Achievable Refined Asymptotics for Successive Refinement Using Gaussian Codebooks

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Abstract

We study the mismatched successive refinement problem where one uses Gaussian codebooks to compress an arbitrary memoryless source with successive minimum Euclidean distance encoding under the quadratic distortion measure. Specifically, we derive achievable refined asymptotics under both the joint excess-distortion probability (JEP) and the separate excess-distortion probabilities (SEP) criteria. For both second-order and moderate deviations asymptotics, we consider two types of codebooks: the spherical codebook where each codeword is drawn independently and uniformly from the surface of a sphere and the i.i.d. Gaussian codebook where each component of each codeword is drawn independently from a Gaussian distribution. We establish the achievable second-order rate-region under JEP and we show that under SEP any memoryless source satisfying mild moment conditions is strongly successively refinable. When specialized to a Gaussian memoryless source (GMS), our results provide an alternative achievability proof with specific code design. We show that under JEP and SEP, the same moderate deviations constant is achievable. For large deviations asymptotics, we only consider the i.i.d. Gaussian codebook since the i.i.d. Gaussian codebook has better performance than the spherical codebook in this regime for the one layer mismatched rate-distortion problem (Zhou, Tan, Motani, TIT, 2019). We derive achievable exponents of both JEP and SEP and specialize our results to a GMS, which appears to be a novel result of independent interest.

I. Introduction

The successive refinement problem [3] was motivated by diverse applications such as image and video compression and clinical diagnosis using X-rays. As shown in Fig. 1, in this problem, there are two layers of encoders and decoders. For each $i \in \{1,2\}$, the encoder f_i has access to a source sequence X^n and compresses it into a message S_i . Two decoders aim to recover the source sequence with different distortion requirements and different access to compressed information. Specifically, decoder ϕ_1 aims to reproduce the source sequence X^n within distortion level D_1 using the compressed information S_1 from encoder f_1 and the decoder ϕ_2 aims to reproduce X^n within a finer distortion level $D_2 < D_1$ with the additional access to the compressed information S_2 from encoder f_2 . Let the reproduced versions at decoders ϕ_1 and ϕ_2 be \hat{X}_1^n and \hat{X}_2^n , respectively.

To evaluate the performance of a coding scheme, there are two different performance criteria for successive refinement: the joint excess-distortion probability (JEP) that measures the probability that either decoder fails to decode X^n within the desired distortion level and the separate excess-distortion probabilities (SEP) that measure the decoding failure of two decoders separately. For any discrete memoryless source (DMS), Rimoldi [3] derived the rate-distortion region that asymptotically characterizes the rate requirements of both encoders to ensure reliable recovery at decoders with vanishing JEP. Subsequently, Kanlis and Narayan [4] refined Rimoldi's result by showing that for any rate pair strictly inside the rate-distortion region, the JEP decays exponentially fast to zero and characterized the optimal decay rate. Koshelev [5] and Equitz-Cover [6] studied the so-called successive refinability property which shows that the rate-distortion region can be reduced to the case where rates of both encoders are bounded by rates of a single layer rate-distortion function with desired distortion levels under certain conditions on the source distribution and the distortion measures. For example, a Gaussian memoryless source (GMS) under the quadratic distortion measure is successively refinable [6]. Recently, Zhou, Tan and Motani [7] refined Rimoldi's results by deriving second-order and moderate deviations asymptotics for any DMS and GMS. Under SEP, Tuncel and Rose [8] derived error exponents for any DMS, which implied that the rate-distortion region remains the same. The authors of [8] also discovered an interesting tradeoff between the exponents of SEP and identified the conditions under which the exponents of the two layers are successively refinable. Under SEP, No, Ingber and Weissman [9] derived achievable second-order asymptotics for both a DMS and a GMS and extended the successive refinability property to the stronger second-order asymptotic case.

The above studies of successive refinement are very insightful. However, all these works assume that the distribution of the source sequence X^n is perfectly known, which we term as the matched case. Such an assumption is impractical since usually it is challenging to obtain the exact source distribution before compressing a source. Thus, one should use coding schemes ignorant of the source distribution. This direction was pioneered by Lapidoth [10] who proposed to compress an arbitrary memoryless source using i.i.d. Gaussian codebooks with minimum Euclidean distance encoding under the quadratic distortion measure for the rate-distortion problem, i.e.,the first layer of successive refinement in Fig. 1. Specifically, Lapidoth [10, Theorem 3] showed that for any ergodic source with finite second moment σ^2 , the minimal compression rate that guarantees a vanishing excess-distortion probability is exactly the rate-distortion function for a GMS. Recently, Zhou, Tan and Motani [11] refined

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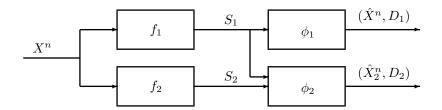


Fig. 1. System Model for Successive Refinement.

Lapidoth's results by deriving ensemble tight refined asymptotics for the same setting using both i.i.d. Gaussian codebooks and spherical codebooks.

One might wonder whether it is possible to generalize the the results in [10], [11] to the successive refinement problem. Specifically, can one propose a coding scheme to compress an arbitrary memoryless source using Gaussian codebooks for the successive refinement problem and ensure universal good performance under the quadratic distortion measure? In this paper, we answer this question affirmatively. Our main contributions are summarized as follows.

A. Main Contributions

We propose to use Gaussian codebooks to compress an arbitrary memoryless source with successive minimum Euclidean distance encoding under the quadratic distortion measure. Specifically, we consider two types of codebooks: the spherical codebook and the i.i.d. Gaussian codebook. The codewords of a spherical codebook are generated independently and uniformly from the surface of a sphere while each codeword of an i.i.d. Gaussian codebook is drawn independently from a product Gaussian distribution. We consider two types of performance criterion: ensemble JEP and ensemble SEP. Similar to [10], [11], by "ensemble", we mean that the probability is calculated not only with respect to the distribution of the source but also over the random codebooks. Under the above setting, we derive achievable refined asymptotics under both ensemble JEP and SEP performance criteria.

We first derive achievable second-order and moderate deviations asymptotics for both spherical and i.i.d. Gaussian codebooks. Our results complement existing literature [7], [9]–[11]. Specifically, our results generalize the refined asymptotics analyses of the mismatched rate-distortion problem [11] to the more complicated successive refinement setting. Similar to [11], we find that both spherical and i.i.d. Gaussian codebooks achieve the same second-order and moderate deviations asymptotics under both JEP and SEP. By letting the blocklength tend to infinity, a corollary of our result states that asymptotically the achievable rate-distortion region for any memoryless source under our mismatched coding scheme equals the rate-distortion region of the GMS in the matched successive refinement problem, which generalizes the classical result of Lapidoth [10, Theorem 3] to the successive refinement setting. Furthermore, our results generalize [9] and [7] for the matched successive refinement problem to the mismatched case. Under SEP, we generalize the concept "strongly successively refinable [9, Definition 4]" from the matched case with perfect knowledge of the source distribution to the mismatched case. We find that any memoryless source is strongly successively refinable under mild conditions using our mismatched coding scheme (cf. Theorem 2). Finally, when specializing to a GMS, our results for second-order asymptotics under JEP and SEP provide alternative achievability proofs with explicit code design for the results in [7, Theorem 20] and [9, Theorem 7], respectively.

We next derive achievable large deviations asymptotics for the i.i.d. Gaussian codebook. We do not consider spherical codebook in this regime because Zhou, Tan and Motani [11, Lemma 4] showed that i.i.d. Gaussian codebook has strictly larger excess-distortion exponent than the spherical codebook for the mismatched rate-distortion problem. We derive the achievable excess-distortion exponents of both JEP and SEP and show the exponents are all positive for any rate pair strictly inside the rate-distortion region of a GMS in the matched case. Note that in our conference version [2], we used a mismatched coding scheme where the power of each codeword is invariant of the coding rates and not able to ensure positive exponent under either SEP or JEP for all rate pairs inside the rate-distortion region. In this paper, we manage to solve the problem by slightly changing the design of the mismatched coding scheme and allowing the number of codewords of both encoders to be a function of their rates (see Section III-C for detailed discussions.) To our best knowledge, our result is the first achievable large deviations analysis for a continuous memoryless source for the successive refinement problem, whose matched counterpart is not even available for a GMS. Furthermore, our results generalize [11, Theorem 3] for the i.i.d. Gaussian codebook to the successive refinement setting and generalize [4], [8] to the mismatched case.

B. Other Related Works

We briefly summarize other works on matched successive refinement and mismatched lossy source coding. Effros [12] generalized Rimoldi's characterization of the rate-distortion region [3] to the case of discrete stationary ergodic and non-ergodic sources. Kostina and Tuncel [13] studied successive refinement for abstract sources and derived non-asymptotic converse

bounds that are tight for second-order asymptotics under both JEP and SEP criteria. Dembo and Kontoviannis [14] derived the asymptotic coding rate for mismatched compression using a lossy version of the asymptotic equipartition property. Kontoyiannis and Zamir [15] applied variable-length source coding, also known as entropy coding, to mismatched compression and proved that asymptotically the rate-distortion function can be achieved with a slight loss for any source with unknown distribution. Zhou, Tan and Motani [16] proposed a mismatched joint source channel coding scheme to transmit an arbitrary memoryless source over an additive arbitrary noise channel and derived the ensemble tight results in second-order and moderate deviation asymptotics for both spherical codebooks and i.i.d. Gaussian codebooks. Different from the mismatched setting considered in this paper where the source distribution is unknown, Lapidoth also considered another case of the mismatched setting for lossy source coding, where the encoder and decoder use different distortion measures [10, Theorem 1]. Kanabar and Scarlett [17] revisited the problem in [10, Theorem 1] and derived the achievable rate-distortion function using superposition coding.

C. Organization of the Rest of the Paper

The rest of the paper is organized as follows. In Section II, we set up the notation, formulate the problem and present necessary definitions. In Section III, we present our achievable refined asymptotic results under both JEP and SEP and discuss the significance of our results. The proofs of our results are presented in Sections IV to VI. Finally, in Section VII, we conclude the paper and discuss future research directions. For smooth presentation, we defer the proofs of all supporting lemmas to the appendices.

II. PROBLEM FORMULATION

Notation

Random variables are in capital (e.g., X) and their realizations are in lower case (e.g., x). Random vectors of length nand their particular realizations are denoted as $X^n := (X_1, \dots, X_n)$ and $x^n = (x_1, \dots, x_n)$, respectively. We use calligraphic font (e.g., \mathcal{X}) to denote all sets. We use \mathbb{R} , \mathbb{R}_+ , \mathbb{N} to denote the set of real numbers, positive real numbers and integers respectively. For any two integers $(a,b) \in \mathbb{N}^2$, we use [a:b] to denote the set of integers between a and b, and we use [a] to denote [1:a]. We use logarithm with base e. Gaussian complementary cumulative distribution function (cdf) and its inverse are denoted as $Q(\cdot)$ and $Q^{-1}(\cdot)$, respectively. We use $\|x^n\| = \sqrt{\sum_i x_i^2}$ to denote the ℓ_2 norm of a vector $x^n \in \mathbb{R}^n$. The quadratic distortion measure for any two sequences $(x^n, y^n) \in (\mathbb{R}^n)^2$ is defined as $d(x^n, y^n) := \frac{1}{n} \|x^n - y^n\|^2 = \frac{1}{n} \sum_{i \in [n]} (x_i - y_i)^2$. We use $1\{\cdot\}$ to denote the indicator function, i.e., $1\{A\} = 1$ if A is true and otherwise $1\{A\} = 0$. For two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ if $\lim_{n \to \infty} \frac{1}{n} \log \frac{a_n}{b_n} = 0$. Given any real number $t \in \mathbb{R}$, we use $\Lambda_X(t)$ to denote its cumulant generating function and use $\Lambda_X^*(t)$ to denote the Fenchel-Legendre transform of the cumulant generating function of the random variable X, i.e., $\Lambda_X(t) = \log \mathsf{E}[\exp(tX)]$ and $\Lambda_X^*(t) = \sup_{\theta > 0} \{\theta t - \Lambda_X(\theta)\}$. Finally, we use $|a|^+$ to denote $\max\{0, a\}.$

A. Problem Formulation

Consider a memoryless source with distribution P_X^{-1} defined on an alphabet \mathcal{X} satisfying the moment constraint

$$\mathsf{E}[X^2] = \sigma^2. \tag{1}$$

In this paper, we generalize the rate-distortion saddle-point problem [10, Theorem 3] to the successive refinement setting [3]. Specifically, we study the successive refinement problem where one is constrained to use Gaussian codebooks (either spherical or i.i.d.) and the minimal Euclidean distance encoding to compress an arbitrary memoryless source X^n under the quadratic distortion measure. Consistent with [10], [11], we name our problem as mismatched successive refinement. Without loss of generality, we assume that the desired distortion levels at the two decoders satisfy $\sigma^2 > D_1 > D_2$. A definition of a code is as follows.

Definition 1. An (n, M_1, M_2) -code for the mismatched successive refinement problem consists of:

- a set of M_1 codewords $\{Y^n(i)\}_{i\in[M_1]}$ known by encoders (f_1, f_2) and decoders (ϕ_1, ϕ_2) ,
- a set of M_2 codewords $\{Z^n(i,j)\}_{j\in[M_2]}$ known by the encoder f_2 and the decoder ϕ_2 for each $i\in[M_1]$,
- \bullet two encoders f_1 and f_2 that use successive minimum Euclidean distance encoding to compresses the source sequence X^n , i.e.,

$$f_1(X^n) := \underset{i \in [M_1]}{\arg \min} d(X^n, Y^n(i)),$$
 (2)

$$f_1(X^n) := \underset{i \in [M_1]}{\arg \min} d(X^n, Y^n(i)),$$

$$f_2(X^n) := \underset{j \in [M_2]}{\arg \min} d(X^n, Z^n(f_1(X^n), j)),$$
(3)

¹The distribution is either discrete or continuous. For a discrete source, P_X is a probability mass function. For a continuous source, P_X is a probability density function.

• two decoders ϕ_1 and ϕ_2 that operate as follows:

$$\phi_1(f_1(X^n)) = Y^n(f_1(X^n)), \tag{4}$$

$$\phi_2(f_1(X^n), f_2(X^n)) = Z^n(f_1(X^n), f_2(X^n)). \tag{5}$$

For simplicity, throughout the paper, we use \hat{X}_1^n and $Y^n(f_1(X^n))$ interchangeably to denote the reproduced source sequence of the first decoder ϕ_1 , and similarly we use \hat{X}_2^n and $Z^n(f_1(X^n), f_2(X^n))$ interchangeably.

To specify the codewords in Definition 1, we need to define the following distributions that are functions of the two parameters: a vector $c^n = (c_1, \dots, c_n) \in \mathbb{R}^n$ and a positive real number $P \in \mathbb{R}_+$.

• We first define a uniform distribution over the surface of a sphere with center c^n and radius \sqrt{nP} , i.e. for any $u^n = (u_1, \dots, u_n) \in \mathbb{R}^n$,

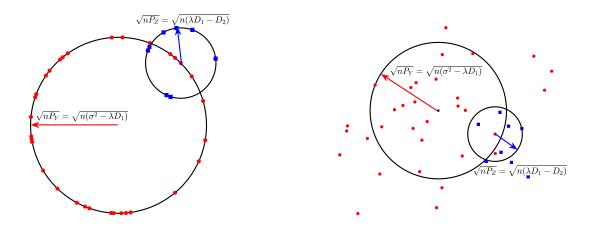
$$f_{\rm sp}(u^n|c^n, P) = \frac{1\{\|u^n - c^n\|^2 - nP\}}{S_n(\sqrt{nP})},\tag{6}$$

where $1\{\cdot\}$ is the indicator function, $S_n(r) = n\pi^{n/2}r^{n-1}/\Gamma(\frac{n+2}{2})$ is the surface area of an n-dimensional sphere with radius r, and $\Gamma(\cdot)$ is the Gamma function.

• We also define a product Gaussian distribution, i.e., for any $u^n=(u_1,\ldots,u_n)\in\mathbb{R}^n$,

$$f_{\text{iid}}(u^n|c^n, P) = \prod_{i \in [n]} \frac{\exp\left(-\frac{(u_i - c_i)^2}{2P}\right)}{\sqrt{2\pi P}}.$$
(7)

Next we specify the codebooks used in our mismatched coding scheme. Let $\lambda \in \mathbb{R}_+$ be a design parameter to be specified, $P_Y := \sigma^2 - \lambda D_1$ and $P_Z := \lambda D_1 - D_2$. Similar to [11], we consider both spherical and i.i.d. Gaussian codebooks. Since both encoders can use either spherical or i.i.d. Gaussian codebooks, there are in total four different combinations of the codebooks. For ease of notation, throughout the paper, we use \dagger to denote the type of the codebook used by encoder f_1 and \ddagger to denote the type of the codebook used by encoder f_2 . Note that $(\dagger, \ddagger) \in \{\text{sp, iid}\}^2$ where "sp" is short for "spherical". Specifically, for any $(\dagger, \ddagger) \in \{\text{sp, iid}\}^2$, encoder f_1 uses a random codebook with M_1 independent codewords $(Y^n(1), \dots, Y^n(M_1))$, where each codeword $Y^n(i)$ is generated according to the distribution $f_\dagger(Y^n(i)|\mathbf{0}^n, P_Y)$ where $\mathbf{0}^n$ denotes the length-n vector with all elements of 0; for each $i \in [M_1]$ and $Y^n(i)$, encoder f_2 uses a random codebook with M_2 independent codewords $(Z^n(i,1),\dots,Z^n(i,M_2))$, where each codeword $Z^n(i,j)$ is generated according to the distribution $f_\ddagger(Z^n(i,j)|Y^n(i),P_Z)$. For n=2, we illustrate encoders' codebooks in Fig. 2.



(a) Both encoders use spherical codebooks

(b) Both encoders use i.i.d. Gaussian codebooks

Fig. 2. Illustration of codewords when both encoders use the same kind of codebooks. The red dots denote the codewords of encoder f_1 and the blue squares denote the codewords of encoder f_2 . In a spherical codebook, the codewords of f_1 distribute uniformly over the surface of the sphere with center of the origin and the radius of $\sqrt{nP_Y} = \sqrt{n(\sigma^2 - \lambda D_1)}$. When $f_1(X^n) = i$, given the codeword $Y^n(i)$, the codewords for f_2 distribute uniformly over the surface of the sphere with center $Y^n(i)$ and the radius $\sqrt{nP_Z} = \sqrt{n(\lambda D_1 - D_2)}$. In an i.i.d. Gaussian codebook, the codewords of f_1 are generated independently and each codeword is generated i.i.d. from the Gaussian distribution with mean 0 and variance $\sigma^2 - \lambda D_1$. When $f_1(X^n) = i$, given the codeword $Y^n(i)$, the codewords for encoder f_2 are generated similarly from a Gaussian product distribution.

To evaluate the performance of the mismatched coding scheme, we consider the ensemble separate excess-distortion probabilities:

$$P_{\dagger}^{n}(D_{1}|M_{1},M_{2}) := \Pr\{d(X^{n},\hat{X}_{1}^{n}) > D_{1}\},\tag{8}$$

$$P_{\dagger}^{n}(D_{2}|M_{1},M_{2}) := \Pr\{d(X^{n},\hat{X}_{2}^{n}) > D_{2}\},\tag{9}$$

and the ensemble joint-excess-distortion probability:

$$P_{\dagger,\pm}^n(D_1, D_2|M_1, M_2) := \Pr\{d(X^n, \hat{X}_1^n) > D_1 \text{ or } d(X^n, \hat{X}_2^n) > D_2\}.$$
(10)

Note that the probability terms average not only over the distribution of the source sequence X^n , but also over the random codebooks. These definitions are consistent with existing works on mismatched communication [10], [11], [18], [19].

B. The Rate-distortion Region

The rate-distortion region collects rate pairs of both encoders with which our mismatched code in Definition 1 ensures vanishing ensemble joint excess-distortion probability and vanishing ensemble separate excess-resolution probabilities. In the following, we define the rate-distortion region under JEP.

Definition 2. A rate pair $(R_1, R_2) \in \mathbb{R}^2_+$ is said to be $(D_1, D_2|\dagger, \ddagger)$ -achievable for mismatched successive refinement if there exists a sequence of (n, M_1, M_2) -codes using (\dagger, \ddagger) codebooks such that

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 \le R_1,\tag{11}$$

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$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 M_2 \le R_1 + R_2, \tag{12}$$

and

$$\lim_{n \to \infty} P_{\dagger, \ddagger}^n(D_1, D_2 | M_1, M_2) = 0.$$
 (13)

The convex closure of the set of all $(D_1, D_2|\dagger, \dagger)$ -achievable rate pairs is called the rate-distortion region and denoted as $\mathcal{R}_{\dagger,\dagger}(D_1,D_2).$

As a corollary of our result in Theorem 1 by letting $n \to \infty$, one has the following inner bound to $\mathcal{R}_{\dagger,\dot{z}}(D_1,D_2)$:

$$\mathcal{R}_{\text{inner}} := \left\{ (R_1, R_2) : \ R_1 \ge \frac{1}{2} \log \frac{\sigma^2}{D_1}, \ R_1 + R_2 \ge \frac{1}{2} \log \frac{\sigma^2}{D_2} \right\} \subseteq \mathcal{R}_{\dagger, \ddagger}(D_1, D_2). \tag{14}$$

Since $P_{\dagger,\dagger}^n(D_1,D_2|M_1,M_2) \geq \max_{i \in [2]} P_{\dagger}^n(D_i|M_1,M_2)$, the same inner bound holds under SEP. Note that \mathcal{R}_{inner} is the ratedistortion region for a GMS under the quadratic distortion measure in the matched successive refinement problem [6], [7]. One might wonder whether the ensemble converse in [10, Theorem 3] implies the ensemble converse for mismatched successive refinement. Unfortunately, the answer doesn't directly follow since the codewords for the second encoder of mismatched successive refinement are not equivalent to the codewords for the encoder in mismatched rate-distortion, a figure illustration of which are available Fig. 3 and Fig. 4 where both encoders use either spherical or i.i.d. Gaussian codebooks. It is worthwhile future work to investigate whether the rate-distortion region $\mathcal{R}_{\mathrm{inner}}$ is ensemble tight.

C. Definitions of Fundamental Limits

In this paper, we interested in refined asymptotics that include the second-order, moderate and large deviations. These analyses reveal the tradeoff between the rates of encoders and the blocklength under different performance criteria beyond the rate-distortion region. In this section, we present explicit definitions of fundamental limits of refined asymptotics.

Fix (R_1^*, R_2^*) as an arbitrary rate pair on the boundary of the region $\mathcal{R}_{inner}(D_1, D_2)$. In second-order asymptotics under the JEP criterion, we characterize how a rate pair approaches (R_1^*, R_2^*) with respect to the blocklength when a non-vanishing JEP is tolerated.

Definition 3. Given any $\varepsilon \in (0,1)$, a pair $(L_1,L_2) \in \mathbb{R}^2_+$ is said to be second-order $(R_1^*,R_2^*,D_1,D_2,\varepsilon|\dagger,\ddagger)$ -achievable under JEP if there exists a sequence of (n, M_1, M_2) -codes in Definition 1 such that

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} (\log M_1 - nR_1^*) \le L_1, \tag{15}$$

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} (\log M_1 - nR_1^*) \le L_1,$$

$$\lim_{n \to \infty} \sup_{n \to \infty} \frac{1}{\sqrt{n}} (\log M_1 M_2 - nR_1^* - nR_2^*) \le L_2,$$
(15)

and

$$\limsup_{n \to \infty} P_{\dagger, \ddagger}^n(D_1, D_2 | M_1, M_2) \le \varepsilon. \tag{17}$$

The closure of the set of all second-order $(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \dagger)$ -achievable rate pairs is denoted as $\mathcal{L}(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \dagger)$.

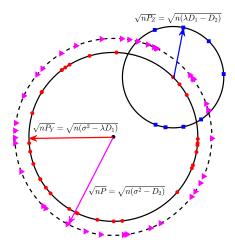


Fig. 3. Illustration of the spherical codebooks for n=2. The magenta triangle denote the codeword of encoder f, the red dots denote the codewords of encoder f_1 and the blue squares denote the codewords of encoder f_2 . In point-to-point source coding, the codewords are generated uniformly over the surface of the sphere denoted as dash circle with center of origin and the radius $\sqrt{nP} = \sqrt{n(\sigma^2 - D_2)}$. However, in successive refinement setting, the codewords of encoder f_1 are generated uniformly over the surface of the sphere with center of origin and the radius $\sqrt{nP_Y} = \sqrt{n(\sigma^2 - \lambda D_1)}$ and the codewords of encoder f_2 are generated uniformly over the surface of the sphere centered on the given codeword of encoder f_1 with radius $\sqrt{nP_Z} = \sqrt{n(\lambda D_1 - D_2)}$. Thus, the codewords of encoder f_2 are not uniformly distributed over the surface of the sphere, which makes it different from the spherical codebook used in the mismatched rate-distortion problem.

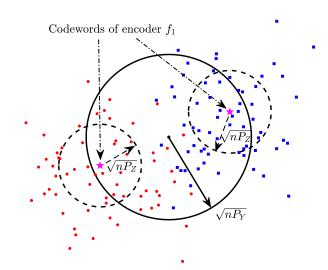


Fig. 4. Illustration of i.i.d. Gaussian codebooks for n=2. The magenta stars denote the codewords of encoder f_1 and the red dots and blue squares denote the codewords of encoder f_2 . Assume that $M_1=2$. The codewords $Y^n(1), Y^n(2)$ of encoder f_1 are generated i.i.d. from the Gaussian distribution with mean 0 and variance P_Y . Given each codeword $Y^n(i)$, the codewords $\{Z^n(i,j)\}_{j\in[M_2]}$ of encoder f_2 are generated i.i.d. from the Gaussian distribution with mean $Y^n(i)$ and variance P_Z . Although the marginal distribution of each codeword $Z^n(i,j)$ is Gaussian with mean 0 and variance $P_Y + P_Z = \sigma^2 - D_2$, the dependence of the codewords make it different from the i.i.d. Gaussian codebook in the mismatched rate-distortion problem.

An equivalent form of (15) and (16) are as follows,

$$\log M_1 \le nR_1^* + \sqrt{nL_1} + o(\sqrt{n}),\tag{18}$$

$$\log M_1 M_2 \le n(R_1^* + R_2^*) + \sqrt{nL_2} + o(\sqrt{n}), \tag{19}$$

where (L_1, L_2) are second-order rate pair.

Under SEP, we only define the second-order coding rate for the corner point of rate-distortion region, which corresponds to the minimal rates of both encoders. This is because the second-order coding rates are not well defined for other points on the boundary of the rate-distortion region for at least one encoder.

Definition 4. Given any $(\varepsilon_1, \varepsilon_2) \in (0, 1)^2$, a pair $(L_1, L_2) \in \mathbb{R}^2_+$ is said to be second-order $(D_1, D_2, \varepsilon_1, \varepsilon_2 | \dagger, \ddagger)$ -achievable under SEP if there exists a sequence of (n, M_1, M_2) -codes such that

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left(\log M_1 - \frac{n}{2} \log \frac{\sigma^2}{D_1} \right) \le L_1, \tag{20}$$

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \left(\log M_1 M_2 - \frac{n}{2} \log \frac{\sigma^2}{D_2} \right) \le L_2, \tag{21}$$

and

$$\limsup_{n \to \infty} P_{\dagger, \ddagger}^{n}(D_1 | M_1, M_2) \le \varepsilon_1,$$

$$\limsup_{n \to \infty} P_{\dagger, \ddagger}^{n}(D_2 | M_1, M_2) \le \varepsilon_2.$$
(22)

$$\limsup_{n \to \infty} P_{\dagger, \ddagger}^n(D_2 | M_1, M_2) \le \varepsilon_2. \tag{23}$$

The closure of the set of all second-order $(D_1, D_2, \varepsilon_1, \varepsilon_2 | \dagger, \dagger)$ -achievable rate pairs is called the $(D_1, D_2, \varepsilon_1, \varepsilon_2 | \dagger, \dagger)$ -achievable rate region and denoted as $\mathcal{L}_{SEP}(D_1, D_2, \varepsilon_1, \varepsilon_2 | \dagger, \ddagger)$.

In the moderate deviations regime, we are interested in a sequence of (n, M_1, M_2) -codes whose rates approach a boundary rate pair (R_1^*, R_2^*) and whose excess-distortion probabilities vanish simultaneously. The definitions of the moderate deviations constants under JEP and SEP are given as follows.

Let θ_i , $i \in [2]$ be two positive real numbers. Consider any sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that

$$\rho_n \to 0 \text{ and } \sqrt{n}\rho_n \to \infty \text{ as } n \to \infty.$$
 (24)

Definition 5. A number $v \in \mathbb{R}_+$ is said to be a $(R_1^*, R_2^*, D_1, D_2, \theta_1, \theta_2 | \dagger, \ddagger)$ -achievable moderate deviations constant under JEP if there exists a sequence of (n, M_1, M_2) -codes such that

$$\limsup_{n \to \infty} \frac{1}{n\rho_n} \left(\log M_1 - nR_1^* \right) \le \theta_1, \tag{25}$$

$$\lim \sup_{n \to \infty} \frac{1}{n\rho_n} \left(\log M_1 - nR_1^* \right) \le \theta_1,$$

$$\lim \sup_{n \to \infty} \frac{1}{n\rho_n} \left(\log M_1 M_2 - n(R_1^* + R_2^*) \right) \le \theta_1 + \theta_2,$$
(25)

and

$$\liminf_{n \to \infty} -\frac{\log \mathcal{P}_{\dagger, \ddagger}^n \left(D_1, D_2 \middle| M_1, M_2\right)}{n\rho_n^2} \ge v.$$
(27)

The supremum of all $(R_1^*, R_2^*, D_1, D_2, \theta_1, \theta_2|\dagger, \dagger)$ -achievable moderate deviations constants is denoted as $v_{\dagger,\dagger}^*(D_1, D_2|R_1^*, R_2^*, \theta_1, \theta_2).$

Definition 6. A pair $(v_1, v_2) \in \mathbb{R}^2_+$ is said to be a $(D_1, D_2, \theta_1, \theta_2 | \dagger, \ddagger)$ -achievable moderate deviations constant pair under SEP if there exists a sequence of (n, M_1, M_2) -codes such that

$$\limsup_{n \to \infty} \frac{1}{n\rho_n} \left(\log M_1 - \frac{n}{2} \log \frac{\sigma^2}{D_1} \right) \le \theta_1, \tag{28}$$

$$\limsup_{n \to \infty} \frac{1}{n\rho_n} \left(\log M_1 M_2 - \frac{n}{2} \log \frac{\sigma^2}{D_2} \right) \le \theta_1 + \theta_2, \tag{29}$$

and

$$\lim_{n \to \infty} \inf \left(-\frac{\log \mathcal{P}_{\uparrow, \ddagger}^n \left(D_1 | M_1, M_2 \right)}{n \rho_n^2} \ge v_1, \right)$$

$$\lim_{n \to \infty} \inf \left(-\frac{\log \mathcal{P}_{\uparrow, \ddagger}^n \left(D_2 | M_1, M_2 \right)}{n \rho_n^2} \ge v_2. \right)$$
(30)

$$\liminf_{n \to \infty} -\frac{\log \mathcal{P}_{\dagger, \ddagger}^n \left(D_2 | M_1, M_2\right)}{n\rho_n^2} \ge v_2.$$
(31)

The convex closure of the set of all $(D_1, D_2, \theta_1, \theta_2 | \uparrow, \ddagger)$ -achievable moderate deviations constant pairs is called the $(D_1, D_2, \theta_1, \theta_2|\dagger, \ddagger)$ -achievable moderate deviations constant region and denoted as $\mathcal{V}(D_1, D_2, \theta_1, \theta_2|\dagger, \ddagger)$.

We next define the exponents of ensemble JEP and SEP when both encoders use i.i.d. Gaussian codebooks.

Definition 7. A number $E \in \mathbb{R}_+$ is said to be a (D_1, D_2, R_1, R_2) -achievable exponent under JEP if there exists a sequence of (n, M_1, M_2) -codes such that

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 \le R_1, \tag{32}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 M_2 \le R_1 + R_2, \tag{33}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 M_2 \le R_1 + R_2, \tag{33}$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log P_{\text{iid}}^n(D_1, D_2 | M_1, M_2) \ge E.$$
(34)

The supremum of all (D_1, D_2, R_1, R_2) achievable exponents is denoted as $E^*(D_1, D_2|R_1, R_2)$.

Definition 8. A pair $(E_1, E_2) \in \mathbb{R}^2_+$ is said to be (D_1, D_2, R_1, R_2) -achievable exponents under SEP if there exists a sequence of (n, M_1, M_2) -codes such that

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 \le R_1,\tag{35}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 \le R_1, \tag{35}$$

$$\limsup_{n \to \infty} \frac{1}{n} \log M_1 M_2 \le R_1 + R_2, \tag{36}$$

and

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mathcal{P}_{\text{iid}}^n(D_1|M_1, M_2) \ge E_1,$$
(37)

$$\liminf_{n \to \infty} -\frac{1}{n} \log P_{\text{iid}}^n(D_2|M_1, M_2) \ge E_2.$$
(38)

The convex closure of the set of all (D_1, D_2, R_1, R_2) -achievable exponents is denoted as $\mathcal{E}(D_1, D_2, R_1, R_2)$.

III. MAIN RESULTS

In this section, we present our main results concerning the achievability analyses of second-order, moderate and large deviations asymptotics under both JEP and SEP. We consider all four combinations of spherical and i.i.d. Gaussian codebooks for both second-order and moderate deviations analyses and study large deviations only when both encoders use i.i.d. Gaussian codebooks.

A. Second-Order Asymptotics

Consider an arbitrary memoryless source that satisfies the moment constraint in (1) and two additional moment constraints:

$$\mathsf{E}[X^4] =: \zeta,\tag{39}$$

$$\mathsf{E}[X^6] < \infty. \tag{40}$$

Recall the definition of the "mismatched" dispersion in [11]:

$$V(\sigma^{2}, \zeta) := \frac{\zeta - \sigma^{4}}{4\sigma^{4}} = \frac{Var[X^{2}]}{4(E[X^{2}]^{2})}.$$
(41)

We first present the result under JEP.

Theorem 1. Given any $\varepsilon \in (0,1)$, for any $\lambda \in (\frac{D_2}{D_1},1]$ and $(\dagger,\dagger) \in \{\text{sp},\text{iid}\}^2$, there exists a sequence of (n,M_1,M_2) -codes using Gaussian codebooks such that

$$\log M_1 = \frac{n}{2} \log \frac{\sigma^2}{\lambda D_1} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n), \tag{42}$$

$$\log M_1 M_2 = \frac{n}{2} \log \frac{\sigma^2}{D_2} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n), \tag{43}$$

and

$$\lim_{n \to \infty} P_{\dagger, \ddagger}^n(D_1, D_2 | M_1, M_2) \le \varepsilon. \tag{44}$$

The proof of Theorem 1 is provided in Section IV. A few remarks are in order.

Theorem 1 generalizes the achievability results in [11, Theorem 1] for the mismatched rate-distortion problem to the successive refinement setting. For any $\lambda \in (\frac{D_2}{D_1},1]$, when the target rate pair is $(R_1^*,R_2^*)=(\frac{1}{2}\log\frac{\sigma^2}{\lambda D_1},\frac{1}{2}\log\frac{\lambda D_1}{D_2})$, the pair $(L_1,L_2)=(\sqrt{V(\sigma^2,\zeta)}Q^{-1}(\varepsilon),\sqrt{V(\sigma^2,\zeta)}Q^{-1}(\varepsilon))$ is second-order $(R_1^*,R_2^*,D_1,D_2,\varepsilon|\dagger,\ddagger)$ -achievable (cf. Definition 3). This result generalizes [11, Theorem 1] by showing that all four combinations of spherical and i.i.d. Gaussian codebooks achieve the same second-order asymptotics.

At the first glance, Theorem 1 is not in the same form when presenting second-order asymptotics as [7, Theorem 20] for a GMS in the matched case. However, with proper choice of the parameter λ in Theorem 1, we have the following inner bound to the second-order coding region $\mathcal{L}(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \ddagger)$.

• Case (i): Let $\lambda \in (\frac{D_2}{D_1}, 1)$, $R_1^* = \frac{1}{2} \log \frac{\sigma^2}{\lambda D_1} > \frac{1}{2} \log \frac{\sigma^2}{D_1}$, $R_1^* + R_2^* = \frac{1}{2} \log \frac{\sigma^2}{D_2}$. It follows that

$$\left\{ (L_1, L_2) : L_2 \ge \sqrt{V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) \right\} \subseteq \mathcal{L}(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \ddagger). \tag{45}$$

• Case (ii): Let $\lambda=1,\ R_1^*=\frac{1}{2}\log\frac{\sigma^2}{D_1}$ and $R_1^*+R_2^*>\frac{1}{2}\log\frac{\sigma^2}{D_2}$. It follows that

$$\left\{ (L_1, L_2) : L_1 \ge \sqrt{V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) \right\} \subseteq \mathcal{L}(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \ddagger). \tag{46}$$

• Case (iii): Let $\lambda=1$, $R_1^*=\frac{1}{2}\log\frac{\sigma^2}{D_1}$ and $R_1^*+R_2^*=\frac{1}{2}\log\frac{\sigma^2}{D_2}$. It follows that

$$\left\{ (L_1, L_2) : \min\{L_1, L_2\} \ge \sqrt{V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) \right\} \subseteq \mathcal{L}(R_1^*, R_2^*, D_1, D_2, \varepsilon | \dagger, \ddagger). \tag{47}$$

Recall that the power of each codeword for encoder f_1 and encoder f_2 are $P_Y = \sigma^2 - \lambda D_1$ and $P_Z = \lambda D_1 - D_2$, respectively. If one uses a codebook with $\lambda = 1$ as done in [1], we could only prove the results for cases (ii) and (iii). When $\lambda < 1$, the power of each codeword for encoder f_1 increases and the power of each codewords for encoder f_2 decreases. In the proof of Theorem 1, we show that $R_1^* = \frac{1}{2}\log\frac{\sigma^2}{\lambda D_1} > \frac{1}{2}\log\frac{\sigma^2}{D_1}$ and $R_2^* = \frac{1}{2}\log\frac{\lambda D_1}{D_2} < \frac{1}{2}\log\frac{D_1}{D_2}$ is asymptotically achievable, as a consequence of our refined second-order asymptotics. By introducing the parameter $\lambda \in (\frac{D_2}{D_1}, 1]$, we manage to cover all boundary points of the rate-distortion region, which allows us to derive achievable second-order asymptotics for all three cases of interest, especially for case (i) when the rate of encoder f_1 is larger than $\frac{1}{2}\log\frac{\sigma^2}{D_1}$.

When specialized to a GMS, the mismatched dispersion satisfies $V(\sigma^2, \zeta) = \frac{1}{2}$. This implies that our proof of Theorem 1 is an

When specialized to a GMS, the mismatched dispersion satisfies $V(\sigma^2, \zeta) = \frac{1}{2}$. This implies that our proof of Theorem 1 is an alternative achievability proof for [7, Theorem 20] with specific code design. Specifically, the achievability proof in [7, Theorem 20] used a covering lemma in [20] that bounds the minimal number of required codewords to cover a ball *without* providing the location of each codeword. In contrast, our proof of Theorem 1 specifies that the same second-order asymptotic performance can be achieved with an arbitrary combination of spherical and i.i.d. Gaussian codebooks using successive minimum Euclidean distance encoding.

It is a pity that we could not derive the ensemble converse result. The intuition is as follows. When either encoder f_1 or f_2 uses a spherical codebook, the codewords for the second pairs of encoder and decoder are not spherically symmetric, which makes it difficult to analyze the excess-distortion probability involving D_2 . When both encoders used i.i.d. Gaussian codebooks, although the codeword distribution for the second pairs of encoder and decoder are i.i.d. Gaussian, with the parameter of $\lambda < 1$, the target distortion for the first encoder-decoder pair is D_1 while the best achievable distortion from our codebook design is λD_1 . This additional mismatch between the target distortion level and the achievable distortion level for the first pair of encoder and decoder prevented us from obtaining ensemble tight results for M_1 . Thus, the ensemble converse for mismatched successive refinement remains challenging and left as future work.

Note that when we ignore the first pair of encoder and decoder, the mismatched successive refinement problem considered in this paper reduces to a mismatched rate-distortion problem with superposition coding. It was recently shown in [17, Lemma 4] by Kanabar and Scarlett that superposition coding could improve the compression rate for the "mismatched" rate-distortion problem where the encoder and the decoder use different distortion measures. It would be of worthwhile to investigate whether superposition coding could improve the refined asymptotics of the mismatched rate-distortion problem in [11].

We next present the result under SEP.

Theorem 2. Given any $\varepsilon_1, \varepsilon_2 \in (0,1)^2$, for any $(\dagger, \ddagger) \in \{\text{sp}, \text{iid}\}^2$,

$$\left\{ (L_1, L_2) : L_1 \ge \sqrt{V(\sigma^2, \zeta)} Q^{-1} \left(\min\{\varepsilon_1, \varepsilon_2\} \right), L_2 \ge \sqrt{V(\sigma^2, \zeta)} Q^{-1} \left(\min\{\varepsilon_1, \varepsilon_2\} \right) \right\} \subseteq \mathcal{L}_{SEP}(D_1, D_2, \varepsilon_1, \varepsilon_2 | \dagger, \ddagger). \quad (48)$$

The proof of Theorem 2 follows from the proof of Theorem 1 with $\lambda = 1$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$.

A few remarks are as follows. Note that the minimum of the tolerable excess-distortion probabilities $(\varepsilon_1, \varepsilon_2)$ dominates the second-order achievable rate of both encoders. This is because under our mismatched coding scheme, only a non-excess-distortion event of decoder ϕ_1 could guarantee a non-excess-distortion event of decoder ϕ_2 with proper rate choices due to our layered coding scheme.

Theorem 2 generalizes the achievability part of the second-order asymptotic results in [9] for the matched case to the mismatched scenario. When specialized to a GMS, we provide an alternative proof for [9, Theorem 7] by constructing a structured codebook with specific code design. It is important to note that the results in [9, Theorem 7] holds also when $\min\{\varepsilon_1, \varepsilon_2\}$ dominates the second-order coding rate since a similar layered coding scheme based on sphere covering was used².

²We remark that the claim in [9, Theorem 7] was flawed since ε_1 and ε_2 were used as the parameter of the $Q^{-1}(\cdot)$ function in the second-order coding rates. This is because, in order not to incur an excess-distortion event at decoder ϕ_2 for a sequence x^n , decoder ϕ_1 should not incur an excess-distortion event since otherwise, the "correct" decoding of decoder ϕ_2 is not guaranteed under the layered coding scheme.

Theorem 2 implies that when $\varepsilon_1 = \varepsilon_2 = \varepsilon$ for some $\varepsilon \in (0,1)$, any memoryless source satisfying (1) is strongly successive refinable [9, Definition 4] under the quadratic distortion measure using our mismatched coding scheme. Following the proof of Theorem 1 and letting $\lambda = 1$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that there exists a sequence of (n, M_1, M_2) codes such that

$$\log M_1 = \frac{n}{2} \log \frac{\sigma^2}{D_1} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n), \tag{49}$$

$$\log M_1 M_2 = \frac{n}{2} \log \frac{\sigma^2}{D_2} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n), \tag{50}$$

and

$$\lim_{n \to \infty} P_{\dagger}^{n}(D_{1}|M_{1}) \le \varepsilon, \tag{51}$$

$$\lim_{n \to \infty} P_{\dagger}^{n}(D_{1}|M_{1}) \leq \varepsilon, \tag{51}$$

$$\lim_{n \to \infty} P_{\dagger}^{n}(D_{2}|M_{1}, M_{2}) \leq \varepsilon. \tag{52}$$

Such a result is named strongly successive refinable since the sum rate in (50) is precisely the rate required to achieve (52) even without the first layer of encoders and decoders, up to second-order asymptotics.

B. Moderate Deviation Asymptotics

Consider an arbitrary memoryless source with distribution P_X satisfying (1) and (39) such that i) $\Lambda_{X^2}(\theta)$ is finite for some positive number θ and ii) the mismatched dispersion $V(\sigma^2, \zeta)$ is finite.

Theorem 3. For any rate pair (R_1^*, R_2^*) on the boundary of \mathcal{R}_{inner} , any $(\dagger, \ddagger) \in \{sp, iid\}^2$ and any real numbers $\lambda \in (\frac{D_2}{D_1}, 1]$, $(\theta_1, \theta_2) \in \mathbb{R}^2_+$,

$$v_{\dagger,\ddagger}^*(D_1, D_2 | R_1^*, R_2^*, \theta_1, \theta_2) \ge \frac{\theta_1^2}{2V(\sigma^2, \zeta)}.$$
 (53)

The proof of Theorem 3 is provided in Sections V-B and V-C. A few remarks are as follows.

We explain why $v_{\dagger,\pm}^*(D_1,D_2|R_1^*,R_2^*,\theta_1,\theta_2)$ depends solely on θ_1 and not θ_2 . We remark that θ_1 is the speed at which $\log M_1$ deviates from R_1^* and θ_2 is the speed at which $\log M_2$ deviates from R_2^* (cf. Definition 5). The dominant excess-distortion probability term related to M_1 scales as $\exp\left\{-\frac{n\theta_1^2\rho_n^2}{2V(\sigma^2,\zeta)} + o(n\theta_1^2\rho_n^2)\right\}$ (cf. (145)-(146)) while the dominant excess-distortion probability term related to M_2 scales $\exp\{-n\theta_2\rho_n + o(\rho_n)\}$ (cf. (147)-(148)). Thus, $v_{\dagger,\dagger}^*(D_1,D_2|R_1^*,R_2^*,\theta_1,\theta_2)$ is naturally dominant by θ_1 .

When specialized to a GMS, $V(\sigma^2,\zeta)=\frac{1}{2}$. Our results in Theorem 3 recover the results of [7, Theorem 21, Cases (ii) and (iii)]. The reason why we could not recover the optimal moderate deviations constant for a GMS in Case (i) where $R_1^* \in \left(\frac{1}{2}\log\frac{\sigma^2}{D_1}, \frac{1}{2}\log\frac{\sigma^2}{D_2}\right)$ and $R_2^* = \frac{1}{2}\log\frac{\sigma^2}{D_2}$ is that under our mismatched coding scheme, the dominant error event is the excess-distortion event at decoder ϕ_1 regardless of the rates of both encoders.

We next present the achievable moderate deviations constants under SEP.

Theorem 4. For any $(\dagger, \ddagger) \in \{\text{sp}, \text{iid}\}^2$, $(\theta_1, \theta_2) \in \mathbb{R}^2_+$ and positive $V(\sigma^2, \zeta)$, the achievable moderate deviations constants under SEP satisfy

$$\left\{ (v_1, v_2) : v_1 \ge \frac{\theta_1^2}{2V(\sigma^2, \zeta)}, v_2 \ge \frac{\theta_1^2}{2V(\sigma^2, \zeta)} \right\} \subseteq \mathcal{V}(D_1, D_2, \theta_1, \theta_2 | \dagger, \ddagger). \tag{54}$$

The proof of Theorem 4 is provided in Section V-E.

C. Large Deviation Asymptotics

We only derive the exponent of ensemble JEP and SEP when i.i.d. Gaussian codebooks are used by both encoders since the large deviations performance of an i.i.d. Gaussian codebook is strictly better than the spherical codebook in the mismatched ratedistortion problem (cf. [11, Lemma 4]). To present our results, we need the following definitions. Given any $(s, w, P, D) \in \mathbb{R}^4_+$, define the following functions

$$R_{\text{iid}}(s, w, P, D) := \frac{1}{2}\log(1+2s) + \frac{sw}{(1+2s)P} - \frac{sD}{P},\tag{55}$$

$$s^*(w, P, D) := \max \left\{ 0, \frac{P - 2D + \sqrt{P^2 + 4wD}}{4D} \right\},\tag{56}$$

$$R_{\text{iid}}(w, P, D) := R_{\text{iid}}(s^*(w, P, D), w, P, D).$$
(57)

We remark that $R_{iid}(w, P_Y, D_1)$ is the exponential decay rate of the non-excess-distortion probability with respect to the distortion level D_1 for a source sequence x^n with power $w = \frac{1}{n}||x^n||^2$ when its reproduction sequence is generated from $f_{\text{iid}}(Y^n|\mathbf{0}^n, P_Y)$ (cf. (7)), i.e.,

$$\lim_{n \to \infty} -\frac{1}{n} \log \Pr_{f_{\text{iid}}(Y^n | \mathbf{0}^n, P_Y)} \{ d(x^n, Y^n) \le D_1 \} = R_{\text{iid}}(w, P_Y, D_1).$$
 (58)

Furthermore, $R_{iid}(l, P_Z, D_2)$ is the the exponential decay rate of the non-excess-distortion probability with respect to the distortion level D_2 for any source sequence x^n whose quadratic distortion with respect to the output codeword \hat{x}_1^n of encoder f_1 is $l:=d(x^n,\hat{x}_1^n)$ when its reproduction sequence is generated from $f_{\rm iid}(Z^n|\hat{x}_1^n,P_Z)$, i.e.,

$$\lim_{n \to \infty} -\frac{1}{n} \log \Pr_{f_{\text{iid}}(Z^n | \hat{x}_1^n, P_Z)} \{ d(x^n, Z^n) \le D_2 \} = R_{\text{iid}}(l, P_Z, D_2).$$
 (59)

Recall that $\Lambda_{X^2}^*(t)$ is the Fenchel-Legendre transform of the cumulant generating function of X^2 , which is also known as the large deviation rate function [21, Chapter 2.3].

We first present the achievable exponent of ensemble JEP under our mismatched coding scheme.

Theorem 5. Given any $(R_1, R_2) \in \mathbb{R}^2_+$, let $\lambda = \min\left\{\frac{D_2 \exp\{2R_2\}}{D_1}, 1\right\}$ and let $\alpha^*(R_1, R_2)$ be the solution of α to $R_1 = R_{iid}(\alpha, P_Y, \lambda D_1)$. The joint excess-distortion exponent $E^*(D_1, D_2 | R_1, R_2)$ satisfies:

$$E^*(D_1, D_2|R_1, R_2) \ge \Lambda_{X^2}^*(\alpha^*(R_1, R_2)). \tag{60}$$

The proof of Theorem 5 is provided in Section VI-B, which generalizes the proof of [11, Theorem 3] for the rate-distortion problem to the successive refinement setting. We make the following remarks.

The joint excess-distortion exponent $\Lambda_{X^2}^*(\alpha^*(R_1,R_2))$ depends solely on R_1 if R_2 is large enough. Intuitively, this is because when the rate R_2 of encoder f_2 is large, the number of codewords used by f_2 is sufficient to D_2 -cover the ball with center \hat{X}_1^n and radius D_1 . In this case, the excess-distortion probability of the second decoder ϕ_2 with respect to the distortion level D_2 vanishes doubly exponentially fast (cf. (171) to (174)) and the joint excess-distortion event is dominated by the excess-distortion event of decoder ϕ_1 (cf. (181)).

The exponent $\Lambda_{X^2}^*(\alpha^*(R_1, R_2))$ is positive if the rate pair (R_1, R_2) is strictly inside the rate region \mathcal{R}_{inner} (cf. (14)). The reason is as follows. An equivalent form of $\mathcal{R}_{\mathrm{inner}}$ is

$$\mathcal{R}_{\text{inner}} = \bigcup_{\eta \in (\frac{D_2}{D_1}, 1]} \left\{ (R_1, R_2) : R_1 \ge \frac{1}{2} \log \frac{\sigma^2}{\eta D_1} \text{ and } R_2 \ge \frac{1}{2} \log \frac{\eta D_1}{D_2} \right\}.$$
 (61)

Note that (61) follows from (14) since i) the constraint on rate R_1 in (61) equals to the one in (14) by taking the union of $\eta \in (\frac{D_2}{D_1}, 1]$ and ii) adding the two inequalities together (61) gives the the sum-rate constraint in (14).

Lemma 6. For any $\eta \in (\frac{D_2}{D_1}, 1]$, $\Lambda_{X^2}^*(\alpha^*(R_1, R_2)) > 0$ if $R_1 > \frac{1}{2} \log \frac{\sigma^2}{\eta D_1}$ and $R_2 > \frac{1}{2} \log \frac{\eta D_1}{D_2}$.

 $\textit{Proof.} \ \ \text{Given any} \ \eta \in (\tfrac{D_2}{D_1}, 1], \ \text{recall that} \ \lambda = \min\left\{\tfrac{D_2 \exp\{2R_2\}}{D_1}, 1\right\}, \ \text{for} \ R_1 > \tfrac{1}{2}\log\tfrac{\sigma^2}{\eta D_1} \ \text{and} \ R_2 > \tfrac{1}{2}\log\tfrac{\eta D_1}{D_2}, \ \text{it follows that} \right\}$

$$\lambda > \min\left\{\frac{D_2}{D_1} \frac{\eta D_1}{D_2}, 1\right\} \tag{62}$$

$$=\eta. \tag{63}$$

Thus, it follows that $R_1 > \frac{1}{2} \log \frac{\sigma^2}{\eta D_1} > \frac{1}{2} \log \frac{\sigma^2}{\lambda D_1}$. Recall that $\alpha^*(R_1, R_2)$ is the solution of α to $R_1 = R_{\mathrm{iid}}(\alpha, P_Y, \lambda D_1)$. It follows that

$$R_{\text{iid}}(\alpha^*(R_1, R_2), P_Y, \lambda D_1) > \frac{1}{2} \log \frac{\sigma^2}{\lambda D_1}$$

$$\tag{64}$$

$$= R_{\rm iid}(\sigma^2, P_Y, \lambda D_1), \tag{65}$$

where (65) follows from the definition of $R_{\text{iid}}(\cdot)$ in (57) and Claim i) of Lemma 12.

Note that $R_{iid}(w, P, D)$ increases in w for $w \ge |D - P|^+$ (cf. Lemma 12, Claim iii)), we have that

$$\alpha^*(R_1, R_2) > \sigma^2. \tag{66}$$

The proof is completed by noting that $\Lambda_{X^2}^*(\alpha) > 0$ if $\alpha > \sigma^2$ (cf. Lemma 12, Claim iv)).

When specialized to a GMS, $\Lambda_{X^2}^*(\alpha) = \sup_{\theta \in \mathbb{R}} \left\{ \alpha \theta + \frac{1}{2} \log(1 - 2\sigma^2 \theta) \right\} = \frac{1}{2} \left(\frac{\alpha}{\sigma^2} - \log \frac{\alpha}{\sigma^2} - 1 \right)$. To our best knowledge, this appears to be the first characterization of the excess-distortion exponent of successive refinement for continuous memoryless sources. To illustrate the exponent, we plot the $\Lambda_{X^2}^*(\alpha^*(R_1,R_2))$ for various values of (R_1,R_2) in Fig. 5 for a GMS. It is

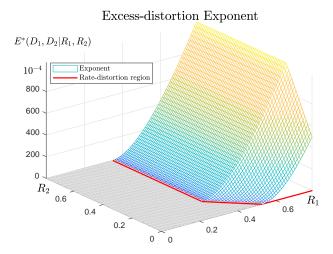


Fig. 5. Illustration of excess-distortion exponent $E^*(D_1, D_2|R_1, R_2)$ when $P_X = \mathcal{N}(0, \sigma^2)$ with $\sigma^2 = 1$. The exponent maintains positive in rate-distortion

left as future work to check whether our achievable exponent is tight or not. To do so, one might generalize the result in [22], especially the converse part, to the successive refinement setting.

One might wonder why we use a coding scheme where the powers of codewords $P_Y = \sigma^2 - \lambda D_1$ and $P_Z = \lambda D_1 - D_2$ both have a parameter λ , in contrast to the mismatched rate-distortion case where one uses codewords of fixed powers $\sigma^2 - D_1$ without any dummy variable [11]. As discussed in the second remark of Theorem 1, such a code design allows us to derive results for all cases of interest in second-order asymptotics. For large deviations analysis, as we demonstrate above, this codebook design allows us to obtain positive exponent under JEP for all rate pairs strictly inside the rate region \mathcal{R}_{inner} , which is impossible for the codebook design with $\lambda = 1$ as done in our previous ISIT paper [2]. To discuss in detail, we first recall the main result in [2].

Theorem 7 ([2, Theorem 1]). Consider the mismatched code in Definition 1 where the codebook is designed with parameter $\lambda = 1$. Given any $(R_1, R_2) \in \mathbb{R}^2_+$, let α_1 be the solution to $R_1 = R_{iid}(\alpha_1, P_Y, D_1)$, γ_2 be the solution to $R_2 = R_{iid}(\gamma_2, P_Z, D_2)$ and α_2 be the solution to $R_1 = R_{iid}(\alpha_2, P_Y, \gamma_2)$. The corresponding joint excess-distortion exponent $E_{\lambda=1}^*(D_1, D_2|R_1, R_2)$

• Case i): If $R_1 > \frac{1}{2} \log \frac{\sigma^2}{D_1}$ and $R_2 > \frac{1}{2} \log \frac{D_1}{D_2}$

$$E_{\lambda=1}^*(D_1, D_2|R_1, R_2) \ge \Lambda_{X^2}^*(\alpha_1) > 0.$$
(67)

• Case ii): If $\frac{1}{2} \log \frac{D_1}{D_2} > R_2 > R_{iid}(|D_2 - P_Z|^+, P_Z, D_2)$ and $R_1 > R_{iid}(\max\{\sigma^2, \gamma_2 - P_Y\}, P_Y, \gamma_2)$,

$$E_{\lambda=1}^*(D_1, D_2|R_1, R_2) \ge \Lambda_{X^2}^*(\alpha_2) > 0.$$
(68)

• Case iii): Otherwise, $E_{\lambda=1}^*(D_1, D_2|R_1, R_2) = 0$.

The condition for Case ii) is valid since $R_{\rm iid}(|D_2-P_Z|^+,P_Z,D_2)<\frac{1}{2}\log\frac{D_1}{D_2}$ and $R_{\rm iid}(\max\{\sigma^2,\gamma_2-P_Y\},P_Y,\gamma_2)\geq\frac{1}{2}\log\frac{\sigma^2}{D_1}$ (cf. Appendix A). For ease of understanding, we illustrate all three cases of Theorem 7 in Fig. 6 where $\mathcal{R}_{\rm inner}$ is denoted as the region above the red line.

Note that if the rate pair $(R_1,R_2) \in \mathcal{R}_{inner}$ such that $R_2 < R_{iid}(|D_2-P_Z|^+,P_Z,D_2) < \frac{1}{2}\log\frac{D_1}{D_2}$ and $R_1 > R_{iid}(\max\{\sigma^2,\gamma_2-P_Z\}^+,P_Z,D_2)$ $P_Y\}, P_Y, \gamma_2) > \frac{1}{2}\log\frac{\sigma^2}{D_1}$, the exponent $E_{\lambda=1}^*(D_1, D_2|R_1, R_2) = 0$. Intuitively, this is because using the $\lambda=1$ code, regardless of the rate $R_1 > \frac{1}{2}\log\frac{\sigma^2}{D_1}$ of encoder f_1 , the distortion between a source sequence x^n and the output \hat{x}_1^n of f_1 is roughly D_1 . In this case, a rate $R_2 < \frac{1}{2} \log \frac{D_1}{D_2}$ of for encoder f_2 is not sufficient to D_2 -cover the distortion ball of the source sequence with radius D_1 . This problem can be resolved by introducing a rate related parameter $\lambda \in (\frac{D_2}{D_1}, 1]$ into the codebook design as we do in the present paper. This way, the average power of codewords for encoder f_1 is $P_Y = \sigma^2 - \lambda D_1$, hence the achievable distortion level of encoder f_1 reduces to λD_1 for $R_1 = \frac{1}{2}\log\frac{\sigma^2}{\lambda D_1} > \frac{1}{2}\log\frac{\sigma^2}{D_1}$. Furthermore, the average power of codewords for encoder f_2 is now $P_Z = \lambda D_1 - D_2$ so that encoder f_2 is able to D_2 -cover the distortion ball of a source sequence with radius λD_1 with rate $R_2 > \frac{1}{2}\log\frac{\lambda D_1}{D_2}$, which is smaller than $\frac{1}{2}\log\frac{D_1}{D_2}$. As shown in Fig. 7, our present rate-dependent coding scheme grantees a positive exponent $E^*(D_1,D_2|R_1,R_2)$ for all $(R_1,R_2) \in \mathcal{R}_{inner}$.

We next present achievable exponents of ensemble SEP. Recall that $\lambda = \min\left\{\frac{D_2\exp\{2R_2\}}{D_1},1\right\}$.

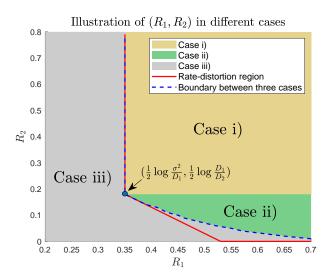


Fig. 6. Illustration of different cases in Theorem 7. The exponent $E_{\lambda=1}^*(D_1,D_2|R_1,R_2)$ is positive in case i) and case ii) but equals to 0 in case iii).

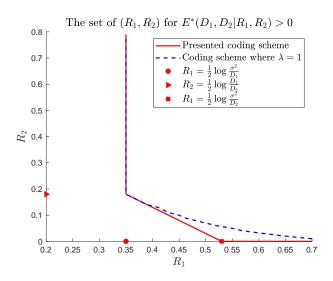


Fig. 7. Comparison of the set of rate pairs (R_1, R_2) that ensure a positive JEP exponent $E^*(D_1, D_2|R_1, R_2)$ for a GMS $P_X = \mathcal{N}(0, 1)$ between our present coding scheme in Theorem 5 and coding scheme in Theorem 7 where $\lambda = 1$. Note that such rate pairs are above the boundary denoted by red and blue dotted lines for the two coding schemes, respectively.

Theorem 8. Given any $(R_1, R_2) \in \mathbb{R}^2_+$, let $\alpha_1^*(R_1)$ be the solution of α to $R_1 = R_{iid}(\alpha, P_Y, D_1)$. The following results hold.

• If $R_2 \leq \frac{1}{2} \log \frac{D_1}{D_2}$, let $\alpha_2^*(R_1)$ be the solution of α to $R_1 = R_{iid}(\alpha, P_Y, \lambda D_1)$. The exponent region satisfies

$$\{(E_1, E_2) : E_1 \ge \Lambda_{X^2}^*(\alpha_1^*(R_1)), E_2 \ge \Lambda_{X^2}^*(\alpha_2^*(R_1))\} \subseteq \mathcal{E}(D_1, D_2, R_1, R_2). \tag{69}$$

• If $R_2 > \frac{1}{2} \log \frac{D_1}{D_2}$, note that $\lambda = 1$, let $\gamma^*(R_2)$ be the solution of γ to $R_2 = R_{iid}(\gamma, P_Z, D_2)$ and let $\alpha_2^*(R_1, R_2)$ be the solution of α to $R_1 = R_{iid}(\alpha, P_Y, \gamma^*(R_2))$. The exponent region satisfies

$$\{(E_1, E_2) : E_1 \ge \Lambda_{X^2}^*(\alpha_1^*(R_1)), E_2 \ge \Lambda_{X^2}^*(\alpha_2^*(R_1, R_2))\} \subseteq \mathcal{E}(D_1, D_2, R_1, R_2). \tag{70}$$

The proof of Theorem 8 is provided in Section VI-C. For ease of notation, let $E^*(D_1|R_1,R_2):=\Lambda_{X^2}^*(\alpha_1^*(R_1))$ and let $E^*(D_2|R_1,R_2)$ be

$$E^*(D_2|R_1, R_2) := \begin{cases} \Lambda_{X^2}^*(\alpha_2^*(R_1)) & \text{if } R_2 \le \frac{1}{2} \log \frac{D_1}{D_2}, \\ \Lambda_{X^2}^*(\alpha_2^*(R_1, R_2)) & \text{otherwise.} \end{cases}$$
(71)

The following lemma states the condition under which $E^*(D_1|R_1,R_2)$ and $E^*(D_2|R_1,R_2)$ are positive.

Lemma 9. The following claims hold.

i) The exponent $E^*(D_1|R_1, R_2) > 0$ if $R_1 > \frac{1}{2} \log \frac{\sigma^2}{D_1}$.

ii) The exponent $E^*(D_2|R_1,R_2)>0$ if the rate pair (R_1,R_2) is strictly inside the rate region \mathcal{R}_{inner} .

iii) Both $E^*(D_1|R_1,R_2)$ and $E^*(D_2|R_1,R_2)$ are positive if (R_1,R_2) is strictly inside the rate region \mathcal{R}_{inner} .

Proof. Claim i) is established as follows. Recall that $E^*(D_1|R_1,R_2)=\Lambda_{X^2}^*(\alpha_1^*(R_1))$ and $\alpha_1^*(R_1)$ be the solution of α to $R_1 = R_{\text{iid}}(\alpha, P_Y, D_1)$. For $R_1 > \frac{1}{2} \log \frac{\sigma^2}{D_1}$, it follows that

$$R_{\text{iid}}(\alpha_1^*(R_1), P_Y, D_1) > \frac{1}{2} \log \frac{\sigma^2}{D_1}.$$
 (72)

Note that $R_{\text{iid}}(\sigma^2, P_Y, D_1) = \frac{1}{2} \log \frac{\sigma^2}{D_1}$ and $R_{\text{iid}}(w, P, D)$ increases in w for $w \ge |D - P|^+$ (cf. Lemma 12, Claim i) and iii)), we have that

$$\alpha_1^*(R_1) > \sigma^2. \tag{73}$$

The proof of Claim i) is completed by noting that $\Lambda_{X^2}^*(a) > 0$ when $a > \sigma^2$ (cf. Lemma 12, Claim iv)). Claim ii) is proved as follows. When $R_2 \leq \frac{1}{2}\log\frac{D_1}{D_2}$, since $E^*(D_2|R_1,R_2) = E^*(D_1,D_2|R_1,R_2)$, it follows from the positivity analysis of $E^*(D_1,D_2|R_1,R_2)$ (cf. Lemma 6) that $E^*(D_2|R_1,R_2)$ is positive if (R_1,R_2) is strictly inside the rate region \mathcal{R}_{inner} .

When $R_2 > \frac{1}{2} \log \frac{D_1}{D_2}$, recall that $\lambda = 1$, the rate pair (R_1, R_2) is strictly inside the rate region \mathcal{R}_{inner} if and only if $R_1 > \frac{1}{2}\log\frac{\sigma^2}{D_1}$. Recall that $\gamma^*(R_2)$ is the solution of γ to $R_2 = R_{\mathrm{iid}}(\gamma, P_Z, D_2)$, invoking that $R_{\mathrm{iid}}(l, P, D)$ increases in l when $l \geq |D - P|^+$ and $R_{\mathrm{iid}}(D_1, P_Z, D_2) = \frac{1}{2}\log\frac{D_1}{D_2}$ (cf. Lemma 12, Claim i) and iii)), it follows that

$$\gamma^*(R_2) > D_1. \tag{74}$$

Recall that $\alpha_2^*(R_1, R_2)$ is the solution of α to $R_1 = R_{iid}(\alpha, P_Y, \gamma^*(R_2))$; it follows that

$$R_{\text{iid}}(\alpha_2^*(R_1, R_2), P_Y, \gamma^*(R_2)) = R_1 \tag{75}$$

$$> \frac{1}{2} \log \frac{\sigma^2}{D_1} \tag{76}$$

$$=R_{\rm iid}(\sigma^2, P_Y, D_1),\tag{77}$$

where (77) follows from Lemma 12, Claim i). Invoking that $R_{iid}(l, P, D)$ increases in l when $l \ge |D - P|^+$, decreases in Dwhen $D \le l + P$ (cf. Lemma 12, Claim iii)) and $\gamma^*(R_2) > D_1$, it follows that

$$\alpha_2^*(R_1, R_2) > \sigma^2.$$
 (78)

Therefore, $E^*(D_2|R_1,R_2) > \Lambda_{X^2}^*(\alpha_2^2(R_1,R_2)) > \Lambda_{X^2}^*(\sigma^2) > 0$. Claim iii) is established as follows. Note that $\frac{1}{2}\log\frac{\sigma^2}{\lambda D_1} \geq \frac{1}{2}\log\frac{\sigma^2}{D_1}$ for $\lambda \in (\frac{D_2}{D_1},1]$, we have that $E^*(D_2|R_1,R_2) > 0$ implies $E^*(D_1|R_1,R_2) > 0$. Combining the above arguments, we find that both $E^*(D_1|R_1,R_2)$ and $E^*(D_2|R_1,R_2)$ are positive if (R_1, R_2) is strictly inside the rate region \mathcal{R}_{inner} .

When specialized to a GMS, $\Lambda_{X^2}^*(\alpha_2) = \frac{1}{2} \left(\frac{\alpha_2}{\sigma^2} - \log \frac{\alpha_2}{\sigma^2} - 1 \right)$. This result appears the first attempt to derive a large deviations result for GMS under SEP in the successive refinement problem. It is interesting future work to check whether our exponents are tight or not. To illustrate the exponent, we plot $E^*(D_2|R_1,R_2)$ in Fig. 8 for a GMS.

IV. PROOF OF SECOND-ORDER ASYMPTOTICS UNDER JEP (THEOREM 1)

A. Preliminaries

Recall that $P_Y = \sigma^2 - \lambda D_1$ and $P_Z = \lambda D_1 - D_2$. For any $\varepsilon \in (0,1)$, let

$$V := Var[X^2] = \zeta - \sigma^4, \tag{79}$$

$$a_n := \sqrt{V \frac{\log n}{n}},\tag{80}$$

$$b_n := \sqrt{\frac{V}{n}} Q^{-1}(\varepsilon). \tag{81}$$

In subsequent analysis, for simplicity, let the random variable W denote the normalized ℓ_2 norm of source sequence, i.e., $\frac{\|X^n\|^2}{n}$. We use f_W to denote the distribution of random variable W and use w as a particular realization of W. Similarly, we define the random variable L as the quadratic distortion of the first layer, i.e., $L = d(X^n, \hat{X}_1^n)$. Furthermore, we use f_L to denote the induced distribution of the random variable L and use l as a particular realization of L.

Excess-distortion Exponent

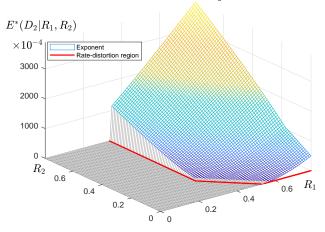


Fig. 8. Illustration of excess-distortion exponent $E^*(D_2|R_1,R_2)$ when $P_X=\mathcal{N}(0,1)$.

For any (x^n, y^n) and $(\dagger, \dagger) \in \{\text{sp}, \text{iid}\}^2$, define following non-excess-distortion probabilities

$$\Psi_{\dagger}(n, w, D) := \Pr_{f_{\dagger}(\cdot | \mathbf{0}^n, P_Y)} \{ d(x^n, Y^n) \le D \}, \tag{82}$$

$$\Phi_{\ddagger}(n, x^n, y^n) := \Pr_{f_{\ddagger}(\cdot | y^n, P_Z)} \{ d(x^n, Z^n) \le D_2 \}, \tag{83}$$

where $f_{\cdot}(\cdot)$ is the distribution of codewords (cf. Definition 1). Define two constants

$$\beta_1 := \sqrt{P_Z} - \sqrt{D_2},\tag{84}$$

$$\beta_2 := \sqrt{P_Z} + \sqrt{D_2}.\tag{85}$$

It follows that

$$\beta_1^2 \le |P_Z - D_2|. \tag{86}$$

We justify (86) as follows. Note that $\beta_1^2 = \lambda D_1 - 2\sqrt{P_Z}\sqrt{D_2}$. If $\lambda D_1 \ge 2D_2$, then $\sqrt{P_Z} \ge \sqrt{D_2}$ and $|P_Z - D_2| = \lambda D_1 - 2D_2$, and thus

$$\beta_1^2 = \lambda D_1 - 2\sqrt{P_Z}\sqrt{D_2} \tag{87}$$

$$<\lambda D_1 - 2D_2 \tag{88}$$

$$= |P_Z - D_2|. (89)$$

If $\lambda D_1 < 2D_2$, then $\sqrt{P_Z} < \sqrt{D_2}$ and $|P_Z - D_2| = 2D_2 - \lambda D_1$, and thus

$$\beta_1^2 - |P_Z - D_2| = \lambda D_1 - 2\sqrt{P_Z}\sqrt{D_2} - 2D_2 + \lambda D_1 \tag{90}$$

$$=2(P_Z-\sqrt{P_Z}\sqrt{D_2})\tag{91}$$

$$=2\sqrt{P_Z}(\sqrt{P_Z}-\sqrt{D_2})\tag{92}$$

$$<0.$$
 (93)

Note that (82) is valid since $\Pr_{f_{\dagger}(\cdot|\mathbf{0}^n,P_Y)}\{d(x^n,Y^n)\leq D\}$ depends on x^n only through its normalized ℓ_2 norm $w=\frac{\|x^n\|^2}{n}$. Similarly to [23, Theorem 37], $\Phi_{\mathrm{sp}}(n,x^n,y^n)$ depends on (x^n,y^n) only through their quadratic distortion $l:=d(x^n,y^n)$ and we obtain the following lemma.

Lemma 10. Recall that $|x|^+ = \max\{x,0\}$. For any (x^n,y^n) such that $\sqrt{l} \in [|\beta_1|^+,\beta_2]$

$$\Phi_{\rm sp}(n, x^n, y^n) \ge \frac{\Gamma(\frac{n+2}{2})}{\sqrt{\pi}n\Gamma(\frac{n+1}{2})} \left(1 - \frac{(l + P_Z - D_2)^2}{4lP_Z}\right)^{\frac{n-1}{2}} \\
=: \underline{h}(n, l), \tag{94}$$

and otherwise $\Phi_{\rm sp}(n,x^n,y^n)=0$.

The proof of Lemma 10 is available in Appendix B.

We also need the following function

$$\kappa(s, w, P) := \frac{(P(1+2s) + 2w)^2}{P(1+2s)^3}.$$
(95)

Recall the definition of $R_{\text{iid}}(w, P, D)$, $s^*(w, P, D)$ and $\kappa(s, w, P)$ in (56), (57) and (95), respectively. Similar to Zhou et. al [11, cf. (89)], using the strong large deviations theorem in [21, Theorem 3.7.4], noting that $\Phi_{\text{iid}}(n, x^n, y^n)$ depends on (x^n, y^n) only through their quadratic distortion $l = d(x^n, y^n)$, we obtain

$$\Phi_{\text{iid}}(n,l) \sim \frac{\exp\{-nR_{\text{iid}}(l, P_Z, D_2)\}}{s^*(l, P_Z, D_2)\sqrt{\kappa(s^*(l, P_Z, D_2), l, P_Z, D_2)}}.$$
(96)

Finally, note that for any $(\dagger, \ddagger) \in \{\text{sp}, \text{iid}\}^2$, the ensemble JEP satisfies

$$P_{\dagger,\pm}^{n}(D_1, D_2|M_1, M_2) \le \Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1 \text{ or } d(X^n, \hat{X}_2^n) > D_2\}$$
(97)

$$= \Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\} + \Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\}. \tag{98}$$

We next further upper bound (98).

B. When Both Encoders Use Spherical Codebooks

We first consider the case where both encoders use spherical codebooks where $(\dagger, \ddagger) = \{\text{sp, sp}\}$. Note that the first term in (98) is similar to the mismatched rate-distortion problem studied in [11, Section IV. B], except that i) the distortion level is replaced from D to λD_1 , and ii) the power of codebook is changed from $P_Y = \sigma^2 - D_1$ to $P_Y = \sigma^2 - \lambda D_1$. Choose M_1 such that

$$\log M_1 = \frac{n}{2} \log \frac{\sigma^2}{\lambda D_1} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n).$$
(99)

Similar to [11, cf. (43)-(58)], we have that

$$\Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\} \le \varepsilon + O\left(\frac{1}{\sqrt{n}}\right). \tag{100}$$

Thus, we focus on the analysis of the second term in (98). For ease of notation, we use \mathbf{Y} to denote the random codebooks $(Y^n(1), \dots, Y^n(M_1))$ of the first encoder and use \mathbf{y} for a particular realization. Recall that given the source sequence x^n and a particular realization of the codebook \mathbf{y} , the compressed index of the encoder f_1 is $f_1(x^n)$ and the reproduced source sequence of the first encoder is

$$\hat{x}_1^n = y^n(f_1(x^n)) = \min_{i \in [M_1]} d(x^n, y^n(i)).$$
(101)

It follows that

$$\Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\}$$

$$= \int_{\substack{x^n, \mathbf{y}:\\ d(x^n, \hat{x}_1^n) \le \lambda D_1}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{sp}}(y^n(i) | \mathbf{0}^n, P_Y) \times \Pr\{\forall \ j \in [M_2]: \ d(x^n, Z^n(f_1(x^n), j)) > D_2\} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$
(102)

$$= \int_{\substack{x^n, \mathbf{y}:\\ d(x^n, \hat{x}_1^n) \le \lambda D_1}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{sp}}(y^n(i) | \mathbf{0}^n, P_Y) \times (1 - \Phi_{\mathrm{sp}}(n, x^n, \hat{x}_1^n))^{M_2} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$
(103)

$$= \int_{x^n,\mathbf{y}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{sp}}(y^n(i) | \mathbf{0}^n, P_Y) \times 1\{d(x^n, \hat{x}_1^n) < (|\beta_1|^+)^2\} \times (1 - \Phi_{\mathrm{sp}}(n, x^n, \hat{x}_1^n))^{M_2} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$

$$+ \int_{x^n, \mathbf{y}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{sp}}(y^n(i) | \mathbf{0}^n, P_Y) \times 1\{d(x^n, \hat{x}_1^n) \in [(|\beta_1|^+)^2, \lambda D_1]\} \times (1 - \Phi_{\mathrm{sp}}(n, x^n, \hat{x}_1^n))^{M_2} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$
(104)

$$= \Pr\{d(X^{n}, \hat{X}_{1}^{n}) < (|\beta_{1}|^{+})^{2}\} + \int_{x^{n}, \mathbf{y}} P_{X}^{n}(x^{n}) \prod_{i \in [M_{1}]} f_{\mathrm{sp}}(y^{n}(i)|\mathbf{0}^{n}, P_{Y}) \times 1\{d(x^{n}, \hat{x}_{1}^{n}) \in [(|\beta_{1}|^{+})^{2}, \lambda D_{1}]\}$$

$$\times (1 - \Phi_{\mathrm{sp}}(n, x^{n}, \hat{x}_{1}^{n}))^{M_{2}} \mathrm{d}x^{n} \mathrm{d}\mathbf{y},$$

$$(105)$$

where (103) follows since given x^n , y and thus \hat{x}_1^n , the codewords $(Z^n(\hat{x}_1^n, 1), \dots, Z^n(\hat{x}_1^n, M_2))$ are independent and generated by the same distribution $f_{\rm sp}(\cdot|\hat{x}_1^n, \lambda D_1 - D_2)$, (104) divides the whole section into two parts and the second term is valid since $(|\beta_1|^+)^2 < \lambda D_1$ and (105) follows from the results in Lemma 10.

Recall that $l=d(x^n,y^n)$ is the quadratic distortion between x^n and y^n . The second term in (105) can be further upper bounded by

$$\int_{(|\beta_1|^+)^2}^{\lambda D_1} f_L(l) (1 - \underline{h}(n, l))^{M_2} dl$$
 (106)

$$\leq \int_{(|\beta_1|^+)^2}^{|P_Z - D_2|} f_L(l) dl + \int_{|P_Z - D_2|}^{\lambda D_1} f_L(l) (1 - \underline{h}(n, l))^{M_2} dl \tag{107}$$

$$\leq \int_{(|\beta_1|+)^2}^{|P_Z-D_2|} f_L(l) dl + \int_{|P_Z-D_2|}^{\lambda D_1} f_L(l) \exp\{-M_2 \underline{h}(n,l)\} dl$$
(108)

$$\leq \int_{(|\beta_1|^+)^2}^{|P_Z - D_2|} f_L(l) dl + \exp\{-M_2 \underline{h}(n, \lambda D_1)\}, \tag{109}$$

where (106) follows from the result in (94), (107) follows since $\beta_1^2 \leq |P_Z - D_2|$ (cf. (86)) and $\underline{h}(n,l) \in [0,1]$, (108) follows from that $(1-a)^M \leq \exp\{-Ma\}$ for any $a \in [0,1)$ and (109) follows since $\underline{h}(n,l)$ is a decreasing function of l if $l \ge |P_Z - D_2|.$

Combining (105) and (109), we have

$$\Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\} \le \Pr\{d(X^n, \hat{X}_1^n) \le |P_Z - D_2|\} + \exp\{-M_2 \underline{h}(n, \lambda D_1)\}. \tag{110}$$

Choose M_2 such that

$$\log M_2 = -\log \underline{h}(n, \lambda D_1) + \log(\log \sqrt{n}) \tag{111}$$

$$= \frac{n}{2}\log\frac{\lambda D_1}{D_2} + O(\log n),\tag{112}$$

where (112) follows from the definition of $\underline{h}(n,l)$ in (94) and the fact that $P_Z = \lambda D_1 - D_2$ and $\Gamma\left(\frac{n+2}{2}\right)/\Gamma\left(\frac{n+1}{2}\right) = O(\sqrt{n})$. With the choice of M_2 in (111), we have

$$\exp\{-M_2\underline{h}(n,\lambda D_1)\} = \frac{1}{\sqrt{n}}.$$
(113)

To upper bound the first term in (110), similarly to [11, Section IV. C], we define the sets

$$\mathcal{P} := \{ r \in \mathbb{R} : b < r - \sigma^2 \le a_n \},\tag{114}$$

$$Q := \{ r \in \mathbb{R} : r + P_Y - |P_Z - D_2| \ge 0 \}, \tag{115}$$

where the value of b is to be specified. Recall that $w = \frac{\|x^n\|^2}{n}$ is the normalized ℓ_2 norm of x^n . Furthermore, we need the following definition:

$$R_{\rm sp}(w, P, D) := -\frac{1}{2} \log \left(1 - \frac{(w + P - D)^2}{4wP} \right). \tag{116}$$

Recall the definition of $\Psi_{\rm sp}(n,w,D)$ in (82). It follows from [11, cf. (63)-(68)] that

$$\Psi_{\rm sp}(n, w, |P_Z - D_2|) \le \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \exp\{-(n-3)R_{\rm sp}(w, P_Y, |P_Z - D_2|)\}
=: \bar{g}(n, w, |P_Z - D_2|),$$
(117)

and similar to [11, cf. (69)-(77)], we have

$$\Pr\{d(X^{n}, \hat{X}_{1}^{n}) \ge |P_{Z} - D_{2}|\}$$

$$= \int_{0}^{\infty} (1 - \Psi_{\rm sp}(n, w, |P_{Z} - D_{2}|))^{M_{1}} f_{W}(w) dw$$
(118)

$$\geq \left(1 - \frac{1}{\sqrt{n}}\right) \Pr\left\{W \in \mathcal{P} \cap \mathcal{Q}, \log M_1 \leq -\log 2 - \log \bar{g}(n, \sigma^2 + b, |P_Z - D_2|) - \frac{1}{2}\log n\right\}. \tag{119}$$

We next further lower bound the probability term in (119) and show that the probability term tends to 1 asymptotically. Note that $R_{\rm sp}(w,P,D)$ is a decreasing function of D and is an increasing function of w. For $w_1=\sigma^2$ and λD_1 , we have $R_{\mathrm{sp}}(w_1, P_Y, \lambda D_1) = \frac{1}{2}\log\frac{\sigma^2}{\lambda D_1}$. Recall that $P_Z = \lambda D_1 - D_2$, since $|\lambda D_1 - 2D_2| < \lambda D_1$, if we choose $w_2 \in \mathbb{R}_+$ such that

$$R_{\rm sp}(w_2, P_Y, |\lambda D_1 - 2D_2|) = \frac{1}{2} \log \frac{\sigma^2}{\lambda D_1},$$
 (120)

it follows that $w_2 < w_1 = \sigma^2$. Set $w_3 = \frac{w_1 + w_2}{2}$. Thus $w_2 < w_3 < \sigma^2$ and

$$R_{\rm sp}(w_3, P_Y, |\lambda D_1 - 2D_2|) > R_{\rm sp}(w_2, P_Y, |\lambda D_1 - 2D_2|)$$
 (121)

$$=\frac{1}{2}\log\frac{\sigma^2}{\lambda D_1}.\tag{122}$$

Set the value of b as

$$b := \frac{w_2 - \sigma^2}{2}. (123)$$

Using the definition of $\bar{g}(n, w)$ in (117), it follows that for some $\delta > 0$,

$$-\log \bar{g}(n, \sigma^2 + b, |P_Z - D_2|)$$

$$= nR_{\rm sp}(w_3, P_Y, |\lambda D_1 - 2D_2|) + O(\log n)$$
(124)

$$= \frac{n}{2}\log\frac{\sigma^2}{\lambda D_1} + n\delta + O(\log n),\tag{125}$$

where (124) follows since $\Gamma\left(\frac{n}{2}\right)/\Gamma\left(\frac{n-1}{2}\right)=O(\sqrt{n})$ and $P_Z=\lambda D_1-D_2$, and (125) follows from the result in (122). Combining (99) and (125) and invoking the weak law of large numbers, we conclude that

$$\lim_{n \to \infty} \Pr \left\{ \log M_1 \le -\log 2 - \log \bar{g}(n, \sigma^2 + b) - \frac{1}{2} \log n \right\} = 1.$$
 (126)

Note that (126) follows since (125) implies that the dominant term in the right-hand side is strictly larger than $\frac{n}{2}\log\frac{\sigma^2}{\lambda D_1}$, which is the dominant term of $\log M_1$ (cf. (99)). Similar to [11, Lemma 5], using the weak law of large numbers, the Berry-Esseen theorem [24], [25] and the definition of b, we conclude that

$$\Pr\left\{W \in \mathcal{P} \cap \mathcal{Q}\right\} \ge 1 - O\left(\frac{1}{\sqrt{n}}\right). \tag{127}$$

Finally, combining (99), (105), (112), (113) and (127), we have

$$\lim_{n \to \infty} \Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\} = 0.$$
(128)

The proof of Theorem 1 when both encoders use spherical codebooks completed by combining (98), (100), (128).

C. When Both Encoders Use i.i.d. Gaussian Codebooks

We next present the proof when both encoders use i.i.d. Gaussian codebooks, i.e., $(\dagger, \ddagger) = \{\text{iid}, \text{iid}\}$. Similar to the case when both encoders use spherical codebooks, the first term in (98) is similar to the mismatched rate-distortion problem studied in [11, Section IV. D]. Choose M_1 such that

$$\log M_1 = \frac{n}{2} \log \frac{\sigma^2}{\lambda D_1} + \sqrt{n V(\sigma^2, \zeta)} Q^{-1}(\varepsilon) + O(\log n).$$
 (129)

Similar to [11, cf. (90)-(103)], we have that

$$\Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\} \le \varepsilon + O\left(\frac{1}{\sqrt{n}}\right). \tag{130}$$

The second term in (98) is upper bounded as follows.

 $\Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\}$

$$= \int_{d(x^n, \hat{x}_1^n) \le \lambda D_1} P_X^n(x^n) \prod_{i \in [M_1]} f_{iid}(y^n(i)|\mathbf{0}^n, P_Y) \times \Pr\{\forall \ j \in [M_2] : \ d(x^n, Z^n(f_1(x^n), j)) > D_2\} dx^n d\mathbf{y}$$
(131)

$$= \int_{\substack{x^n, \mathbf{y}:\\ d(x^n, \hat{x}_1^n) \le \lambda D_1}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{iid}}(y^n(i)|\mathbf{0}^n, P_Y) \times (1 - \Phi_{\mathrm{iid}}(n, x^n, \hat{x}_1^n))^{M_2} \mathrm{d}x^n \mathrm{d}\mathbf{y}, \tag{132}$$

$$= \int_0^{\lambda D_1} f_L(l) (1 - \Phi_{iid}(n, l))^{M_2} dl$$
 (133)

$$\leq \int_{0}^{|P_{Z}-D_{2}|} f_{L}(l) dl + \int_{|P_{Z}-D_{2}|}^{\lambda D_{1}} f_{L}(l) (1 - \Phi_{iid}(n, l))^{M_{2}} dl \tag{134}$$

$$\leq \int_{0}^{|P_{Z}-D_{2}|} f_{L}(l) dl + \int_{|P_{Z}-D_{2}|}^{\lambda D_{1}} f_{L}(l) \exp\{-M_{2}\Phi_{iid}(n,l)\} dl \tag{135}$$

$$\leq \Pr\{d(X^n, \hat{X}_1^n) \leq |P_Z - D_2|\} + \exp\{-M_2\Phi_{\text{iid}}(n, \lambda D_1)\},\tag{136}$$

where (132) follows since given x^n , \mathbf{y} and thus \hat{x}_1^n , the codewords $(Z^n(\hat{x}_1^n,1),\ldots,Z^n(\hat{x}_1^n,M_2))$ are independent and generated by the same distribution $f_{\mathrm{iid}}(\cdot|\hat{x}_1^n,\lambda D_1-D_2)$, (133) follows from (96), (134) follows since $\Phi_{\mathrm{iid}}(n,l)\leq 1$, (135) follows since $(1-a)^M\leq \exp\{-Ma\}$ for any $a\in[0,1)$ and (136) since $\Phi_{\mathrm{iid}}(n,l)$ is a decreasing function of l if $l\geq |P_Z-D_2|$. Choose M_2 such that

$$\log M_2 = -\log \Phi_{\text{iid}}(n, \lambda D_1) + \log(\log \sqrt{n}) \tag{137}$$

$$= \frac{n}{2}\log\frac{\lambda D_1}{D_2} + O(\log n). \tag{138}$$

It follows that

$$\exp\{-M_2\Phi_{\mathrm{iid}}(n,\lambda D_1)\} = \frac{1}{\sqrt{n}}.$$
(139)

The analysis of the first term in (136) is similar to spherical codebook (cf. (114) to (127)) except the following two points: i) replace $\bar{g}(n,w)$ with $\Psi_{\rm iid}(n,w,|P_Z-D_2|)$ and ii) replace $\mathcal{P}\cap\mathcal{Q}$ with \mathcal{P} . Hence, we have

$$\lim_{n \to \infty} \Pr\{d(X^n, \hat{X}_1^n) \le \lambda D_1, \ d(X^n, \hat{X}_2^n) > D_2\} = 0.$$
(140)

The proof of Theorem 1 when both encoders use i.i.d. Gaussian codebooks is completed by combining (98), (130), (140).

D. When Two Encoders Use Different Codebooks

We next prove Theorem 1 when different types of codebooks are used in encoders f_1 and f_2 . When $(\dagger, \ddagger) = \{\text{iid}, \text{sp}\}$, to bound the first term in (98), the steps are exactly the same as the case $(\dagger, \ddagger) = \{\text{iid}, \text{iid}\}$ until (130). To upper bound the second term in (98), the steps are exactly the same as (102)-(128). When $(\dagger, \ddagger) = \{\text{sp}, \text{iid}\}$, the proof is exactly the same as the case $(\dagger, \ddagger) = \{\text{sp}, \text{sp}\}$ until (100). To upper bound the second term in (98), the proof is exactly the same as (131)-(140).

V. PROOF OF MODERATE DEVIATION ASYMPTOTICS

A. Preliminaries

Recall the following version of the Chernoff bound [26, 25, Th. B.4.1]

Lemma 11. Given an i.i.d. sequence X^n , suppose that the cumulant generating function $\Lambda_X^*(\theta)$ is finite for some positive number θ . For any $t > \mathsf{E}[X]$,

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n}X_{i}>t\right\} \leq \exp\{-n\Lambda_{X}^{*}(t)\},\tag{141}$$

and for any t < E(X),

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n}X_{i} < t\right\} \le \exp\{-n\Lambda_{X}^{*}(t)\}. \tag{142}$$

In both cases, $\Lambda_X^*(t) > 0$.

In other words, the probability decays exponentially fast if the threshold t deviates from the mean by a constant.

B. When Both Encoders Use Spherical Codebook Under JEP

For any $\theta_1 > 0$, let

$$c_n := 2\sigma^2 \theta_1 \rho_n. \tag{143}$$

The proof follows from Section IV-B until (110) with c_n taking place b_n . Combining (98) and (110), we have that

$$P_{\text{sp,sp}}^{n}(D_{1}, D_{2}|M_{1}, M_{2}) \leq \Pr\{d(X^{n}, \hat{X}_{1}^{n}) > \lambda D_{1}\} + \Pr\{d(X^{n}, \hat{X}_{1}^{n}) \leq |P_{Z} - D_{2}|\} + \exp\{-M_{2}\underline{h}(n, \lambda D_{1})\}.$$
(144)

Note that the first term in (144) is similar to the mismatched rate-distortion problem studied in [11, Section V. B] except that the distortion level is replaced from D_1 to λD_1 and the power of codebook is replaced from $P_Y = \sigma^2 - D_1$ to $P_Y = \sigma^2 - \lambda D_1$. Choose M_1 such that

$$\log M_1 = n \left(\frac{1}{2} \log \frac{\sigma^2}{\lambda D_1} + \theta_1 \rho_n \right), \tag{145}$$

Similar to [11, cf. (107)-(115)], we have that

$$\Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\} \le \exp\{-nt_1\} + \exp\left\{-\frac{n\theta_1^2 \rho_n^2}{2V(\sigma^2, \zeta)} + o(n\theta_1^2 \rho_n^2)\right\} + \exp\{-n(\theta_1 \rho_n)^{3/2} + o(\rho_n)\}$$
(146)

for some $t_1 > 0$.

To upper bound the third term in (144), for any $\theta_2 > 0$, we choose M_2 such that

$$\log M_2 = n \left(\frac{1}{2} \log \frac{\lambda D_1}{D_2} + \theta_2 \rho_n \right). \tag{147}$$

With the conditions on ρ_n in (24) and the choice of M_2 in (147), it follows that

$$\exp\{-M_2\underline{h}(n,\lambda D_1)\} = \exp\{-n\theta_2\rho_n + o(\rho_n)\}. \tag{148}$$

To upper bound the second term in (144), similarly to [11, Section V. C], we define the set

$$\mathcal{P}' := \{ r \in \mathbb{R} : b < r - \sigma^2 \le 2c_n \}. \tag{149}$$

Following the similar proof in (119), with \mathcal{P}' in place of \mathcal{P} , $\exp\{-n(\theta_1\rho_n)^{3/2}\}$ in place of $\frac{1}{\sqrt{n}}$ and $n(\theta_1\rho_n)^{3/2}$ in place of $\frac{1}{2}\log n$, we have that

$$\Pr\{d(X^{n}, \hat{X}_{1}^{n}) \geq |P_{Z} - D_{2}|\}$$

$$\geq \left(1 - \exp\{-n(\theta_{1}\rho_{n})^{3/2}\}\}\right) \Pr\{W \in \mathcal{P}' \cap \mathcal{Q}, \log M_{1} \leq -\log 2 - \log \bar{g}(n, \sigma^{2} + b, |P_{Z} - D_{2}|) - n(\theta_{1}\rho_{n})^{3/2}\}.$$
(150)

Similarly from (120) to (126), with the same choice of b in (123), we show that

$$\lim_{n \to \infty} \Pr \left\{ \log M_1 \le -\log 2 - \log \bar{g}(n, \sigma^2 + b) - n(\theta_1 \rho_n)^{3/2} \right\} = 1.$$
 (151)

Invoking Lemma 11, and moderate deviations theorem in [21, Th. 3.7.1], we have that

$$\Pr\left\{W \in \mathcal{P}' \cap \mathcal{Q}\right\} \ge \Pr\{W \in \mathcal{P}'\} - \Pr\{W \notin \mathcal{Q}\}$$

$$= \Pr\left\{\frac{1}{n}\sum_{i=1}^{n} X_{i}^{2} > \sigma^{2} + b\right\} - \Pr\left\{\frac{1}{n}\sum_{i=1}^{n} X_{i}^{2} > \sigma^{2} + 2c_{n}\right\} - \Pr\left\{\frac{1}{n}\sum_{i=1}^{n} X_{i}^{2} < |P_{Z} - D_{2}| - P_{Y}\right\}$$
(152)

$$\geq \left(1 - \exp\{-nt_2\}\right) - \exp\left\{-\frac{4n\theta_1^2 \rho_n^2}{2V(\sigma^2, \zeta)} + o(n\rho_n^2)\right\} - \Pr\left\{\frac{1}{n} \sum_{i=1}^n X_i^2 < \sigma^2 - 2P_Y\right\}$$
(154)

$$\geq \left(1 - \exp\{-nt_2\}\right) - \exp\left\{-\frac{4n\theta_1^2 \rho_n^2}{2V(\sigma^2, \zeta)} + o(n\rho_n^2)\right\} - \exp\{-nt_3\}$$
 (155)

for some $t_2 > 0$ and $t_3 > 0$. Combining the results in (144)-(148), (150), (151) and (155), we conclude that

$$\liminf_{n \to \infty} -\frac{1}{n\rho_n^2} \log P_{\text{sp,sp}}^n(D_1, D_2 | M_1, M_2) \ge \frac{\theta_1^2}{2V(\sigma^2, \zeta)}.$$
 (156)

C. When Both Encoders Use the i.i.d. Gaussian codebook Under JEP

The proof is omitted since it is similar to Section V-B except for following two points: i) replace $\bar{g}(n,w)$ with $\Psi_{\mathrm{iid}}(n,w,|P_Z-D_2|)$, ii) replace $\mathcal{P}'\cap\mathcal{Q}$ with \mathcal{P}' .

D. When Two Encoders Use Different Codebooks Under JEP

We next prove Theorem 3 when different types of codebooks are used by encoders f_1 and f_2 . When $(\dagger, \ddagger) = \{\text{sp}, \text{sid}\}$, the proof is exactly the same as the case $(\dagger, \ddagger) = \{\text{sp}, \text{sp}\}$ until (146). The rest of the proof is exactly the same as the case $(\dagger, \ddagger) = \{\text{iid}, \text{iid}\}$. When $(\dagger, \ddagger) = \{\text{iid}, \text{sp}\}$, to bound the term $\Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\}$, the steps are exactly the same as the case $(\dagger, \ddagger) = \{\text{iid}, \text{iid}\}$. The rest of the proof are exactly the same as (147)-(155).

E. Under SEP

Under ensemble SEP, each layer of encoder and decoder only consider their own performance. Set $\lambda = 1$. Thus, $P_Y = \sigma^2 - D_1$ and $P_Z = D_1 - D_2$. Note that v_{\dagger}^* is exactly the moderate deviations constant in rate-distortion problem [11, Section V.B], thus we only calculate v_{\dagger}^* in following section.

The excess-distortion probability $P_{\dagger,\dagger}^n(D_2|M_1,M_2)$ satisfies

$$P_{\dagger,\ddagger}^{n}(D_2|M_1, M_2)
= \Pr\{d(X^n, \hat{X}_2^n) > D_2\}$$
(157)

$$\leq \Pr\{d(X^n, \hat{X}_1^n) > D_1 \text{ or } d(X^n, \hat{X}_2^n) > D_2\}$$
 (158)

$$= P_{\dagger,\dagger}^n(D_1, D_2|M_1, M_2). \tag{159}$$

Recalling the results in V-B, with the choice of M_1 in (145) and M_2 in (147), we have that

$$\liminf_{n \to \infty} -\frac{1}{n\rho_n^2} \log \mathcal{P}_{\dagger,\ddagger}^n(D_2|M_1, M_2) \ge \frac{\theta_1^2}{2\mathcal{V}(\sigma^2, \zeta)}.$$
(160)

VI. PROOF OF LARGE DEVIATION ASYMPTOTICS

A. Preliminaries

Recall the definition of $\Psi_{iid}(\cdot)$ in (82), $\Phi_{iid}(\cdot)$ in (83) and the fact that Φ_{iid} depends on (x^n, y^n) only through their quadratic distortion $l = d(x^n, y^n)$ (cf. (96)). Similarly, we have that $\Psi_{iid}(n, x^n, D)$ depends on x^n only through its normalized ℓ_2 -norm $w = \frac{\|x^n\|^2}{n}$:

$$\Psi_{\text{iid}}(n, w, D) \sim \frac{\exp\{-nR_{\text{iid}}(w, P_Y, D)\}}{s^*(w, P_Y, D)\sqrt{\kappa(w, P_Y, D)}}.$$
(161)

We use the following properties of $R_{iid}(l, P, D)$ (cf. (56) and cf. (57)) in the proof.

Lemma 12. The following results hold.

- i) Given $\lambda \in (\frac{D_2}{D_1}, 1]$, $R_{\text{iid}}(l, P, D) = 0$ if $l = |D P|^+$, $R_{\text{iid}}(\sigma^2, P_Y, \lambda D_1) = \frac{1}{2} \log \frac{\sigma^2}{\lambda D_1}$ and $R_{\text{iid}}(\lambda D_1, P_Z, D_2) = \frac{1}{2} \log \frac{\lambda D_1}{D_2}$.
- ii) $s^*(l, P, D) > 0$ if and only if $l > |D P|^+$;
- iii) $R_{\mathrm{iid}}(l,P,D) = \sup_{s \geq 0} R_{\mathrm{iid}}(l,P,D)$ and thus $R_{\mathrm{iid}}(l,P,D)$ increases in l when $l \geq |D-P|^+$ and decreases in D when $D \leq l+P$.
- iv) $\Lambda_{X^2}^{*}(t) > 0$ if and only if $t > \sigma^2 = E[X^2]$.

The proof of Lemma 12 is similar to [11, Lemma 7] and thus omitted.

B. Proof under JEP (Theorem 5)

Fix $\lambda \in (\frac{D_2}{D_1}, 1]$. It follows that $\lambda D_1 > |D_2 - P_Z|^+$. Using (96), for any $l \leq \lambda D_1$, any positive δ and sufficiently large n, we have

$$\Phi_{\text{iid}}(n,l) \ge \Phi_{\text{iid}}(n,\lambda D_1) \tag{162}$$

$$\geq \exp\{-n(1+\delta)R_{\text{iid}}(\lambda D_1, P_Z, D_2)\}. \tag{163}$$

It follows from the definition of the ensemble JEP $P_{\dagger,\pm}^n(D_1,D_2|M_1,M_2)$ in (10) that

$$P_{\text{iid iid}}^n(D_1, D_2 | M_1, M_2) = \Pr\{d(X^n, \hat{X}_1^n) > D_1\} + \Pr\{d(X^n, \hat{X}_1^n) \le D_1, \ d(X^n, \hat{X}_2^n) > D_2\}. \tag{164}$$

For $\lambda \in (\frac{D_2}{D_1}, 1)$, the second term in (164) can be further upper bounded as follows,

 $\Pr\{d(X^n, \hat{X}_1^n) \le D_1, \ d(X^n, \hat{X}_2^n) > D_2\}$

$$= \int_{\substack{x^n, \mathbf{y}:\\ d(x^n, \hat{x}_1^n) \le D_1}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{iid}}(y^n(i) | \mathbf{0}^n, P_Y) \times \Pr\{\forall \ j \in [M_2]: \ d(x^n, Z^n(f_1(x^n), j)) > D_2\} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$
(165)

$$= \int_{\substack{x^n, \mathbf{y}:\\ d(x^n, \hat{x}^n) < D_1}} P_X^n(x^n) \prod_{i \in [M_1]} f_{\mathrm{iid}}(y^n(i) | \mathbf{0}^n, P_Y) \times (1 - \Phi_{\mathrm{iid}}(n, x^n, \hat{x}_1^n))^{M_2} \mathrm{d}x^n \mathrm{d}\mathbf{y}$$
(166)

$$= \int_{0}^{D_{1}} f_{L}(l) (1 - \Phi_{iid}(n, l))^{M_{2}} dl, \tag{167}$$

$$\leq \int_0^{\lambda D_1} f_L(l) (1 - \Phi_{iid}(n, l))^{M_2} dl + \Pr\{D_1 \geq d(X^n, \hat{X}_1^n) > \lambda D_1\}$$
(168)

$$\leq \int_{0}^{\lambda D_{1}} f_{L}(l) (1 - \exp\{-n(1+\delta)R_{iid}(\lambda D_{1}, P_{Z}, D_{2})\})^{M_{2}} dl + \Pr\{D_{1} \geq d(X^{n}, \hat{X}_{1}^{n}) > \lambda D_{1}\}$$
(169)

$$\leq \exp\left\{-M_2 \exp\{-n(1+\delta)R_{\text{iid}}(\lambda D_1, P_Z, D_2)\}\right\} + \Pr\{D_1 \geq d(X^n, \hat{X}_1^n) > \lambda D_1\},\tag{170}$$

where (166) follows from the definition of $\Phi_{\mathrm{iid}}(\cdot)$, (167) follows since $\Phi_{\mathrm{iid}}(n,x^n,y^n)$ depends on x^n and y^n only through their quadratic distortion $l=\frac{1}{n}\|x^n-y^n\|^2$, (168) follows since $\Phi_{\mathrm{iid}}(n,x^n,y^n)\geq 0$, (169) follows from (163) and (170) follows since $(1-a)^M\leq \exp\{-Ma\}$ for any $a\in[0,1)$.

Combining the result in (164) and (170), we have

$$P_{\text{iid,iid}}^{n}(D_{1}, D_{2}|M_{1}, M_{2}) \leq \exp\left\{-M_{2} \exp\{-n(1+\delta)R_{\text{iid}}(\lambda D_{1}, P_{Z}, D_{2})\}\right\} + \Pr\{d(X^{n}, \hat{X}_{1}^{n}) > \lambda D_{1}\}.$$
(171)

Choose M_2 such that

$$\log M_2 = n(1+2\delta)R_{\text{iid}}(\lambda D_1, P_Z, D_2) \tag{172}$$

$$=\frac{n(1+2\delta)}{2}\log\frac{\lambda D_1}{D_2}. (173)$$

It follows that

$$\exp\{-M_2 \exp\{-n(1+\delta)R_{iid}(\lambda D_1, P_Z, D_2)\}\} = \exp\{-n\delta R_{iid}(\lambda D_1, P_Z, D_2)\}\},$$
(174)

which vanishes doubly exponentially fast since $\lambda D_1 > |D_2 - P_Z|^+$ and the Claims i), ii) and iii) of Lemma 12 ensures that $R_{\text{iid}}(\lambda D_1, P_Z, D_2) > 0$.

Finally, we bound the second term in (171). Fix α_2 such that $\alpha_2 > |\lambda D_1 - P_Y|^+$. Using (161), for any $w \le \alpha_2$, any positive δ and sufficiently large n, we have

$$\Psi_{\text{iid}}(n, w, \lambda D_1) \ge \Psi_{\text{iid}}(n, \alpha_2, \lambda D_1) \tag{175}$$

$$\geq \exp\left\{-n(1+\delta)R_{\text{iid}}(\alpha_2, P_Y, \lambda D_1)\right\}. \tag{176}$$

It follows that

$$\Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\} = \mathsf{E}_{f_X^n} \left[(1 - \Pr\{d(X^n, Y^n) \le \lambda D_1 | X^n\}) \right]$$
(177)

$$= \int_{0}^{\infty} (1 - \Psi_{iid}(n, w, \lambda D_1))^{M_1} f_W(w) dw$$
 (178)

$$\leq \int_0^{\alpha_2} (1 - \Psi_{iid}(n, \alpha_2, \lambda D_1))^{M_1} f_W(w) dw + \Pr\left\{ \frac{1}{n} \sum_{i=1}^n X_i^2 \geq \alpha_2 \right\}$$
(179)

$$\leq \int_0^{\alpha_2} (1 - \exp\{-n(1+\delta)R_{iid}(\alpha_2, P_Y, \lambda D_1)\})^{M_1} f_W(w) dw + \Pr\left\{\frac{1}{n} \sum_{i=1}^n X_i^2 \geq \alpha_2\right\} \tag{180}$$

$$\leq \exp\left\{-M_1 \exp\left\{-n(1+\delta)R_{\mathrm{iid}}(\alpha_2, P_Y, \lambda D_1)\right\}\right\} + \Pr\left\{\frac{1}{n} \sum_{i=1}^n X_i^2 \geq \alpha_2\right\},\tag{181}$$

where (178) follows from the definition of $\Psi_{\text{iid}}(\cdot)$ in (82), (179) follows since $(1 - \Psi_{\text{iid}}(n, w, \lambda D_1)) \le 1$, (180) follows from (176), and (181) follows since $(1 - a)^M \le \exp\{-Ma\}$ for any $a \in [0, 1)$.

Choose M_1 such that

$$\log M_1 = n(1+2\delta)R_{\text{iid}}(\alpha_2, P_Y, \lambda D_1). \tag{182}$$

The first term of (181) vanishes doubly exponentially since $\alpha_2 > |\lambda D_1 - P_Y|^+$ and Lemma 12 implies that $R_{\text{iid}}(\alpha_2, P_Y, \lambda D_1) > 1$ 0. Using Cramér's Theorem [21, Th. 2.2.3] and the definition of $\Lambda_{X^2}^*(\cdot)$, we have

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n}X_{i}^{2} \ge \alpha_{2}\right\} \le \exp\{-n\Lambda_{X^{2}}^{*}(\alpha_{2})\}.$$
(183)

Combining the results in (171), (181) and (183), we have

$$P_{\text{iid,iid}}^{n}(D_{1}, D_{2}|M_{1}, M_{2}) \leq \exp\left\{-M_{2} \exp\left\{-n(1+\delta)R_{\text{iid}}(\lambda D_{1}, P_{Z}, D_{2})\right\}\right\} \\
+ \exp\left\{-M_{1} \exp\left\{-n(1+\delta)R_{\text{iid}}(\alpha_{2}, P_{Y}, \lambda D_{1})\right\}\right\} + \exp\left\{-n\Lambda_{X^{2}}^{*}(\alpha_{2})\right\}, \tag{184}$$

where the first two terms in (184) vanish doubly exponentially fast with the choice of M_2 in (173) and M_1 in (182). Recall that $\Lambda_{X^2}^*(t) = 0$ if $t \leq \sigma^2$, $R_{\mathrm{iid}}(\sigma^2, P_Y, \lambda D_1) = \frac{1}{2}\log\frac{\sigma^2}{\lambda D_1}$ and $R_{\mathrm{iid}}(\lambda D_1, P_Z, D_2) = \frac{1}{2}\log\frac{\lambda D_1}{D_2}$. Letting $\delta \downarrow 0$, we conclude that for any $\lambda \in (\frac{D_2}{D_1}, 1]$,

$$E^*(D_1, D_2|R_1, R_2) \ge \Lambda_{X^2}^*(\alpha_2),$$
 (185)

where R_1 is determined from $R_1 = R_{iid}(\alpha_2, P_Y, \lambda D_1)$ and R_2 is determined from $R_2 = \frac{1}{2} \log \frac{\lambda D_1}{D_2}$. The proof Theorem 5 is now completed.

C. Proof under SEP (Theorem 8)

We start with the proof of $P_{iid}^n(D_1|M_1,M_2)$. Fix $\alpha_1 > |D_1 - P_Y|^+$. Using (161), for any $w \le \alpha_1$, any positive δ and sufficiently large n, we have

$$\Psi_{\text{iid}}(n, w, D_1) \ge \Psi_{\text{iid}}(n, \alpha_1, D_1) \tag{186}$$

$$\geq \exp\{-n(1+\delta)R_{\text{iid}}(\alpha_1, P_Y, D_1)\}.$$
 (187)

Recall the definition of $P_{iid}^n(D_1|M_1,M_2)$ in (8). Similar to [11, cf. (154)-(158)],

$$P_{iid}^{n}(D_1|M_1, M_2) = \Pr\{d(X^n, \hat{X}_1^n) > D_1\}$$
(188)

$$\leq \exp\left\{-M_1 \exp\left\{-n(1+\delta)R_{\text{iid}}(\alpha_1, P_Y, D_1)\right\}\right\} + \Pr\left\{\frac{1}{n}\sum_{i=1}^n X_i^2 \geq \alpha_1\right\}.$$
 (189)

Choose M_1 such that

$$\log M_1 = n(1+2\delta)R_{\text{iid}}(\alpha_{\cdot}P_Y, D_1). \tag{190}$$

The first term of (189) vanishes doubly exponentially fast which follows from Lemma 12 and the fact that $\alpha_1 > |D_1 - P_Y|^+$. Invoking the definition of Cramér's Theorem [21, Th. 2.2.3] and the definition of $\Lambda_{X^2}^*(\cdot)$, we obtain that

$$\Pr\left\{\frac{1}{n}\sum_{i=1}^{n}X_{i}^{2} \ge \alpha_{1}\right\} \le \exp\{-n\Lambda_{X^{2}}^{*}(\alpha_{1})\}.$$
(191)

Recall that $\Lambda_{X^2}^*(t)=0$ if $t\leq \sigma^2$ and $|D_1-P_Y|^+<\sigma^2$. Letting $\delta\downarrow 0$, we conclude that for any $\alpha>|D_1-P_Y|^+$,

$$\liminf_{n \to \infty} -\frac{1}{n} \log P_{\text{iid}}^n(D_1|M_1, M_2) \ge \Lambda_{X^2}^*(\alpha_1), \tag{192}$$

where R_1 is determined by of $R_1 = R_{\text{iid}}(\alpha_1, P_Y, D_1)$. We next analyze $P_{\dagger}^n(D_2|M_1, M_2)$ when $R_2 \leq \frac{1}{2}\log\frac{D_1}{D_2}$. Fix $\lambda \in (\frac{D_2}{D_1}, 1]$, it follows that $\lambda D_1 > |D_2 - P_Z|^+$. Using (96), for any $l \leq \lambda D_1$, any positive δ and sufficiently large n, we have

$$\Phi_{\text{iid}}(n,l) > \Phi_{\text{iid}}(n,\lambda D_1) \tag{193}$$

$$\geq \exp\{-n(1+\delta)R_{\text{iid}}(\lambda D_1, P_Z, D_2)\}. \tag{194}$$

Recall the definition of $P_{\dagger}^{n}(D_{2}|M_{1},M_{2})$ in (9), we have

$$P_{iid}^n(D_2|M_1,M_2)$$

$$= \Pr\{d(X^n, \hat{X}_2^n) > D_2\}$$
(195)

$$= \int_{x^n, \mathbf{y}} P_X^n(x^n) \prod_{i \in [M_1]} f_{iid}(y^n(i) | \mathbf{0}^n, P_Y) \times \Pr\{\forall \ j \in [M_2] : \ d(x^n, Z^n(f_1(x^n), j)) > D_2\} dx^n d\mathbf{y}$$
 (196)

$$= \int_{x^n, \mathbf{y}} P_X^n(x^n) \prod_{i \in [M_1]} f_{iid}(y^n(i) | \mathbf{0}^n, P_Y) \times (1 - \Phi_{iid}(n, x^n, \hat{x}_1^n))^{M_2} dx^n d\mathbf{y}$$
(197)

$$= \int_{0}^{\infty} f_{L}(l) (1 - \Phi_{iid}(n, l))^{M_{2}} dl$$
 (198)

$$\leq \int_0^{\lambda D_1} f_L(l) (1 - \Phi_{iid}(n, l))^{M_2} dl + \Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\}$$
(199)

$$\leq \int_{0}^{\lambda D_{1}} f_{L}(l) \left(1 - \exp\{-n(1+\delta)R_{iid}(\lambda D_{1}, P_{Z}, D_{2})\}\right)^{M_{2}} dl + \Pr\{d(X^{n}, \hat{X}_{1}^{n}) > \lambda D_{1}\}$$
(200)

$$\leq \exp\left\{-M_2 \exp\{-n(1+\delta)R_{iid}(\lambda D_1, P_Z, D_2)\}\right\} + \Pr\{d(X^n, \hat{X}_1^n) > \lambda D_1\},\tag{201}$$

where (197) follows from the definition of $\Phi_{\mathrm{iid}}(\cdot)$, (198) follows since $\Phi_{\mathrm{iid}}(n,x^n,y^n)$ depends on x^n and y^n only through $l=\frac{1}{n}\|x^n-y^n\|^2$, (199) follows since $\Phi_{\mathrm{iid}}(n,x^n,y^n)\geq 0$, (200) follows from (194) and (201) follows since $(1-a)^M\leq \exp\{-Ma\}$ for any $a\in[0,1)$.

Note that (201) is the same as (171), the rest of the proof follows from (173) to (185) and thus omitted.

Finally, we study the case when $R_2 > \frac{1}{2}\log\frac{D_1}{D_2}$. Since parameter λ in introduced to adapt the situation that R_1 is large but R_2 is small, we set $\lambda = 1$ in this case. Fix $\gamma > D_1$. Using (96), for any $l \leq \gamma$, any positive δ and sufficiently large n, we have

$$\Omega(n,l) \ge \Omega(n,\gamma_2) \tag{202}$$

$$\geq \exp\{-n(1+\delta)R_{\text{iid}}(\gamma, P_Z, D_2)\}. \tag{203}$$

Similar to the case where $R_2 \leq \frac{1}{2} \log \frac{D_1}{D_2}$, the excess-distortion probability $P_{\ddagger}^n(D_2|M_1,M_2)$ can be upper bounded as follows,

$$P_{iid}^{n}(D_2|M_1, M_2) = \int_0^\infty f_L(l)(1 - \Phi_{iid}(n, l))^{M_2} dl$$
(204)

$$\leq \int_0^{\gamma} f_L(l) (1 - \Omega(n, l))^{M_2} dl + \Pr\{d(X^n, \hat{X}_1^n) > \gamma\}$$
 (205)

$$\leq \int_0^{\gamma} f_L(l) (1 - \exp\{-n(1+\delta)R_{iid}(\gamma, P_Z, D_2)\})^{M_2} dl + \Pr\{d(X^n, \hat{X}_1^n) > \gamma\}$$
 (206)

$$\leq \exp\left\{-M_2 \exp\{-n(1+\delta)R_{\text{iid}}(\gamma, P_Z, D_2)\}\right\} + \Pr\{d(X^n, \hat{X}_1^n) > \gamma\}. \tag{207}$$

Choose M_2 such that

$$\log M_2 = n(1+2\delta)R_{iid}(\gamma, P_Z, D_2). \tag{208}$$

It follows that

$$\exp\{-M_2 \exp\{-n(1+\delta)R_{iid}(\gamma, P_Z, D_2)\}\} = \exp\{\exp\{-n\delta R_{iid}(\gamma, P_Z, D_2)\}\},\tag{209}$$

which vanishes doubly exponentially fast since $\gamma > |D_2 - P_Z|^+$ and Lemma 12 ensure that $R_{iid}(\gamma, P_Z, D_2) > 0$.

Note that $\Pr\{d(X^n, \hat{X}_1^n) > \gamma\}$ is similar to the second term in (171), except we change the distortion level to from λD_1 to γ . Fix α_2 such that $\alpha_2 > |\gamma - P_Y|^+$. Similarly to (178)-(181), we have

$$\Pr\{d(X^n, \hat{X}_1^n) > \gamma\} \le \exp\{-M_1 \exp\{-n(1+\delta)R_{\text{iid}}(\alpha_2, P_Y, \gamma)\}\} + \Pr\left\{\frac{1}{n}\sum_{i=1}^n X_i^2 \ge \alpha_2\right\}.$$
 (210)

Choose M_1 such that

$$\log M_1 = n(1+2\delta)R_{\text{iid}}(\alpha_2, P_Y, \gamma). \tag{211}$$

It follows that the first term of (210) vanishes doubly exponentially since $\alpha_2 > |\gamma - P_Y|^+$ and Lemma 12 implies that $R_{\text{iid}}(\alpha_2, P_Y, \gamma) > 0$. The rest of the proof is similar to (183)-(185) and thus omitted.

VII. CONCLUSION

We derived achievable refined asymptotics for the mismatched successive refinement problem where one uses Gaussian codebooks and successive minimum Euclidean distance encoding to compress an arbitrary memoryless source satisfying mild moment constraints. We studied the performance of all four combinations of spherical and i.i.d. Gaussian codebooks in both second-order and moderate deviations asymptotics. For large deviations, we only studied the case where both encoders use i.i.d. Gaussian codebooks and showed that our derived exponent under JEP and exponents under SEP are all positive for rate pairs strictly inside the rate-distortion region of a GMS in the matched case.

There are several future research directions. Firstly, one can derive the ensemble converse results of the rate-distortion region and the second-order, moderate and large deviations, especially when either encoder uses a spherical codebook, in order to check whether our achievability results are ensemble tight. Secondly, for a GMS under quadratic distortion measures in the matched successive refinement problem, it is worthwhile to derive exact large deviations asymptotics under both JEP and SEP and check whether our result is optimal when specialized to a GMS. Finally, one could generalize the results in this paper to mismatched settings of other multiterminal lossy source coding problems, e.g., the Kaspi problem [27]–[29], the lossy Gray-Wyner problem [30], [31] and the multiple descriptions problem [32]–[34].

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APPENDIX

A. Justification of the Conditions in Case (ii) of Theorem 7

We first prove that $R_{iid}(|D_2 - P_Z|^+, P_Z, D_2) < \frac{1}{2} \log \frac{D_1}{D_2}$ as follows.

Proof. Recall that $\lambda = 1$, $P_Y = \sigma^2 - D_1$ and $P_Z = D_1 - D_2$. It follows that

$$|D_2 - P_Z|^+ = \max\{0, 2D_2 - D_1\}$$
(212)

$$< D_1, (213)$$

where (213) follows since $D_2 < D_1$. Recall that $R_{\text{iid}}(l, P, D)$ increases in l for $l \ge |D - P|^+$ (cf. Lemma 12, Claim iii)), we have that

$$R_{\text{iid}}(|D_2 - P_Z|^+, P_Z, D_2) < R_{\text{iid}}(D_1, P_Z, D_2)$$
 (214)

$$=\frac{1}{2}\log\frac{D_1}{D_2},\tag{215}$$

where (215) follows from Claim i), Lemma 12.

We next prove that $R_{\text{iid}}(\max\{\sigma^2, \gamma_2 - P_Y\}, P_Y, \gamma_2) > \frac{1}{2}\log\frac{\sigma^2}{D_1}$ as follows.

Proof. Recall that γ_2 is the solution to $R_2 = R_{\text{iid}}(\gamma_2, P_Z, D_2)$. It follows that

$$R_{\text{iid}}(\gamma_2, P_Z, D_2) = R_2 \tag{216}$$

$$<\frac{1}{2}\log\frac{D_1}{D_2}$$
 (217)

$$= R_{iid}(D_1, P_Z, D_2), \tag{218}$$

where (217) follows from the conditions on R_2 in Case (ii) in Theorem 7. Thus, we have that $\gamma_2 < D_1$ since $R_{\text{iid}}(l, P, D)$ increases in l for $l \ge |D - P|^+$ (cf. Lemma 12, Claim iii)). Finally, we have that

$$R_{\text{iid}}(\max\{\sigma^2, \gamma_2 - P_Y\}, P_Y, \gamma_2) > R_{\text{iid}}(\sigma^2, P_Y, \gamma_2)$$

$$\tag{219}$$

$$> R_{\rm iid}(\sigma^2, P_Y, D_1) \tag{220}$$

$$=\frac{1}{2}\log\frac{\sigma^2}{D_1},\tag{221}$$

where (220) follows since $R_{iid}(l, P, D)$ decreases in D for $D \le l + P$ (cf. Claim iii) of Lemma 12) and (221) follows from Claim i) of Lemma 12.

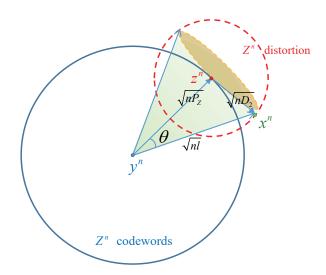


Fig. 9. Geometry of the excess-distortion probability calculation of encoder f_2 .

B. Proof of Lemma 10

Recall the definition of $\Phi_{\rm sp}(n,x^n,y^n)$ in (83), β_1 in (84) and β_2 in (85).

Note that $\Phi_{\rm sp}(n,x^n,y^n)$ depends on (x^n,y^n) only through their quadratic distortion $l=d(x^n,y^n)$. Similarly to [23, Theorem 37], $\Phi_{\rm sp}(n,x^n,y^n)=0$ if $\sqrt{l} \leq |\beta_1|^+$ or $\sqrt{l} \geq \beta_2$. Let $S_n(r)=\frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}r^{n-1}$ be the surface area of an n-dimensional sphere of radius r and $S_n(r,\theta)$ be the surface area of an n-dimensional polar cap of radius r and polar angle θ . For $\sqrt{l} \in [\max\{0,\beta_1\},\beta_2]$, similarly to [23, Theorem 37],

$$\Phi_{\rm sp}(n,l) = \frac{S_n(l,\theta)}{S_n(l)} \tag{222}$$

$$\geq \frac{\Gamma(\frac{n}{2}+1)}{\sqrt{\pi}n\Gamma(\frac{n-1}{2}+1)}(\sin\theta)^{n-1} \tag{223}$$

$$= \frac{\Gamma(\frac{n+2}{2})}{\sqrt{\pi n}\Gamma(\frac{n+1}{2})} \left(1 - \frac{(l+P_Z - D_2)^2}{4lP_Z}\right)^{\frac{n-1}{2}}$$
(224)

$$=\underline{h}(n,l),\tag{225}$$

where (223) follows since $S_n(r,\theta) \ge \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2}+1)} (r\sin\theta)^{n-1}$ (cf. Fig. 9) and (224) follows from the law of cosines.

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