL decoders, permutation-based decoders have been proposed in [5] for SC; a similar approach was proposed in [6] for belief propagation (BP) and in [7] for soft cancellation (SCAN). In a permutation-based decoder, M instances of the same decoder are run in parallel on permuted factor graphs of the code, or alternatively on permuted versions of the received signal. Even if the decoding delay is reduced, the error correction performance gain of permutation-based decoders is quite poor due to the alteration of the frozen set caused by the permutation. Research towards permutations not altering the frozen set were carried out [8]; such permutations are the *automorphisms* of the code and form the group of permutations mapping a codeword into another codeword. This new decoding approach, i.e. the use of automorphisms in a permutation decoder, is referred to as *automorphism ensemble* (AE) decoding.

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Given their affinity with polar codes, the analysis of the automorphisms group of Reed-Muller (RM) codes is providing a guidance for this search. In fact, the automorphism group of binary Reed-Muller code is known to be the general affine group [9], and an AE decoder for RM codes has been proposed in [10] using BP as component decoder. The boolean code nature of RM codes permitted authors in [11] to propose a monomial code description of polar codes. Through this definition, in the same papers the authors proved that the group of lower-triangular affine (LTA) transformations form a subgroup of the automorphisms group of polar codes. However, it has been proved in [10] that LTA transformations commute with SC decoding, in the sense that LTA transformations do not alter the result of SC decoding process; this property leads to no gain under AE decoding when these automorphisms are used. Fortunately, the LTA transformation group is not always the full automorphisms group of polar codes [12]; if carefully designed, a polar code exhibits a richer automorphisms group. In [13], [14], the complete affine automorphisms group of a polar code is proved to be the block-lower-triangular affine (BLTA) group. Authors in [12], [13] showed that automorphisms in BLTA group (not belonging to LTA) can be successfully used in an AE decoder; in practice, these automorphisms are not absorbed by the SC decoder. Unfortunately, the group of affine automorphisms of polar codes asymptotically is not much bigger than LTA [15]. These discoveries paved the way for the analysis of the automorphism group of polar codes, and in general on the struggle of constructing polar codes having "good" automorphism groups, i.e. a set of automorphisms to be used in an AE decoder. Author in [16] followed another approach, focusing on the use of polar subcodes in conjunction with a peculiar choice of the permutation set to design polar codes for AE decoding. However, this approach is less systematic than the analysis of the automorphism group of the polar code, leading to less predictable gains.

In this paper, we propose an analysis of the automorphisms group of polar codes, with the scope of designing codes exhibiting good error correction performance under AE decod-

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ing. To begin with, in Section II we delve into the monomial codes description of polar codes proposed in [11], proving the equivalence between the notion of decreasing monomial codes and the universal partial order (UPO) framework proposed in [17] for the description of polar codes. This parallelism will allow us to analyze the algebraic properties of the affine automorphisms group of polar codes in Section III, namely a sub-group of the complete automorphisms group of polar codes whose elements can be described through affine transformations. Thanks to this analysis, we will provide a novel proof of the structure of the affine automorphism group of polar codes, that is different from the ones provided in [13], [14] and is connected to the UPO framework. Next, in Section IV we study the nature of AE decoders introducing the notion of *decoder equivalence*, namely a classification of automorphism providing the same results under permutation decoding. Thanks to this new formalism, we prove that LTA is not always the complete SC absorption group, namely that a larger set of automorphisms may be absorbed under SC decoding. This result is an extension of our preliminary analysis of decoder equivalence published in [18]. We provide an alternative proof of a very recent result presented in [19], namely that the complete SC absorption group has a BLTA structure. Finally, we propose a method to list all the automorphisms that may lead to different candidates under permutation decoding; by introducing the concept of redundant automorphisms, we find the maximum number of permutations providing possibly different codeword candidates under AE-SC. Our method permits to easily list all these automorphisms, greatly simplifying the search for automorphisms to be used in AE-SC decoding. All the results presented in the paper are correlated with examples to guide the reader in the process of constructing and using automorphisms for AE decoding of polar codes. Section V, including a numerical analysis of the performance of AE algorithm for the decoding of polar codes, concludes the paper.

#### II. POLAR CODES AND MONOMIAL CODES

In this section we analyze the parallelism between polar codes and decreasing monomial codes. Monomial codes have been introduced in [11] as a family of codes including polar and RM codes. In the same paper, decreasing monomial codes are introduced to provide a monomial description of polar codes. Through this description, authors in [11] were able to prove that lower triangular affine (LTA) transformations form a sub-group of the automorphisms group of polar codes. In this section, we extend the results in [11] proving the equivalence between the notion of *decreasing monomial codes* and the universal partial order (UPO) framework proposed in [17]. UPO represents a method to partially sort virtual polarized channels independently of the actual channel used for the transmission. Due to the structure of the polarization phenomenon, some of the virtual channels are inherently better than others, and the UPO framework is able to catch this nuance and creates a structure of virtual channels. To the best of our knowledge, this is the first time that this equivalence is proved.

# A. Polar Codes

To begin with, we provide a definition of polar code. In the following, the binary field with elements  $\{0, 1\}$  is denoted by  $\mathbb{F}_2$ , while the set of non-negative integers smaller than N is written as  $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$ .

**Definition 1** (Polar codes). A polar code of length  $N = 2^n$ and dimension K is defined by a transformation matrix  $T_n = T_2^{\otimes n}$ , where  $T_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ , and a frozen set  $\mathcal{F} \subset \mathbb{Z}_{2^n}$ , or inversely by an information set  $\mathcal{I} = \mathcal{F}^c = \mathbb{Z}_{2^n} \setminus \mathcal{F}$ . Encoding is performed as  $x = u \cdot T_N$ , where  $u_{\mathcal{F}} = 0$ ; the code is then given by

$$\mathcal{C} = \{ x = u \cdot T_N : u \in \mathbb{F}_2^N, u_{\mathcal{F}} = 0 \}.$$
(1)

Elements of the information set  $\mathcal{I}$  are usually chosen according to their reliability; in practice, virtual channels originated by the transformation matrix are sorted in reliability order, and the indices of the K most reliable ones form the information set of the code. The reliability of virtual channels depend on the characteristics of the transmission channel, forcing the code designer to recalculate the reliability order every time the channel parameters change. However, even if the polarization effect is non-universal, the structure of transformation matrix  $T_N$  permits to define an *universal partial order* (UPO) among virtual channels. In practice, we say that  $i \leq j$  if virtual channel represented by index *i* is always weaker (i.e. less reliable) than the channel represented by index *j*, independently of the original channel [17]. A polar code fulfills the UPO if this partial reliability order is followed.

**Definition 2** (Universal partial order). A polar code is said to fulfill the universal partial order (UPO) if  $\forall i, j \in \mathbb{Z}_{2^n}$  such that  $i \leq j$  then  $i \in \mathcal{I} \Rightarrow j \in \mathcal{I}$ .

Figure 1 provides a graphical representation of the relations among the virtual channels for a polar code of length N = 32. A node in the diagram represents a virtual channel, while a directed edge represents the order relation between two virtual channels. Nodes that are connected by a directed path can be compared through the UPO, while the order relation between unconnected nodes depends on the transmission channel and should be verified. As an example, since nodes 5 and 9 are connected, we know that  $5 \leq 9$  and virtual channel 5 is always weaker than virtual channel 9; on the other hand, nodes 6 and 9 are not connected, and hence the order between these two virtual channels depends on the original channels. A polar code fulfilling the UPO can be described through few generators composing the minimum information set  $\mathcal{I}_{min}$  [13] as

$$\mathcal{I} = \bigcup_{j \in \mathcal{I}_{min}} \{ i \in [N], j \le i \}.$$
 (2)

It is worth noticing that a polar code designed to be decoded under SCL may not fulfill the UPO; the universal reliability sequence standardized for 5G is an example of this exception [4]. However, in the following we will focus on polar codes fulfilling the UPO, since we are interested in AE decoders having SC component decoders.

The UPO is induced by noticing that the binary expansion of the virtual channel index actually provides the sequence



Fig. 1: Hasse diagram of UPO for N = 32; integers and their binary expansions are depicted in black, corresponding monomials are depicted in blue.

of channel modifications, where a 0 represents a channel degradation and a 1 represents a channel upgrade.

Definition 3 (Binary expansion). The canonical binary expansion connecting  $\mathbb{F}_2^n$  and  $\mathbb{Z}_{2^n}$ ,



maps an integer v to its binary expansion  $\hat{v}$  with least significant bit (LSB) on the left, namely with  $v_0$  representing the LSB of v; zero padding is performed if the number of bits representing v is smaller than n, namely if  $v < 2^{n-1}$ .

So, given two virtual channel indices  $i, j \in \mathbb{Z}_{2^n}$  with i <j, if  $\hat{i}$  and  $\hat{j}$  represent their binary expansions, we have that the corresponding nodes in the Hasse diagram are directly connected, and hence  $i \leq j$ , when:

- if  $\hat{i}$  and  $\hat{j}$  only differ for a single entry t, where  $\hat{i}_t = 0$
- and  $\hat{j}_t = 1$ ; if  $\hat{i}$  and  $\hat{j}$  only differ for two consecutive entries t and t+1, where  $\hat{i}_t = 1$ ,  $\hat{i}_{t+i} = 0$  and  $\hat{j}_t = 0$ ,  $\hat{j}_{t+1} = 1$ .

These rules are iterated to generate the UPO among the virtual channels, and hence among the integers representing these channels.

# B. Monomial Codes

Monomial codes are a family of codes of length  $N = 2^n$ defined by evaluations of boolean functions of n variables. A boolean function can be described as a map from a string of *n* bits to a single bit defined by a specific truth table; this map can always be rewritten as a polynomial f in the polynomial ring in n indeterminates over  $\mathbb{F}_2$ , i.e.  $f \in \mathbb{F}_2[V_0, \ldots, V_{n-1}] \triangleq$  $\mathbb{F}_2^{[n]}$ . Since an element of  $\mathbb{F}_2^n$  can be interpreted as the binary representation of an integer, boolean functions can be seen as maps associating a bit to each integer in  $\mathbb{Z}_{2^n}$ . Every boolean function can be written as a linear combination of monomials in variables  $V_i$ , forming a basis for the space of boolean functions. A monomial in  $V_i$  is a product of powers of variables  $V_i$ , with  $0 \le i < n$ , having non-negative exponents,

e.g.  $V_0^2 V_2$ . However, since  $V_i^2 = V_i$  in  $\mathbb{F}_2^{[n]}$ , variables either appear or not in monomials. Similarly, *negative monomials*, namely monomials in  $\overline{V}_i = \neg V_i = (1 \oplus V_i)$ , form a basis of the same space; in the following, we will use negative monomials to smoothly map polar codes to monomial codes, and we call  $\mathcal{M}^{[n]} \subset \mathbb{F}_2^{[n]}$  the set of monomials in n indeterminates over  $\mathbb{F}_2$ . Given their importance in the definition of boolean functions, we introduce here a notation to connect each monomial in  $\mathcal{M}^{[n]}$  to an integer in  $\mathbb{Z}_{2^n}$ .

Definition 4 (Monomials canonical map). The canonical map connecting  $\mathbb{Z}_{2^n}$  and  $\mathcal{M}^{[n]}$ ,

$$\mathbb{Z}_{2^n} \leftrightarrow \mathcal{M}^{[n]}$$
$$t \leftrightarrow m_t,$$

connects integer  $t \in \mathbb{Z}_{2^n}$ , where for some  $Q \subset \mathbb{Z}_n$ 

$$t = \sum_{i \in Q} 2^i, \tag{3}$$

to monomial  $m_t \in \mathcal{M}^{[n]}$ , defined by

$$m_t = \prod_{i \notin Q} \bar{V}_i \tag{4}$$

In practice, the monomials canonical map transforms a monomial into an integer having zeroes in its binary expansion corresponding to the variable indices, and ones elsewhere. Equipped with this map, we can now define the monomial evaluation function.

Definition 5 (Evaluation function). The evaluation function eval :  $\mathbb{F}_2^{[n]} \to \mathbb{F}_2^N$  checks the output of f for all elements of  $\mathbb{F}_2^n$  in increasing order; in practice,

$$x^{(f)} = \operatorname{eval}(f) \triangleq (f(\widehat{0}), f(\widehat{1}), \dots, f(\widehat{N-1})).$$
(5)

As a consequence, every boolean function f can be naturally associated to a binary vector  $x^{(f)}$  of length N through the evaluate function eval(f). An example of this construction can be found in Appendix A. Given that high degree monomials are defined as products of degree-one monomials, namely single variables, knowing the evaluations of single variables it is easy to calculate the evaluation of high-degree monomials as xor of the associated binary strings. This property permits to generate a family of block codes based on the evaluation of monomial functions.

**Definition 6** (Monomial codes). A monomial code of length  $N = 2^n$  and dimension K is given by the evaluations of linear combinations of the monomials included in a generating monomial set  $\mathcal{G} \subset \mathcal{M}^{[n]}$ , with  $|\mathcal{G}| = K$ .

Given n variables, it is possible to form  $2^n$  different monomials, of which  $\binom{n}{r}$  are of degree r. A monomial code of length  $N = 2^n$  and dimension K is generated by picking K monomials out of the N. Reed-Muller codes are monomial codes defined by picking all monomials up to a certain degree: in particular,  $\mathcal{R}(r, n)$  code is generated by all monomials of degree smaller or equal to r. The family of monomial codes being so rich, in order to introduce a parallelism between these codes and the polar codes, a notion of partial order among monomials is needed. Authors in [11] propose such a partial order. Given two monomials  $m_{t_1}, m_{t_2} \in \mathcal{G}$  of degrees  $s_1$  and  $s_2$  respectively, where  $m_{t_1} = \bar{V}_{i_0} \cdot \ldots \cdot \bar{V}_{i_{s_1-1}}$  and  $m_{t_2} = \bar{V}_{j_0} \cdot \ldots \cdot \bar{V}_{j_{s_2-1}}$ , we say that  $m_{t_1} \leq m_{t_2}$ :

- when  $s_1 = s_2 = s$ , then if and only if  $i_l \leq j_l$  for all  $l = 0, \ldots, s 1$ .
- when  $s_1 < s_2$ , then if and only if there exists a monomial  $m_{t'}$  such that  $m_{t'}|m_{t_2}$ ,  $deg(m_{t_1}) = deg(m_{t'})$  and  $m_{t_1} \le m_{t'}$ .

It is worth noticing that we used the same symbol to define the partial order relations for integers and for monomials; in fact, we will see that these two orders are equivalent. Equipped with the notion of partial order among monomials, authors in [11] define the family of *decreasing monomial codes* as the monomial codes for which for every monomial in  $\mathcal{G}$ , all its sub-monomial factors are also included in  $\mathcal{G}$ .

**Definition 7** (Decreasing monomial codes). A monomial code is *decreasing* if for every monomial  $m_{t_1}, m_{t_2} \in \mathcal{M}^{[n]}$  such that  $m_{t_1} \leq m_{t_2}$  then  $m_{t_2} \in \mathcal{G} \Rightarrow m_{t_1} \in \mathcal{G}$ .

## C. Monomial and Polar Codes Equivalence

TABLE I: Canonical map between integers and monomials for n = 3.

degree	monomial	evaluation	row of $T_8$	expansion
0	$m_7 = 1$	111111111	7	111
	$m_6 = V_0$	10101010	6	011
1	$m_5 = \overline{V}_1$	11001100	5	101
	$m_3 = \overline{V}_2$	11110000	3	110
	$m_4 = \bar{V}_0 \bar{V}_1$	10001000	4	001
2	$m_2 = \bar{V}_0 \bar{V}_2$	10100000	2	010
	$m_1 = \bar{V}_1 \bar{V}_2$	11000000	1	100
3	$m_0 = \bar{V}_0 \bar{V}_1 \bar{V}_2$	10000000	0	000

Polar codes can be described as monomial codes. In fact, the kernel matrix  $T_2$  can be seen as the evaluation of monomials in  $\mathbb{F}_2[\bar{V}_0]$ ;  $\operatorname{eval}(\bar{V}_0) = [1, 0]$  represents the first row of  $T_2$ , while  $\operatorname{eval}(1) = [1, 1]$  represents its second row. An example of this parallelism is provided in Table I for n = 3. This parallelism can be extended to polar codes of any length  $N = 2^n$ , such that each monomial in  $\mathcal{M}^{[n]}$  is represented by an integer in  $\mathbb{Z}_{2^n}$  through the monomials canonical map. This map permits

to connect row t of transformation matrix  $T = T_2^{\otimes n}$  to a unique monomial  $m_t$ ; Then, polar codes can be equivalently defined in terms of information set  $\mathcal{I}$  or generating monomial set  $\mathcal{G}$ , since  $m_t \in \mathcal{G} \Leftrightarrow t \in \mathcal{I}$ . To summarize, polar codes can be seen as monomial codes where the generating monomials are chosen according to polarization effect.

In the following, we prove the main result of this section, namely that UPO property for polar codes is equivalent to decreasing property for monomial codes. As a consequence, every polar code fulfilling the UPO is also a decreasing monomial code, and vice versa. The fact that UPO is a sufficient condition for the generation of decreasing monomial codes has been proved in [11], while, for the best of our knowledge, this is the first time that it is proven that it is also a necessary condition. Before proving the main theorem of the section, we need to prove the equivalence between the partial order defined on integers and the partial order defined on monomials.

# **Lemma 1.** For every $a, b \in \mathbb{Z}_{2^n}$ , then $b \leq a \Leftrightarrow m_a \leq m_b$

**Proof.** Sufficient condition: Given  $a, b \in \mathbb{Z}_{2^n}$  such that  $b \leq a$ , we want to prove that  $m_a \leq m_b$  by checking if the two conditions for the partial ordering of the monomials are satisfied. If we call  $HW(\hat{t})$  the Hamming weight of the binary expansion of integer t, then  $HW(\hat{t}) = n - deg(m_t)$  by Definition 4, and the degree conditions of monomial partial orders can be rewritten as conditions on the Hamming weights of binary expansions of integers.

If  $HW(\hat{a}) = HW(b) = n-s$ , then  $deg(m_a) = deg(m_b) = s$ ; if we call  $\bar{S}$  the set of variable indices composing monomial  $m_t$ , namely  $m_t = \prod_{i \in \bar{S}} \bar{V}_i \in \mathcal{M}^{[n]}$ , then  $m_a \leq m_b$  if and only if  $i_l^{(a)} \leq j_l^{(b)}$  for all  $l = 0, \ldots, s-1$ , where  $i_l^{(t)}$  is the l-th bit in the binary expansion of integer t. Since  $b \leq a$  and  $HW(\hat{a}) = HW(\hat{b})$ , then it is possible to create a chain of integers  $t_0, \ldots, t_r$  such that  $b = t_r \leq t_{r-1} \leq \ldots \leq t_1 \leq t_0 = a$  such that they all have the same Hamming weight and each couple of subsequent integers in the chain only differ for two consecutive entries. By definition, we have that  $m_{t_i} \leq m_{t_{i+1}}$  for the monomials partial order definition, and then the chain can be rewritten as  $m_a = m_{t_0} \leq m_{t_1} \leq \ldots \leq m_{t_{r-1}} \leq m_{t_r-1} \leq m_{t_r} = m_b$ .

Alternatively, if  $HW(\hat{b}) < HW(\hat{a})$ , than there is at least one chain of integers such that  $b = t_r \leq t_{r-1} \leq \ldots \leq$  $t_1 \leq t_0 = a$ , where each couple of subsequent integers only differ for a single or two consecutive entries. Now, let us focus on three elements of the chain  $t_{i+1} \leq t_i \leq t_{i-1}$ such that  $t_{i+1}$  and  $t_i$  differ for a single entry, while  $t_i$  and  $t_{i-1}$  differ for two consecutive entries; in this case, there exists another integer  $t'_i$  such that  $t_{i+1} \leq t'_i \leq t_{i-1}$  and  $t_{i+1}$  and  $t'_i$  differ for two consecutive entries, while  $t'_i$  and  $t_{i-1}$  differ for a single entry. In practice, it is always possible to invert the application of two different rules in the chain. As a consequence, it is always possible to create a chain  $b = t'_r \leq t'_{r-1} \leq \ldots \leq t'_1 \leq t'_0 = a$  such that integers from  $b = t'_r$  to a certain integer  $t'_c$  differ for a single entry, while from  $t'_c$  to  $t'_0 = a$  two consecutive integers differ for two consecutive entries. Then, from this chain we extract element  $t'_c$  with  $b \leq t'_c \leq a$ ; by construction, the set of the indices of the positions of zeroes in the binary expansion of  $t'_c$  is a subset of the same set of b, and hence  $m_{t'_c}|m_b$ . Moreover,  $HW(t'_c) = HW(\hat{a})$ , and then  $deg(m_{t'_c}) = deg(m_a)$  and thus  $m_a \leq m_b$ .

Sufficient condition: Given  $m_a, m_b \in \mathcal{M}^{[n]}$  such that  $m_a \leq$  $m_b$ , we want to prove that  $b \leq a$  by checking if the two conditions for the partial ordering of the integers are satisfied. If  $deg(m_a) = deg(m_b)$ , we can create a chain of intermediate monomials  $m_a = m_{t_r} \leq m_{t_{r-1}} \leq \ldots \leq m_{t_1} \leq m_{t_0} = m_b$ such that they all have the same degree and each couple of subsequent monomials in the chain only differ by a variable; in other words, there exists a variables swap chain passing from  $m_a$  to  $m_b$  where each step of the chain can be sorted according to the partial order. This monomials chain can be mirrored to the corresponding integers chain  $b = t_0 \leq t_1 \leq \ldots \leq t_{r-1} \leq$  $t_r = a$ , where consecutive integers differ for two consecutive entries, and thus  $b \leq a$ . Alternatively, If  $deg(m_a) < deg(m_b)$ , then there exists a monomial  $m_t$  dividing  $m_b$  and having the same degree of  $m_a$  such that  $m_a \leq m_t \leq m_b$ . Then,  $t \leq a$ for the previous case, and  $b \leq t$ . 

Equipped with Lemma 1, we can now prove the main result of this section, namely the equivalence between UPO polar codes and decreasing monomial codes.

**Theorem 1.** A polar code design is compliant with the UPO framework if and only if it is a decreasing monomial code.

*Proof.* First, we assume that the information set of the polar code is compliant with the UPO framework. We need to prove that, if  $m_t \in \mathcal{G}$ , then also  $m_{t'} \in \mathcal{G}$  for every  $m_{t'} \leq m_t$ . According to Lemma 1,  $t \leq t'$ , and since  $t \in \mathcal{I}$  we have that also  $t' \in \mathcal{I}$  for the UPO hypothesis, and hence  $m_{t'} \in \mathcal{G}$ . Second, we assume the code to be decreasing monomial; now we need to prove that for every  $t \in \mathcal{I}$ , then also  $m_{t'} \in \mathcal{G}$  for every  $t \leq t'$ . Again, Lemma 1 says that  $m_{t'} \leq m_t$ , and since  $m_t \in \mathcal{G}$  then also  $m_{t'} \in \mathcal{G}$  for the decreasing monomial hypothesis, and hence  $t' \in \mathcal{I}$ .

#### **III. POLAR CODE AUTOMORPHISMS**

Automorphisms are permutations of code bit positions that are invariant to the code, namely that map codewords into codewords. The analysis of the automorphism group of a code permits to discover hidden symmetries of the codewords, and can be used to find new properties of the code. In this paper, our study of the automorphism group of polar codes is driven by the will of improving AE decoding algorithms for this family of codes. In this section, we first revise permutations defined by general affine transforms and then discuss properties of such permutations that are automorphisms of polar codes. Moreover, we provide all the tools to help the reader to map an affine transformation to the related code bit permutation, by explicitly showing how to pass from one to the other. Equipped with this map, it will be easier for the reader to understand the main results of following sections and to reproduce the results presented in Section V.

# A. Permutations as affine transformations

**Definition 8** (Permutation). A permutation  $\pi$  over the set  $\mathbb{Z}_{2^n}$ ,

$$\pi: \mathbb{Z}_{2^n} \to \mathbb{Z}_{2^n}$$
$$i \mapsto \pi(i),$$

is a bijection of  $\mathbb{Z}_{2^n}$  onto itself.

The *trivial (identity) permutation* is written as 1 and maps every integer to itself. Permutations can be applied to vectors in different ways; in the following, for permutations of vectors we will use the *functional passive notation*, where the element in position *i* is replaced by element in position  $\pi(i)$  after the permutation and permutations are concatenated giving priority to the right [20].

**Definition 9** (Vector permutation). Given a permutation  $\pi$ , the vector  $y = (y_0, y_1, \dots, y_{2^n-1})$  is called the permuted vector of vector  $x = (x_0, x_1, \dots, x_{2^n-1})$ , if and only if

$$y_i = x_{\pi(i)}$$

for all  $i \in \mathbb{Z}_{2^n}$ . For convenience we may write  $y = \pi(x)$  and call y the permutation of x.

According to the introduced notation, the concatenation of two permutations  $\pi_1$  and  $\pi_2$  is written as  $\pi_2 \circ \pi_1 = \pi_2 \pi_1$ , to be applied from right to left, i.e.,  $\pi_1$  first and  $\pi_2$  second. To apply this to vectors, assume three vectors  $x, y = \pi_1(x)$ , and  $z = \pi_2(y) = \pi_2(\pi_1(x))$ ; then  $y_i = x_{\pi_1(i)}$  and  $z_j = y_{\pi_2(j)}$  from Definition 9, and  $z_j = x_{\pi_1(\pi_2(j))}$  by substituting  $i = \pi_1(j)$ . Next, we define the group of affine transformations over binary vector, and we show how these transformations are related to permutations.

**Definition 10** (Affine transformations). The General Affine (GA) group GA(n) is the group of affine transformations of binary vectors  $v \in \mathbb{F}_2^n$ ,

$$T_{(A,b)}: \mathbb{F}_2^n \to \mathbb{F}_2^n$$
$$v \mapsto Av + b.$$

with invertible matrices  $A \in \mathbb{F}_2^{n \times n}$  and arbitrary vectors  $b \in \mathbb{F}_2^n$ . Each element  $T_{(A,b)}$  of this group is uniquely identified by a matrix-vector pair (A, b).

Every affine transformation gives rise to a permutation, and it does this in natural way for codewords of monomial codes through Definition 3 as shown as follows.

**Definition 11** (GA permutations). The GA permutations group is the group of permutations over  $\mathbb{Z}_{2^n}$  defined by affine transformations as  $\pi_{(A,b)}(v) = \widetilde{T_{(A,b)}}\hat{v}$ ; the mapping between  $\mathbb{Z}_{2^n}$  and  $\mathbb{F}_2^n$  is given as

$$\begin{array}{cccc} \mathbb{Z}_{2^n} & \xrightarrow{\hat{\cdot}} & \mathbb{F}_2^n & v & \xrightarrow{\hat{\cdot}} & \hat{v} \\ & & & \downarrow^{\pi_{(A,b)}} & & \downarrow^{T_{(A,b)}} & & \downarrow^{\pi_{(A,b)}} & \downarrow^{T_{(A,b)}} \\ \mathbb{Z}_{2^n} & \xleftarrow{\hat{\cdot}} & \mathbb{F}_2^n & & \pi_{(A,b)}(v) & \xleftarrow{\hat{\cdot}} & A\hat{v} + b \end{array}$$

<sup>1</sup>The meaning of  $\pi$  becomes clear from the context.

It is worth noticing that the vice versa is not true, namely that not every permutation can be expressed as an affine transformation. Assuming a boolean function  $f \in \mathbb{F}_2^{[n]}$ , an affine transformation  $T_{(A,b)}$  can be applied to f, obtaining boolean function  $g = T_{(A,b)}(f)$  defined by

$$g(V) = f(A \cdot V + b)$$

It is possible to concatenate affine transformations; given  $g = T_{(A_1,b_1)}(f)$  and  $h = T_{(A_2,b_2)}(g)$ , we have that  $h = T_{(A,b)}(f) = T_{(A_2,b_2)} \circ T_{(A_1,b_1)}(f)$ . By definition, we have that

$$h(V) = g(A_2 \cdot V + b_2) = f(A_1(A_2 \cdot V + b_2) + b_1) = (6)$$
  
=  $f(A_1A_2 \cdot V + A_1b_2 + b_1).$  (7)

As a consequence,  $T_{(A,b)}$  is defined by matrix  $A = A_1A_2$  and vector  $b = A_1b_2 + b_1$ . Note that the order is of the matrices is reversed, as compared to the order of the permutations.

Let us consider now  $g = T_{(A,b)}(f)$  and its evaluation  $x^{(g)} = eval(g) = (x_0^{(g)}, \ldots, x_{2^n-1}^{(g)})$ ; this represents a binary vector that is connected to  $x^{(f)}$  by a permutation as follows.

**Lemma 2.** Assume a GA transform  $T_{(A,b)}$  over  $\mathbb{F}_2^n$  and a boolean functions  $f \in \mathbb{F}_2^{[n]}$ , then

$$x^{(T_{(A,b)}(f))} = \pi_{(A,b)} \left(x^{(f)}\right),$$

where  $\pi_{(A,b)}$  follows Definition 11.

*Proof.* Let us consider boolean function  $g = T_{(A,b)}(f)$ ; by Definitions 5 and 9, we have that

$$x_{i}^{(T_{(A,b)}(f))} = x_{i}^{(g)} = g(\hat{i}) = f\left(A\hat{i} + b\right) =$$
(8)

$$= f\left(\widehat{\pi_{(A,b)}(i)}\right) = x_{\pi_{(A,b)}(i)}^{(f)}.$$
 (9)

 $\pi_2$ 

As a consequence, binary vector  $x^{(g)}$  is a permutation of  $x^{(f)}$ ; this permutation is the one induced by affine transformation  $T_{(A,b)}$ , as expressed in this scheme:

$$\begin{array}{c} f \xrightarrow{eval} x^{(f)} \\ \downarrow_{T_{(A,b)}} & \downarrow_{\pi_{(A,b)}} \\ g \xrightarrow{eval} x^{(g)} \end{array}$$

For convenient (though imprecise) notation we may simply write  $\pi_{A,b}(i) = Ai + b$ , presuming the equivalence between an integer *i* and its binary expansions  $\hat{i}$ ; in practice, we will apply affine transformations or the equivalent permutation on both vectors and boolean functions, the meaning becoming clear from the context. An example of this construction can be found in Appendix A.

Operating over binary vectors, Lemma 2 associates a codeword permutation to each GA transform, and we will refer to these as GA *permutations*; note, however, that not all permutations can be represented by GA transforms. The GA permutations group is obviously isomorphic to the group of affine transformations GA(n). For convenience we refer to GA(n) and the group of permutations simply as the group GA. The relation between GA transforms and permutations as given in Lemma 2 gives rise to the question how this translates to concatenation. This is answered in the following lemma.

**Lemma 3** (Concatenation of GA permutations). Given two GA permutations  $\pi_{(A_1,b_1)}$  and  $\pi_{(A_2,b_2)}$ , their concatenation

$$\pi_{(A,b)} = \pi_{(A_2,b_2)} \circ \pi_{(A_1,b_1)},$$

namely when  $\pi_{A_1,b_1}$  is applied first and  $\pi_{A_2,b_2}$  second, is the GA permutation  $T_{(A,b)}$  defined by

$$A = A_1 A_2$$
,  $b = A_1 b_2 + b_1$ .

*Proof.* As GA permutations form a group, their composition is obviously also a GA transform. The scheme of the concatenation is:



If we call  $\pi_j = \pi_{(A_j,b_j)}$  and  $T_j = T_{(A_j,b_j)}$  for j = 1, 2, by Lemma 2 we have that

$$\circ \pi_1(x_i^{(f)}) = x_{\pi_1(\pi_2(i))}^{(f)} = x_{\pi_2(i)}^{(T_1(f))} = x_{\pi_2(i)}^{(g)} =$$
(10)

$$=x_i^{(12(g))} = x_i^{(12(11(g)))}.$$
 (11)

Given the equivalence between affine transformations and affine permutations, in the following we may use  $\pi$  to define an affine transformation.

In addition to the group GA itself, we introduce several sub-groups of GA and the corresponding permutations. These sub-groups will be used in the following sections to prove various results concerning the automorphisms group of polar codes. The following sub-groups are defined by transformations  $T_{(A,b)}$ , where A and B assume a peculiar format:

- the General Linear group GL, for which b = 0;
- the *Lower-Triangular Affine group* LTA, for which A is lower-triangular;
- the Block-Lower-Triangular Affine group BLTA(S), for which A is block-lower-triangular with profile  $S = (s_1, s_2, ..., s_l)$  with  $s_1 + s_2 + ... + s_l = n$ , i.e. a block diagonal matrix having non-zero elements below the diagonal;
- the *Upper-Triangular Linear group* UTL, for which A is upper-triangular;
- the *Permutation Linear group* PL, for which A is a permutation matrix;
- the *Translation group* T, for which A is the identity matrix.

Similarly to GA we may use GL, LTA, BLTA, UTL, and PL to refer to the corresponding groups of permutations.

# B. The affine automorphisms group of polar codes

**Definition 12** (Automorphism). A permutation  $\pi$  is called an automorphism of code C if  $\pi(C) = C$ , i.e., if  $\pi(x) \in C$  for all  $x \in C$ . The set of automorphisms of a code C is denoted by Aut(C) and forms a group.

For monomial codes, the permutations from GA are of particular interest, since they represent linear operations on the variables. If a GA permutation is an automorphism, we call it a GA (or affine) automorphism. We further denote the *group of affine automorphisms* of a code C by

$$\mathcal{A} \triangleq \mathsf{GA} \cap \mathsf{Aut}(\mathcal{C}). \tag{12}$$

We consider binary (N, K) polar codes C of length  $N = 2^n$ and dimension K, having information set I and monomial set G and following UPO framework. In this section, we will prove that the affine automorphisms group of such a polar code is a BLTA group. This property has been proved in [13], [14]; in this paper, we provide an alternative proof based on the introduction of elementary permutations and on the algebraic structure of the BLTA group.

To begin with, we notice that GA permutations map monomials of  $\mathcal{G}$  to linear combinations of monomials in  $\mathcal{M}^{[n]}$ . A property of GA automorphisms is their capacity to map the generating monomial set into itself.

**Lemma 4.**  $\pi \in \mathcal{A}$  if and only if for every  $m_t \in \mathcal{G}$ , then for every  $m_{t_1}, \ldots, m_{t_s}$  such that

$$\pi(m_t) = m_{t_1} + \ldots + m_{t_s}, \tag{13}$$

we have that  $m_{t_i} \in \mathcal{G}$ .

*Proof.* This condition is obviously necessary: if all  $m_{t_i} \in \mathcal{G}$ , then also  $\pi(m_t) \in \mathcal{G}$  for every  $m_t \in \mathcal{G}$ . Conversely, if  $\pi \in \mathcal{A}$ , then also  $\pi(m_t) \in \mathcal{G}$ . However, in order for the polynomial  $m_{t_1} + \ldots + m_{t_s}$  to be included into  $\langle \mathcal{G} \rangle$ , all its addends must belong to  $\mathcal{G}$  because monomials form a base for the code space.

Equipped with these definitions, we now focus on the characterization of  $\mathcal{A}$ . To begin with, we prove that LTA  $\subseteq \mathcal{A}$  for any polar code, namely that all the LTA transformations are automorphisms; it is worth noticing that this property holds only for polar codes fulfilling the UPO framework. In order to prove it, we denote the row-addition elementary matrix by  $E_{(i,j)}$ , namely the  $n \times n$  square matrix with ones on the diagonal and an additional one at row *i* and column *j*. This matrix is associated to the elementary linear transformation

$$E_{(i,j)}: \overline{V}_i \to \overline{V}_i + \overline{V}_j + 1, \tag{14}$$

defined by permutation  $\epsilon_{(i,j)} = \pi_{(E_{(i,j)},0)}$ . This elementary matrix is of particular interest since every LTA transformation can be decomposed as the product of elementary linear transformations having their one below the diagonal, plus a translation.

**Lemma 5.** Every  $\pi_{(A,b)} \in LTA$  can be decomposed as

$$\pi_{(A,b)} = \epsilon_{(i_q,j_q)} \circ \ldots \circ \epsilon_{(i_1,j_1)} \circ \tau, \tag{15}$$

where  $\epsilon_{(i,j)}$  is an elementary linear transformation,  $\tau \in T$  and q is the number of nonzero entries of A below the diagonal.

*Proof.* To begin with, we prove that every lower triangular matrix A can be written as a product of q row-addition elementary matrices. To do it, we sort the nonzero entries of A below the diagonal from top to bottom and then from left to right: in practice, if  $a_{i_l,j_l}$  and  $a_{i_{l+1},j_{l+1}}$  are nonzero entries of A, then  $i_l \leq i_{l+1}$  and, if  $i_l = i_{l+1}$ , then  $j_l < j_{l+1}$ ; in this way, we have that

$$A = \prod_{l=1}^{q} E_{(i_l, j_l)}.$$
 (16)

In fact, given a matrix B, we have that  $B \cdot E_{(i,j)}$  is the matrix produced from B by adding column i to column j; we have that  $i_l > j_l$  since A is lower triangular, hence realizing the product from left to right results in adding at every step a nonzero entry in position  $(i_l, j_l)$ , resulting in matrix A. Next, if we define  $\tau = \pi_{I,b}$ , the lemma is proved.

When applied to monomials, however, these elementary transformations may lead to polynomials, making it difficult to connect monomial sets. In fact, given  $m_t \in \mathcal{M}^{[n]}$ , then

$$\epsilon_{(i,j)}(m_t) = \begin{cases} m_t & \text{if } i \notin Q \text{ or } i, j \in Q \\ m_t + m_{t'} + m_{t''} & \text{if } i \in Q \text{ and } j \in Q \end{cases}$$
(17)

where t' is obtained from t by swapping entries i and j of its binary expansion and t'' by adding a one in position i of the binary expansion of t. Lemma 4 shows that  $\epsilon_{(i,j)} \in \mathcal{A}$  if and only if for every  $m_t \in \mathcal{G}$ , then  $m_{t'}$  and  $m_{t''}$  obtained from (17) belong to  $\mathcal{G}$ . In order to prove the first main result of this section, we need to prove that we can focus our analysis only on elementary linear transformations, neglecting the effect of translations. The following lemma gives us this possibility.

**Lemma 6.** If C follows UPO framework, then  $T \subset A$ .

*Proof.* Translation  $\tau_i \in \mathsf{T}$  maps  $\bar{V}_i$  to  $\bar{V}_i + 1$ . Then, it maps each monomial in  $\mathcal{G}$  including  $\bar{V}_i$  into the sum of the monomial itself and the same monomial without  $\bar{V}_i$ ; since the last monomial is included in  $\mathcal{G}$  due to the UPO hypothesis, then  $\mathsf{T} \subset \mathcal{A}$ .

Since translations are always automorphisms, we can focus on automorphisms in GL, and more in details we can use elementary linear transformations to prove the following lemma, that is the first main result of this section.

# **Theorem 2.** LTA $\subseteq A$ if and only if C follows UPO framework.

*Proof.* To begin with, we show that this is a necessary condition. The matrix A of a given LTA transformation can be decomposed as a product of elementary matrices having their extra one below the diagonal. for any of these elementary transformations  $\epsilon_{(i,j)}$ , any  $m_t \in \mathcal{G}$  is transformed into a polynomial  $m_t + m_{t'} + m_{t''}$  (or remains the same). By definition,  $m_{t''} \leq m_t$ , and then, by UPO hypothesis, also  $m_{t''} \in \mathcal{G}$ ; moreover, since by construction i < j, also  $m_{t'} \leq m_t$ , and again  $m_{t'} \in \mathcal{G}$  by UPO hypothesis. Since all the monomials forming it belong to  $\mathcal{G}$ , then also

 $\epsilon_{(i,j)}(m_t) \in \mathcal{G}$  and  $\epsilon_{(i,j)} \in Aut(\mathcal{C})$  when i < j; since the product of automorphisms is still an automorphism, we have that LTA  $\subseteq \mathcal{A}$ ).

To show that it is a sufficient condition, let us proceed by absurd. We suppose C not following the UPO framework, namely there exists some  $a \in \mathcal{I}$  for which at least one integer  $b, a \leq b$ , such that  $b \notin \mathcal{I}$ . It is then possible to create a chain of integers  $a = t_0 \leq t_1 \leq \ldots \leq t_{r-1} \leq t_r = b$  such that the passage from an integer to the next one is performed by one of the UPO basic rules. By absurd hypothesis, there exists an index s such that  $t_s \in \mathcal{I}$  and  $t_{s+1} \notin \mathcal{I}$ . Then, it is possible to find two integers i, j, with i > j, such that  $\epsilon_{(i,j)}(m_{t_s}) = m_{t_{s+1}} + m_{t'}$ , either by taking the indices of the swapped entries if  $HW(t_s) = HW(t_{s+1})$  or setting i as the index of the added one otherwise. By construction,  $\epsilon_{(i,j)} \notin \mathcal{A}$ and then LTA  $\notin \mathcal{A}$ .

Theorem 2 proves that, if the polar code follows the UPO framework, then LTA  $\subseteq A$ . Now we expand this result by proving that the affine automorphisms group of a polar code fulfilling UPO has a BLTA structure. To begin with, we prove a lemma regarding elementary linear transformations.

**Lemma 7.** If C follows UPO framework, then  $\epsilon_{(i,j)} \in A$  implies that also  $\epsilon_{(i+1,j)}, \epsilon_{(i,j-1)} \in A$ .

*Proof.* According to the hypothesis, for every  $m_t \in \mathcal{G}$  and  $\epsilon_{(i,j)} \in \mathcal{A}$  such that  $\epsilon_{(i,j)}(m_t) = m_t + m_{t'} + m_{t''}$ , then  $m_{t'}, m_{t''} \in \mathcal{G}$ . Then if  $\epsilon_{(i+1,j)}(m_t) = m_t + m_{t_1} + m_{t_2}$ , by construction we have that  $m_{t_1} \leq m_{t'}$  and  $m_{t_2} \leq m_{t''}$ , and hence by UPO hypothesis  $m_{t_1}, m_{t_2} \in \mathcal{G}$  and  $\epsilon_{(i+1,j)} \in \mathcal{A}$  by Lemma 4. Similarly, if  $\epsilon_{(i,j-1)}(m_t) = m_t + m_{t_3} + m_{t''}$ , by construction we have that  $m_{t_3} \leq m_{t'}$  and hence by UPO hypothesis  $m_{t_3} \in \mathcal{G}$  and  $\epsilon_{(i,j-1)} \in \mathcal{A}$  by Lemma 4.

Lemma 7 will be used to prove the second main result of this section.

**Theorem 3.** A polar code C is compliant with the UPO framework if and only if for some profile s we have that  $\mathcal{A} = \mathsf{BLTA}(S)$ .

*Proof.* The condition is necessary since  $LTA \subseteq BLTA(S)$  for any profile S, and for Theorem 2  $LTA \subseteq A$  implies that C is compliant with the UPO framework.

To prove that the condition is sufficient, we begin from the observation that Lemma 7 implies that the affine automorphism group of a code C following UPO framework has an overlapping block triangular (OBLT) structure. In practice, this structure is defined by blocks over the diagonal that can overlap. In the following, we show that such a structure cannot be a group, and hence a BLTA is the only "blocky" matrix structure compliant with Lemma 7. In the following, we restrict our analysis to the case depicted in Figure 2, namely an OBLT $(s_1, s_2)$  structure with  $s_1 + s_2 > n$  and  $s_1 > s_2$ ; the results of this case study can be easily extended to more general OBLT structures. In the following, we will show that it is possible to generate a matrix having a nonzero entry in row i and column j, for every  $0 \le i < n - s_2$ and  $n - s_1 \leq j < n$ , as the result of the multiplication of two matrices belonging to  $OBLT(s_1, s_2)$ . This would prove



Fig. 2: Structure of OBLT matrix.

that  $OBLT(s_1, s_2)$  is not a group, and the closure of this set represents the group structure of the automorphisms. Given i, jdefined above, elementary matrices  $E_{(i,s_1-1)}$  and  $E_{(s_1-1,j)}$ belong to  $OBLT(s_1, s_2)$  by construction: the first is included in the upper block, while the other is included in the lower block. Matrix  $A = E_{(i,s_1-1)} \cdot E_{(s_1-1,j)}$  is given by  $E_{(s_1-1,j)}$ but adding row  $s_1 - 1$  to row i. As a result,  $a_{i,j} = 1$ , and  $OBLT(s_1, s_2)$  is not a group.

#### IV. CLASSIFICATION OF AFFINE AUTOMORPHISMS

In this section, we describe a new framework for the analysis of automorphisms for permutation decoding of polar codes. We introduce the concept of *permutation decoding equivalence* for automorphisms; this notion permits to cluster code automorphisms into classes of automorphisms always providing the same codeword candidate under permutation decoding, no matter the received signal. As a consequence, the selection of automorphisms for AE decoding should avoid automorphisms belonging to the same class. These classes are generated as cosets of the absorption group of the polar code, which plays a capital role in the definition of the classes. This result is an extension of our preliminary analysis of decoder equivalence published in [18].

Next, we focus on the analysis of the absorption group under SC decoding, providing an alternative proof of a very recent result presented in [19], namely that the complete SC absorption group has a BLTA structure. Thanks to the knowledge of the structure of the SC absorption group, we calculate the number of the equivalence classes, representing the maximum number of permutations providing possibly different codeword candidates under AE-SC decoding. Finally, we propose a practical method to find one automorphism for each equivalence class, in order to avoid redundant automorphisms.

#### A. Permutation decoding equivalence

We begin by formally defining what is a decoding function of polar codes. We consider transmission over a binary



Fig. 3: Structure of the automorphism ensemble (AE) decoder.

symmetric memoryless channel with output y such that

$$p(y_i|x_i = 0) = p(-y_i|x_i = 1)$$
(18)

for i = 0, 1, ..., N - 1. An example of such a channel is the BI-AWGN channel,

$$y = \tilde{x} + w,$$

where  $\tilde{x} = bpsk(x)$  with BPSK mapping bpsk(0) = +1, bpsk(1) = -1, and w is i.i.d. Gaussian noise with distribution  $\mathcal{N}(0, \sigma^2)$ . We assume a decoding algorithm for the AWGN channel, operating on the channel outputs<sup>2</sup> y.

Definition 13 (Decoding function). The function

$$\det : \mathbb{R}^N \to \mathcal{C}$$
$$y \mapsto x = \det(y) \tag{19}$$

denotes the decoding function for polar codes.

This definition can be generalized to the notion of decoding function wrapped by an automorphism. An *automorphism decoder* is a decoder run on a received signal that is scrambled according to a code automorphism; the result is then scrambled back to retrieve the original codeword estimation.

**Definition 14** (Automorphism decoding function). For an automorphism  $\pi \in Aut(\mathcal{C})$ , the function

$$\operatorname{adec}: \mathbb{R}^N \times \operatorname{Aut}(\mathcal{C}) \to \mathcal{C}$$
 (20)

 $y \mapsto x = \operatorname{adec}(y; \pi)$  (21)

with

$$\operatorname{adec}(y;\pi) \triangleq \pi^{-1}(\operatorname{dec}(\pi(y))).$$
 (22)

is called the automorphism decoding function of the polar code.

Note that this function may be seen as a decoding function with parameter  $\pi$ . Moreover, we focus our analysis on affine automorphisms in  $\mathcal{A} \subseteq \operatorname{Aut}(\mathcal{C})$ . An *automorphism ensemble* (AE) decoder, originally proposed in [10] for Reed-Muller codes, consists of M automorphism decoders running in parallel, as depicted in Figure 3, where the codeword candidate is selected using a least-squares metric. Starting from these definitions, we analyze the effects of automorphism decoding on polar codes.

**Definition 15** (Decoder equivalence). Two automorphisms  $\pi_1, \pi_2 \in \mathcal{A}$  are called equivalent with respect to dec, written as  $\pi_1 \sim \pi_2$ , if

$$\operatorname{adec}(y;\pi_1) = \operatorname{adec}(y;\pi_2) \quad \text{for all } y \in \mathbb{R}^N.$$
 (23)

This is an equivalence relation, since it is reflexive, symmetric and transitive. The equivalence classes are defined as

$$[\pi] \triangleq \{\pi' \in \mathcal{A} : \pi \sim \pi'\}.$$
 (24)

The equivalence class [1] of the trivial permutation 1 is also called the set of *decoder-absorbed automorphisms*. In fact, for all  $\pi \in [1]$ ,

$$\operatorname{adec}(y;\pi) = \operatorname{dec}(y) \quad \text{for all } y \in \mathbb{R}^N,$$
 (25)

i.e., these permutations are absorbed by the decoder, or in other words, the decoding function is invariant to these permutations. This happens also when an absorbed permutation is concatenated to other permutations. In fact, given two automorphisms  $\pi, \sigma \in A$ , then

$$\operatorname{adec}(y; \pi \circ \sigma) = \sigma^{-1}(\operatorname{adec}(\sigma(y); \pi)),$$
 (26)

and if  $\pi \in [1]$ , we obtain

$$\operatorname{adec}(y; \pi \circ \sigma) = \sigma^{-1}(\operatorname{adec}(\sigma(y); \pi)) =$$
 (27)

$$= \sigma^{-1}(\operatorname{dec}(\sigma(y))) = \operatorname{adec}(y;\sigma), \qquad (28)$$

i.e., the component  $\pi$  of the composition  $\pi \circ \sigma$  is absorbed. In the following, we generalize these properties by showing that [1] forms a sub-group of  $\mathcal{A}$ . To begin with, we prove that the inverse of an absorbed automorphism is also absorbed.

**Lemma 8.** If  $\pi \in [1]$ , then  $\pi^{-1} \in [1]$ .

*Proof.* From the definition of the equivalence class, we have that

$$\operatorname{dec}(y) = \pi^{-1}(\operatorname{dec}(\pi(y))) \iff \pi(\operatorname{dec}(y)) = \operatorname{dec}(\pi(y))$$
(29)

(with the latter denoting an equivariance); and so

$$\operatorname{adec}(y;\pi^{-1}) = \pi(\operatorname{dec}(\pi^{-1}(y))) =$$
 (30)

$$= \det(\pi(\pi^{-1}(y))) = \det(y).$$
(31)

Now we can prove that  $[1] \leq A$ ;

**Lemma 9.** The equivalence class [1] (set of decoder-absorbed automorphisms) is a subgroup of  $\mathcal{A}$ , i.e.,  $[1] \leq \mathcal{A}$ .

*Proof.* We prove this lemma using the subgroup test, stating that  $[1] \leq A$  if and only if  $\forall \pi, \sigma \in [1]$  then  $\pi^{-1}\sigma \in [1]$ :

$$\operatorname{adec}(y; \pi^{-1}\sigma) = (\pi^{-1}\sigma)^{-1}(\operatorname{dec}((\pi^{-1}\sigma)(y))) = (32)$$

$$= \sigma^{-1} \left( \pi \left( \operatorname{dec} \left( \pi^{-1} \left( \sigma \left( y \right) \right) \right) \right) \right) = \qquad (33)$$

$$= \sigma^{-1} \left( \operatorname{dec} \left( \sigma \left( y \right) \right) \right) = \tag{34}$$

$$= \operatorname{dec}\left(y\right). \tag{35}$$

 $<sup>^{2}</sup>$ If the decoder operates on LLRs, we may include the LLR computation into the channel model.

Now, we use cosets of [1] to classify automorphisms into equivalence classes (ECs) containing permutations providing the same results under AE decoding.

**Lemma 10.** Equivalence classes  $[\sigma]$  defined by decoder equivalence correspond to right cosets of [1].

*Proof.* Right cosets of [1] are defined as

$$[1]\sigma \triangleq \{\pi \circ \sigma : \pi \in [1]\}.$$
(36)

Hence, permutations  $\sigma_1, \sigma_2 \in [1]\sigma$  if and only if there exist two permutations  $\pi_1, \pi_2 \in [1]$  such that  $\sigma_1 = \pi_1 \circ \sigma$  and  $\sigma_2 = \pi_2 \circ \sigma$ , where the second implies  $\sigma = \pi_2^{-1} \circ \sigma_2$ . Then the proof is concluded by

$$\operatorname{adec}(y;\sigma_1) = \operatorname{adec}(y;\pi_1\sigma) =$$
 (37)

$$= \operatorname{adec}(y; \pi_1 \pi_2^{-1} \sigma_2) = \operatorname{adec}(y; \sigma_2).$$
(38)

According to our notation, two automorphisms in the same EC always provide the same candidate under adec decoding. The number of non-redundant automorphisms for AE-dec, namely the maximum number of different candidates listed by an AE-dec decoder, is then given by the number of equivalence classes of our relation.

**Lemma 11.** There are  $E = \frac{|\mathcal{A}|}{|[1]|}$  equivalence classes  $[\pi], \pi \in \mathcal{A}$ , all having the same size.

# *Proof.* Follows from Lemma 10 and Lagrange's Theorem.

The number E of equivalence classes provides an upper bound on the number of different candidates of an adec decoder. In fact, all the permutations included in an equivalence class give the same result under adec decoding. Then, when selecting permutations for the adec decoder, it is fundamental to select permutations of different equivalence classes. Moreover, it is useless to select more permutations than the number of equivalence classes. This observation permits to control the list size of an adec decoder. However, the proposed relation does not assure that, for some value of y, two ECs do not produce the same candidate; in practice, our relation permits to calculate the maximum number of different results under adec, providing an upper bound of parameter M of an AE-dec decoder.

Equipped with the definition of equivalence classes under a given decoder, it is worth analyzing the *redundancy* of an automorphism set  $\mathcal{L}$  used in an AE decoder. Let us denote by  $M = |\mathcal{L}|$  the number of automorphisms used in the AE decoder; if the elements of  $\mathcal{L}$  are randomly drawn from  $\mathcal{A} \setminus [1]$ , we call  $P_{\geq 1}(M)$  the probability of having redundant automorphisms in the set, i.e. that at least two automorphisms belong to the same equivalence class and hence provably provide always the same candidates. The probability of having non-redundant sets is connected to the birthday problem; if we denote by  $E = |\mathsf{EC}|$  the number of equivalence classes under the chosen decoder, this probability is given by

$$P_{\geq 1}(M) = 1 - P_0(M) = 1 - \prod_{i=0}^{M-1} \frac{E-i}{E},$$
 (39)

where  $P_m(M)$  represents the probability of having exactly mredundant automorphisms. The probability  $P_m(M)$  is more difficult to compute, since the m automorphism may belong to the same equivalence classes or from different equivalence classes. The second possibility being more likely to happen, we provide a lower bound of  $P_m(M)$  as the probability of having m EC with exactly 2 representatives as

$$P_m(M) \ge {\binom{E}{m}} \prod_{k=0}^{m-1} {\binom{M-2k}{2}} \prod_{k=0}^{M-2m-1} \frac{E-i-m}{E}.$$
(40)

The effect of the redundancy of  $\mathcal{L}$  will be shown in Section V. In the next section, we will analyze ECs under SC decoding.

#### B. Absorption group of Successive Cancellation decoders

We assume the standard algorithm for SC decoding for the AWGN channel, operating with the min-approximation of the boxplus operation, with the kernel decoding equations

$$l^{-} = \operatorname{sgn}(l_0) \cdot \operatorname{sgn}(l_1) \cdot \min\{|l_0|, |l_1|\}, l^{+} = l_0 + l_1.$$

This decoder is independent<sup>3</sup> of the SNR and can directly operate on the channel outputs y rather than (properly scaled) channel LLRs. Here we propose again Definitions 13,14 for SC decoding.

Definition 16 (SC decoding function). The function

$$SC : \mathbb{R}^N \to \mathcal{C}$$
$$y \mapsto x = SC(y) \tag{41}$$

denotes the decoding function implemented by the SC algorithm (with min-approximation).

**Definition 17** (Automorphism SC decoding function). For an automorphism  $\pi \in A$ , the function

$$\mathsf{aSC}: \mathbb{R}^N \times \mathcal{A} \to \mathcal{C} \tag{42}$$

$$y \mapsto x = \mathsf{aSC}(y; \pi)$$
 (43)

with

$$\mathsf{aSC}(y;\pi) \triangleq \pi^{-1}(\mathsf{SC}(\pi(y))). \tag{44}$$

is called the automorphism SC decoding function.

Again, aSC function may be seen as a decoding function with parameter  $\pi$ . The notion of SC-absorbed automorphisms group [1] has been introduced in [10] as the set of permutations that are SC decoding invariant. In the same paper, authors proved that LTA automorphisms are absorbed under SC decoding; this result is reported in the following lemma:

**Lemma 12** (SC-absorption of LTA). The group LTA is SC-absorbed,  $LTA \leq [1]$ .

*Proof.* The proof is provided in [10].

In [10] it was also conjectured that [1] = LTA, however this statement is not true. In fact, the full absorption group of a polar code may be larger than LTA:

<sup>3</sup>The decoder with the exact boxplus operation has similar properties, but depends on the SNR.

**Lemma 13.** If BLTA(S) is the affine permutation group of a polar code with  $s_1 > 1$ , then BLTA(2, 1, ..., 1) < [1].

*Proof.* Given  $\pi_{(A,b)} \in \mathcal{A}$ , to set  $a_{0,1} = 1$  in matrix A corresponds to map variable  $X_0$  to variable  $a_{0,0} + X_1$ ; in practice it represents a permutation that identically scrambles 4-uples of the codeword. To be more precise, the entries  $a_{0,0}, a_{0,1}$  of A represent a permutation of the first 4 entries of the codeword, which is repeated identically for every subsequent block of 4 entries of the vector. Under SC decoding, this represents a permutation of the entries of the last  $4 \times 4$  decoding block; in practice, the difference between the aSC decoding of two codewords permuted according two permutations differing only for entries  $a_{0,0}, a_{0,1}$  is that entries of the leftmost  $4 \times 4$ decoding block are permuted. Since the polar code follows UPO as stated by Theorem 3, each block of 4 entries of the input vector can be described as one of the following sequences of frozen (F) and information (I) bits, listed in increasing rate order with the notation introduced for fast-SC decoders [21];

- [*FFFF*]: this represents a rate-zero node, and returns a string of four zeroes no matter the input LLRs; this is independent of the permutation.
- [*FFFI*]: this represents a repetition node, and returns a string of four identical bits given by the sign of the sum of the LLRs; this is independent of the permutation.
- [FFII]: this case is not possible if  $s_1 > 1$ .
- [FIII]: this represents a single parity check node, and returns the bit representing the sign of each LLR while the smallest LLR may be flipped if the resulting vector has even Hamming weight; permuting them back gives the same result for the two decoders.
- [*IIII*]: this represents a rate-one node, and returns the bit representing the sign of each LLR; permuting them back gives the same result for the two decoders.

As a consequence, changing entries  $a_{0,0}, a_{0,1}$  of the permutation matrix (while keeping it invertible) does not change the result of the aSC decoder.

Recently, authors in [19] extended this result to other BLTA structures, proving that, under SC decoding, [1] is a BLTA space, and providing guidance to find the profile of such a group. Here we reconsider this result, providing a non-constructive proof based on algebraic reasoning that does not consider the frozen set of the code.

**Theorem 4.** If  $\mathcal{A} = \text{BLTA}(S)$ , then  $[1] = \text{BLTA}(S_1)$ , with  $S_1 = (s_{1,1}, s_{1,2}, \dots, s_{1,j_1}, s_{2,1}, \dots, s_{2,j_2}, \dots, s_{s_l,1}, \dots, s_{s_l,j_l})$  where  $s_{i,1} + \dots + s_{i,j_i} = s_i$  for every  $1 \le i \le l$ .

*Proof.* The proof follows from Lemma 13 and from the same line of reasoning of the proof of sufficient condition of Theorem 3.

In practice, to construct the profile  $S_1$ , the block of size  $s_i$ of S is divided into sub-blocks of size  $s_{i,1}, \ldots, s_{i,j_i}$ . Given that both  $\mathcal{A}$  and [1] have a BLTA structure under SC decoding, in order to count the number of ECs under this decoder we need to calculate the size of such a group. **Lemma 14.** The size of BLTA(S), with  $S = (s_1, \ldots, s_t)$  and  $\sum_{i=1}^{t} s_i = n$ , is:

$$|\mathsf{BLTA}(S)| = 2^{\frac{n(n+1)}{2}} \cdot \prod_{i=1}^{t} \left( \prod_{j=2}^{s_i} \left( 2^j - 1 \right) \right).$$
(45)

*Proof.* It is well known that  $|\mathsf{GL}(m)| = \prod_{i=0}^{m-1} (2^m - 2^i)$ . From this, we have that

$$\begin{aligned} |\mathsf{GL}(m)| &= \prod_{i=0}^{m-1} \left( 2^m - 2^i \right) = \\ &= \prod_{i=0}^{m-1} 2^i \left( 2^{m-i} - 1 \right) = \\ &= \left( \prod_{j=0}^{m-1} 2^j \right) \cdot \left( \prod_{i=0}^{m-1} \left( 2^{m-i} - 1 \right) \right) = \\ &= 2^{\sum_{j=0}^{m-1} j} \cdot \prod_{i=1}^m \left( 2^i - 1 \right) = \\ &= 2^{\frac{m(m-1)}{2}} \cdot \prod_{i=2}^m \left( 2^i - 1 \right). \end{aligned}$$

It is worth noting that we rewrote the size of GL(m) as the product of the number of lower-triangular matrices and the product of the first n powers of two, diminished by one. This property can be used to simplify the calculation of BLTA(S). In fact, each block of the BLTA structure forms an independent  $GA(s_i)$  space, having size  $|GA(s_i)|$ . All the entries above the block diagonal are set to zero, so they are not taken into account in the size calculation, while the entries below the block diagonal are free, and can take any binary value. Then, the size of BLTA(S) can be calculated as  $|LTA(n)| = 2^{\frac{n(n+1)}{2}}$  multiplied by the product of the first  $s_i$  powers of two, diminished by one, for each size block  $s_i$ , which concludes the proof.

Now we can prove the last result of this section, namely the number of ECs under SC decoding.

**Lemma 15.** A polar code of length  $N = 2^n$  with  $\mathcal{A} = BLTA(S)$ ,  $S = (s_1, \ldots, s_t)$  and  $[\mathbb{1}] = BLTA(S_1)$  defined in Theorem 4 has

$$E = \frac{|\mathsf{BLTA}(S)|}{|\mathsf{BLTA}(S_1)|} = \frac{\prod_{i=1}^{t} \left(\prod_{j=2}^{s_i} \left(2^j - 1\right)\right)}{\prod_{i=1}^{t} \prod_{l=1}^{j_i} \left(\prod_{r=2}^{s_{i,l}} \left(2^r - 1\right)\right)}.$$
 (46)

equivalence classes under SC decoding.

*Proof.* This follows from the application of Lemma 11 and Lemma 14.

Lemma 15 also provides an upper bound on the number of ECs for a polar code under SC decoding; in fact, since LTA < [1], we have that

$$E = \frac{|\mathsf{BLTA}(S)|}{|[\mathbb{1}]|} \leqslant \frac{|\mathsf{BLTA}(S)|}{|\mathsf{LTA}|} = \prod_{i=1}^{t} \left( \prod_{j=2}^{s_i} \left( 2^j - 1 \right) \right).$$
(47)

In the scenario of Lemma 13, the upper bound provided by Lemma 15 on the number of ECs for a polar code under SC decoding can be rewritten as

$$E = \frac{|\mathsf{BLTA}(S)|}{|\mathsf{BLTA}(2,1,\dots,1)|} = \frac{1}{3} \prod_{i=1}^{t} \left( \prod_{j=2}^{s_i} \left( 2^j - 1 \right) \right).$$
(48)

#### C. Decomposition of equivalence classes under SC decoding

Lemma 11 provides the number of ECs of the aSC decoding relation, however without providing a method to calculate them. In this section we show how, thanks to the structure of the automorphism group of polar codes, EC representatives can be decomposed in blocks, which can be calculated separately. This decomposition greatly simplifies the task of listing all the ECs of the relation for practical application of our proposal.

To begin with, we define the block diagonal matrix B = $diag(B_1,\ldots,B_l)$  where  $B_i$  is an invertible square matrix in  $GL(s_i)$ . By definition, B is the transformation matrix of an automorphism of the code; in the following, we will prove that every EC contains at least an automorphism defined by such a matrix. In particular, we call the equivalence sub-class  $EC_i =$  $GL(s_i)/BLTL(s_{i,1},\ldots,s_{i,j_i})$ , namely the set of the cosets of  $BLTL(s_{i,1},\ldots,s_{i,j_i})$  in  $GL(s_i)$ . By definition, we have that  $\prod_{i=1}^{l} |\mathsf{EC}_i| = |\mathsf{EC}|$  under SC decoding; in the following, we prove that a representative of any EC can be expressed as the juxtaposition of representatives of equivalence sub-classes.

**Lemma 16.** For every  $B_i, D_i \in GL(s_i)$  such that B = $diag(B_1,\ldots,B_l)$  and  $D = diag(D_1,\ldots,D_l)$  for all i =1,...,l, then  $[\pi_{(B,0)}] = [\pi_{(D,0)}]$  if and only if  $B_i$  and  $D_i$ belong to the same coset of  $GL(s_i)/BLTA(s_{i,1},\ldots,s_{i,j_i})$ .

*Proof.* First, assume that  $B_i$  and  $D_i$  belong to the same equivalence sub-class for all i = 1, ..., l; since each block is independent,

$$BD^{-1} = diag(B_1D_1^{-1}, \dots, B_lD_l^{-1}) \in \mathsf{BLTA}(S_1)$$
 (49)

since for every block  $B_i D_i^{-1} \in \mathsf{BLTL}(s_{i,1}, \ldots, s_{i,j_i})$ . Conversely, assume  $[\pi_{(B,0)}] = [\pi_{(D,0)}]$ . This means that  $BD^{-1} \in$  $BLTA(S_1)$ , and given that each block is independent,  $B_i D_i^{-1} \in$  $BLTL(s_{i,1}, ..., s_{i,j_i})$  for all i = 1, ..., l. 

Now we can prove that every EC can be represented by an affine automorphism whose matrix can be expressed as a block diagonal matrix.

**Theorem 5.** Every EC includes at least an affine automorphism whose matrix is block diagonal.

*Proof.* Lemma 16 proves the connection between equivalence sub-classes and ECs; in particular, we have seen that different compositions of sub-classes lead to different ECs. The proof can be conluded by proving that the number of ways to

compose block diagonal matrices using representatives of different sub-classes is equal to the number of ECs; in fact,

$$\prod_{i=1}^{l} |\mathsf{EC}_{i}| = \prod_{i=1}^{l} |\mathsf{GL}(s_{i})/\mathsf{BLTA}(s_{i,1},\dots,s_{i,j_{i}})| =$$
(50)

$$=\prod_{i=1}^{l} \frac{|\mathsf{GL}(s_i)|}{|\mathsf{BLTA}(s_{i,1},\dots,s_{i,j_i})|} =$$
(51)

$$=\prod_{i=1}^{l} \frac{2^{\frac{s_i(s_i-1)}{2}} \prod_{j=2}^{s_i} (2_j-1)}{2^{\frac{s_i(s_i-1)}{2}} \prod_{i=1}^{t} (\prod_{j=2}^{s_{i,j_t}} (2^j-1))} = (52)$$

$$= \frac{\prod_{i=1}^{t} \left( \prod_{j=2}^{t} (2^{j} - 1) \right)}{\prod_{i=1}^{t} \prod_{l=1}^{j_{i}} \left( \prod_{r=2}^{s_{i,l}} (2^{r} - 1) \right)} =$$
(53)  
= |EC|. (54)

It is worth noticing that this property holds when LTA  $\leq$ [1]; if the absorption group is smaller, the property may not be true. This property can be used to efficiently list all the equivalence classes under SC decoding. In fact, each block of the BLTA automorphism group can be searched independently, and the resulting sub-classes can be merged to find all the equivalence classes.

Focusing on a sub-class  $EC_i$ , we propose to use PUL matrix decomposition to further simplify the representatives search.

**Definition 18** (PUL decomposition). The matrices  $P \in PL$ ,  $U \in \mathsf{UTL}$  and  $L \in \mathsf{LTL}$  form the PUL decomposition of invertible matrix A if A = PUL.

The PUL decomposition exists for all  $A \in GL$  and is not necessarily unique; this is due to the non-uniqueness of the LU decomposition, of which the PUL decomposition is merely a variation. We extend the notion of PUL decomposition to automorphisms  $\pi_{(A,b)} \in \mathcal{A}$ , corresponding to the concatenation of the permutations as

$$\pi_{(A,b)} = \pi_{(L,b)} \circ \pi_{(U,0)} \circ \pi_{(P,0)}, \tag{55}$$

applied from right to left. By Lemma 12, we have that under SC decoding  $\pi_{L,b} \in [1]$ . In the following, we show that representatives of ECs can be decomposed as the product of two automorphisms, one belonging to UTL and the other belonging to PL. If we call  $\mathcal{A}_{\mathcal{P}}$  and  $\mathcal{A}_{\mathcal{U}}$  the subgroups of A containing only PL and UTL automorphisms respectively, we can always find an EC representative composing elements from these two sets.

**Lemma 17.** Each EC contains at least an automorphism  $P \cdot U$ with  $P \in \mathcal{A}_{\mathcal{P}}$  and  $U \in \mathcal{A}_{\mathcal{U}}$ .

*Proof.* By Theorem 5 we know that every EC includes an affine automorphism  $\pi_{(D,b)}$  such that  $D = diag(D_1, \ldots, D_l)$ . Every sub-matrix  $D_i$  has PUL decomposition  $D_i = P_i U_i L_i$ , and we define  $P = diag(P_1, \ldots, P_l), U = diag(U_1, \ldots, U_l)$  and  $L = diag(L_1, \ldots, L_l)$ , with  $P \in \mathcal{A}_{\mathcal{P}}$  and  $U \in \mathcal{A}_{\mathcal{U}}$ . then, we have that

$$\pi_{(D,b)} = \pi_{(diag(D_1,...,D_l),b)} =$$
(56)

$$=\pi_{(diag(P_1U_1L_1,...,P_lU_lL_l),b)} =$$
(57)

$$=\pi_{(PUL,b)}=\tag{58}$$

$$=\pi_{(L,b)}\circ\pi_{(U,0)}\circ\pi_{(P,0)},$$
(59)

and since  $\pi_{(L,b)} \in [1]$  we have that  $[\pi_{(D,b)}] = [\pi_{(U,0)} \circ \pi_{(P,0)}]$ .

The proof of the previous lemma is constructive, in the sense that it proposes a method to list all ECs by mixing UTL and PL automorphisms that are block diagonal. In fact, in order to generate an EC candidate matrix A, it is sufficient to randomly draw l UTL matrices  $U_1, \ldots, U_l$  and l PL matrices  $P_1, \ldots, P_l$ , of size  $s_1, \ldots, s_l$  respectively, create the block diagonal matrices  $U = diag(U_1, \ldots, U_l)$  and  $P = diag(P_1, \ldots, P_l)$  and generate  $A = P \cdot U$ . By listing all the possible matrices P and U, Lemma 16 ensures that all the ECs will be found.

However, since the PUL decomposition of a matrix is not unique, this method is redundant, in the sense that it may generate multiple representatives of the same EC. A further check is required to assure that the EC representative does not belong to an already calculated EC. This check is done by multiplying the calculated  $P \cdot U$  matrix and the  $P \cdot U$  matrices of the previously calculated ECs as stated in Lemma 18.

**Lemma 18.** Given  $\pi_1, \pi_2 \in \mathcal{A}$  having transformation matrices  $A_1, A_2$ , then  $\pi_1 \in [\pi_2]$  if and only if  $A_1 \cdot A_2^{-1} \in \mathsf{BLTA}(S_1)$ . *Proof.* This follows directly from the definitions of EC and [1].

We can now propose a practical method to create the automorphisms list  $\mathcal{L}$ , including an automorphism for every EC. The idea is to start by dividing the affine transformation matrix in l blocks of size  $s_1, \ldots, s_l$ . For each block, all the matrices belonging to  $\mathcal{U}(s_i)$  and  $\mathcal{P}(s_i)$ , namely the group of upper triangular and permutation matrices of size  $s_i$ , are calculated. Next, all the block upper diagonal matrices  $U \in \mathcal{A}_{\mathcal{U}}$ having matrices in  $\mathcal{U}(s_i)$  on the diagonal are calculated, along with all the block permutation matrices  $P \in \mathcal{A}_{\mathcal{P}}$ having matrices in  $\mathcal{P}(s_i)$  on the diagonal; each block diagonal matrix is considered ad the affine transformation matrix of a representative of an EC and included in L, after checking that the new automorphisms is not includes in any coset already included in the list as stated in Lemma 17. Next, matrices in the form  $P \cdot U$  are included in the list if they pass the check of Lemma 17. When |EC| representatives are found, the searching process stops.

In order to evaluate the complexity of the proposed method, we need to evaluate the number of matrices that are generated during the process. By construction, we have that

$$|\mathcal{U}(s_i)| = 2^{\frac{s_i(s_i-1)}{2}}$$
,  $|\mathcal{P}(s_i)| = s_i!$ . (60)

For a block structure  $S = (s_1, \ldots, s_t)$ , the number of UTL and PL automorphisms are:

$$|\mathcal{A}_{\mathcal{U}}| = \prod_{i=1}^{t} 2^{\frac{s_i(s_i-1)}{2}} \quad , \quad |\mathcal{A}_{\mathcal{P}}| = \prod_{i=1}^{t} s_i! \; . \tag{61}$$



Fig. 4: Error-correction performance of (128,85) polar codes under AE-SC decoding.

Hence, the number of generated matrix is given by

$$|\mathcal{A}_{\mathcal{U}}| + |\mathcal{A}_{\mathcal{P}}| + |\mathcal{A}_{\mathcal{U}}| \cdot |\mathcal{A}_{\mathcal{P}}|.$$
(62)

It is worth noticing that if [1] = LTA then every EC include only one UTL or PL matrix, such that  $\mathcal{A}_{\mathcal{U}} \cup \mathcal{A}_{\mathcal{P}} \subset \mathcal{L}$ . An example of this construction can be found in Appendix A.

## V. NUMERICAL RESULTS

In this section, we present a numerical analysis of error correction performance of AE decoders. Simulation results are obtained under BPSK modulation over the AWGN channel. We analyze polar codes having meaningful affine automorphism groups, decoded under AE-M-dec where M represents the number of parallel adec decoders. For each code, we provide the minimum information set  $\mathcal{I}_{min}$ , the affine automorphism group  $\mathcal{A}$  and the SC absorption group [1]. Automorphism set  $\mathcal{L}$  are generated according to the method described in Lemma 18; the results obtained are compared to a random selection of automorphisms. We compare errorcorrection performance of proposed polar codes to 5G polar codes under CRC-aided SCL [4]; ML bounds depicted in the figures are retrieved with the truncated union bound, computed with the minimum distance  $d_{min}$  and the number of minimum distance codewords of the code [11].

Figure 4 shows the performance of AE-SC decoding for the (128,85) code defined by  $\mathcal{I}_{min} = \{23,25\}$  and having  $\mathcal{A} = \mathsf{BLTA}(3, 1, 3)$ . Following the procedure of Lemma 13, it is possible to prove that this code has sequences of length 8 of frozen and information bits that are invariant under SC decoding, thus [1] = BLTA(3, 1, 1, 1, 1) [19]. This code has E = 21 equivalence classes, so a set  $\mathcal{L}$  composed of 21 EC representatives provides a bound on the decoding performance of AE-SC decoding. AE with a set of M = 3 randomly drawn automorphisms suffers from a significant loss with respect to AE having a set composed of M = 3 EC representatives; this loss is eliminated by drawing an additional automorphism, setting M = 4. In this case, a set of 21 random automorphisms is essentially matching the AE bound; the impact of new nonredundant automorphisms seems to be reducing with the list size M, and a small number of equivalence classes E reduces



Fig. 5: Error-correction performance of (256, 95) polar codes under AE-SC and AE-SCAN decoding.

the effect of the AE decoder. Performance of 5G polar code decoded under CA-SCL with L = 4 is plotted as a reference; proposed polar code under AE decoding outperforms CA-SCL 5G polar code for BLER $\ge 10^{-3}$ , while 5G polar code shows better performance for lower BLER, however at a larger decoding cost.

Next, in Figure 5 we investigate the error-correction performance of the (256, 95) code defined by  $\mathcal{I}_{min} = \{55, 120, 228\}$ and having  $\mathcal{A} = \mathsf{BLTA}(2, 1, 1, 1, 3)$ . The number of ECs under SC decoding for this code can be calculated as E = 21 by knowing that [1] = BLTA(2, 1, 1, 1, 1, 1, 1). For M = 7, AE-SC decoding with a random automorphism set suffers from a loss with respect to AE-SC decoding performed with automorphisms from 7 distinct ECs. The bound under AE-SC decoding is obtained by using M = 21 non-redundant automorphisms, one representative from each EC. However, this AE bound is far away from the ML bound of the code; then, the errorcorrection performance under AE-SCAN decoding is analyzed. We observe that error-correction performance of AE-2-SCAN with a set composed of two LTA automorphisms is not equivalent to the error-correction performance of SCAN. Thus, the absorption group of SCAN is smaller than LTA, permitting to use additional automorphisms without redundancy; the characterization of the absorption set under SCAN decoding, that we conjecture to be limited to the trivial permutation, is still an open problem. Decoding performance of AE-64-SCAN designed with a set of 64 random automorphism from BLTA(2, 1, 1, 1, 3) matches the ML bound for low BLER.

Figure 6 shows the error-correction of (128, 60) code defined by  $\mathcal{I}_{min} = \{27\}$ . This code has  $\mathcal{A} = BLTA(3, 4)$  and SC absorption group  $[1]_{SC} = BLTA(2, 1, 1, 1, 1, 1)$ . Under SC decoding, the code exhibits E = 2205 ECs permitting to match the ML bound under AE-SC when M = 2205 non-redundant automorphisms are used. Given the large number of ECs, the probability for a random set of automorphisms to include redundant permutation is quite small; as an example,  $P_{\leq 1}(8) = 0.0126$ . A more accurate analysis shows that for large sets  $\mathcal{L}$ , some aSC decoding units returns the correct codeword, whereas AE-SC decoder selects another codeword based on the least-square metric. To overcome this problem,



Fig. 6: Error-correction performance of (128, 60) polar code with  $\mathcal{A} = \mathsf{BLTA}(3, 4)$  under AE-SC.

we introduced a short CRC of 3 bits; simulation results show that in this way it is possible to beat the ML bound of the code with only M = 64 automorphisms. However, the introduction of a CRC is not always useful; in fact, its introduction for the previously analyzed codes did not provide any benefit. How and when to introduce a CRC under AE-SC decoding is still an open problem. Finally, we note that the code investigated has a good ML bound allowing the error correction performance of AE decoding with M = 8 to outperform 5G polar code decoded under CA-SCL-8 decoding.

# VI. CONCLUSIONS

In this paper, we introduced the notion of redundant automorphisms of polar codes under AE decoding. This notion permits to greatly reduce the number of automorphisms that can be used in AE-SC decoding of polar codes. Moreover, by analyzing the number of non-redundant automorphisms it is possible to have an idea of the impact of an AE-SC dedoder: in fact, even if a polar code has a large affine automorphism group, if it SC absorption group is too large, the number of distinct codeword candidate under AE-SC decoding may be too small. Then, we introduced a method to generate a set if non-redundant automorphisms to be used in AE-SC decoding. All these results were made possible by a preliminar analysis of the structure of the affine automorphism group of polar codes. We proposed a novel approach to prove the most recent results in this field, and we provide a proof of the equivalence between decreasing monomial codes and polar codes following UPO. Simulation results show the goodness of our approach, however leaving open various problems: the structure of the absorption group of decoding algorithms other than SC, e.g. BP or SCAN, is still unknown, while it is not clear if it is possible to further reduce the automorphism set size by eliminating automorphisms providing the same result under a given received signal y. With our paper, we hope to provide new tools to help the researcher to answer to these and to other question related to AE decoding of polar codes, a really promising decoding algorithm for high parallel implementations.

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#### APPENDIX

#### **BOOLEAN FUNCTIONS AND PERMUTATIONS**

Boolean function  $f \in \mathbb{F}_2^{[3]}$  defined by

$$f = \bar{V}_2 \oplus \bar{V}_0 \bar{V}_1 \oplus \bar{V}_0 \bar{V}_2 \oplus \bar{V}_1 \bar{V}_2 = m_3 + m_4 + m_2 + m_1$$
(63)

TABLE II: Truth table for boolean function f in (63).

i		$\widehat{i}$		$eval(\bar{V}_0)$	$eval(\bar{V}_1)$	$eval(\bar{V}_2)$
0	0	0	0	1	1	1
1	1	0	0	0	1	1
2	0	1	0	1	0	1
3	1	1	0	0	0	1
4	0	0	1	1	1	0
5	1	0	1	0	1	0
6	0	1	1	1	0	0
7	1	1	1	0	0	0

can be easily evaluated on the basis of single variable evaluations given by

$$\begin{aligned} x^{(m_6)} &= \operatorname{eval}(\bar{V}_0) = 10101010 \\ x^{(m_5)} &= \operatorname{eval}(\bar{V}_1) = 11001100 \\ x^{(m_3)} &= \operatorname{eval}(\bar{V}_2) = 11110000 \end{aligned}$$

Each single variable evaluation is described by a column of the Truth Table II. In fact, the evaluation of function f is given by

$x^{(m_3)} = \operatorname{eval}(\bar{V}_2)$	11110000	$\oplus$
$x^{(m_4)} = \operatorname{eval}(\bar{V}_0 \bar{V}_1)$	10001000	$\oplus$
$x^{(m_2)} = \operatorname{eval}(\bar{V}_0 \bar{V}_2)$	10100000	$\oplus$
$x^{(m_1)} = \operatorname{eval}(\bar{V}_1 \bar{V}_2)$	11000000	=
$x^{(f)} = \operatorname{eval}(f)$	00011000	

Now let us apply left shift permutation  $\sigma_j$  to this vector, where  $\sigma_j(i) = [i+j]_{2^n}$ , with  $[\cdot]_{2^n}$  representing modulo  $2^n$  operation; according to our notation, element in *i* is replaced by element in  $[i + j]_{2^n}$ . Double left shift  $\sigma_2$  can be expressed as an affine transformation, while  $\sigma_3$  cannot; for n = 3,  $\sigma_2$  can be expressed as an affine transformation defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \qquad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let us take boolean function  $g \in \mathbb{F}_2^{[3]}$  defined by  $g = \bar{V}_0 \bar{V}_2 \oplus \bar{V}_1 \bar{V}_2 = m_2 + m_1$ , having evaluation  $x^{(g)} = 01100000$ . According to Lemma 2, we have that  $\sigma_2(x^{(f)}) = x^{(g)}$  when  $g = \sigma_2(f)$ . In fact, according to its affine transformation definition,  $\sigma_2$  maps

$$\bar{V}_0 \mapsto \bar{V}_0, \bar{V}_1 \mapsto \bar{V}_1 \oplus 1, \qquad \bar{V}_2 \mapsto \bar{V}_1 \oplus \bar{V}_2 \oplus 1.$$

It is worth noticing that the use of negative monomials adds a " $\oplus$ 1" addend compared to positive monomials when the number of output variables is even. It is easy to verify that applying this variables substitutions to boolean function fleads to boolean function g.

#### EC REPRESENTATIVES CALCULATION

Here we provide an example of the generation of automorphisms list  $\mathcal{L}$  according to the proposed method. We analyze the (256,95) code defined by  $\mathcal{I}_{min} = \{55, 120, 228\}$ and having  $\mathcal{A} = \text{BLTA}(2, 1, 1, 1, 3)$ , whose error correction performance are depicted in Figure 5. According to Lemma 14, this code has

$$|\mathcal{A}| = 2^{\frac{8(8+1)}{2}} \cdot \prod_{i=1}^{5} \left( \prod_{j=2}^{s_i} \left( 2^j - 1 \right) \right) = 2^{36} \cdot (3) \cdot (3 \cdot 7) = 63 \cdot 2^{36}$$
(64)

affine automorphisms. This code has SC absorption group [1] = BLTA(2, 1, 1, 1, 1, 1), and again for Lemma 14 the number of SC absorbed affine automorphisms is given by

$$|[\mathbb{1}]| = 2^{\frac{8(8+1)}{2}} \cdot \prod_{i=1}^{7} \left( \prod_{j=2}^{s_i} \left( 2^j - 1 \right) \right) = 3 \cdot 2^{36}.$$
(65)

According to Lemma 11, there are  $E = \frac{|\mathcal{A}|}{|[1]|} = 21$  equivalence classes under SC decoding, as confirmed by Lemma 15. So, there are only 21 possible different outcomes under AE-SC decoding, greatly reducing the maximum size M of the AE decoder.

In the following, we use the result of Lemma 17 to create a list of 21 affine automorphisms belonging to different equivalence classes under SC decoding. In this case, there are  $|\mathcal{A}_{\mathcal{U}}| = 16$  UTL automorphisms and  $|\mathcal{A}_{\mathcal{P}}| = 12$  PL automorphisms; it is worth noticing that  $|\mathcal{A}_{\mathcal{U}}| + |\mathcal{A}_{\mathcal{P}}| > 21$ since  $[\mathbb{1}] \subsetneq$  LTA, so automorphisms in  $\mathcal{A}_{\mathcal{U}}$  and  $\mathcal{A}_{\mathcal{P}}$  are not all included in  $[\mathbb{1}]$ . To begin with, we generate all the 16 UTL automorphisms composing  $\mathcal{A}_{\mathcal{U}}$ ; each of them has the form  $\pi_{(U,0)}$  where

$$U = \begin{bmatrix} U_1 & & 0 \\ & 1 & & \\ & & 1 & \\ & & & 1 & \\ 0 & & & U_5 \end{bmatrix},$$
(66)

and  $U_1$  is a 2 × 2 UTL matrix, while  $U_5$  is a 3 × 3 UTL matrix. There are only 2 UTL matrices of size 2 and 8 UTL matrices of size 3; combining them in U it is possible to list all the 16 UTL affine automorphisms of the code. Similarly, a PL automorphism of the code is described by where

$$P = \begin{bmatrix} P_1 & & 0 \\ & 1 & & \\ & & 1 & \\ & & & 1 & \\ 0 & & & P_5 \end{bmatrix}$$
(67)

and  $P_1$  is a 2×2 PL matrix and  $P_5$  is a 3×3 PL matrix. Again, there are only 2 PL matrices of size 2 and 6 PL matrices of size 3, for a total of 12 PL automorphisms. In order to generate  $\mathcal{L} = \{\pi_i, \ldots, \pi_{21}\}$ , where  $\pi_i = \pi_{A_i,0}$ , we begin by setting  $A_1 = I$ , such that  $[\pi_1] = [\mathbb{1}]$ . Next, we star adding to  $\mathcal{L}$ the elements of  $\mathcal{A}_{\mathcal{U}}$ ; only 7 of them can be added, an they are listed as  $A_2, \ldots, A_8$ . This happens because the first block  $U_1$  of an UTL matrix U is always included in the absorption group, and hence only the matrices having different last block  $U_5$  can be included, as stated by Lemma 18. Next, elements of  $\mathcal{A}_{\mathcal{P}}$  are added; only 5 of them are independent from the already included ones, and they are listed as  $A_9, \ldots, A_{13}$ . Finally, the remaining elements of  $\mathcal{L}$  need to be calculated as  $P \cdot U$ , with  $U \in \mathcal{A}_{\mathcal{U}}$  and  $P \in \mathcal{A}_{\mathcal{P}}$ , as stated in Lemma 17. There are  $|\mathcal{A}_{\mathcal{U}}| \cdot |\mathcal{A}_{\mathcal{P}}| = 192$  possible combinations of UTL an PL automorphisms; however, it is not mandatory to calculate all of them, since when the remaining 8 independent matrices are calculated, the process can stop. Finally, we can generate the automorphism list  $\mathcal{L} = \{\pi_i, \ldots, \pi_{21}\}$ , where  $\pi_i = \pi_{A_i,0}$  and

. . . . . . . .

$A_1 =$	$\left[\begin{array}{c} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$	,	$A_2 =$	$ \left[ \begin{array}{c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], $
$A_3 =$	$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$	,	$A_4 =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$
$A_5 =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_6 =$	$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$
$A_7 =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_8 =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$
$A_9 =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{10} =$	$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right],$
$A_{11} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{12} =$	$ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0$
$A_{13} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{14} =$	$ \left[ \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] ,$
$A_{15} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{16} =$	$\left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \end{bmatrix},$
$A_{17} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{18} =$	$ \left[ \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] ,$
$A_{19} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$	,	$A_{20} =$	$\left[\begin{array}{c}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0$
$A_{21} =$	$\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$			