# Bregman divergence based em algorithm and its application to classical and quantum rate distortion theory

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#### Abstract

We formulate em algorithm in the framework of Bregman divergence, which is a general problem setting of information geometry. That is, we address the minimization problem of the Bregman divergence between an exponential subfamily and a mixture subfamily in a Bregman divergence system. Then, we show the convergence and its speed under several conditions. We apply this algorithm to rate distortion and its variants including the quantum setting, and show the usefulness of our general algorithm. In fact, existing applications of Arimoto-Blahut algorithm to rate distortion theory make the optimization of the weighted sum of the mutual information and the cost function by using the Lagrange multiplier. However, in the rate distortion theory, it is needed to minimize the mutual information under the constant constraint for the cost function. Our algorithm directly solves this minimization. In addition, we have numerically checked the convergence speed of our algorithm in the classical case of rate distortion problem.

#### **Index Terms**

em algorithm, Bregman divergence, information geometry, rate distortion

### I. INTRODUCTION

Em algorithm is known as a useful algorithm in various areas including machine learning and neural network [1], [2], [3]. Its basic idea can be backed to the reference [4]. In information theory, the Arimoto-Blahut algorithm [5], [6] is known as a powerful tool to calculate various information-theoretical optimization problems including mutual information. Both algorithms are composed of iterative steps. In this paper, we apply em algorithm to rate distortion and its variants including the quantum setting.

Although em algorithm has several variants, the most general form is given as the minimum divergence between a mixture family and an exponential family [1]. However, the convergence speed of em algorithm is not known in general. Moreover, it has a possibility to converge to a local minimum [1], [2], [3]. Therefore, it is needed to guarantee the convergence to the global minimum and clarify the convergence speed. In this paper, to address these problems in a unified viewpoint, similar to the paper [2], we formulate em algorithm in a framework of Bregman divergence, which is given from a general smooth convex function as a general problem setting of information geometry [7], [8]. In this general framework, we derive a necessary condition for the global convergence, and discuss the convergence speed. When an additional condition is satisfied, this algorithm has exponential convergence. This additional condition is easily satisfied when the iteration is close to the true value. Hence, this algorithm rapidly converges around the true value under a certain condition.

When an information-theoretical optimization problem is written in the above form, em algorithm can be applied to it. As a typical example, we consider the rate distortion problem, which is written as a minimization of the mutual information under a linear constraint to a given distribution. That is, the

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objective distribution of this problem belongs to a certain mixture family. Mutual information is written as the minimum divergence between a given distribution and the set of independent distributions, which forms an exponential family. Hence, this minimization is given as the minimization of the divergence between the given mixture family and the exponential family composed of independent distributions. The minimization for the rate distortion problem was studied by Blahut [5] and various papers [9], [10], [11]. However, to remove the constraint, they change the objective function by using a Lagrange multiplier. That is, they minimize the weighted sum of the original objective function and the cost function, whereas the

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The remaining part of this paper is organized as follows. Section II formulates general basic properties for Bregman divergence. Section III explains how the set of probability distributions and the set of quantum states satisfy the condition for Bregman divergence. Section IV states em algorithm in the framework of Bregman divergence, and derives its various properties. Section V applies the above general results to classical rate distribution and its variants. Section VI applies them to its quantum extension.

#### II. Bregman divergence: Information geometry based on convex function

In this section, we prepare general basic properties for Bregman divergence. Originally, information geometry was studied as the geometry of probability distributions. This structure can be generalized as a geometry of a smooth strictly convex function, which is called Bregman Divergence. This section discusses several useful properties of Bregman Divergence.

#### A. Legendre transform

In this paper, a sequence  $a=(a^i)_{i=1}^k$  with an upper index expresses an vertical vector and a sequence  $b=(b_i)_{i=1}^k$  with an lower index expresses an horizontal vector as

$$a = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^k \end{pmatrix}, \quad b = (b_1, b_2, \dots, b_k). \tag{1}$$

Let  $\Theta$  be an open convex set in  $\mathbb{R}^d$  and  $F:\Theta\to\mathbb{R}$  be a  $C^{\infty}$ -class strictly convex function. We introduce another parametrization  $\eta=(\eta_1,\ldots,\eta_d)\in\mathbb{R}^d$  as

$$\eta_i := \partial_i F(\theta), \tag{2}$$

where  $\partial_j$  expresses the partial derivative for the j-th variable. We introduce the vector  $\nabla^{(e)}[F](\theta) := (\partial_j F(\theta))_{j=1}^d$ . Hence, the relation (2) is rewritten as

$$\eta = \nabla^{(e)}[F](\theta). \tag{3}$$

Therefore,  $\nabla^{(e)}$  can be considered as a horizontal vector.

Since F is a  $C^{\infty}$ -class strictly convex function, this conversion is one-to-one. the parametrization  $\eta_j$  is called the mixture parameter. We denote the open set of vectors  $\eta(\theta) = (\eta_1, \dots, \eta_d)$  given in (2), by  $\Xi$ . For  $\eta \in \Xi$ , we define the *Legendre transform*  $F^* = \mathcal{L}[F]$  of F

$$F^*(\eta) = \sup_{\theta \in \Theta} \langle \eta, \theta \rangle - F(\theta). \tag{4}$$

We have [2, Section 3][15, Section 2.2]

$$\partial^j F^*(\eta(\theta)) = \theta^j, \tag{5}$$

where  $\partial^j$  expresses the partial derivative for the j-th variable under the mixture parameter. We introduce the vector  $\nabla^{(m)}[F^*](\eta) := (\partial^j F^*(\eta))_{i=1}^d$ . Hence, the relation (5) is rewritten as

$$\theta = \nabla^{(m)}[F^*](\eta(\theta)). \tag{6}$$

In later discussion, we address subfamilies related to m vectors  $v_1, \ldots, v_m \in \mathbb{R}^d$ . For a preparation for such cases, we prepare the following two equations, which will be used for calculations based on mixture parameters. Then, we define a  $d \times m$  matrix V as  $(v_1 \ldots v_m)$ . The multiplication function of V from the left (right) hand side is denoted by L[V] (R[V]). Since

$$\partial_{j}(F \circ L[V])(\theta) = \frac{\partial F}{\partial \theta^{j}}(V\theta) = \sum_{i} v_{j}^{i} \partial_{i} F(V\theta) = (R[V] \circ (\nabla^{(e)}[F]) \circ L[V](\theta))_{j}, \tag{7}$$

we have

$$\nabla^{(e)}[F \circ L[V]] = R[V] \circ (\nabla^{(e)}[F]) \circ L[V]. \tag{8}$$

In the same way, we can show

$$\nabla^{(m)}[F^* \circ R[V]] = L[V] \circ \nabla^{(m)}[F^*] \circ R[V]. \tag{9}$$

Also, we have

$$(F^* \circ R[V])^*(\theta') = \sup_{\eta} \langle \eta, \theta' \rangle - \sup_{\theta \in \Theta} \langle \eta V, \theta \rangle - F(\theta)$$

$$= \sup_{\eta} \inf_{\theta \in \Theta} \langle \eta, \theta' - V\theta \rangle + F(\theta) = \inf_{\theta : \theta' = V\theta} F(\theta).$$
(10)

When V is one-to-one, we define the function  $F \circ L[V^{-1}]$  on  $L[V](\Theta)$ . Since

$$F^* \circ R[V](\eta) = \sup_{\theta \in \Theta} \langle \eta V, \theta \rangle - F(\theta) = \sup_{\theta \in \Theta} \langle \eta, V \theta \rangle - F(\theta)$$
$$= \sup_{\theta' \in L[V](\Theta)} \langle \eta, \theta \rangle - F(V^{-1}\theta') = (F \circ L[V^{-1}])^*(\eta), \tag{11}$$

we have

$$(F \circ L[V^{-1}])^* = F^* \circ R[V] \tag{12}$$

Combining the above two relations, we have

$$\nabla^{(m)}[(F \circ L[V^{-1}])^*] = \nabla^{(m)}[F^* \circ R[V]] = L[V] \circ \nabla^{(m)}[F^*] \circ R[V]. \tag{13}$$

# B. Exponential subfamily

A subset  $\mathcal{E} \subset \Theta$  is called an *exponential subfamily* generated by l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  at  $\theta_0 \in \Theta$  when the subset  $\mathcal{E}$  is given as

$$\mathcal{E} = \left\{ \phi_{\mathcal{E}}^{(e)}(\bar{\theta}) \in \Theta \,\middle|\, \bar{\theta} \in \Theta_{\mathcal{E}} \right\}. \tag{14}$$

In the above definition,  $\phi_{\mathcal{E}}^{(e)}(\bar{\theta})$  is defined for  $\bar{\theta} = (\bar{\theta}^1, \dots, \bar{\theta}^l) \in \mathbb{R}^l$  as

$$\phi_{\mathcal{E}}^{(e)}(\bar{\theta}) := \theta_0 + \sum_{j=1}^l \bar{\theta}^j v_j \tag{15}$$

and the set  $\Theta_{\mathcal{E}}$  is defined as

$$\Theta_{\mathcal{E}} := \{ \bar{\theta} \in \mathbb{R}^l | \phi_{\mathcal{E}}^{(e)}(\bar{\theta}) \in \Theta \}. \tag{16}$$

Since  $\Theta$  is an open set, the set  $\Theta_{\mathcal{E}}$  is an open set. In the following, we restrict the domain of  $\phi_{\mathcal{E}}^{(e)}$  to  $\Theta_{\mathcal{E}}$ . We define the inverse map  $\psi_{\mathcal{E}}^{(e)} := (\phi_{\mathcal{E}}^{(e)})^{-1} : \mathcal{E} \to \Theta_{\mathcal{E}}$ .

For an exponential subfamily  $\mathcal{E}$ , we define the function  $F_{\mathcal{E}}$  as

$$F_{\mathcal{E}}(\bar{\theta}) := F(\phi_{\mathcal{E}}^{(e)}(\bar{\theta})). \tag{17}$$

In fact, even in an exponential subfamily  $\mathcal{E}$ , we can employ the mixture parameter  $\psi_{\mathcal{E},j}^{(m)}(\phi_{\mathcal{E}}^{(e)}(\bar{\theta})):=\partial_j F_{\mathcal{E}}(\bar{\theta})$  because the map  $\bar{\theta}\mapsto F_{\mathcal{E}}(\bar{\theta})$  is also a  $C^{\infty}$ -class strictly convex function. We define the set  $\Xi_{\mathcal{E}}:=\{(\partial_j F_{\mathcal{E}}(\bar{\theta}))_{j=1}^l\}_{\bar{\theta}\in\Theta_{\mathcal{E}}}$ . We define the inverse map  $\phi_{\mathcal{E}}^{(m)}:=(\psi_{\mathcal{E}}^{(m)})^{-1}:\Xi_{\mathcal{E}}\to\mathcal{E}$ .

## C. Mixture subfamily

For d linearly independent vectors  $u_1, \ldots, u_d \in \mathbb{R}^d$ , and a vector  $a = (a_1, \ldots, a_{d-k})^T \in \mathbb{R}^{d-k}$ , a subset  $\mathcal{M} \subset \Theta$  is called a *mixture subfamily* generated by the constraint

$$\sum_{i=1}^{d} u_{k+j}^{i} \partial_{i} F(\theta) = a_{j}$$
(18)

for j = 1, ..., d - k when the subset  $\mathcal{M}$  is written as

$$\mathcal{M} = \{ \theta \in \Theta \mid \text{ Condition (18) holds.} \}. \tag{19}$$

We define a  $d \times d$  matrix U as  $(u_1 \dots u_d)$ . To make a parametrization in the above mixture subfamily  $\mathcal{M}$ , we set the new natural parameter  $\bar{\theta} = (\bar{\theta}^1, \dots, \bar{\theta}^d)$  as  $\theta = U\bar{\theta}$ , and introduce the new mixture parameter

$$\bar{\eta}_i = \partial_i (F \circ U)(\bar{\theta}) \tag{20}$$

Since  $\bar{\eta}_{k+i} = a_i$  for  $i = 1, \ldots, d-k$  in  $\mathcal{M}$ , the initial k elements  $\bar{\eta}_1, \ldots, \bar{\eta}_k$  gives a parametrization for  $\mathcal{M}$ . For the parametrization, we define the map  $\psi_{\mathcal{M}}^{(m)}$  as  $\psi_{\mathcal{M}}^{(m)}(U\bar{\theta}) := (\partial_j(F \circ U)(\bar{\theta}))_{j=1}^k$ . We define the set  $\Xi_{\mathcal{M}} := \{\psi_{\mathcal{M}}^{(m)}(\theta)|\theta \in \mathcal{M}\}$  of the new mixture parameters, and the inverse map  $\phi_{\mathcal{M}}^{(m)} := (\psi_{\mathcal{M}}^{(m)})^{-1} : \Xi_{\mathcal{M}} \to \mathcal{M}$ . Since  $\Theta$  is an open set, the set  $\Xi_{\mathcal{M}}$  is an open subset of  $\mathbb{R}^k$ . When an element  $\bar{\eta} \in \Xi_{\mathcal{M}}$  satisfies  $\bar{\eta}_i = \partial_i(F \circ U)(\bar{\theta})$  for  $j = 1, \ldots, k$ , we have

$$\partial^{i}(F \circ U)^{*}(\bar{\eta}, a) = \bar{\theta}^{i} \tag{21}$$

for  $i=1,\ldots,d$ . Since  $\bar{\eta}\mapsto (F\circ U)^*(\bar{\eta},a)$  is strictly convex, the map  $\bar{\eta}\mapsto (\partial^i(F\circ U)^*(\bar{\eta},a))_{i=1}^k$  is one-to-one. Hence, the initial k elements  $\bar{\theta}^1,\ldots,\bar{\theta}^k$  give a parametrization for  $\mathcal{M}$ . That is, we have

$$((U^{-1}\theta)^i)_{i=1}^k = (\partial^i (F \circ U)^* (\psi_{\mathcal{M}}^{(m)}(\theta), a))_{i=1}^k.$$
(22)

We define the set  $\Theta_{\mathcal{M}} := \{((U^{-1}\theta)^i)_{i=1}^k | \theta \in \mathcal{M}\}$ . This set is written as

$$\Theta_{\mathcal{M}} = \left\{ (\theta^1, \dots, \theta^k) \in \mathbb{R}^k \middle| \begin{array}{l} \exists (\theta^{k+1}, \dots, \theta^d) \in \mathbb{R}^{d-k} \text{ such that} \\ \sum_{i=1}^d u^i_{k+j} \partial_i F(U(\theta^1, \dots, \theta^d)) = a_j \\ \text{for } j = 1, \dots, d-k. \end{array} \right\}.$$
 (23)

When the mixture subfamily  $\mathcal{M}$  is an exponential subfamily generated by  $u_1, \ldots, u_k$ , we retake  $\theta_0$  such that  $(U^{-1}\theta_0)^i=0$  for  $i=1,\ldots,k$ . Then, the subsets  $\Theta_{\mathcal{M}}$  and  $\Xi_{\mathcal{M}}$  are the same subsets defined in Subsection II-B.

## D. Bregman Divergence and e- and m- projections

Definition 1 (Bregman Divergence): Let  $\Theta$  be an open set in  $\mathbb{R}^d$  and  $F: \Theta \to \mathbb{R}$  be a  $C^{\infty}$ -class strictly convex function. The Bregman divergence  $D^F$  is defined by

$$D^{F}(\theta_{1} \| \theta_{2}) := \langle \nabla^{(e)}[F](\theta_{1}), \theta_{1} - \theta_{2} \rangle - F(\theta_{1}) + F(\theta_{2}) \ (\theta_{1}, \theta_{2} \in \Theta). \tag{24}$$

We call the triplet  $(\Theta, F, D^F)$  a Bregman divergence system. In the one-parameter case, we have the following lemma.

Lemma 1: Assume that d=1.  $\frac{\partial}{\partial \theta_1} D^F(\theta_1 \| \theta_2) = \frac{d^2}{d\theta^2} F(\theta_1)(\theta_1-\theta_2)$ . Hence, when  $D^F(\theta_1 \| \theta_2)$  is monotonically increasing for  $\theta_1$  in  $(\infty,\theta_2]$ , and is monotonically decreasing for  $\theta_1$  in  $(\theta_2,-\infty)$ . By using the Hesse matrix  $J_{i,j}(\theta) := \frac{\partial^2 F}{\partial \theta^i \partial \theta^j}(\theta)$ , this quantity can be written as

$$D^{F}(\theta_1 \| \theta_2) = \int_0^1 \sum_{i,j} (\theta_1^i - \theta_2^i)(\theta_1^j - \theta_2^j) J_{i,j}(\theta_2 + t(\theta_1 - \theta_2)) t dt.$$
 (25)

This expression shows the inequality

$$D^{F}(\theta_{1}\|\theta_{2}) > D^{F}(\theta_{1}\|\theta_{2} + t(\theta_{1} - \theta_{2})) + D^{F}(\theta_{2} + t(\theta_{1} - \theta_{2})\|\theta_{2})$$
(26)

for  $t \in (0, 1)$ .

For an invertible matrix U, we have

$$D^{F}(\theta_1 \| \theta_2) = D^{F \circ U}(U^{-1}(\theta_1) \| U^{-1}(\theta_2)). \tag{27}$$

Since

$$\frac{\partial}{\partial \theta_2^i} \frac{\partial}{\partial \theta_2^j} D^F(\theta_1 \| \theta_2) = J_{i,j}(\theta_2), \tag{28}$$

 $D^F(\theta_1 \| \theta_2)$  is convex function with respect to the second parameter  $\theta_2$ .

When  $\theta_2$  is given as  $\theta_1 + \Delta \theta$ , and the norm of  $\Delta \theta$  is small, The relation (25) shows that

$$D^{F}(\theta_1 \| \theta_1 + \Delta \theta) = \sum_{i,j} \frac{1}{2} J_{i,j}(\theta_1) (\Delta \theta)^i (\Delta \theta)^j + o(\|\Delta \theta\|^2). \tag{29}$$

Since the relations (2) and (4) imply

$$F^*(\eta) = \sum_{i=1}^d \theta^i \eta(\theta_i) - F(\theta) = \langle \eta(\theta), \theta \rangle - F(\theta), \tag{30}$$

we have

$$D^{F^*}(\nabla^{(e)}[F](\theta_2) \| \nabla^{(e)}[F](\theta_1)) = D^{F^*}(\eta(\theta_2) \| \eta(\theta_1))$$

$$= \langle \eta(\theta_2) - \eta(\theta_1), \theta_2 \rangle - F^*(\eta(\theta_2)) + F^*(\eta(\theta_1))$$

$$= \langle \eta(\theta_1), \theta_1 - \theta_2 \rangle - F(\theta_1) + F(\theta_2) = D^F(\theta_1 \| \theta_2).$$
(31)

Therefore, when  $\theta_2$  is fixed and  $D^F(\theta_1||\theta_2)$  is a convex function for a mixture parameter  $\eta(\theta_1)$ . We define the matrix  $J^*(\theta) := (J^{i,j,*}(\theta))_{i,j}$  as

$$J^{i,j,*}(\theta) := \frac{\partial^2 F^*}{\partial \eta_i \partial \eta_j}(\eta) \tag{32}$$

with  $\eta = \eta(\theta)$ , which is the inverse matrix  $J(\theta)^{-1}$  of  $J(\theta)$ . Applying the formula (25) to  $F^*$ , we have

$$D^{F}(\theta_{1} \| \theta_{2}) = D^{F^{*}}(\eta(\theta_{2}) \| \eta(\theta_{1}))$$

$$= \int_0^1 \sum_{i=1}^d \sum_{j=1}^d (\eta(\theta_2) - \eta(\theta_1))_i (\eta(\theta_2) - \eta(\theta_1))_j J^{i,j,*}(\theta(s)) s ds, \tag{33}$$

where  $\theta(s)$  is defined as  $\eta(\theta(s)) = \eta(\theta_1) + s(\eta(\theta_2) - \eta(\theta_1))$ . Similar to (26), we have

$$D^{F^*}(\eta(\theta_2)||\eta(\theta_1)) = D^F(\theta_2||\theta_1) \ge D^{F^*}(\eta(\theta_2)||\eta(\theta(s))) + D^{F^*}(\eta(\theta(s))||\eta(\theta_1))$$

$$= D^F(\theta_2||\theta(s)) + D^F(\theta(s)||\theta_1).$$
(34)

In fact, when we restrict both inputs into an exponential subfamily  $\mathcal{E}$ , we have the following characterization. That is, the restriction of the Bregman divergence system  $(\Theta, F, D^F)$  to  $\mathcal{E}$  can be considered as the Bregman divergence system  $(\Theta_{\mathcal{E}}, F_{\mathcal{E}}, D^{F_{\mathcal{E}}})$  because we have

$$D^{F}((\phi_{\mathcal{E}}^{(e)}(\bar{\theta}_{1})\|(\phi_{\mathcal{E}}^{(e)}(\bar{\theta}_{2})) = D^{F_{\mathcal{E}}}(\bar{\theta}_{1}\|\bar{\theta}_{2})$$

$$\tag{35}$$

for  $\bar{\theta}_1, \bar{\theta}_2 \in \Theta_{\mathcal{E}}$ .

Using a simple calculation, we can show the following proposition.

Proposition 1 (Pythagorean Theorem [7]): Let  $\mathcal{E} \subset \Theta$  be an exponential subfamily generated by l vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  at  $\theta_0 \in \Theta$ , and  $\mathcal{M} \subset \Theta$  be a mixture subfamily generated by the constraint  $\sum_{i=1}^d v_j^i \eta_i(\theta) = a_j$  for  $j = 1, \ldots, l$ . Assume that an intersection  $\theta^*$  of  $\mathcal{E}$  and  $\mathcal{M}$  exists. For any  $\theta \in \mathcal{E}$  and  $\theta' \in \mathcal{M}$ , we have

$$D^{F}(\theta \| \theta') = D^{F}(\theta \| \theta^*) + D^{F}(\theta^* \| \theta'). \tag{36}$$

*Proof:* To show the relation (36), we choose an invertible matrix  $U = (u_1 \dots u_d)$  such that  $u_i = v_i$  for  $i = 1, \dots, l$ . Using the formula (27), we have

$$D^{F}(\theta \| \theta') = D^{F \circ U}(U^{-1}(\theta) \| U^{-1}(\theta'))$$

$$= \sum_{i=1}^{d} \frac{\partial}{\partial \theta^{i}} F \circ U(\theta) ((U^{-1}\theta)^{i} - (U^{-1}\theta')^{i}) - F(\theta) + F(\theta')$$

$$\stackrel{(a)}{=} \sum_{i=1}^{d} \frac{\partial}{\partial \theta^{i}} F \circ U(\theta) ((U^{-1}\theta)^{i} - (U^{-1}\theta^{*})^{i}) - F(\theta) + F(\theta^{*})$$

$$+ \sum_{i=1}^{l} \frac{\partial}{\partial \theta^{i}} F \circ U(\theta^{*}) ((U^{-1}\theta^{*})^{i} - (U^{-1}\theta')^{i}) - F(\theta^{*}) + F(\theta')$$

$$= D^{F}(\theta \| \theta^{*}) + D^{F}(\theta^{*} \| \theta'), \tag{37}$$

where (a) follows from the following facts; Since  $\theta^*$  and  $\theta'$  belong to the same exponential family  $\mathcal{E},\ (U^{-1}\theta^*)^i=(U^{-1}\theta')^i$  for  $i=l+1,\ldots,d$ . Since  $\theta^*$  and  $\theta$  belong to the same mixture family  $\mathcal{M},\ \frac{\partial}{\partial \theta^i}F\circ U(\theta)=\frac{\partial}{\partial \theta^i}F\circ U(\theta^*)$  for  $i=1,\ldots,l$ .

Lemma 2: Let  $\mathcal{E}$  be an exponential family generated by l vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$ . The following conditions are equivalent for the exponential subfamily  $\mathcal{E}$ ,  $\theta^* \in \mathcal{E}$ , and  $\theta_0 \in \Theta$ .

- (E0) The element  $\theta^* \in \mathcal{E}$  achieves a local minimum for the minimization  $\min_{\hat{\theta} \in \mathcal{E}} D^F(\theta_0 || \hat{\theta})$ .
- (E1) The element  $\theta^* \in \mathcal{E}$  achieves the minimum value for the minimization  $\min_{\hat{\theta} \in \mathcal{E}} D^F(\theta_0 || \hat{\theta})$ .
- (E2) Let  $\mathcal{M} \subset \Theta$  be the mixture subfamily generated by the constraint  $\sum_{i=1}^{d} v_j^i \eta_i(\theta) = \sum_{i=1}^{d} v_j^i \eta_i(\theta_0)$  for  $j = 1, \dots, l$ . The element  $\theta^* \in \mathcal{E}$  belongs to the intersection  $\mathcal{M} \cap \mathcal{E}$ .

Further, when there exists an element  $\theta^* \in \mathcal{E}$  to satisfy the above condition, such an element is unique.

In the following, we denote the above mixture family  $\mathcal{M}$  by  $\mathcal{M}_{\theta_0 \to \mathcal{E}}$ . Then,  $\theta^* \in \mathcal{E}$  is called the *e-projection* of  $\theta$  onto an exponential subfamily  $\mathcal{E}$ , and is denoted by  $\Gamma_{\mathcal{E}}^{(e),F}(\theta)$  because the points  $\theta$  and  $\theta^*$  are connected via the mixture family  $\mathcal{M}_{\theta_0 \to \mathcal{E}}$ . We call the minimum  $\min_{\hat{\theta} \in \mathcal{E}} D^F(\theta \| \hat{\theta})$  the projected Bregman divergence between  $\theta$  and  $\mathcal{E}$ .

*Proof:* Assume that (E0) holds. When an element  $\hat{\theta} \in \mathcal{E}$  belongs to the neighbor hood of  $\theta^*$ , we have

$$D^{F}(\theta_{0}||\hat{\theta}) - D^{F}(\theta_{0}||\theta^{*})$$

$$= \sum_{i=1}^{l} \frac{\partial}{\partial \theta^{i}} F \circ U(\theta_{0}) ((U^{-1}\theta^{*})^{i} - (U^{-1}\hat{\theta})^{i}) - F(\theta^{*}) + F(\hat{\theta})$$

$$= \sum_{i=1}^{l} \left( \frac{\partial}{\partial \theta^{i}} F \circ U(\theta^{*}) - \frac{\partial}{\partial \theta^{i}} F \circ U(\theta_{0}) \right) ((U^{-1}\theta^{*})^{i} - (U^{-1}\hat{\theta})^{i})$$

$$+ \sum_{i=1}^{l} \frac{\partial}{\partial \theta^{i}} F \circ U(\theta_{0}) ((U^{-1}\theta^{*})^{i} - (U^{-1}\hat{\theta})^{i}) - F(\theta^{*}) + F(\hat{\theta})$$

$$= \sum_{i=1}^{l} \left( \frac{\partial}{\partial \theta^{i}} F \circ U(\theta^{*}) - \frac{\partial}{\partial \theta^{i}} F \circ U(\theta_{0}) \right) ((U^{-1}\theta^{*})^{i} - (U^{-1}\hat{\theta})^{i})$$

$$+ D^{F}(\theta^{*}||\hat{\theta}). \tag{38}$$

In the following, assuming  $(\frac{\partial}{\partial \theta^i}F \circ U(\theta^*))_{i=1}^l \neq (\frac{\partial}{\partial \theta^i}F \circ U(\theta_0))_{i=1}^l$ , we derive the contradiction. Since  $\theta^*$  is an inner element of  $\mathcal{E}$ , we choose an element  $\hat{\theta} \in \mathcal{E}$  as  $\theta^* + x\Delta\theta$  such that  $T := \sum_{i=1}^l \left(\frac{\partial}{\partial \theta^i}F \circ U(\theta^*) - \frac{\partial}{\partial \theta^i}F \circ U(\theta_0)\right)(\Delta\theta)^i < 0$ . Then, due to (29), the divergence  $D^F(\theta^*\|\hat{\theta})$  behaves as the order  $O(x^2)$ . Hence, choosing sufficiently small x, we have  $D^F(\theta_0\|\hat{\theta}) - D^F(\theta_0\|\theta^*) = Tx + O(x^2) < 0$ , which implies contradiction. Hence, we have  $(\frac{\partial}{\partial \theta^i}F \circ U(\theta^*))_{i=1}^l = (\frac{\partial}{\partial \theta^i}F \circ U(\theta_0))_{i=1}^l$ , which implies that  $\theta^*$  is an intersection between  $\mathcal{M}$  and  $\mathcal{E}$ . Hence, (E2) holds.

Assume that (E2) holds. Let  $\theta^*$  an intersection between  $\mathcal{M}$  and  $\mathcal{E}$ . Then, the relation (36) guarantees that the element  $\theta^*$  realizes the minimum  $\min_{\hat{\theta} \in \mathcal{E}} D^F(\theta_0 || \hat{\theta})$ . Hence, (E1) holds. Further, (E1) implies (E0).

When there are two different intersections between  $\mathcal{M}$  and  $\mathcal{E}$ , the above discussion and the relation (36) guarantee that the divergence between two intersections must be zero, which yields contradiction. Thus, the intersection between  $\mathcal{M}$  and  $\mathcal{E}$  should be unique.

Exchanging the roles of the exponential family and the mixture family, we have the following lemma. Lemma 3: We choose l vectors  $v_1,\ldots,v_l\in\mathbb{R}^d$ . Let  $\mathcal{M}$  be an mixture family generated by generated by the constraint  $\sum_{i=1}^d v_j^i \eta_i(\theta) = \sum_{i=1}^d v_j^i \eta_i(\theta_0)$  for  $j=1,\ldots,l$ . The following conditions are equivalent for the mixture subfamily  $\mathcal{M}$ ,  $\theta^{**}\in\mathcal{M}$ , and  $\theta_0\in\Theta$ .

- (M0) The element  $\theta^{**} \in \mathcal{M}$  achieves a local minimum for the minimization  $\min_{\hat{\theta} \in \mathcal{M}} D^F(\hat{\theta} || \theta_0)$ .
- (M1) The element  $\theta^{**} \in \mathcal{M}$  achieves the minimum value for the minimization  $\min_{\hat{\theta} \in \mathcal{M}} D^F(\hat{\theta} \| \theta_0)$ .
- (M2) Let  $\mathcal{E} \subset \Theta$  be the mixture subfamily generated by l vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  at  $\theta_0 \in \Theta$ . The element  $\theta^{**} \in \mathcal{M}$  belongs to the intersection  $\mathcal{M} \cap \mathcal{E}$ .

Further, when there exists an element  $\theta^{**} \in \mathcal{M}$  to satisfy the above condition, such an element is unique. In the following, we denote the above exponential family  $\mathcal{E}$  by  $\mathcal{E}_{\theta_0 \to \mathcal{M}}$ . Then,  $\theta^{**} \in \mathcal{M}$  is called the m-projection of  $\theta$  onto an mixture subfamily  $\mathcal{M}$ , and is denoted by  $\Gamma_{\mathcal{M}}^{(m),F}(\theta)$  because the points  $\theta$  and  $\theta^{**}$  are connected via the exponential family  $\mathcal{E}_{\theta_0 \to \mathcal{M}}$ . When  $\mathcal{M}$  is an exponential subfamily and a mixture subfamily, we can define both projections  $\Gamma_{\mathcal{M}}^{(e),F}$  and  $\Gamma_{\mathcal{M}}^{(m),F}$ , and these projections are different maps. Hence, the subscripts (e) and (m) are needed.

Lemma 4: Let  $\mathcal{E} \subset \Theta$  be an exponential subfamily generated by l vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  at  $\theta_0 \in \Theta$ . For  $\theta_* \in \Theta$ , the element  $\Gamma_{\mathcal{E}}^{(e),F}(\theta_*) = \theta^* \in \mathcal{E}$  is uniquely characterized as  $\sum_{j=1}^d v_i^j \partial_j F(\theta^*) = \sum_{j=1}^d v_i^j \partial_j F(\theta_*)$ , i.e.,  $R[V] \circ \nabla[F](\theta^*) = R[V] \circ \nabla[F](\theta_*)$ . That is, the mixture parameter of the element  $\Gamma_{\mathcal{E}}^{(e),F}(\theta_*) = \theta^* \in \mathcal{E}$  is given by the above condition.

*Proof:* We choose the mixture subfamily  $\mathcal{M}$  generated by the constraint

$$\sum_{j=1}^{d} v_i^j \partial_j F(\theta) = \sum_{j=1}^{d} v_i^j \partial_j F(\theta_*)$$
(39)

for i = 1, ..., l. Due to Pythagorean theorem (Proposition 1), the point  $\theta^*$  is characterized by the intersection between  $\mathcal{M}$  and  $\mathcal{E}$ . Hence, the constraint (39) for  $\mathcal{M}$  guarantees the desired statement.

Lemma 5: Let l vectors  $u_1, \ldots, u_d \in \mathbb{R}^d$  be linearly independent. Let  $\mathcal{M} \subset \Theta$  be a mixture subfamily generated by the constraint

$$\sum_{i=1}^{d} u_j^i \partial_i F(\theta) = a_j \tag{40}$$

for  $j=k+1,\ldots,d$ . When the maximum  $\max_{\theta\in\mathcal{M}}D^F(\theta\|\theta_{**})$  exists, we obtain the following characterizations for  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{**})$ .

(A1) The point  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{**})=\theta^{**}\in\mathcal{M}$  is uniquely characterized as

$$(U^{-1}\theta^{**})^i = (U^{-1}\theta_{**})^i \tag{41}$$

for i = 1, ..., k, where U is defined in the same way as Subsection II-C.

- We choose the exponential subfamily  $\mathcal{E}$  generated by d-k vectors  $u_{k+1},\ldots,u_d\in\mathbb{R}^d$  at  $\theta_{**}$ . The intersection between  $\mathcal{M}$  and  $\mathcal{E}$  is composed of the unique element  $\Gamma^{(m),F}_{\mathcal{M}}(\theta_{**})$ . The point  $\Gamma^{(m),F}_{\mathcal{M}}(\theta_{**})=\theta^{**}\in\mathcal{M}$  is uniquely characterized as  $\theta_{**}+\sum_{j'=1}^{d-k}\bar{\tau}^{j'}u_{k+j'}$ , where  $(\bar{\tau}^1,\ldots,\bar{\tau}^{d-k})$  is the unique element to satisfy

$$\frac{\partial}{\partial \tau^j} F\left(\theta_* + \sum_{j'=1}^l \tau^{j'} u_{k+j'}\right) = a_j \tag{42}$$

for j = 1, ..., d - k.

*Proof:* To characterize elements of  $\mathcal{M}$ , we employ the parameter  $\bar{\eta}$  defined in (20). Then, the set  $\mathcal{M}$  is given as  $\{(\bar{\eta}_1,\ldots,\bar{\eta}_k,a_1,\ldots,a_{d-k})|(\bar{\eta}_1,\ldots,\bar{\eta}_k)\in\mathbb{R}^k\}$  under this parameterization. Then, using (31), we have

$$D^{F}(\phi_{\mathcal{M}}^{(m)}(\bar{\eta}_{1},\ldots,\bar{\eta}_{k},a_{1},\ldots,a_{d-k})\|\theta_{**})$$

$$=D^{(F\circ U)^{*}}(\psi_{\mathcal{M}}^{(m)}(\theta_{**})\|(\bar{\eta}_{1},\ldots,\bar{\eta}_{k},a_{1},\ldots,a_{d-k}))$$
(43)

Since the map  $(\bar{\eta}_1,\ldots,\bar{\eta}_k)\mapsto D^{(F\circ U)^*}(\psi_{\mathcal{M}}^{(m)}(\theta_{**})\|(\bar{\eta}_1,\ldots,\bar{\eta}_k,a_1,\ldots,a_{d-k}))$  is smooth and convex, the minimum  $\min_{(\bar{\eta}_1,\ldots,\bar{\eta}_k)}D^{(F\circ U)^*}(\psi_{\mathcal{M}}^{(m)}(\theta_{**})\|(\bar{\eta}_1,\ldots,\bar{\eta}_k,a_1,\ldots,a_{d-k}))$  is realized when

$$\partial^{i}(F \circ U)^{*}(\bar{\eta}_{1}, \dots, \bar{\eta}_{k}, a_{1}, \dots, a_{d-k}) = \partial^{i}(F \circ U)^{*}(\psi_{\mathcal{M}}^{(m)}(\theta_{**})). \tag{44}$$

for i = 1, ..., k. Since (44) is equivalent to (41) due to (21), we obtain (A1).

The exponential subfamily  $\mathcal{E}$  is characterized as  $\{\theta|(U^{-1}\theta)^i=(U^{-1}\theta_{**})^i \text{ for } i=1,\ldots,k\}$ . Then, we find that the intersection between  $\mathcal{M}$  and  $\mathcal{E}$  is not empty and contains  $\theta^{**}$ . Further, when an element  $\theta$ belongs to the intersection between  $\mathcal{M}$  and  $\mathcal{E}$ , the Pythagorean theorem (Proposition 1) guarantees that the element  $\theta$  realizes the maximum  $\max_{\theta \in \mathcal{M}} D^F(\theta \| \theta_{**})$ . Hence, the intersection between  $\mathcal{M}$  and  $\mathcal{E}$  is composed of the unique element  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{**})$ . Hence, we obtain (A2). Due to (A2), the unique element  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{**})$  is characterized as an element in  $\mathcal{E} = \{\theta_{**} + \sum_{j'=1}^{d-k} \tau^{j'} u_{k+j'} | (\tau^1, \dots, \tau^{d-k}) \in \mathbb{R}^{d-k} \}$  to satisfy (42). Hence, we obtain (A3).

Due to Lemmas 2 and 3, it is important to find a sufficient condition for (E2) and (M2). To discuss this issue for a convex function F and  $\Theta$ , we fix l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$ . Then, we consider the following conditions;

- (M3) We denote the exponential family generated by the l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  at  $\theta_0 \in \Theta$  by  $\mathcal{E}(\theta_0)$ . The l-dimensional parameter space  $\Theta_{\mathcal{E}(\theta_0)}$  does not depend on  $\theta_0 \in \Theta$ . while the space  $\Theta_{\mathcal{E}(\theta_0)}$  is defined in the way as (16). In this case, this set is denoted by  $\Xi(v_1, \ldots, v_l)$ .
- (E3) We denote the mixture family generated by the constraint  $\sum_{i=1}^{d} v_j^i \partial_i F(\theta) = a_j$  for  $j = 1, \ldots, l$  by  $\mathcal{M}(a_1, \ldots, a_l)$ . The d-l-dimensional parameter space  $\Theta_{\mathcal{M}(a_1, \ldots, a_l)}$  does not depend on  $(a_1, \ldots, a_l) \in \mathbb{R}^l$  unless  $\mathcal{M}(a_1, \ldots, a_l)$  is empty while the space  $\Theta_{\mathcal{M}(a_1, \ldots, a_l)}$  is defined in the way as (23). In this case, this set is denoted by  $\Theta(v_1, \ldots, v_l)$ .

Under the above condition, we have the following lemmas.

Lemma 6: Assume that the l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  satisfy Condition (M3). Given  $(a_1, \ldots, a_l) \in \Xi(v_1, \ldots, v_l)$ , we define the mixture family  $\mathcal{M}(a_1, \ldots, a_l)$  by using the condition (40). Then, for  $\theta_0 \in \Theta$ , the projected point  $\Gamma^{(m),F}_{\mathcal{M}(a_1,\ldots,a_l)}(\theta_0)$  exists. Proof: When the assumption holds, for  $\theta_0 \in \Theta$ , the exponential family  $\mathcal{E}(\theta_0)$  contains an element whose

*Proof:* When the assumption holds, for  $\theta_0 \in \Theta$ , the exponential family  $\mathcal{E}(\theta_0)$  contains an element whose mixture parameter is  $(a_1, \ldots, a_l)$ . Hence, due to Lemma 3, the exponential family  $\mathcal{E}(\theta_0)$  and the mixture family  $\mathcal{M}(a_1, \ldots, a_l)$  have a unique intersection. Therefore, the projected point  $\Gamma^{(m),F}_{\mathcal{M}(a_1,\ldots,a_l)}(\theta_0)$  exists unless  $\mathcal{M}(a_1,\ldots,a_l)$  is empty.

Lemma 7: Assume that the l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  satisfy Condition (E3). Then, for  $(b^1, \ldots, b^{d-l}) \in \mathbb{R}^{d-l}$  and  $\theta_0 \in \Theta$ , the projected point  $\Gamma^{(e), F}_{\mathcal{E}(b^1, \ldots, b^{d-l})}(\theta_0)$  exists unless  $\mathcal{E}(b^1, \ldots, b^{d-l})$  is empty where the exponential family  $\mathcal{E}(b^1, \ldots, b^{d-l})$  is defined as  $\{(\sum_{i=1}^{d-l} u_i^j b^i + \sum_{i=1}^l u_i^j \theta^i)_{j=1}^d | (\theta^1, \ldots, \theta^l) \in \mathbb{R}^l \} \cap \Theta$ .

*Proof:* Assume that the assumption holds. For  $\theta_0 \in \Theta$ , we define the mixture family  $\mathcal{M}(\theta_0)$  by using the constraint;  $\sum_{j=1}^d v_i^j \partial_j F(\theta) = \sum_{j=1}^d v_i^j \partial_j F(\theta_0)$  for  $i=1,\ldots,l$ . Then, the mixture family  $\mathcal{M}(\theta_0)$  contains an element whose natural parameter is  $(b^1,\ldots,b^{d-l})$ . Hence, due to Lemma 2 the mixture  $\mathcal{M}(\theta_0)$  and the exponential family  $\mathcal{E}(b^1,\ldots,b^{d-l})$  have a unique intersection. Therefore, the projected point  $\Gamma^{(e),F}_{\mathcal{E}(b^1,\ldots,b^{d-l})}(\theta_0)$  exists unless  $\mathcal{E}(b^1,\ldots,b^{d-l})$  is empty.

In addition, we introduce the following conditions for the Bregman divergence system  $(\Theta, F, D^F)$ .

- (M4) Any l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  satisfy Condition (M3) for  $l = 1, \ldots, d-1$ .
- (E4) Any l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^d$  satisfy Condition (E3) for  $l = 1, \ldots, d-1$ . When (M4) holds, the m-projection  $\Gamma_{\mathcal{M}}^{(m),F}$  can be defined for any mixture subfamily  $\mathcal{M}$ . Also, when (E4) holds, the e-projection  $\Gamma_{\mathcal{E}}^{(e),F}$  can be defined for any exponential subfamily  $\mathcal{E}$ . Therefore, these two conditions are helpful for the analysis of these projections.

TABLE I SUMMARY OF DIMENSIONS

Symbol	Space
d	Dimension of the whose space
l	Dimension of Exponential family $\mathcal{E}$
k	Dimension of Mixture family $\mathcal{M}$

#### E. Evaluation of Bregman divergence without Pythagorean theorem

Next, we evaluate Bregman divergence when we cannot use Pythagorean theorem. For this aim, we focus on  $J(\theta)^{-1}$ , i.e., the inverse of the Hesse matrix  $J(\theta)$  defined for the parameters of  $\Theta$ . Then, we introduce the following quantity  $\gamma(\hat{\Theta}|\Theta)$  for a subset  $\hat{\Theta}$  of  $\Theta$ .

$$\gamma(\hat{\Theta}|\Theta) := \inf\{\gamma | \gamma J(\theta_1)^{-1} \ge J(\theta_2)^{-1} \text{ for } \theta_1, \theta_2 \in \hat{\Theta}\}$$
(45)

(46)

We say that a subset  $\hat{\Theta}$  of  $\Theta$  is a *star subset* for an element  $\theta_1 \in \hat{\Theta}$  when  $\lambda \eta(\theta) + (1 - \lambda) \eta(\theta_1) \in \eta(\hat{\Theta})$  for  $\theta \in \hat{\Theta}$  and  $\lambda \in (0, 1)$ .

Then, we have the following theorem.

Theorem 1: We assume that Condition (M4) holds. Then, for a star subset with  $\hat{\Theta}$  for  $\theta_1 \in \hat{\Theta}$ ,  $\theta_2 \in \hat{\Theta}$ , and  $\theta_3 \in \Theta$ , we have

$$D^{F}(\theta_{1}\|\theta_{2}) \leq D^{F}(\theta_{1}\|\theta_{3}) + \gamma(\hat{\Theta}|\Theta)D^{F}(\theta_{2}\|\theta_{3}) + 2\gamma(\hat{\Theta}|\Theta)\sqrt{D^{F}(\theta_{1}\|\theta_{3})D^{F}(\theta_{2}\|\theta_{3})}.$$
(47)

The proof of Theorem 1 is given in Appendix B.

# F. Bregman divergence system for mixture subfamily

When  $\mathcal{E}$  is an exponential subfamily, the triplet  $(\Theta_{\mathcal{E}}, F_{\mathcal{E}}, D^{F_{\mathcal{E}}})$  is a Bregman divergence system as explained in (35). However, when  $\mathcal{M}$  is a mixture subfamily and it is not an exponential subfamily, it is not so trivial to recover a Bregman divergence system. We use the symbols defined in Subsection II-C. Any element in  $\mathcal{M}$  can be parameterized by an element  $\bar{\theta} \in \Theta_{\mathcal{M}}$ . Therefore, there uniquely exists an vector  $\kappa(\bar{\theta}) \in \mathbb{R}^{d-k}$  such that  $U(\bar{\theta}, \kappa(\bar{\theta})) \in \mathcal{M}$ . Then, we define the map  $\phi_{\mathcal{M}}^{(e)} : \Theta_{\mathcal{M}} \to \mathcal{M}$  as  $\phi_{\mathcal{M}}^{(e)}(\bar{\theta}) := U(\bar{\theta}, \kappa(\bar{\theta}))$ , and its inverse map  $\psi_{\mathcal{M}}^{(e)} := (\phi_{\mathcal{M}}^{(e)})^{-1} : \mathcal{M} \to \Theta_{\mathcal{M}}$ .

A convex function  $F_{\mathcal{M}}(\bar{\theta})$  is defined as

$$F_{\mathcal{M}}(\bar{\theta}) := (F \circ U)(\bar{\theta}, \kappa(\bar{\theta})) - \sum_{i=k+1}^{d} \partial_{i}(F \circ U)(\bar{\theta}, \kappa(\bar{\theta}))\kappa^{i-k}(\bar{\theta})$$

$$= (F \circ U)(\bar{\theta}, \kappa(\bar{\theta})) - \sum_{i=k+1}^{d} a_{i}\kappa^{i-k}(\bar{\theta}). \tag{48}$$

 $(F \circ U)^*|_{\Xi_{\mathcal{M}}}$  is a convex function. Due to (22), the Legendre transform of  $(F \circ U)^*|_{\Xi_{\mathcal{M}}}$  is  $F_{\mathcal{M}}(\bar{\theta})$ . Hence,  $F_{\mathcal{M}}(\bar{\theta})$  is a convex function.

Also, we have

$$\frac{\partial}{\partial \bar{\theta}^{j}} F_{\mathcal{M}}(\bar{\theta})$$

$$= \partial_{j} (F \circ U)(\bar{\theta}, \kappa(\bar{\theta})) + \sum_{i=k+1}^{d} \partial_{i} (F \circ U)(\bar{\theta}, \kappa(\bar{\theta})) \partial_{j} \kappa^{i-k}(\bar{\theta}) - \sum_{i=k+1}^{d} a_{i} \kappa^{i-k}(\bar{\theta})$$

$$= \partial_{j} (F \circ U)(\bar{\theta}, \kappa(\bar{\theta})).$$
(49)

Thus,

$$D^{F_{\mathcal{M}}}(\bar{\theta}_{1}||\bar{\theta}_{2})$$

$$=F_{\mathcal{M}}(\bar{\theta}_{1}) - F_{\mathcal{M}}(\bar{\theta}_{2}) - \sum_{j=1}^{k} \frac{\partial}{\partial \bar{\theta}^{j}} F_{\mathcal{M}}(\bar{\theta})(\bar{\theta}_{1}^{j} - \bar{\theta}_{2}^{j})$$

$$=F \circ U(\bar{\theta}_{1}, \kappa(\bar{\theta}_{1})) - F \circ U(\bar{\theta}_{2}, \kappa(\bar{\theta}_{2}))$$

$$- \sum_{j=1}^{d} \partial_{j} (F \circ U)((\bar{\theta}_{1}, \kappa(\bar{\theta}_{1}))^{j} - (\bar{\theta}_{2}, \kappa(\bar{\theta}_{2}))^{j})$$

$$=D^{F \circ U}((\bar{\theta}_{1}, \kappa(\bar{\theta}_{1}))||(\bar{\theta}_{2}, \kappa(\bar{\theta}_{2}))) = D^{F}(U(\bar{\theta}_{1}, \kappa(\bar{\theta}_{1}))||U(\bar{\theta}_{2}, \kappa(\bar{\theta}_{2})))$$

$$=D^{F}(\phi_{\mathcal{M}}^{(e)}(\bar{\theta}_{1})||\phi_{\mathcal{M}}^{(e)}(\bar{\theta}_{2})). \tag{50}$$

Therefore, the Bregman divergence in the Bregman divergence system  $(\Theta_{\mathcal{M}}, F_{\mathcal{M}}, D^{F_{\mathcal{M}}})$  equals the Bregman divergence in the Bregman divergence system  $(\Theta, F, D^F)$  for two elements in  $\mathcal{M}$ .

A subset  $\mathcal{E} \subset \mathcal{M}$  is called an l-dimensional exponential subfamily of  $\mathcal{M}$  generated by l linearly independent vectors  $v_1, \ldots, v_l \in \mathbb{R}^k$  at  $\theta_0 \in \Theta_M$  with  $l \leq k$  when the subset  $\mathcal{E}$  is given as

$$\mathcal{E} = \left\{ \phi_{\mathcal{M}}^{(e)} \left( \theta_0 + \sum_{i=1}^l \bar{\theta}^i v_i \right) \middle| \bar{\theta} \in \mathbb{R}^l \right\} \cap \mathcal{M}.$$
 (51)

A subset  $\mathcal{M}_1 \subset \Theta$  is called an *l*-dimensional *mixture subfamily* of  $\mathcal{M}$  generated by the additional constraints

$$\sum_{i=1}^{k} v_j^i \eta_i = a_j \tag{52}$$

for j = 1, ..., l with  $v_1, ..., v_l \in \mathbb{R}^k$  when the subset  $\mathcal{M}_1$  is written as

$$\mathcal{M}_1 = \left\{ \phi_{\mathcal{M}}^{(m)}(\eta) \in \mathcal{M} \middle| \eta \in \Xi_{\mathcal{M}} \text{ satisfies Condition (52).} \right\}$$
 (53)

# G. Closed convex mixture subfamily

A closed subset  $\mathcal{M}$  of a mixture subfamily  $\hat{\mathcal{M}}$  is called a closed mixture subfamily. The mixture subfamily  $\hat{\mathcal{M}}$  is called the *extended mixture family* of  $\mathcal{M}$  when  $\hat{\mathcal{M}}$  and  $\mathcal{M}$  have the same dimension. When a closed mixture subfamily  $\mathcal{M}$  is a convex set with respect to the mixture parameter, it is called a closed convex mixture subfamily.

We define the boundary set  $\partial \mathcal{M} := \mathcal{M} \setminus \operatorname{int} \mathcal{M}$ , where  $\operatorname{int} \mathcal{M}$  is the interior of  $\mathcal{M}$ . For an element  $\theta \in \partial \mathcal{M}$ , a d-1-dimensional mixture family  $\mathcal{M}'$  is called a tangent space of  $\mathcal{M}$  at  $\theta$  when  $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$ and  $\mathcal{M}' \cap \operatorname{int} \mathcal{M} = \emptyset$ . When  $\mathcal{M}$  is a closed convex mixture subfamily, any element  $\theta \in \partial \mathcal{M}$  has a tangent space. When  $\mathcal{M}$  is composed of one element, we consider that  $\hat{\mathcal{M}} := \mathcal{M}$ ,  $\partial \mathcal{M} = \emptyset$ , and  $\operatorname{int} \mathcal{M} = \mathcal{M}$ .

Lemma 8: Assume that the Bregman divergence system  $(\Theta, F, D^F)$  satisfies Condition (M4). For any element  $\theta \in \Theta$  and any closed convex mixture subfamily  $\mathcal{M}$ , there uniquely exists a minimum point

$$\Gamma_{\mathcal{M}}^{(m),F}(\theta) := \underset{\theta' \in \mathcal{M}}{\operatorname{argmin}} D^F(\theta' \| \theta). \tag{54}$$

In addition, any element  $\theta' \in \mathcal{M}$  satisfies the inequality

$$D^{F}(\theta'\|\theta) \ge D^{F}(\theta'\|\Gamma_{\mathcal{M}}^{(m),F}(\theta)) + D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta)\|\theta). \tag{55}$$

Further, we denote the extended mixture family of  $\mathcal{M}$  by  $\hat{\mathcal{M}}$ . When  $\theta$  belongs to  $\hat{\mathcal{M}} \setminus \mathcal{M}$ , then,

$$\Gamma_{\mathcal{M}}^{(m),F}(\theta) \in \partial \mathcal{M}. \tag{56}$$

*Proof:* Step 1: We choose a sequence  $\theta^{(n)} \in \partial \mathcal{M}$  such that

$$\lim_{n \to \infty} D^F(\theta^{(n)} \| \theta) = \inf_{\theta' \in \mathcal{M}} D^F(\theta' \| \theta). \tag{57}$$

Since  $\{\theta' \in \mathcal{M} | D^F(\theta' \| \theta) \leq D^F(\theta^{(1)} \| \theta) \}$  is a compact subset, there exists a subsequence of  $n_m$  such that  $\theta^{(n_m)}$  converges. Since  $\mathcal{M}$  is a closed subset,  $\theta^* := \lim_{m \to \infty} \theta^{(n_m)}$  belongs to  $\mathcal{M}$ .

We define the vector  $v := (\theta^* - \theta) \in \mathbb{R}^d$  and the real numbers  $a^* := \sum_{i=1}^d v^i \partial_i F(\theta^*) \in \mathbb{R}$  and

 $b^* := \sum_{i=1}^d v^i \partial_i F(\theta) \in \mathbb{R}$ . When  $a^* > b^*$ , as shown in Step 2, any element  $\theta' \in \mathcal{M}$  satisfies

$$\sum_{i=1}^{d} v^{i} \partial_{i} F(\theta') \ge a^{*}. \tag{58}$$

Otherwise, any element  $\theta' \in \mathcal{M}$  satisfies

$$\sum_{i=1}^{d} v^{i} \partial_{i} F(\theta') \le a^{*}. \tag{59}$$

**Step 2:** We show only (58) by contradiction because the relation (59) can be shown in the same way. We choose an element  $\theta' \in \mathcal{M}$  such that (58) does not hold. We denote the mixture parameters of  $\theta^*$  and  $\theta'$  by  $\eta^*$  and  $\eta'$ . Since  $\mathcal{M}$  is convex with respect to the mixture parameter,  $\theta(\eta^* + t(\eta' - \eta^*))$  belongs to  $\mathcal{M}$  for  $t \in [0, 1]$ .

We denote the one-dimensional exponential subfamily  $\{\theta + t(\theta^* - \theta)\}_{t \in \mathbb{R}} \cap \Theta$  by  $\mathcal{E}_1$ . We define  $a(t) := \sum_{i=1}^d v^i \partial_i F(\theta(\eta^* + t(\eta' - \eta^*)))$ . We denote the d-1-dimensional mixture subfamily  $\{\theta'' \in \Theta | \sum_{i=1}^d v^i \partial_i F(\theta'') = a(t)\}$  by  $\mathcal{M}(t)$ . Condition (M4) guarantees that the intersection  $\mathcal{M}(t) \cap \mathcal{E}_1$  is composed of only one element. We denote the element by  $\theta(t)$ . Then, we have

$$D^{F}(\theta(\eta^* + t(\eta' - \eta^*)) \| \theta) = D^{F}(\theta(\eta^* + t(\eta' - \eta^*)) \| \theta(t)) + D^{F}(\theta(t) \| \theta).$$
(60)

We assume that t > 0 is sufficiently small. Since (58) does not hold, the formula (25) implies  $D^F(\theta^* \| \theta) - D^F(\theta(t) \| \theta) = O(t)$ . However,  $D^F(\theta(\eta^* + t(\eta' - \eta^*)) \| \theta(t)) = O(t)$ . The combination of these relations shows that

$$D^{F}(\theta(\eta^* + t(\eta' - \eta^*)) \| \theta) < D^{F}(\theta^* \| \theta), \tag{61}$$

which yields the contradiction.

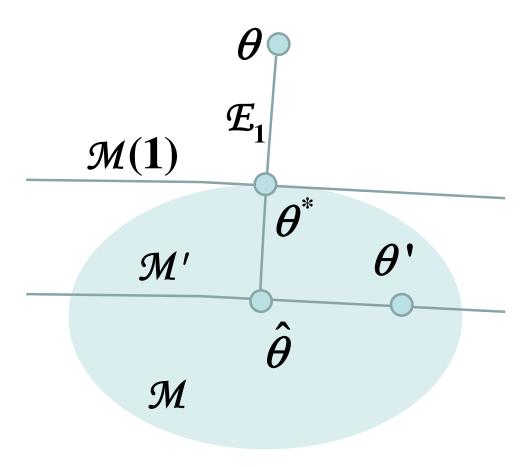


Fig. 1. Figure for Step 3 of the proof of Lemma 8.

**Step 3:** We show the uniqueness of the minimum point and (55) only in the case when  $a^* > b^*$ . We can show the other case in the same way.

We define  $a' := \sum_{i=1}^d v^i \partial_i F(\theta')$ . We denote the d-1-dimensional mixture subfamily  $\{\theta'' \in \Theta | \sum_{i=1}^d v^i \partial_i F(\theta'') = a'\}$  by  $\mathcal{M}'$ . Condition (M4) guarantees that the intersection  $\mathcal{M}' \cap \mathcal{E}_1$  is composed of only one element. We denote the element by  $\hat{\theta}$ . Hence, we have

$$D^{F}(\theta'\|\theta) \stackrel{(a)}{=} D^{F}(\theta'\|\hat{\theta}) + D^{F}(\hat{\theta}\|\theta)$$

$$\stackrel{(b)}{\geq} D^{F}(\theta'\|\hat{\theta}) + D^{F}(\hat{\theta}\|\theta^{*}) + D^{F}(\theta^{*}\|\theta)$$

$$\stackrel{(c)}{=} D^{F}(\theta'\|\theta^{*}) + D^{F}(\theta^{*}\|\theta),$$
(62)

where (a) and (c) follow from Proposition 1, and (b) follows from (26). The relation (62) implies that  $D^F(\theta'\|\theta) > D^F(\theta^*\|\theta)$ . Hence, we obtain the uniqueness of the minimum point. Also, (62) implies (55). **Step 4:** We show (56) by contradiction only in the case when  $a^* > b^*$  because we can show the other case in the same way. We assume that (56) does not hold. We parameterize the exponential family  $\mathcal{E}_1$  as  $\{\theta_t\}$  such that  $\theta_0 = \theta$  and  $\theta_1 = \theta_* = \Gamma_{\mathcal{M}}^{(m),F}(\theta)$ . Since  $\theta_1 \in \int \mathcal{M}$ , there is an element  $t_0 \in (0,1)$  such that  $\theta_t \in \partial \mathcal{M}$ . Hence, Lemma 1 guarantees that  $D^F(\theta_t\|\theta_0) > D^F(\theta_1\|\theta_0)$ , which contradicts that  $\theta_1 = \Gamma_{\mathcal{M}}^{(m),F}(\theta)$ .

We say that a set of closed convex mixture subfamilies  $\{\mathcal{M}_{\lambda}\}_{{\lambda}\in\Lambda}$  covers the boundary  $\partial\mathcal{M}$  of a closed convex mixture family  $\mathcal{M}$  with subsets  $\Lambda_{\lambda}\subset\Lambda$  and  $\lambda\in\Lambda_{*}:=\Lambda\cup\{0\}$  when the following two conditions hold; The relation

$$\partial \mathcal{M}_{\lambda} = \cup_{\lambda' \in \Lambda_{\lambda}} \mathcal{M}_{\lambda'} \tag{63}$$

holds unless  $\partial \mathcal{M}_{\lambda} = \emptyset$ . That is, when  $\partial \mathcal{M}_{\lambda} = \emptyset$ ,  $\Lambda_{\lambda}$  is the empty set. The relation

$$\mathcal{M}_{\lambda'} \not\subset \mathcal{M}_{\lambda''}$$
 (64)

holds for two elements  $\lambda', \lambda'' \in \Lambda_{\lambda}$ . That is,  $0 \in \Lambda_*$  is considered as the index to express  $\mathcal{M}$ . Hence, we define  $\mathcal{M}_0 := \mathcal{M}$ .

Also, we define the subset  $\bar{\Lambda}_{\lambda} := \{\lambda' \in \Lambda | \exists \lambda_2, \dots, \lambda_{x-1} \text{ such that } \lambda_{i+1} \in \Lambda_{\lambda_i} \text{ with } \lambda_1 = \lambda, \lambda_x = \lambda' \}$ . In addition, we define the depth  $D(\lambda)$  of an element  $\lambda \in \Lambda$  as follows. The depth  $D(\lambda)$  of an element  $\lambda$  is zero when  $\Lambda_{\lambda}$  is the empty set. Otherwise, the depth  $D(\lambda)$  of an element  $\lambda$  is defined as  $1 + \max_{\lambda' \in \Lambda_{\lambda}} D(\lambda')$ . Then, the depth of  $\mathcal{M}$  is defined to be the depth D(0).

Lemma 9: The sets int  $M_{\lambda}$  are disjoint, i.e.,

$$int M_{\lambda} \cap int M_{\lambda'} = \emptyset \text{ for } \lambda \neq \lambda' \in \Lambda_*.$$
 (65)

Also, we have

$$\partial \mathcal{M} = \bigcup_{\lambda' \in \Lambda} \operatorname{int} \mathcal{M}_{\lambda'}. \tag{66}$$

*Proof of Lemma 9:* We show the following statement by induction for depth  $D(\lambda)$ ; The relations

$$\operatorname{int} \mathcal{M}_{\lambda'} \cap \operatorname{int} \mathcal{M}_{\lambda''} = \emptyset \text{ for } \lambda' \neq \lambda'' \in \Lambda_{\lambda}. \tag{67}$$

$$\partial \mathcal{M}_{\lambda} = \cup_{\lambda' \in \bar{\Lambda}_{\lambda}} \operatorname{int} \mathcal{M}_{\lambda'}. \tag{68}$$

The relations (67) and (68) are trivial when  $D(\lambda) = 0$ . In the following, we show the relations (67) and (68) for  $D(\lambda) = k$  when they hold for  $D(\lambda) \le k - 1$ .

The convexity of  $\mathcal{M}_{\lambda}$  guarantees that  $\mathcal{M}_{\lambda'} \cap \mathcal{M}_{\lambda''} \in \partial \mathcal{M}_{\lambda'}, \partial \mathcal{M}_{\lambda''}$  for  $\lambda', \lambda'' \in \Lambda_{\lambda}$ . Hence, we have (67). For  $\lambda' \in \Lambda_{\lambda}$ , the assumption of induction implies

$$\partial \mathcal{M}_{\lambda'} = \cup_{\lambda'' \in \bar{\Lambda}_{\lambda'}} \operatorname{int} \mathcal{M}_{\lambda''}. \tag{69}$$

Thus,

$$\partial \mathcal{M}_{\lambda} = \bigcup_{\lambda' \in \Lambda_{\lambda}} \mathcal{M}_{\lambda'} = \bigcup_{\lambda' \in \Lambda_{\lambda}} \left( \operatorname{int} \mathcal{M}_{\lambda'} \cup \partial \mathcal{M}_{\lambda'} \right)$$

$$= \bigcup_{\lambda' \in \Lambda_{\lambda}} \left( \operatorname{int} \mathcal{M}_{\lambda'} \cup \left( \bigcup_{\lambda'' \in \bar{\Lambda}_{\lambda'}} \operatorname{int} \mathcal{M}_{\lambda''} \right) \right)$$

$$= \bigcup_{\lambda' \in \bar{\Lambda}_{\lambda}} \operatorname{int} \mathcal{M}_{\lambda'}. \tag{70}$$

Under the above case, the point  $\Gamma_{\mathcal{M}}^{(m),F}(\theta)$  for  $\theta \in \Theta$  can be characterized as follows.

*Lemma 10:* Assume that a set of closed convex mixture subfamilies  $\{\mathcal{M}_{\lambda}\}_{{\lambda}\in\Lambda}$  covers the boundary  $\partial \mathcal{M}$  of a closed convex mixture family  $\mathcal{M}$  with subsets  $\Lambda_{\lambda} \subset \Lambda$  and  $\lambda \in \Lambda_{*} := \Lambda \cup \{0\}$ . We denote the extended mixture subfamily of  $\mathcal{M}_{\lambda}$  by  $\hat{\mathcal{M}}_{\lambda}$  for  $\hat{\lambda} \in \Lambda_{*}$ . For  $\theta \in \Theta$ , we define  $\lambda_{0} := \underset{\lambda' \in \Lambda_{*}}{\operatorname{argmin}} \{ D^{F}(\Gamma_{\hat{\mathcal{M}}_{\lambda'}}^{(m),F}(\theta) \| \theta) | \Gamma_{\hat{\mathcal{M}}_{\lambda'}}^{(m),F}(\theta) \in \operatorname{int} \mathcal{M}_{\lambda'} \}$ . Then, we have  $\Gamma_{\mathcal{M}}^{(m),F}(\theta) = \Gamma_{\hat{\mathcal{M}}_{\lambda_{0}}}^{(m),F}(\theta)$ .

*Proof:* Due to Lemma 9, there uniquely exists  $\lambda_0 \in \Lambda$  such that  $\Gamma_{\mathcal{M}}^{(m),F}(\theta) \in \operatorname{int} \mathcal{M}_{\lambda_0}$ . Then,  $\Gamma_{\mathcal{M}}^{(m),F}(\theta) = \underset{\theta' \in \operatorname{int} \mathcal{M}_{\lambda_0}}{\operatorname{argmin}} D^F(\theta'\|\theta)$ . Since Lemma 3 guarantees that  $\underset{\theta' \in \operatorname{int} \mathcal{M}_{\lambda_0}}{\operatorname{argmin}} D^F(\theta'\|\theta) = \underset{\theta' \in \hat{\mathcal{M}}_{\lambda_0}}{\operatorname{argmin}} D^F(\theta'\|\theta)$ , we have  $\Gamma_{\mathcal{M}}^{(m),F}(\theta) = \Gamma_{\hat{\mathcal{M}}_{\lambda_0}}^{(m),F}(\theta).$ 

When  $\lambda \in \Lambda$  satisfies the condition  $\Gamma_{\hat{\mathcal{M}}_{\lambda}}^{(m),F}(\theta) \in \operatorname{int} \mathcal{M}_{\lambda}$ , we have  $D^{F}(\Gamma_{\hat{\mathcal{M}}_{\lambda_{0}}}^{(m),F}(\theta) \| \theta) = D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta) \| \theta) \leq$  $D^F(\Gamma_{\hat{\mathcal{M}}}^{(m),F}(\theta)\|\theta)$  because  $\Gamma_{\hat{\mathcal{M}}}^{(m),F}(\theta) \in \mathcal{M}$ . Hence, we obtain the desired statement.

#### III. EXAMPLES OF BREGMAN DIVERGENCE

# A. Classical system

We consider the set of probability distributions on the finite set  $\mathcal{X} = \{1, \dots, n\}$ . We focus on d linearly independent functions  $f_1, \ldots, f_d$  defined on  $\mathcal{X}$ , where the linear space spanned by  $f_1, \ldots, f_d$  does not contain a constant function and  $d \leq n-1$ . Then, we define the  $C^{\infty}$  strictly convex function  $\mu$  on  $\mathbb{R}^d$  as  $\mu(\theta) := \log(\sum_{x \in \mathcal{X}} \exp(\sum_{j=1}^d \theta^j f_j(x)))$ , which yields the Bregman divergence system  $(\mathbb{R}^d, \mu, D^\mu)$ . When d = n - 1, any probability distribution with full support on  $\mathcal{X}$  can be written as  $P_\theta$ , which is defined as  $P_{\theta}(x) := \exp\left(\left(\sum_{j=1}^{n-1} \theta^j f_j(x)\right) - \mu(\theta)\right)$ . It is known that the KL divergence equals the Bregman divergence of the potential function  $\mu$  [7, Section 3.4], i.e., we have

$$D^{\mu}(\theta \| \theta') = D(P_{\theta} \| P_{\theta'}) \tag{71}$$

for  $\theta \in \mathbb{R}^d$ , where the KL divergence D(q||p) is defined as

$$D(q||p) = \sum_{\omega} p(\omega)(\log p(\omega) - \log q(\omega)). \tag{72}$$

Examle 1: When d=n-1, the Bregman divergence system  $(\mathbb{R}^d,\mu,D^\mu)$  describes the set  $\mathcal{P}_{\mathcal{X}}$  of distributions on  $\mathcal{X}$  with full support and the KL divergence.

Examle 2: When  $\mathcal{X}$  is given as  $\mathcal{X}_1 \times \mathcal{X}_2$  with  $n_i = |\mathcal{X}_i|$ ,  $f_i$  is a function on  $\mathcal{X}_1$  or  $\mathcal{X}_2$ , and  $d = n_1 + n_2 - 2$ , the Bregman divergence system  $(\mathbb{R}^d, \mu, D^{\mu})$  describes the set  $\mathcal{P}_{\mathcal{X}_1} \times \mathcal{P}_{\mathcal{X}_2}$  of independent distributions on  $\mathcal{X}_1 \times \mathcal{X}_2$ .

Examle 3: When  $\mathcal{X}$  is given as  $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$  with  $n_i = |\mathcal{X}_i|$ ,  $f_i$  is a function on  $\mathcal{X}_1, \mathcal{X}_2$  or  $\mathcal{X}_2, \mathcal{X}_3$ , and  $d = n_2(n_1 + n_3 - 2) + n_2 - 1$ , the Bregman divergence system  $(\mathbb{R}^d, \mu, D^\mu)$  describes the set  $\mathcal{P}_{X_1 - X_2 - X_3}$ of distributions on  $\mathcal{X}_1 \times \mathcal{X}_2 \times X_3$  to satisfy the Markovian condition  $X_1 - X_2 - X_3$ . When the parameter  $\theta$  is limited to  $(\bar{\theta}, \underbrace{0, \dots, 0})$  with  $\bar{\theta} \in \mathbb{R}^l$ , the set of distributions  $P_{\theta}$  forms an

exponential subfamily. Also, when the linear space spanned by d-k linearly independent functions

 $g_1, \ldots, g_{d-k}$  does not contain a constant function, for d-k constants  $a_1, \ldots, a_{d-k}$ , the following set of distributions forms a mixture subfamily;

$$\left\{ P_{\theta} \middle| \sum_{x \in \mathcal{X}} g_i(x) P_{\theta}(x) = a_i \text{ for } i = 1, \dots, d - k \right\}.$$
 (73)

When we make linear constraints as explained in Subsection II-F, changing the potential function  $\mu$  in the way as (48), we can recover (71).

For the possibility of the projection, we have the following lemma.

Lemma 11: The Bregman divergence system  $(\mathbb{R}^d, \mu, D^{\mu})$  defined in this subsection satisfies Conditions (E4) and (M4).

To show this lemma, we prepare the following lemma.

Lemma 12: For  $(\theta^1, \dots, \theta^{d-l}) \in \mathbb{R}^{d-l}$  and  $\xi := (\xi^1, \dots, \xi^l) \in \mathbb{R}^l$ , we define

$$\tau_{(\theta^1,\dots,\theta^{d-l})}(\xi) := \left(\frac{\sum_{x \in \mathcal{X}} f_{d-j}(x) \exp\left(\sum_{i=1}^d \theta^i f_i(x)\right)}{\mu(\theta)}\right)_{j=1}^l \in \mathbb{R}^l$$
 (74)

with  $\theta^{d-l+i} = \xi^i$ . Then, the set  $\mathcal{T}_{1,(\theta^1,\dots,\theta^{d-l})} := \{\tau_{(\theta^1,\dots,\theta^{d-l})}(\xi)|\xi\in\mathbb{R}^l\}$  equals the inner  $\mathcal{T}_2$  of the convex full of  $\{(f_{d-j}(x))_{j=1}^l\}_{x\in\mathcal{X}}$ .

*Proof of Lemma 12:* In this proof,  $\mathcal{T}_{1,(\theta^1,\ldots,\theta^{d-l})}$  and  $\tau_{(\theta^1,\ldots,\theta^{d-l})}(\xi)$  are simplified to  $\mathcal{T}_1$  and  $\tau(\xi)$ . Since  $\mathcal{T}_1 \subset \mathcal{T}_2$  is trivial, we show only the opposite relation.

Step 1: Any element in the boundary of the convex full of  $\{(f_{d-j}(x))_{j=1}^l\}_{x\in\mathcal{X}}$  is written as  $\left(\sum_{i=1}^{l'}p_if_{d-j}(x_i)\right)_{j=1}^l$  with extremal points  $(f_{d-j}(x_i))_{j=1}^l$  with at most l elements  $x_i\in\mathcal{X}$  and at most l positive numbers  $p_i$ , where  $i=1,\ldots,l'\leq l$ . There exists an element  $\xi_*\in\mathbb{R}^l$  such that  $\max_{x\in\mathcal{X}}\sum_{j=1}^l\xi_*^jf_{d-j}(x)=\sum_{j=1}^l\xi_*^jf_{d-j}(x_i)=1$  and  $\sum_{j=1}^l\xi_*^jf_{d-j}(x)<1$  unless  $(f_{d-j}(x))_{j=1}^l$  is written as a convex combination of  $\{(f_{d-j}(x_i))_{j=1}^l\}_{i=1}^{l'}$ . For any  $x_i$ , there exists an element  $\xi(x_i)_*\in\mathbb{R}^l$  such that  $\max_{x\in\mathcal{X}}\sum_{j=1}^l\xi(x_i)_*^jf_{d-j}(x)=\sum_{j=1}^l\xi(x_i)_*^jf_{d-j}(x_i)=1$ ,  $\sum_{j=1}^l\xi(x_i)_*^jf_{d-j}(x)<1$  for  $x\neq x_i$  and  $\sum_{j=1}^l\xi(x_i)_*^jf_{d-j}(x_i)>0$  for  $i'\neq i$ . Then, there exist elements  $t_i>0$  such that

$$\frac{\exp\left(\sum_{i'=1}^{l'}\sum_{j=1}^{l}t_{i'}\xi(x_{i'})_{*}^{j}f_{d-j}(x_{i}) + \sum_{j=1}^{d-l}\theta^{j}f_{j}(x_{i})\right)}{\sum_{i''=1}^{l'}\exp\left(\sum_{i'=1}^{l'}\sum_{j=1}^{l}t_{i'}\xi(x_{i'})_{*}^{j}f_{d-j}(x_{i''}) + \sum_{j=1}^{d-l}\theta^{j}f_{j}(x_{i''})\right)} = p_{i}.$$
(75)

Hence, we have

$$\tau \left( t\xi(x_0) + \sum_{i'=1}^{l'} t_{i'}\xi(x_{i'})_* \right) \to \left( \sum_{i=1}^{l'} p_i f_{d-j}(x_i) \right)_{j=1}^l$$
 (76)

as  $t \to \infty$ .

**Step 2:** Conversely, for any  $\xi \in \mathbb{R}^l$ , we can choose at most l elements  $x_1, \ldots, x_{l'} \in \mathcal{X}$  such that  $\max_{x \in \mathcal{X}} \sum_{j=1}^l \xi^j f_{d-j}(x) = \sum_{j=1}^l \xi^j f_{d-j}(x_i)$  and  $\sum_{j=1}^l \xi^j f_{d-j}(x) < \sum_{j=1}^l \xi^j f_{d-j}(x_i)$  for  $x \notin \{x_1, \ldots, x_{l'}\}$ . Then, we have

$$\tau(t\xi) \to \left(\sum_{i=1}^{l'} \frac{\exp\left(\sum_{j=1}^{d-l} \theta^{j} f_{j}(x_{i})\right)}{\sum_{i'=1}^{l'} \exp\left(\sum_{j=1}^{d-l} \theta^{j} f_{j}(x_{i'})\right)} f_{d-j}(x_{i})\right)_{j=1}^{l}$$
(77)

as  $t \to \infty$ .

**Step 3:** We consider the compact set  $\mathcal{T}(t) := \{\tau(\xi)\}_{\max_{x \in \mathcal{X}} \sum_{j=1}^{l} \xi^j f_{d-j}(x) = t}$  for a large real number t > 0. The analysis on Steps 1 and 2 shows that the set  $\mathcal{T}(t)$  approaches to the boundary of the convex

full of  $\{(f_{d-j}(x))_{j=1}^l\}_{x\in\mathcal{X}}$  when t approaches infinity. Since map  $\tau$  is continuous, the image  $\mathcal{D}(t)$  of  $\{\xi\in\mathbb{R}^l|\max_{x\in\mathcal{X}}\sum_{j=1}^l\xi^jf_{d-j}(x)\leq t\}$  for the map  $\tau$  is a compact subset whose boundary is close to the boundary of the convex full of  $\{(f_{d-j}(x))_{j=1}^l\}_{x\in\mathcal{X}}$ . Therefore,  $\cup_{t>0}\mathcal{D}(t)$  equals the convex full of  $\{(f_{d-j}(x))_{j=1}^l\}_{x\in\mathcal{X}}$ .

Proof of Lemma 11: It is sufficient to show Conditions (E3) and (M3) for any set of vectors  $v_1,\ldots,v_l$ , where  $l=1,\ldots,d-1$ . This fact can be shown as follows. First, we show (E3). For this aim, we choose an invertible matrix U such that  $u_{d-i}=v_i$  for  $i=1,\ldots,l$ . For simplicity, we rewrite  $\sum_{i=1}^d u_j^i f_i$  by  $f_j$ . Also, we choose  $(a_1,\ldots,a_l)\in\mathbb{R}^l$  such that  $\mathcal{M}(a_1,\ldots,a_l)$  is not empty. We show that  $\mathcal{M}(a_1,\ldots,a_l)$  is  $\mathbb{R}^{d-l}$ . Due to Lemma 12, for  $(\theta^1,\ldots,\theta^{d-l})\in\mathbb{R}^{d-l}$ , there exists  $(\theta^{d-l+1},\ldots,\theta^d)\in\mathbb{R}^l$  such that

$$\frac{\sum_{x \in \mathcal{X}} f_{d-j}(x) \exp\left(\sum_{i=1}^{d} \theta^{i} f_{i}(x)\right)}{\mu(\theta)} = a_{j}$$
(78)

for  $j = 1, \dots, l$ . The above condition is equivalent to

$$\frac{\partial \mu}{\partial \theta^j}(\theta) = a_j. \tag{79}$$

This condition implies the relation  $\mathcal{M}(a_1,\ldots,a_l)=\mathbb{R}^{d-l}$ . Hence, we have Condition (E3).

Next, we show (M3). The relation (78) means that the set  $\Xi_{\mathcal{E}(\theta_0)}$  does not depend on  $\theta_0 \in \Theta$  because the choice of  $(\theta^1, \dots, \theta^{d-l}) \in \mathbb{R}^{d-l}$  corresponds to the choice of  $\theta_0 \in \Theta$  in the relation (79). Hence, we have Condition (M3).

# B. Classical system with fixed marginal distribution

We consider the set of probability distributions on the finite set  $\mathcal{X} \times \mathcal{Y}$  with  $n_1 = |\mathcal{X}|$  and  $n_2 = |\mathcal{Y}|$ . In particular, the marginal distribution on  $\mathcal{X}$  is restricted as  $P_X(x) = p_x$ . We focus on d linearly independent functions  $\bar{f}_1, \ldots, \bar{f}_{n_2-1}$  defined on  $\mathcal{Y}$ , where the linear space spanned by  $\bar{f}_1, \ldots, \bar{f}_{n_2-1}$  does not contain a constant function. Then, we define the  $C^{\infty}$  strictly convex function  $\bar{\mu}$  on  $\mathbb{R}^{n_1(n_2-1)}$  as  $\bar{\mu}(\bar{\theta}) := \sum_{x \in \mathcal{X}} p_x \mu_x(\bar{\theta})$ , where  $\mu_x(\bar{\theta}) := \log(\sum_{y \in \mathcal{Y}} \exp(\sum_{j=1}^{n_2-1} \bar{\theta}^{(x-1)(n_2-1)+j} \bar{f}_j(y))$ , which yield the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$ .

A probability distribution with full support on  $\mathcal{X} \times \mathcal{Y}$  with the marginal distribution  $p_x$  can be written as  $P_{\theta}$ , which is defined as as  $P_{\bar{\theta}}(x,y) := p_x \exp\left((\sum_{j=1}^{n_2-1} \bar{\theta}^{(x-1)(n_2-1)+j} \bar{f}_j(y)) - \mu_x(\bar{\theta})\right)$ . The KL divergence equals the Bregman divergence of the potential function  $\bar{\mu}$ , i.e., we have

$$D^{\bar{\mu}}(\bar{\theta}||\bar{\theta}_{0})$$

$$= \sum_{x,j} p_{x} \left( \frac{\partial \mu_{x}(\bar{\theta})}{\partial \bar{\theta}^{(x-1)(n_{2}-1)+j}} (\bar{\theta}^{(x-1)(n_{2}-1)+j} - \bar{\theta}_{0}^{(x-1)(n_{2}-1)+j}) - \mu_{x}(\bar{\theta}) + \mu_{x}(\bar{\theta}_{0}) \right)$$

$$= D(P_{\bar{\theta}}||P_{\bar{\theta}_{0}})$$
(80)

for  $\theta \in \mathbb{R}^d$ .

Next, we consider the Bregman divergence system  $(\mathbb{R}^{n_1n_2-1}, \mu, D^{\mu})$  defined in Subsection III-A with  $f_1, \ldots, f_{n_1n_2-1}$  defined as follows;  $f_{(i-1)(n_2-1)+j}(x,y) := \delta_{i,x}\bar{f}_x(y)$  for  $i=1,\ldots,n_1$  and  $j=1,\ldots,n_2-1$ .  $f_{n_1(n_2-1)+i}(x,y) := \delta_{i,x}$  for  $i=1,\ldots,n_1-1$ . We define the mixture subfamily  $\mathcal{M}$  by the constraint

$$\frac{\partial \mu}{\partial \theta^{n_1(n_2-1)+i}} = p_i \tag{81}$$

for  $i = 1, ..., n_1 - 1$ . We apply the discussion given in Subsection II-F to the mixture subfamily  $\mathcal{M}$ . The matrix U is the identity matrix. The mixture subfamily  $\mathcal{M}$  is parameterized by the natural parameter

 $\bar{\theta} = (\bar{\theta}^1, \dots, \bar{\theta}^{n_1(n_2-1)})$ . The function  $\kappa$  is chosen as  $\kappa^{n_1(n_2-1)+i}(\bar{\theta}) := \mu_x(\bar{\theta})$ . Then, the parameter  $(\bar{\theta}, \kappa(\bar{\theta}))$  satisfies the condition (81). Hence, the mixture subfamily  $\mathcal{M}$  coincides with the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$ .

As an extension of Lemma 11, we have the following lemma.

*Lemma 13:* The Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$  defined in this subsection satisfies Conditions (E4) and (M4).

*Proof:* Condition (E4) holds because the parameter set is  $\mathbb{R}^{n_1(n_2-1)}$ . Since the Bregman divergence system  $(\mathbb{R}^{n_1n_2-1}, \mu, D^{\mu})$  satisfies Condition (M4), its mixture subfamily  $\mathcal{M}$  satisfies Condition (M4). Hence,  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$  satisfies Condition (M4).

### C. Quantum system

In the quantum system, we focus on the n-dimensional Hilbert space  $\mathcal{H}$  [15]. We choose d linearly independent Hermitian matrices  $X_1,\ldots,X_d$  on  $\mathcal{H}$ , where the linear space spanned by  $X_1,\ldots,X_d$  does not contain the identify matrix. Then, we define the  $C^{\infty}$  strictly convex function  $\mu$  on  $\mathbb{R}^d$  as  $\mu(\theta) := \log(\operatorname{Tr}\exp(\sum_{j=1}^d \theta^j X_j))$ . A quantum state on  $\mathcal{H}$  is given as a positive semi definite Hermitian matrix  $\rho$  with the condition  $\operatorname{Tr} \rho = 1$ , which is called a density matrix. We denote the set of density matrices by  $\mathcal{S}(\mathcal{H})$ . Any density matrix with full support on  $\mathcal{H}$  can be written as  $\rho_{\theta}$ , which is defined as as  $\rho_{\theta} := \exp\left((\sum_{j=1}^d \theta^j X_j) - \mu(\theta)\right)$ . It is known that the relative entropy equals the Bregman divergence of the potential function  $\mu$  [7, Section 7.2], i.e., we have

$$D^{\mu}(\theta \| \theta') = D(\rho_{\theta} \| \rho_{\theta'}) \tag{82}$$

for  $\theta \in \mathbb{R}^d$ , where the relative entropy  $D(\rho \| \rho')$  is defined as

$$D(\rho \| \rho') = \operatorname{Tr} \rho(\log \rho - \log \rho'). \tag{83}$$

Examle 4: When  $d = n^2 - 1$ , the Bregman divergence system  $(\mathbb{R}^d, \mu, D^{\mu})$  describes the set  $\mathcal{S}(\mathcal{H})$  of density matrices on  $\mathcal{H}$  with full support and the relative entropy.

Examle 5: When  $\mathcal{H}$  is given as  $\mathcal{H}_1 \otimes \mathcal{H}_2$  with  $n_i = \dim \mathcal{H}_i$ ,  $X_i$  is an Hermitian matrices with the form  $A \otimes I$  or  $I \otimes B$ , and  $d = n_1^2 + n_2^2 - 2$ , the Bregman divergence system  $(\mathbb{R}^d, \mu, D^{\mu})$  describes the set  $\mathcal{S}(\mathcal{H}_1) \otimes \mathcal{S}(\mathcal{H}_2)$  of product density matrices on  $\mathcal{H}_1 \otimes \mathcal{H}_2$ .

When the parameter  $\theta$  is limited to  $(\bar{\theta}, \underbrace{0, \dots, 0})$  with  $\bar{\theta} \in \mathbb{R}^l$ , the set of distributions  $\rho_{\theta}$  forms an

exponential family. Also, when the linear space spanned by d-k linearly independent Hermitian matrices  $Y_1, \ldots, Y_{d-k}$  does not contain a constant function, for d-k constants  $a_1, \ldots, a_{d-k}$ , the following set of distributions forms a mixture family;

$$\left\{ \rho_{\theta} \middle| \operatorname{Tr} Y_i \rho_{\theta} = a_i \text{ for } i = 1, \dots, d - k \right\}.$$
 (84)

For the possibility of the projection, we have the following lemma.

Lemma 14: The Bregman divergence system  $(\mathbb{R}^d, \mu, D^{\mu})$  defined in this section satisfies Conditions (E4) and (M4).

To show this lemma, we prepare the following lemma in a way similar to Lemma 12.

Lemma 15: For  $(\theta^1, \dots, \theta^{d-l}) \in \mathbb{R}^{d-l}$  and  $\xi := (\xi^1, \dots, \xi^l) \in \mathbb{R}^l$ , we define

$$\tau_{(\theta^1,\dots,\theta^{d-l})}(\xi) := \left(\frac{\operatorname{Tr} X_{d-j} \exp\left(\sum_{i=1}^d \theta^j X_{d-i}\right)}{\mu(\theta)}\right)_{j=1}^l \tag{85}$$

with  $\theta^{d-l+i} = \xi^i$ . Then, the set  $\mathcal{T}_{1,(\theta^1,\dots,\theta^{d-l})} := \{\tau_{(\theta^1,\dots,\theta^{d-l})}(\xi)|\xi\in\mathbb{R}^l\}$  equals the inner  $\mathcal{T}_2$  of the convex full of  $\{(\operatorname{Tr} X_{d-j}\rho)_{j=1}^l\}_{\rho\in\mathcal{P}}$ , where  $\mathcal{P}$  is the set of pure states.

*Proof of Lemma 15:* In this proof,  $\mathcal{T}_{1,(\theta^1,\ldots,\theta^{d-l})}$  and  $\tau_{(\theta^1,\ldots,\theta^{d-l})}(\xi)$  are simplified to  $\mathcal{T}_1$  and  $\tau(\xi)$ . Since  $\mathcal{T}_1 \subset \mathcal{T}_2$  is trivial, we show only the opposite relation.

Step 1: Any element in the boundary of the convex full of  $\{(\operatorname{Tr} X_{d-j}\rho)_{j=1}^l\}_{\rho\in\mathcal{P}}$  is written as  $\left(\sum_{i=1}^{l'}p_i\operatorname{Tr}\rho_iX_{n-1-j}\right)_{j=1}^l$  with extremal points  $(\operatorname{Tr}\rho X_{d-j})_{j=1}^l$  with at most l orthogonal elements  $\rho_i\in\mathcal{P}$  and at most l positive numbers  $p_i$ , where  $i=1,\ldots,l'\leq l$ . There exists an element  $\xi_*\in\mathbb{R}^l$  such that  $\max_{\rho\in\mathcal{P}}\sum_{j=1}^l\xi_*^j\operatorname{Tr}\rho X_{d-j}=\sum_{j=1}^l\xi_*^j\operatorname{Tr}\rho_iX_{d-j}=1$  and  $\sum_{j=1}^l\xi_*^j\operatorname{Tr}\rho X_{d-j}<1$  unless  $(\operatorname{Tr}\rho X_{d-j})_{j=1}^l$  is written as a convex combination of  $\{(\operatorname{Tr}\rho_iX_{d-j})_{j=1}^l\}_{i=1}^{l'}$ . For any  $\rho_i$ , there exists an element  $\xi(\rho_i)_*\in\mathbb{R}^l$  such that  $\max_{\rho\in\mathcal{P}}\sum_{j=1}^l\xi(\rho_i)_*^j\operatorname{Tr}\rho X_{d-j}=\sum_{j=1}^l\xi(\rho_i)_*^j\operatorname{Tr}\rho_iX_{d-j}=1$ ,  $\sum_{j=1}^l\xi(x_i)_*^j\operatorname{Tr}\rho X_{d-j}<1$  for  $\rho(\neq\rho_i)\in\mathcal{P}$  and  $\sum_{j=1}^l\xi(x_i)_*^j\operatorname{Tr}\rho_{i'}X_{d-j}>0$  for  $i'\neq i$ . Then, there exists elements  $t_i>0$  such that

$$\frac{\operatorname{Tr} \rho_{i} \exp \left(\sum_{i'=1}^{l'} \sum_{j=1}^{l} t_{i'} \xi(x_{i'})_{*}^{j} X_{n-1-j} + \sum_{j=1}^{d-l} \theta^{j} X_{j}\right)}{\sum_{i''=1}^{l'} \operatorname{Tr} \rho_{i''} \exp \left(\sum_{i'=1}^{l'} \sum_{j=1}^{l} t_{i'} \xi(x_{i'})_{*}^{j} X_{d-j} + \sum_{j=1}^{d-l} \theta^{j} X_{j}\right)} = p_{i}.$$
(86)

Hence, we have

$$\tau \left( t\xi(x_0) + \sum_{i'=1}^{l'} t_{i'}\xi(x_{i'})_* \right) \to \left( \sum_{i=1}^{l'} p_i \operatorname{Tr} \rho_i X_{d-j} \right)_{j=1}^{l}$$
 (87)

as  $t \to \infty$ .

**Step 2:** Conversely, for any  $\xi \in \mathbb{R}^l$ , we can choose at most l orthogonal pure states  $\rho_1, \ldots, \rho_{l'} \in \mathcal{X}$  such that  $\sum_{j=1}^{l} \xi^j X_{d-j}$  is commutative with  $\rho_1, \ldots, \rho_{l'}$ ,  $\max_{\rho \in \mathcal{P}} \sum_{j=1}^{l} \xi^j \operatorname{Tr} \rho X_{d-j} = \sum_{j=1}^{l} \xi^j \operatorname{Tr} \rho_i X_{d-j}$  and  $\sum_{j=1}^{l} \xi^j \operatorname{Tr} \rho X_{d-j} < \sum_{j=1}^{l} \xi^j \operatorname{Tr} \rho_i X_{d-j}$  unless  $(\operatorname{Tr} \rho X_{d-j})_{j=1}^{l}$  is written as a convex combination of  $\{(\operatorname{Tr} \rho_i X_{d-j})_{j=1}^{l}\}_{i=1}^{l'}$ . Then, we have

$$\tau(t\xi) \to \left(\sum_{i=1}^{l'} \frac{\operatorname{Tr} \rho_i \exp\left(\sum_{j=1}^{d-l} \theta^j X_j\right)}{\sum_{i'=1}^{l'} \operatorname{Tr} \rho_{i'} \exp\left(\sum_{j=1}^{d-l} \theta^j X_j\right)} X_{d-j}\right)_{j=1}^{l}$$
(88)

as  $t \to \infty$ .

Step 3: We consider the compact set  $\mathcal{T}(t) := \{\tau(\xi)\}_{\|(\sum_{j=1}^{l} \xi^{j} X_{d-j})_{+}\| = t}$  for large real number t > 0, where  $(X)_{+}$  is an operator composed of the positive part. The analysis on Steps 1 and 2 shows that the set  $\mathcal{T}(t)$  approaches to the boundary of the convex full of  $\{(\operatorname{Tr} \rho X_{d-j})_{j=1}^{l}\}_{\rho \in \mathcal{P}}$  when t approaches infinity. Since the map  $\tau$  is continuous, the image  $\mathcal{D}(t)$  of  $\{\xi \in \mathbb{R}^{l} \mid \|(\sum_{j=1}^{l} \xi^{j} X_{d-j})_{+}\| \leq t\}$  for the map  $\tau$  is a compact subset whose boundary is close to the boundary of the convex full of  $\{(\operatorname{Tr} \rho X_{d-j})_{j=1}^{l}\}_{\rho \in \mathcal{P}}$ . Therefore,  $\cup_{t>0} \mathcal{D}(t)$  equals the convex full of  $\{(\operatorname{Tr} \rho X_{d-j})_{j=1}^{l}\}_{\rho \in \mathcal{P}}$ .

*Proof of Lemma 14:* Lemma 14 can be shown in the same way as Lemma 11 by replacing the role of Lemma 12 by Lemma 15.

#### IV. EM-ALGORITHM

#### A. Basic description for algorithm

In this section, we address a minimization problem for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$  although the paper [2] discussed a similar problem setting based on Bregman divergence. Here, we employ notations  $u_{k+j}^i$ ,  $a_j$ , etc, for a k-dimensional mixture

subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$  that are introduced in Subsections II-B and II-C. We assume the following condition;

(B0) The Bregman divergence system  $(\Theta, F, D^F)$  satisfies Conditions (E4) and (M4). Hence, the minimums  $\min_{\theta' \in \mathcal{E}} D^F(\theta \| \theta')$  and  $\min_{\theta \in \mathcal{M}} D^F(\theta \| \theta')$  exist. We consider the following minimization problem;

$$C_{\inf}(\mathcal{M}, \mathcal{E}) := \inf_{\theta \in \mathcal{M}} D^F(\theta \| \Gamma_{\mathcal{E}}^{(e), F}(\theta)) = \inf_{\theta \in \mathcal{M}} \min_{\theta' \in \mathcal{E}} D^F(\theta \| \theta').$$
 (89)

The first task is to clarify whether the minimum exists in (89). If the minimum exists, our second task is to find the minimization point

$$\theta^*(\mathcal{M}, \mathcal{E}) := \operatorname*{argmin}_{\theta \in \mathcal{M}} D^F(\theta \| \Gamma_{\mathcal{E}}^{(e), F}(\theta)). \tag{90}$$

When we define  $\theta_*(\mathcal{M}, \mathcal{E}) := \Gamma_{\mathcal{E}}^{(e),F}(\theta^*(\mathcal{M}, \mathcal{E}))$ , we have the opposite relation  $\theta^*(\mathcal{M}, \mathcal{E}) = \Gamma_{\mathcal{M}}^{(m),F}(\theta_*(\mathcal{M}, \mathcal{E}))$  because  $\theta^*(\mathcal{M}, \mathcal{E})$  achieves the maximum. Hence, we have the relation  $\mathcal{M}_{\theta_* \to \mathcal{E}} = \mathcal{E}_{\theta^* \to \mathcal{M}}$ . If there is no risk of confusion,  $\theta^*(\mathcal{M}, \mathcal{E})$  and  $\theta_*(\mathcal{M}, \mathcal{E})$  are simplified to  $\theta^*$  and  $\theta_*$ , respectively. If the minimum does not exit, our second task is to find a sequence of elements  $\{\theta_n^*(\mathcal{M}, \mathcal{E})\}$  in  $\mathcal{M}$  to achieve the infimum (89).

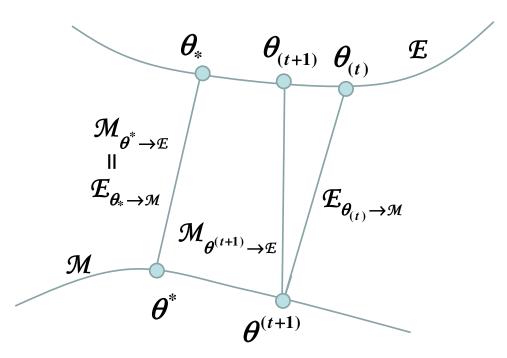


Fig. 2. Algorithms 1 and 2: This figure shows the topological relation among  $\theta_*$ ,  $\theta^*$ ,  $\theta_{(t+1)}$ ,  $\theta^{(t+1)}$ , and  $\theta_{(t)}$ , which is used in the application of Phythagorean theorem (Proposition 1).  $\mathcal{M}_{\theta_* \to \mathcal{E}} = \mathcal{E}_{\theta^* \to \mathcal{M}}$  and  $\mathcal{M}_{\theta^{(t+1)} \to \mathcal{E}}$  are the mixture subfamilies to project  $\theta(\epsilon_1)$  and  $\theta^{(t+1)}$  to the exponential subfamily  $\mathcal{E}$ , respectively.  $\mathcal{E}_{\theta_{(t)} \to \mathcal{M}}$  is the exponential subfamily to project  $\theta_{(t)}$  to the mixture subfamily  $\mathcal{M}$ .

Although the above minimization problem is very common in machine learning and statistics, many kinds of minimization problems in information theory can be written in the above form as explained in Section I. The above minimization asks to minimize the divergence between two points in the mixture and exponential subfamilies  $\mathcal{E}$  and  $\mathcal{M}$ . Algorithm 1 shows an algorithm to calculate the element  $\theta^*(\mathcal{M}, \mathcal{E})$  to achieve the minimum. This algorithm is called the em algorithm, and is illustrated in Fig. 2. By describe the m-step in a concrete form, Algorithm 1 is rewritten as Algorithm 2, which follows from (A3) of Lemma 5.

When the mixture family  $\mathcal{M}$  has to many parameters, the optimization in m-step takes a long time. In this case, m-step can be replaced by another optimization problem with d-k parameters. This replacement is useful when k>d-k.

## Algorithm 1 em-algorithm

Assume that  $\mathcal{M}$  is characterized by (40). Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ; **repeat** 

**m-step:** Calculate  $\theta^{(t+1)} := \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$ . That is,  $\theta^{(t+1)}$  is given as  $\underset{\theta \in \mathcal{M}}{\operatorname{argmin}} D^F(\theta \| \theta_{(t)})$ , i.e., the unique element in  $\mathcal{M}$  to realize the minimum of the smooth convex function  $\theta \mapsto D^F(\theta \| \theta_{(t)})$ . **e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)})$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\theta^{(t+1)} \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the smooth convex function  $\theta' \mapsto D^F(\theta^{(t+1)} \| \theta')$ . **until** convergence.

# Algorithm 2 em-algorithm

Assume that  $\mathcal{M}$  is characterized by (40). Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ; **repeat** 

**m-step:** Calculate  $\theta^{(t+1)} := \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$ . That is,  $\theta^{(t+1)}$  is given as  $\theta_{(t)} + \sum_{j=k+1}^{d} \tau_o^j u_j$ , where  $(\tau_o^{k+1}, \dots, \tau_o^d)$  is the unique element to satisfy

$$\frac{\partial}{\partial \tau^{\bar{j}}} F\left(\theta_{(t)} + \sum_{j=1}^{l} \tau^{j} u_{j}\right) \Big|_{\tau^{j} = \tau_{o}^{\bar{j}}} = a_{\bar{j}}$$

$$(91)$$

for  $\bar{j} = k + 1, \dots, d$ . The above choice is equivalent to the following;

$$(\tau_o^{k+1}, \dots, \tau_o^d) := \underset{\bar{\tau}^{k+1}, \dots, \bar{\tau}^d}{\operatorname{argmin}} F\left(\theta_{(t)} + \sum_{j=1}^l \bar{\tau}^j u_j\right) - \sum_{j=k+1}^d \bar{\tau}_j a_j.$$
(92)

**e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\hat{\theta}^{(t+1)})$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\theta^{(t+1)} \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the smooth convex function  $\theta' \mapsto D^F(\theta^{(t+1)} \| \theta')$ . **until** convergence.

The em-algorithm repetitively applies the function  $\Gamma_{\mathcal{M}}^{(m),F} \circ \Gamma_{\mathcal{E}}^{(e),F}|_{\mathcal{M}}$  for an element  $\theta \in \mathcal{M}$ . Since the application of  $\Gamma_{\mathcal{M}}^{(m),F} \circ \Gamma_{\mathcal{E}}^{(e),F}|_{\mathcal{M}}$  monotonically decreases the minimum Bregman divergence from the exponential family  $\mathcal{E}$ , when we apply the updating rule  $\theta^{(t+1)} := \Gamma_{\mathcal{M}}^{(m),F} \circ \Gamma_{\mathcal{E}}^{(e),F}|_{\mathcal{M}}(\theta^{(t)})$ , it is expected that the outcome  $\theta^{(t)}$  of the repetitive application converges to  $\theta^*(\mathcal{M},\mathcal{E})$ . However, it is not guaranteed that the converged point gives the global minimum in general [1], [2], [3]. To get a global minimum by this algorithm, we introduce the following condition for an exponential subfamily  $\mathcal{E}$ .

#### (B1) The relation

$$D^{F}(\theta'\|\theta) \ge D^{F}(\Gamma_{\mathcal{E}}^{(e),F}(\theta')\|\Gamma_{\mathcal{E}}^{(e),F}(\theta)) \tag{93}$$

holds for any  $\theta, \theta' \in \Theta$ .

Also, as its weak version, we consider the following condition.

 $(B1\mathcal{M})$  The relation

$$D^{F}(\theta'\|\theta) \ge D^{F}(\Gamma_{\mathcal{E}}^{(e),F}(\theta')\|\Gamma_{\mathcal{E}}^{(e),F}(\theta)) \tag{94}$$

holds for any  $\theta, \theta' \in \mathcal{M}$ .

Then, we have the following theorem.

Theorem 2: Assume Conditions (B0), (B1 $\mathcal{M}$ ), and  $\sup_{\theta \in \mathcal{E}} D^F(\theta \| \theta_{(1)}) < \infty$  for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$ . Then, in Algorithms 1 and 2, the quantity  $D^F(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)}))$  converges to the minimum  $C_{\inf}(\mathcal{M},\mathcal{E})$  with the speed

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) = o(\frac{1}{t}).$$

$$(95)$$

Also, we have another type evaluation

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \frac{\sup_{\theta \in \mathcal{M}} D^{F}(\theta \| \theta_{(1)})}{t-1}.$$
(96)

Further, when  $t-1 \geq \frac{\sup_{\theta \in \mathcal{M}} D^F(\theta \| \theta_{(1)})}{\epsilon}$ , the parameter  $\theta^{(t)}$  satisfies

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \epsilon.$$
(97)

In particular, when the minimum in (89) exists, i.e.,  $\theta^*(\mathcal{M}, \mathcal{E})$  exists, the supremum  $\sup_{\theta \in \mathcal{E}} D^F(\theta \| \theta_{(1)})$  in the above evaluation is replaced by  $D^F(\theta^*(\mathcal{M}, \mathcal{E}) \| \theta_{(1)})$ .

The proof of Theorem 2 is given in Appendix C.

To improve the above evaluation, we introduce a strength version of Condition (B1) as a condition for  $\mathcal{M}, \mathcal{E}$ , and  $\theta' \in \mathcal{E}$ .

(B1+) The minimizer  $\theta^* = \theta^*(\mathcal{M}, \mathcal{E})$  exists. There exists a constant  $\beta(\theta') < 1$  to satisfy the following condition. When an element  $\theta \in \operatorname{Im} \Gamma_{\mathcal{M}}^{(m),F}|_{\mathcal{E}} \subset \mathcal{M}$  satisfies the condition  $D^F(\theta^* \| \theta) \leq D^F(\theta_* \| \theta')$ , the relation

$$\beta(\theta')D^F(\theta^*\|\theta) \ge D^F(\theta_*\|\Gamma_{\mathcal{E}}^{(e),F}(\theta)) \tag{98}$$

holds.

Then, we have the following theorem.

Theorem 3: Assume that Conditions (B0) and (B1+) hold for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$ , an l-dimensional exponential subfamily  $\mathcal{E}$ , and  $\theta' = \theta_{(1)} \in \mathcal{E}$ . Then, in Algorithms 1 and 2, the quantity  $D^F(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)}))$  converges to the minimum  $C_{\inf}(\mathcal{M},\mathcal{E})$  with the speed

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) = \beta(\theta_{(1)})^{t-2} D^{F}(\theta_{*} \| \theta_{(1)}).$$
(99)

Further, when  $t-2 \geq \frac{\log D^F(\theta_* \| \theta_{(1)}) - \log \epsilon}{-\log \beta(\theta_{(1)})}$ , the parameter  $\theta^{(t)}$  satisfies

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \epsilon.$$
(100)

The proof of Theorem 3 is given in Appendix D.

In fact, it is not so easy to find  $\theta_{(1)}$  to satisfy Condition (B1+). However, when we apply Algorithm 2,  $\theta_{(t)}$  becomes close to  $\theta_*$  with sufficiently large t. When  $\theta_{(t)} \in \mathcal{E}$  belongs to the neighborhood of  $\theta_*$ , Condition (B1+) holds by substituting  $\theta_{(t)}$  into  $\theta_{(1)}$  so that Theorem 3 can be applied with sufficiently t. That is, once  $\theta_{(t)} \in \mathcal{E}$  belongs to the neighborhood of  $\theta_*$ , we have an exponential convergence.

Further, it is not easy to implement e- and m- projections perfectly, in general. Hence, we need an alternative algorithm instead of Algorithms 1 and 2. Now, we consider the case when only e-step can be perfectly implemented and m-step is approximately done with  $\epsilon$  error. Examples of such a case will be discussed later sections. Algorithm 1 is modified as follows.

Then, we have the following theorem.

Theorem 4: Assume Conditions (B0), (B1), and the existence of the minimizer  $\theta^* = \theta^*(\mathcal{M}, \mathcal{E})$  in (90) for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$ . In addition, we define the set  $\mathcal{E}_0 := \{\theta \in \mathcal{E} | D^F(\theta_* || \theta) \leq D^F(\theta_* || \theta_{(1)})\} \subset \mathcal{E}$ .

# **Algorithm 3** em-algorithm with $\epsilon$ approximated m-step in the mixture subfamily $\mathcal{M}$

Assume that  $\mathcal{M}$  is characterized by (40). Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ;

## repeat

**m-step:** Calculate  $\theta^{(t+1)}$ . That is, we choose an element  $\theta^{(t+1)} \in \mathcal{M}$  such that

$$D^{F}(\theta^{(t+1)} \| \theta_{(t)}) \le \min \left( D^{F}(\theta^{(t)} \| \theta_{(t)}), \min_{\theta \in \mathcal{M}} D^{F}(\theta \| \theta_{(t)}) + \epsilon \right), \tag{101}$$

where  $D^F(\theta^{(1)}||\theta_{(1)})$  is defined as  $\infty$ .

**e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\hat{\theta}^{(t+1)})$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\theta^{(t+1)} \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the smooth convex function  $\theta' \mapsto D^F(\theta^{(t+1)} \| \theta')$ . **until** convergence.

Then, in Algorithm 3, the quantity  $D^F(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)}))$  converges to the minimum  $C_{\inf}(\mathcal{M},\mathcal{E})$  with the speed

$$D^{F}(\theta^{(t+1)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq \frac{D^{F}(\theta_{*} \| \theta_{(1)})}{t} + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon.$$
(102)

where  $\gamma := \gamma(\mathcal{E}_0|\mathcal{E})$ . Further, when  $t \geq \frac{2D^F(\theta_{*,1}||\theta_{(1)})}{\epsilon'} + 1$  and  $\epsilon \leq \frac{{\epsilon'}^2}{4(3\gamma+1)^2D^F(\theta_*||\theta_{(1)})}$ , the parameter  $\theta^{(t)}$  satisfies

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \epsilon'. \tag{103}$$

The proof of Theorem 4 is given in Appendix E.

# **Algorithm 4** em-algorithm with $\epsilon$ approximated m-step in the exponential subfamily

Assume that  $\mathcal{M}$  is characterized by (40). We choose two parameters  $\epsilon_1 < \epsilon_2$ . Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ;

# repeat

**m-step:** We choose  $\theta^{(t+1)} \in \mathcal{M}$  and  $\bar{\theta}^{(t+1)} = \theta_{(t)} + \sum_{j=k+1}^d \tau_o^j u_j$  such that

$$F\left(\theta_{(t)} + \sum_{j=k+1}^{d} \tau_{o}^{j} u_{j}\right) - \sum_{j=k+1}^{d} \tau_{o}^{j} a_{j}$$

$$\leq \min_{\bar{\tau}^{k+1}, \dots, \bar{\tau}^{d}} F\left(\theta_{(t)} + \sum_{j=k+1}^{d} \bar{\tau}^{j} u_{j}\right) - \sum_{j=k+1}^{d} \bar{\tau}^{j} a_{j} + \epsilon_{1}. \tag{104}$$

and

$$D(\theta^{(t+1)} \| \bar{\theta}^{(t+1)}) \le \epsilon_2. \tag{105}$$

**e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\bar{\theta}^{(t+1)}).$ 

**until**  $t = t_1 - 1$ .

final step: We output the final estimate  $\theta_f^{(t_1)} := \theta^{(t_2)} \in \mathcal{M}$  by using  $t_2 := \underset{t=2,...,t_1}{\operatorname{argmin}} D^F(\theta^{(t)} \| \theta_{(t-1)}) - D^F(\theta^{(t)} \| \bar{\theta}^{(t)})$ .

Since m-step has two conditions, Algorithm 4 seems complicated. This step can be realized as follows. The condition (104) simply shows the error for the minimization of the convex function  $F(\theta_{(t)} + \theta_{(t)})$ 

 $\sum_{j=k+1}^{d} \bar{\tau}^{j} u_{j}$   $-\sum_{j=k+1}^{d} \bar{\tau}^{j} a_{j}$ . The condition (105) is related to the choice of  $\theta^{(t+1)} \in \mathcal{M}$ . As one possible choice, we choose  $\theta^{(t+1)}$  as follows. Next, we choose the element  $\kappa^{j'}$  by solving the equations

$$u_{j'}^{i'}\Big(\eta_{i'}(\bar{\theta}^{(t+1)}) + \sum_{i=1}^{d} J_{i',i}(\bar{\theta}^{(t+1)}) \sum_{j=k+1}^{d} u_{j}^{i} \kappa^{j}\Big) = a_{j'}$$
(106)

for  $j'=k+1,\ldots,d$ . Then, we choose the element  $\theta^{(t+1)}$  by  $\eta_j(\theta^{(t+1)})=\eta_j(\bar{\theta}^{(t+1)})+\sum_{i=1}^d\sum_{j'=k+1}^dJ_{j,i}(\bar{\theta}^{(t+1)})u^i_{j'}\kappa^{j'}$  for  $j=1,\ldots,d$ . If  $\theta^{(t+1)}$  does not satisfy (105), we retake  $\bar{\theta}^{(t+1)}$  such that the value  $F\left(\theta_{(t)}+\sum_{j=k+1}^d\tau^j_ou_j\right)-\sum_{j=k+1}^d\tau^j_oa_j$  is smaller than the previous one.

In this way, the m-step of Algorithm 3 requires the approximate calculation of the minimum  $\min_{\theta \in \mathcal{M}} D^F(\theta \| \theta_{(t)})$ , which can be done as the convex minimization with respect to the mixture parameter in  $\mathcal{M}$ . However, this minimization needs to handle d-k parameters. If k < d-k, the alternative minimization given in (92) has a smaller number of parameters. As an approximate version of Algorithm 2, we have Algorithm 4. Indeed, if we can calculate the derivative of the convex function  $F\left(\theta_{(t)} + \sum_{j=k+1}^{d} \bar{\tau}^{j} u_{j}\right) - \sum_{j=k+1}^{d} \bar{\tau}^{j} a_{j}$ , we can employ algorithms explained in Appendix A.

In Algorithm 4, we use the relation

$$\Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)}) = \Gamma_{\mathcal{E}}^{(e),F}(\bar{\theta}^{(t+1)}). \tag{107}$$

In fact, the point  $\Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)})$  is characterized by the intersection between the exponential subfamily  $\mathcal{E}$  and the mixture subfamily whose mixture parameters  $\eta_1,\ldots,\eta_l$  are fixed to  $\eta_1(\theta^{(t+1)}),\ldots,\eta_l(\theta^{(t+1)})$ . Hence, the above relation (107) holds.

Then, we have the following theorem.

Theorem 5: Assume Conditions (B0), (B1), and the existence of the minimizer  $\theta^* := \theta^*(\mathcal{M}, \mathcal{E})$  in (90) for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$ . Then, in Algorithm 4, we have

$$D^{F}(\theta^{(t+1),*} \| \bar{\theta}^{(t+1)}) \le \epsilon_1 \tag{108}$$

for  $t=1,\ldots,t_1-1$ , where  $\theta^{(t+1),*}$  is defined as  $\theta_{(t)}+\sum_{j=k+1}^d \tau_*^j u_j$  by using  $(\tau_*^{k+1},\ldots,\tau_*^d):=\operatorname*{argmin}_{\bar{\tau}^{k+1},\ldots,\bar{\tau}^d} F\Big(\theta_{(t)}+\sum_{j=k+1}^d \tau_*^j u_j\Big)$ 

 $\sum_{j=k+1}^{d} \bar{\tau}^{j} u_{j} \Big) - \sum_{j=k+1}^{d} \bar{\tau}^{j} a_{j}. \text{ Also, the quantity } D^{F}(\theta_{f}^{(t_{1})} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t_{1})})) \text{ converges to the minimum } C_{\inf}(\mathcal{M}, \mathcal{E}) \text{ with the speed}$ 

$$D^{F}(\theta_{f}^{(t_{1})} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t_{1})})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq \frac{1}{t_{1} - 1} D^{F}(\theta_{*} \| \theta_{(1)}) + \epsilon_{1} + \epsilon_{2}.$$

$$(109)$$

The proof of Theorem 5 is given in Appendix F.

Considering Taylor expansion, we have

$$a_{j'} = \sum_{j'} u_{j'}^{i'} \eta_{i'}(\theta_{(t_2-1)} + \sum_{j=k+1}^{d} \tau_*^j u_j)$$

$$\cong u_{j'}^{i'} \left( \eta_{i'}(\bar{\theta}^{(t_2)}) + \sum_{i=1}^{d} J_{i',i}(\bar{\theta}^{(t_2)}) \sum_{j=k+1}^{d} u_j^i (\tau_*^j - \bar{\tau}^j) \right)$$
(110)

for  $j' = k + 1, \dots, d$ . Hence,

$$\kappa^j \cong (\tau^j_* - \bar{\tau}^j) \tag{111}$$

Using (29), we have

$$D^{F}(\theta^{(t_2),*} \| \bar{\theta}^{(t_2)}) \cong \frac{1}{2} \sum_{j=1}^{d} \sum_{i=1}^{d} \sum_{j'=k+1}^{d} \sum_{i'=k+1}^{d} J_{j,i}(\bar{\theta}^{(t_2)}) u_{j'}^{i}(\tau_*^{j'} - \bar{\tau}^{j'}) u_{i'}^{j}(\tau_*^{i'} - \bar{\tau}^{i'}).$$
(112)

Using (29), we have

$$D^{F}(\theta^{(t_{2})} \| \bar{\theta}^{(t_{2})}) = D^{F^{*}}(\eta(\bar{\theta}^{(t_{2})}) \| \eta(\theta^{(t_{2})}))$$

$$\cong \frac{1}{2} \sum_{j=1}^{d} \sum_{\bar{j}=1}^{d} (J(\bar{\theta}^{(t_{2})})^{-1})^{j,\bar{j}} \sum_{\bar{i}=1}^{d} \sum_{\bar{j}'=k+1}^{d} J_{\bar{j},\bar{i}}(\bar{\theta}^{(t_{2})}) u_{\bar{j}'}^{i} \kappa^{\bar{j}'} \sum_{i=1}^{d} \sum_{j'=k+1}^{d} J_{j,i}(\bar{\theta}^{(t_{2})}) u_{j'}^{i} \kappa^{j'}$$

$$= \frac{1}{2} \sum_{j=1}^{d} \sum_{i=1}^{d} \sum_{j'=k+1}^{d} \sum_{i'=k+1}^{d} J_{j,i}(\bar{\theta}^{(t_{2})}) u_{j'}^{i} \kappa^{j'} u_{i'}^{j} \kappa^{i'}.$$
(113)

Combining (108), (113), and (112), we have

$$D^F(\theta^{(t_2)} \| \bar{\theta}^{(t_2)}) \lessapprox \epsilon_1 \tag{114}$$

Therefore, (109) is rewritten as

$$D^{F}(\theta_{f}^{(t_{1})} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t_{1})})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\lesssim \frac{1}{t_{1} - 1} D^{F}(\theta_{*} \| \theta_{(1)}) + 2\epsilon_{1}.$$
(115)

Hence, when  $t_1 - 1 \ge \frac{3D^F(\theta_{*,1}||\theta_{(1)})}{\epsilon}$ , and  $\epsilon_1 \le \frac{\epsilon}{3}$ , the parameter  $\theta^{(t)}$  satisfies

$$D^{F}(\theta_f^{(t_1)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_f^{(t_1)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \lessapprox \epsilon.$$
(116)

## B. Closed convex mixture family

In this section, we address a similar minimization problem for a pair of a k-dimensional closed convex mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$  under the following condition (B0). That is, we discuss a closed convex mixture subfamily instead of a mixture subfamily  $\mathcal M$  while we consider an exponential subfamily  $\mathcal{E}$ . Under this condition, we employ the same e-projection  $\Gamma_{\mathcal{E}}^{(e),F}$  defined in Lemma 7 as in the previous subsection, but, we use the m-projection  $\Gamma_{\mathcal{M}}^{(m),F}$  defined in Lemma 8. Hence, we consider Algorithm 5 instead of Algorithm 2.

#### **Algorithm 5** em-algorithm with closed convex mixture family

Assume that  $\mathcal{M}$  is characterized by the mixture parameter  $\eta$ . Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ;

**m-step:** Calculate  $\eta^{(t+1)}$ . That is,  $\eta^{(t+1)}$  is given as  $\underset{\eta \in \Xi_{\mathcal{M}}}{\operatorname{argmin}} D^F(\phi_{\mathcal{M}}^{(m)}(\eta) \| \theta_{(t)})$ , i.e., the unique

element in  $\mathcal{M}$  to realize the minimum of the smooth convex function  $\eta \mapsto D^F(\phi_{\mathcal{M}}^{(m)}(\eta) \| \theta_{(t)})$ . **e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}))$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}) \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the

smooth convex function  $\theta' \mapsto D^F(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}) || \theta').$ 

until convergence.

When the boundary  $\partial \mathcal{M}$  is composed of a finite number of closed mixture families, due to Lemmas 5 and 10, Algorithm 5 can be simplified to Algorithm 6 because Lemma 10 guarantees that  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$  is **Algorithm 6** em-algorithm with closed convex mixture family whose boundary is composed of finite number of closed mixture families

Assume the following conditions; A set of closed convex mixture subfamilies  $\{\mathcal{M}_{\lambda}\}_{\lambda\in\Lambda}$  covers the boundary  $\partial\mathcal{M}$  of a closed convex mixture family  $\mathcal{M}$  with subsets  $\Lambda_{\lambda}\subset\Lambda$  and  $\lambda\in\Lambda_{*}:=\Lambda\cup\{0\}$ . Each closed convex mixture subfamily  $\mathcal{M}_{\lambda}$  is generated by the constraint by  $\sum_{i=1}^{d}u_{j,\lambda}^{i}\partial_{i}F(\theta)=a_{j,\lambda}$  for  $j=k_{\lambda}+1,\ldots,d$  for  $\lambda\in\Lambda_{*}$ . Choose the initial value  $\theta_{(1)}\in\mathcal{E}$ ;

repeat

**m-step:** Calculate  $\theta^{(t+1)} := \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$  in the following way. For  $\lambda \in \Lambda_*$ , we calculate  $\theta^{(t+1),\lambda}$  is given as  $\theta_{(t)} + \sum_{j=k_{\lambda}+1}^{d} \tau^{j,\lambda} u_j$ , where  $(\tau^{k_{\lambda}+1,\lambda},\ldots,\tau^{d,\lambda})$  is the unique element to satisfy

$$\frac{\partial}{\partial \tau^{\bar{j},\lambda}} F\left(\theta_{(t)} + \sum_{j=k_{\lambda}+1}^{d} \tau^{j,\lambda} u_j\right) = a_{\bar{j},\lambda}$$
(117)

for  $\bar{j} = k_{\lambda} + 1, \dots, d$ . We set  $\theta^{(t+1)}$  as  $\theta^{(t+1),\lambda_0}$ , where

$$\lambda_0 := \underset{\lambda \in \Lambda_*}{\operatorname{argmin}} \{ D^F(\theta^{(t+1),\lambda} || \theta_{(t)}) | \theta^{(t+1),\lambda} \in \mathcal{M} \}.$$
(118)

**e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\hat{\theta}^{(t+1)})$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\theta^{(t+1)} \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the smooth convex function  $\theta' \mapsto D^F(\theta^{(t+1)} \| \theta')$ . **until** convergence.

given as  $\Gamma_{\hat{\mathcal{M}}_{\lambda_0}}^{(m),F}(\theta_{(t)})$ , where we denote the extended mixture subfamily of  $\mathcal{M}_{\lambda}$  by  $\hat{\mathcal{M}}_{\lambda}$  for  $\lambda \in \Lambda_*$ , and  $\lambda_0$  is given in (118).

Then, in the same way as Theorem 2, we have the following theorem.

Theorem 6: Assume Conditions (B0), (B1), and  $\sup_{\theta \in \mathcal{E}} D^F(\theta \| \theta_{(1)}) < \infty$  for a pair of a k-dimensional closed convex mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$ . Then, Algorithms 5 and 6 have the same conclusion as Theorem 2.

Also, in the same way as Theorem 3, we have the following theorem;

Theorem 7: Assume that Conditions (B0) and (B1+) hold for a pair of a k-dimensional close convex mixture subfamily  $\mathcal{M}$ , an l-dimensional exponential subfamily  $\mathcal{E}$ , and  $\theta' = \theta_{(1)} \in \mathcal{E}$ . Then, the quantity  $D^F(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)}))$  converges to the minimum  $C_{\inf}(\mathcal{M},\mathcal{E})$  with the speed

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) = \beta(\theta_{(1)})^{t-2} D^{F}(\theta_{*} \| \theta_{(1)}). \tag{119}$$

Further, when  $t-2 \geq \frac{\log D^F(\theta_* \| \theta_{(1)}) - \log \epsilon}{\log \beta(\theta_{(1)})}$ , the parameter  $\theta^{(t)}$  satisfies

$$D^{F}(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \epsilon.$$
(120)

Theorems 6 and 7 are shown in Appendix G.

When we need to care the error in the m-step, as an error version of Algorithm 5, we have Algorithm 7 in the same way as Algorithm 3.

Then, we have the following theorem.

Theorem 8: Assume Conditions (B0), (B1), and the existence of the minimizer  $\theta^* := \theta^*(\mathcal{M}, \mathcal{E})$  in (90) for a pair of a k-dimensional mixture subfamily  $\mathcal{M}$  and an l-dimensional exponential subfamily  $\mathcal{E}$ . In addition, we define the set  $\mathcal{E}_0 := \{\theta \in \mathcal{E} | D^F(\theta_* \| \theta) \leq D^F(\theta_* \| \theta_{(1)})\} \subset \mathcal{E}$  and  $\theta_* := \Gamma_{\mathcal{E}}^{(e),F}(\theta^*)$ . Then, Algorithm 7 has the same conclusion as Theorem 4.

Theorem 8 is shown in Appendix G.

When k < d - k, we need an alternative minimization for the m-step for Algorithm 7 in a way similar to Algorithm 4. However, although we can consider a modification of Algorithm 6 in a way similar to

## **Algorithm 7** em-algorithm with $\epsilon$ approximated m-step

Assume that  $\mathcal{M}$  is characterized by the mixture parameter  $\eta$ . Choose the initial value  $\theta_{(1)} \in \mathcal{E}$ ; **repeat** 

**m-step:** Calculate  $\eta^{(t+1)}$ . That is, we choose  $\eta^{(t+1)} \in \mathcal{M}$  such that

$$D^{F}(\theta^{(t+1)} \| \theta_{(t)}) \le \min \left( D^{F}(\theta^{(t)} \| \theta_{(t)}), \min_{\theta \in \mathcal{M}} D^{F}(\theta \| \theta_{(t)}) + \epsilon \right), \tag{121}$$

where  $D^F(\theta^{(1)} || \theta_{(1)})$  is defined as  $\infty$ .

**e-step:** Calculate  $\theta_{(t+1)} := \Gamma_{\mathcal{E}}^{(e),F}(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}))$ . That is,  $\theta_{(t+1)}$  is given as  $\underset{\theta' \in \mathcal{E}}{\operatorname{argmin}} D^F(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}) \| \theta')$ , i.e., the unique element in  $\mathcal{E}$  to realize the minimum of the

smooth convex function  $\theta' \mapsto D^F(\phi_{\mathcal{M}}^{(m)}(\eta^{(t+1)}) \| \theta').$ 

until convergence.

Algorithm 4, it is not so easy to evaluate the error or the modified algorithm. Hence, to take into account the error in the m-step, we propose another method to modify Algorithm 6 as Algorithm 8.

# **Algorithm 8** em-algorithm with $\epsilon$ approximated m-step in the exponential subfamily

We assume the same conditions as Algorithm 6. We denote the extended mixture subfamily of  $\mathcal{M}_{\lambda}$  by  $\hat{\mathcal{M}}_{\lambda}$  for  $\lambda \in \Lambda_*$ .

**1st-step:** For  $\lambda \in \Lambda_*$ , we apply Algorithm 4 to the pair of the exponential subfamily  $\mathcal{E}$  and the mixture subfamily  $\hat{\mathcal{M}}_{\lambda}$ . As the result with t iteration, we denote the number  $t_2$  in this application of Algorithm 4 by  $t_2(\lambda)$ . Then, we denote  $\theta^{(t_2(\lambda))}$ ,  $\bar{\theta}^{(t_2(\lambda))}$ , and  $\theta_{(t_2(\lambda)-1)}$ , in this application by  $\theta^{(t_2(\lambda)),\lambda}$ ,  $\bar{\theta}^{(t_2(\lambda)),\lambda}$ , and  $\theta_{(t_2(\lambda)-1),\lambda}$ , respectively

**2nd-step:** We output the final estimate  $\theta_f^{(t)} := \theta^{(t_2(\lambda_0)),\lambda_0} \in \mathcal{M}$ , where

$$\lambda_0 := \underset{\lambda \in \Lambda_n}{\operatorname{argmin}} \left\{ D^F \left( \theta^{(t_2(\lambda)), \lambda} \middle\| \theta_{(t_2(\lambda) - 1), \lambda} \right) \middle| \theta^{(t_2(\lambda)), \lambda} \in \mathcal{M}_{\lambda} \right\}. \tag{122}$$

To evaluate the error of Algorithm 8, we prepare the following lemma. Therefore, using Theorem 5, we obtain the following theorem for the error evaluation of Algorithm 8.

Theorem 9: Assume the same assumption as Algorithm 8 and Conditions (B0) and (B1) for  $\mathcal{E}$ . Also, we assume the existence of the minimizer  $\theta^* := \theta^*(\mathcal{M}_\lambda, \mathcal{E})$  in (90) for  $\lambda \in \Lambda_*$ . Then, in Algorithm 8, the quantity  $D^F(\theta_f^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_f^{(t)}))$  converges to the minimum  $C_{\inf}(\mathcal{M}, \mathcal{E})$  with the speed

$$D^{F}(\theta_{f}^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq (D(0) + 1) \max_{\lambda \in \Lambda_{*}} \left( \frac{1}{t_{1} - 1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)),\lambda} \| \bar{\theta}^{(t_{2}(\lambda)),\lambda}) \right). \tag{123}$$

Notice that  $D(\lambda)$  is defined before Lemma 9.

The proof of Theorem 9 is given in Appendix H.

#### V. CLASSICAL RATE DISTORTION

## A. Classical rate distortion without side information

Let  $\mathcal{X} := \{1, \dots, n_1\}$  and  $\mathcal{Y} := \{1, \dots, n_2\}$  be finite sets. We call a map  $W : \mathcal{X} \to \mathcal{P}_{\mathcal{Y}}$  a channel from  $\mathcal{X}$  to  $\mathcal{Y}$ . We denote the set of the above maps by  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}$ . We use the notation  $W_x(y) := W(y|x)$ . For  $q \in \mathcal{P}_{\mathcal{X}}$  and  $q \in \mathcal{P}_{\mathcal{Y}}$ ,  $W \cdot q \in \mathcal{P}_{\mathcal{Y}}$ ,  $W \cdot q \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$ , and  $q \times r \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}}$  are defined by  $(W \cdot q)(y) := \sum_{x \in \mathcal{X}} W(y|x)q(x)$ ,  $(W \times q)(x,y) := W(y|x)q(x)$ , and  $(q \times r)(x,y) := q(x)r(y)$  respectively.

Given a distortion measure d(x, y) on  $\mathcal{X} \times \mathcal{Y}$  and a distribution  $P_X$  on  $\mathcal{X}$ , we define the following sets;

$$\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D} := \left\{ W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}} \middle| \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y)W \times P_X(x,y) = D \right\}$$
 (124)

$$\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq} := \left\{ W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}} \middle| \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y)W \times P_X(x,y) \leq D \right\}.$$
 (125)

We define  $\bar{d}(x,y)$  as

$$\bar{d}(x,y) := d(x,y) - d(x,n_2), \quad \bar{d}(x,n_2) := 0$$
 (126)

for  $x \in \mathcal{X}$  and  $y = 1, \dots, n_2 - 1$ . Then, the condition

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x, y)W \times P_X(x, y) \le D \tag{127}$$

is equivalent to

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \bar{d}(x, y)W \times P_X(x, y) \le D - \sum_{x \in \mathcal{X}} P_X(x)d(x, n_2). \tag{128}$$

Hence, for simplicity, we assume that  $d(x,n_2)=0$  in the following. Also, we define the vector  $d=(d_j)_{j=1}^{n_1(n_2-1)}$  as  $d_{(x-1)(n_2+1)+y}:=d(i,j)$  for  $x\in\mathcal{X}$  and  $y=1,\ldots,n_2-1$ .

The standard rate distortion function is given as

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}} I(X;Y)_{W \times P_X} = \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}} D(W \times P_X \| (W \cdot P_X) \times P_X)$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}} \min_{q \in \mathcal{P}_{\mathcal{Y}}} D(W \times P_X \| q \times P_X). \tag{129}$$

In the following, we use the notation  $W_* := \underset{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}}{\operatorname{argmin}} I(X;Y)_{W \times P_X}.$ 

When there exists a distribution  $Q_Y$  on  $\mathcal{Y}$  such that

$$\sum_{x,y} P_X(x)Q_Y(y)d(x,y) \le D,\tag{130}$$

the above minimum (129) is zero. The existence of  $Q_Y$  to satisfy the condition (130) is equivalent to

$$\min_{y} d_Y(y) \le D,\tag{131}$$

where  $d_Y(y) := \sum_{x \in \mathcal{X}} P_X(x) d(x, y)$ .

Then, we consider the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$  defined in Subsection III-B, which coincides with the set of distributions  $W \times P_X$ . The set of distributions  $q \times P_X$  forms an exponential subfamily  $\mathcal{E}$ , and the subset  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D} \times P_X$  forms a mixture subfamily  $\mathcal{M}$ .

Then, we have the following theorem.

Lemma 16: When (131) holds,  $\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}} I(X;Y)_{W \times P_X} = 0$ . Otherwise,

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}} I(X;Y)_{W \times P_X} = \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D}} I(X;Y)_{W \times P_X}$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D}} \min_{q \in \mathcal{P}_{\mathcal{Y}}} D(W \times P_X || q \times P_X). \tag{132}$$

*Proof:* The first statement has been already shown. We show the second statement by contradiction. Assume that (131) nor the first equation in (132) does not hold. We define  $Q_{Y,1}$  as  $Q_{Y,1} \times P_X = \Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(W_* \times P_X)$ .

Since  $Q_{Y,1} \times P_X$  does not belong to  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}$ , applying (56) in Lemma 8 to the closed convex mixture subfamily  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}$ , we find that  $\Gamma_{\mathcal{M}}^{(m),\bar{\mu}}(Q_{Y,1} \times P_X) = \Gamma_{\mathcal{M}}^{(m),\bar{\mu}} \circ \Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(W_* \times P_X)$  belongs to  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D}$ . We choose  $W_1$  such that  $(W_1 \times P_X) = \Gamma_{\mathcal{M}}^{(m),\bar{\mu}}(Q_{Y,1} \times P_X)$ . Hence, we have

$$I(X;Y)_{W_* \times P_X} = D(W_* \times P_X || Q_{Y,1} \times P_X)$$
  
 
$$\geq D(W_1 \times P_X || Q_{Y,1} \times P_X) \geq I(X;Y)_{W_1 \times P_X},$$
 (133)

which contradicts 
$$W_* = \underset{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D,\leq}}{\operatorname{argmin}} I(X;Y)_{W \times P_X}.$$

Due to Lemma 16, when (131) does not hold, it is sufficient to address the minimization (132). In the following, we address the minimization problem (132), which is a special case of the minimization (89) with the formulation given in Subsection IV-A. The mixture family  $\mathcal{M}$  has  $n_1(n_2-1)-1$  parameters.

Since the total dimension is  $n_1(n_2-1)$ , we employ Algorithm 2 instead of Algorithm 1. Since Lemma 13 guarantees Condition (B0) for this problem, Algorithm 2 works and is rewritten as Algorithm 9.

# Algorithm 9 em-algorithm for rate distortion

Choose the initial distribution  $P_Y^{(1)}$  on  $\mathcal{Y}$ . Then, we define the initial joint distribution  $P_{XY,(1)}$  as  $P_Y^{(1)} \times P_X$ ;

repeat

**m-step:** Calculate  $P_{XY}^{(t+1)}$  as  $P_{XY}^{(t+1)}(x,y) := P_X(x)P_Y^{(t)}(y)e^{\bar{\tau}d(x,y)}\left(\sum_{y'}P_Y^{(t)}(y')e^{\bar{\tau}d(x,y')}\right)^{-1}$ , where  $\bar{\tau}$  is the unique element  $\tau$  to satisfy

$$\frac{\partial}{\partial \tau} \sum_{x} P_X(x) \log \left( \sum_{y} P_Y^{(t)}(y) e^{\tau d(x,y)} \right) = D \tag{134}$$

This choice can be written in the way as (92).

**e-step:** Calculate  $P_Y^{(t+1)}(y)$  as  $\sum_{x \in \mathcal{X}} P_{XY}^{(t+1)}(x,y)$ .

until convergence.

To check Condition (B1), we set  $\theta$  and  $\theta'$  be elements of  $\mathbb{R}^{n_1n_2-1}$  corresponding to  $W \times P_X$  and  $W' \times P_X$  in the sense of the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$  defined in Subsection III-B. Then, the relation

$$D^{\bar{\mu}}(\Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(\theta') \| \Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(\theta)) = D((W' \cdot P_X) \times P_X \| (W' \cdot P_X) \times P_X)$$
  
=  $D(W' \cdot P_X \| W' \cdot P_X) \le D(W' \times P_X \| W' \times P_X) = D^{\bar{\mu}}(\theta' \| \theta)$  (135)

guarantees condition (B1). When the initial value  $\theta_{(1)}$  is chosen as the case that W has full support,  $\sup_{\theta \in \mathcal{E}} D^{\bar{\mu}}(\theta \| \theta_{(1)})$  has a finite value. Hence, Theorem 6 guarantees the convergence to the global minimum. Now, we set  $\theta_{(1)}$  to be the product of  $P_X$  and the uniform distribution on  $\mathcal{Y}$ . Then, we have

$$D^{\bar{\mu}}(\theta_* \| \theta_{(1)}) \le \sup_{\theta \in \mathcal{M}} D^{\bar{\mu}}(\theta \| \theta_{(1)}) = \log n_2.$$
 (136)

Hence, the inequality (96) is rewritten as

$$I(X;Y)_{P_{XY}^{(t)}} - \min_{W \in \mathcal{P}_{Y|X}^{d,P_X,D,\leq}} I(X;Y)_{W \times P_X} \le \frac{\log n_2}{t-1}$$
(137)

In particular, when  $t \geq \frac{\log n_2}{\epsilon} + 1$ , the above value is bounded by  $\epsilon$ .

The original problem (132) is written as a concave optimization with respect to  $n_1(n_2 - 1)$  mixture parameters because the mutual information is concave with respect to the conditional distribution. Although our protocol contains a convex optimization in m-step, the convex optimization in m-step has only one

variable. Therefore, our method is considered to convert a complicated concave optimization with a larger size to iterative applications of a convex optimization with one variable.

Next, we consider the case when we cannot exactly calculate the unique element  $\bar{\tau}$  to satisfy (134). Alternatively, we need to use  $\epsilon$  approximation for the solution. We employ Algorithm 4, which requires to solve the minimization of the one-variable smooth convex function  $\hat{F}[P_Y](\tau) := \sum_x P_X(x) \log \left( \sum_y P_Y(y) e^{\tau(D-d(x,y))} \right)$ . That is, it is needed to find the minimizer  $\tau_*[P_Y] := \operatorname{argmin} \hat{F}[P_Y](\tau)$ .

To consider this minimization, we focus on the one-parameter exponential subfamily  $P_{X,Y|\tau}[P_Y](x,y) :=$  $P_X(x) \frac{P_Y(y)e^{\tau(D-d(x,y))}}{\sum_{y'} P_Y(y')e^{\tau(D-d(x,y'))}}$ . The first and second derivatives are calculated as

$$\frac{d}{d\tau}\hat{F}[P_Y](\tau) = \mathbb{E}_{P_{X,Y|\tau}[P_Y]}[D - d(X,Y)] \tag{138}$$

$$\frac{d^2}{d\tau^2} \hat{F}[P_Y](\tau) = \mathbb{E}_{P_{X,Y|\tau}[P_Y]}[(D - d(X,Y))^2] - \mathbb{E}_{P_{X,Y|\tau}[P_Y]}[D - d(X,Y)]^2. \tag{139}$$

Defining  $\zeta_+ := \max_{x,y} |D - d(x,y)|^2$ , we have

$$\frac{d^2}{d\tau^2}\hat{F}[P_Y](\tau) \le \zeta_+. \tag{140}$$

The condition (140) guarantees that

$$\hat{F}[P_Y](\tau) \le \hat{F}[P_Y](0) + \frac{d}{d\tau}\hat{F}[P_Y](0)\tau + \frac{1}{2}\zeta_+\tau^2$$
(141)

for  $\tau > 0$ . To solve  $\min_{\tau} \hat{F}[P_Y](\tau)$ , we employ the bisection method explained in Appendix A-A. Since (131) holds, the relation  $\mathbb{E}_{P_X \times P_Y}[d(X,Y)] > D$ , i.e.,  $\frac{d}{d\tau}\hat{F}[P_Y](0) < 0$  holds for any distribution  $P_Y$ . Hence,  $\frac{d}{d\tau}\hat{F}[P_Y](-\frac{\frac{d}{d\tau}\hat{F}[P_Y](0)}{\zeta_-}) \geq 0$ . For the application of the bisection method, we consider the following condition for the convex function

 $F[P_Y](\tau);$ 

$$\frac{d^2}{d\tau^2} \hat{F}[P_Y](\tau) \ge \zeta_- \text{ for } \tau \in [0, \tau_*[P_Y]]. \tag{142}$$

Since the condition (142) guarantees  $0 \le \tau_*[P_Y] \le -\frac{\frac{d}{d\tau}\hat{F}[P_Y](0)}{\zeta_-}$ , we can apply the bisection method, Algorithm 15 with a=0 and  $b=-\frac{\frac{d}{d\tau}\hat{F}[P_Y](0)}{\zeta_-}$ . Under the condition (142), we have

$$\hat{F}[P_Y](0) - \hat{F}[P_Y](\tau_*[P_Y]) \le -\frac{d}{d\tau}\hat{F}[P_Y](0)\tau_*[P_Y] \le \frac{1}{\zeta_-} \left(\frac{d}{d\tau}\hat{F}[P_Y](0)\right)^2. \tag{143}$$

We choose the estimate  $\tau_k[P_Y]$  as  $b_k$  of Algorithm 15, which requires k iterations. Then, we have  $\frac{d}{d\tau}\hat{F}[P_Y](\tau_k[P_Y]) > 0$ . The relation (202) guarantees that

$$\hat{F}[P_{Y}](\tau_{k}[P_{Y}]) - \hat{F}[P_{Y}](\tau_{*}[P_{Y}]) 
\leq \frac{1}{2^{k-1}} \max \left(\hat{F}[P_{Y}](0) - \hat{F}[P_{Y}](\tau_{*}[P_{Y}]), \hat{F}[P_{Y}](-\frac{\frac{d}{d\tau}\hat{F}[P_{Y}](0)}{\zeta_{-}}) - \hat{F}[P_{Y}](\tau_{*}[P_{Y}])\right) 
\leq \frac{1}{2^{k-1}} \max \left(\frac{1}{\zeta_{-}} \left(\frac{d}{d\tau}\hat{F}[P_{Y}](0)\right)^{2}, \right. 
\left. \frac{1}{\zeta_{-}} \left(\frac{d}{d\tau}\hat{F}[P_{Y}](0)\right)^{2} + \frac{d}{d\tau}\hat{F}[P_{Y}](0) \cdot \frac{-\frac{d}{d\tau}\hat{F}[P_{Y}](0)}{\zeta_{-}} + \frac{1}{2}\zeta_{+} \cdot \left(\frac{-\frac{d}{d\tau}\hat{F}[P_{Y}](0)}{\zeta_{-}}\right)^{2}\right) 
= \frac{1}{2^{k}} \left(\frac{d}{d\tau}\hat{F}[P_{Y}](0)\right)^{2} \frac{\zeta_{+}}{\zeta_{-}^{2}}.$$
(144)

The relation (204) guarantees that

$$0 \le \tau_k[P_Y] - \tau_*[P_Y] \le -\frac{1}{2^k} \frac{\frac{d}{d\tau} \hat{F}[P_Y](0)}{\zeta_-}.$$
(145)

Hence,

$$0 \le \frac{d}{d\tau} \hat{F}[P_Y](\tau_k[P_Y]) \le -\frac{\zeta_+}{2^k} \frac{\frac{d}{d\tau} \hat{F}[P_Y](0)}{\zeta_-}.$$
 (146)

We can choose  $\kappa[P_Y] \geq 0$  as  $0 = (1 - \kappa[P_Y]) \mathbb{E}_{P_{X,Y} | \tau_k[P_Y]}[D - d(X,Y)] + \kappa[P_Y] \mathbb{E}_{P_X \times P_Y}[D - d(X,Y)]$ . Then,

$$0 \le \kappa[P_Y] = \frac{\frac{d}{d\tau}\hat{F}[P_Y](\tau_k[P_Y])}{\frac{d}{d\tau}\hat{F}[P_Y](\tau_k[P_Y]) - \frac{d}{d\tau}\hat{F}[P_Y](0)} \le \frac{\frac{d}{d\tau}\hat{F}[P_Y](\tau_k[P_Y])}{-\frac{d}{d\tau}\hat{F}[P_Y](0)} \le \frac{\zeta_+}{2^k\zeta_-}.$$
 (147)

Then, we choose  $P_{XY|k}[P_Y]$  as follows.

$$P_{XY|k}[P_Y] := (1 - \kappa[P_Y]) P_{X,Y|\tau_k[P_Y]}[P_Y] + \kappa[P_Y] P_X \times P_Y. \tag{148}$$

Since

$$D(P_Y \times P_X || P_{X,Y|\tau_k[P_Y]}[P_Y]) = \hat{F}[P_Y](\tau_k[P_Y]) - \hat{F}[P_Y](0) - \frac{d}{d\tau}\hat{F}[P_Y](0)\tau_k[P_Y]$$

$$\leq -\frac{d}{d\tau}\hat{F}[P_Y](0)\tau_k[P_Y] \leq \frac{\left(\frac{d}{d\tau}\hat{F}[P_Y](0)\right)^2}{\zeta_-},$$
(149)

we have

$$D(P_{XY|k}[P_Y]||P_{X,Y|\tau_k[P_Y]}[P_Y])$$

$$\leq (1 - \kappa[P_Y])D(P_{X,Y|\tau_k[P_Y]}[P_Y]||P_{X,Y|\tau_k[P_Y]}[P_Y]) + \kappa[P_Y]D(P_X \times P_Y||P_{X,Y|\tau_k[P_Y]}[P_Y])$$

$$= \kappa[P_Y]D(P_X \times P_Y||P_{X,Y|\tau_k[P_Y]}[P_Y])$$

$$\leq \kappa[P_Y]\frac{\left(\frac{d}{d\tau}\hat{F}[P_Y](0)\right)^2}{\zeta_-} \leq \frac{\zeta_+}{2^k\zeta_-^2}\left(\frac{d}{d\tau}\hat{F}[P_Y](0)\right)^2. \tag{150}$$

Given  $\epsilon' > 0$ , we choose k as

$$k[P_Y, \epsilon'] := \log_2\left(\left(\frac{d}{d\tau}\hat{F}[P_Y](0)\right)^2 \frac{\zeta_+}{\zeta_-^2}\right) - \log_2\epsilon' \le \log_2\left(\frac{\zeta_+^2}{\zeta_-^2}\right) - \log_2\epsilon'. \tag{151}$$

The relations (144) and (150) guarantee

$$\hat{F}[P_Y](\tau_k[P_Y]) - \hat{F}[P_Y](\tau_*[P_Y]) \le \epsilon'$$
(152)

$$D(P_{XY|k}[P_Y]||P_{X,Y|\tau_k[P_Y]}[P_Y]) \le \epsilon'.$$
(153)

Combining the above discussion for the bisection method and Algorithm 4, we obtain Algorithm 10. Since  $\epsilon'$  is chosen as  $\epsilon/2$  in Algorithm 10, and the conditions (B0) and (B1) hold, Theorem 5 guarantees the precision (109) with  $\epsilon_1 = \epsilon_2 = \epsilon/3$ . For its calculation complexity, we have the following lemma .

Lemma 17: Assume the conditions  $\zeta_- = O(1)$ , (142), and  $\zeta_+ = O(n_2^2)$ . We choose  $P_Y^{(1)}$  as the uniform distribution on  $\mathcal{Y}$ . To guarantee

$$I(X;Y)_{P_{XY}^{(t)}} - \min_{W \in \mathcal{P}_{V|X}^{d,P_X,D,\leq}} I(X;Y)_{W \times P_X} \le \epsilon, \tag{154}$$

Algorithm 10 needs calculation complexity  $O(\frac{n_1 n_2 \log n_2}{\epsilon} (\log_2 n_2 + \log_2 \epsilon))$ .

# Algorithm 10 em-algorithm for rate distortion

Choose the initial distribution  $P_Y^{(1)}$  on  $\mathcal{Y}$ . Then, we define the initial joint distribution  $P_{XY,(1)}$  as  $P_Y^{(1)} \times P_X$ ;

repeat

**m-step:** Calculate  $P_{XY}^{(t+1)}$  and  $\bar{P}_{XY}^{(t+1)}$  as follows. We apply Algorithm 15 with a:=0 and  $b:=-\frac{\frac{d}{d\tau}\hat{F}[P_Y^{(t)}](0)}{\zeta}$  with  $k=k[P_Y^{(t)},\frac{\epsilon}{3}]$  iterations; We choose  $\bar{P}_{XY}^{(t+1)}$  and  $\bar{P}_{XY}^{(t+1)}$  as  $P_{X,Y|\tau_k[P_Y]}[P_Y]$  and  $P_{XY|k}[P_Y]$ , respectively.

**e-step:** Calculate  $P_Y^{(t+1)}(y)$  as  $\sum_{x \in \mathcal{X}} \bar{P}_{XY}^{(t+1)}(x,y)$ .

**until**  $t = t_1 - 1$ .

final step: We output the final estimate  $P_{XY,f}^{(t_1)} := P_{XY}^{(t_2)} \in \mathcal{M}$  by using  $t_2 := \underset{t=2,\dots,t_1}{\operatorname{argmin}} D(P_{XY}^{(t)} \| P_X \times P_{XY}^{(t_1)} \| P_X \times P_{XY}^{(t_2)} \| P_X \times P_X^{(t_1)} \| P_X \times P_X^{(t_2)} \| P_X \times P_X^{(t_1)} \| P_X^{(t_1)} \| P_X \| P_X^{(t_1)} \| P_X^{(t_1)$ 

$$P_Y^{(t-1)}) - D(P_{XY}^{(t)} || \bar{P}_{XY}^{(t)}).$$

*Proof:* Each iteration in the bisection method needs calculation complexity  $O(n_1n_2)$ . Each application of the bisection method has  $O(\log_2 n_2 + \log_2 \epsilon)$  iterations. Hence, one application of the bisection method has  $O(n_1n_2(\log_2 n_2 + \log_2 \epsilon))$  calculation complexity.

Since  $D(\theta_* || \theta_1) = D(W_* \times P_X || P_Y \times P_X) \le \log n_2$ , the number  $t_1 = \frac{3 \log n_2}{\epsilon} + 1$  satisfies

$$\frac{1}{t_1 - 1} D(\theta_* \| \theta_1) \le \frac{\epsilon}{3}. \tag{155}$$

Since  $\epsilon_1$  and  $\epsilon_2$  are chosen as  $\epsilon_1 = \epsilon_2 = \epsilon/3$  in Algorithm 10, the RHS of (109) is upper bounded by  $\epsilon$ , which implies (154). In this case, the calculation complexity of Algorithm 10 is  $(\frac{3 \log n_2}{\epsilon} + 1) \cdot O(n_1 n_2 (\log_2 n_2 + \log_2 \epsilon)) = O(\frac{n_1 n_2 \log n_2}{\epsilon} (\log_2 n_2 + \log_2 \epsilon))$ .

Next, we compare Algorithm 15 and a simple application of accelerated proximal gradient method whose performance is evaluated as (208). In this application of accelerated proximal gradient method, we treat  $I(X;Y)_{W\times P_X}$  as a convex function for the mixture parameter, which is composed of  $(n_2-1)n_1$  parameters. In this case, L in (207) is  $\zeta_+^{\frac{1}{2}}$  and  $\|x_0-x_*\|^2$  in (208) is  $O((n_2-1)n_1)$ . Hence, to achieve the same precision as (154), the number of iteration is  $O(n_2^{\frac{1}{2}}n_1^{\frac{1}{2}}\zeta_+^{\frac{1}{4}}\frac{1}{\epsilon})$ . Each iteration has calculation complexity  $O(n_1n_2)$ . Hence, in total, this method has calculation complexity  $O(\frac{1}{\epsilon}n_2^{\frac{3}{2}}n_1^{\frac{3}{2}}\zeta_+^{\frac{1}{4}}) = O(\frac{1}{\epsilon}n_2^2n_1^{\frac{3}{2}})$ . This is larger than the calculation complexity given in Lemma 17.

Remark 1: Next, we see what Blahut algorithm [5] solved in the relation to (129). For this aim, we focus on the function  $f(D) := \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{d,P_X,D}} I(X;Y)_{W \times P_X}$  Instead of f(D), using Lagrange multiplier  $\tau_0$ , Blahut [5] focused on the minimization

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}} \tau_0 D + I(X;Y)_{W \times P_X} - \tau_0 \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x,y)(W \times P_X)(x,y). \tag{156}$$

When  $\frac{d}{dD}f(D) = \tau_0$ , the minimum (156) equals f(D). However, finding such  $\tau_0$  is not so easy. The algorithm to find such  $\tau_0$  was not given in [5]. The algorithm by [5] to solves (156) is the same as Algorithm 9 with replacing  $\bar{\tau}$  by  $\tau_0$ . That is, his algorithm does not consider the condition (134). Attaching the condition (134), our algorithm guarantees the following constraint condition (157) in each iteration.

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d(x, y) P_{Y|X} \times P_X(x, y) = D.$$
(157)

The algorithm by [5] has calculation complexity  $O(\frac{n_1 n_2 \log n_2}{\epsilon})$ . While our algorithm has the additional factor  $-\log \epsilon$ , this factor can be considered as the additional cost to satisfy (157).

## B. Numerical analysis for classical rate distortion without side information

To see how our algorithm works, we make numerical analysis for the case when  $n_1 = n_2 = 3$  and D = 1.5. We choose the cost function d as

$$\begin{pmatrix} d(1,1) & d(1,2) & d(1,3) \\ d(2,1) & d(2,2) & d(2,3) \\ d(3,1) & d(3,2) & d(3,3) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 1 \end{pmatrix},$$
 (158)

and the distribution  $P_X$  as

$$P_X(1) = 0.5, P_X(2) = 0.3, P_X(3) = 0.2.$$
 (159)

We set the initial marginal distribution  $P_Y^{(1)}$  to be the uniform distribution. By applying Algorithm 9, the mutual information  $I(X;Y)_{P_{XY}^{(t)}}$  converges to

$$I(X;Y)_{P_{XY}^*} := 0.100039,$$
 (160)

and the conditional distribution  $P_{Y\mid X}^{(t)}$  converges to

$$P_{Y|X}^* = \begin{pmatrix} 0.0855598 & 0.188594 & 0.430983 \\ 0.22431 & 0.494433 & 0.139579 \\ 0.69013 & 0.316974 & 0.429438 \end{pmatrix}.$$
 (161)

In particular, the marginal distribution  $P_{Y}^{\left(t\right)}$  converges to

$$P_Y^* = \begin{pmatrix} 0.185555\\ 0.288401\\ 0.526045 \end{pmatrix}. \tag{162}$$

Also, the parameter  $\bar{\tau}$  appearing in Algorithm 9 converges to 0.522814. Fig. 4 shows the behavior of the parameter  $\bar{\tau}$ . In addition, Fig. 3 shows that the error  $I(X;Y)_{P_{X,Y}^{(t)}} - I(X;Y)_{P_{X,Y}^*}$  is much smaller than the upper bound given in (137), which suggests the existence of a much better evaluation than (137).

# C. Another approach to classical rate distortion without side information

To see the exponential decay, we discuss another approach to classical rate distortion without side information. To apply Theorem 3, we need to satisfy Condition (B1+) holds. For this aim, we apply the model given in Section III-A to the case when  $\mathcal{X}$  is  $\mathcal{X} \times \mathcal{Y}$ . Then, we consider the Bregman divergence system  $(\mathbb{R}^{n_1n_2-1}, \mu, D^{\mu})$  given in Section III-A. The set of distributions  $q \times P_X$  forms an exponential family  $\mathcal{E}$  and the set of distributions  $W \times P_X$  forms a mixture family  $\mathcal{M}$ . Hence, the minimization problem (129) is a special case of the minimization (89) with the formulation given in Subsection IV-B.

Since the mixture family  $\mathcal{M}$  has  $n_1(n_2-1)-1$  parameters, and the total dimension is  $n_1n_2-1$ , Algorithm 2 is rewritten as Algorithm 11. In this case Conditions (B0) and (B1) hold in the same way as Subsection V-A.

The m-step in Algorithm 11 has optimization with  $n_1$ -variable convex function  $\log\left(\sum_{x',y'}P_{XY,(t)}(x',y')e^{\tau_{x'}+\tau_0d(x',y')}\right)$ . However, this case can satisfy Condition (B1+), which leads the exponential decay as follows. That is, the above evaluation for the convergence can be improved by using Theorem 3, i.e., the same precision (100) can be realized with  $t-2 \geq \frac{(\log\log n)-\log\epsilon}{\log\beta}$ . In fact, when an element  $\theta'$  close to  $\theta^*$  satisfies Condition (B1+), the iterated point  $\theta^{(t)}$  converges to the true value exponentially after the iterated point  $\theta^{(t)}$  is close to the true value.

In the following, we discuss a necessary condition for (B1+) with an element  $\theta'$  close to  $\theta^*$ . When two elements are close to each other, the divergence can be approximated by the Fisher information. Hence, we consider the Fisher information version of (B1+). For this aim, we consider the exponential family

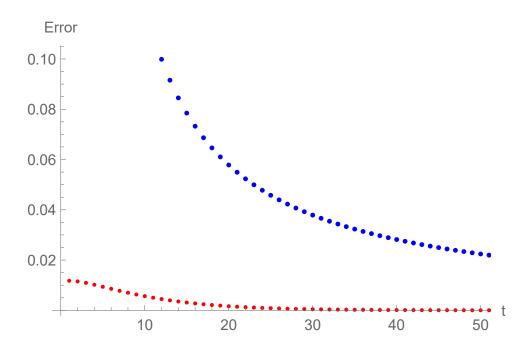


Fig. 3. Behavior of the error  $I(X;Y)_{P_{X,Y}^{(t)}} - I(X;Y)_{P_{X,Y}^*}$  of the minimum mutual information. Red points show the value of the  $I(X;Y)_{P_{X,Y}^{(t)}} - I(X;Y)_{P_{X,Y}^*}$  depending on the number of iteration t. The blue points show its upper bound given in (137).

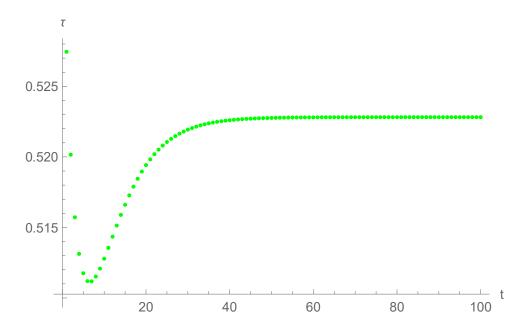


Fig. 4. The behavior of the parameter  $\tau$  depending on the number of iteration t. The green points show the parameter  $\tau$  in algorithm (9).

# Algorithm 11 em-algorithm for rate distortion

Choose the initial distribution  $P_Y^{(1)}$  on  $\mathcal{Y}$ . Then, we define the initial joint distribution  $P_{XY,(1)}$  as  $P_Y^{(1)}$  ×  $P_X$ ;

### repeat

**m-step:** Calculate  $P_{XY}^{(t+1)}$  as  $P_{XY}^{(t+1)}(x,y) := P_{XY,(t)}(x,y)e^{\bar{\tau}_x + \bar{\tau}_0 d(x,y)} \Big( \sum_{x',y'} P_{XY,(t)}(x',y')e^{\bar{\tau}_{x'} + \bar{\tau}_0 d(x',y')} \Big)^{-1}$ , where  $(\bar{\tau}_x)_{x\in\mathcal{X}}$  and  $\bar{\tau}_0$  are the unique elements  $(\tau_x)_{x\in\mathcal{X}}$  and  $\tau_0$  to satisfy

$$\frac{\partial}{\partial \tau_x} \log \left( \sum_{x', y'} P_{XY,(t)}(x', y') e^{\tau_{x'} + \tau_0 d(x', y')} \right) = P_X(x)$$
(163)

$$\frac{\partial}{\partial \tau_0} \log \left( \sum_{x',y'} P_{XY,(t)}(x',y') e^{\tau_{x'} + \tau_0 d(x',y')} \right) = D \tag{164}$$

for  $x \in X \setminus \{n_1\}$  and  $\tau_{n_1} = \bar{\tau}_{n_1}$  is fixed to 0. This choice can be written in the way as (92). **e-step:** Calculate  $P_{XY,(t+1)}$  as  $P_Y^{(t+1)} \times P_X$  where  $P_Y^{(t+1)}(y) := \sum_{x \in \mathcal{X}} P_{XY}^{(t+1)}(x,y)$ . until convergence.

 $\{P_{\theta,Y}\}$  defined in Subsection III-A with  $d=n_2-1$  by replacing  $\mathcal{X}$  by  $\mathcal{Y}$ . Let  $J_{\theta,1}$  and  $J_{\theta,2}$  be the Fisher information matrices of  $\{\Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y}\times P_X)\}$  and  $\{\Gamma_{\mathcal{E}}^{(e),\mu}\circ\Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y}\times P_X)\}$ . We choose  $\theta^*\in\mathbb{R}^{n_1n_2-1}$ corresponding to  $\Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta_0^*,Y}\times P_X)$  in the sense of the Bregman divergence system  $(\mathbb{R}^{n_1n_2-1},\mu,D^\mu)$ given in Section III-A. The local version of Condition (B1+) is written as

$$\beta J_{\theta_0^*,1} \ge J_{\theta_0^*,2}$$
 (165)

with a constant  $0 < \beta < 1$ . In this case, when the iterated point  $\theta^{(t)}$  is close to the true minimum point, the difference  $D^F(\theta^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})) - C_{\inf}(\mathcal{M},\mathcal{E})$  approaches to zero exponential rate  $\log \beta^{-1}$ . Therefore, our algorithm has such an exponential convergence at the neighborhood when the inequality

$$J_{\theta_0^*,1} > J_{\theta_0^*,2} \tag{166}$$

holds, i.e.,  $J_{\theta_0^*,1} - J_{\theta_0^*,2}$  is strictly positive-semidefinite.

Then, we have the following theorem.

Theorem 10: The matrix  $J_{\theta,1} - J_{\theta,2}$  is a strictly positive semi-definite matrix when the linear space spanned by the distributions  $\{W_{\theta,x}\}_{x\in X}$  has dimension at least  $n_2$  as a function space on  $\mathcal{Y}$ .

Therefore, when the condition for Theorem 10 holds, Algorithm 11 has such an exponential convergence at the neighborhood.

*Proof of Theorem 10:* To show Theorem 10, we define the parametric family  $\{P_{XY,\theta,\tau}\}_{\theta,\tau}$  with  $\theta=(\theta^i)_{i=1}^{n_2-1}$  and  $\tau=(\tau^i)_{i=0}^{n_1-1}$  as

$$P_{XY,\theta,\tau}(x,y) := \frac{P_{\theta,Y}(y)P_X(x)e^{\sum_{i=0}^{n_1-1}g_i(x,y)\tau_i}}{\sum_{x'y'}P_{\theta,Y}(y')P_X(x')e^{\sum_{i=0}^{n_1-1}g_i(x',y')\tau_i}},$$
(167)

where  $g_i(x,y) := \delta_{i,x}$  and  $g_0(x,y) := d(x,y)$ . We define the Fisher information matrix  $J_{\theta,\tau,3}$  of the parametric family  $\{P_{XY,\theta,\tau}\}_{\theta,\tau}$ . We define the channel  $W_{\theta}$  from  $\mathcal{X}$  to  $\mathcal{Y}$  as  $W_{\theta} \times P_X = \Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y} \times P_X)$ . Also, we choose  $\tau(\theta)$  as  $W_{\theta} \times P_X = P_{XY,\theta,\tau(\theta)}$ . In the  $n_1 + n_2 - 1$  dimensional vector space, we denote the projections to the first  $n_2 - 1$ -dimensional space corresponding to  $\theta$  and the latter  $n_1$ -dimensional space corresponding to  $\tau$  by  $P_1$  and  $P_2$ , respectively.

Then, Theorem 10 follows from the following two lemmas.

Lemma 18: The relation  $\operatorname{Ker} P_2 J_{\theta,\tau(\theta),3} P_1 = \{0\}$  holds when the linear space spanned by the distributions  $\{W_{\theta,x}\}_{x\in X}$  has dimension at least  $n_2$  as a function space on  $\mathcal{Y}$ .

Lemma 19: The matrix  $J_{\theta,1}-J_{\theta,2}$  is a strictly positive semi-definite matrix if and only if  $\operatorname{Ker} P_2 J_{\theta,\tau(\theta),3} P_1 = \{0\}$ .

Lemmas 18 and 19 is shown in Appendix I.

Remark 2: Now, we can explain why we cannot show Condition (B1+) for Algorithm 9 in the above method. If we apply the same discussion to Algorithm 9 the projection  $P_2$  is the projection to the one-dimensional space. Hence, the condition  $\operatorname{Ker} P_2 J_{\theta,\tau(\theta),3} P_1 = \{0\}$  does not hold unless  $n_2 = 2$ .

# D. Classical rate distortion with multiple distortion constraint without side information

Recently, the paper [13, Theorem 1] considers a rate-distortion problem motivated by the consideration of semantic information. That is, it considers two sets  $\hat{\mathcal{X}}$  and  $\hat{\mathcal{S}}$  in addition to the set  $\mathcal{X}$ , and focus on two distortion measures  $d_s(x,\hat{s})$  and  $d_a(x,\hat{x})$  for  $x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}$  and  $\hat{s} \in \hat{\mathcal{S}}$ . Then, we define the following set for channels  $W: \mathcal{X} \to \hat{\mathcal{X}} \times \hat{\mathcal{S}}$  as

$$\mathcal{P}_{\hat{\mathcal{X}} \times \hat{\mathcal{S}} \mid \mathcal{X}}^{d_{\mathbf{a}}, d_{\mathbf{s}}, P_{X}, D_{\mathbf{a}}, D_{\mathbf{s}}, \leq}$$

$$:= \left\{ W \middle| \sum_{x \in \mathcal{X}, \hat{s} \in \hat{\mathcal{S}}, \hat{x} \in \hat{\mathcal{X}}} d_{i}(x, \hat{x}) W \times P_{X}(x, \hat{x}, \hat{s}) \leq D_{i} \text{ for } i = \mathbf{a}, \mathbf{s} \right\}.$$

$$(168)$$

The paper [13, Theorem 1] addresses the following minimization problem;

$$\min_{\substack{W \in \mathcal{P}^{d_{\mathbf{a}}, d_{\mathbf{s}}, P_X, D_{\mathbf{a}}, D_{\mathbf{s}}, \leq \\ \hat{X} \times \hat{S} \mid \mathcal{X}}} I(X; \hat{X}, \hat{S})_{W \times P_X}. \tag{169}$$

For its generalization, we consider a set  $\mathcal{Y}$  and m distortion measures  $d_i(x,y)$  for  $x \in \mathcal{X}, y \in \mathcal{Y}$  and i = 1, ..., m. We define the following set for channels  $W : \mathcal{X} \to \mathcal{Y}$  as

$$\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{(d_i)_{i=1}^m, P_X, (D_i)_{i=1}^m, \leq}$$

$$:= \left\{ W \middle| \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} d_i(x, y) W \times P_X(x, y) \leq D_i \text{ for } i = 1, \dots, m \right\}.$$

$$(170)$$

Then, the following minimization problem can be regarded a generalization of (169) by considering the case with  $Y = (\hat{X}, \hat{S})$  and m = 2;

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{(d_i)_{i=1}^m, P_X, (D_i)_{i=1}^m, \leq}} I(X; Y)_{W \times P_X}$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{(d_i)_{i=1}^m, P_X, (D_i)_{i=1}^m, \leq}} \min_{q \in \mathcal{P}_Y} D(W \times P_X || q \times P_X). \tag{171}$$

The minimization problem (171) can be considered as rate distortion with multiple distortion functions. Now, we focus on the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)}, \bar{\mu}, D^{\bar{\mu}})$  defined in Subsection III-B, which coincides with the set of distributions  $W \times P_X$ . The set of distributions  $q \times P_X$  forms an exponential subfamily  $\mathcal{E}$ , and the subset  $\mathcal{P}_{\mathcal{Y}|\mathcal{X}}^{(d_i)_{i=1}^m, P_X, (D_i)_{i=1}^m, \leq} \times P_X$  forms a closed convex mixture subfamily  $\mathcal{M}$ . Then, the minimization problem (171) is a special case of the minimization (89) with the formulation given in Subsection IV-B

Since Lemma 11 guarantees Condition (B0) for this problem, Algorithm 6 works for the minimization problem (171) and is rewritten as Algorithm 12. Condition (B1) can be checked in the same way as (135). In addition, similar to Algorithm 4 in Subsection V-A, when we cannot solve the equations (172), Algorithm 8 works in this model.

# Algorithm 12 em-algorithm for rate distortion with multiple distortion functions

Choose the initial distribution  $P_Y^{(1)}$  on  $\mathcal{Y}$ . Then, we define the initial joint distribution  $P_{XY,(1)}$  as  $P_Y^{(1)} \times P_X$ ;

### repeat

**m-step:** For any subset  $A \subset \{1,\ldots,m\}$ , Calculate  $P_{XY}^{(t+1),A}$  as  $P_{XY}^{(t+1),A}(x,y) := P_X(x)P_{Y|X,(t)}(y|x)e^{\sum_{i\in A}\bar{\tau}_{A,i}d_i(x,y)}\Big(\sum_{y'}P_{Y|X,(t)}(y'|x)e^{\sum_{i\in A}\bar{\tau}_{A,i}d_i(x,y')}\Big)^{-1}$ , where  $(\bar{\tau}_{A,i})_{i\in A}$  are the unique elements  $(\tau_{A,i})_{i\in A}$  to satisfy

$$\frac{\partial}{\partial \tau_{A,i}} \sum_{x} P_X(x) \log \left( \sum_{y'} P_{Y|X,(t)}(y'|x) e^{\tau_{x'} + \sum_{i' \in A} \tau_{A,i'} d_{i'}(x,y')} \right) = D_i$$

$$(172)$$

for  $i \in A$ . Choose  $P_{XY}^{(t+1)}$  to be  $P_{XY}^{(t+1),A_0}$ , where

$$A_0 := \underset{A \subset \{1, \dots, m\}}{\operatorname{argmin}} \left\{ D(P_{XY}^{(t+1), A} || P_{XY, (t+1)}) \middle| \begin{array}{c} \sum_{x, y} P_{XY}^{(t+1), A}(x, y) d_i(x, y) \leq D_i \\ \text{for } i = 1, \dots, m \end{array} \right\}.$$
 (173)

**e-step:** Calculate  $P_{XY,(t+1)}$  as  $P_Y^{(t+1)} \times P_X$  where  $P_Y^{(t+1)}(y) := \sum_{x \in \mathcal{X}} P_{XY}^{(t+1)}(x,y)$ . **until** convergence.

### E. Classical rate distortion with side information

Next, we consider the rate distortion problem when the side information state  $S \in \mathcal{S} = \{1, \dots, n_3\}$  is available to both the encoder and the decoder [12]. Hence, our channel W is given as a map  $\mathcal{X} \times \mathcal{S} \to \mathcal{P}_{\mathcal{Y}}$ . Given a distortion measure d(x,y) on  $\mathcal{X} \times \mathcal{Y}$  and a distribution  $P_{XS}$  on  $\mathcal{X} \times \mathcal{S}$ , we define the following sets;

$$\mathcal{P}_{\mathcal{Y}|\mathcal{X}\times\mathcal{S}}^{d,P,\chi_SD} := \left\{ W \middle| \sum_{x \in \mathcal{X}, s \in \mathcal{S}, y \in \mathcal{Y}} d(x,y)W \times P_{XS}(x,s,y) = D \right\}$$
(174)

$$\mathcal{P}_{\mathcal{Y}|\mathcal{X}\times\mathcal{S}}^{d,P_{XS},D,\leq} := \left\{ W \middle| \sum_{x\in\mathcal{X},s\in\mathcal{S},y\in\mathcal{Y}} d(x,y)W \times P_{XS}(x,s,y) \leq D \right\}.$$
 (175)

We define the set  $\mathcal{P}_{X-S-Y}$  of distributions on  $\mathcal{X} \times \mathcal{S} \times \mathcal{Y}$  to satisfy the Markov chain X-S-Y with the marginal distribution  $P_{XS}$ . The rate distortion function is given as

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d,P,D,\leq}} I(Y;X|S)_{W \times P_{XS}}$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d,P,D,\leq}} \sum_{s \in \mathcal{S}} P_{S}(s) D(W \times P_{X|S=s} || (W \cdot P_{X|S=s}) \times P_{X|S=s})$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d,P,D,\leq}} \min_{Q \in \mathcal{P}_{X-S-Y}} D(W \times P_{XS} || Q), \tag{176}$$

where  $P_{X|S=s}$  is the conditional distribution on X with the condition S=s of  $P_{XS}$ .  $P_S$  is the marginal distribution on S of  $P_{XS}$ . Now, we apply the discussion in Subsection III-B to the joint system  $(\mathcal{X} \times \mathcal{S}) \times \mathcal{Y}$ . Then, we consider the Bregman divergence system  $(\mathbb{R}^{n_1 n_3 (n_2 - 1)}, \bar{\mu}, D^{\bar{\mu}})$ , which coincides with the set of distributions  $W \times P_{XS}$ . The set  $\mathcal{P}_{X-S-Y}$  forms an exponential subfamily  $\mathcal{E}$ , and the subset  $\mathcal{P}_{\mathcal{Y}|X\times\mathcal{S}}^{d,P,\chi_S D} \times P_{XS}$  forms a mixture subfamily  $\mathcal{M}$ . Similar to (131), there exists a distribution  $P_{XSY} \in \mathcal{E}$  such that

$$\sum_{x,y,s} P_{XYS}(x,y,s)d(x,y) \le D \tag{177}$$

if and only if

$$\sum_{s} P_S(s) \min_{y} d_{YS}(y, s) \le D \tag{178}$$

where  $d_{YS}(y,s) := \sum_{x} P_{X|S=s} d(x,y)$ . Therefore, in the same way as Lemma 16, we can show the following lemma.

Lemma 20: When (178) holds,  $\min_{W \in \mathcal{P}_{\mathcal{Y}|X \times S}^{d,P_{XS},D,\leq}} I(X;Y|S)_{W \times P_{XS}} = 0$ . Otherwise,

$$\min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d, P_{XS}, D, \leq}} I(X; Y|S)_{W \times P_{XS}} = \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d, P_{XS}, D}} I(X; Y|S)_{W \times P_{XS}}$$

$$= \min_{W \in \mathcal{P}_{\mathcal{Y}|\mathcal{X} \times \mathcal{S}}^{d, P, D}} \min_{Q \in \mathcal{P}_{X - S - Y}} D(W \times P_{XS} || Q). \tag{179}$$

Due to Lemma 20, when (178) does not hold, it is sufficient to address the minimization (179). In the following, we discuss the minimization problem (179), which is a special case of the minimization (89) with the formulation given in Subsection IV-A. The mixture family  $\mathcal{M}$  has  $n_1n_3(n_2-1)-1$  parameters. Since the total dimension is  $n_1n_3(n_2-1)$ , we employ Algorithm 2 instead of Algorithm 1. Since Lemma 13 guarantees Condition (B0) for this problem, Algorithm 2 works and is rewritten as Algorithm 13.

### **Algorithm 13** em-algorithm for rate distortion with side information

Choose the initial conditional distribution  $P_{Y|S}^{(1)}$  on  $\mathcal{Y}$  with the condition on  $\mathcal{S}$ . Then, we define the initial joint distribution  $P_{XYS,(1)}$  as  $P_{Y|S}^{(1)} \times P_{XS}$ ;

repeat

**m-step:** Calculate  $P_{XYS}^{(t+1)}$  as  $P_{XYS}^{(t+1)}(x,y,s) := P_{XS}(x,s)P_{Y|S,(t)}(y|s)e^{\bar{\tau}d(x,y)}\left(\sum_{y'}P_{Y|S,(t)}(y'|s)e^{\bar{\tau}d(x,y')}\right)^{-1}$ , where  $\bar{\tau}$  is the unique element  $\tau$  to satisfy

$$\frac{\partial}{\partial \tau} \sum_{x,s} P_{XS}(x,s) \log \left( \sum_{y'} P_{Y|S,(t)}(y'|s) e^{\tau d(x,y')} \right) = D.$$
 (180)

**e-step:** Calculate 
$$P_{XYS,(t+1)}$$
 as  $P_{Y|S}^{(t+1)}$   $\times$   $P_{XS}$  where  $P_{Y|S}^{(t+1)}(y|s)$  :=  $\sum_{x \in \mathcal{X}} P_{XYS}^{(t+1)}(x,y,s) / \sum_{x' \in \mathcal{X}, y' \in \mathcal{Y}} P_{XYS}^{(t+1)}(x',y',s)$ . **until** convergence.

To check condition (B1), we set  $\theta$  and  $\theta'$  be elements of  $\mathbb{R}^{n_1n_3(n_2-1)}$  corresponding to  $W\times P_{XS}$  and  $W'\times P_{XS}$ . We define the distribution  $Q_W$  on  $\mathcal{X}\times\mathcal{S}\times\mathcal{Y}$  as  $Q_W(x,s,y)=\sum_{x'}W(y|x',s)P_{X|S=s}(x')P_S(s)$ . In the same way, we define  $Q_{W'}$  on  $\mathcal{X}\times\mathcal{S}\times\mathcal{Y}$  by replacing W by W'. Then, the relations

$$D^{\bar{\mu}}(\Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(\theta') \| \Gamma_{\mathcal{E}}^{(e),\bar{\mu}}(\theta)) = D(Q_{W'} \| Q_{W})$$

$$= \sum_{s \in \mathcal{S}} P_{S}(s) D((W'_{Y|X,S=s} \cdot P_{X|S=s}) \times P_{X|S=s} \| (W_{Y|X,S=s} \cdot P_{X|S=s}) \times P_{X|S=s})$$

$$= \sum_{s \in \mathcal{S}} P_{S}(s) D(W'_{Y|X,S=s} \cdot P_{X|S=s} \| W_{Y|X,S=s} \cdot P_{X|S=s})$$

$$\leq \sum_{s \in \mathcal{S}} P_{S}(s) D(W'_{Y|X,S=s} \times P_{X|S=s} \| W_{Y|X,S=s} \times P_{X|S=s})$$

$$= D(W' \times P_{XS} \| W \times P_{XS}) = D^{\bar{\mu}}(\theta' \| \theta)$$
(181)

guarantee Condition (B1). When the initial value  $\theta_{(1)}$  is chosen as the case that W has full support,  $\sup_{\theta \in \mathcal{E}} D^{\bar{\mu}}(\theta \| \theta_{(1)})$  has a finite value. Hence, Theorem 2 guarantees the convergence to the global minimum

as follows. When we choose the initial value  $\theta_{(1)}$  in the same way as the above case, the precision (97) holds with  $t \ge \frac{\log n_2}{\epsilon} + 1$ . In addition, in the same way as Subsection IV-A, we can apply Algorithm 4.

Next, we consider the case when we cannot exactly calculate the unique element  $\bar{\tau}$  to satisfy (180). Alternatively, we need to use Algorithm 4, which can be rewritten in the same way as Algorithm 10. That is, it is sufficient to replace X by XS and define  $\hat{F}^{(t)}(\tau)$  by  $\sum_{x,s} P_{XS}(x,s) \log \left( \sum_{y'} P_{Y|S,(t)}(y'|s) e^{\tau(D-d(x,y')))} \right)$ in Algorithm 10. When we fix the precision level  $\epsilon > 0$  and choose  $\epsilon_1 := \frac{\epsilon}{3}$ , this algorithm achieves the precision condition (116) with  $\frac{2 \log n_2}{2} + 1$  rounds due to (136). The calculation complexity can be evaluated in the same way as Algorithm 10.

### VI. QUANTUM ENTANGLEMENT-ASSISTED RATE DISTORTION

Consider two quantum systems  $\mathcal{H}_A$  and  $\mathcal{H}_B$  with dimension  $d_A$  and  $d_B$ . Let  $\mathcal{H}_R$  be the reference system of  $\mathcal{H}_A$  with the dimension  $d_A$ . We focus on a density matrix  $\rho$  on  $\mathcal{H}_A$  and a Hermitian matrix  $\Delta$  on  $\mathcal{H}_R \otimes \mathcal{H}_B$ , which expresses our distortion measure. Using a purification  $\Psi$  of  $\rho$  on  $\mathcal{H}_A \otimes \mathcal{H}_R$ , we define the following sets of TP-CP maps with the input system  $\mathcal{H}_A$  and the output system  $\mathcal{H}_B$ .

$$\mathcal{P}_{A\to B}^{\Delta,\rho,D} := \left\{ \mathcal{N} \middle| \operatorname{Tr} \Delta(id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|) = D \right\}$$
(182)

$$\mathcal{P}_{A\to B}^{\Delta,\rho,D,\leq} := \Big\{ \mathcal{N} \Big| \operatorname{Tr} \Delta(id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|) \leq D \Big\}.$$
 (183)

The entanglement-assisted rate distortion function is given as [14, Theorem 2]

$$\min_{\mathcal{N} \in \mathcal{P}_{A \to B}^{\Delta, \rho, D, \leq}} D((id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|) \| (id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|)_R \otimes (id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|)_B)$$

$$= \min_{\mathcal{N} \in \mathcal{P}_{A \to B}^{\Delta, \rho, D, \leq}} \min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D((id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|) \| \rho_R \otimes \sigma_B). \tag{184}$$

where  $\rho_R := \operatorname{Tr}_A |\Psi\rangle\langle\Psi|$ . Essentially, the above minimization handles the state  $(id_R \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|)$ . Hence, we introduce the following sets of states on  $\mathcal{H}_R \otimes \mathcal{H}_B$ ;

$$S_{RB}^{\Delta,\Psi,D} := \left\{ \bar{\rho}_{RB} \middle| \operatorname{Tr} \Delta \bar{\rho}_{RB} = D, \quad \bar{\rho}_{R} = \rho_{R} \right\}$$
(185)

$$S_{RB}^{\Delta,\Psi,D,\leq} := \left\{ \bar{\rho}_{RB} \middle| \operatorname{Tr} \Delta \bar{\rho}_{RB} \leq D, \quad \bar{\rho}_{R} = \rho_{R} \right\}. \tag{186}$$

The minimization (184) is rewritten as

$$\min_{\bar{\rho}_{RB} \in \mathcal{S}_{RB}^{\Delta,\Psi,D,\leq}} \min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D(\bar{\rho}_{RB} \| \rho_R \otimes \sigma_B). \tag{187}$$

Now, we apply the discussion in Section III-C to the case when  $\mathcal{H}$  is  $\mathcal{H}_R \otimes \mathcal{H}_B$ . Then, we consider the Bregman divergence system  $(\mathbb{R}^{d_A^2 d_B^2 - 1}, \mu, D^{\mu})$ . The set of states  $\rho_R \otimes \sigma_B$  forms an exponential family  $\mathcal{E}$ , and the set  $\mathcal{S}_{RB}^{\Delta,\Psi,D}$  forms a mixture family  $\mathcal{M}$ .

Similar to (131), there exists a state  $\sigma_B$  such that

$$\operatorname{Tr} \Delta \rho_R \otimes \sigma_B \le D \tag{188}$$

if and only if

$$\lambda_{\min}(\Delta_B) \le D \tag{189}$$

where  $\Delta_B := \operatorname{Tr}_R \Delta \rho_R \otimes I_B$  and  $\lambda_{\min}(\Delta_B)$  expresses the minimum eigenvalue of  $\Delta_B$ . Therefore, in the same way as Lemma 16, we can show the following lemma.

Lemma 21: When (189) holds, the minimum (187) equals zero. Otherwise,

$$\min_{\bar{\rho}_{RB} \in \mathcal{S}_{RB}^{\Delta,\Psi,D,\leq}} \min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D(\bar{\rho}_{RB} \| \rho_R \otimes \sigma_B)$$
(190)

$$\min_{\bar{\rho}_{RB} \in \mathcal{S}_{RB}^{\Delta,\Psi,D,\leq}} \min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D(\bar{\rho}_{RB} \| \rho_R \otimes \sigma_B) 
= \min_{\bar{\rho}_{RB} \in \mathcal{S}_{RB}^{\Delta,\Psi,D}} \min_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} D(\bar{\rho}_{RB} \| \rho_R \otimes \sigma_B). \tag{190}$$

Due to Lemma 21, when (189) does not hold, it is sufficient to address the minimization (191). In the following, we discuss the minimization problem (191). To address it as a special case of the minimization (89) with the formulation given in Subsection IV-A, we choose  $d_B^2-1$  linearly independent Hermitian matrices  $X_{1,R},\ldots,X_{d_B^2-1}$  on  $\mathcal{H}_R$ , and set  $\mathcal{H}$  to be  $\mathcal{H}_R\otimes\mathcal{H}_B$ . Then, we consider the Bregman divergence system  $(\mathbb{R}^{n_1(n_2-1)},\mu,D^\mu)$  defined in Subsection III-C, where  $X_{d_R^2(d_B^2-1)}=\Delta$  and  $X_{d_R^2(d_B^2-1)+1},\ldots,X_{d_A^2d_B^2-1}$  are  $\theta^iX_{1,R}\otimes I_B,\ldots,\theta^iX_{d_R^2-1,R}\otimes I_B$ . Then,  $\mathcal{S}_{RB}^{\Delta,\Psi,D}$  is given as

$$\mathcal{M} := \{ \theta \in \mathbb{R}^{d_R^2 d_B^2 - 1} | \text{Conditions (193) and (194) hold.} \}, \tag{192}$$

where

$$Tr \,\rho_{\theta} X_{d_{\mathbf{p}}^2(d_{\mathbf{p}}^2 - 1)} = D, \tag{193}$$

$$\operatorname{Tr} \rho_{\theta} X_{d_R^2(d_B^2 - 1) + j} = \operatorname{Tr} \rho_R X_{d_R^2(d_B^2 - 1) + j}, \tag{194}$$

for  $j = 1, ..., d_B^2 - 1$ . Also, we choose the set  $\mathcal{E}$  as

$$\mathcal{E} := \{ \theta \in \mathbb{R}^{d_R^2 d_B^2 - 1} | \rho_\theta = \rho_R \otimes \sigma_B \}. \tag{195}$$

The mixture family  $\mathcal{M}$  has  $d_A^2(d_B^2-1)$  parameters. Since the total dimension is  $d_A^2d_B^2-1$ , we employ Algorithm 2 instead of Algorithm 1. Since Lemma 14 guarantees Condition (B0) for this problem, Algorithm 2 works and is rewritten as Algorithm 14.

Since

$$D(\rho_{\theta} \| \rho_{R} \otimes \sigma_{B}) = D(\rho_{\theta} \| \rho_{R} \otimes \operatorname{Tr}_{R} \rho_{\theta}) + D(\operatorname{Tr}_{R} \rho_{\theta} \| \sigma_{B})$$

$$= D(\rho_{\theta} \| \rho_{R} \otimes \operatorname{Tr}_{R} \rho_{\theta}) + D(\rho_{R} \otimes \operatorname{Tr}_{R} \rho_{\theta} \| \rho_{R} \otimes \sigma_{B}), \tag{196}$$

we find that

$$\rho_{\Gamma_{\mathcal{E}}^{(e),\mu}(\theta)} = \rho_R \otimes \operatorname{Tr}_R \rho_{\theta}. \tag{197}$$

Therefore, we have

$$D^{\mu}(\Gamma_{\mathcal{E}}^{(e),\mu}(\theta') \| \Gamma_{\mathcal{E}}^{(e),\mu}(\theta)) = D(\rho_R \otimes \operatorname{Tr}_R \rho_{\theta'} \| \rho_R \otimes \operatorname{Tr}_R \rho_{\theta}) = D(\operatorname{Tr}_R \rho_{\theta'} \| \operatorname{Tr}_R \rho_{\theta})$$

$$\leq D(\rho_{\theta'} \| \rho_{\theta}) = D^{\mu}(\theta' \| \theta), \tag{198}$$

which guarantees Condition (B1). Hence, Theorem 6 guarantees the convergence to the global minimum. Since Conditions (B0) and (B1) hold, Theorem 5 guarantees that Algorithm 4 works when m-step has an error. Since m-step of this case has  $d_R^2$  parameters, it requires more calculation amount as a convex optimization than Algorithms 10 and 13. However, it still has small smaller calculation amount of the case when the original problem (187) is treated as a convex optimization because (187) has  $d_R^2(d_B^2-1)$  variables.

### VII. CONCLUSION

We have formulated em algorithm in the general framework of Bregman divergence, and have shown the convergence to the true value and the convergence speed under conditions that match informationtheoretical problem settings. Then, we have applied them to the rate distortion problem and its variants including the quantum settings.

Our em algorithm in the general framework contains two types of minimization processes in e- and m- steps. Due to the above property of our em algorithm, our em algorithm has merit only when the optimizations in the e- and m- step are written in a form without optimization, or are converted to simpler optimizations with a smaller number of parameters than the original minimization problem. Fortunately, rate distortion problem and its variants satisfy this condition. In particular, classical rate distortion problem with and without side information need only a one-parameter convex optimization in each iteration.

### Algorithm 14 em-algorithm for Quantum entanglement-assisted rate distortion

Choose the state  $\rho_B^{(1)}$ , and set  $\rho_{RB,(1)}$  to be  $\rho_R \otimes \rho_B^{(1)}$ .

repeat

**m-step:** Calculate  $\rho_{RB}^{(t+1)}$  as  $\rho_{RB}^{(t+1)} := \exp(\log \rho_{RB,(t)} + \sum_i \theta^i X_i \otimes I_B + \theta^0 \Delta) / \operatorname{Tr} \exp(\log \rho_{RB,(t)} + \sum_i \theta^i X_{i,R} \otimes I_B + \theta^0 \Delta)$ , where  $(\theta^i)$  are the unique elements to satisfy

$$\frac{\partial}{\partial \theta^i} \log \operatorname{Tr} \exp(\log \rho_{RB,(t)} + \sum_i \theta^i X_i \otimes I_B + \theta^0 \Delta) = \operatorname{Tr} X_i \rho_R$$
(199)

$$\frac{\partial}{\partial \theta^0} \log \operatorname{Tr} \exp(\log \rho_{RB,(t)} + \sum_i \theta^i X_i \otimes I_B + \theta^0 \Delta) = D$$
 (200)

for  $i = 1, \dots, d_R^2 - 1$ .

**e-step:** Calculate  $\rho_{RB,(t+1)}$  as  $\rho_R \otimes \rho_B^{(t+1)}$ , where  $\rho_B^{(t+1)} := \operatorname{Tr}_R \rho_{RB}^{(t+1)}$ . **until** convergence.

To remove the constraint (157), existing papers for the rate distortion problem and its variants changed the objective function by using a Lagrange multiplier, and no preceding paper showed how to choose the Lagrange multiplier [5], [9], [10], [11]. Indeed, the number of studies for this topic is limited while more papers studied channel capacities [5], [6], [10], [11], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]. Since the set of conditional distributions with the linear constraint (157) forms a mixture family, our method can be directly applied to the original objective function with the linear constraint (157). To handle the linear constraint, each iteration has a convex optimization only with one variable in m-step. Due to this convex optimization, our algorithm has a larger calculation complexity than the algorithm by [5]. However, this difference is not so large, and can be considered as the additional cost to exactly solve the original minimization (129) instead of the modified minimization (156).

Further, since our result is written in a form of Bregman divergence, we can expect large applicability. That is, our results have the advantage with respect to its generality over existing methods. To emphasize our advantage, we need to apply our method to other problems because the problems discussed in this paper are limited. Hence, it is an interesting open problem to apply our em algorithm to other optimization problems. For example, it can be expected to extend our result to the case with memory [10], [27], [28] because various information quantities in the Markovian setting can be written in a form of Bregman divergence [29], [30], [31], [32], [33], [34]. As another future problem, it is interesting to extend our method to the optimization of the exponential decreasing rate in various settings, which requires the optimization of Rényi mutual information by using Rényi version of Pythagorean theorem [35, Lemma 3 in Suppl. Mat.][36, Lemma 2.11].

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## APPENDIX A REVIEW OF CONVEX OPTIMIZATION

In this appendix, we review several existing algorithms for the minimization of a differentiable convex function F defined on a closed convex set C. In the following, we use the notation  $x_* := \operatorname{argmin} F(x)$ .

#### A. Bisection method

First we consider the bisection method, which works with one-variable differentiable convex function F defined on an interval [a, b] [16].

### **Algorithm 15** Bisection method

Set  $a_0 := a$  and  $b_0 := b$ ;

#### repeat

k+1**th-step:** Set  $x_k:=\frac{a_k+b_k}{2}$ . If  $\frac{d}{dx}F(x_k)>0$ , we set  $a_{k+1}:=a_k$  and  $b_{k+1}:=x_k$ . Otherwise, we set  $a_{k+1}:=x_k$  and  $b_{k+1}:=b_k$ . This construction guarantees the conditions  $\frac{d}{dx}F(a_{k+1})\leq 0$ .  $\frac{d}{dx}F(b_{k+1})\geq 0$ .

until convergence.

To see the precision, we define the parameter  $V_0 := \max_{x,y \in [a,b]} |F(x) - F(y)|$ . When we use the bisection method, i.e., Algorithm 15, we have

$$F(x_k) - F(x_*) \le \frac{V_0}{2^k} \tag{201}$$

$$F(a_k) - F(x_*), F(b_k) - F(x_*) \le \frac{V_0}{2^{k-1}}$$
 (202)

$$|x_k - x_*| \le \frac{b - a}{2^{k+1}} \tag{203}$$

$$x_* - a_k, b_k - x_* \le \frac{b - a}{2^k}. (204)$$

That is, to guarantee  $|F(x_k) - F(x_*)| \le \epsilon$ , the number of iteration k needs to satisfy  $k \ge \log_2 \frac{V_0}{\epsilon}$ .

### B. Gradient method

Next, we consider the gradient method, which works for a differentiable d-variable convex with the uniform Lipschitz condition. We consider a differentiable d-variable convex function F defined on a convex set  $C \subset \mathbb{R}^d$ , and assume the uniform Lipschitz condition with a constant L;

$$\|\nabla F(x) - \nabla F(y)\| \le L\|x - y\| \tag{205}$$

for  $x, y \in C$ .

### Algorithm 16 Gradient method

Set an initial value  $x_0 \in C$ ;

#### repeat

k+1**th-step:** Set  $x_{k+1}$  as

$$x_{k+1} := x_k - \frac{1}{L} \nabla F(x_k). \tag{206}$$

until convergence.

When we use the gradient method, i.e., Algorithm 16, we have [37, Chapter 10] [38], [39]

$$|F(x_k) - F(x_*)| \le \frac{L}{2k} ||x_* - x_0||^2.$$
(207)

That is, to guarantee  $|F(x_k) - F(x_*)| \le \epsilon$ , the number of iteration k needs to satisfy  $k \ge \frac{L||x_* - x_0||^2}{2\epsilon}$ . When we employ accelerated proximal gradient methods, the evaluation (207) is improved as [38], [40], [39], [41], [42], [43]

$$|F(x_k) - F(x_*)| \le \frac{L}{2(k+2)^2} ||x_* - x_0||^2.$$
 (208)

## APPENDIX B PROOF OF THEOREM 1

In this proof, we simplify  $\gamma(\hat{\Theta}|\Theta)$  to  $\gamma$ . We consider the mixture subfamily  $\mathcal{M}:=\{\theta\in\Theta|\exists\lambda\in\mathbb{R},\eta(\theta)=(1-\lambda)\eta(\theta_1)+\lambda\eta(\theta_2)\}$ . Due to Condition (M4), we can define the m-projection  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_3)\in\mathcal{M}$ . We choose  $\lambda$  such that  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_3)=(1-\lambda)\eta(\theta_1)+\lambda\eta(\theta_2)$  We consider three cases; (i)  $\lambda<0$ . (ii)  $0\leq\lambda\leq1$ . (iii)  $1<\lambda$ .

Case (i); Since the subset  $\hat{\Theta} \subset \Theta$  is a star subset for  $\theta_1 \in \hat{\Theta}$ , and  $\theta_2 \in \hat{\Theta}$ , we have  $\theta(s) \in \hat{\Theta}$  for  $s \in [0, 1]$ . Hence, we have the matrix inequality

$$J(\theta(s))^{-1} \le \gamma J(\theta(1-s))^{-1}. \tag{209}$$

Thus, we have

$$D^{F}(\theta_{1}\|\theta_{2}) \stackrel{(a)}{=} \int_{0}^{1} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} s ds$$

$$\stackrel{(b)}{\leq} \gamma \int_{0}^{1} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(1-s))^{-1})^{i,j} s ds$$

$$\stackrel{(c)}{=} \gamma D^{F}(\theta_{2}\|\theta_{1})$$
(210)

where (a), (b), and (c) follow from (33), (209), and (33), respectively. Also, we have

$$D^{F}(\theta_{2}\|\theta_{1}) \leq D^{F}(\theta_{2}\|\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3}))$$
  
$$\leq D^{F}(\theta_{2}\|\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) + D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})\|\theta_{3}) = D^{F}(\theta_{2}\|\theta_{3}).$$
(211)

The combination of (210) and (211) yields (47).

Case (iii); We have

$$D^{F}(\theta_{1} \| \theta_{2}) \leq D^{F}(\theta_{1} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3}))$$

$$\leq D^{F}(\theta_{1} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) + D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3}) \| \theta_{3}) = D^{F}(\theta_{1} \| \theta_{3}). \tag{212}$$

Case (ii); We use the quantity  $M := \Big( \max_{s \in [0,1]} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_2) - \eta(\theta_1))_i (\eta(\theta_2) - \eta(\theta_1))_j (J(\theta(s))^{-1})^{i,j} \Big)$ . Then, we have

$$D^{F}(\theta_{1} \| \theta_{3})$$

$$=D^{F}(\theta_{1} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) + D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3}) \| \theta_{3})$$

$$\geq D^{F}(\theta_{1} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) = D^{F}(\theta_{1} \| \theta(\lambda))$$

$$= \int_{0}^{\lambda} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} s ds$$

$$\geq \left( \int_{0}^{\lambda} s ds \right) \left( \min_{s \in [0,1]} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} \right)$$

$$\geq \frac{\lambda^{2}}{2\gamma} M, \tag{213}$$

and

$$D^{F}(\theta_{2} \| \theta_{3})$$

$$=D^{F}(\theta_{2} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) + D^{F}(\Gamma_{\mathcal{M}}^{(m),F}(\theta_{3}) \| \theta_{3})$$

$$\geq D^{F}(\theta_{2} \| \Gamma_{\mathcal{M}}^{(m),F}(\theta_{3})) = D^{F}(\theta_{2} \| \theta(\lambda))$$

$$= (1 - \lambda)^{2} \int_{0}^{1} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(1 - s(1 - \lambda)))^{-1})^{i,j} s ds$$

$$\geq \frac{(1 - \lambda)^{2}}{2} \Big( \min_{s \in [0,1]} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} \Big)$$

$$\geq \frac{(1 - \lambda)^{2}}{2\gamma} M.$$
(214)

That is, we obtain

$$\lambda \le \sqrt{\frac{2\gamma D^F(\theta_1 \| \theta_3)}{M}}, \quad 1 - \lambda \le \sqrt{\frac{2\gamma D^F(\theta_2 \| \theta_3)}{M}}. \tag{215}$$

Therefore, we have

$$D^{F}(\theta_{1}\|\theta_{2}) \stackrel{(a)}{=} \int_{0}^{1} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} s ds$$

$$\stackrel{(a)}{=} \int_{0}^{\lambda} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} s ds$$

$$+ \int_{\lambda}^{1} \sum_{i=1}^{d} \sum_{j=1}^{d} (\eta(\theta_{2}) - \eta(\theta_{1}))_{i} (\eta(\theta_{2}) - \eta(\theta_{1}))_{j} (J(\theta(s))^{-1})^{i,j} s ds$$

$$\stackrel{(b)}{\leq} D^{F}(\theta_{1}\|\theta(\lambda)) + \left(\int_{\lambda}^{1} s ds\right) M$$

$$\stackrel{(c)}{\leq} D^{F}(\theta_{1}\|\theta_{3}) + \frac{1 - \lambda^{2}}{2} M$$

$$\stackrel{(c)}{=} D^{F}(\theta_{1}\|\theta_{3}) + ((1 - \lambda)^{2} + \lambda(1 - \lambda)) \frac{M}{2}$$

$$\stackrel{(c)}{\leq} D^{F}(\theta_{1}\|\theta_{3}) + \frac{2\gamma}{M} \left(D^{F}(\theta_{2}\|\theta_{3}) + \sqrt{D^{F}(\theta_{1}\|\theta_{3})D^{F}(\theta_{2}\|\theta_{3})}\right) \frac{M}{2}, \tag{216}$$

which implies (47).

### APPENDIX C PROOF OF THEOREM 2

Remember that  $\theta_{(t)}$  is  $\Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t)})$ , which implies that  $\Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)}) = \theta^{(t+1)}$ . For any  $\epsilon_1 > 0$ , we choose an element  $\theta(\epsilon_1)$  of  $\mathcal{M}$  such that  $D^F(\theta(\epsilon_1) \| \Gamma_{\mathcal{E}}^{(e),F}(\theta(\epsilon_1))) \leq C_{\inf}(\mathcal{M},\mathcal{E}) + \epsilon_1$ . Also, let  $\theta(\epsilon_1)_*$  be  $\Gamma_{\mathcal{E}}^{(e),F}(\theta(\epsilon_1))$ . As explained in Fig. 5, Phythagorean theorem (Proposition 1) guarantees that the divergence  $D^F(\theta(\epsilon_1) \| \theta_{(t)})$ 

can be written in the following two ways (as two equations (a) and (b));

$$D^{F}(\theta(\epsilon_{1})\|\theta^{(t+1)}) + D^{F}(\theta^{(t+1)}\|\theta_{(t)}) \stackrel{(a)}{=} D^{F}(\theta(\epsilon_{1})\|\theta_{(t)})$$

$$\stackrel{(b)}{=} D^{F}(\theta(\epsilon_{1})\|\theta(\epsilon_{1})_{*}) + D^{F}(\theta(\epsilon_{1})_{*}\|\theta_{(t)}). \tag{217}$$

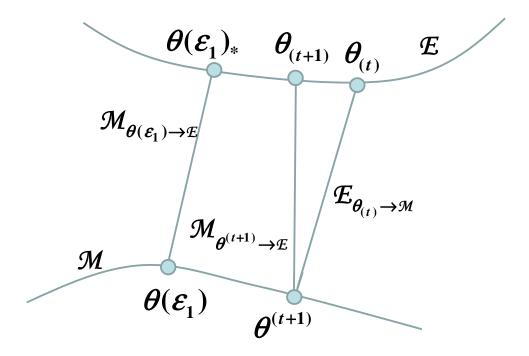


Fig. 5. Algorithms 1 and 2: This figure shows the topological relation among  $\theta(\epsilon_1)_*$ ,  $\theta(\epsilon_1)$ ,  $\theta(\epsilon_1)$ ,  $\theta(t+1)$ , and  $\theta(t)$ , which is used in the application of Phythagorean theorem (Proposition 1).  $\mathcal{M}_{\theta(\epsilon_1)\to\mathcal{E}}$  and  $\mathcal{M}_{\theta(t+1)\to\mathcal{E}}$  are the mixture subfamilies to project  $\theta(\epsilon_1)$  and  $\theta(t+1)$  to the exponential subfamily  $\mathcal{E}$ , respectively.  $\mathcal{E}_{\theta(t)\to\mathcal{M}}$  is the exponential subfamily to project  $\theta(t)$  to the mixture subfamily  $\mathcal{M}$ .

Hence,

$$D^{F}(\theta^{(t+1)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) - \epsilon_{1}$$

$$\leq D^{F}(\theta^{(t+1)} \| \theta_{(t)}) - D^{F}(\theta(\epsilon_{1}) \| \theta(\epsilon_{1})_{*})$$

$$\stackrel{(a)}{=} D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(t)}) - D^{F}(\theta(\epsilon_{1}) \| \theta^{(t+1)})$$

$$\stackrel{(b)}{\leq} D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(t)}) - D^{F}(\Gamma_{\mathcal{E}}^{(e),F}(\theta(\epsilon_{1})) \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)}))$$

$$= D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(t)}) - D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(t+1)}), \tag{218}$$

where the steps (a) and (b) follows from (217) and Condition (B1 $\mathcal{M}$ ), respectively. Thus,

$$\sum_{i=2}^{t} D^{F}(\theta^{(i)} \| \theta_{(i-1)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) - \epsilon_{1}$$

$$\leq \sum_{i=2}^{t} D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(i-1)}) - D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(i)})$$

$$= D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(1)}) - D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(t)}) \leq D^{F}(\theta(\epsilon_{1})_{*} \| \theta_{(1)})$$

$$\leq \sup_{\theta \in \mathcal{E}} D^{F}(\theta \| \theta_{(1)}). \tag{219}$$

Taking the limit  $\epsilon_1 \to 0$  in (219), we have

$$\sum_{i=2}^{t} D^{F}(\theta^{(i)} \| \theta_{(i-1)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) \le \sup_{\theta \in \mathcal{E}} D^{F}(\theta \| \theta_{(1)}).$$
 (220)

Since the relations

$$D^{F}(\theta^{(i+1)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(i+1)})) \leq D^{F}(\theta^{(i+1)} \| \theta_{(i)}) \leq D^{F}(\theta^{(i)} \| \theta_{(i)})$$

$$= D^{F}(\theta^{(i)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(i)}))$$
(221)

for i = 2, ..., t, (220) implies

$$(t-1)(D^F(\theta^{(t)}\|\theta_{(t-1)}) - C_{\inf}(\mathcal{M}, \mathcal{E})) \le \sup_{\theta \in \mathcal{E}} D^F(\theta\|\theta_{(1)}). \tag{222}$$

Thus, we have

$$D^{F}(\theta^{(t)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) \leq D^{F}(\theta^{(t)} \| \theta_{(t-1)}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq \frac{1}{t-1} \sup_{\theta \in \mathcal{E}} D^{F}(\theta \| \theta_{(1)}), \tag{223}$$

which implies (96) and (97).

When the inequality

$$D^{F}(\theta^{(t)} \| \theta_{(t-1)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) \ge D^{F}(\theta^{(t)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) \ge c(\frac{1}{t})$$

$$(224)$$

holds with a constant c > 0, the relation (220) implies

$$\infty = \sum_{t=1}^{\infty} c(\frac{1}{t}) \le \sup_{\theta \in \mathcal{E}} D^F(\theta \| \theta_{(1)}), \tag{225}$$

which yields the contradiction. Hence, we have

$$D^{F}(\theta^{(t)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E}) = o(\frac{1}{t}).$$
(226)

Combining (221), we obtain (95).

Indeed, when the minimum in (89) exists, i.e.,  $\theta_*(\mathcal{M}, \mathcal{E})$  exists, the supremum  $\sup_{\theta \in \mathcal{E}} D^F(\theta || \theta_{(1)})$  in the above evaluation is replaced by  $D^F(\theta_*(\mathcal{M}, \mathcal{E}) || \theta_{(1)})$  because  $\theta(\epsilon_1)$  is replaced by  $\theta_*(\mathcal{M}, \mathcal{E})$ .

## APPENDIX D PROOF OF THEOREM 3

We use the same notation as the proof of Theorem 2. We set  $\beta := \beta(\theta^{(1)})$ . The relations (218) is rewritten as the following way for the case with  $\epsilon_1 = 0$ ;

$$0 \leq D^{F}(\theta^{(t+1)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta^{(t+1)})$$

$$\leq D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\Gamma_{\mathcal{E}}^{(e),F}(\theta^{*}) \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)}))$$

$$= D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}).$$
(228)

Thus, we have  $D^F(\theta^* \| \theta^{(t+1)}) \overset{(a)}{\leq} D^F(\theta_* \| \theta_{(t)}) \overset{(b)}{\leq} D^F(\theta_* \| \theta_{(t-1)}) \overset{(c)}{\leq} D^F(\theta_* \| \theta_{(1)})$ , where (a) and (b) follow from (227) and (228), respectively, and (c) follows from multiple use of (227). Thus, Condition (B1+) implies  $\beta D^F(\theta^* \| \theta^{(t+1)}) \geq D^F(\theta_* \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)})) = D^F(\theta_* \| \theta_{(t+1)})$ . Combining (227), we have  $\beta D^F(\theta_* \| \theta_{(t)}) \geq D^F(\theta_* \| \theta_{(t+1)})$ . Thus, we have

$$D^{F}(\theta_{*} \| \theta_{(t+1)}) \le \beta^{t} D^{F}(\theta_{*} \| \theta_{(1)}). \tag{229}$$

Using (227), we have

$$D^{F}(\theta^{(t+1)} \| \theta_{(t+1)}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq D^{F}(\theta^{(t+1)} \| \theta_{(t)}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\leq D^{F}(\theta_{*} \| \theta_{(t)}) \leq \beta^{t-1} D^{F}(\theta_{*} \| \theta_{(1)}).$$
(230)

Hence, we obtain (99).

### APPENDIX E PROOF OF THEOREM 4

**Step 1:** In this proof, we use the notations  $\theta^{(t+1),*} := \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$  and  $\theta_{(t+1),*} := \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1),*})$ . From the construction,  $D^F(\theta^{(t)} \| \theta_{(t)})$  is monotonically decreasing for t as

$$D^{F}(\theta^{(t+1)} \| \theta_{(t+1)}) \le D^{F}(\theta^{(t+1)} \| \theta_{(t)}) \stackrel{(a)}{\le} D^{F}(\theta^{(t)} \| \theta_{(t)}), \tag{231}$$

where (a) follows from (101).

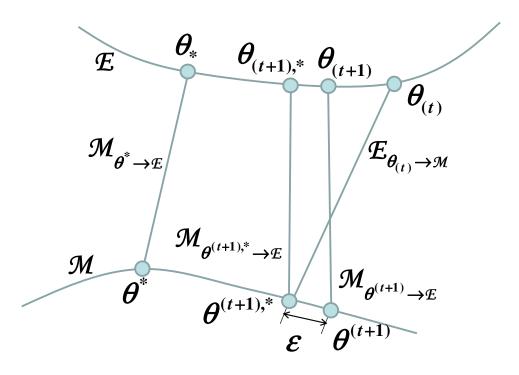


Fig. 6. Algorithm 3: This figure shows the topological relation among  $\theta_*$ ,  $\theta^*$ ,  $\theta_{(t+1)}$ ,  $\theta^{(t+1)}$ ,  $\theta_{(t+1),*}$ ,  $\theta^{(t+1),*}$ , and  $\theta_{(t)}$ , which is used in the application of Phythagorean theorem (Proposition 1).  $\mathcal{M}_{\theta^* \to \mathcal{E}}$ ,  $\mathcal{M}_{\theta^{(t+1),*} \to \mathcal{E}}$ , and  $\mathcal{M}_{\theta^{(t+1)} \to \mathcal{E}}$  are the mixture subfamilies to project  $\theta^*$ ,  $\theta^{(t+1),*}$ , and  $\theta^{(t+1)}$  to the exponential subfamily  $\mathcal{E}$ , respectively.  $\mathcal{E}_{\theta_{(t)} \to \mathcal{M}}$  is the exponential subfamily to project  $\theta_{(t)}$  to the mixture subfamily  $\mathcal{M}$ .

### **Step 2:** The aim of this step is the derivation of the relation;

$$D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)})$$

$$\geq D^{F}(\theta^{(t+1)} \| \theta_{(t+1)}) - D^{F}(\theta^{*} \| \theta_{*}) - 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(t)})\epsilon} - (\gamma + 1)\epsilon.$$
(232)

We notice that

$$D^{F}(\theta^{(t+1)} \| \theta^{(t+1),*}) \stackrel{(a)}{=} D^{F}(\theta^{(t+1)} \| \theta_{(t)}) - D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) \stackrel{(b)}{\leq} \epsilon, \tag{233}$$

where (a) and (b) follow from Phythagorean theorem (Proposition 1) and (101), respectively. Since  $\theta^{(t+1),*} = \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$ , we have

$$D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) \le D^{F}(\theta^{(t)} \| \theta_{(t)}). \tag{234}$$

Since the set  $\mathcal{E}_0$  is a star subset of  $\mathcal{E}$  for  $\theta_*$ , we can apply Theorem 1 to the set  $\mathcal{E}_0$  as a star subset for  $\theta_*$ , and obtain

$$D^{F}(\theta_{*}\|\theta_{(t+1)})$$

$$\stackrel{(a)}{\leq} D^{F}(\theta_{*}\|\theta_{(t+1),*}) + 2\gamma\sqrt{D^{F}(\theta^{*}\|\theta_{(t+1),*})}D^{F}(\theta_{(t+1)}\|\theta_{(t+1),*})$$

$$+ \gamma D^{F}(\theta_{(t+1)}\|\theta_{(t+1),*})$$

$$\stackrel{(b)}{\leq} D^{F}(\theta_{*}\|\theta_{(t+1),*}) + 2\gamma\sqrt{D^{F}(\theta^{*}\|\theta_{(t+1),*})}D^{F}(\theta^{(t+1)}\|\theta^{(t+1),*})$$

$$+ \gamma D^{F}(\theta^{(t+1)}\|\theta^{(t+1),*})$$

$$\stackrel{(c)}{\leq} D^{F}(\theta_{*}\|\theta^{(t+1),*}) + 2\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(t+1),*})\epsilon} + \gamma\epsilon,$$
(235)

where (a), (b), and (c) follow from Theorem 1, Condition (B1), and (233), respectively. The definition (89) implies

$$D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) \ge C_{\inf}(\mathcal{M}, \mathcal{E}) = D^{F}(\theta^{*} \| \theta_{*}).$$
 (236)

Also, applying Phythagorean theorem (Proposition 1) to  $D^F(\theta^* || \theta_{(t)})$ , we have

$$D^{F}(\theta^{*}\|\theta^{(t+1),*}) + D^{F}(\theta^{(t+1),*}\|\theta_{(t)}) \stackrel{(a)}{=} D^{F}(\theta^{*}\|\theta_{(t)}) \stackrel{(b)}{=} D^{F}(\theta_{*}\|\theta_{(t)}) + D^{F}(\theta^{*}\|\theta_{*}). \tag{237}$$

That is, steps (a) and (b) in (237) follow from Phythagorean theorem. Using (237), we have

$$0 \stackrel{(a)}{\leq} D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\stackrel{(b)}{=} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta^{(t+1),*})$$

$$\stackrel{(c)}{\leq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1),*})$$

$$\stackrel{(d)}{\leq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(t+1),*})\epsilon} + \gamma\epsilon$$

$$\stackrel{(e)}{\leq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(t)})\epsilon} + \gamma\epsilon, \tag{238}$$

where (a), (b), (c), and (d) follow from (236), (237), Condition (B1), and (235), respectively. The final step (e) is derived by the inequality  $D^F(\theta_* || \theta_{(t)}) - D^F(\theta_* || \theta_{(t+1),*}) \ge 0$ , which can be shown from (a) and (b). Comparing the RHS of (a) and the final term, we have

$$D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\leq D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(t)})\epsilon} + \gamma \epsilon.$$
(239)

In addition,  $D^F(\theta^{(t+1),*}||\theta_{(t)})$  can be evaluated as

$$D^{F}(\theta^{(t+1)} \| \theta_{(t+1)}) \stackrel{(a)}{\leq} D^{F}(\theta^{(t+1)} \| \theta_{(t)}) \stackrel{(b)}{=} D^{F}(\theta^{(t+1)} \| \theta^{(t+1),*}) + D^{F}(\theta^{(t+1),*} \| \theta_{(t)})$$

$$\stackrel{(c)}{\leq} \epsilon + D^{F}(\theta^{(t+1),*} \| \theta_{(t)}), \tag{240}$$

where (a), (b), and (c) follow from the relation  $\theta_{(t+1)} = \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t+1)})$ , the relation  $\theta^{(t+1),*} = \Gamma_{\mathcal{M}}^{(m),F}(\theta_{(t)})$ , and (233), respectively.

Combining the above relations, we have

$$D^{F}(\theta_{*}\|\theta_{(t)}) - D^{F}(\theta_{*}\|\theta_{(t+1)})$$

$$\geq D^{F}(\theta^{(t+1),*}\|\theta_{(t)}) - D^{F}(\theta^{*}\|\theta_{*}) - 2\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(t)})\epsilon} - \gamma\epsilon$$

$$\geq D^{F}(\theta^{(t+1)}\|\theta_{(t+1)}) - D^{F}(\theta^{*}\|\theta_{*}) - 2\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(t)})\epsilon} - (\gamma + 1)\epsilon.$$
(241)

where (a) and (b) follow from (239) and (240), respectively. Hence, we obtain (232).

**Step 3:** The aim of this step is showing

$$D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) \ge 0$$
(242)

for  $t \le t_0$  by induction when we assume that  $t_0$  satisfies the following condition with  $t \le t_0$ ;

$$D^{F}(\theta^{(t)} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*}) \ge 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon.$$
(243)

Due to the assumption of induction, we have

$$D^{F}(\theta_* \| \theta_{(t)}) \le D^{F}(\theta_* \| \theta_{(1)}). \tag{244}$$

The combination of (232), (243), and (244) implies the relation (242).

**Step 4:** The aim of this step is showing

$$D^{F}(\theta^{(t_{0}+1)} \| \theta_{(t_{0}+1)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\leq \frac{D^{F}(\theta_{*} \| \theta_{(1)})}{t_{0}} + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon.$$
(245)

If there exists a number  $t \le t_0$  that does not satisfy the condition (243), we have (245) as

$$D^{F}(\theta^{(t_{0}+1)} \| \theta_{(t_{0}+1)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\leq D^{F}(\theta^{(t)} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$< 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon$$

$$\leq \frac{D^{F}(\theta_{*} \| \theta_{(1)})}{t_{0}} + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon.$$
(246)

Hence, it is sufficient to show (245) under the assumption (243) with  $t \leq t_0$ .

Using the facts shown above, under this assumption, we have

$$D^{F}(\theta^{(t+1)} \| \theta_{(t+1)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\stackrel{(a)}{\leq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(t)})\epsilon} + (\gamma + 1)\epsilon$$

$$\stackrel{(b)}{\leq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)}) + 2\gamma \sqrt{D^{F}(\theta_{*} \| \theta_{(1)})\epsilon} + (\gamma + 1)\epsilon,$$
(247)

where (a) and (b) follow from (232) and (242), respectively.

Taking the sum for (247), we have

$$t_{0}\left(D^{F}(\theta^{(t_{0}+1)}\|\theta_{(t_{0}+1)}) - D^{F}(\theta^{*}\|\theta_{*})\right)$$

$$\stackrel{(a)}{\leq} \sum_{t=1}^{t_{0}} \left(D^{F}(\theta^{(t+1)}\|\theta_{(t+1)}) - D^{F}(\theta^{*}\|\theta_{*})\right)$$

$$\stackrel{(b)}{\leq} \sum_{t=1}^{t_{0}} \left(D^{F}(\theta_{*}\|\theta_{(t)}) - D^{F}(\theta_{*}\|\theta_{(t+1)}) + 2\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(1)})\epsilon} + (\gamma + 1)\epsilon\right)$$

$$= D^{F}(\theta_{*}\|\theta_{(1)}) - D^{F}(\theta_{*}\|\theta_{(t_{0}+1)}) + 2t_{0}\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(1)})\epsilon} + t_{0}(\gamma + 1)\epsilon$$

$$\leq D^{F}(\theta_{*}\|\theta_{(1)}) + 2t_{0}\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(1)})\epsilon} + t_{0}(\gamma + 1)\epsilon, \tag{248}$$

where (a) and (b) follow from (231) and (247), respectively. Hence, we have (245).

Step 5: Finally, we derive (103) from (102). The condition  $t \geq \frac{2D^{F}(\theta_{*,1}\|\theta_{(1)})}{\epsilon'} + 1$  implies  $\frac{D^{F}(\theta_{*}\|\theta_{(1)})}{t} \leq \epsilon'$ . The condition  $\epsilon \leq \frac{\epsilon'^{2}}{4(3\gamma+1)^{2}D^{F}(\theta_{*}\|\theta_{(1)})}$  implies  $(3\gamma+1)\sqrt{D^{F}(\theta_{*}\|\theta_{(1)})\epsilon} \leq \frac{\epsilon'}{2}$ . Since  $D^{F}(\theta_{*}\|\theta_{(1)}) \geq \epsilon$  and  $\gamma > 1$ , we have  $+2\gamma\sqrt{D^{F}(\theta_{*}\|\theta_{(1)})\epsilon} + (\gamma+1)\epsilon \leq \frac{\epsilon'}{2}$ . Thus, we obtain (103).

### APPENDIX F PROOF OF THEOREM 5

**Step 1:** We define  $\theta_* := \Gamma_{\mathcal{E}}^{(e),F}(\theta^*)$ . The aim of this step is showing the inequality (108). The condition (104) implies that

$$F(\bar{\theta}^{(t+1)}) - \sum_{j=k+1}^{d} (\bar{\theta}^{(t+1)})^j a_j \le F(\theta^{(t+1,*)}) - \sum_{j=k+1}^{d} (\theta^{(t+1,*)})^j a_j + \epsilon_1.$$
(249)

Hence,

$$D^{F}(\theta^{(t+1),*} \| \bar{\theta}^{(t+1)}) = \sum_{i=1}^{d} \eta_{i}(\theta^{(t+1),*}) (\theta^{(t+1),*} - \bar{\theta}^{(t+1)})^{i} - F(\theta^{(t+1),*}) + F(\bar{\theta}^{(t+1)}) \le \epsilon_{1}.$$
 (250)

**Step 2:** The aim of this step is showing

$$D^{F}(\theta^{(t_3),*} \| \theta_{(t_3-1)}) - D^{F}(\theta^* \| \theta_*) \le \frac{1}{t_1 - 1} D^{F}(\theta_* \| \theta_{(1)}) + \epsilon_1$$
(251)

under the choice of  $t_3 := \underset{\bullet}{\operatorname{argmin}} D^F(\theta^{(t),*} || \theta_{(t-1)}).$ 

Pythagorean theorem (Proposition 1) implies that

$$D^{F}(\theta^* \| \theta^{(t+1),*}) + D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) = D^{F}(\theta^* \| \theta_{(t)}) = D^{F}(\theta^* \| \theta_*) + D^{F}(\theta_* \| \theta_{(t)}). \tag{252}$$

Using the result of Step 1 and various formulas, we have

$$D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)})$$

$$\stackrel{(a)}{\geq} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \bar{\theta}^{(t+1)}) \stackrel{(b)}{=} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta^{(t+1),*}) - D^{F}(\theta^{(t+1),*} \| \bar{\theta}^{(t+1)})$$

$$\stackrel{(c)}{=} D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*}) - D^{F}(\theta^{(t+1),*} \| \bar{\theta}^{(t+1)})$$

$$\stackrel{(d)}{\geq} D^{F}(\theta^{(t+1),*} \| \theta_{(t)}) - D^{F}(\theta^{*} \| \theta_{*}) - \epsilon_{1}, \tag{253}$$

where each step is derived as follows. Step (a) follows from Condition (B1). Step (b) follows from Pythagorean theorem (Proposition 1). Step (c) follows from (252). Step (d) follows from (250).

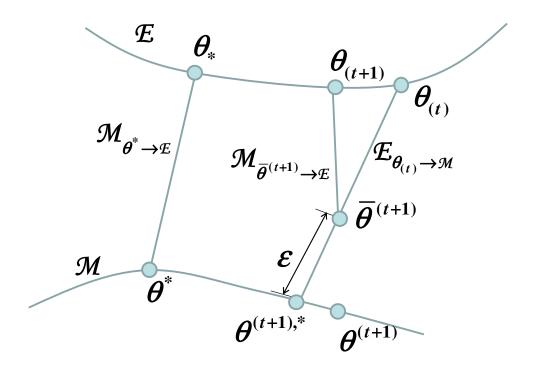


Fig. 7. Algorithm 4: This figure shows the topological relation among  $\theta_*$ ,  $\theta^*$ ,  $\theta_{(t+1)}$ ,  $\theta^{(t+1)}$ ,  $\bar{\theta}^{(t+1)}$ ,  $\theta^{(t+1),*}$ , and  $\theta_{(t)}$ , which is used in the application of Phythagorean theorem (Proposition 1).  $\mathcal{M}_{\theta^* \to \mathcal{E}}$ ,  $\mathcal{M}_{\theta^{(t+1),*} \to \mathcal{E}}$ , and  $\mathcal{M}_{\theta^{(t+1)} \to \mathcal{E}}$  are the mixture subfamilies to project  $\theta^*$ ,  $\theta^{(t+1),*}$ , and  $\theta^{(t+1)}$  to the exponential subfamily  $\mathcal{E}$ , respectively.  $\mathcal{E}_{\theta_{(t)} \to \mathcal{M}}$  is the exponential subfamily to project  $\theta_{(t)}$  to the mixture subfamily  $\mathcal{M}$ .

We choose 
$$t_3 := \underset{2 \le t \le t_1}{\operatorname{argmin}} D^F(\theta^{(t),*} \| \theta_{(t-1)})$$
. Hence, for  $t \le t_1 - 1$ , we have 
$$D^F(\theta^{(t_3),*} \| \theta_{(t_3-1)}) - D^F(\theta^* \| \theta_*) - \epsilon_1 \le D^F(\theta_* \| \theta_{(t)}) - D^F(\theta_* \| \theta_{(t+1)}). \tag{254}$$

Taking the sum for (254), we have

$$D^{F}(\theta^{(t_{3}),*} \| \theta_{(t_{3}-1)}) - D^{F}(\theta^{*} \| \theta_{*}) - \epsilon_{1}$$

$$\leq \frac{1}{t_{1}-1} \sum_{t=1}^{t=t_{1}-1} D^{F}(\theta_{*} \| \theta_{(t)}) - D^{F}(\theta_{*} \| \theta_{(t+1)})$$

$$= \frac{1}{t_{1}-1} (D^{F}(\theta_{*} \| \theta_{(1)}) - D^{F}(\theta_{*} \| \theta_{(t_{1})})) \leq \frac{1}{t_{1}-1} D^{F}(\theta_{*} \| \theta_{(1)}).$$
(255)

Therefore, we obtain (251).

**Step 3:** The aim of this step is showing the following inequality;

$$D^{F}(\theta^{(t_{2})} \| \theta_{(t_{2}-1)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\leq \frac{1}{t_{1}-1} D^{F}(\theta_{*} \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2})} \| \bar{\theta}^{(t_{2})}).$$
(256)

Remember that the final estimate  $\theta_f^{(t_1)}$  is defined as  $\theta^{(t_2)} \in \mathcal{M}$  by using  $t_2 = \operatorname*{argmin}_{t=2,\dots,t_1} D^F(\theta^{(t)} \| \theta_{(t-1)}) - \theta_{(t-1)}$ 

 $D^F(\theta^{(t)}||\bar{\theta}^{(t)})$ . Then, Eq. (256) is shown as follows.

$$D^{F}(\theta^{(t_{2})} \| \theta_{(t_{2}-1)}) - D^{F}(\theta^{(t_{2})} \| \bar{\theta}^{(t_{2})})$$

$$\leq D^{F}(\theta^{(t_{3})} \| \theta_{(t_{3}-1)}) - D^{F}(\theta^{(t_{3})} \| \bar{\theta}^{(t_{3})})$$

$$\stackrel{(b)}{=} D^{F}(\theta^{(t_{3})} \| \theta^{(t_{3}),*}) + D^{F}(\theta^{(t_{3}),*} \| \theta_{(t_{3}-1)}) - D^{F}(\theta^{(t_{3})} \| \bar{\theta}^{(t_{3})})$$

$$\leq D^{F}(\theta^{(t_{3})} \| \theta^{(t_{3}),*}) + D^{F}(\theta^{(t_{3}),*} \| \bar{\theta}^{(t_{3})}) + D^{F}(\theta^{(t_{3}),*} \| \theta_{(t_{3}-1)}) - D^{F}(\theta^{(t_{3})} \| \bar{\theta}^{(t_{3})})$$

$$\stackrel{(c)}{=} D^{F}(\theta^{(t_{3}),*} \| \theta_{(t_{3}-1)})$$

$$\stackrel{(d)}{\leq} \frac{1}{t_{1}-1} D^{F}(\theta_{*} \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{*} \| \theta_{*}), \tag{257}$$

where each step is derived as follows. Step (a) follows from the definition of  $t_2$ . Steps (b) and (c) follow from Pythagorean theorem (Proposition 1) for  $D^F(\theta^{(t_3)} \| \theta_{(t_3-1)})$  and  $D^F(\theta^{(t_3)} \| \bar{\theta}^{(t_3)})$ , respectively. Step (d) follows from (251).

Step 4: The aim of this step is showing Eq. (109). Eq. (109) is shown as follows;

$$D^{F}(\theta_{f}^{(t_{1})} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t_{1})})) - D^{F}(\theta^{*} \| \theta_{*})$$

$$= D^{F}(\theta^{(t_{2})} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t_{2})})) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\stackrel{(a)}{\leq} D^{F}(\theta^{(t_{2})} \| \theta_{(t_{2}-1)}) - D^{F}(\theta^{*} \| \theta_{*})$$

$$\stackrel{(b)}{\leq} \frac{1}{t_{1}-1} D^{F}(\theta_{*} \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2})} \| \bar{\theta}^{(t_{2})}), \tag{258}$$

where Step (a) follows from the definition of  $\Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t_2)})$  and Step (b) follows from (256).

## APPENDIX G PROOFS OF THEOREMS 6, 7, AND 8

*Proof of Theorem 6:* Theorem 2 is shown by application of Phythagorean theorem (Proposition 1) to m-projection to  $\mathcal{M}$ . We can show Theorem 6 in the same way as the proof of Theorem 2 by replacing the role of Proposition 1 by Lemma 8 In this case, the proof of Theorem 6 is completed by replacing the equations at (a) of (217) and (a) of (218) by the inequality  $\leq$ .

*Proof of Theorem 7:* In the proof of Theorem 3, Phythagorean theorem is applied to m-projection to  $\mathcal{M}$ . However, this theorem is used only in the derivation for (227), which is essentially given in (217). In the current setting, the step (a) of (217) is derived by Lemma 8 instead of Proposition 1. Hence, the proof of Theorem 7 is completed.

*Proof of Theorem 8:* We can show Theorem 8 in the same way as the proof of Theorem 4 by replacing the role of Proposition 1 by Lemma 8 for Phythagorean theorem to the projection to m-projection to  $\mathcal{M}$ . In this case, the proof of Theorem 8 is completed by replacing the equations at (a) of (233), (a) of (237), and (b) of (239) by the inequality  $\leq$ .

# APPENDIX H PROOF OF THEOREM 9

**Step 1:** To show Theorem 9, we prepare the following lemma.

Lemma 22: Assume the same assumption as Algorithm 8. Also, we assume Conditions (B0) and (B1) for  $\mathcal{E}$ . When the relation  $C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) = C_{\inf}(\hat{\mathcal{M}}_{\lambda}, \mathcal{E})$  holds for  $\lambda \in \Lambda_*$ , for  $\theta_0 \in \hat{\mathcal{M}}_{\lambda} \setminus \mathcal{M}_{\lambda}$ , we have

$$\min_{\lambda' \in \Lambda_{\lambda}} C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \le D^{F}(\theta_0 \| \Gamma_{\mathcal{E}}^{(e), F}(\theta_0)). \tag{259}$$

Proof of Lemma 22: Lemma 2 guarantees that there is no local minimum for the minimization  $\min_{\theta \in \mathcal{M}_{\lambda}} D^{F}(\theta \| \Gamma_{\mathcal{E}}^{(e),F}(\theta))$ . Hence, there exists a one-parameter continuous curve  $\theta(s) \in \mathcal{M}_{\lambda}$  such that  $\theta(0) = \theta_{0}$ ,

$$\lim_{s \to 1} D^F(\theta(s) \| \Gamma_{\mathcal{E}}^{(e),F}(\theta(s))) = C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}), \tag{260}$$

and  $D^F(\theta(s)||\Gamma_{\mathcal{E}}^{(e),F}(\theta(s)))$  is monotonically increasing for s. Then, there exits  $s_0 \in (0,1)$  such that  $\theta(s_0) \in \partial \mathcal{M}_{\lambda}$ . We choose  $\lambda'' \in \Lambda_{\lambda}$  such that  $\theta(s_0) \in \mathcal{M}_{\lambda''}$ . Then, we obtain

$$\min_{\lambda' \in \Lambda_{\lambda}} C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \leq C_{\inf}(\mathcal{M}_{\lambda''}, \mathcal{E}) \leq D^{F}(\theta(s_{0}) \| \Gamma_{\mathcal{E}}^{(e), F}(\theta(s_{0})))$$

$$\leq D^{F}(\theta_{0} \| \Gamma_{\mathcal{E}}^{(e), F}(\theta_{0})). \tag{261}$$

In the following, we show Theorem 9 by using Lemma 22 and Eq. (256) in the proof of Theorem 5. **Step 2:** The aim of this step is showing the following relation by induction for  $D(\lambda)$ ;

$$\min_{\substack{\lambda' \in \bar{\Lambda}_{\lambda} \cup \{\lambda\}: \theta^{(t_{2}(\lambda')), \lambda'} \in \mathcal{M}_{\lambda'}}} D^{F}(\theta^{(t_{2}(\lambda')), \lambda'} \| \theta_{(t_{2}(\lambda') - 1), \lambda'}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E})$$

$$\leq \frac{1}{t_{1} - 1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)), \lambda} \| \bar{\theta}^{(t_{2}(\lambda)), \lambda})$$

$$+ \sum_{k=0}^{D(\lambda) - 1} \max_{\lambda' \in \bar{\Lambda}_{\lambda}: D(\lambda') = k} \left( \frac{1}{t_{1} - 1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda'}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda')), \lambda'} \| \bar{\theta}^{(t_{2}(\lambda')), \lambda'}) \right). \tag{262}$$

Eq. (256) in the proof of Theorem 5 implies (262) with the condition D(0)=0. In the following, we show (262) with the condition  $D(\lambda)=k$  by assuming (262) with the condition  $D(\lambda)\leq k-1$ . When the relation

$$C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) = C_{\inf}(\hat{\mathcal{M}}_{\lambda}, \mathcal{E})$$
(263)

does not hold, there exists  $\lambda' \in \bar{\Lambda}_{\lambda}$  such that  $C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) = C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E})$ . Since  $D(\lambda) \leq k - 1$ , the assumption of induction implies (262). When the relation

$$\theta^{(t),\lambda} \in \hat{\mathcal{M}}_{\lambda} \setminus \mathcal{M}_{\lambda} \tag{264}$$

dos not hold, i.e.,  $\theta^{(t),\lambda} \in \mathcal{M}_{\lambda}$ , Theorem 5 implies (262). Hence, it is sufficient to show (262) when (263) and (264) hold.

Due to these two conditions, Lemma 22 implies that

$$\min_{\lambda' \in \Lambda_{\lambda}} C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \le D^{F}(\theta^{(t_{2}(\lambda)), \lambda} \| \Gamma_{\mathcal{E}}^{(e), F}(\theta^{(t_{2}(\lambda)), \lambda})). \tag{265}$$

Thus,

$$\min_{\lambda' \in \Lambda_{\lambda}} C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) 
\leq D^{F}(\theta^{(t_{2}(\lambda)), \lambda} \| \Gamma_{\mathcal{E}}^{(e), F}(\theta^{(t_{2}(\lambda)), \lambda})) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) 
\stackrel{(a)}{\leq} D^{F}(\theta^{(t_{2}(\lambda)), \lambda} \| \theta_{(t_{2}(\lambda) - 1), \lambda}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) 
\stackrel{(b)}{\leq} \frac{1}{t_{1} - 1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)), \lambda} \| \bar{\theta}^{(t_{2}(\lambda)), \lambda}),$$
(266)

where (a) follows from the definition of the e-projection  $\Gamma_{\mathcal{E}}^{(e),F}$  and (b) follows from Eq. (256) in the proof of Theorem 5. Hence, we have

$$\min_{\lambda' \in \bar{\Lambda}_{\lambda} \cup \{\lambda\}: \theta^{(t),\lambda'} \in \mathcal{M}_{\lambda'}} D^{F}(\theta^{(t_{2}(\lambda')),\lambda'} \| \theta_{(t_{2}(\lambda')-1),\lambda'}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E})$$

$$\leq \min_{\lambda' \in \bar{\Lambda}_{\lambda}} \left( \min_{\lambda'' \in \Lambda_{\lambda'} \cup \{\lambda'\}: \theta^{(t),\lambda''} \in \mathcal{M}_{\lambda''}} D^{F}(\theta^{(t_{2}(\lambda'')),\lambda''} \| \theta_{(t_{2}(\lambda'')-1),\lambda''}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) \right)$$

$$= \min_{\lambda' \in \Lambda_{\lambda}} \left( \min_{\lambda'' \in \Lambda_{\lambda'} \cup \{\lambda'\}: \theta^{(t),\lambda''} \in \mathcal{M}_{\lambda''}} D^{F}(\theta^{(t_{2}(\lambda'')),\lambda''} \| \theta_{(t_{2}(\lambda'')-1),\lambda''}) - C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \right)$$

$$+ \left( C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) - C_{\inf}(\mathcal{M}_{\lambda}, \mathcal{E}) \right)$$

$$\leq \max_{\lambda' \in \Lambda_{\lambda}} \left( \min_{\lambda'' \in \Lambda_{\lambda'} \cup \{\lambda'\}: \theta^{(t),\lambda''} \in \mathcal{M}_{\lambda''}} D^{F}(\theta^{(t_{2}(\lambda'')),\lambda''} \| \theta_{(t_{2}(\lambda'')-1),\lambda''}) - C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \right)$$

$$+ \min_{\lambda' \in \Lambda_{\lambda}} \left( \sum_{\lambda'' \in \Lambda_{\lambda'} \cup \{\lambda'\}: \theta^{(t),\lambda''} \in \mathcal{M}_{\lambda''}} D^{F}(\theta^{(t_{2}(\lambda'')),\lambda''} \| \theta_{(t_{2}(\lambda'')-1),\lambda''}) - C_{\inf}(\mathcal{M}_{\lambda'}, \mathcal{E}) \right)$$

$$+ \min_{\lambda' \in \Lambda_{\lambda}} \left( \sum_{\lambda'' \in \Lambda_{\lambda'} \cup \{\lambda'\}: \theta^{(t),\lambda''} \in \mathcal{M}_{\lambda''}} \left( \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda''}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda')),\lambda'}) \right)$$

$$+ \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)),\lambda} \| \bar{\theta}^{(t_{2}(\lambda)),\lambda})$$

$$\stackrel{(c)}{\leq} \sum_{k=0}^{D(\lambda)-1} \max_{\lambda' \in \Lambda_{\lambda}: D(\lambda')=k} \left( \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda'}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)),\lambda}) \| \bar{\theta}^{(t_{2}(\lambda')),\lambda'} \| \bar{\theta}^{(t_{2}(\lambda'$$

where Step (a) follows from the definition of  $\bar{\Lambda}_{\lambda}$ . The second line of (b) follows from (266). The first line of (b) follows from the substitution of  $\lambda'$  into  $\lambda$  in the relation (262) as the assumption of induction. Step (c) follows from the following fact. For  $\lambda' \in \Lambda_{\lambda}$ , we have the relations  $\bar{\Lambda}_{\lambda'} \subset \bar{\Lambda}_{\lambda}$  and  $D(\lambda) - 1 \ge D(\lambda') > D(\lambda') - 1$ . Hence, we obtain (262) in the general case.

**Step 3:** The aim of this step is showing (123) by using (262). We apply (262) to the case with  $\lambda = 0$ .

We have

$$D^{F}(\theta_{f}^{(t)} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta_{f}^{(t)})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\stackrel{(a)}{=} D^{F}(\theta^{(t_{2}(\lambda_{0})),\lambda_{0}} \| \Gamma_{\mathcal{E}}^{(e),F}(\theta^{(t_{2}(\lambda_{0})),\lambda_{0}})) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\stackrel{(b)}{\leq} D^{F}(\theta^{(t_{2}(\lambda_{0})),\lambda_{0}} \| \theta_{(t_{2}(\lambda_{0})-1),\lambda_{0}}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\stackrel{(c)}{=} \min_{\lambda' \in \Lambda_{*}: \theta^{(t_{2}(\lambda')),\lambda'} \in \mathcal{M}_{\lambda'}} D^{F}(\theta^{(t_{2}(\lambda')),\lambda'} \| \theta_{(t_{2}(\lambda')-1),\lambda'}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\stackrel{(d)}{=} \min_{\lambda' \in \bar{\Lambda}_{0} \cup \{0\}: \theta^{(t_{2}(\lambda')),\lambda'} \in \mathcal{M}_{\lambda'}} D^{F}(\theta^{(t_{2}(\lambda')),\lambda'} \| \theta_{(t_{2}(\lambda')-1),\lambda'}) - C_{\inf}(\mathcal{M}, \mathcal{E})$$

$$\stackrel{(e)}{\leq} \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{0}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(0)),0} \| \bar{\theta}^{(t_{2}(0)),0})$$

$$+ \sum_{k=0}^{D(0)-1} \max_{\lambda' \in \bar{\Lambda}_{\lambda}: D(\lambda') = k} \left( \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda'}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda')),\lambda'} \| \bar{\theta}^{(t_{2}(\lambda')),\lambda'}) \right)$$

$$= (D(0) + 1) \max_{\lambda \in \Lambda_{*}} \left( \frac{1}{t_{1}-1} D^{F}(\theta_{*}(\mathcal{M}_{\lambda}, \mathcal{E}) \| \theta_{(1)}) + \epsilon_{1} + D^{F}(\theta^{(t_{2}(\lambda)),\lambda} \| \bar{\theta}^{(t_{2}(\lambda)),\lambda}) \right), \tag{268}$$

where each step is shown as follows. (a) follows from the definition of  $\theta_f^{(t)}$ . (b) follows from the definition of the e-projection  $\Gamma_{\mathcal{E}}^{(e),F}$ . (c) follows from the definition of  $\lambda_0$ . (d) follows from the relation  $\bar{\Lambda}_0 \cup \{0\} = \Lambda_*$ . (e) follows from the application of (262) to the case with  $\lambda = 0$ .

### APPENDIX I PROOFS OF LEMMAS 18 AND 19

Proof of Lemma 18: The assumption implies  $n_1 \ge n_2$ . It is sufficient to show that the matrix  $((P_2J_{\theta,\tau(\theta),3}P_1)_{i,j})_{i=1,\dots,n}$  has at least rank  $n_2-1$  under the given condition. For  $i=1,\dots,n_1-1,j=1,\dots,n_2-1$ , we choose  $c_{1,i}$  and  $c_{2,j}$  as

$$\sum_{x,y} P_X(x) W_{\theta,x}(y) f_j(y) = c_{1,j}$$
(269)

$$\sum_{x,y} P_X(x) W_{\theta,x}(y) \delta_{i,x} = c_{2,i}, \tag{270}$$

where  $f_i(y)$  is defined in Subsection III-A. Then, we have

$$(P_2 J_{\theta,\tau(\theta),3} P_1)_{i,j} = \sum_{x,y} P_X(x) W_{\theta,x}(y) (\delta_{i,x} - c_{2,i}) (f_j(y) - c_{1,j})$$

$$= \sum_x P_X(x) (\delta_{i,x} - c_{2,i}) \Big( \sum_y W_{\theta,x}(y) (f_j(y) - c_{1,j}) \Big). \tag{271}$$

When  $(f_j(y)-c_{1,j})_{y,j}$  is considered as a matrix, its rank is  $n_2-1$ . Also,  $(W_{\theta,x}(y))_{x,y}$  can be regarded as a rank- $n_2-1$  matrix. Hence,  $\left(\sum_y W_{\theta,x}(y)(f_j(y)-c_{1,j})\right)_{x,j}$  can be regarded as a rank- $n_2-1$  matrix. Also,  $(P_X(x)(\delta_{i,x}-c_{2,i}))_{x,i}$  can be regarded as a rank- $n_1-1$  matrix. Since  $n_1\geq n_2$ ,  $(P_2J_{\theta,\tau(\theta),3}P_1)_{i,j}$  is a rank- $n_1-1$  matrix.

*Proof of Lemma 19*: To show Lemma 19, we prepare the following lemma;

Lemma 23: We consider a one-parameterized family of channels  $\{\bar{W}_s\}_{s\in\mathbb{R}}$  We denote the Fisher information of  $\{\bar{W}_s\times P_X\}_s$  by  $\bar{J}_{s,1}$ . We denote the Fisher information of  $\{\bar{W}_s\cdot P_X\}_s$  by  $\bar{J}_{s,2}$ . Then,

 $\bar{J}_{s_0,1} \geq \bar{J}_{s_0,2}$ . The equality hold if and only if the function  $(x,y) \mapsto \frac{\frac{d}{ds}\bar{W}_s(y|x)\Big|_{s=s_0}}{\bar{W}_{s_0}(y|x)}$  can be written as a function of y.

We denote the mixture parameter of the exponential family  $\{P_{XY,\theta,\tau}\}_{\theta,\tau}$  by  $(\eta_1(\theta,\tau),\eta_2(\theta,\tau))$ . The condition (164) implies

$$\eta_{2,0}(\theta, \tau(\theta)) = D,\tag{272}$$

and the construction of  $P_{XY}^{(t+1)}$  implies

$$\eta_{2,x}(\theta,\tau(\theta)) = P_X(x) \tag{273}$$

for  $x \in \mathcal{X} \setminus \{n_1\}$ . We choose a one-parameter family  $c(t) \in \mathbb{R}^{n_2-1}$  such that  $c(0) = \theta_0$  and  $v_1 := \frac{d}{dt}c(t)|_{t=0} \neq 0$ . Then, we have

$$\frac{d}{dt}\eta_2(c(t), \tau(\theta_0)) + \frac{d}{dt}\eta_2(\theta_0, \tau(c(t))) = 0.$$
(274)

We denote  $\frac{d}{dt}\tau(c(t))|_{t=0}$  by  $v_2$ . The condition (274) is equivalent to the condition;

$$P_2 J_{\theta_0, \tau(\theta_0), 3} P_1 v_1 + P_2 J_{\theta_0, \tau(\theta_0), 3} P_2 v_2 = 0.$$
(275)

That is,

$$v_2 = -(P_2 J_{\theta_0, \tau(\theta_0), 3} P_2)^{-1} P_2 J_{\theta_0, \tau(\theta_0), 3} P_1 v_1.$$
(276)

Hence, the vector  $v_2$  is not zero for any  $v_1 \neq 0$  if and only if  $\operatorname{Ker} P_2 J_{\theta,\tau(\theta),3} P_1 = \{0\}$ . In addition,

$$\frac{d}{dt} \left. \Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y} \times P_X)(x,y) \right|_{t=0} 
= \Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta_0,Y} \times P_X)(x,y) \left( \sum_{i=1}^{n_2-1} v_1^i f_i(y) + \sum_{j=0}^{n_1-1} v_2^j g_j(x,y) \right).$$
(277)

Therefore,  $\frac{\frac{d}{dt} \, \Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y} \times P_X)(x,y) \Big|_{t=0}}{\Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta_0,Y} \times P_X)(x,y)} \text{ cannot be written as a function of } y \text{ for any } v_1 \neq 0 \text{ if and only if } \operatorname{Ker} P_2 J_{\theta,\tau(\theta),3} P_1 = \{0\}.$ 

We define  $W_{\theta}$  as  $W_{\theta} \times P_X = \Gamma_{\mathcal{M}}^{(m),\mu}(P_{\theta,Y} \times P_X)$ . Applying Lemma 23 with substitution of  $W_{c(t)}$  into  $\bar{W}_s$ , we obtain the desired statement of Lemma 19 from the above equivalence relation.

Proof of Lemma 23:

$$\bar{J}_{s_{0},1} = \sum_{x,y} \left( \frac{d}{ds} \bar{W}_{s}(y|x)|_{s=s_{0}} \right)^{2} \bar{W}_{s_{0}}(y|x)^{-1} P_{X}(x) 
= \sum_{y} \left( \sum_{x'} \frac{d}{ds} \bar{W}_{s}(y|x')|_{s=s_{0}} P_{X}(x') \right)^{2} \left( \sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x') \right)^{-1} 
+ \sum_{y} \left( \sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x') \right) 
\cdot \sum_{x} \left( \frac{d}{ds} \frac{P_{X}(x) \bar{W}_{s}(y|x)}{\sum_{x'} \bar{W}_{s}(y|x') P_{X}(x')} \Big|_{s=s_{0}} \right)^{2} \left( \frac{P_{X}(x) \bar{W}_{s_{0}}(y|x)}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right)^{-1}.$$
(278)

Hence,

$$\begin{split} & \bar{J}_{s_{0},1} - \bar{J}_{s_{0},2} \\ & = \sum_{y} \left( \sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x') \right) \\ & \cdot \sum_{x} \left( \frac{d}{ds} \frac{P_{X}(x) \bar{W}_{s}(y|x)}{\sum_{x'} \bar{W}_{s}(y|x') P_{X}(x')} \Big|_{s=s_{0}} \right)^{2} \left( \frac{P_{X}(x) \bar{W}_{s_{0}}(y|x)}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right)^{-1} \\ & = \sum_{y} \left( \sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x') \right) \\ & \cdot \sum_{x} \left( \frac{d}{ds} \log \left( \frac{P_{X}(x) \bar{W}_{s}(y|x)}{\sum_{x'} \bar{W}_{s}(y|x') P_{X}(x')} \right) \Big|_{s=s_{0}} \right)^{2} \left( \frac{P_{X}(x) \bar{W}_{s_{0}}(y|x)}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right) \\ & = \sum_{x,y} \bar{W}_{s_{0}}(y|x) P_{X}(x) \\ & \cdot \left( \frac{d}{ds} \log P_{X}(x) \bar{W}_{s}(y|x) \right) - \log \left( \sum_{x'} \bar{W}_{s}(y|x') P_{X}(x') \right) \Big|_{s=s_{0}} \right)^{2} \\ & = \sum_{x,y} \bar{W}_{s_{0}}(y|x) P_{X}(x) \\ & \cdot \left( \frac{d}{ds} \log \bar{W}_{s}(y|x) \right) \Big|_{s=s_{0}} - \frac{d}{ds} \log \left( \sum_{x'} \bar{W}_{s}(y|x') P_{X}(x') \right) \Big|_{s=s_{0}} \right)^{2} \\ & = \sum_{x,y} \bar{W}_{s_{0}}(y|x) P_{X}(x) \left( \frac{\frac{d}{ds} \bar{W}_{s}(y|x)}{\bar{W}_{s_{0}}(y|x)} - \frac{\frac{d}{ds} \left( \sum_{x'} \bar{W}_{s}(y|x') P_{X}(x') \right)}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right)^{2} \\ & = \sum_{x,y} \bar{W}_{s_{0}}(y|x) P_{X}(x) \left( l(x,y) - \frac{\left( \sum_{x'} l(x',y) \bar{W}_{s_{0}}(y|x') P_{X}(x')}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right)^{2} \\ & = \sum_{x,y} \bar{W}_{s_{0}}(y|x) P_{X}(x) \left( l(x,y) - \frac{\left( \sum_{x'} l(x',y) \bar{W}_{s_{0}}(y|x') P_{X}(x')}{\sum_{x'} \bar{W}_{s_{0}}(y|x') P_{X}(x')} \right)^{2}, \end{cases}$$

$$(279)$$

where  $l(x,y):=\frac{\frac{d}{ds}\bar{W}_s(y|x)}{\bar{W}_{s_0}(y|x)}$ . Hence, we have  $\bar{J}_{s_0,1}-\bar{J}_{s_0,2}\geq 0$ . The equality holds if and only if l(x,y) depends only on y. The desired statement is obtained.

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