Martingale Methods for Sequential Estimation of Convex Functionals and Divergences

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March 14, 2023

Abstract

We present a unified technique for sequential estimation of convex divergences between distributions, including integral probability metrics like the kernel maximum mean discrepancy, φ -divergences like the Kullback-Leibler divergence, and optimal transport costs, such as powers of Wasserstein distances. This is achieved by observing that empirical convex divergences are (partially ordered) reverse submartingales with respect to the exchangeable filtration, coupled with maximal inequalities for such processes. These techniques appear to be complementary and powerful additions to the existing literature on both confidence sequences and convex divergences. We construct an offline-to-sequential device that converts a wide array of existing offline concentration inequalities into time-uniform confidence sequences that can be continuously monitored, providing valid tests or confidence intervals at arbitrary stopping times. The resulting sequential bounds pay only an iterated logarithmic price over the corresponding fixed-time bounds, retaining the same dependence on problem parameters (like dimension or alphabet size if applicable). These results are also applicable to more general convex functionals—like the negative differential entropy, suprema of empirical processes, and V-Statistics—and to more general processes satisfying a key leave-one-out property.

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1 Introduction

Divergences between probability distributions arise pervasively in information theory, statistics, and machine learning (Liese and Vajda, 2006; Sriperumbudur et al., 2012). Common examples include φ -divergences, such as the Kullback-Leibler divergence, integral probability metrics (IPMs), such as the kernel maximum mean discrepancy, and optimal transport costs, such as powers of Wasserstein distances. Increasingly many applications employ such quantities as methodological tools, in which it is of interest to estimate a given divergence between unknown probability distributions on the basis of random samples thereof. Statistical inference is well-studied for many divergences when the data is available in a fixed batch ahead of the user's analysis. When data is instead collected sequentially in time, such methods are typically invalid for repeatedly assessing uncertainty of a divergence estimate, and may therefore provide overly optimistic confidence.

The aim of this paper is to develop rigorous sequential uncertainty quantification methods for a large class of divergences between probability distributions. Throughout the sequel, we denote by $D: \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+$ a generic convex divergence, where $\mathcal{P}(\mathcal{X})$ denotes the set of Borel probability measures over a set $\mathcal{X} \subseteq \mathbb{R}^d$. Recall that the functional D is said to be convex if for all measures $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X})$ and all $\lambda \in [0, 1]$,

$$D(\lambda \mu_1 + (1 - \lambda)\mu_2 \parallel \lambda \nu_1 + (1 - \lambda)\nu_2) \le \lambda D(\mu_1 \parallel \nu_1) + (1 - \lambda)D(\mu_2 \parallel \nu_2). \tag{1}$$

This property can be verified for IPMs, φ -divergences, and optimal transport costs (see Section 2.1).

Given two independent sequences $(X_t)_{t=1}^{\infty}$ and $(Y_s)_{s=1}^{\infty}$ of i.i.d. observations arising respectively from unknown distributions $P, Q \in \mathcal{P}(\mathcal{X})$, we aim to construct a sequence of confidence intervals $(C_{ts})_{t,s=1}^{\infty}$ with the uniform coverage property

$$\mathbb{P}(\forall t, s \ge 1 : D(P||Q) \in C_{ts}) \ge 1 - \delta, \tag{2}$$

for some pre-specified level $\delta \in (0,1)$. Such a sequence $(C_{ts})_{t,s=1}^{\infty}$ is called a *confidence sequence*, and differs from the standard notion of confidence interval through the uniformity in times t, s of the probability in equation (2). As stated precisely in Section 3.3, the guarantee (2) is equivalent to the requirement that for all stopping times (τ, σ) ,

$$\mathbb{P}(D(P||Q) \in C_{\tau\sigma}) \ge 1 - \delta. \tag{3}$$

The scope of potential applications of such confidence sequences is far-reaching. For instance, a confidence sequence C_{ts} directly gives rise to a sequential two-sample test for the null hypothesis $H_0: P = Q$, where the null is rejected when $0 \notin C_{ts}$. Fixed-time, two-sample testing is a classical problem which continues to receive a wealth of attention, but sequential, nonparametric two-sample testing is relatively less explored; two exceptions include Balsubramani and Ramdas (2016), Lhéritier and Cazals (2018). We also note that confidence sequences for divergences can sometimes lead to confidence sequences for other estimands of interest, as illustrated in Section 4.5. Though our work is motivated by such practical applications, our results can also be viewed from a purely theoretical standpoint as deriving concentration inequalities for divergences which hold uniformly over time. We are not aware of analogous time-uniform concentration inequalities in the literature for the majority of divergences studied explicitly in this paper.

Our Contributions. The primary contribution of our work is to provide a general recipe for deriving confidence sequences for convex divergences D. Our key observation is that the process

$$M_{ts} = D(P_t || Q_s) - D(P || Q), \quad t, s \ge 1,$$
 (4)

is a partially-ordered reverse submartingale, with respect to the so-called exchangeable filtration introduced below. Here, $P_t = (1/t) \sum_{i=1}^t \delta_{X_i}$ and $Q_s = (1/s) \sum_{i=1}^s \delta_{Y_j}$ denote empirical measures. A related property was previously identified by Pollard (1981) for suprema of empirical processes. This reverse submartingale property allows us to apply maximal inequalities to (functions of) $(M_{ts})_{t,s=1}^{\infty}$, which will lead to confidence sequences for D(P||Q) based on the plug-in estimator $D(P_t||Q_s)$. We note that this estimator is inconsistent for divergences requiring the absolute continuity of the probability measures being compared, such as φ -divergences for distributions supported over \mathbb{R}^d . We therefore extend our results by showing that the process in equation (4) continues to be a reverse submartingale in each of its indices when P_t and Q_s are replaced by their smoothed counterparts, $P_t \star \mathcal{K}_{\sigma}$ and $Q_s \star \mathcal{K}_{\sigma}$, where \mathcal{K}_{σ} denotes a kernel with bandwidth σ .

We illustrate these findings by deriving explicit confidence sequences for the kernel Maximum Mean Discrepancy (MMD), Wasserstein distances, Total Variation distance, and Kullback-Leibler divergence, among others, some for distributions over finite alphabets and others for arbitrary distributions. In all cases, we take care to track the effect of dimensionality, matching the best known rates is non-sequential settings. To the best of our knowledge, there are no other existing confidence sequences for these quantities, apart from the (statistically suboptimal but computationally efficient) linear-time kernel MMD (Balsubramani and Ramdas, 2016). We also derive a sequential analogue of the celebrated Dvoretzky-Kiefer-Wolfowitz inequality (Dvoretzky et al., 1956; Massart, 1990) quite differently from both Howard and Ramdas (2022) and a recent preprint by Maillard (2021), and demonstrate how these results can be used to obtain confidence sequences for convex functionals which do not necessarily arise from divergences.

Outline. We organize this manuscript as follows. In Section 2, we provide background on martingales and maximal inequalities, which form the main technical tools used throughout this paper.

We also provide background on sequential analysis and on several classes of convex divergences. In Section 3, we state our main results and derive a general confidence sequence for D(P||Q) on the basis of the process (M_{ts}) , which also applies to more general convex functionals. We explicitly illustrate its application to a wide range of commonly-used divergences in Section 4. We close with a conclusion in Section 5. Most proofs of our results are relegated to Appendices A–E of the Supplementary Material.

Notation. For a vector $x \in \mathbb{R}^d$, and for some integer $p \geq 1$, $||x||_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ denotes the ℓ_p norm of x, and $||x||_{\infty} = \max_{1 \leq i \leq d} |x_i|$ denotes its ℓ_{∞} norm. We also denote the L^{∞} norm of a function $f : \mathbb{R}^d \to \mathbb{R}$ by $||f||_{\infty} = \operatorname{esssup}_{x \in \mathbb{R}^d} |f(x)|$. We abbreviate sequences $(A_t)_{t=1}^{\infty}$ of sets, functions, or real numbers by (A_t) when doing so causes no confusion. For any $a,b \in \mathbb{R}$, $a \vee b := \max\{a,b\}$ and $a \wedge b := \min\{a,b\}$. The diameter of a set $A \subseteq \mathbb{R}^d$ is given by $\operatorname{diam}(A) = a \vee b = a \vee b$ $\sup\{\|x-y\|_2:x,y\in A\}$. The Dirac measure placing mass at a point $x\in\mathbb{R}^d$ is denoted δ_x . The Borel σ -algebra on $\mathcal{X} \subseteq \mathbb{R}^d$ is denoted $\mathbb{B}(\mathcal{X})$. The floor and ceiling of a real number $x \in \mathbb{R}$ are respectively denoted |x| and [x]. We write $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}, \mathbb{R}_+ = [0, \infty), \mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{1, 2, \dots\}$ $\{0,1,\ldots\}$. The convolution of functions $f,g:\mathbb{R}^d\to\mathbb{R}$ is denoted by $(f\star g)(x)=\int f(x-y)g(y)dy$. Furthermore, the convolution of two Borel probability measures P, Q is the measure $(P \star Q)(B) =$ $\int I_B(x+y)dP(x)dQ(y)$ for all $B\in\mathbb{B}(\mathbb{R}^d)$, where $I_B(x)=I(x\in B)$ is the indicator function of B. The convolution of P with f is the function $(P \star f)(x) = \int f(x-y)dP(y)$. For a convex function $\varphi : \mathbb{R} \to \overline{\mathbb{R}}$, its Legendre-Fenchel transform is denoted $\varphi^*(y) = \sup_{x \in \mathbb{R}} \{x^\top y - \varphi(x)\}$. The logarithm of base b>1 is denoted $\log_b(x)=\log x/\log b$, where \log is the natural logarithm. An \mathbb{R}^d -valued random variable Y is said to be σ^2 -sub-Gaussian if $\mathbb{E}\exp(\lambda^\top(Y-\mathbb{E}Y))\leq \exp(\|\lambda\|_2^2\sigma^2/2)$ for all $\lambda \in \mathbb{R}^d$. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the Lebesgue spaces of equivalence classes are denoted $L^p(\mathbb{P})$ for $p \geq 1$. For any two sub- σ -algebras $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$, we use the standard notation for joins and intersections of σ -algebras, respectively given by $\mathcal{G} \vee \mathcal{H} := \sigma(\mathcal{G} \cup \mathcal{H})$ and $\mathcal{G} \wedge \mathcal{H} := \mathcal{G} \cap \mathcal{H}$ to emphasize that they both result in σ -algebras.

2 Background

2.1 IPMs, Optimal Transport Costs, and φ -Divergences

Let $\mathcal{X} \subseteq \mathbb{R}^d$ and let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$ be a set of pairs of probability measures. Throughout this paper, we use the term divergence to refer to any map $D: \mathcal{A} \to \mathbb{R}$ which is nonnegative and satisfies D(P||Q) = 0 if P = Q, for all $(P,Q) \in \mathcal{A}$. When the divergence D is convex, we extend the domain of D from \mathcal{A} to the entire set $\mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X})$ by letting D take the value ∞ wherever it is not defined—as such, convex divergences will always be understood as maps $D: \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}$.

We consider sequential estimation of generic convex divergences, but the following three classes will be used as recurring examples throughout our development. Let $P, Q \in \mathcal{P}(\mathcal{X})$ in the sequel.

• Integral Probability Metrics (IPMs). Let \mathcal{J} denote a set of Borel-measurable, real-valued functions on \mathcal{X} . The IPM (Müller, 1997) associated with \mathcal{J} is given by

$$D_{\mathcal{J}}(P||Q) = \sup_{f \in \mathcal{J}} \int f d(P - Q). \tag{5}$$

For instance, when P and Q have supports contained in \mathbb{R} , the class of indicator functions $\mathcal{J} = \{I_{(-\infty,x]} : x \in \mathbb{R}\}$ gives rise to the Kolmogorov-Smirnov distance, $D_{\mathcal{J}}(P\|Q) = \|F - G\|_{\infty}$. When \mathcal{J} is the unit ball of a reproducing kernel Hilbert space, $D_{\mathcal{J}}$ is called the (kernel) Maximum Mean Discrepancy (Gretton et al. (2012); see also Section 4.2). When \mathcal{J} is the set of Borel-measurable maps $f: \mathbb{R}^d \to \mathbb{R}$ satisfying $\|f\|_{\infty} \leq 1$, $D_{\mathcal{J}}$ becomes the total variation distance,

$$||P - Q||_{\text{TV}} = \sup_{A \in \mathbb{B}(\mathcal{X})} |P(A) - Q(A)|. \tag{6}$$

When the function space \mathcal{J} is sufficiently small, $D_{\mathcal{J}}(P_t||Q_s)$ is a consistent estimator of $D_{\mathcal{J}}(P||Q)$, and will form the basis of our confidence sequences. We refer to Sriperumbudur et al. (2012) for a study of convergence rates for such plug-in estimators.

• Optimal Transport Costs. Let $\Pi(P,Q)$ denote the set of joint probability distributions $\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ with marginals P,Q, that is, satisfying $\pi(B \times \mathcal{X}) = P(B)$ and $\pi(\mathcal{X} \times B) = Q(B)$ for all $B \in \mathbb{B}(\mathcal{X})$. Given a nonnegative cost function $c : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, the optimal transport cost between P and Q is given by

$$\mathcal{T}_c(P,Q) = \inf_{\pi \in \Pi(P,Q)} \int c(x,y) d\pi(x,y). \tag{7}$$

 $\mathcal{T}_c(P,Q)$ admits the natural interpretation of measuring the work required to couple the measures P and Q—we refer to Panaretos and Zemel (2019); Villani (2003) for surveys. When c takes the form $c=d^p$ for some $p \geq 1$ and some metric $d: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, the quantity $W_p := \mathcal{T}_{d^p}^{1/p}$ becomes a metric on $\mathcal{P}_p(\mathcal{X}) = \{P \in \mathcal{P}(\mathcal{X}) : \int d(x,x_0)^p dP(x) < \infty, x_0 \in \mathcal{X}\}$ known as the p-Wasserstein distance.

The minimization problem (7) is an infinite-dimensional linear program, and admits a dual formulation known as the Kantorovich dual problem. When c is lower semi-continuous, strong duality holds (Villani (2003), Theorem 1.3), leading to the following representation,

$$\mathcal{T}_c(P,Q) = \sup_{(f,g)\in\mathcal{M}_c} \int f dP + \int g dQ, \tag{8}$$

where $\mathcal{M}_c \equiv \mathcal{M}_c(P,Q)$ denotes the set of pairs (f,g) of functions $f \in L^1(P)$ and $g \in L^1(Q)$ such that $f(x) + g(y) \leq c(x,y)$ for P-almost all $x \in \mathcal{X}$ and Q-almost all $y \in \mathcal{X}$. Moreover, when the cost function c is bounded, the supremum in the above display is achieved and the pairs $(f,g) \in \mathcal{M}_c$ may be restricted to take the form $0 \leq f \leq ||c||_{\infty}, -||c||_{\infty} \leq g \leq 0$. Finally, when c = d is itself a metric, the Kantorovich-Rubinstein formula implies that \mathcal{M}_c may be further reduced to the set of pairs $\{(f,-f): f \in \mathcal{J}\}$, with \mathcal{J} denoting the set of 1-Lipschitz functions with respect to d. In this case, \mathcal{T}_d is precisely the IPM generated by \mathcal{J} . We refer to Fournier and Guillin (2015), Sommerfeld and Munk (2018), Niles-Weed and Rigollet (2022), Chizat et al. (2020), Manole and Niles-Weed (2021), Hundrieser et al. (2022), and references therein for convergence rates and fixed-time concentration inequalities for plug-in estimators of optimal transport costs and Wasserstein distances.

• φ -Divergences. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function, and let $\nu \in \mathcal{P}(\mathcal{X})$ be a σ -finite measure which dominates both P and Q (for instance, $\nu = (P+Q)/2$). Let $p = dP/d\nu$ and $q = dQ/d\nu$

be the respective densities. Then, the φ -divergence (Ali and Silvey, 1966) between P and Q is given by

$$D_{\varphi}(P||Q) = \int_{q>0} \varphi\left(\frac{p}{q}\right) dQ + P(q=0) \lim_{x \to \infty} \frac{\varphi(x)}{x}, \tag{9}$$

with the convention that the second term of the above display is equal to zero whenever P(q=0)=0, which is in particular the case if $P\ll Q$. For instance, assuming the latter condition holds, the Kullback-Leibler divergence $\mathrm{KL}(P\|Q)=\int\log\left(\frac{dP}{dQ}\right)dP$ corresponds to the map $\varphi(x)=x\log x$. The total variation distance in equation (6) is the unique nontrivial IPM which is also a φ -divergence (Sriperumbudur et al., 2012). φ -divergences also admit the variational representation

$$D_{\varphi}(P||Q) \ge \sup_{g \in \mathcal{J}} \int \left[gdQ - (\varphi^* \circ g)dP \right], \tag{10}$$

for any collection \mathcal{J} of functions mapping \mathcal{X} to \mathbb{R} . Equality holds in the above display if and only if the subdifferential $\partial \varphi(dQ/dP)$ contains an element of \mathcal{J} (Nguyen et al., 2010).

Unlike most common IPMs and optimal transport costs, φ -divergences are typically uninformative when P is not absolutely continuous with respect to Q, as the expression (9) becomes dominated by its second term. This fact sometimes prohibits the estimation of φ -divergences via the plug-in estimator $D(P_t||Q_s)$ —for instance, P_t is almost surely not absolutely continuous with respect to Q_s when P and Q are both absolutely continuous with respect to the Lebesgue measure. One exception is the situation where P and Q are supported on countable sets, in which case Berend and Kontorovich (2013), Agrawal and Horel (2020), Guo and Richardson (2020), Cohen et al. (2020), Han et al. (2015), Kamath et al. (2015), study concentration and convergence rates of the empirical measure P_t under the Kullback-Leibler and Total Variation divergences. We develop time-uniform bounds which build upon these results in Section 4.4. For distributions P and Q which are not countably-supported, distinct estimators have been developed by Nguyen et al. (2010), Póczos and Schneider (2012), Sricharan et al. (2010), Krishnamurthy et al. (2015), Rubenstein et al. (2019), Singh and Póczos (2014), Wang et al. (2005), Berrett and Samworth (2019), and references therein.

The following Lemma is standard, and stated without proof.

Lemma 1. For any class \mathcal{J} of Borel-measurable functions from \mathbb{R}^d to \mathbb{R} , the IPM generated by \mathcal{J} is convex. Furthermore, the φ -divergence generated by any convex function $\varphi : \mathbb{R} \to \mathbb{R}$ is convex. Finally, for any nonnegative cost function c, the optimal transport cost \mathcal{T}_c is convex.

Though our main focus is on the above divergences, our results also apply to generic convex functionals $\Phi: \mathcal{P}(\mathcal{X}) \to \mathbb{R}$, which satisfy $\Phi(\lambda \mu_1 + (1-\lambda)\mu_2) \leq \lambda \Phi(\mu_1) + (1-\lambda)\Phi(\mu_2)$ for all $\lambda \in [0,1]$ and $\mu_1, \mu_2 \in \mathcal{P}(\mathcal{X})$. Notice that for any fixed distribution $Q \in \mathcal{P}(\mathcal{X})$, the map $\Phi(P) = D(P||Q)$ is a convex functional—additional examples include the negative differential entropy (cf. Section 4.5) and certain expectation functionals (cf. Section 4.2).

Maximal martingales inequalities form a key tool in the development of confidence sequences, thus we provide an overview in what follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, over which all processes hereafter will be taken. Before discussing time-reversed concepts, it is useful to first briefly overview standard martingales and filtrations.

2.2 Forward filtrations, martingales and maximal inequalities

A forward filtration is a sequence of σ -algebras $(\mathcal{F}_t)_{t=1}^{\infty}$ contained in \mathcal{F} which is nondecreasing:

$$\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$$
, for all $t \geq 1$.

Any process $(S_t)_{t=1}^{\infty}$ on Ω is adapted to its canonical (forward) filtration $(C_t)_{t=1}^{\infty}$ defined by $C_t = \sigma(S_1, \ldots, S_t)$, for all $t \geq 1$. A (forward) martingale with respect to a (forward) filtration $(\mathcal{F}_t)_{t=1}^{\infty}$ is a stochastic process $(S_t)_{t=1}^{\infty}$ such that for all $t \geq 1$, S_t is \mathbb{P} -integrable, \mathcal{F}_t -measurable, and satisfies

$$\mathbb{E}[S_{t+1}|\mathcal{F}_t] = S_t, \quad \text{ for all } t \ge 1.$$

Supermartingales and submartingales are respectively defined by replacing the equality in the above display by \leq and \geq . When it causes no confusion, we frequently abbreviate $(S_t)_{t=1}^{\infty}$ and $(\mathcal{F}_t)_{t=1}^{\infty}$ by (S_t) and (\mathcal{F}_t) . The construction of confidence sequences typically relies on maximal inequalities. A prominent example is Ville's inequality (Ville, 1939), which states that any nonnegative supermartingale (S_t) satisfies

$$\mathbb{P}(\exists t \ge t_0 : S_t \ge u) \le \frac{\mathbb{E}[S_{t_0}]}{u}, \quad \text{for all } u > 0 \text{ and all integers } t_0 \ge 1.$$
 (11)

Inequality (11) is a time-uniform extension of Markov's inequality for nonnegative random variables. The unbounded range in Ville's inequality is made possible by nonnegativity, and the fact that supermartingales have nonincreasing expectations—see for instance Howard et al. (2020) for a formal proof. In contrast, submartingales admit nondecreasing expectations, and do not generally satisfy an infinite-horizon inequality such as (11). Nonnegative submartingales $(S_t)_{t=1}^{\infty}$ instead satisfy Doob's submartingale inequality (e.g. Durrett (2019), Theorem 4.4.2):

$$\mathbb{P}(\exists t \le T : S_t \ge u) \le \frac{\mathbb{E}[S_T]}{u} \quad \text{for all } u > 0 \text{ and any integer } T \ge 1.$$
 (12)

The prototypical example of a martingale is a sum $S_t = \sum_{i=1}^t X_i$ of i.i.d. random variables $(X_t)_{t=1}^{\infty} \subseteq \mathcal{X} \subseteq \mathbb{R}^d$, with respect to its canonical filtration $\mathcal{C}_t = \sigma(X_1, \dots, X_t)$. As described in Section 2.5, a wealth of existing works have developed confidence sequences for the expected value of a sequence of i.i.d. random variables, on the basis of inequalities such as (11) and (12).

2.3 Reverse filtrations, martingales and maximal inequalities

An alternate approach is to apply maximal inequalities to the sample mean itself, namely to the process $R_t = (1/t) \sum_{i=1}^t X_i$. This approach is relatively underexplored but turns out to be well-suited to our goals. Unlike (S_t) , the process (R_t) is a reverse martingale. To elaborate, a reverse filtration is a nonincreasing sequence of σ -algebras $(\mathcal{F}_t)_{t=1}^{\infty}$ contained in \mathcal{F} :

$$\mathcal{F}_t \supseteq \mathcal{F}_{t+1}$$
 for all $t \ge 1$.

A \mathbb{P} -integrable process $(R_t)_{t=1}^{\infty}$ is then said to be a reverse martingale with respect to $(\mathcal{F}_t)_{t=1}^{\infty}$ if for all $t \geq 1$, R_t is \mathcal{F}_t -measurable and

$$\mathbb{E}[R_t|\mathcal{F}_{t+1}] = R_{t+1}, \quad \text{ for all } t \geq 1.$$

Reverse submartingales and supermartingales are defined analogously, with the above equality respectively replaced by \geq and \leq . The sample average $R_t = (1/t) \sum_{i=1}^t X_i$ defines a reverse martingale with respect to its canonical reverse filtration $(\sigma(R_t, R_{t+1}, \dots))_{t=1}^{\infty}$. (R_t) is also a reverse martingale with respect to a richer filtration known as the *symmetric* or *exchangeable filtration* (Pollard, 2002; Durrett, 2019).

Definition 1 (Exchangeable Filtration). Given a sequence of random variables $(X_t)_{t=1}^{\infty}$, the exchangeable filtration is the reverse filtration $(\mathcal{E}_t)_{t=1}^{\infty}$, where \mathcal{E}_t denotes the σ -algebra generated by all real-valued Borel-measurable functions of X_1, X_2, \ldots which are permutation-symmetric in their first t arguments.

Informally, one may think of \mathcal{E}_t as $\sigma(P_t, X_{t+1}, X_{t+2}, \dots)$ or $\sigma(P_t, P_{t+1}, P_{t+2}, \dots)$, where $P_t = (1/t) \sum_{i=1}^t \delta_{X_i}$ is the empirical measure—while we do not formally define such σ -algebras, we present them here because we have found this intuition to be useful. Forward filtrations are reflective of the worlds that humans inhabit, in which information increases with time, but the exchangeable filtration can be an odd object to initially gain intuition about, so we provide another informal analogy for the unfamiliar reader. One can imagine \mathcal{E}_t to be the information accessible to an "amnesic oracle", who at any time t knows the entire future, but is confused about the ordering of events in the past. The amnesia causes the oracle to know less and less as time passes, because events in the initially clear future occur, and get transformed to a muddled past.

Given a sequence $(X_t)_{t=1}^{\infty}$ of exchangeable random variables, recall that $P_t = (1/t) \sum_{i=1}^t \delta_{X_i}$ denotes their empirical measure. It can be seen that the process $(P_t(B))_{t\geq 1}$ is a reverse martingale with respect to the exchangeable filtration $(\mathcal{E}_t)_{t=1}^{\infty}$, for any fixed Borel set $B \subseteq \mathbb{R}^d$. This property makes $(P_t)_{t=1}^{\infty}$ into a so-called measure-valued reverse martingale (Kallenberg, 2006). In fact, the converse nearly holds true: if (P_t) is a measure-valued reverse martingale, and if (X_t) is stationary, then (X_t) is exchangeable (Bladt, 2019). Note that the exchangeability condition is weaker than the i.i.d. assumption which we shall assume for the majority of this paper. We will later see that, with some care, this measure-valued reverse martingale gets effectively translated into a (real-valued) reverse submartingale property for convex functionals.

A key technical tool for handling real-valued, reverse submartingales will be the following analogue of Ville's inequality (11), first proved by Doob (1940; Theorem 1.1, p. 458) for reverse martingales; see also Lee (1990; Theorem 3, p. 112).

Theorem 2 (Ville's Inequality for Nonnegative Reverse Submartingales). Let $(R_t)_{t=1}^{\infty}$ be a nonnegative reverse submartingale with respect to a reverse filtration $(\mathcal{F}_t)_{t=1}^{\infty}$. Then, for any integer $t_0 \geq 1$ and real number u > 0,

$$\mathbb{P}(\exists t \ge t_0 : R_t \ge u) \le \frac{\mathbb{E}[R_{t_0}]}{u}.$$

Since Theorem 2 plays a central role in our development, we provide two self-contained proofs in Appendix B for completeness. As an example, it can be deduced from Theorem 2 that for any sequence of exchangeable random variables $(X_t)_{t=1}^{\infty}$,

$$\mathbb{P}\left(\sup_{t\geq 1} \left| \frac{1}{t} \sum_{i=1}^{t} X_i \right| \geq u \right) \leq \frac{\mathbb{E}|X_1|}{u}, \tag{13}$$

for all u > 0. This can be seen as a strengthening of Markov's inequality, whose left-hand side is $\mathbb{P}(|X_1| \geq u)$.

2.4 Partially ordered martingales

In order to handle the two-sample process (M_{ts}) in equation (4), we also employ (reverse) martingales indexed by \mathbb{N}^2 . We endow \mathbb{N}^2 with the standard partial ordering, that is we write $(t, s) \leq (t', s')$ for all $(t, s), (t', s') \in \mathbb{N}^2$ such that $t \leq t'$ and $s \leq s'$. The notation (t, s) < (t', s') indicates that at least one of the componentwise inequalities holds strictly. The definitions of filtration and martingale extend to this setting in the natural way: a forward (or reverse) filtration is a sequence $(\mathcal{F}_{ts})_{t,s\geq 1}$ of σ -algebras which is nondecreasing (or nonincreasing) with respect to the partial ordering, and a forward (or reverse) martingale $(S_{ts})_{t,s\geq 1}$ (or $(M_{ts})_{t,s\geq 1})$ is an $L^1(\mathbb{P})$ process adapted to (\mathcal{F}_{ts}) which satisfies $\mathbb{E}[S_{ts}|\mathcal{F}_{t's'}] = S_{t's'}$ for all (t,s) > (t',s') (or $\mathbb{E}[R_{ts}|\mathcal{F}_{t's'}] = R_{t's'}$ for all (t,s) < (t',s')). When the latter equalities are replaced by the inequalities \leq and \geq , (M_{ts}) is respectively called a (reverse) supermartingale and submartingale. We refer to Ivanoff and Merzbach (1999) for a survey.

Cairoli (1970) showed that a direct analogue of Ville's inequality cannot hold for partially ordered nonnegative martingales—the inequality $\mathbb{P}(\exists t, s \geq 1 : S_{ts} \geq u) \leq \mathbb{E}[S_{11}]/u$ does not generally hold for all u > 0. Distinct maximal inequalities for partially ordered forward and reverse martingales have nevertheless been established by Christofides and Serfling (1990b) under suitable moment assumptions, and under the so-called conditional independence (CI) assumption introduced by Cairoli and Walsh (1975). A reverse filtration $(\mathcal{F}_{ts})_{t,s\geq 1}$ is said to satisfy the CI property if for all $t,t',s,s'\geq 1$,

$$\mathbb{E}\left\{\mathbb{E}[\ \cdot\ |\ \mathcal{F}_{ts'}]\ |\ \mathcal{F}_{t's}\right\} = \mathbb{E}\left\{\ \cdot\ |\ \mathcal{F}_{(t\vee t')(s\vee s')}\right\},\tag{14}$$

or equivalently, that $\mathcal{F}_{ts'}$ and $\mathcal{F}_{t's}$ are conditionally independent given $\mathcal{F}_{(t\vee t')(s\vee s')}$ (Merzbach, 2003). The following Ville-type inequality for partially ordered reverse submartingales can be obtained by employing Corollary 2.9 of Christofides and Serfling (1990b).

Proposition 3. Let $(R_{ts})_{t,s\geq 1}$ be a nonnegative reverse submartingale with respect to a reverse filtration $(\mathcal{F}_{ts})_{t,s\geq 1}$ satisfying the conditional independence assumption. Assume that for some $\alpha > 1$, $R_{ts} \in L^{\alpha}(\mathbb{P})$ for all $t, s \geq 1$. Then, for all u > 0,

$$\mathbb{P}(\exists t \ge t_0, s \ge s_0 : R_{ts} \ge u) \le \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \frac{\mathbb{E}[R_{t_0 s_0}^{\alpha}]}{u^{\alpha}}.$$

The special case $\alpha = 2$ was stated in Corollary 2.10 of Christofides and Serfling (1990b). A proof and further discussion of this result is given in Appendix B.2, and forms the basis for our two-sample results.

2.5 Time-uniform confidence sequences

Confidence sequences are defined similarly as in (2) for estimands other than divergences. Given a functional $\theta \equiv \theta(P)$ of interest, and an error level $\delta \in (0,1)$, a $(1-\delta)$ -confidence sequence $(C_t)_{t=1}^{\infty}$ based on an i.i.d. sequence of random variables $(X_t)_{t=1}^{\infty}$ from P is a sequence of sets $C_t \in \sigma(X_1, \ldots, X_t)$ satisfying $\mathbb{P}(\exists t \geq 1 : \theta \not\in C_t) \leq \delta$. When θ is real-valued, we say two sequences (ℓ_t) and (u_t) are lower and upper confidence sequences if $\mathbb{P}(\exists t \geq 1 : \theta \leq \ell_t) \leq \delta$ and $\mathbb{P}(\exists t \geq 1 : \theta \geq u_t) \leq \delta$ respectively. Confidence sequences were pioneered by Robbins, Darling, Siegmund and Lai (Darling and Robbins, 1967; Robbins and Siegmund, 1969; Lai, 1976), and new techniques have

been recently developed that enable their extensions to new, nonparametric settings (Kaufmann and Koolen, 2021; Howard et al., 2021). This resurgence of interest in sequential analysis has been driven in part by its applications to best-arm identification algorithms for multi-armed bandits (Jamieson et al., 2014; Kaufmann et al., 2016; Shin et al., 2021) and reinforcement learning (Karampatziakis et al., 2021), to name a few.

The mean of a distribution is perhaps the target of inference which has received the most attention in prior work on confidence sequences. As described in Section 2.2, the process $S_t = \sum_{i=1}^t (X_i - \mu)$ forms a canonical example of a forward martingale, where μ denotes the common (finite) mean of the i.i.d. random variables X_i . To obtain a confidence sequence for μ , a maximal inequality such as that of Ville (see equation (11)) cannot be directly be applied to (S_t) , however, since it is not a nonnegative process. Inspired by the Cramér-Chernoff method for deriving concentration inequalities, it is instead natural to consider the nonnegative process

$$U_t(\lambda) = \exp(\lambda S_t), \quad t \ge 1,$$

for $\lambda > 0$. Here, we assume a tail assumption is placed on X_1 , such that it admits a finite cumulant generating function satisfying $\log \{\mathbb{E}[\exp(\lambda(X_1 - \mu))]\} \leq \phi(\lambda)$, for some (say, known) map $\phi : [0, \lambda_{\max}) \to \mathbb{R}$, where $\lambda_{\max} > 0$. The exponential process $(U_t(\lambda))$ is a nonnegative submartingale by Jensen's inequality, and forms the basis of several confidence sequences described below. A distinct line of work constructs confidence sequences for μ by downweighting this process to recover a (super)martingale. For instance, it can be verified that

$$L_t(\lambda) = \exp{\{\lambda S_t - t\phi(\lambda)\}}, \quad t \ge 1,$$

is a nonnegative supermartingale with respect to the canonical filtration. Variants of the process $(L_t(\lambda))$ appear in a long line of work aimed at deriving sequential concentration inequalities for means—see Howard et al. (2020) for a comprehensive review of such approaches.

Applying a maximal martingale inequality to either $(U_t(\lambda))$ or $(L_t(\lambda))$ does not, on its own, lead to satisfactory confidence sequences for μ . Infinite-horizon maximal inequalities are not available for submartingales $(U_t(\lambda))$, and even for the supermartingale $(L_t(\lambda))$, a direct application of Ville's inequality leads to a confidence sequence for μ with nonvanishing length. To obtain confidence sequences with lengths scaling at the optimal rate $O(\sqrt{\log \log t/t})$, implied by the law of the iterated logarithm, it is instead common to use a variant of the "method of mixtures", for example by repeatedly applying a maximal inequality over geometrically-spaced epochs in time—such methods are often known as peeling, chaining, or stitching. Jamieson et al. (2014); Zhao et al. (2016) use stitching arguments based on $(U_t(\lambda))$ and Doob's submartingale inequality, while Garivier (2013); Kaufmann et al. (2016); Howard et al. (2021), use $(L_t(\lambda))$ and Ville's inequality. The resulting confidence sequences decay at similar rates, though with varying constants and tail assumptions—see Howard et al. (2021); Waudby-Smith and Ramdas (2021) for a comparison of such approaches.

In Appendix E, we describe the difficulties in extending the above framework to targets of inference distinct from means. We provide therein a naive approach to deriving confidence sequences for some divergences between probability distributions, using an analogue of the process $(U_t(\lambda))$. However, our main results which follow provide a far-reaching improvement, which hinges upon reverse submartingales—a rarely used tool for deriving confidence sequences.

3 Confidence sequences for convex functionals

Let $(X_t)_{t=1}^{\infty}$ and $(Y_s)_{s=1}^{\infty}$ respectively denote independent sequences of i.i.d. observations from two distributions $P, Q \in \mathcal{P}(\mathcal{X})$, with support contained in a set $\mathcal{X} \subseteq \mathbb{R}^d$. Given convex functionals $\Phi: \mathcal{P}(\mathcal{X}) \to \overline{\mathbb{R}}$ and $\Psi: \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \overline{\mathbb{R}}$, the goal of this section is to derive confidence sequences for $\Phi(P)$ and $\Psi(P,Q)$. Our primary interest is in the special case where Ψ is a convex divergence D, and $\Phi = D(\cdot || Q)$ when Q is known, but we formulate our results for arbitrary convex functionals in the interest of generality. We shall make use of the processes

$$N_t = \Phi(P_t) - \Phi(P), \quad M_{ts} = \Psi(P_t, Q_s) - \Psi(P, Q), \quad t, s \ge 1.$$
 (15)

We prove in Section 3.1, that $(M_{ts})_{t,s\geq 1}$ and $(N_t)_{t\geq 1}$ are reverse submartingales with respect to suitable filtrations, whose choice is further discussed in Section 3.3. We then derive maximal inequalities for these processes, from which lower confidence sequences will follow using an epoch-based analysis. We follow a distinct strategy to obtain upper confidence sequences in Section 3.2.

3.1 Lower confidence sequences via reverse submartingales

Let $(\mathcal{E}_t^X)_{t=1}^{\infty}$ and $(\mathcal{E}_s^Y)_{s=1}^{\infty}$ denote the exchangeable filtrations associated with the sequences $(X_t)_{t=1}^{\infty}$ and $(Y_s)_{s=1}^{\infty}$ respectively, and define $\mathcal{E}_{ts} = \mathcal{E}_t^X \bigvee \mathcal{E}_s^Y$ for all $t,s \geq 1$. As stated in Section 2.2, the sequences of empirical measures (P_t) and (Q_s) form measure-valued reverse martingales with respect to (\mathcal{E}_t^X) and (\mathcal{E}_s^Y) respectively. This fact suggests that any convex functional evaluated at (P_t, Q_s) is a reverse submartingale, as we now show.

Theorem 4. Let Φ, Ψ be convex functionals such that $\Phi(P_t), \Psi(P_t, Q_s) \in L^1(\mathbb{P})$ for all $t, s \geq 1$. Then,

- (i) $(\Phi(P_t))_{t\geq 1}$ is a reverse submartingale with respect to (\mathcal{E}_t^X) .
- (ii) $(\Psi(P_t, Q_s))_{t,s\geq 1}$ is a partially ordered reverse submartingale with respect to (\mathcal{E}_{ts}) .

Theorem 4(i) is known in the special case where Φ is the supremum of an empirical processes—see for instance Pollard (1981) and Lemma 2.4.5 of van der Vaart and Wellner (1996). Our proof below is inspired by these works, and in particular extends them to general convex functionals and to partially ordered filtrations.

Notice that the processes in Theorem 4 lie in $L^1(\mathbb{P})$, and are therefore assumed to be measurable. When this assumption is removed, it can be shown that there exist measurable *covers* of the processes in Theorem 4 for which the result continues to hold—this is the approach taken by van der Vaart and Wellner (1996) for suprema of empirical processes; see also Pollard (1981) and Strobl (1995). With this modification, we believe that all subsequent claims in this paper can be made to hold in outer probability, but we prefer to retain the measurability condition to avoid being overwhelmed by technicalities. In particular, (N_t) and (M_{ts}) are tacitly assumed to be measurable throughout the sequel.

Proof of Theorem 4. We will prove Theorem 4(ii) and a similar argument can be used to prove Theorem 4(i). For all $t, s \ge 1 \Psi(P_t, Q_s)$ is invariant to permutations of X_1, \ldots, X_t and of Y_1, \ldots, Y_s .

It follows that $(\Psi(P_t, Q_s))$ is adapted to (\mathcal{E}_{ts}) , thus it suffices to prove the reverse submartingale property. Fix $t, s \geq 1$. Define the (t+1) different "leave-one-out" empirical measures

$$P_t^i = \frac{1}{t} \left[\sum_{j=1}^{i-1} \delta_{X_j} + \sum_{j=i+1}^{t+1} \delta_{X_j} \right], \quad i = 1, 2, \dots, t+1.$$

Then $P_{t+1} = \frac{1}{t+1} \sum_{i=1}^{t+1} P_t^i$, and the convexity of Ψ implies

$$\Psi(P_{t+1}, Q_s) \le \frac{1}{t+1} \sum_{i=1}^{t+1} \Psi(P_t^i, Q_s).$$

Since $\Psi(P_{t+1}, Q_s)$ is $\mathcal{E}_{(t+1)s}$ -measurable, we deduce that

$$\Psi(P_{t+1}, Q_s) \le \frac{1}{t+1} \sum_{i=1}^{t+1} \mathbb{E}[\Psi(P_t^i, Q_s) | \mathcal{E}_{(t+1)s}]. \tag{16}$$

Since $P_t^{t+1} = P_t$ by definition, the claim will follow upon proving the key identity

$$\mathbb{E}[\Psi(P_t^i, Q_s)|\mathcal{E}_{(t+1)s}] = \mathbb{E}[\Psi(P_t, Q_s)|\mathcal{E}_{(t+1)s}], \quad i = 1, \dots, t.$$

$$(17)$$

Notice that we can write $\mathcal{E}_{(t+1)s} = \mathcal{E}_{t+1}^X \bigvee \mathcal{E}_s^Y = \sigma(\mathcal{I})$ where

$$\mathcal{I} = \{ A^X \cap A^Y : A^X \in \mathcal{E}_{t+1}^X, A^Y \in \mathcal{E}_s^Y \}.$$

 \mathcal{I} is clearly a π -system. To prove (17), it will thus suffice to prove that for any set $A = A^X \cap A^Y$, with $A^X \in \mathcal{E}^X_{t+1}$ and $A^Y \in \mathcal{E}^Y_s$, and for all $1 \leq i \leq t$, we have $\mathbb{E}[\Psi(P_t^i,Q_s)I_A] = \mathbb{E}[\Psi(P_t,Q_s)I_A]$, where $I_A : \Omega \to \{0,1\}$ is the indicator function of A and $I_A = I_{A^X}I_{A^Y}$.

Notice that I_{A^X} is \mathcal{E}^X_{t+1} -measurable, thus it is a function f_{A^X} of X_1, X_2, \ldots which is permutation symmetric in X_1, \ldots, X_{t+1} . For convenience, we write $I_{A^X} = f_{A^X}(X_1, X_2, \ldots)$, whence we may also write $I_A = I_{A^Y} f_{A^X}(X_1, X_2, \ldots)$. Now, let $\tau : \mathbb{N} \to \mathbb{N}$ be the permutation such that $\tau(j) = j$ if $j \notin \{t+1, i\}$ and $\tau(t+1) = i$, $\tau(i) = t+1$. By exchangeability of X_1, X_2, \ldots , and by their independence from Y_1, Y_2, \ldots , we have $(X_1, Y_1, X_2, Y_2, \ldots) \stackrel{d}{=} (X_{\tau(1)}, Y_1, X_{\tau(2)}, Y_2, \ldots)$, whence,

$$\mathbb{E}[\Psi(P_t,Q_s)I_A] = \mathbb{E}[\Psi(P_t^{t+1},Q_s)f_{A^X}(X_1,X_2,\dots)I_{A^Y}] = \mathbb{E}[\Psi(P_t^i,Q_s)f_{A^X}(X_{\tau(1)},X_{\tau(2)},\dots)I_{A^Y}].$$

Since f_{AX} is permutation-symmetric in its first t+1 arguments, and the permutation τ fixes all natural numbers greater or equal to t+2, we obtain

$$f_{AX}(X_{\tau(1)}, X_{\tau(2)}, \dots,) = f_{AX}(X_1, X_2, \dots),$$

implying that

$$\mathbb{E}[\Psi(P_t, Q_s) f_{A^X}(X_1, X_2, \dots) I_{A^Y}] = \mathbb{E}[\Psi(P_t^i, Q_s) f_{A^X}(X_1, X_2, \dots) I_{A^Y}].$$

Equation (17) now follows. Returning to equation (16) we deduce that

$$\Psi(P_{t+1}, Q_s) \le \mathbb{E}[\Psi(P_t, Q_s) | \mathcal{E}_{(t+1)s}].$$

A symmetric argument shows that $\Psi(P_t, Q_{s+1}) \leq \mathbb{E}[\Psi(P_t, Q_s) | \mathcal{E}_{t(s+1)}]$, implying that $(\Psi(P_t, Q_s))$ is a partially ordered reverse submartingale with respect to (\mathcal{E}_{ts}) .

It is apparent from the proof that the convexity requirement is stronger than necessary. For example, the following more general statement can be inferred from the proof of Theorem 4. We record it formally as it may be of independent interest.

Proposition 5. Suppose $(R_t)_{t=1}^{\infty}$ is an $L^1(\mathbb{P})$ process of the form $R_t = f_t(X_1, \dots, X_t)$, for some sequence of permutation invariant maps $f_t : \mathcal{X}^t \to \mathbb{R}$. Suppose further that for all $t \geq 1$,

$$R_{t+1} \le \frac{1}{t+1} \sum_{i=1}^{t+1} f_t(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_{t+1}).$$
 (18)

Then, (R_t) is a reverse submartingale with respect to (\mathcal{E}_t^X) . Furthermore, if the above display holds with equality, (R_t) is a reverse martingale with respect to (\mathcal{E}_t^X) .

We refer to condition (18) as the *leave-one-out property*. In Sections 4.2 and 4.6 we will briefly make use of processes which cannot be expressed as evaluations of a convex functional at the empirical measure, but which satisfy the leave-one-out property.

Theorem 4 permits the use of maximal inequalities discussed in Sections 2.3 and 2.4 for reverse submartingales. In view of generalizing the Cramér-Chernoff technique (Boucheron et al., 2013) to our sequential setup, we shall state our bounds under the assumption that N_t and M_{ts} admit finite cumulant generating functions over an interval $[0, \lambda_{\text{max}})$, and are upper bounded by known convex functions $\psi_{ts}, \psi_t : [0, \lambda_{\text{max}}) \to \mathbb{R}$,

$$\log \left\{ \mathbb{E} \left[\exp(\lambda M_{ts}) \right] \right\} \le \psi_{ts}(\lambda), \quad \log \left\{ \mathbb{E} \left[\exp(\lambda N_t) \right] \right\} \le \psi_t(\lambda), \quad t, s \ge 1, \ \lambda \in [0, \lambda_{\max}).$$
 (19)

We shall discuss in Section 4 how such tail assumptions may be replaced by tail assumptions on the distributions P and Q themselves, for various special cases of functionals. Under equation (19), we obtain the following result.

Proposition 6. Let Φ, Ψ be convex functionals such that the processes (M_{ts}) and (N_t) satisfy the bounds of equation (19). Then, for all u > 0, and all integers $t_0, s_0 \ge 1$, the following hold.

- (i) (One-Sample) $\mathbb{P}(\exists t \geq t_0 : N_t \geq u) \leq \exp(-\psi_{t_0}^*(u)).$
- (ii) (Two-Sample) $\mathbb{P}(\exists t \ge t_0, s \ge s_0 : M_{ts} \ge u) \le e \cdot \exp(-\psi_{t_0 s_0}^*(u)).$

A proof of Proposition 6 appears in Appendix C. In view of Theorem 4, Proposition 6(i) is obtained through an application of the Cramér-Chernoff technique, together with Ville's inequality for reverse submartingales (Theorem 2). It can thus also be seen as an extension of the supermartingale techniques in Howard and Ramdas (2022) to the reversed setting. In Proposition 6(ii), an analogous bound is obtained for the two-sample case, though with the additional factor e in the probability bound. The presence of such a factor greater than 1 is necessary due to the aforementioned counterexample of Cairoli (1970) regarding maximal inequalities for partially ordered martingales, though we do not know if the value e is sharp. The bound itself is obtained via Proposition 3.

Inverting the probability inequalities of Proposition 6 leads to one- and two-sample confidence sequences for $\Phi(P)$ or $\Psi(P,Q)$, though with lengths which are constant with respect to t,s. To obtain confidence sequences scaling at rate-optimal lengths, we employ a stitching construction inspired by those described in Section 2.5, together with Proposition 6. Our result will depend on user-specified functions $\ell, g : [0, \infty) \to [1, \infty)$, known as stitching functions, which dictate the shape of the resulting confidence sequences below. We construct these functions to satisfy

$$\sum_{k=1}^{\infty} \frac{1}{\ell(k)} \le 1, \quad \sum_{j,k=1}^{\infty} \frac{e}{g(k+j)} \le 1, \tag{20}$$

as well as $\ell(j) = \ell(1)$ for all $j \in [0,1]$, and g(k) = g(2) for all $k \in [0,2]$. Typical choices include $\ell(k) = (1 \vee k)^{\alpha} \zeta(\alpha)$ and $g(k) = e(2 \vee k)^{\alpha+1} (\zeta(\alpha) - \zeta(\alpha+1))$ (Borwein and Borwein (1987), p. 305), where $\alpha > 1$ and $\zeta(\alpha) = \sum_{k=1}^{\infty} (1/k^{\alpha})$. For ease of exposition, we also insist that the sequences $2^{-u} \log \ell(u)$ and $(2^{-u} + 2^{-v}) \log g(u+v)$ are chosen to be decreasing in each of the indices $u, v \geq 1$, implying that $\ell(u), g(u) = o(\exp(\exp(u)))$. Our main result is below.

Theorem 7. Let Φ , Ψ denote convex functionals for which the processes (M_{ts}) and (N_t) satisfy the bounds of equation (19). For any integer $t \geq 1$, let $\bar{t} = \lceil t/2 \rceil$, and fix $\delta \in (0,1)$.

(i) (One-Sample) Assume ψ_t^* is invertible for all $t \geq 1$, and that the sequence

$$\gamma_t = (\psi_{\bar{t}}^*)^{-1} \Big(\log \ell(\log_2 t) + \log(2/\delta) \Big), \quad t \ge 1$$

is nonincreasing. Then,

$$\mathbb{P}\Big\{\exists t \ge 1 : \Phi(P_t) \ge \Phi(P) + \gamma_t\Big\} \le \delta/2.$$

(ii) (Two-Sample) Assume ψ_{ts}^* is invertible for all $t, s \geq 1$, and that the sequence

$$\gamma_{ts} = (\psi_{\bar{t}\bar{s}}^*)^{-1} \Big(\log g(\log_2 t + \log_2 s) + \log(2/\delta) \Big), \quad t, s \ge 1$$

is nonincreasing with respect to the partial order on \mathbb{N}^2 . Then,

$$\mathbb{P}\Big\{\exists t, s \ge 1 : \Psi(P_t, Q_s) \ge \Psi(P, Q) + \gamma_{ts}\Big\} \le \delta/2.$$

We begin by noting that for a fixed sample size n, if we denote $\bar{\gamma}_n = (\psi_n^*)^{-1}(\log(2/\delta))$, then the fixed-time Cramér-Chernoff concentration bound corresponding to part (i) is given by (Boucheron et al., 2013),

$$\mathbb{P}\Big\{\Phi(P_n) \ge \Phi(P) + \bar{\gamma}_n\Big\} \le \delta/2,$$

and an analogous statement can also be made for part (ii) above. Thus, our time-uniform bounds are essentially an iterated logarithm factor worse than the usual fixed-time bounds, but now also apply at arbitrary stopping times.

Before further commenting on the above result, we instantiate it in the special case where N_t and M_{ts} are sub-Gaussian for all $t, s \geq 1$. In Section 4, we illustrate how such a condition can be satisfied under tail assumptions on the distributions P and Q themselves.

Corollary 8. Fix $\delta \in (0,1)$, and recall that $\bar{t} = \lceil t/2 \rceil$.

(i) (One-Sample) Assume N_t is κ_t^2 -sub-Gaussian for some $\kappa_t > 0$, and for all $t \ge 1$. Choose ℓ so that $(\kappa_t^2 \log \ell(\log_2 t))_{t \ge 1}$ is nonincreasing. Then,

$$\mathbb{P}\left\{\exists t \geq 1 : N_t \geq \mathbb{E}\left(N_{\bar{t}}\right) + \sqrt{2\kappa_{\bar{t}}^2 \left[\log \ell(\log_2 t) + \log(2/\delta)\right]}\right\} \leq \delta/2,$$

(ii) (Two-Sample) Assume M_{ts} is σ_{ts}^2 -sub-Gaussian for some $\sigma_{ts} > 0$, and for all $s, t \ge 1$. Choose g so that $(\sigma_{t\bar{s}}^2 \log g(\log_2 t + \log_2 s))_{t,s\ge 1}$ is nonincreasing. Then,

$$\mathbb{P}\left\{\exists t, s \geq 1 : M_{ts} \geq \mathbb{E}\left(M_{\bar{t}\bar{s}}\right) + \sqrt{2\sigma_{\bar{t}\bar{s}}^2 \left[\log g(\log_2 t + \log_2 s) + \log(2/\delta)\right]}\right\} \leq \delta/2.$$

Theorem 7(i) is proved by dividing time $t \geq 1$ into geometrically increasing epochs of the form $[2^j, 2^{j+1}], j \geq 0$, over each of which we construct confidence boundaries at the level $\delta_j/2 = \delta/(2\ell(j+1)) \in (0,1)$ using Proposition 6. Taking a union bound over these boundaries leads to a miscoverage probability of at most $\sum_{j=0}^{\infty} (\delta_j/2) \leq \delta/2$. The two-sample process (M_{ts}) is handled similarly, by instead forming two sequences of epochs. In Appendix C, we also state and prove a more general version of Theorem 7 in terms of epoch sizes different than 2, which will be needed in Section 4.8.

Remark. Given a convex divergence D, Theorem 7(i) implies that $(1 - \delta/2)$ -upper confidence sequences for the processes $N_t^X = D(P_t || P)$ and $N_s^Y = D(Q_s || Q)$ are respectively given by

$$\gamma_t^X = (\psi_{X,\bar{t}}^*)^{-1} \Big(\log \ell(\log_2 t) + \log(2/\delta) \Big), \quad \gamma_s^Y = (\psi_{Y,\bar{s}}^*)^{-1} \Big(\log \ell(\log_2 s) + \log(2/\delta) \Big),$$

where $\psi_{X,t}$ is an upper bound on the cumulant generating function of N_t^X , and similarly for $\psi_{Y,s}$. When D satisfies the triangle inequality, one may deduce the following two-sided confidence sequence for D(P||Q),

$$\mathbb{P}(\forall t, s \ge 1 : |D(P_t || Q_s) - D(P || Q)| \le \gamma_t^X + \gamma_s^Y)
\ge 1 - \mathbb{P}(\exists t \ge 1 : N_t^X > \gamma_t^X) - \mathbb{P}(\exists s \ge 1 : N_s^Y > \gamma_s^Y) \ge 1 - \delta.$$
(21)

Equation (21) is significant in that it provides a time-uniform bound for a partially ordered reverse submartingale on the basis of two totally ordered reverse submartingales. Doing so bypasses the nearly unavoidable factor e in condition (20), but may nevertheless be looser in general due to the application of the triangle inequality. We also remark that equation (21) does not require P_t to be independent of Q_s , unlike Theorem 7(ii).

3.2 Upper confidence sequences via affine minorants

The lower confidence sequences derived in Theorem 7 hinged upon the reverse submartingale property of the processes (N_t) and (M_{ts}) —an inherently one-sided condition. We show in this section how a different approach can be used to derive upper confidence sequences, motivated both by technical and statistical considerations.

- On the technical side, it would seem natural to repeat the steps of Theorem 7 with respect to the process $(-N_t)$ to obtain an upper confidence sequence. However, $(-N_t)$ is a reverse supermartingale and thus cannot satisfy infinite-horizon (Ville-type) maximal inequalities. Furthermore, the exponential process $(\exp(-N_t))$ may generally be neither a reverse supermartingale nor a submartingale, thus an analogue of Proposition 6 cannot be derived.
- On the statistical side, the plug-in estimators $\Phi(P_t)$ and $\Psi(P_t, Q_s)$ are typically upward biased in estimating $\Phi(P)$ and $\Psi(P,Q)$ respectively. This fact can be deduced from Corollary 10 below, but can already be anticipated from the fact that $\mathbb{E}[\Phi(P_t)]$ and $\mathbb{E}[\Psi(P_t, Q_s)]$ are nonincreasing sequences, since $(\Phi(P_t))$ and $(\Psi(P_t, Q_s))$ are reverse submartingales (Theorem 4). This upward bias suggests that confidence sequences for D(P||Q) of the form $[D(P_t, Q) \ell_t, D(P_t, Q) + u_t]$ should typically be asymmetric, with the sequence (u_t) potentially decaying at a faster rate than (ℓ_t) .

Our approach is summarized as follows. The convexity of Φ guarantees that it can be minorized by an affine functional on $\mathcal{P}(\mathcal{X})$. Notice that an affine functional evaluated at the empirical measure is a sample average, and therefore a reverse martingale. Furthermore, when a convex duality result guarantees that Φ is equal to the supremum over a set of minorizing affine functionals, it can be shown that the difference $\Phi(P_t) - \Phi(P)$ is in fact minorized by a mean-zero sample average, for which confidence sequences of length $O(\sqrt{\log \log t/t})$ can be obtained in a standard way under appropriate tail conditions. Doing so leads to a time-uniform lower bound on $\Phi(P_t) - \Phi(P)$, which is easily rephrased as an upper confidence sequence for $\Phi(P)$ scaling at a near-parametric rate.

While this intuition can be made rigorous for a broad collection of convex functionals, we shall avoid doing so in full generality to avoid introducing additional terminology. We shall instead assume that Φ and Ψ take the following form, which is sufficiently general to cover the divergences of primary interest in our development,

$$\Phi(\mu) = \sup_{f \in \mathcal{F}_{\Phi}} \int f d\mu, \quad \Psi(\mu, \nu) = \sup_{(f,g) \in \mathcal{H}_{\Psi}} \int f d\mu + \int g d\nu, \tag{22}$$

for all $\mu, \nu \in \mathcal{P}(\mathcal{X})$. Here, \mathcal{F}_{Φ} denotes a set of Borel-measurable functions on \mathcal{X} , \mathcal{H}_{Ψ} a set of pairs of such functions, and we also write $\mathcal{F}_{\Psi} = \{f : (f,g) \in \mathcal{H}_{\Psi}\}$ and $\mathcal{G}_{\Psi} = \{g : (f,g) \in \mathcal{H}_{\Psi}\}$. It is clear from equations (5), (8) and (10) that if D is an IPM, φ -divergence, or optimal transport cost, then the functionals $\Psi = D$ and $\Phi = D(\cdot || Q)$ (for a fixed measure $Q \in \mathcal{P}(\mathcal{X})$) admit the representation (22)—for instance, in the case of IPMs generated by a function class \mathcal{J} , one may take $\mathcal{F}_{\Phi} = \{f - \int f dQ : f \in \mathcal{J}\}$ and $\mathcal{H}_{\Psi} = \{(f, -f) : f \in \mathcal{J}\}$. With this notation in place, the following observation is straightforward.

Proposition 9. If Φ and Ψ admit the representation (22), and the suprema therein are achieved for $\mu = P$ and $\nu = Q$, respectively by $f_{\Phi} \in \mathcal{F}_{\Phi}$ and by $(f_{\Psi}, g_{\Psi}) \in \mathcal{H}_{\Psi}$, then

1. The process $(N_t)_{t\geq 1}$ is bounded below by

$$R_t = \int f_{\Phi} d(P_t - P),$$

which is a (mean-zero) reverse martingale with respect to the exchangeable filtration (\mathcal{E}_t^X) .

2. The process $(M_{ts})_{t,s\geq 1}$ is bounded below by $(R_t^X + R_s^Y)_{t,s\geq 1}$, where

$$R_t^X = \int f_{\Psi} d(P_t - P), \ t \ge 1, \quad and \quad R_s^Y = \int g_{\Psi} d(Q_s - Q), \ s \ge 1,$$

are (mean-zero) reverse martingales with respect to (\mathcal{E}_t^X) and (\mathcal{E}_s^Y) respectively.

We begin by noting that Proposition 9 implies the aforementioned upward bias of the plug-in estimators $\Phi(P_t)$, $\Psi(P_t, Q_s)$, including at arbitrary stopping times. Indeed, by the optional stopping theorem (see for instance Durrett (2019), Theorem 4.8.3.), we can easily infer the following fact that we record formally for reference.

Corollary 10. Assume the same conditions as Proposition 9, and that the processes $(R_t), (R_t^X), (R_s^Y)$ therein are uniformly integrable. Then, for any stopping times τ and σ with respect to the canonical forward filtrations $(\sigma(X_1, \ldots, X_t))_{t=1}^{\infty}$ and $(\sigma(Y_1, \ldots, Y_s))_{s=1}^{\infty}$ respectively, we have

$$\mathbb{E}[\Phi(P_{\tau})] \ge \Phi(P), \quad \mathbb{E}[\Psi(P_{\tau}, Q_{\sigma})] \ge \Psi(P, Q). \tag{23}$$

Furthermore, Proposition 9 can readily be used to form upper confidence sequences for $\Phi(P)$ or $\Psi(P,Q)$ on the basis of the reverse martingales (R_t) , or (R_t^X) and (R_s^Y) . These processes are simply sample averages, hence they can already be controlled using the existing literature on sequential mean estimation (summarized in Section 2.5). Nevertheless, for the purpose of being self-contained, we use reverse martingale techniques to derive such upper confidence sequences under tail conditions on \mathcal{F}_{Φ} and \mathcal{H}_{Ψ} , in Proposition 29 of Appendix C.2. The following special case of this result will be used repeatedly in Section 4.

Corollary 11. Assume the same conditions as Proposition 9. Assume further that for any $h \in \mathcal{F}_{\Psi} \cup \mathcal{G}_{\Psi}$, diam $(h(\mathcal{X})) \leq B < \infty$, and define

$$\kappa_t = \sqrt{\frac{\log \ell(\log_2 t) + \log(4/\delta)}{t}}, \quad \kappa_{ts} = \kappa_t + \kappa_s.$$
 (24)

Then, for any $\delta \in (0,1)$, we have $\mathbb{P}(\exists t, s \geq 1 : \Psi(P_t, Q_s) \leq \Psi(P, Q) - B\kappa_{ts}) \leq \delta/2$.

To summarize, under the assumptions and notation of Theorem 7 and Proposition 11, we deduce that a two-sided, two-sample, $(1 - \delta)$ -confidence sequence for $\Psi(P, Q)$ is given by

$$C_{ts} = \left[\Psi(P_t, Q_s) - \kappa_{ts}, \Psi(P_t, Q_s) + \gamma_t^X + \gamma_s^Y\right], \tag{25}$$

and similarly for the functional Φ .

3.3 On the choice of filtrations and stopping times

The majority of confidence sequences derived in past literature, such as those described in Section 2.5, employ martingales with respect to the canonical, or "data-generating" filtration. A notable exception is the work of Vovk (2021), which shows that the power of certain sequential tests can be increased by coarsening the canonical filtration. It was similarly fruitful in our work

to distinguish the data-generating filtration, with respect to which our processes do not appear to admit any martingale-type property, from a different filtration with respect to which our processes do admit a (reverse) martingale property. To elaborate, let

$$\mathcal{D}_t^X = \sigma(X_1, X_2, \dots, X_t), \quad \mathcal{D}_s^Y = \sigma(Y_1, Y_2, \dots, Y_s), \quad t, s = 1, 2, \dots$$

denote the canonical filtrations associated with each sequence of samples. Our bounds have implicitly assumed that at any pair of times (t, s), the practitioner has access to the information encoded by the data-generating filtration

$$\mathcal{D}_{ts} = \mathcal{D}_t^X \bigvee \mathcal{D}_s^Y, \quad t, s = 1, 2, \dots$$
 (26)

The process (M_{ts}) is naturally adapted to (\mathcal{D}_{ts}) , but we are not aware of it satisfying a martingale-type property with respect to this filtration in general¹. It is, however, also adapted to the exchangeable filtration $\mathcal{E}_{ts} = \mathcal{E}_t^X \bigvee \mathcal{E}_s^Y$, but unlike before, (M_{ts}) is also a reverse submartingale with respect to (\mathcal{E}_{ts}) . Our paper reinforces the somewhat underappreciated view in sequential analysis that filtrations should not be viewed as being "inherent" to the problem, or as tedious formalism for ensuring measurability, but instead viewed as design tools—a nonstandard choice of filtration can yield a powerful design tool.

Validity at Stopping Times. To better understand the underlying role of the filtration (\mathcal{D}_{ts}) , we shall now prove that the results of Theorem 7 can equivalently be stated as bounds which hold at arbitrary stopping times with respect to (\mathcal{D}_{ts}) . We focus on the two-sample case in what follows.

In order to define a notion of stopping time which is suitable for our purposes, define the set

$$\overline{\mathbb{N}}^2 = \mathbb{N}^2 \cup \{(t, \infty) : t \ge 1\} \cup \{(\infty, s) : s \ge 1\} \cup \{(\infty, \infty)\},$$

for some symbols $(t, \infty), (\infty, s), (\infty, \infty)$. We endow $\overline{\mathbb{N}}^2$ with the natural partial order, given by that of \mathbb{N}^2 (described in Section 2.4), together with the following additional relations: $(t, s) \leq (t', \infty)$ whenever $t \leq t'$ and $s \in \mathbb{N}$; $(t, s) \leq (\infty, s')$ whenever $s \leq s'$ and $t \in \mathbb{N}$; $u \leq (\infty, \infty)$ for all $u \in \overline{\mathbb{N}}^2$. A map $\eta : \Omega \to \overline{\mathbb{N}}^2$ is said to be a stopping time with respect to a filtration (\mathcal{F}_{ts}) if $\{\eta = (t, s)\} \in \mathcal{F}_{ts}$ for all $(t, s) \in \mathbb{N}^2$.

Intuitively, the event $\{\eta = (t, s)\}$ indicates that the data collection from each of P and Q was terminated at times (t, s), whereas the event $\{\eta = (t, \infty)\}$ indicates that data was collected from P until time t, but indefinitely so from Q. Likewise, the event $\{\eta = (\infty, \infty)\}$ indicates that neither of the two data collections were halted. With these definitions in place, we arrive at the following general equivalence.

Proposition 12. Let $(A_{ts})_{t,s=1}^{\infty}$ be a sequence of events adapted to a forward filtration $(\mathcal{F}_{ts})_{t,s=1}^{\infty}$. Define for all $t, s \geq 1$,

$$A_{t\infty} = \limsup_{s \to \infty} A_{ts}, \quad A_{\infty s} = \limsup_{t \to \infty} A_{ts}, \quad A_{\infty \infty} = \left(\limsup_{t \to \infty} A_{t\infty}\right) \cup \left(\limsup_{s \to \infty} A_{\infty s}\right). \tag{27}$$

Then, for all $\delta \in (0,1)$, the following statements are equivalent.

¹Nevertheless, under some conditions, it can be deduced from Theorem 4 that there exists a bivariate canonical filtration (\mathcal{F}_{ts}) and a process (\widetilde{M}_{ts}) , which has the same distribution as (M_{ts}) , such that (\widetilde{M}_{ts}) is a reverse submartingale with respect to (\mathcal{F}_{ts}) rather than (\mathcal{E}_{ts}) —see Theorem B of Rzeszut and Trojan (2020).

- (i) $\mathbb{P}\left(\bigcup_{t,s=1}^{\infty} A_{ts}\right) \leq \delta$.
- (ii) For any stopping time (τ, σ) with respect to (\mathcal{F}_{ts}) , we have $\mathbb{P}(A_{\tau\sigma}) \leq \delta$.
- (iii) For any random time (T, S), not necessarily a stopping time, we have $\mathbb{P}(A_{TS}) \leq \delta$.

The proof of Proposition 12 is given in Appendix C.3. Analogues of Proposition 12 for one-sample processes have previously been given by Howard et al. (2021), Ramdas et al. (2020), and Zhao et al. (2016), so our result is an extension of theirs to partially ordered processes. In our setting, recall that M_{ts} is $(\mathcal{D}_{ts} \wedge \mathcal{E}_{ts})$ -measurable. While Proposition 12 could be reformulated in reverse time, so that (\mathcal{F}_{ts}) can be taken to be the modeling filtration (\mathcal{E}_{ts}) , it is most interpretable to take it to be the data-generating filtration (\mathcal{D}_{ts}) . Doing so, under the assumptions of Theorem 7, leads for instance to the bound

$$\mathbb{P}\{\Psi(P,Q) \in C_{\tau\sigma}\} \ge 1 - \delta \quad \text{for all stopping times } \eta = (\tau,\sigma) \text{ with respect to } (\mathcal{D}_{ts}), \tag{28}$$

where C_{ts} denotes the two-sided interval (25), understood with conventions for infinities which can be deduced from equation (27).

Alternate Data-Generating Filtrations. Though we presumed the data-generating filtration (26) throughout our development, slightly tighter confidence sequences can be obtained if the user has access to additional information. For instance, our confidence sequences hold uniformly over arbitrary pairs of time (t, s), but such flexibility is unnecessary if the practitioner knows the order in which sample points from P and Q arrive. We illustrate two such examples below, focusing on lower confidence sequences:

(i) **Paired Samples.** When the observations X_t and Y_t are presumed to arrive at the same time, in pairs (X_t, Y_t) , the data-generating filtration may be replaced by

$$\mathcal{D}_t = \sigma(X_t, Y_t), \quad t = 1, 2, \dots$$

In this case, following along similar lines as before, the following two-sample bound may be established, and is tighter than that of Theorem 7(ii),

$$\mathbb{P}\left\{\exists t \ge 1 : \Psi(P_t, Q_t) \ge \Psi(P, Q) + (\psi_{tt}^*)^{-1} \left(\log \ell(\log_2 t) + \log(1/\delta)\right)\right\} \le \delta.$$
 (29)

Unlike Theorem 7, we note that the bound (29) can be taken to hold without assuming that (X_t) and (Y_s) are independent of each other.

(ii) Samples Ordered by External Randomization. As a generalization of the previous point, assume the observations X_t and Y_s arrive in a possibly random order which is independent of the data. Specifically, let $(\iota_n)_{n\geq 1}$ denote a sequence of random variables taking values in $\{0,1\}$, which are independent of (X_t) and (Y_s) , but possibly dependent on an external source of randomness U, say distributed uniformly on [0,1]. Let $t(n) = \sum_{i=1}^n \iota_n$, and $s(n) = n - t_n$ so that at any time $n \geq 1$, the practitioner observes $\iota_n X_{t(n)} + (1 - \iota_n) Y_{s(n)}$. In this case, one has access to the filtration $\mathcal{I}_n = \sigma(U, \iota_1, \iota_2, \dots, \iota_n), n \geq 1$, which determines the order in which the sample points X_t, Y_s arrive, as well as to the data-generating filtration

$$\overline{\mathcal{D}}_n = \overline{\mathcal{D}}_{t(n)}^X \bigvee \overline{\mathcal{D}}_{s(n)}^Y, \quad n \ge 1, \tag{30}$$

where $\overline{\mathcal{D}}_{t(n)}^{X}$ and $\overline{\mathcal{D}}_{s(n)}^{Y}$ are defined similarly as follows:

$$\overline{\mathcal{D}}_{t(n)}^{X} = \{ A \in \mathcal{F} : A \cap \{ t(n) = t \} \in \overline{\mathcal{D}}_{t}^{X}, \ \forall t \ge 1 \}, \quad \text{where} \quad \overline{\mathcal{D}}_{t}^{X} = \sigma(U, X_{1}, \dots, X_{t}). \quad (31)$$

Note that we could have assumed that the sequence (ι_n) is fully deterministic, in exchange for simpler notation. However, there are many situations, like clinical trials, in which we may wish to use external randomization (encoded by U) to determine how to obtain the next data point; for example, Efron (1971) shows how to adaptively randomize participants while encouraging balance between t(n) and s(n). Under this setting, it can be shown that $(\Psi(P_{t(n)}, Q_{s(n)}))_{n\geq 1}$ is a reverse submartingale, and assuming for simplicity that $\psi_n = \psi_{n0} = \psi_{0n}$, one may obtain the confidence sequence

$$\mathbb{P}\left\{\exists n \ge 1 : \Psi(P_{t(n)}, Q_{s(n)}) \ge \Psi(P, Q) + (\psi_{\bar{n}}^*)^{-1} \left(\log \ell(\log_2 n) + \log(1/\delta)\right)\right\} \le \delta.$$

In contrast to the above two settings, our confidence sequence C_{ts} satisfies the guarantee (2), which is uniform over pairs $(t,s) \in \mathbb{N}^2$, and therefore yields valid coverage even if the orderings t(n) and s(n) depend arbitrarily on the samples observed from P and Q.

4 Explicit corollaries for common divergences

We now specialize the confidence sequence C_{ts} to several examples of divergences including IPMs (Sections 4.1, 4.2), optimal transport costs (Section 4.3), φ -divergences (Section 4.4), and divergences smoothed by convolution (Section 4.5). Moving beyond divergences, we also derive time-uniform generalization error bounds for binary classification problems (Section 4.6), and confidence sequences for multivariate means (Section 4.7). These special cases will illustrate how our framework can be used to port existing fixed-time concentration inequalities to time-uniform ones, typically at the expense of iterated logarithmic factors. In these cases, any improvements to existing fixed-time concentration inequalities would typically carry over to our time-uniform setting. Though our focus is on non-asymptotic bounds, in Section 4.8, we also show how Theorem 7 can be used to derive a one-sided analogue of the classical law of the iterated logarithm, for several divergences between an empirical and true underlying measure. We defer all proofs to Appendix D.

4.1 Kolmogorov-Smirnov Distance

Theorem 7 leads to a sequential analogue of the classical Dvoretzky-Kiefer-Wolfowitz (DKW) inequality (Dvoretzky et al., 1956; Massart, 1990), based on distinct techniques than those of Howard and Ramdas (2022) and Maillard (2021). Let P be any distribution over \mathbb{R} with cumulative distribution function (CDF) F. Let $F_t(x) = (1/t) \sum_{i=1}^n I(X_i \leq x)$ denote the empirical CDF of F.

Corollary 13. For any $\delta \in (0,1)$,

$$\mathbb{P}\left(\exists t \ge 1 : \|F_t - F\|_{\infty} \ge \sqrt{\frac{\pi}{t}} + 2\sqrt{\frac{2}{t} \left[\log \ell(\log_2 t) + \log(1/\delta)\right]}\right) \le \delta.$$

Notice that Corollary 13 involves the term $\log(1/\delta)$, as opposed to the term $\log(2/\delta)$ which appears in the classical DKW inequality. This is due to the one-sidedness of the bounds in Theorem 7. The price to pay is the additional additive term $\sqrt{\pi/t}$, which is an upper bound on the expectation of $||F_{\lceil t/2 \rceil} - F||_{\infty}$.

Corollary 13 and Proposition 29 give rise to a sequential analogue of the celebrated Kolmogorov-Smirnov two-sample test. In what follows, Q denotes a second distribution over \mathbb{R} with CDF G, and empirical CDF $G_s(y) = (1/s) \sum_{i=1}^s I(Y_i \leq y)$. Recall also the sequence (κ_{ts}) defined in equation (24).

Corollary 14. Given $\delta \in (0, 1)$, set $\gamma_{ts} = \sqrt{\pi/t} + \sqrt{\pi/s} + 2\sqrt{\frac{2ts}{t+s}} \left[\log g(\log_2 t + \log_2 s) + \log(2/\delta) \right]$.

$$\mathbb{P}(\forall t, s \ge 1 : -\gamma_{ts} \le ||F - G||_{\infty} - ||F_t - G_s||_{\infty} \le \kappa_{ts}) \ge 1 - \delta.$$

In particular, the sequential Kolmogorov-Smirnov test which rejects the null hypothesis $H_0: P = Q$ when $||F_t - G_s||_{\infty} > \gamma_{ts}$ has type-I error controlled at $\delta/2$.

We now turn our attention to another popular IPM that is based on reproducing kernels.

4.2 Maximum Mean Discrepancy, V-Statistics, and U-Statistics

The kernel Maximum Mean Discrepancy (MMD) is an IPM measuring the distance between embeddings of distributions in a reproducing kernel Hilbert space (RKHS). We provide a brief definitions in what follows, and refer the reader to Scholkopf and Smola (2018) for further details. Let $K: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be a Mercer kernel, that is a symmetric and continuous function such that for any finite set of points $x_1, \ldots, x_n \in \mathbb{R}^d$, the matrix $(K(x_i, x_j))_{i,j=1}^n$ is positive semidefinite. The RKHS \mathcal{H} corresponding to K is the closure of the set

$$\mathcal{H}_0 = \left\{ \sum_{i=1}^k \alpha_i K(\cdot, x_i) : \alpha_i \in \mathbb{R}, \ x_i \in \mathbb{R}^d, k \ge 1 \right\},\,$$

endowed with the inner product and norm

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{i=1}^{k} \sum_{j=1}^{k'} \alpha_i \beta_j K(x_i, y_j), \quad \|f\|_{\mathcal{H}} = \sqrt{\langle f, f \rangle_{\mathcal{H}}},$$

where $f = \sum_{i=1}^k \alpha_i K(\cdot, x_i)$ and $g = \sum_{j=1}^{k'} \beta_j K(\cdot, y_j)$ denote the expansions of any two functions $f, g \in \mathcal{H}_0$. The MMD is defined as the IPM over the unit ball $\mathcal{J} = \{f \in \mathcal{H} : ||f||_{\mathcal{H}} \leq 1\}$ of \mathcal{H} . The plug-in estimator of the MMD admits the following representation, which makes its computation straightforward

$$MMD(P_t, Q_s) = \sqrt{\frac{1}{t^2} \sum_{i,j=1}^t K(X_i, X_j) + \frac{1}{s^2} \sum_{i,j=1}^s K(Y_i, Y_j) - \frac{2}{st} \sum_{i=1}^t \sum_{j=1}^s K(X_i, Y_j)}.$$
 (32)

In particular, if s = t and the data are available in pairs $Z_t = (X_t, Y_t)$ for all $t \ge 1$, as in Section 3.3(1), the above expression may be rewritten as a square root of a second-order V-Statistic,

$$MMD(P_t, Q_t) = \sqrt{\frac{1}{t^2} \sum_{i,j=1}^t J(Z_i, Z_j)},$$
(33)

where J((x,y),(x',y'))=K(x,x')+K(y,y')-2K(x,y'), for any $x,x',y,y'\in\mathbb{R}^d$. Assuming the kernel K is bounded, we derive a sequential concentration bound for this statistic as follows.

Corollary 15. Let $P,Q \in \mathcal{P}(\mathbb{R}^d)$. Assume that $\sup\{K(x,y) : x,y \in \mathbb{R}^d\} \leq B < \infty$. For any $\delta \in (0,1)$, define

$$\gamma_{ts} = 2\sqrt{2B}(t^{-\frac{1}{2}} + s^{-\frac{1}{2}}) + 4\sqrt{\frac{B(t+s)}{ts} \left[\log g(\log_2 t + \log_2 s) + \log(2/\delta)\right]},$$

and let (κ_{ts}) be the sequence defined in equation (24). Then,

$$\mathbb{P}\Big(\forall t, s \ge 1 : -\gamma_{ts} \le \text{MMD}(P, Q) - \text{MMD}(P_t, Q_s) \le 2\sqrt{B}\kappa_{ts}\Big) \ge 1 - \delta.$$

Assuming that the stitching function g is bounded above by a polynomial, Corollary 15 provides a confidence sequence for $\mathrm{MMD}(P,Q)$ scaling at the rate $O(\sqrt{\log\log(t\vee s)/(t\vee s)})$. Up to the iterated logarithmic factor, we recover the fixed-time rate obtained by Gretton et al. (2012) (Theorem 7), which was shown to be minimax optimal by Tolstikhin et al. (2016). In the setting of equal sample sizes t=s (cf. Section 3.3(1)), it is well-known that the V-Statistic $\mathrm{MMD}^2(P_t,Q_t)$ has first-order degeneracy, so that $\mathrm{MMD}^2(P_t,Q_t)=O_p(1/t)$ when P=Q (Lee, 1990). On the other hand, the bound $|\mathrm{MMD}^2(P_t,Q_t)-\mathrm{MMD}^2(P,Q)|=O_p(1/\sqrt{t})$ is tight when P,Q are fixed and $P\neq Q$. On the squared scale, the bound of Corollary 15 adapts to these distinct rates of convergence, since it implies that with probability at least $1-\delta$,

$$\forall t \ge 1, \quad |\mathrm{MMD}^2(P_t, Q_t) - \mathrm{MMD}^2(P, Q)| = O\left(\mathrm{MMD}(P, Q)\sqrt{\frac{\log\log t}{t}} + \frac{\log\log t}{t}\right). \tag{34}$$

Above, the right-hand side decays at the rate $O(\sqrt{(\log \log t)/t})$ in general, but improves to $O((\log \log t)/t)$ when P = Q. Similar considerations are discussed by Gretton et al. (2012).

While the plug-in estimator $\mathrm{MMD}(P_t,Q_s)$ is typically upwards biased, the squared MMD also admits a widely-used unbiased estimator (Gretton et al., 2012) obtained by replacing the V-Statistics in equations (32) and (33) by U-Statistics. We derive confidence sequences for $\mathrm{MMD}^2(P,Q)$ based on this estimator in Appendix D.2. The bounds therein do not adapt to the distinct rates of convergence described above, however, therefore we recommend those of Corollary 15 in practice.

We conclude this section with a more general discussion of sequential inference based on U- and V-Statistics, for expectation functionals of the form

$$\Phi: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}, \quad \Phi(P) = \iint h(x, y) dP(x) dP(y),$$

where $h: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a symmetric function. Following Lee (1990); Serfling (2009), the U- and V-Statistics corresponding to Φ are respectively defined by

$$U_t = \mathbb{E}[h(X_1, X_2) | \mathcal{E}_t^X] = \frac{2}{t(t-1)} \sum_{1 \le i < j \le t} h(X_i, X_j), \quad V_t = \Phi(P_t) = \frac{1}{t^2} \sum_{i,j=1}^t h(X_i, X_j).$$

The above representation immediately implies that (U_t) is a reverse martingale whenever it is integrable—a fact which can equivalently be derived from the leave-one-out property (18), which holds for (U_t) with equality. Notice that this property holds irrespective of the kernel h, so long as it is symmetric. On the other hand, the following can be said about (V_t) .

Proposition 16. Assume that $h: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is a continuous, symmetric, positive definite kernel over a compact set $\mathcal{X} \subseteq \mathbb{R}^d$. Then $\sqrt{\Phi(\cdot)}$ is a convex functional, thus $(\sqrt{V_t})$ and (V_t) are reverse submartingales with respect to (\mathcal{E}_t^X) .

The proof is in Appendix D.2. We do not generally expect the above result to hold for any symmetric kernel h, because the functional Φ is akin to a quadratic form, which may be nonconvex if its kernel is not positive semidefinite. Under the above results, a straightforward generalization of Theorem 7 can be used to derive two-sided confidence sequences for $\Phi(P)$ centered at U_t for all symmetric h, or lower confidence sequences for $\sqrt{\Phi(P)}$ and $\Phi(P)$ based on (V_t) for all positive definite h (which may be coupled with upper confidence sequences similarly as in Section 3.2). While these considerations make (U_t) -based confidence sequences seem attractive, we recall that (V_t) -based confidence sequences sometimes have the advantage of providing rate-optimal inference even when Φ is degenerate.

4.3 Optimal Transport Costs

Let $c: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a nonnegative cost function, and assume for simplicity that c is bounded above over \mathcal{X} by $\Delta := \sup_{x,y \in \mathcal{X}} c(x,y) < \infty$. We derive confidence sequences for the optimal transport cost $\mathcal{T}_c(P,Q)$, which will depend on an upper bound $\alpha_{c,ts}$ for the bias of the empirical plug-in estimator,

$$\mathbb{E}[\mathcal{T}_c(P_t, Q_s)] - \mathcal{T}_c(P, Q) \le \alpha_{c,ts}. \tag{35}$$

Such bounds have been derived in the literature for various choices of the cost c. For instance, one may take $\alpha_{c,ts}$ to scale as $(t \wedge s)^{-\alpha/d}$ for $d \geq 5$ when c is an α -Hölder smooth cost for some $\alpha \in [1,2]$ (Manole and Niles-Weed, 2021; Niles-Weed and Rigollet, 2022; Chizat et al., 2020), and as $(t \wedge s)^{-1/2}$ when \mathcal{X} is finite or one-dimensional, with suitable costs c (Sommerfeld and Munk, 2018; Forrow et al., 2018; Munk and Czado, 1998; Manole et al., 2022). We refer to Hundrieser et al. (2022) for further examples. The leading constants in these rates cannot always be made explicit, thus their use below will be of primarily theoretical interest.

Corollary 17. Let $c: \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a lower semi-continuous cost function bounded above by Δ , assume that equation (35) is satisfied, and recall that $\bar{t} = \lceil t/2 \rceil$. Furthermore, let (κ_{ts}) be the

sequence defined in equation (24). Then, for any $\delta \in (0,1)$,

$$\mathbb{P}\left(\forall t, s \ge 1 : -\alpha_{c,\bar{t}\bar{s}} - 2\Delta\sqrt{\frac{2(t+s)}{ts}} \left[\log g(\log_2 t + \log_2 s) + \log(2/\delta)\right] \\ \le \mathcal{T}_c(P,Q) - \mathcal{T}_c(P_t,Q_s) \le \Delta\kappa_{ts}\right) \ge 1 - \delta.$$

Corollary 17 exhibits a confidence sequence for $\mathcal{T}_c(P,Q)$ whose length is typically dominated by the expected deviation bound $\alpha_{c,\bar{t}\bar{s}}$. The potentially severe dependence on dimensionality in these rates can limit the applicability of Corollary 17 in high-dimensional problems. Section 4.5 derives confidence sequences for smoothed 1-Wasserstein distances, which admit significantly improved dimension dependence.

4.4 φ -Divergences over finite sets

Let P be a probability distribution supported on a set $\mathcal{X} = \{a_1, \ldots, a_k\}$ of finite cardinality $k \geq 2$. Set $p_j = P(\{a_j\})$ for all $j = 1, \ldots, k$. In this setting, we write the empirical measure as

$$P_t = \frac{1}{t} \sum_{i=1}^t \delta_{X_i} = \frac{1}{t} \sum_{j=1}^k C_j \delta_{a_j}, \text{ where } C_j = \sum_{i=1}^t I(X_i = a_j), j = 1, \dots, k.$$

The vector (C_1, \ldots, C_k) can be viewed as a random sample from a multinomial experiment with t trials and k categories with probabilities (p_1, \ldots, p_k) . Concentration inequalities for φ -divergences D_{φ} between the empirical measure and the finitely-supported true distribution P have received significant attention in the offline setting. Here, we show how such results can be used together with Theorem 7 to obtain time-uniform bounds on $D_{\varphi}(P_t||P)$. We focus on the Kullback-Leibler divergence and Total Variation distance in what follows.

Kullback-Leibler Divergence. Tight upper bounds on the moment generating function of the scaled Kullback-Leibler divergence $t\text{KL}(P_t||P)$ have recently been derived by Guo and Richardson (2020) (see also Agrawal (2020)), who prove

$$\mathbb{E}[\exp(\lambda t \text{KL}(P_t || P))] \le G_{k,t}(\lambda) := \sum_{x_1, \dots, x_k} {t \choose x_1, \dots, x_k} \prod_{i=1}^k [\lambda x_i / t + (1-\lambda)p_i]^{x_i}, \tag{36}$$

for all $\lambda \in [0,1]$. Guo and Richardson (2020) show that this upper bound is nearly tight, in the sense that it nearly matches the scaling in k and t of the moment generating function of the limiting distribution of $t\text{KL}(P_t||P)$. Furthermore, the value of $G_{k,t}(\lambda)$ does not depend on P (Guo and Richardson (2020), Proposition 1). Nevertheless, $G_{k,t}$ cannot easily be used in Theorem 7, since the optimization problem $\sup_{\lambda \in [0,1]} \left\{ \lambda u - G_{k,t}(\lambda) \right\}$, for any u > 0, is non-convex. Guo and Richardson (2020) instead derive several closed-form sequences (λ_t) which approximately solve this maximization problem. We derive a sequential analogue of their bounds in terms of a generic choice of such a sequence. The following result is obtained by repeating a similar stitching argument as that of the proof of Theorem 7.

Proposition 18. Let $\delta \in (0,1)$, and let P be a distribution supported on a finite set of size $k \geq 2$. Let $(\lambda_t)_{t=1}^{\infty} \subseteq [0,1/2]$ be a sequence of real numbers such that

$$\gamma_t = \frac{2}{\lambda_t t} \log \left(\frac{G_{k, \lfloor t/2 \rfloor}(2\lambda_t) \ell(\log_2 t)}{\delta} \right), \quad t \ge 1,$$

is a nonincreasing sequence in t. Then, $\mathbb{P}\left\{\exists t \geq 1 : \mathrm{KL}(P_t || P) \geq \gamma_t\right\} \leq \delta$.

It can be seen that the fitted probability vector $(C_1/t, \ldots, C_k/t)$ is precisely the maximum likelihood estimator of (p_1, \ldots, p_k) , and that the scaled Kullback-Leibler divergence $t\text{KL}(P_t||P)$ is a multiple of the log-likelihood ratio of (p_1, \ldots, p_k) . Proposition 18 therefore leads to a confidence sequence for the probability vector p on the basis of the classical likelihood ratio statistic.

Total Variation Distance. We now similarly derive time-uniform bounds for the discrete Total Variation distance $||P_t - P||_{\text{TV}} := \frac{1}{2} \sum_{j=1}^k |(C_j/t) - p_j|$. The following Corollary follows from Theorem 7 using elementary tail bounds for the Total Variation distance (see for instance Berend and Kontorovich (2013)), together with an expectation bound due to Kamath et al. (2015).

Corollary 19. For all $\delta \in (0,1)$, we have

$$\mathbb{P}\left\{\exists t \geq 1: \|P_t - P\|_{\text{TV}} \geq \sqrt{\frac{k}{\pi t}} + 2\left(\frac{2k}{t}\right)^{\frac{3}{4}} + 2\sqrt{\frac{2}{t}\left[\log\ell(\log_2 t) + \log(1/\delta)\right]}\right\} \leq \delta.$$

Up to a polylogarithmic factor, the bound of Corollary 19 scales at the parametric rate of convergence when the alphabet size k is fixed. For general distributions with uncountable support, such rates are not achievable under the Total Variation distance due to the lack of absolute continuity of P_t with respect to P. The following subsection studies a notable exception, in which parametric rates are retained when the measures are smoothed by convolution with a kernel admitting fixed bandwidth.

4.5 Smoothed divergences and differential entropy

Let $K: \mathbb{R}^d \to \mathbb{R}_+$ denote a smoothing kernel, that is, a nonnegative and continuous function satisfying $\int_{\mathbb{R}^d} K(x) dx = 1$. Given a bandwidth $\sigma > 0$, let \mathcal{K}_{σ} be the probability measure admitting density $K_{\sigma}(x) = (1/\sigma^d)K(x/\sigma)$ with respect to the Lebesgue measure. Let D denote a convex divergence, and define its smoothed counterpart by

$$D^{\sigma}: (P,Q) \longmapsto D(P \star \mathcal{K}_{\sigma} || Q \star \mathcal{K}_{\sigma}).$$

It can be directly verified that D^{σ} is itself a convex divergence, due to the linearity of the convolution operator. Theorem 7 can therefore be used to derive a confidence sequence for $D^{\sigma}(P||Q)$ based on the plug-in estimator $D^{\sigma}(P_t||Q_s)$. We emphasize that this estimator is sensible even if the original divergence D requires absolute continuity of the distributions being compared, as is the case for φ -divergences. In such cases, D^{σ} forms a proxy of D which can be estimated using the empirical plug-in estimator. We refer to Goldfeld et al. (2020b) for upper bounds on $\mathbb{E}D^{\sigma}(P_t, P)$ under a wide range of divergences D. Smoothing by Gaussian convolution has also recently been studied as a means of regularizing optimal transport problems, and thereby reducing the curse of dimensionality

in estimating Wasserstein distances. For instance, Goldfeld and Greenewald (2020), Goldfeld et al. (2020a) show that that the empirical measure converges to P at the parametric rate under W_1^{σ} , in expectation, contrasting the unavoidable $t^{-1/d}$ rate for this problem under W_1 itself (Singh and Póczos, 2019).

Motivated by these two applications, we show in what follows how Theorem 7 can be used to derive confidence sequences for the smoothed Total Variation distance, and for the smoothed 1-Wasserstein distance. The results which follow are obtained by first deriving upper bounds on the moment generating functions of $D^{\sigma}(P_t||P)$, and second, invoking the upper bounds of Goldfeld et al. (2020b) on the expectation of this quantity. In order to appeal to their results, we assume in what follows that $K(x) = e^{-||x||_2^2/2}/\sqrt{2\pi}$ is taken to be the standard Gaussian kernel.

We first recall a tail assumption which will be used in the sequel. Given a metric d on \mathcal{X} , we say a measure $P \in \mathcal{P}_1(\mathcal{X})$ satisfies the $T_1(\tau^2)$ inequality with respect to d, for some $\tau > 0$, if

$$\mathcal{T}_d(\mu, P) \leq \sqrt{2\tau^2 \mathrm{KL}(\mu \| P)}, \quad \text{for all } \mu \in \mathcal{P}_1(\mathcal{X}).$$

Such inequalities are at the heart of the transportation method for deriving fixed-time concentration inequalities—we refer to Gozlan and Léonard (2010) for a survey. For our purposes, transportation inequalities are known to provide a natural tail assumption on P in order to guarantee sub-Gaussian concentration of empirical Wasserstein distances: Niles-Weed and Rigollet (2022) (Theorem 6) prove that P satisfies $T_1(\tau^2)$ if and only if $W_1(P_t, P)$ is (τ^2/t) -sub-Gaussian. By extending their result to smoothed Wasserstein distances, we arrive at the following.

Proposition 20 (Smoothed Divergences). Let $\delta \in (0,1)$, $\sigma > 0$, and let $P \in \mathcal{P}(\mathbb{R}^d)$.

(i) (Total Variation Distance) Assume P is τ^2 -sub-Gaussian for some $\tau > 0$. Then,

$$\mathbb{P}\left(\exists t \ge 1 : \|P_t - P\|_{\text{TV}}^{\sigma} \ge \frac{c_d}{\sqrt{t}} + 4\sqrt{\frac{2}{t} \left[\log \ell(\log_2 t) + \log(1/\delta)\right]}\right) \le \delta,$$

where
$$c_d = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{\tau}{\sigma} \right)^{\frac{d}{2}} e^{\frac{3d}{16}}$$
.

(ii) (W₁ Distance) Assume P satisfies the $T_1(\tau^2)$ inequality with respect to $\|\cdot\|_2$, for some $\tau > 0$. Then, $W_1^{\sigma}(P_t, P)$ is (τ^2/t) -sub-Gaussian, and for all $\delta \in (0, 1)$,

$$\mathbb{P}\left(\exists t \ge 1 : W_1^{\sigma}(P_t, P) \ge \frac{C_d}{\sqrt{t}} + 2\sqrt{\frac{\tau^2}{t} \left[\log \ell(\log_2 t) + \log(1/\delta)\right]}\right) \le \delta,\tag{37}$$

where
$$C_d = 2\sqrt{d\sigma^2} \left(\frac{1}{\sqrt{2}} + \frac{\tau}{\sigma}\right) c_d$$
.

Extensions of Proposition 20 to the two-sample setting are straightforward and omitted for brevity. Proposition 20(i) yields a confidence sequence for the smoothed Total Variation under a mere moment condition. Such a result could not have been obtained by our framework in the absence of smoothing, except in the special case of Section 4.4 where P was assumed to have countable support. Proposition 20(ii) contrasts our earlier Corollary 17, which implied a confidence sequence for the 1-Wasserstein distance scaling at the rate $O(t^{-1/d})$ for $d \ge 3$. Smoothing removes the

dimension dependence from the rate itself when σ is held fixed, although the constant C_d continues to grow exponentially in d. It can be deduced from Theorem 1 of Goldfeld et al. (2020b), combined with equation (47), below that exponential dimension dependence in this constant is necessary, although the optimal constant is not known. Any sharpening of these constants in future work could directly be used to update the time-uniform bound in Proposition 20.

Inspired by Weed (2018), we briefly close this subsection by illustrating how Proposition 20 can further be used to obtain sequential bounds for the smoothed differential entropy of P, $h(P \star \mathcal{K}_{\sigma}) = -\int \log(P \star K_{\sigma}) dP \star \mathcal{K}_{\sigma}$, using the fact that it is Lipschitz with respect to the W_1 metric (Polyanskiy and Wu, 2016).

Corollary 21. Let $\delta \in (0,1)$, $\sigma > 0$, and $P \in \mathcal{P}([-1,1]^d)$. Then,

$$\mathbb{P}\left(\forall t \geq 1: |h(P_t \star \mathcal{K}_{\sigma}) - h(P \star \mathcal{K}_{\sigma})| \leq \frac{3}{\sigma^2} \sqrt{\frac{d}{t} \left[\log \ell(\log_2 t) + \log(4/\delta)\right]} + \frac{\sqrt{d}C_d}{\sqrt{t}\sigma^2}\right) \geq 1 - \delta.$$

Notice that the mapping $P \mapsto -h(P \star \mathcal{K}_{\sigma})$ is convex, therefore a confidence sequence for the smoothed differential entropy could also have been obtained by directly appealing to Theorem 7, assuming a bound on the cumulant generating function of $h(P_t \star \mathcal{K}_{\sigma})$ were available. Beyond this approach and the bound of Corollary 21, we are not aware of other existing sequential concentration inequalities for this problem.

4.6 Sequential generalization error bounds for binary classification

Theorem 7 can be used to derive generalization error bounds for classification or regression problems that are valid at stopping times. We illustrate the special case of binary classification. Let \mathcal{X} be a topological space, P be a Borel probability distribution over $\mathcal{X} \times \{-1, 1\}$, and $(X_t, Y_t)_{t=1}^{\infty}$ a sequence of i.i.d. observations from P. Let \mathcal{F} be a collection of Borel-measurable functions from \mathcal{X} to $\{-1, 1\}$, and define the population and empirical classification risks by

$$R(f) = \mathbb{P}(f(X) \neq Y), \text{ and } R_t(f) = \frac{1}{t} \sum_{i=1}^t I(f(X_i) \neq Y_i), f \in \mathcal{F}.$$

High-probability bounds on the supremum of the empirical process $\sup_{f \in \mathcal{F}} |R(f) - R_t(f)|$ for fixed times $t \geq 1$ are well-studied, and can lead to conservative confidence intervals for the generalization error $R(\hat{f}_t)$ of any data-dependent classifier $\hat{f}_t \in \mathcal{F}$, such as an empirical risk minimizer, or an approximate one obtained by stochastic optimization. Such bounds necessarily depend on the complexity of \mathcal{F} , as measured for instance by population or empirical Rademacher complexities, respectively defined by

$$\mathcal{R}_t(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\varepsilon}_t, \mathbf{X}_t} \left[\sup_{f \in \mathcal{F}} \frac{1}{t} \left| \sum_{i=1}^t \epsilon_i f(X_i) \right| \right], \quad \widehat{\mathcal{R}}_t(\mathcal{F}) = \mathbb{E}_{\boldsymbol{\varepsilon}_t} \left[\sup_{f \in \mathcal{F}} \frac{1}{t} \left| \sum_{i=1}^t \epsilon_i f(X_i) \right| \right].$$

Here, we denote $\mathbf{X}_t = (X_1, \dots, X_t)$ and $\boldsymbol{\varepsilon}_t = (\epsilon_1, \dots, \epsilon_t)$, where $(\epsilon_t)_{t=1}^{\infty}$ denotes a sequence of i.i.d. Rademacher random variables (taking values ± 1 with equal probability). We obtain the following bound which is uniform both over hypotheses $f \in \mathcal{F}$ and over time $t \geq 1$.

Corollary 22. Let $\delta \in (0,1)$, and recall that $\bar{t} = |t/2|$ for all $t \geq 1$.

1. The population Rademacher complexity provides a time-uniform generalization error bound:

$$\mathbb{P}\left(\forall t \geq 1 : \sup_{f \in \mathcal{F}} |R_t(f) - R(f)| \leq \mathcal{R}_{\bar{t}}(\mathcal{F}) + 2\sqrt{\frac{2}{t} \left[\log \ell(\log_2 t) + \log(1/\delta)\right]}\right) \geq 1 - \delta.$$

2. The empirical Rademacher complexity $(\widehat{\mathcal{R}}_t(\mathcal{F}))$ is a reverse submartingale with respect to (\mathcal{E}_t^X) , and we have:

$$\mathbb{P}\Big(\forall t \geq 1 : \mathcal{R}_{\bar{t}}(\mathcal{F}) \geq \widehat{\mathcal{R}}_t(\mathcal{F}) - 4\sqrt{\frac{2}{t} \Big[\log \ell(\log_2 t) + \log(1/\delta)\Big]}\Big) \geq 1 - \delta.$$

In particular, if τ is an arbitrary stopping time and \hat{f}_t is an arbitrary data-dependent classifier, then $\mathbb{P}(|R_{\tau}(\hat{f}_{\tau}) - R(\hat{f}_{\tau})| \leq \mathcal{R}_{\bar{\tau}}(\mathcal{F}) + 2\sqrt{(2/\tau)[\log \ell(\log_2 \tau) + \log(1/\delta)]}) \geq 1 - \delta$. We are not aware of other such generalization bounds that hold at stopping times.

Corollary 22 is comparable to the following well-known fixed-time bound which can be deduced, for instance, from the proof of Theorem 3.5 of Mohri et al. (2018):

$$\mathbb{P}\left(\sup_{f\in\mathcal{F}}|R_t(f)-R(f)|\leq \mathcal{R}_t(\mathcal{F})+\sqrt{\log(2/\delta)/2t}\right)\geq 1-\delta.$$

Once again, we observe that our time-uniform bound only loses iterated logarithmic factors and small universal constants in comparison to the above display. When the population Rademacher complexity $\mathcal{R}_t(\mathcal{F})$ is unavailable in closed form, Corollary 22(ii) may be used to provide a time-uniform lower bound on this quantity in terms of its empirical counterpart. We obtain this result in Appendix D.6 by noting that $\hat{\mathcal{R}}_t$ satisfies the leave-one-out property in equation (18), although it cannot easily be written as the evaluation of a convex functional at the empirical measure. We leave open the question of providing upper confidence sequences on $\mathcal{R}_t(\mathcal{F})$, which combined with Corollary 22(i) would lead to a fully empirical bound for the classification risk.

The proof of Corollary 22(i) shows that analogous time-uniform concentration inequalities can be obtained for general suprema of empirical processes over uniformly bounded function classes, up to modifying the expectation bound $\mathcal{R}_{\bar{t}}(\mathcal{F})$, which yields time-uniform inference for the risk of arbitrary estimators with respect to a bounded loss function, in terms of their empirical risk.

4.7 Sequential estimation of multivariate means

We next show how Theorem 7 can be used to derive confidence sequences for the mean μ of a multivariate distribution $P \in \mathcal{P}(\mathbb{R}^d)$. In the special case d = 1, our results show how our reverse submartingale techniques can recover known confidence sequences for univariate sequential mean estimation (summarized in Section 2.5), up to constant factors.

Let $(X_t)_{t=1}^{\infty}$ be a sequence of i.i.d. random variables with mean μ , and let $\mu_t = (1/t) \sum_{i=1}^t X_i$. We state our bounds in terms of a general norm $\|\cdot\|$ on \mathbb{R}^d , whose dual norm is denoted by $\|\cdot\|_{\star} =$

 $\sup_{\|\lambda\|=1}\langle\lambda,\cdot\rangle$. Assume further that there exists $\lambda_{\max}>0$ and a convex function $\psi:[0,\lambda_{\max})\to\mathbb{R}$ such that

$$\sup_{\nu \in \mathbb{S}_{+}^{d-1}} \log \left(\mathbb{E}_{X \sim P} \left[\exp \left(\lambda \langle \nu, X - \mu \rangle \right) \right] \right) \le \psi(\lambda), \quad \lambda \in [0, \lambda_{\max}), \tag{38}$$

where $\mathbb{S}^{d-1}_{\star} = \{x \in \mathbb{R}^d : \|x\|_{\star} = 1\}$. For instance, when $\psi(\lambda) = \lambda^2 \sigma^2/2$, the above definition reduces to that of a $(\sigma^2, \lambda_{\max}^{-1})$ -sub-exponential random vector given in Vershynin (2018) when $\lambda_{\max} < \infty$, or of a σ^2 -sub-Gaussian random vector when $\lambda_{\max} = \infty$. Finally, for any $\gamma > 0$, let N_{γ} denote the γ -covering number (van der Vaart and Wellner, 1996) of \mathbb{S}^{d-1}_{\star} with respect to the norm $\|\cdot\|_{\star}$.

Corollary 23. Assume P satisfies the tail assumption (38). Then, for all $\gamma \in [0,1)$, $\delta \in (0,1)$,

$$\mathbb{P}\left\{\exists t \geq 1: \|\mu_t - \mu\| \geq \frac{1}{1 - \gamma} (\psi^*)^{-1} \left(\frac{\log \ell(\log_2 t) + \log(1/\delta) + \log N_\gamma}{\lceil t/2 \rceil}\right)\right\} \leq \delta.$$

Above, $\gamma = 1/2$ is a reasonable default value. We first illustrate the result of Corollary 23 in the special case when d=1 and P is 1-sub-Gaussian. If the norm $\|\cdot\|$ is taken to be the absolute value, notice that one may choose $\gamma = 0$ and $N_{\gamma} = 2$, thus Corollary 23 implies

$$\forall t \ge 1: |\mu_t - \mu| \le 2\sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(2/\delta) \right]}, \quad \text{with probability at least } 1 - \delta.$$
 (39)

Equation (39) is comparable to state-of-the-art confidence sequences for univariate means, summarized in Section 2.5. For example, Theorem 1 of Howard et al. (2021) provides a one-sided bound for means of 1-sub-Gaussian random variables, which together with a union bound leads to the two-sided confidence sequence

$$\forall t \ge 1: |\mu_t - \mu| \le k_1 \sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(2/\delta) \right]}, \quad \text{with probability at least } 1 - \delta, \tag{40}$$

where $k_1 = \frac{2^{1/4} + 2^{-1/4}}{\sqrt{2}} \approx 1.8$. It can be seen from the preceding two displays that our confidence sequence is wider by a mere factor of $2/1.8 \approx 1.1$ compared to that of Howard et al. (2021). In fact, Corollary 23 is a special case of a more general result that can be deduced from Theorem 28 in Appendix C.1, for which there are tuning parameters that we did not optimize here, so the factor 1.1 could presumably be lowered further. More importantly though, our result applies more generally to means of multivariate distributions, for which we do not know of any other confidence sequences in the literature beyond those of Abbasi-Yadkori et al. (2011). The latter paper only has sub-Gaussian bounds decaying at the $\sqrt{\log t/t}$ rate, instead of our $\sqrt{\log \log t/t}$ rate.

As a multivariate example, suppose now that P is (σ^2, α) -sub-exponential, so that $\psi(\lambda) = \lambda^2 \sigma^2/2$ with $\lambda_{\max} = 1/\alpha$. When $\|\cdot\|$ is taken to be the Euclidean norm $\|\cdot\|_2$, one may derive the bound

$$\forall t \geq 1: \|\mu_t - \mu\|_2 \leq 2 \begin{cases} \sqrt{2\sigma^2 \gamma_t}, & t: 0 \leq \gamma_t < \frac{\sigma^2}{2\alpha^2}, \\ \gamma_t \alpha + \frac{\sigma^2}{2\alpha}, & t: \frac{\sigma^2}{2\alpha^2} \leq \gamma_t, \end{cases}, \text{ with probability at least } 1 - \delta,$$

where $\gamma_t = (2/t)[\log \ell(\log_2 t) + \log(1/\delta) + d\log 5]$. In particular, we recover the optimal dependence on both d and t from the fixed-time setting, up to iterated logarithmic factors.

4.8 An upper law of the iterated logarithm for sub-Gaussian divergences

We now show how our finite-sample results can be used to derive an asymptotic statement which mirrors the classical (upper) law of the iterated logarithm (LIL; Stout (1970)) for sums of i.i.d. random variables.

Corollary 24. Let D be a convex divergence such that $D(P_t||P)$ is (σ^2/t) -sub-Gaussian for all $t \ge 1$ and some $\sigma > 0$. Assume $\mathbb{E}D(P_t||P) = o(\sqrt{(\log \log t)/t})$. Then,

$$\limsup_{t \to \infty} \frac{tD(P_t || P)}{\sqrt{2t\sigma^2 \log \log t}} \le 1, \quad a.s.$$

Corollary 24 establishes an upper LIL for convex divergences admitting the same constant as the classical LIL, which states that for any sequence of mean-zero i.i.d. random variables $(X_t)_{t=1}^{\infty}$ admitting finite variance $\sigma^2 > 0$,

$$\limsup_{t \to \infty} \frac{1}{\sqrt{2t\sigma^2 \log \log t}} \sum_{i=1}^{t} X_i = 1, \quad \text{a.s.}$$

Obtaining a matching lower bound in Corollary 24 would, for instance, necessitate anti-concentration bounds on the process $D(P_t||P)$, and is therefore beyond the scope of this work. The sub-Gaussianity assumption can also likely be weakened, but given again that our purpose was not asymptotics, we leave this for future work. Though results analogous to Corollary 24 have possibly appeared in past literature for various divergences, we are only aware of the LILs for the Kolmogorov-Smirnov statistic derived by Smirnov (1944), for the 1-Wasserstein distance in dimension d = 1 (del Barrio et al., 1999), and for certain von Mises differentiable functionals (Serfling, 2009).

Adaptations of the proofs of Corollaries 13, 15, 19, and 20 respectively imply that Corollary 24 holds when D is taken to be the Kolmogorov-Smirnov distance, the Maximum Mean Discrepancy with bounded kernel, the Total Variation distance for distributions supported on a finite set, or the Total Variation and 1-Wasserstein distances smoothed by Gaussian convolution under suitable tail assumptions on P. The conditions of Corollary 24 can also be verified for the transportation cost W_p^p , for any $p \geq 1$ satisfying p > d/2, under suitable moment assumptions on P (Fournier and Guillin, 2015; Niles-Weed and Rigollet, 2022). To the best of our knowledge, this functional is not known to be von Mises differentiable without further assumptions on the connectedness of the support or absolute continuity of P (cf. Goldfeld et al. (2022) and Manole et al. (2021, Section 5.2)).

5 Summary

Existing approaches to anytime-valid sequential inference, both classical ideas (Darling and Robbins, 1967) and modern nonparametric treatments (Howard et al., 2020), are centered around the identification or design of certain nonnegative supermartingales, coupled with a maximal inequality due to Ville (1939). A primary technical contribution of this current work is to recognize that these techniques are unfortunately not well-suited to dealing with general convex divergences. Instead,

we build on the elegant fact that the empirical measure is a measure-valued reverse martingale with respect to the exchangeable filtration, and utilize it to infer that every convex divergence is a nonnegative reverse (partially ordered) submartingale. Once this connection is made, the literature uncovers a little-known maximal inequality that is well-suited for our purposes—a time-reversed version of Ville's maximal inequality, together with its analogues for partially ordered processes. These tools act as batch-to-sequential devices: they allow us to port existing fixed-time Chernoff-type bounds that have been developed for a large variety of convex divergences into the sequential setting, at the loss of only iterated logarithm factors. As a result of our modular approach, if the fixed-time bounds have optimal rates in all problem-dependent parameters, then so do our sequential bounds, and if some of those bounds are improved in the future, then those improvements are immediately obtained in the sequential setting.

Acknowledgments

TM would like to thank Ian Waudby-Smith for several conversations related to this work. TM was supported in part by the Natural Sciences and Engineering Research Council of Canada, through a PGS D Scholarship. AR would like to acknowledge NSF DMS grant 1916320 and an Adobe faculty research award for funding work on this topic.

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A Additional Lemmas

We begin by recalling McDiarmid's inequality (Wainwright, 2019). We say that a map $G : \mathbb{R}^t \to \mathbb{R}$ satisfies the bounded differences property with parameters $(L_1, \ldots, L_t) \in \mathbb{R}^t_+$ if for every $x_1, \ldots, x_t, x_1', \ldots, x_t' \in \mathbb{R}$ and all $i = 1, \ldots, t$,

$$|G(x_1,\ldots,x_t)-G(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_t)| \leq L_i.$$

Theorem 25 (McDiarmid's Inequality). Assume G satisfies the bounded differences property with parameters (L_1, \ldots, L_t) and that X_1, \ldots, X_t are independent random variables. Then, for all $u \geq 0$,

$$\mathbb{P}\Big(\big|G(X_1,\ldots,X_t) - \mathbb{E}G(X_1,\ldots,X_t)\big| \ge u\Big) \le 2\exp\left\{-\frac{2u^2}{\sum_{i=1}^t L_i^2}\right\}.$$

In several of the proofs for Section 4, we will show that the processes $\Phi(P_t, Q)$ or $\Psi(P_t, Q_s)$ satisfy the bounded differences property, when viewed as functions of the samples therein. The following standard result will then imply that these processes are sub-Gaussian (see for instance Rigollet and Hütter (2015), Lemma 1.5, for a statement with the exact constants used below).

Lemma 26. Let P be a distribution over \mathbb{R} such that for $X \sim P$ and $\sigma > 0$, $\mathbb{E}[X] = 0$ and

$$\mathbb{P}(|X| > u) \le 2e^{-u^2/2\sigma^2}, \quad u > 0.$$

Then, P is $(8\sigma^2)$ -sub-Gaussian.

B Proofs from Section 2

B.1 Proofs of Theorem 2

Theorem 2 was proven for instance by Lee (1990), Theorem 3, p. 112. Due to its centrality in our work, we provide two self-contained proofs of this result below. The first proof follows directly from Doob's submartingale inequality (see equation (12)). The second proof is a restatement of Lee's original proof, which we include for reference in the following subsection.

First Proof (via Doob's submartingale inequality). For any integer $T \geq t_0$, define the process $S_t = R_{T-t+t_0}$, for all $t_0 \leq t \leq T$, as well as the forward filtration $\mathcal{G}_t = \mathcal{F}_{T-t+t_0}$. Since (R_t) is a reverse submartingale, we have $\mathbb{E}[R_t|\mathcal{F}_{t+1}] \geq R_{t+1}$, whence for all $t_0 + 1 \leq t \leq T$,

$$\mathbb{E}[S_t|\mathcal{G}_{t-1}] = \mathbb{E}[R_{T-t+t_0}|\mathcal{F}_{T-t+t_0+1}] \ge R_{T-t+t_0+1} = S_{t-1}.$$

It follows that $(S_t)_{t=t_0}^T$ forms a forward submartingale with respect to $(\mathcal{G}_t)_{t=t_0}^T$. Applying Doob's submartingale inequality, we therefore obtain

$$\mathbb{P}(\exists t_0 \le t \le T : S_t \ge u) \le \frac{\mathbb{E}[S_T]}{u},$$

for all u > 0. Equivalently,

$$\mathbb{P}(\exists t_0 \le s \le T : R_s \ge u) \le \frac{\mathbb{E}[R_{t_0}]}{u}.$$

Notice that the event within the probability on the left-hand side of the above display is monotonically increasing with T, converging to the event $\{\exists s \geq t_0 : R_s \geq u\}$. Taking $T \to \infty$, we thus have

$$\mathbb{P}(\exists s \ge t_0 : R_s \ge u) \le \frac{\mathbb{E}[R_{t_0}]}{u},$$

which proves the claim.

Second Proof (first principles). Let $T \geq t_0$, u > 0, and define the disjoint sets

$$A_t = \{R_t \ge u\} \cap \bigcap_{j=t+1}^T \{R_j < u\}, \quad t = t_0, t_0 + 1, \dots, T,$$

where t represents the last time (in the range $t_0, t_0 + 1, ..., T$) at which R_t was larger than u. We have,

$$\mathbb{P}(\exists t_0 \le t \le T : R_t \ge u) = \sum_{t=t_0}^T \mathbb{P}(A_t) \le \frac{1}{u} \sum_{t=t_0}^T \int_{A_t} R_t d\mathbb{P} \le \frac{1}{u} \sum_{t=t_0}^T \int_{A_t} \mathbb{E}[R_{t_0} | \mathcal{F}_t] d\mathbb{P},$$

where the first inequality follows because $R_t > u$ on A_t , and the second inequality follows by the reverse submartingale property of (R_t) . Note that R_j is \mathcal{F}_j -measurable and hence also \mathcal{F}_t -measurable for $t \leq j$ due to the reversed nature of the filtrations. Thus, $A_t \in \mathcal{F}_t$, whence $\int_{A_t} \mathbb{E}[R_{t_0}|\mathcal{F}_t] d\mathbb{P} = \int_{A_t} R_{t_0} d\mathbb{P}$ and we obtain

$$\mathbb{P}(\exists t_0 \le t \le T : R_t \ge u) \le \frac{1}{u} \sum_{t=t_0}^{T} \int_{A_t} R_{t_0} d\mathbb{P} \le \frac{1}{u} \mathbb{E}[R_{t_0}],$$

where the last step utilizes the nonnegativity of R_t and the fact that the events $\{A_t\}_{t=t_0}^T$ are disjoint by construction. The claim now follows as before by taking $T \to \infty$, noting that the right-hand side remains fixed while the left-hand side is the probability of an increasing sequence of events whose limit is $\{\exists t \geq t_0 : R_t \geq u\}$.

B.2 Proof of Proposition 3

Let $(\mathcal{F}_{ts})_{t,s\geq 1}$ be a reverse filtration. Under the conditional independence condition (14) on (\mathcal{F}_{ts}) , Christofides and Serfling (1990b) establish a maximal inequality for partially ordered reverse submartingales, which we state below before proving Proposition 3. We will require the following notation. Let $1 \leq t_0 \leq T$, $1 \leq s_0 \leq S$, and let $\{C_{ts}: t, s \geq 0\}$ be a nondecreasing array of nonnegative numbers. Given a reverse submartingale $(R_{ts})_{t,s\geq 1}$ with respect to (\mathcal{F}_{ts}) , a general bound will be given on $\mathbb{P}(A)$, where for any u > 0,

$$A = \left\{ \max_{\substack{t_0 \le t \le T \\ s_0 \le s \le S}} C_{ts} R_{ts} \ge u \right\} = \bigcup_{t=t_0}^{T} \bigcup_{s=s_0}^{S} A_{ts}, \text{ where } A_{ts} = \{R_{ts} C_{ts} \ge u\} \cap \bigcap_{j=t+1}^{T} \bigcap_{k=s+1}^{S} \{R_{jk} C_{jk} < u\}.$$
(41)

This decomposition into the sets A_{ts} is analogous to that in the proof of Ville's inequality for reverse submartingales (Appendix B.1). Unlike that result, however, the lack of total ordering on

 \mathbb{N}^2 prevents the sets A_{ts} from being disjoint; for instance A_{43} and A_{34} could both potentially happen. Christofides and Serfling (1990b) instead form a partition $(B_{ts}^{(1)})_{t,s\geq 1}$ of A, defined recursively by the following algorithm.

Let
$$D_0 = \emptyset$$
, $m := 1$
For $j = t_0$ to T
For $k = s_0$ to S

$$B_{jk}^{(1)} := A_{jk} \setminus \bigcup_{l < m} D_l$$

$$D_m := A_{jk}, \ m := m + 1.$$

A second partition $(B_{ts}^{(2)})_{t,s\geq 1}$ is further formed by changing the order of the for-loops in the above display. Specifically,

$$B_{ts}^{(1)} = A_{ts} \setminus \left\{ \left(\bigcup_{j=t_0}^{t-1} \bigcup_{k=s_0}^{S} A_{jk} \right) \cup \left(\bigcup_{k=s_0}^{s-1} A_{tk} \right) \right\}, \quad B_{ts}^{(2)} = A_{ts} \setminus \left\{ \left(\bigcup_{k=s_0}^{s-1} \bigcup_{j=t_0}^{T} A_{jk} \right) \cup \left(\bigcup_{j=t_0}^{t-1} A_{js} \right) \right\},$$

with the convention that an empty union is equal to the empty set. Notice that for j = 1, 2, the sets $(B_{ts}^{(j)})_{t,s\geq 1}$ are mutually disjoint, and $\bigcup_{t,s\geq 1} B_{ts}^{(j)} = A$. Further, unlike (A_{ts}) , the sequence $(B_{ts}^{(j)})$ is not adapted to (\mathcal{F}_{ts}) , but instead satisfies $B_{ts}^{(1)} \in \mathcal{F}_{ts_0}$ and $B_{ts}^{(2)} \in \mathcal{F}_{t_0s}$. We are now in a position to state their bound.

Lemma 27 (Christofides and Serfling (1990b), Corollary 2.9). Let (R_{ts}) be a nonnegative reverse submartingale with respect to (\mathcal{F}_{ts}) , and assume (\mathcal{F}_{ts}) satisfies the conditional independence condition (14). Furthermore, let $\{C_{ts}: t, s \geq 0\}$ be a nondecreasing array of nonnegative numbers. Then, for all u > 0,

$$u\mathbb{P}\left\{\max_{\substack{t_0 \leq t \leq T\\s_0 \leq s \leq S}} C_{ts} R_{ts} \geq u\right\} \leq \left\{\sum_{t=t_0}^T \sum_{s=s_0}^S (C_{ts} - C_{(t-1)s})\mathbb{E}[R_{ts}] - \sum_{s=s_0}^S C_{t_0s} \int_{(\bigcup_{t=t_0}^T B_{ts}^{(1)})^c} R_{t_0s} d\mathbb{P}\right\}$$

$$\wedge \left\{\sum_{t=t_0}^T \sum_{s=s_0}^S (C_{ts} - C_{t(s-1)})\mathbb{E}[R_{ts}] - \sum_{t=t_0}^T C_{ts_0} \int_{(\bigcup_{s=s_0}^S B_{ts}^{(2)})^c} R_{ts_0} d\mathbb{P}\right\}.$$

In the special case where $C_{ts} = I(t \ge t_0, s \ge s_0)$ for all $0 \le t \le T, 0 \le s \le S$, Lemma 27 reduces to the following bound

$$\mathbb{P}\left\{\max_{\substack{t_0 \le t \le T\\s_0 \le s \le S}} R_{ts} \ge u\right\} \le \frac{1}{u} \left[\sum_{s=s_0}^{S} C_{t_0 s} \mathbb{E}[R_{t_0 s}] - \sum_{s=s_0}^{S} C_{t_0 s} \int_{(\bigcup_{t=t_0}^{T} B_{ts}^{(1)})^c} R_{t_0 s} d\mathbb{P}\right] \\
= \frac{1}{u} \sum_{s=s_0}^{S} \int_{\bigcup_{t=t_0}^{T} B_{ts}^{(1)}} R_{t_0 s} d\mathbb{P}. \tag{42}$$

This simplification of Lemma 27 turns out to be simple to show, and we provide a self-contained proof before using it to prove Proposition 3 below.

Proof of Inequality (42). We have for any $s_0 \le s \le S$,

$$u\mathbb{P}\left(\bigcup_{t=t_{0}}^{T}B_{ts}^{(1)}\right) = u\sum_{t=t_{0}}^{T}\mathbb{P}(B_{ts}^{(1)})$$

$$\leq \sum_{t=t_{0}}^{T}\int_{B_{ts}^{(1)}}R_{ts}d\mathbb{P}$$

$$\leq \sum_{t=t_{0}}^{T}\int_{B_{ts}^{(1)}}\mathbb{E}[R_{t_{0}s}|\mathcal{F}_{ts}]d\mathbb{P} \qquad \text{(By the reverse submartingale property)}$$

$$= \sum_{t=t_{0}}^{T}\int_{B_{ts}^{(1)}}\mathbb{E}\Big\{\mathbb{E}[R_{t_{0}s}|\mathcal{F}_{t_{0}s}] \mid \mathcal{F}_{ts_{0}}\Big\}d\mathbb{P} \qquad \text{(By the CI condition)}$$

$$= \sum_{t=t_{0}}^{T}\int_{B_{ts}^{(1)}}\mathbb{E}[R_{t_{0}s}|\mathcal{F}_{t_{0}s}]d\mathbb{P} \qquad \text{(Since } B_{ts}^{(1)} \in \mathcal{F}_{ts_{0}})$$

$$= \sum_{t=t_{0}}^{T}\int_{B_{ts}^{(1)}}R_{t_{0}s}d\mathbb{P} = \int_{\bigcup_{t=t_{0}}^{T}B_{ts}^{(1)}}R_{t_{0}s}d\mathbb{P}.$$

The claim follows by taking a summation over $s_0 \leq s \leq S$ on both sides.

Lemma 27 leads to the following proof of Proposition 3, which generalizes Corollary 2.10 of Christofides and Serfling (1990b).

Proof of Proposition 3. Let $T \ge t_0, S \ge s_0$. By inequality (42),

$$u^{\alpha} \mathbb{P} \left\{ \sup_{\substack{t_0 \leq t \leq T \\ s_0 \leq s \leq S}} R_{ts} \geq u \right\} \leq \sum_{s=s_0}^{S} \int_{\bigcup_{t=t_0}^{T} B_{ts}^{(1)}} R_{t_0s}^{\alpha} d\mathbb{P}$$

$$\leq \sum_{s=s_0}^{S} \int_{\bigcup_{t=t_0}^{T} B_{ts}^{(1)}} \left(\max_{s_0 \leq s \leq S} R_{t_0s}^{\alpha} \right) d\mathbb{P}$$

$$= \int_{A} \left(\max_{s_0 \leq s \leq S} R_{t_0s}^{\alpha} \right) d\mathbb{P}$$

$$\leq \mathbb{E} \left(\max_{s_0 \leq s \leq S} R_{t_0s}^{\alpha} \right)$$

$$\leq \left(\frac{\alpha}{\alpha - 1} \right)^{\alpha} \mathbb{E}[R_{t_0s_0}^{\alpha}],$$

where the last inequality follows from Doob (1953), Theorem 3.4, page 317. Taking $T, S \to \infty$ on both sides of the above display leads to the claim.

C Proofs from Section 3

C.1 Proofs from Subsection 3.1

Proof of Proposition 6. To prove part (i), Theorem 4 implies that (N_t) forms a reverse sub-

martingale with respect to (\mathcal{E}_t^X) . Furthermore, the map $x \in \mathbb{R} \mapsto \exp\{\lambda x\}$ is convex and monotonic for any fixed $\lambda \in [0, \lambda_{\max})$, so Jensen's inequality implies that the process

$$L_t(\lambda) = \exp(\lambda N_t), \quad t \ge 1,$$

is also a reverse submartingale with respect to (\mathcal{E}_t^X) . By Theorem 2, we obtain for all u>0,

$$\mathbb{P}\Big(\exists t \ge t_0 : N_t \ge u\Big) \le \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{P}\Big(\exists t \ge t_0 : L_t(\lambda) \ge e^{\lambda u}\Big) \le \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{E}\left[\exp(-\lambda u)L_{t_0}(\lambda)\right] \\
\le \inf_{\lambda \in [0, \lambda_{\max})} \exp\left\{-\lambda u + \psi_{t_0}(\lambda)\right\} = \exp\left\{-\psi_{t_0}^*(u)\right\}, \tag{43}$$

as claimed. To prove Proposition 6(ii), recall that $\mathcal{E}_{ts} = \mathcal{E}_t^X \bigvee \mathcal{E}_s^Y$ is a σ -algebra generated by a union of independent σ -algebras. It follows that the filtration (\mathcal{E}_{ts}) satisfies the conditional independence property (14), by Cairoli and Walsh (1975), example (a), page 114 (see also Christofides and Serfling (1990a)). Furthermore, similarly as in part (i), the process

$$L_{ts}(\lambda, \alpha) = \exp(\lambda M_{ts}/\alpha), \quad t, s \ge 1,$$

is a partially ordered reverse submartingale for any fixed choice of $\lambda \in [0, \lambda_{\text{max}})$ and $\alpha > 1$. Notice also that $L_{ts}(\lambda, \alpha) \in L^{\alpha}(\mathbb{P})$, thus we may apply Proposition 3 to obtain

$$\mathbb{P}(\exists t \geq t_0, s \geq s_0 : M_{ts} \geq u) = \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{P}\left(\exists t \geq t_0, s \geq s_0 : L_{ts}(\lambda, \alpha) \geq \exp(\lambda u / \alpha)\right) \\
\leq \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \inf_{\lambda \in [0, \lambda_{\max})} \exp(-\lambda u) \mathbb{E}\left[L_{t_0 s_0}^{\alpha}(\lambda, \alpha)\right] \\
= \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \inf_{\lambda \in [0, \lambda_{\max})} \exp(-\lambda u) \mathbb{E}\left[\exp(\lambda M_{t_0 s_0})\right] \\
\leq \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \exp(-\psi_{t_0 s_0}^*(u)).$$

Taking the infimum over $\alpha > 1$ on both sides of the above display leads to the claim.

We now turn to proving a more general version of Theorem 7. Let $\eta, \xi > 1$ be fixed constants which determine the sizes of the geometric epochs used in the proofs. Furthermore, given $t, s \geq 1$, we use the shorthand notation $\bar{t} = \lceil t/\lceil \eta \rceil \rceil$ and $\bar{s} = \lceil s/\lceil \xi \rceil \rceil$.

Theorem 28. Let Φ, Ψ be convex functionals, and let $\delta \in (0,1)$.

(i) (One-Sample) Assume ψ_t^* is invertible for all $t \geq 1$, and if η is not an integer, assume $(\psi_t^*)^{-1}(\lambda)$ is a decreasing (resp. increasing) sequence in t (resp. λ). Set

$$\gamma_t = (\psi_{\bar{t}}^*)^{-1} \Big(\log \ell(\log_{\eta} t) + \log(1/\delta) \Big).$$

Assume further that (γ_t) is a nonincreasing sequence. Then,

$$\mathbb{P}\big\{\exists t \ge 1 : \Phi(P_t) \ge \Phi(P) + \gamma_t\big\} \le \delta.$$

(ii) (Two-Sample) Assume ψ_{ts}^* is invertible for all $t, s \geq 1$, and if η (resp ξ) is not an integer, assume $\psi_{ts}^*(\lambda)$ is decreasing in t (resp. in s), and increasing in λ . Set

$$\gamma_{ts} = (\psi_{\bar{t}\bar{s}}^*)^{-1} \Big(\log g(\log_{\eta} t + \log_{\xi} s) + \log(1/\delta) \Big).$$

Assume further that (γ_{ts}) is a nonincreasing sequence in each of its indices. Then,

$$\mathbb{P}\left\{\exists t, s \ge 1 : \Psi(P_t, Q_s) \ge \Psi(P, Q) + \gamma_{ts}\right\} \le \delta.$$

Proof of Theorems 7 and 28. The proofs of claims (i) and (ii) of Theorem 28 are similar, thus we only prove (ii). Theorem 7 will then follow by setting $\eta = \xi = 2$. Let $u_j = \lceil \eta^j \rceil$ and $v_k = \lceil \xi^k \rceil$ for all $j, k \in \mathbb{N}_0$. Since γ_{ts} is decreasing in t and s, we have

$$\mathbb{P}\left(\exists t, s \geq 1 : M_{ts} \geq \gamma_{ts}\right) \\
\leq \mathbb{P}\left(\bigcup_{j \in \mathbb{N}_0} \bigcup_{k \in \mathbb{N}_0} \left\{ \exists t \in \{u_j, \dots, u_{j+1}\}, s \in \{v_k, \dots, v_{k+1}\} : M_{ts} \geq \gamma_{ts} \right\} \right) \\
\leq \mathbb{P}\left(\bigcup_{j \in \mathbb{N}_0} \bigcup_{k \in \mathbb{N}_0} \left\{ \exists t \in \{u_j, \dots, u_{j+1}\}, s \in \{v_k, \dots, v_{k+1}\} : M_{ts} \geq \gamma_{u_{j+1}v_{k+1}} \right\} \right).$$

Now, $u_{j+1} \leq u_j \lceil \eta \rceil$ (resp. $v_{k+1} \leq v_k \lceil \xi \rceil$), with equality if η (resp. ξ) is an integer. Therefore, by definition of \bar{t}, \bar{s} , and by the fact that $(\psi_{ts}^*)^{-1}$ is decreasing in t (resp. s) when η (resp. ξ) is not an integer, we have

$$\gamma_{u_{j+1}v_{k+1}} \ge (\psi_{u_jv_k}^*)^{-1} \Big(\log g \Big(\log_{\eta}(u_{j+1}) + \log_{\xi}(v_{k+1})\Big) + \log(1/\delta)\Big).$$

Since $(\psi_{ts}^*)^{-1}(\lambda)$ is increasing in λ when η or ξ are not integers, we deduce

$$\gamma_{u_{j+1}v_{k+1}} \ge (\psi_{u_jv_k}^*)^{-1} \Big(\log g(j+k+2) + \log(1/\delta)\Big).$$

Applying a union bound together with Proposition 6 then leads to

$$\mathbb{P}(\exists t, s \ge 1 : M_{ts} \ge \gamma_{ts}) \le e \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \exp\left\{-\psi_{u_{j}v_{k}}^{*} \left((\psi_{u_{j}v_{k}}^{*})^{-1} \left(\log g(j+k+2) + \log(1/\delta)\right)\right)\right\} \\
\le e \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \exp\left\{-\left(\log g(j+k+2) + \log(1/\delta)\right)\right\} \\
= \delta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{e}{g(j+k+2)} \le \delta,$$

as claimed. \Box

Proof of Corollary 8. We only prove the claim for $(M_{ts})_{t,s=1}^{\infty}$, as the proof for $(N_t)_{t=1}^{\infty}$ is similar. Since M_{ts} is σ_{ts}^2 -sub-Gaussian, we have for all $\lambda \in \mathbb{R}_+$,

$$\mathbb{E}\left\{\exp\left(\lambda M_{ts}\right)\right\} = \mathbb{E}\left\{\exp\left[\lambda(M_{ts} - \mathbb{E}(M_{ts}))\right]\right\} \exp\left\{\lambda\mathbb{E}(M_{ts})\right\} \le \exp\left\{\frac{\lambda^2 \sigma_{ts}^2}{2}\right\} \exp\left\{\lambda\mathbb{E}(M_{ts})\right\},$$

whence, an upper bound on the CGF of M_{ts} is given by $\psi_{ts}(\lambda) = \frac{\lambda^2 \sigma_{ts}^2}{2} + \lambda \mathbb{E}(M_{ts})$. Thus, for any $x \geq \mathbb{E}(M_{ts})$ and any $\gamma \in \mathbb{R}_+$,

$$\psi_{ts}^*(x) = \frac{(x - \mathbb{E}(M_{ts}))^2 \sigma_{ts}^{-2}}{2}, \quad (\psi_{ts}^*)^{-1}(\gamma) = \mathbb{E}(M_{ts}) + \sqrt{2\gamma \sigma_{ts}^2}.$$

The claim now follows from Theorem 7.

C.2 Proofs from Subsection 3.2

We state and prove the following stronger version of Corollary 11.

Proposition 29. Assume that the processes (N_t) and (M_{ts}) satisfy the conditions of Proposition 9, and fix $\delta \in (0,1)$. Assume further that there exists $\lambda_{\max} > 0$ and convex functions $\psi_{\Phi}, \psi_{\Psi}, \phi_{\Psi}$ such that for all $\lambda \in [0, \lambda_{\max})$,

$$\sup_{f \in \mathcal{F}_{\Phi}} \log \left\{ \mathbb{E} \left[e^{\lambda (f(X) - \mathbb{E}f(X))} \right] \right\} \le \psi_{\Phi}(\lambda),$$

$$\sup_{f \in \mathcal{F}_{\Psi}} \log \left\{ \mathbb{E} \left[e^{\lambda(f(X) - \mathbb{E}f(X))} \right] \right\} \leq \psi_{\Psi}(\lambda), \quad \sup_{g \in \mathcal{G}_{\Psi}} \log \left\{ \mathbb{E} \left[e^{\lambda(g(Y) - \mathbb{E}g(Y))} \right] \right\} \leq \phi_{\Psi}(\lambda).$$

Assume further that the Legendre-Fenchel transforms $\psi_{\Phi}^*, \psi_{\Psi}^*, \phi_{\Psi}^*$ are invertible, with nondecreasing inverses.

(i) (One-Sample) We have

$$\mathbb{P}\left\{\exists t \geq 1: \Phi(P_t) \leq \Phi(P) - (\psi_{\Phi}^*)^{-1} \left(\frac{2}{t} \left[\log \ell(\log_2 t) + \log(2/\delta)\right]\right)\right\} \leq \delta/2.$$

(ii) (Two-Sample) Define

$$\kappa_t^X = (\psi_{\Psi}^*)^{-1} \left(\frac{2}{t} \left[\log \ell(\log_2 t) + \log \left(\frac{4}{\delta} \right) \right] \right), \quad \kappa_s^Y = (\phi_{\Psi}^*)^{-1} \left(\frac{2}{s} \left[\log \ell(\log_2 s) + \log \left(\frac{4}{\delta} \right) \right] \right).$$

Then, we have

$$\mathbb{P}(\exists t, s \ge 1 : \Psi(P_t, Q_s) \le \Psi(P, Q) - \kappa_t^X - \kappa_s^Y) \le \delta/2. \tag{44}$$

When it exists, the cumulant generating function of any mean-zero random variable Z scales quadratically near zero—specifically, it is easy to check that $\lim_{\lambda\to 0} \log(\mathbb{E}\exp(\lambda Z))/(\lambda^2/2) = \operatorname{Var}(Z)$. Thus, the inverse of its Legendre-Fenchel transform typically scales as the square root function near zero. The upper confidence sequences in Proposition 29 thus typically scale at the parametric rate up to a necessary iterated logarithmic factor.

Proof of Proposition 29. To prove part (i), define for this proof only,

$$\eta_t = (\psi_{\Phi}^*)^{-1} \left(\frac{2}{t} \left[\log \ell(\log_2 t) + \log(2/\delta) \right] \right).$$

Furthermore, let $u_j = 2^j$ for all $j \in \mathbb{N}_0$. By Proposition 9, (R_t) minorizes (N_t) thus

$$\mathbb{P}(\exists t \geq 1 : \Phi(P_t) \leq \Phi(P) - \eta_t) \leq \mathbb{P}\left(\exists t \geq 1 : -R_t \geq \eta_t\right)
\leq \mathbb{P}\left(\bigcup_{j \in \mathbb{N}_0} \left\{\exists t \in \{u_j, \dots, u_{j+1}\} : -R_t \geq \eta_t\right\}\right)
\leq \mathbb{P}\left(\bigcup_{j \in \mathbb{N}_0} \left\{\exists t \in \{u_j, \dots, u_{j+1}\} : -R_t \geq (\psi_{\Phi}^*)^{-1} \left(\frac{2}{u_{j+1}} \left[\log \ell(j+1) + \log(2/\delta)\right]\right)\right\}\right), \quad (45)$$

where on the last line, we used the fact that $(\psi_{\Phi}^*)^{-1}$ is nondecreasing. Now, (R_t) is a reverse martingale with respect to (\mathcal{E}_t^X) , whence $(\exp(-\lambda R_t))_{t=1}^{\infty}$ is a reverse submartingale for any fixed $\lambda \in [0, t_0 \lambda_{\max})$. Applying Theorem 2 similarly as in the proof of Proposition 6, we therefore obtain for all u > 0 and $t_0 \ge 1$,

$$\mathbb{P}\Big(\exists t \geq t_0 : -R_t \geq u\Big) = \inf_{\lambda \in [0, t_0 \lambda_{\max})} \mathbb{P}\Big(\exists t \geq t_0 : \exp(-\lambda R_t) \geq e^{\lambda u}\Big)
\leq \inf_{\lambda \in [0, t_0 \lambda_{\max})} \exp(-\lambda u) \mathbb{E}\left[\exp(-\lambda R_{t_0})\right]
\leq \inf_{\lambda \in [0, t_0 \lambda_{\max})} \exp\left\{-\lambda u + t_0 \psi_{\Phi}(\lambda/t_0)\right\}
\leq \inf_{\lambda \in [0, t_0 \lambda_{\max})} \exp\left\{-t_0[(\lambda/t_0)u - \psi_{\Phi}(\lambda/t_0)]\right\}
= \inf_{\lambda \in [0, \lambda_{\max})} \exp\left\{-t_0[\lambda u - \psi_{\Phi}(\lambda)]\right\} = \exp\left\{-t_0 \psi_{\Phi}^*(u)\right\}.$$

Returning to equation (45), we deduce

$$\mathbb{P}(\exists t \ge 1 : \Phi(P_t) \le \Phi(P) - \eta_t) \le \sum_{j=0}^{\infty} \exp\left\{-\frac{2u_j}{u_{j+1}} \left[\log \ell(j+1) + \log(2/\delta)\right]\right\}$$
$$\le \sum_{j=0}^{\infty} \exp\left\{-\left[\log \ell(j+1) + \log(2/\delta)\right]\right\} = \sum_{j=0}^{\infty} \frac{\delta/2}{\ell(j+1)} \le \frac{\delta}{2}.$$

The proof of claim (i) follows. The proof of part (ii) follows by a similar probability bound for each of $-R_t^X$ and $-R_s^Y$ at level $\delta/4$, combined with a union bound.

Proof of Corollary 11. By Hoeffding's Lemma, we may take $\psi_{\Phi}(\lambda) = \lambda^2 B^2/8$ for all $\lambda \in \mathbb{R}_+$, thus, $(\psi_{\Phi}^*)^{-1}(\lambda) = \sqrt{B^2 \lambda/2}$, and similarly for the two-sample case. The claim follows directly from Proposition 29.

C.3 Proofs from Subsection 3.3

Proof of Proposition 12. Assume first that $\mathbb{P}(\bigcup_{t,s=1}^{\infty} A_{ts}) \leq \delta$. Let $\eta = (T,S)$ be any random

time. Then

$$A_{TS} = \left(\bigcup_{t=1}^{\infty} \bigcup_{s=1}^{\infty} A_{ts} \cap \{\eta = (t, s)\}\right)$$

$$\cup \left(\bigcup_{t=1}^{\infty} A_{t\infty} \cap \{\eta = (t, \infty)\}\right) \cup \left(\bigcup_{s=1}^{\infty} A_{\infty s} \cap \{\eta = (\infty, s)\}\right) \cup (A_{\infty \infty} \cap \{\eta = (\infty, \infty)\}).$$

Since $A_{\infty\infty}$, $A_{t\infty}$, $A_{\infty s} \subseteq \bigcup_{t,s=1}^{\infty} A_{ts}$, we deduce that $A_{TS} \subseteq \bigcup_{t,s=1}^{\infty} A_{ts}$, implying that $\mathbb{P}(A_{TS}) \leq \delta$. Thus (i) implies (iii). It is also clear that (iii) implies (ii), thus it remains to prove that (ii) implies (i). To this end, assume $\mathbb{P}(A_{\tau\sigma}) \leq \delta$ for any stopping time (τ, σ) . For any $\omega \in \Omega$, let

$$I(\omega) = \left\{ (t, s) \in \mathbb{N}^2 : \omega \in A_{ts} \text{ and } \omega \notin A_{t's'}, \forall (t', s') \in \mathbb{N}^2, (t', s') < (t, s) \right\}.$$

We may then define

$$(\tau(\omega), \sigma(\omega)) = \begin{cases} (\infty, \infty), & I(\omega) = \emptyset \\ \underset{(t,s) \in I(\omega)}{\operatorname{argmin}} t, & |I(\omega)| \ge 1 \end{cases}.$$

The minimizer in the above display is unique and unambiguous, because when $I(\omega)$ has cardinality greater or equal to 2, any of its distinct elements (t,s) and (t',s') must have $t \neq t'$ and $s \neq s'$ by construction; for example, (t,s) and (t,s') cannot both be elements of $I(\omega)$ for $s \neq s'$. Notice that (τ,σ) is a stopping time with respect to (\mathcal{F}_{ts}) because $A_{t's'} \in \mathcal{F}_{ts}$ for all $(t',s') \leq (t,s)$. Furthermore, its definition guarantees $\bigcup_{t,s=1}^{\infty} A_{ts} \subseteq A_{\tau\sigma}$. We deduce by assumption that $\mathbb{P}(\bigcup_{t,s=1}^{\infty} A_{ts}) \leq \delta$, as claimed.

We note that our precise definitions of the events $A_{t\infty}$ and $A_{\infty s}$, for $t, s \ge 1$, was not crucial in the preceding argument.

D Proofs from Section 4

D.1 Proofs from Subsection 4.1

Proof of Corollary 13. By the DKW inequality, for all u > 0, we have

$$\mathbb{P}\Big(\|F_t - F\|_{\infty} \ge u\Big) \le 2e^{-2tu^2}.$$

Therefore,

$$\mathbb{E}[\|F_t - F\|_{\infty}] = \int_0^{\infty} \mathbb{P}(\|F_t - F\|_{\infty} \ge u) du \le 2 \int_0^{\infty} e^{-2tu^2} du = \sqrt{\frac{\pi}{2t}}.$$

Furthermore, it is a straightforward observation that the map

$$(x_1, \dots, x_t) \in \mathbb{R}^t \mapsto \sup_{x \in \mathbb{R}} \left| \frac{1}{t} \sum_{i=1}^t I(x_i \le x) - F(x) \right|$$

satisfies the bounded differences property with parameters $L_1 = \cdots = L_t = 1/t$. McDiarmid's inequality (Theorem 25) therefore implies the bound

$$\mathbb{P}(|\|F_t - F\|_{\infty} - \mathbb{E}\|F_t - F\|_{\infty}| \ge u) \le 2e^{-2tu^2}, \quad u > 0.$$

Lemma 26 then implies that $||F_t - F||_{\infty}$ is a (2/t)-sub-Gaussian random variable.

Now, notice that $D_{\mathcal{J}}(P_t||P) = ||F_t - F||_{\infty}$ is an IPM over the class $\mathcal{J} = \{(-\infty, x] : x \in \mathbb{R}\}$, and in particular $D_{\mathcal{J}}$ is convex by Lemma 1. We may therefore apply Corollary 8 to the process $N_t = D_{\mathcal{J}}(P_t||P)$, $t \geq 1$, to obtain the claim.

Proof of Corollary 14. By the triangle inequality,

$$M_{ts} := \|F_t - G_s\|_{\infty} - \|F - G\|_{\infty} \le \|G_s - G\|_{\infty} + \|F_t - F\|_{\infty},$$

so that $\mathbb{E}M_{ts} \leq \sqrt{\pi/2t} + \sqrt{\pi/2s}$, by the same argument as in the proof of Corollary 13. Furthermore, the map

$$(x_1,\ldots,x_t,y_1,\ldots,y_s) \mapsto \sup_{x\in\mathbb{R}} \left| \frac{1}{t} \sum_{i=1}^t I(x_i \leq x) - \frac{1}{s} \sum_{i=1}^s I(y_i \leq x) \right|,$$

satisfies the bounded differences property with parameters $L_1 = \cdots = L_t = 1/t$ and $L_{t+1} = \cdots = L_{t+s} = 1/s$. Therefore, McDiarmid's inequality implies

$$\mathbb{P}(|M_{ts} - \mathbb{E}M_{ts}| \ge u) \le 2\exp(-2tsu^2/(s+t)), \quad u > 0.$$

It now follows similarly as in the proof of Corollary 13 that M_{ts} is sub-Gaussian with parameter 2(t+s)/ts. Applying Corollary 8 leads to the bound $\mathbb{P}(\exists t, s \geq 1 : M_{ts} \geq \gamma_{ts}) \leq \delta/2$. Furthermore, from Corollary 11, $\mathbb{P}(\exists t, s \geq 1 : M_{ts} \leq -\kappa_{ts}) \leq \delta/2$. Applying a union bound leads to the claim.

To prove the validity of the test, notice simply that under the null F = G, the aforementioned bound reduces to $\mathbb{P}(\exists t, s \geq 1 : ||F_t - G_s||_{\infty} \geq \gamma_{ts}) \leq \delta/2$.

D.2 Proofs from Subsection 4.2

Proof of Corollary 15 The proof of Theorem 7 (equation (16)) of Gretton et al. (2012) yields the expectation bound

$$\mathbb{E} \big| \text{MMD}(P_t, Q_s) - \text{MMD}(P, Q) \big| \le 2 \Big[(B/t)^{1/2} + (B/s)^{1/2} \Big].$$

Further, the following concentration bound follows from equation (15) of Gretton et al. (2012):

$$\mathbb{P}(\left| \text{MMD}(P_t, Q_s) - \mathbb{E}[\text{MMD}(P, Q)] \right| \ge u) \le 2 \exp\left\{ -\frac{tsu^2}{2B(t+s)} \right\}, \quad u > 0.$$

Therefore, Lemma 26 implies that $MMD(P_t, Q_s)$ is $\frac{8B(t+s)}{ts}$ -sub-Gaussian. Finally, MMD is an IPM by its definition, and is therefore convex by Lemma 1. Combining these facts with Corollary 8, applied to the process $M_{ts} = MMD(P_t, Q_s) - MMD(P, Q)$, leads to the bound

$$\mathbb{P}(\exists t, s \geq 1 : \text{MMD}(P_t, Q_s) - \text{MMD}(P, Q) \geq \gamma_{ts}) \leq \delta/2.$$

To obtain an upper confidence sequence, notice that the set $\mathcal{J} = \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq 1\}$ consists of functions taking values in the interval $[-\sqrt{B}, \sqrt{B}]$. Indeed, for all $f \in \mathcal{J}$, $x \in \mathbb{R}^d$,

$$|f(x)| = |\langle f, K(x, \cdot) \rangle_{\mathcal{H}}| \le ||f||_{\mathcal{H}} ||K(x, \cdot)||_{\mathcal{H}} \le \sqrt{K(x, x)} \le \sqrt{B}.$$

Applying Corollary 11, we thus have

$$\mathbb{P}(\exists t, s \geq 1 : \text{MMD}(P_t, Q_s) - \text{MMD}(P, Q) \leq -2\sqrt{B}\kappa_{ts}) \leq \delta/2.$$

Applying a union bound leads to the claim.

Thus far we have studied the plug-in estimator $\text{MMD}^2(P_t, Q_s)$, which is known to be a biased estimator of $\text{MMD}^2(P,Q)$. The following unbiased estimator, which we state only in the case t=s, is widely-used and is obtained by replacing the V-statistic in (33) by the following U-Statistic

$$\widehat{M}_t^2 = \frac{1}{t(t-1)} \sum_{i \neq j} \widetilde{J}(Z_i, Z_j), \tag{46}$$

where $\widetilde{J}((x,y),(x',y')) = K(x,x') + K(y,y') - K(x,y') - K(x',y)$. The process \widehat{M}_t does not admit a simple characterization as a convex functional of the empirical measures P_t and Q_t , thus Theorem 7 and Proposition 29 cannot be directly applied. U-Statistics are, however, known to be reverse martingales, as discussed in Section 4.2, implying that \widehat{M}_t^2 is a reverse martingale. While this does not imply that $\widehat{M}_t - \text{MMD}(P,Q)$ is a reverse submartingale, Theorem 7 can be applied directly to the mean-zero process $\widehat{M}_t^2 - \text{MMD}^2(P,Q)$.

Proposition 30. Under the same conditions as Corollary 15, we have for all $\delta \in (0,1)$,

$$\mathbb{P}\left(\exists t \geq 1 : \widehat{M}_t^2 \geq \text{MMD}^2(P, Q) + 16B\sqrt{\frac{1}{t-1}\Big[\log\ell(\log_2 t) + \log(1/\delta)\Big]}\right) \leq \delta.$$

Unlike equation (34), Proposition 30 does not lead to a confidence sequence for $\text{MMD}^2(P, Q)$ scaling at the rate $O(\log \log t/t)$ when P = Q. We therefore recommend the use of the plug-in estimator $\text{MMD}(P_t, Q_s)$ and Corollary 15 when a confidence sequence is needed in practice.

Proof of Proposition 30. By Theorem 10 of Gretton et al. (2012) and Lemma 26, one can infer similarly as in the proof of Corollary 15 that \widehat{M}_t^2 is $(32B^2/\underline{t})$ -sub-Gaussian, where $\underline{t} = \lfloor t/2 \rfloor$. The claim then follows by the same proof technique as Theorem 7 and Corollary 8, using the fact that \widehat{M}_t^2 is a reverse martingale, and using the inequality $\lceil \lfloor t/2 \rfloor / 2 \rceil \ge (t-1)/4$.

We close this section with a proof of Proposition 16.

Proof of Proposition 16. Let $\nu \in \mathcal{P}(\mathcal{X})$ be any fixed reference measure. By Mercer's Theorem (see for instance Christmann and Steinwart (2008), Theorem 4.49), a continuous, symmetric, and positive definite kernel h admits the representation

$$h(x,y) = \sum_{i=0}^{\infty} \lambda_i \psi_i(x) \psi_i(y), \quad x, y \in \mathcal{X},$$

where $(\lambda_i)_{i\geq 0} \subseteq \ell^2 \equiv \ell^2(\mathbb{N}_0)$ is the sequence of eigenvalues corresponding to the Hilbert-Schmidt operator $f \in L^2(\nu) \mapsto \int h(\cdot,y)f(y)d\nu(y)$, and $(\psi_i)_{i\geq 0} \subseteq L^2(\nu)$ is a corresponding sequence of eigenfunctions. It follows that one may write for any $\mu \in \mathcal{P}(\mathcal{X})$,

$$\sqrt{\Phi(\mu)} = \sqrt{\iint h(x,y) d\mu(x) d\mu(y)} = \sqrt{\sum_{i=0}^{\infty} \lambda_i \left(\int \psi_i d\mu \right)^2} = \left\| \left(\sqrt{\lambda_i} \int \psi_i d\mu \right)_{i \geq 0} \right\|_{\ell^2},$$

The right-hand side of the above display is a composition of the convex map $\|\cdot\|_{\ell^2}$ with the affine map $\mu \in V \mapsto \left(\sqrt{\lambda_i} \int \psi_i d\mu\right)_{i\geq 0} \in \ell^2$, where V is the vector space of finite signed measures on $(\mathcal{X}, \mathbb{B}(\mathcal{X}))$. It follows that the functional $\sqrt{\Phi(\cdot)}$ is itself convex. Since this functional is also nonnegative, and the square function is convex and increasing on \mathbb{R}_+ , it is straightforward to verify that the functional $\Phi(\cdot)$ is likewise convex. The claim then follows from Theorem 4.

D.3 Proofs from Subsection 4.3

Proof of Corollary 17. We will make use of the Kantorovich duality (cf. Section 2.1) to show that the map

$$G: (x_1, \dots, x_t, y_1, \dots, y_s) \in \mathcal{X}^{t+s} \mapsto \mathcal{T}_c \left(\frac{1}{t} \sum_{i=1}^t \delta_{x_i}, \frac{1}{s} \sum_{j=1}^s \delta_{y_j} \right) = \sup_{(f,g) \in \mathcal{M}_c} \frac{1}{t} \sum_{i=1}^t f(x_i) + \frac{1}{s} \sum_{j=1}^s g(y_j),$$

satisfies the bounded differences property. This generalizes the one-sample analogue proven for instance by Weed and Bach (2019) (Proposition 20). Let $1 \le k \le t$, and $x_1, \tilde{x}_1, \ldots, x_t, \tilde{x}_t \in \mathcal{X}$ be such that $\tilde{x}_i = x_i$ for all $i \ne k$. Furthermore, let $y_1, \ldots, y_s \in \mathcal{X}$. Let $(f_0, g_0) \in \mathcal{M}_c$ denote optimal Kantorovich potentials satisfying

$$\mathcal{T}_c\left(\frac{1}{t}\sum_{i=1}^t \delta_{x_i}, \frac{1}{s}\sum_{j=1}^s \delta_{y_j}\right) = \frac{1}{t}\sum_{i=1}^t f_0(x_i) + \frac{1}{s}\sum_{j=1}^s g_0(y_j).$$

Furthermore, recall from Section 2.1 that we may (and do) choose (f_0, g_0) such that $0 \le f_0 \le \Delta$ and $-\Delta \le g_0 \le 0$. Then,

$$G(x_1, \dots, x_t, y_1, \dots, y_s) - G(\widetilde{x}_1, \dots, \widetilde{x}_t, y_1, \dots, y_s)$$

$$\leq \frac{1}{t} \sum_{i=1}^t f_0(x_i) + \frac{1}{s} \sum_{j=1}^s g_0(y_j) - \frac{1}{t} \sum_{i=1}^t f_0(\widetilde{x}_i) - \frac{1}{s} \sum_{j=1}^s g_0(y_j) \leq \frac{1}{t} [f_0(x_k) - f_0(\widetilde{x}_k)] \leq \Delta/t.$$

Repeating a symmetric argument, we obtain

$$|G(x_1,\ldots,x_t,y_1,\ldots,y_s)-G(\widetilde{x}_1,\ldots,\widetilde{x}_t,y_1,\ldots,y_s)| \leq \Delta/t.$$

We similarly have that for all $\widetilde{y}_1, \ldots, \widetilde{y}_s \in \mathcal{X}$, satisfying $\widetilde{y}_i = y_i$ for all $i \neq k$,

$$|G(x_1,\ldots,x_t,y_1,\ldots,y_s)-G(x_1,\ldots,x_t,\widetilde{y}_1,\ldots,\widetilde{y}_s)| < \Delta/s.$$

We deduce that G satisfies the bounded differences property with parameters $L_1 = \cdots = L_t = \Delta/t$ and $L_{t+1} = \ldots L_{t+s} = \Delta/s$. McDiarmid's inequality then implies

$$\mathbb{P}\big(|\mathcal{T}_c(P_t,Q_s) - \mathbb{E}\mathcal{T}_c(P_t,Q_s)| \ge u\big) \le 2\exp\left\{-\frac{2tsu^2}{(t+s)\Delta^2}\right\}, \quad u > 0.$$

It follows that $\mathcal{T}_c(P_t, Q_s)$ is $\frac{2\Delta^2(t+s)}{ts}$ -sub-Gaussian by Lemma 26. Since \mathcal{T}_c is convex, applying Corollary 8 yields

$$\mathbb{P}\left(\exists t, s \ge 1 : \mathcal{T}_c(P_t, Q_s) - \mathcal{T}_c(P, Q) \ge \alpha_{c, ts} + 2\Delta \sqrt{\frac{2(t+s)}{ts} \left[\log g(\log_2 t + \log_2 s) + \log(2/\delta)\right]}\right) \le \delta/2.$$

Furthermore, Corollary 11 and the Kantorovich duality immediately lead to the bound

$$\mathbb{P}(\exists t, s \ge 1 : \mathcal{T}_c(P_t, Q_s) - \mathcal{T}_c(P, Q) \le -\Delta \kappa_{ts}) \le \delta/2.$$

The claim follows. \Box

D.4 Proofs from Subsection 4.4

Proof of Proposition 18. We repeat a similar stitching argument as that of the proof of Theorem 7. Let $N_t = \mathrm{KL}(P_t || P)$, $t \geq 1$. (N_t) forms a reverse submartingale by Lemma 1 and Theorem 4, implying by Jensen's inequality that for any integer $t_1 \geq 1$, $\left(\exp(t_1 \lambda_{t_1} N_t)\right)_{t=1}^{\infty}$ is a reverse submartingale. Therefore, following along similar lines as the proof of Theorem 7, and applying Theorem 2, we have for all y > 0 and all integers $t_0 \geq 1$,

$$\mathbb{P}\Big(\exists t \ge t_0 : N_t \ge y\Big) = \mathbb{P}\Big(\exists t \ge t_0 : \exp(\lambda_{t_1} t_1 N_t) \ge \exp(\lambda_{t_1} t_1 y)\Big)$$

$$\le \mathbb{E}[\exp(-yt_1\lambda_{t_1} + t_1\lambda_{t_1} N_{t_0})] \le \exp(-yt_1\lambda_{t_1})G_{k,t_0}(t_1\lambda_{t_1}/t_0).$$

Now, letting $u_j = 2^j$ for all integers $j \ge 0$, and $\gamma_t = \frac{1}{\lambda_t t} \log \left(\delta^{-1} G_{k,|t/2|}(2\lambda_t) \ell(\log_2 t) \right)$, we obtain

$$\mathbb{P}\left(\exists t \geq 1 : N_t \geq \gamma_t\right) \leq \mathbb{P}\left(\bigcup_{j=0}^{\infty} \left\{\exists t \in \{u_j, \dots, u_{j+1}\} : N_t \geq \gamma_t\right\}\right)$$

$$\leq \mathbb{P}\left(\bigcup_{j=0}^{\infty} \left\{\exists t \in \{u_j, \dots, u_{j+1}\} : N_t \geq \gamma_{u_{j+1}}\right\}\right)$$

$$\leq \sum_{j=0}^{\infty} \exp\left\{-u_{j+1}\gamma_{u_{j+1}}\lambda_{u_{j+1}}\right\} G_{k,u_j}(2\lambda_{u_{j+1}}) \leq \sum_{j=0}^{\infty} \frac{\delta}{\ell(j+1)} \leq \delta,$$

where on the final line, we used the fact that $G_{k,t}(\lambda)$ increases with t for all fixed $k \geq 2, \lambda \in [0,1]$ (Guo and Richardson (2020), Lemma 1). The claim follows.

Proof of Corollary 19. By Berend and Kontorovich (2013), Eq. (17), and references therein, we have

$$\mathbb{P}\left(\|P_t - P\|_{\text{TV}} - \mathbb{E}[\|P_t - P\|_{\text{TV}}] \ge u\right) \le 2\exp(-2tu^2), \quad u > 0,$$

implying that $||P_t - P||_{\text{TV}}$ is (2/t)-sub-Gaussian. Furthermore, Lemma 7 of Kamath et al. (2015) implies

$$\mathbb{E}[\|P_t - P\|_{\text{TV}}] \le \sqrt{\frac{k}{2\pi t}} + 2\left(\frac{k}{t}\right)^{\frac{3}{4}}.$$

Finally, $\|\cdot\|_{TV}$ forms a convex divergence by Lemma 1. The claim now follows by Corollary 8.

D.5 Proofs from Subsection 4.5

We shall make use of the following result due to Bobkov and Götze (1999).

Lemma 31 (Bobkov and Götze (1999), Theorem 1.3). Let d denote a metric on \mathcal{X} . Then, a measure $\mu \in \mathcal{P}_1(\mathcal{X})$ satisfies the $T_1(\sigma^2)$ inequality with respect to d if and only if f(X) is σ^2 -sub-Gaussian for all functions $f: \mathcal{X} \to \mathbb{R}$ which are 1-Lipschitz with respect to d.

Proof of Proposition 20. To prove part (i), it is straightforward to show that the map

$$G: (x_1, \dots, x_t) \in \mathbb{R}^{t \times d} \longmapsto \left\| \frac{1}{t} \sum_{i=1}^t (\delta_{x_i} - P) \right\|_{\text{TV}}^{\sigma},$$

satisfies the bounded differences property. Indeed, given $1 \leq j \leq t$, let $x_1, \widetilde{x}_1, \ldots, x_t, \widetilde{x}_t \in \mathbb{R}^d$, such that $x_i = \widetilde{x}_i$ for all $i \neq j$. Then, the triangle inequality implies

$$|G(x_1, \dots, x_t) - G(\widetilde{x}_1, \dots, \widetilde{x}_t)| \leq \sup_{A \in \mathbb{B}(\mathbb{R}^d)} \left| \left(\frac{1}{t} \sum_{i=1}^t \delta_{x_i} \star \mathcal{K}_{\sigma} \right) (A) - \left(\frac{1}{t} \sum_{i=1}^t \delta_{\widetilde{x}_i} \star \mathcal{K}_{\sigma} \right) (A) \right|$$

$$\leq \frac{1}{t} \sup_{A \in \mathbb{B}(\mathbb{R}^d)} \int_A \left| K_{\sigma}(x - x_j) - K_{\sigma}(x - \widetilde{x}_j) \right| dx$$

$$\leq \frac{1}{t} \sup_{A \in \mathbb{B}(\mathbb{R}^d)} \int_A \left[K_{\sigma}(x - x_j) + K_{\sigma}(x - \widetilde{x}_j) \right] dx \leq 2/t.$$

Therefore, by McDiarmid's Inequality (Theorem 25), we have

$$\mathbb{P}\left(\left|\|P_t - P\|_{\text{TV}}^{\sigma} - \mathbb{E}\|P_t - P\|_{\text{TV}}^{\sigma}\right| \ge u\right) \le 2\exp(-tu^2/2), \quad u > 0.$$

It follows from Lemma 26 that $||P_t - P||_{\text{TV}}^{\sigma}$ is 8t-sub-Gaussian. Furthermore, Goldfeld et al. (2020b) show that $\mathbb{E} ||P_t - P||_{\text{TV}}^{\sigma} \le c_d t^{-1/2} / \sqrt{2}$. Finally, the Total Variation distance is convex by Lemma 1, thus $||\cdot||_{\text{TV}}^{\sigma}$ is also convex. The first claim now follows from Corollary 8.

To prove the second claim, we show similarly as Niles-Weed and Rigollet (2022) that the map

$$G: (x_1, \dots, x_t) \in \mathbb{R}^{t \times d} \longmapsto tW_1^{\sigma} \left(\frac{1}{t} \sum_{i=1}^t \delta_{x_i}, P\right),$$

is Lipschitz with respect to the metric $c_t(x,y) := \sum_{i=1}^t \|x_i - y_i\|_2$ on $\mathbb{R}^{d \times t}$, where $x = (x_1, \dots, x_t), y = (y_1, \dots, y_t) \in \mathbb{R}^{d \times t}$. Let \mathcal{J} denote the set of 1-Lipschitz functions on \mathbb{R}^d , and recall that the W_1 distance coincides with the IPM generated by \mathcal{J} , by the Kantorovich-Rubinstein duality. We thus have, by the triangle inequality for W_1 ,

$$|G(x) - G(y)| \le tW_1 \left(\left(\frac{1}{t} \sum_{i=1}^t \delta_{x_i} \right) \star \mathcal{K}_{\sigma}, \left(\frac{1}{t} \sum_{i=1}^t \delta_{y_i} \right) \star \mathcal{K}_{\sigma} \right)$$

$$= t \sup_{f \in \mathcal{J}} \int f d \left(\frac{1}{t} \sum_{i=1}^t (\delta_{x_i} \star \mathcal{K}_{\sigma} - \delta_{y_i} \star \mathcal{K}_{\sigma}) \right)$$

$$= \sup_{f \in \mathcal{J}} \sum_{i=1}^t \left[(f \star K_{\sigma})(x_i) - (f \star K_{\sigma})(y_i) \right]$$

$$= \sup_{f \in \mathcal{J}} \sum_{i=1}^t \int [f(x_i - z) - f(y_i - z)] K_{\sigma}(z) dz$$

$$\le \sum_{i=1}^t ||x_i - y_i||_2 \int K_{\sigma}(z) dz = c_t(x, y).$$

We deduce that G is 1-Lipschitz with respect to c_t . Furthermore, by Gozlan and Léonard (2010), Proposition 1.9, the product measure $P^{\otimes t}$ satisfies the $T_1(t\tau^2)$ inequality over $\mathbb{R}^{d\times t}$ with respect to c_t . Therefore, $G(X_1,\ldots,X_t)=tW_1^{\sigma}(P_t,P)$ is $(t\tau^2)$ -sub-Gaussian by Lemma 31, i.e. $W_1^{\sigma}(P_t,P)$ is (τ^2/t) -sub-Gaussian. Furthermore, $\mathbb{E}W_1^{\sigma}(P_t,P) \leq C_d t^{-1/2}/\sqrt{2}$ by Goldfeld et al. (2020b). Applying Corollary 8 leads to the claim.

Proof of Corollary 21. Since P is supported in $[-1,1]^d$, Proposition 5 of Polyanskiy and Wu (2016) implies

$$|h(P_t \star \mathcal{K}_\sigma) - h(P \star \mathcal{K}_\sigma)| \le \frac{1}{2\sigma^2} \left(|\mu_t - \mu| + 2\sqrt{dW_1^\sigma}(P_t, P) \right), \tag{47}$$

where $\mu_t = \int x dP_t(x)$ and $\mu = \int x dP(x)$. Notice that P is 1-sub-Gaussian by Hoeffding's Lemma, and thus also satisfies the $T_1(1)$ inequality (by Lemma 31). By Corollary 23 (see also the discussion thereafter), we have

$$\forall t \ge 1 : |\mu_t - \mu| \le 2\sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(4/\delta) \right]},$$
 with probability at least $1 - \delta/2$,

and by Corollary 20,

$$\forall t \ge 1: W_1^{\sigma}(P_t, P) \le \frac{C_d}{\sqrt{t}} + 2\sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(2/\delta)\right]}, \text{ with probability at least } 1 - \delta/2.$$

By a union bound and equation (47), it follows that with probability at least $1 - \delta$, we have

uniformly in $t \geq 1$,

$$|h(P_t \star \mathcal{K}_{\sigma}) - h(P \star \mathcal{K}_{\sigma})| \leq \frac{1}{2\sigma^2} \left\{ |\mu_t - \mu| + 2\sqrt{d}W_1^{\sigma}(P_t, P) \right\}$$

$$\leq \frac{1}{2\sigma^2} \left\{ (2 + 4\sqrt{d})\sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(4/\delta) \right]} + \frac{2\sqrt{d}C_d}{\sqrt{t}} \right\}$$

$$\leq \frac{3\sqrt{d}}{\sigma^2} \sqrt{\frac{1}{t} \left[\log \ell(\log_2 t) + \log(4/\delta) \right]} + \frac{\sqrt{d}C_d}{\sqrt{t}\sigma^2},$$

as claimed. \Box

D.6 Proofs from Subsection 4.6

Proof of Corollary 22. Notice that

$$\sup_{f \in \mathcal{F}} |R(f) - R_t(f)| = \sup_{f \in \mathcal{F}} \left| \int I(f(x) \neq y) d(P - P_t)(x, y) \right| = D_{\mathcal{J}}(P_t || P),$$

where $D_{\mathcal{J}}$ is the IPM generated by the class $\mathcal{J} = \{(x,y) \mapsto I(f(x) \neq y) : f \in \mathcal{F}\}$. Since the functions in \mathcal{J} are uniformly bounded by 1, if follows by the same argument as in the proof of Corollary 13 that that $D_{\mathcal{J}}(P_t||P)$ is (2/t)-sub-Gaussian. Furthermore, a standard symmetrization argument (see for instance equation (4.18) of Wainwright (2019)) implies

$$\mathbb{E}[D_{\mathcal{J}}(P_t||P)] \le 2\mathcal{R}_t(\mathcal{J}) = \mathcal{R}_t(\mathcal{F}),$$

where the final equality follows from Lemma 3.4 of Mohri et al. (2018). By Corollary 8, we deduce

$$\mathbb{P}\left(\exists t \geq 1: D_{\mathcal{J}}(P_t || P) \geq \mathcal{R}_{\bar{t}}(\mathcal{F}) + 2\sqrt{\frac{2}{t} \Big[\log \ell(\log_2 t) + \log(1/\delta)\Big]}\right) \leq \delta,$$

which readily implies the first claim. To prove the second, abbreviate $\widehat{\mathcal{R}}_t(\mathcal{F})$ by $\widehat{\mathcal{R}}_t$, and let

$$\widehat{\mathcal{R}}_t^i = \mathbb{E}_{\boldsymbol{\varepsilon}_{t+1}} \left[\sup_{f \in \mathcal{F}} \frac{1}{t} \left| \sum_{\substack{j=1 \ j \neq i}}^{t+1} \epsilon_j f(X_j) \right| \right], \quad i = 1, \dots, t+1.$$

Then,

$$\widehat{\mathcal{R}}_{t+1} = \mathbb{E}_{\boldsymbol{\varepsilon}_{t+1}} \left[\sup_{f \in \mathcal{F}} \frac{1}{t+1} \left| \sum_{j=1}^{t+1} \epsilon_j f(X_j) \right| \right] = \mathbb{E}_{\boldsymbol{\varepsilon}_{t+1}} \left[\sup_{f \in \mathcal{F}} \frac{1}{t+1} \left| \sum_{i=1}^{t+1} \frac{1}{t} \sum_{j=1}^{t+1} \epsilon_j f(X_j) \right| \right]$$

$$\leq \frac{1}{t+1} \sum_{i=1}^{t+1} \mathbb{E}_{\boldsymbol{\varepsilon}_{t+1}} \left[\sup_{f \in \mathcal{F}} \frac{1}{t} \left| \sum_{j=1}^{t+1} \epsilon_j f(X_j) \right| \right] = \frac{1}{t+1} \sum_{i=1}^{t+1} \widehat{\mathcal{R}}_t^i,$$

implying that $(\widehat{\mathcal{R}}_t)$ satisfies the leave-one-out property in equation (18). It follows from Proposition 5 that $(\widehat{\mathcal{R}}_t)$ is a reverse submartingale with respect to the exchangeable filtration (\mathcal{E}_t^X) . Thus, Theorem 7 and Corollary 8 can be invoked with (N_t) replaced by $(\widehat{\mathcal{R}}_t)$. Furthermore, it can again be deduced as before that $\widehat{\mathcal{R}}_t$ is (8/t)-sub-Gaussian, and has mean \mathcal{R}_t . Corollary 8 thus leads to the claim.

D.7 Proofs from Subsection 4.7

Let \mathcal{A}_{γ} be a γ -cover of \mathbb{S}_{\star}^{d-1} of size N_{γ} . By a straightforward covering argument, notice that for any $\nu \in \mathbb{S}_{\star}^{d-1}$, there exists $\nu_0 \in \mathcal{A}_{\gamma}$ such that $\|\nu - \nu_0\|_* \leq \gamma$ thus

$$\nu^{\top} X = (\nu - \nu_0)^{\top} X + \nu_0^{\top} X \le \gamma \|X\| + \nu_0^{\top} X,$$

whence

$$||X|| = \sup_{\nu \in \mathbb{S}_{+}^{d-1}} \nu^{\top} X \le \gamma ||X|| + \max_{\nu \in \mathcal{A}_{\gamma}} \nu^{\top} X,$$

implying that $||X|| \leq \frac{1}{1-\gamma} \max_{\nu \in \mathcal{A}_{\gamma}} \nu^{\top} X$. We deduce that for all $\lambda \geq 0$,

$$\mathbb{E}\left[\exp\left(\lambda \|\mu_{t} - \mu\|\right)\right] \leq \mathbb{E}\left[\exp\left(\frac{\lambda}{1 - \gamma} \max_{\nu \in \mathcal{A}_{\gamma}} \nu^{\top}(\mu_{t} - \mu)\right)\right]$$

$$\leq \sum_{\nu \in \mathcal{A}_{\gamma}} \mathbb{E}\left[\exp\left(\frac{\lambda}{1 - \gamma} \nu^{\top}(\mu_{t} - \mu)\right)\right]$$

$$= \sum_{\nu \in \mathcal{A}_{\gamma}} \left(\mathbb{E}\left[\exp\left(\frac{\lambda}{t(1 - \gamma)} \nu^{\top}(X - \mu)\right)\right]\right)^{t}$$

$$\leq N_{\gamma} \exp\left(t\psi\left(\frac{\lambda}{t(1 - \gamma)}\right)\right).$$

where we extend the definition of the convex function ψ to \mathbb{R}_+ by setting $\psi(\lambda) = \infty$ for all $\lambda \geq \lambda_{\max}$. We deduce that an upper bound on the cumulant generating function of $\|\mu_t - \mu\|$ is given by

$$\overline{\psi}_t(\lambda) = \log N_{\gamma} + t\psi\left(\frac{\lambda}{t(1-\gamma)}\right), \quad \lambda \ge 0,$$

It is readily seen that for all $x \in \mathbb{R}$ and all $\lambda \geq 0$,

$$\overline{\psi}_t^*(x) = -\log N_\gamma + t\psi^* \left((1 - \gamma)x \right), \quad \left(\overline{\psi}_t^* \right)^{-1}(\lambda) = \frac{1}{1 - \gamma} (\psi^*)^{-1} \left(\frac{\lambda + \log N_\gamma}{t} \right).$$

Finally, notice that the functional $\Phi(Q) = \|\int x dQ(x) - \mu\|$ is convex, thus we may apply Theorem 7 to deduce that for all $\delta \in (0,1)$,

$$\mathbb{P}\left\{\exists t \geq 1: \|\mu_t - \mu\| \geq \frac{1}{1 - \gamma} (\overline{\psi}^*)^{-1} \left(\frac{\log \ell(\log_2 t) + \log(1/\delta) + \log N_{\gamma}}{\lceil t/2 \rceil}\right)\right\} \leq \delta.$$

The claim follows. \Box

D.8 Proofs from Subsection 4.8

Proof of Corollary 24. Let $\eta, \alpha > 1$, and set $u_k = \lceil \eta^k \rceil$ for all $k \ge 0$. Define $\ell(k) = (1 \lor k^\alpha)\zeta(\alpha)$, where $\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ and $\alpha > 1$. From the proof of Theorem 28, for the process $N_t = D(P_t || P)$ in the special case of sub-Gaussian tails $\psi_t(\lambda) = \lambda \mathbb{E}(N_t) + \lambda^2 \sigma^2 / 2t$, it can be seen that $\mathbb{P}(A_k) \le \delta/\ell(k+1)$, where, for all $k = 0, 1, \ldots$,

$$A_k = \left\{ \exists u_k \le t \le u_{k+1} : N_t > \mathbb{E}\left(N_{\lceil t/\lceil \eta \rceil \rceil}\right) + \sqrt{\frac{2\sigma^2}{\lceil t/\lceil \eta \rceil \rceil} \left[\log \ell(\log_{\eta} t) + \log(1/\delta)\right]} \right\}.$$

Thus, by definition of ℓ and by the first Borel-Cantelli Lemma, we have $\mathbb{P}(\limsup_{k\to\infty} A_k)=0$. Therefore,

$$\mathbb{P}\left\{N_t \leq \mathbb{E}\left(N_{\lceil t/\lceil \eta \rceil \rceil}\right) + \sqrt{\frac{2\sigma^2}{\lceil t/\lceil \eta \rceil \rceil} \left[\log \ell(\log_{\eta} t) + \log(1/\delta)\right]} \text{ eventually}\right\} = 1.$$

Note that for all $t \geq \eta$,

$$\log \ell(\log_{\eta} t) = \alpha \log \log_{\eta} t + \log \zeta(\alpha) = \alpha \log \log t - \alpha \log \log \eta + \log \zeta(\alpha).$$

Therefore, we have almost surely,

$$\limsup_{t \to \infty} \frac{D(P_t || P)}{\mathbb{E}\left(N_{\lceil t/\lceil \eta \rceil \rceil}\right) + \sqrt{\frac{2\sigma^2}{\lceil t/\lceil \eta \rceil \rceil} \left[\alpha \log \log t - \alpha \log \log \eta + \log \zeta(\alpha) + \log(1/\delta)\right]}} \leq 1,$$

whence, by assumption on $\mathbb{E}N_t$ and by the fact that δ, α, η are fixed, we obtain

$$\limsup_{t \to \infty} \frac{D(P_t || P)}{\sqrt{\frac{2\sigma^2}{\lceil t/\lceil n \rceil \rceil} \alpha \log \log t}} \le 1 \quad \text{a.s.}$$

Since we may choose η and α arbitrarily close to 1, the claim follows.

E The failure of forward submartingales and canonical filtrations

Our main results in Section 3 derived confidence sequences for convex divergences on the basis of their empirical plug-in estimators, which form reverse submartingales with respect to the exchangeable filtration. The latter are rarely used in sequential analysis. The aim of this section is to illustrate our difficulty in using more typical tools from sequential analysis.

Let D be a convex divergence, and $(X_t)_{t=1}^{\infty}$ a sequence of i.i.d. observations from a distribution $P \in \mathcal{P}(\mathcal{X})$. We shall focus on deriving time-uniform concentration inequalities for the process

$$S_t = tD(P_t || P), \quad t > 1.$$

Specifically, given a level $\delta \in (0,1)$, we derive sequences of nonnegative real numbers $(u_t)_{t=1}^{\infty}$ such that

$$\mathbb{P}\Big(\exists t \ge 1 : S_t \ge u_t\Big) \le \delta. \tag{48}$$

Assuming D is additionally symmetric, satisfies the triangle inequality, and that equation (48) can also be applied to the process $(sD(Q_s||Q))_{s=1}^{\infty}$, one also obtains the bound

$$\mathbb{P}\Big(\exists t, s \ge 1 : |D(P_t || Q_s) - D(P || Q)| \ge u_t / t + u_s / s\Big) \le 2\delta. \tag{49}$$

As was discussed in Section 3, the passage from equation (48) to (49) may lead to confidence sequences with lengths of sub-optimal rate in general.

In Section 2.5, we discussed two prominent stitching constructions used in past work to derive confidence sequences for the common mean μ of (X_t) . Assuming $X_1 - \mu$ admits a finite cumulant generating function $\phi: [0, \lambda_{\max}) \to \mathbb{R}$ for some $\lambda_{\max} > 0$, these approaches are either based on

- (A) applying Ville's inequality to the forward nonnegative supermartingale $(L_t(\lambda))$, or
- (B) applying Doob's submartingale inequality to the forward submartingale $(U_t(\lambda))$.

While concrete applications of these approaches admit various nuances which lead to confidence sequences with distinct constants, under varying assumptions on ϕ , they both hinge upon repeated applications of a maximal forward martingale inequality over geometrically-increasing epochs of time, and lead to confidence sequences of length typically scaling at the rate $O(\sqrt{\log \log t/t})$. We show in what follows that analogues of constructions (A) and (B) can be obtained for the process (S_t) , but may be unsatisfactory in general.

We begin with the following simple result, which provides an analogue of the exponential supermartingale in (A) for the process (S_t) . In what follows, we let $S_t = \sigma(X_1, \ldots, X_t)$, $t \ge 1$, denote the canonical filtration.

Lemma 32. Let D denote a convex divergence, and set $S_t = tD(P_t||P)$ for all $t \ge 1$. Assume there exists $\lambda_{\max} > 0$ and a convex function ψ such that

$$\log \left\{ \mathbb{E}[\exp(\lambda D(\delta_{X_1} || P))] \right\} \le \psi(\lambda), \quad \lambda \in [0, \lambda_{\max}). \tag{50}$$

Then, for any $\lambda \in [0, \lambda_{\max})$, the process

$$\mathcal{L}_t(\lambda) = \exp \left\{ \lambda S_t - t\psi(\lambda) \right\}, \quad t \ge 1$$

is a nonnegative supermartingale with respect to $(S_t)_{t=1}^{\infty}$.

Lemma 32 is proven below. The process $(\mathcal{L}_t(\lambda))$ is in direct analogy with the supermartingale in (A), where the cumulant generating function ϕ of $(X_1 - \mu)$ is replaced by that of $D(\delta_{X_1} \| P)$. Unlike (A), however, the quantity $D(\delta_{X_1} \| P)$ has non-zero mean; for example if D is the KL divergence and P has finite support, then $\mathbb{E}[D(\delta_{X_1} \| P)]$ is the entropy of P. This in turn implies that the upper bound ψ may typically decay linearly as $\lambda \downarrow 0$, as opposed to the typical quadratic rate of decay for cumulant generating functions (Howard et al., 2020). It can be seen that standard stitching constructions applied to this process would then only lead to a confidence sequence for $S_t/t = D(P_t \| P)$ with non-vanishing length, which is overly conservative in most examples of interest.

This intractibility of the process $(L_t(\lambda))$ arises from the fact that, unlike the setting (A) for sums of i.i.d. random variables, the term $t\psi(\lambda)$ can be an excessively loose upper bound on the true cumulant generating function of $tD(P_t||P)$. While the form of this upper bound is pivotal for deriving the supermartingale property of $(L_t(\lambda))$, it would ideally be replaced by a tighter upper bound $\psi_t : [0, \lambda_{\text{max}}) \to \mathbb{R}$, which is not necessarily linear in t, satisfying

$$\log \left\{ \mathbb{E} \left[\exp(\lambda S_t) \right] \right\} \le \psi_t(\lambda), \quad \lambda \in [0, \lambda_{\max}). \tag{51}$$

Without restrictions on the form of ψ_t , however, it is unclear how to enforce a supermartingale property for the process ($\exp{\{\lambda S_t - \psi_t(\lambda)\}}$) for a general divergence D.

These considerations motivate approach (B), though unlike Lemma 32, however, we are not aware of a general result guaranteeing that $(\exp(\lambda S_t))$ is a forward submartingale for any convex divergence D. The following straightforward result presents a notable exception.

Lemma 33. Let \mathcal{J} be any class of Borel-measurable functions from \mathbb{R}^d to \mathbb{R} , let D be the IPM generated by \mathcal{J} , and recall $S_t = tD(P_t||P)$. Assume $S_t \in L^1(\mathbb{P})$ for all $t \geq 1$. Then, $(S_t)_{t=1}^{\infty}$ is a forward nonnegative <u>sub</u>martingale with respect to $(S_t)_{t=1}^{\infty}$. In particular, for any $\lambda \in [0, \lambda_{\max})$, the process $(\exp(\lambda S_t))_{t=1}^{\infty}$ is also a forward nonnegative submartingale with respect to (S_t) whenever it lies in $L^1(\mathbb{P})$.

We use Lemma 33 to derive a confidence sequence for the process (S_t) , in the case where D is an IPM, or any other convex divergence for which $(\exp(\lambda S_t))$ forms a forward martingale. Our result will depend on a stitching function $\ell:[0,\infty)\to[1,\infty)$, which satisfies $\sum_{k=0}^{\infty}1/\ell(k)\leq 1$. We also assume that the upper bound ψ_t is convex. The main result of this section is stated as follows.

Theorem 34. Let D be a convex divergence, and let $S_t = tD(P_t||P)$ for all $t \ge 0$. Assume that for any $\lambda \in [0, \lambda_{\max})$, the process $(\exp(\lambda S_t))_{t=0}^{\infty}$ is a forward submartingale with respect to a filtration $(S_t)_{t=0}^{\infty}$, and satisfies inequality (51). Assume ψ_t^* is invertible in λ and that the sequence

$$\gamma_t = (\psi_t^*)^{-1} (\log \ell (\log_2 t) + \log(2/\delta)), \quad t \ge 1$$

is increasing in t. Then, for all $\delta \in (0,1)$,

$$\mathbb{P}\Big(\exists t \ge 1 : S_t \ge 2\gamma_t\Big) \le \delta. \tag{52}$$

Theorem 34 is proved by dividing time into geometrically increasing epochs of the form $[2^j, 2^{j+1}-1]$, $j \geq 0$, with the goal of applying Doob's submartingale inequality over each epoch, at level $\delta/\ell(j)$. Applying a union bound over these epochs leads to the claim. Our proof makes use of the convexity of D, which guarantees the following bound on the deviations of (S_t) ,

$$S_{t_1} - S_{t_0} \le S_{t_0, t_1} := (t_1 - t_0) D\left(\frac{1}{t_1 - t_0} \sum_{i=t_0+1}^{t_1} \delta_{X_i} \middle\| P\right), \quad \forall t_1 > t_0.$$
 (53)

Inequality (53) is used in our proof to relate upcrossing probabilities over the interval $[2^j, 2^{j+1} - 1]$ to those over its translation at zero, $[0, 2^{j+1} - 1 - 2^j]$, leading to improved constants in Theorem 34, and can be viewed as a generalization of the proof technique employed in Lemma 3 of Jamieson et al. (2014) for processes (S_t) which are submartingales rather than martingales.

While the bound of Theorem 34 is useful, it presents two key shortcomings. On the one hand, it does not directly lead to a confidence sequence for the quantity of interest D(P||Q), as discussed at the beginning of this Appendix. On the other hand, Theorem 34 can only be applied when $(\exp(\lambda S_t))$ forms a forward submartingale. We have shown in Lemma 33 that $(\exp(\lambda S_t))$, is a submartingale when D is an IPM, but we are not aware of a generic result of this kind for other convex divergences. In contrast, we showed in Section 3 that for any convex divergence, this process, and the more general process (M_{ts}) in equation (4), form reverse submartingales with respect to the exchangeable filtration, which turned out to be more natural to handle.

We close this Appendix by proving Lemmas 32, 33, and Theorem 34.

E.1 Proofs

Proof of Lemma 32. $(\mathcal{L}_t(\lambda))$ is clearly adapted to (\mathcal{S}_t) . Now, notice that $P_t = \frac{1}{t}\delta_{X_t} + \frac{t-1}{t}P_{t-1}$. Therefore, by convexity of D,

$$D(P_t||P) \le \frac{t-1}{t}D(P_{t-1}||P) + \frac{1}{t}D(\delta_{X_t}||P), \text{ thus, } S_t \le S_{t-1} + D(\delta_{X_t}||P).$$

It follows that for all $t \geq 1$,

$$\mathbb{E}[\mathcal{L}_{t}(\lambda)|\mathcal{F}_{t-1}] = \mathbb{E}\left\{\exp\left(\lambda S_{t} - t\psi(\lambda)\right) \mid \mathcal{S}_{t-1}\right\}$$

$$\leq \mathbb{E}\left\{\exp\left(\lambda S_{t-1} + \lambda D(\delta_{X_{t}}||P) - t\psi(\lambda)\right) \mid \mathcal{S}_{t-1}\right\}$$

$$= \mathcal{L}_{t-1}(\lambda)\mathbb{E}\left\{\exp[\lambda D(\delta_{X_{t}}||P) - \psi(\lambda)]\right\} \leq \mathcal{L}_{t-1}(\lambda),$$

where the last inequality follows from the definition of ψ .

Proof of Lemma 33. The claim is straightforward. (S_t) is clearly adapted to (S_t) . Furthermore, for all $t \geq 1$, notice that

$$\mathbb{E}[S_t|\mathcal{S}_{t-1}] = \mathbb{E}\left[t\sup_{f\in\mathcal{J}}\int fd(P_t - P)\bigg|\mathcal{S}_{t-1}\right]$$

$$\geq \sup_{f\in\mathcal{J}}\mathbb{E}\left[\sum_{i=1}^t [f(X_i) - \mathbb{E}f(X_i)]\bigg|\mathcal{S}_{t-1}\right] = \sup_{f\in\mathcal{J}}\sum_{i=1}^{t-1} [f(X_i) - \mathbb{E}f(X_i)] = S_{t-1}.$$

The final claim follows from the convexity and monotonicity of the exponential function. \Box

Proof of Theorem 34. The proof is inspired by Jamieson et al. (2014) (Lemma 3). By assumption, $(\exp(\lambda S_t))$ forms a submartingale with respect to the filtration (S_t) , for any $\lambda \in [0, \lambda_{\max})$. By Doob's submartingale inequality, we therefore have for any integer $T \geq 1$ and real number y > 0 the Cramér-Chernoff bound

$$\mathbb{P}\Big(\exists t \leq T : S_t \geq y\Big) = \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{P}\Big(\exists t \leq T : \exp(\lambda S_t) \geq \exp(\lambda y)\Big)$$

$$\leq \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{E}\Big[\exp\big(\lambda S_T - \lambda y\big)\Big]$$

$$\leq \inf_{\lambda \in [0, \lambda_{\max})} \mathbb{E}\Big[\exp\big(\psi_T(\lambda) - \lambda y\big)\Big] = \exp\{-\psi_T^*(y)\}.$$

Now, set $u_k = 2^k$ for all $k \ge 0$. Since (γ_t) is an increasing sequence, we have

$$\mathbb{P}\left(\exists t \geq 1 : S_{t} \geq 2\gamma_{t}\right) \leq \mathbb{P}\left(\bigcup_{k=0}^{\infty} \left\{\exists t \in \{u_{k}, \dots, u_{k+1} - 1\} : (S_{t} - S_{u_{k}}) + S_{u_{k}} \geq 2\gamma_{u_{k}}\right\}\right) \\
\leq \mathbb{P}\left(\bigcup_{k=0}^{\infty} \left\{\exists t \in \{u_{k}, \dots, u_{k+1} - 1\} : S_{u_{k}, t} + S_{u_{k}} \geq 2\gamma_{u_{k}}\right\}\right), \quad \text{(by equation (53))} \\
\leq \mathbb{P}\left(\bigcup_{k=0}^{\infty} \left\{S_{u_{k}} \geq \gamma_{u_{k}}\right\}\right) + \mathbb{P}\left(\bigcup_{k=0}^{\infty} \left\{\exists t \in \{u_{k}, \dots, u_{k+1} - 1\} : S_{u_{k}, t} \geq \gamma_{u_{k}}\right\}\right). \quad (54)$$

We will upper bound the terms in (54) separately. For the first term, by a union bound and another standard application of the Cramér-Chernoff technique, we have

$$\mathbb{P}\left(\bigcup_{k=0}^{\infty} \left\{ S_{u_k} \ge \gamma_{u_k} \right\} \right) \le \sum_{k=0}^{\infty} \mathbb{P}\left(S_{u_k} \ge \gamma_{u_k} \right) \le \sum_{k=0}^{\infty} \exp\left\{ -\psi_{u_k}^*(\gamma_{u_k}) \right\} = \frac{\delta}{2} \sum_{k=0}^{\infty} \frac{1}{\ell(k)} \le \frac{\delta}{2}.$$

For the second term in equation (54), notice that $S_t = S_{0,t} \stackrel{d}{=} S_{r,t+r}$ for all $t, r \geq 1$. Therefore, recalling that $u_{k+1} = 2u_k$ we have for all $k \in \mathbb{N}$,

$$\sum_{k=0}^{\infty} \mathbb{P} \Big(\exists t \in \{u_k, \dots, u_{k+1} - 1\} : S_{u_k, t} \ge \gamma_{u_k} \Big) = \sum_{k=0}^{\infty} \mathbb{P} \Big(\exists t \in \{u_k, \dots, u_{k+1} - 1\} : S_{0, t - u_k} \ge \gamma_{u_k} \Big)$$

$$= \sum_{k=0}^{\infty} \mathbb{P} \Big(\exists u \in \{1, \dots, u_{k+1} - u_k\} : S_{0, u} \ge \gamma_{u_k} \Big)$$

$$= \sum_{k=0}^{\infty} \mathbb{P} \Big(\exists u \in \{1, \dots, u_k\} : S_u \ge \gamma_{u_k} \Big)$$

$$\leq \sum_{k=0}^{\infty} \exp \Big\{ -\psi_{u_k}^* (\gamma_{u_k}) \Big\} = \frac{\delta}{2} \sum_{k=0}^{\infty} \frac{1}{\ell(k)} \le \frac{\delta}{2}.$$

The claim thus follows.