Commitment capacity of classical-quantum channels

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Abstract

We study commitment scheme for classical-quantum channels. To accomplish this we define various notions of commitment capacity for these channels and prove matching upper and lower bound on it in terms of the conditional entropy. Our achievability (lower bound) proof is quantum generalisation of the work of one of the authors (arXiv:2103.11548) which studied the problem of secure list decoding and its application to bit-string commitment. The techniques we use in the proof of converse (upper bound) is similar in spirit to the techniques introduced by Winter, Nascimento and Imai (Cryptography and Coding 2003) to prove upper bound on the commitment capacity of classical channels. However, generalisation of this technique to the quantum case is not so straightforward and requires some new constructions, which can be of independent interest.

I. Introduction

Most of the modern protocols which are used to securely encrypt a message are based on the notion of commitment. Commitment with respect to this secure encryption means that one of the party (Alice) involved in the protocol is able to choose a message from a set and be committed to her choice. Her commitment to this choice of message should be in such a way that while revealing this choice of message to the other party (Bob), she should not be able to reveal something to which she didn't choose and commit. To understand this intuitively, we consider the following example:

- 1) Alice wants to commit a message m chosen from a finite set. She does this by writing the message on a paper and then locking it inside an envelope.
- 2) Alice then gives the locked envelope to Bob. At a later point of time, when Bob wants to read the message m, he asks for the key from Alice so that he can open the envelop and read the message.

The procedure discussed in the above example needs to satisfy the following two properties:

- i **Concealing**: After receiving the locked envelope from Alice, Bob should have no idea about what is written on the paper locked inside the envelope until Alice reveals him the key to open the envelope and read the message.
- ii **Binding**: After locking the message in the envelope, Alice should not be able to change it after she hands over the locked envelope to Bob.

This problem was first introduced and studied by Blum [1]. However, [2] and [3] showed that if there are no computational constraints on the sender and receiver then bit commitment is not possible. Crépeau [4] pointed out that bit commitment can be realized when a binary noisy channel is available. That is, a noisy channel (modeled as $p_{Y|X}$) can help in achieving commitment scheme. Studying this problem from information theory perspective, Winter et al. [5], [6] gave a probabilistic definition of commitment and used information theoretic notion for secrecy (concealing). Using these tools, they defined the commitment capacity of a channel and showed that it is equal to $\max_{p_X} H(X \mid Y)$. Although their direct part is sound, they wrote only the sketch of the converse part. In addition, their converse proof contains an analysis that cannot be extended to the quantum setting, as explained later. The reference [7] discussed the same issue

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in a similar way. Later, Yamamoto et al. [8] studied this problem under the problem setting of multiplex coding. Although this paper also considers the converse part as well as the direct part, their converse part is a weaker statement than the converse part of the original problem setting, as explained later.

Further, the papers [5], [6] also introduced the commitment scheme when the parties have access to a classical-quantum channel (cq-channel). Even though they claimed that the commitment capacity with cq-channel is in terms of conditional quantum entropy, they didn't provide the complete proof. In fact, this generalization is not so straight forward. Also, they didn't explore the possibility that the parties involved have more options in terms of cheating the other party in the quantum case.

Recently, the reference [9] pointed out that a code to achieve the commitment capacity can be constructed by using a special type of list decoding. Originally, list decoding was proposed by Elias [20] and Wozencraft [21] independently. Hamming distance takes a important role in the code construction by [9] similar to the preceding studies [5], [6], [7], [8].

We explore all these issues in this manuscript. In particular, we generalize the notion of interactive protocol for implementing commitment scheme introduced in [5], [6] to the case of classical-quantum channel $W_{X\to Y}$ (in our future discussions we will omit the subscript $X\to Y$.) Towards this aim, we define the notion of active and passive attacks. Using these two notions, we give two types of definitions for the commitment capacity of a classical-quantum channel and denote them as $C_a(W)$ and $C_p(W)$, respectively. In this manuscript, we observe that finding $C_a(W)$ and $C_p(W)$ is a difficult problem. Therefore, we study a simpler version of the interactive protocol. In this simpler version, we restrict Alice and Bob to only use invertible operations to accomplish the commitment scheme. Therefore, to study this special case, we introduce $C_{a,inv}(W)$ and $C_{p,inv}(W)$ which represent the commitment capacity of a classical-quantum channel when the parties are allowed to use only invertible operations. Further, we also explore the case when Alice and Bob implement commitment scheme by only using non-interactive protocol over a classical-quantum channel. We define $C_{a,non}(W)$, $C_{p,non}(W)$ as the capacities under this setting.

We show a relation between these notions of the commitment capacity of classical-quantum channel defined in this manuscript. In particular, the following relationship is one of the main result of this manuscript:

$$C_{a,non}(\boldsymbol{W}) = C_{p,non}(\boldsymbol{W}) = C_{a,inv}(\boldsymbol{W}) = C_{p,inv}(\boldsymbol{W}) = \sup_{P \in \boldsymbol{P}(\mathcal{X})} H(X|Y)_{P}.$$
 (1)

We obtain (1) by first showing that $C_{p,inv}(\mathbf{W}) \leq \sup_{P \in P(\mathcal{X})} H(X|Y)_P$. This is the converse part and it requires construction of some functions which helps in proving the converse part by using the Fano's inequality. However, as explained in Subsection V-C, the references [5], [6], [7] have a problem in the construction of the above type of function. This paper concretely writes down the construction of such a function from a general interactive quantum protocol as Proposition 1 when the protocol satisfies the invertible condition. Since any interactive protocol in the classical setting satisfies the invertible condition, our converse proof covers the classical setting without any condition. In addition, since the reference [8] considered the converse part only for non-interactive protocols, it did not discuss the above type of function.

To show the direct part, we prove that $C_{a,non}(W) \ge \sup_{P \in P(\mathcal{X})} H(X|Y)_P$ by showing the existence of a non-interactive protocol which satisfies the binding and concealing property even for a cq-channel. While our protocol construction is quite similar to the protocol proposed by the reference [9], our code construction is different from that by [9] in the following point. The reference [9] considered only the classical channel, and introduced the special class of list decoding, so called secure list decoding. Then, the reference [9] converts secure list decoding to a non-interactive protocol. In this conversion, Bob applies the list decoder and gets the list of messages in the commitment phase. Bob checks whether the information revealed by Alice is contained in the list in the reveal phase. However, in the case with cq-channel, it is not so easy to construct the list decoder due to the non-commutativity of the density operators. Therefore, in this paper, we construct Alice's encoder of the commitment phase in the same way as the paper [9]. In

our constructed non-interactive protocol, Bob does nothing in the commitment phase. In the reveal phase, he applies the projection corresponding to the information revealed by Alice to check whether Alice is honest or not.

The rest of the manuscript is as follows. Section II prepares notations and the definitions used in this manuscript. Section III mathematically formulates the commitment scheme and gives a formal mathematical definition for it. We also define several notions of the commitment capacity in this section. Section IV applies our result to the case when the channel has symmetry. Section V proves the converse for commitment scheme when Alice and Bob are only using reversible operations to accomplish the commitment scheme. Section VI gives a protocol when Alice and Bob are only allowed to use non-interactive protocol for accomplishing commitment scheme. The protocol is given by a conversion from a specific type of code, and Section VII is devoted to its construction. Section VIII makes conclusion and discusses future studies.

II. PREPARATION

A. Notations and Information quantities

This paper focuses on a noisy classical-quantum (cq-) channel $W = \{W_x\}_{x \in \mathcal{X}}$ from an input classical system \mathcal{X} composed of finite elements to a quantum system \mathcal{H}_Y , where W_x is the density operator on the output quantum system \mathcal{H}_Y with input $x \in \mathcal{X}$. Also, we define the density operator W_P on Y as $W_P := \sum_{x \in \mathcal{X}} P(x)W_x$. Then, the joint cq-state $W \times P$ is defined as

$$\mathbf{W} \times P = \sum_{x \in \mathcal{X}} P(x)|x\rangle\langle x| \otimes W_x. \tag{2}$$

We denote the set of probability distributions on \mathcal{X} and the set of density operators on \mathcal{H}_Y by $\mathbf{P}(\mathcal{X})$ and $\mathcal{S}(\mathcal{H}_Y)$, respectively.

 $D(\rho \| \sigma)$ is the relative entropy between two density operators ρ and σ , which is defined as

$$D(\rho \| \sigma) := \operatorname{Tr} \rho(\log \rho - \log \sigma). \tag{3}$$

The sandwitched realtive entropy $\tilde{D}_{\alpha}(\rho \| \sigma)$ is defined as

$$\tilde{D}_{\alpha}(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr}(\sigma^{-\frac{\alpha - 1}{2\alpha}} \rho \sigma^{-\frac{\alpha - 1}{2\alpha}})^{\alpha}. \tag{4}$$

Given a state ρ_{XY} on XY, we consider various information quantities like mutual information and conditional entropy. When we need to clarify the state on the quantum system for these quantities, we add the symbol like $[\rho]$ after the information quantity. For example, the entropy is defined as $H(XY)[\rho_{XY}] := -\operatorname{Tr} \rho_{XY} \log \rho_{XY}$. Various type of conditional entropies are defined as

$$H(X|Y)[\rho_{XY}] := H(XY)[\rho_{XY}] - H(Y)[\rho_{XY}]$$
 (5)

$$\tilde{H}_{\alpha}(X|Y)[\rho_{XY}] := \max_{\sigma \in \mathcal{S}(\mathcal{H}_Y)} -\tilde{D}_{\alpha}(\rho_{XY} || I_X \otimes \sigma). \tag{6}$$

Various type of mutual informations are defined as

$$I(X;Y)[\rho_{XY}] := H(XY)[\rho_{XY}] - H(X)[\rho_{XY}] - H(Y)[\rho_{XY}]$$
(7)

$$\tilde{I}_{\alpha}(X|Y)[\rho_{XY}] := \max_{\sigma \in \mathcal{S}(\mathcal{H}_Y)} -\tilde{D}_{\alpha}(\rho_{XY} \| \rho_X \otimes \sigma). \tag{8}$$

Now, we consider the case when the joint state ρ_{XY} is given as $W \times P$ by using a distribution P on \mathcal{X} . Under the state $W \times P$, we change the symbol added to various information quantities, $[W \times P]$ to P. That is, we define

$$H(XY)_P := H(XY)[\mathbf{W} \times P], \quad \tilde{H}_{\alpha}(X|Y)_P := \tilde{H}_{\alpha}(X|Y)[\mathbf{W} \times P]$$
(9)

$$I(X;Y)_P := I(X;Y)[\boldsymbol{W} \times P], \quad \tilde{I}_{\alpha}(X|Y)_P := \tilde{I}_{\alpha}(X|Y)[\boldsymbol{W} \times P]. \tag{10}$$

In addition, we denote the trace norm of a operator C and the von Neumann entropy of the density ρ by $\|C\|_1$ and $S(\rho)$, respectively.

B. Quantum measurements

To formulate our general adaptive method for the discrimination of cq-channels, we prepare a general notation for quantum measurements with state changes. A general quantum state evolution from A to B is written as a completely positive trace-preserving (cptp) map \mathcal{M} from the space \mathcal{T}^A to the space \mathcal{T}^B of trace class operators on A and B, respectively. When we make a measurement on the initial system A, we obtain the measurement outcome K and the resultant state on the output system B. To describe this situation, we use a set $\{\kappa_k\}_{k\in\mathcal{K}}$ of cp maps from the space \mathcal{T}^A to the space \mathcal{T}^B such that $\sum_{k\in\mathcal{K}}\kappa_k$ is trace preserving. In this paper, since the classical feed-forward information is assumed to be a discrete variable, K is a discrete (finite or countably infinite) set. Since it is a decomposition of a cptp map, it is often called a cp-map valued measure, and an instrument if their sum is cptp.\(^1 In this case, when the initial state on A is ρ and the outcome k is observed with probability $\mathrm{Tr}\,\kappa_k(\rho)$, where the resultant state on B is $\kappa_k(\rho)/\mathrm{Tr}\,\kappa_k(\rho)$. A state on the composite system of the classical system K and the quantum B is written as $\sum_{k\in\mathcal{K}}|k\rangle\langle k|\otimes\rho_{B|k}$, which belongs to the vector space $\mathcal{T}^{KB}:=\sum_{k\in\mathcal{K}}|k\rangle\langle k|\otimes\mathcal{T}^B$. The above measurement process can be written as the following cptp \mathcal{E} map from \mathcal{T}^A to \mathcal{T}^{KB} .

$$\mathcal{E}(\rho) := \sum_{k \in \mathcal{K}} |k\rangle \langle k| \otimes \kappa_k(\rho). \tag{11}$$

In the following, both of the above cptp map \mathcal{E} and a cp-map valued measure are called a quantum instrument.

III. PROBLEM FORMULATION

A. General protocol description

There are two parties Alice and Bob. Alice wants to communicate a message M chosen uniformly from the set $\{1, \cdots, 2^{nR}\}$ using n uses of a noisy classical-quantum (cq-) channel $\mathbf{W} = \{W_x\}_{x \in \mathcal{X}}$ from an input classical system \mathcal{X} composed of finite elements to a quantum system \mathcal{H}_Y . We also assume the following condition for our cq-channel \mathbf{W} ;

(NR) Any element $x \in \mathcal{X}$ satisfies

the outcome V_1 to Alice.

$$\min_{x \in \mathcal{X}} \min_{P \in \mathbf{P}(\mathcal{X} \setminus \{x\})} D\left(\sum_{x' \in \mathcal{X} \setminus \{x\}} P(x') W_{x'} \middle\| W_x \right) > 0.$$
 (12)

This condition is called the non-redundant condition [5], [6], [7].

They are also allowed to use a noiseless channel any number of times. However, this whole communication process consists of two phases:

Phase 1. (Commit phase) Based on Alice's choice of message $m \in \{1, \cdots, 2^{nR}\}$, there are n rounds of a multi-round of communication from Alice to Bob and Bob to Alice. In all the discussions below one round of communication ends when first Alice communicates to Bob and then Bob communicates to Alice. Further, Alice has a classical memory Z and Bob has a quantum memory Y'. In the first round, Alice communicates $U_1 = f_1(m, Z)$ to Bob over a noiseless channel, and also communicates $X_1 = g_1(m, Z)$ over a classical-quantum channel, which Bob receives as quantum state W_{X_1} on the quantum system Y_1 . Then, Bob has the state $\rho_{U_1Y_1}(m)$. After receiving the quantum system Y_1 and the classical information U_1 from Alice, dependently on $U_1 = u_1$, Bob applies the first quantum instrument $\{\Gamma_{v_1|u_1}^{(1)}\}_{v_1\in\mathcal{V}_1}: U_1Y_1 \to Y_1'V_1$. Then, Bob has the state $\rho_{Y_1'V_1}(m)$. Bob sends

 1 For simplicity, here and in the rest of the paper, we assume the set \mathcal{K} to be discrete. In fact, if the Hilbert spaces A, B, etc, on which the cp maps act are finite dimensional, then every instrument is a convex combination, i.e. a probabilistic mixture, of instruments with only finitely many non-zero elements; this carries over to instruments defined on a general measurable space \mathcal{K} . Thus, in the finite-dimensional case the assumption of discrete \mathcal{K} is not really a restriction.

In the same way as the above, the i-th round is given as follows. Alice communicates $U_i = f_i(m, Z, V_{i-1})$ to Bob over a noiseless channel by using additional classical information V^{i-1} . Also, she communicates $X_i = T_i(m, Z, V_{i-1})$ over a classical-quantum channel, which Bob receives as quantum state W_{X_i} on the quantum system Y_i . Then, Bob has the state $\rho_{Y'_{i-1}U_iY_i}(m)$. After receiving the quantum system Y_i and the classical information U_i from Alice, Dependently on u^i, v^{i-1} , Bob applies the i-th quantum instrument $\{\Gamma^{(i)}_{v_i|u^i,v^{i-1}}\}_{v_i\in\mathcal{V}_i}:Y'_{i-1}U_iY_i\to Y'_iV_i$, where $u^i=(u_1,\ldots,u_i)$ and $v^{i-1}=(v_1,\ldots,v_{i-1})$. Then, Bob has the state $\rho_{Y'_iV_i}(m)$. Bob sends the outcome V_i to Alice. We denote honest Bob's behavior in Phase 1 and Bob's arbitrary behavior in Phase 1 by \mathcal{B} and \mathcal{B}' , respectively. We denote the set of Alice's honest operations in Phase 1 by $\mathcal{A}_1=\{A_1(m)\}_m$, where $A_1(m)$ is Alice's honest operations in Phase 1 with M=m. After n-th round, Bob's state is written as $W^{U^nV^nY'_n}_{\mathcal{B},A_1(m),X^n=x^n}$ or $W^{U^nV^nY'_n}_{\mathcal{B},A_1(m),X^n=x^n}$ dependently on Alice's and Bob's operations and $X^n=x^n$. Similarly, we define $W^{U^nV^nY'_n}_{\mathcal{B},A_1(m),X^n=x^n}$ and and $W^{U^nV^nY'_n}_{\mathcal{B},A_1(m)}$ Also, we define $W^{U^nV^nY'_n}_{\mathcal{B},A_1(m)} = \sum_m 2^{-nR}W^{U^nV^nY'_n}_{\mathcal{B},A_1(m)}$. It is required that the whole communication process at the end of Phase 1 doesn't reveal anything about the message m to Bob until Alice reveals him the message in the reveal phase mentioned below. This property of Phase 1 is called the concealing property and is defined as follows: We call Phase 1 as ε concealing for passive Bob if we have

$$\frac{1}{2} \left\| W_{\mathcal{B}, A_1(m)}^{U^n V^n Y_n'} - W_{\mathcal{B}, A_1(m')}^{U^n V^n Y_n'} \right\|_1 \le \varepsilon. \tag{13}$$

for any message pair (m, m') with $m \neq m'$. We call Phase 1 as ε concealing for active Bob if we have

$$\frac{1}{2} \left\| W_{\mathcal{B}', A_1(m)}^{U^n V^n Y_n'} - W_{\mathcal{B}', A_1(m')}^{U^n V^n Y_n'} \right\|_1 \le \varepsilon. \tag{14}$$

for any message pair (m, m') with $m \neq m'$ and Bob's arbitrary behavior \mathcal{B}' in Phase 1. The concealing property for active Bob is a stronger condition than the concealing property for passive Bob.

Phase 2. (Reveal phase) In this phase, Alice reveals her message M=m and her private randomness Z to Bob via a noiseless channel. Bob tries to answer the question "is the message revealed by Alice correct or not?" For this aim, Bob applies binary valued measurements $\mathcal{T} = \{\{T_{mz}, I - T_{mz}\}\}_{mz}$ on the system $U^nV^nY'_n$, where T_{mz} corresponds to the "Accept". To ensure that Alice doesn't cheat in the reveal phase, i.e., she is not able to reveal some wrong message to Bob, we hope that Phase 2 has this property which we call as the *binding* property. Further, the following condition is required; if both Alice and Bob don't cheat, then the measurement outcome at the end of Phase 2 should ask to Bob to accept the message revealed by Alice. This property of the protocol is called *correctness*. Both these properties of Phase 2 are defined mathematically as follows. We denote the set of Alice's honest operations in Phase 2 by $\mathcal{A}_2 = \{A_2(m)\}_m$, where $A_2(m)$ is Alice's honest operations in Phase 2 with M=m. The correctness condition is given as

Pr
$$\{m \text{ is accepted } | \text{ Alice performs } A_1(m), A_2(m), \text{ Bob performs } \mathcal{B}, \mathcal{T}\} \ge 1 - \delta.$$
 (15)

There are two kinds of binding property. In biding property, we always assume that Bob is honest. The following is the binding property for passive Alice; Any Alice's operation A'_2 for Phase 2 satisfies

$$\Pr\{m' \text{ is accepted } | \text{ Alice performs } A_1(m), A_2', \text{ Bob performs } \mathcal{B}, \mathcal{T}\} \leq \delta$$
 (16)

for $m' \neq m$.

The following is the biding property for active Alice; When Alice's operations A_1 and A_2 for Phases 1 and 2 satisfy the condition

$$\Pr\{m \text{ is accepted } | \text{ Alice performs } A_1, A_2, \text{ Bob performs } \mathcal{B}, \mathcal{T}\} \ge 1 - \delta$$
 (17)

with an element m, any Alice's operation A'_2 satisfies

Pr
$$\{m' \text{ is accepted } | \text{ Alice performs } A_1, A_2', \text{ Bob performs } \mathcal{B}, \mathcal{T}\} \leq \delta.$$
 (18)

By combining the correctness and the binding property, the set of the above conditions is called a δ -binding condition.

The above protocol is written as a combination of the four parts A_1, A_2, \mathcal{B} , and \mathcal{T} . The tuple $(A_1, A_2, \mathcal{B}, \mathcal{T})$ is called a protocol with n rounds and is denoted by \mathcal{P} . We denote the minimum value ε to satisfy the condition (14) (the condition (13)) under the protocol $\mathcal{P} = (A_1, A_2, \mathcal{B}, \mathcal{T})$ by $\varepsilon_a(\mathcal{P})$ ($\varepsilon_p(\mathcal{P})$). The value $\varepsilon_a(\mathcal{P})$ ($\varepsilon_p(\mathcal{P})$) depends only on Alice's operation A_1 in Phase 1 (Alice's operation A_1 and Bob's operation \mathcal{B} in Phase 1). The value $\varepsilon_a(\mathcal{P})$ ($\varepsilon_p(\mathcal{P})$) is called the active concealing parameter (the passive concealing parameter). We denote the minimum value δ to satisfy the conditions (15) and (16) under the protocol $\mathcal{P} = (A_1, A_2, \mathcal{B}, \mathcal{T})$ by $\delta_p(\mathcal{P})$, which is called the passive binding parameter. We denote the minimum value δ to satisfy the conditions (15), (17), and (18) under the protocol $\mathcal{P} = (A_1, A_2, \mathcal{B}, \mathcal{T})$ by $\delta_a(\mathcal{P})$, which is called the active binding parameter. Also, the value R is called the rate of the protocol \mathcal{P} and is denoted by $R(\mathcal{P})$.

B. Two subclass of protocols

Since it is not so easy to discuss a general protocol, we introduce the invertible condition for \mathcal{A}_1 as follows. That is, we introduce the class of invertible protocols as the first subclass. Alice's honest operation \mathcal{A}_1 in Phase 1 is called invertible when there exist set of TP-CP maps $\Lambda_{Y_1'V_1 \to U_1Y_1}, \Lambda_{Y_2'V_2 \to Y_1'U_2Y_2}, \ldots, \Lambda_{Y_n'V_n \to Y_{n-1}'U_nY_n}$ such that the relations

$$\Lambda_{Y_{1}'V_{1} \to U_{1}Y_{1}}(\rho_{Y_{1}'V_{1}}(m)) = \rho_{U_{1}Y_{1}}(m),
\Lambda_{Y_{2}'V_{2} \to Y_{1}'U_{2}Y_{2}}(\rho_{Y_{2}'V_{2}}(m)) = \rho_{Y_{1}'U_{2}Y_{2}}(m),
\vdots
\Lambda_{Y_{n}'V_{n} \to Y_{n-1}'U_{n}Y_{n}}(\rho_{Y_{n}'V_{n}}(m)) = \rho_{Y_{n-1}'U_{n}Y_{n}}(m)$$
(19)

hold for any m. When all densities W_x are commutative with each other, any protocol $\mathcal P$ is invertible. Next, as another subclass, we introduce the class of non-interactive protocols. When the classical communication for the variable U_i nor V_i is communicated in Phase 1, the protocol $\mathcal P=(\mathcal A_1,\mathcal A_2,\mathcal B,\mathcal T)$ by $\delta_p(\mathcal P)$ is called non-interactive. Clearly, the class of non-interactive protocols is included in the class of invertible protocols .

C. Asymptotic analysis

To study the asymptotic limitation of the performance, we focus on the rate R. The rate R is called achievable with active attack (passive attack) under the cq-channel W when there exists a sequence of protocols $\{\mathcal{P}_n\}_{n=1}^{\infty}$ such that \mathcal{P}_n is a protocol with n rounds, $R = \lim_{n \to \infty} R(\mathcal{P}_n)$, $\lim_{n \to \infty} \varepsilon_a(\mathcal{P}_n) = 0$ ($\lim_{n \to \infty} \varepsilon_p(\mathcal{P}_n) = 0$) and $\lim_{n \to \infty} \delta_a(\mathcal{P}_n) = 0$ ($\lim_{n \to \infty} \delta_p(\mathcal{P}_n) = 0$). The supremum of achievable rate under the cq-channel W with active attack (passive attack) is called the commitment capacity of W with active attack (passive attack) and is denoted by $C_a(W)$ ($C_p(W)$).

The rate R is called achievable with invertible protocols and with active attack (passive attack) under the cq-channel W when there exists a sequence of invertible protocols $\{\mathcal{P}_n\}_{n=1}^{\infty}$ such that \mathcal{P}_n is an invertible protocol with n rounds, $R = \lim_{n \to \infty} R(\mathcal{P}_n)$, and $\lim_{n \to \infty} \varepsilon_a(\mathcal{P}_n) = 0$ ($\lim_{n \to \infty} \varepsilon_p(\mathcal{P}_n) = 0$) and $\lim_{n \to \infty} \delta_a(\mathcal{P}_n) = 0$ ($\lim_{n \to \infty} \delta_p(\mathcal{P}_n) = 0$). The supremum of achievable rate with invertible protocols and with active attack (passive attack) under the cq-channel W is called the invertible commitment capacity with active attack (passive attack) of W and is denoted by $C_{a,inv}(W)$ ($C_{p,inv}(W)$). In the same way, we define an achievable rate with non-interactive protocols, and the non-interactive commitment capacity with active attack (passive attack) $C_{a,non}(W)$ ($C_{p,non}(W)$).

From these definitions, we have the following inequalities

$$C_{a}(\mathbf{W}) \leq C_{p}(\mathbf{W})$$

$$| \vee | \vee |$$

$$C_{a,inv}(\mathbf{W}) \leq C_{p,inv}(\mathbf{W})$$

$$| \vee | \vee |$$

$$C_{a,non}(\mathbf{W}) \leq C_{p,non}(\mathbf{W}).$$
(20)

When all densities W_x are commutative with each other, we have $C_a(\mathbf{W}) = C_{a,inv}(\mathbf{W})$ and $C_p(\mathbf{W}) =$ $C_{p,inv}(\boldsymbol{W}).$

Then, we have the following theorem.

Theorem 1: Assume Condition (NR). Then, we have the following relations;

$$C_{a,non}(\boldsymbol{W}) = C_{p,non}(\boldsymbol{W}) = C_{a,inv}(\boldsymbol{W}) = C_{p,inv}(\boldsymbol{W}) = \sup_{P \in \boldsymbol{P}(\mathcal{X})} H(X|Y)_{P}.$$
 (21)

This theorem is composed of two parts because of (20).

$$C_{a,non}(\boldsymbol{W}) \ge \sup_{P \in \boldsymbol{P}(\mathcal{X})} H(X|Y)_P \tag{22}$$

$$C_{a,non}(\mathbf{W}) \ge \sup_{P \in \mathbf{P}(\mathcal{X})} H(X|Y)_{P}$$

$$C_{p,inv}(\mathbf{W}) \le \sup_{P \in \mathbf{P}(\mathcal{X})} H(X|Y)_{P}.$$
(22)

That is, separating active and passive scenarios, we clarify what properties are used in the direct and converse parts in the above way.

IV. SYMMETRIC CHANNEL

A. Formulation

As a typical example of cq-channel, we consider symmetric channel. We consider a finite group \mathcal{G} as the input classical system \mathcal{X} , and a state ρ on the quantum system \mathcal{H}_Y . Also, we consider a unitary representation U of \mathcal{G} on $\mathcal{H}_Y[10]$. That is, for an element $g \in \mathcal{G}$, the unitary U_g is defined to satisfy the following conditions; $U_e = I$ and $U_g U_{g'} = U_{gg'}$, where $e \in \mathcal{G}$ is the unit element. When $U_e = I$ and there exists a complex number $e^{i\theta(g,g')}$ for $g,g' \in \mathcal{G}$ such that $U_g U_{g'} = e^{i\theta(g,g')} U_{gg'}$, the set of unitaries $\{U_g\}_{g \in \mathcal{G}}$ is called a projective unitary representation [10]. In the following, we assume that $\{U_g\}_{g \in \mathcal{G}}$ forms a projective unitary representation.

Then, we define the cq-channel as $W_g := U_g \rho U_g^{\dagger}$. This channel is called a symmetric channel. When $\{U_g\}_{g\in\mathcal{G}}$ forms a projective unitary representation, we have $W_{gg'}:=U_gU_{g'}\rho U_{g'}^{\dagger}U_g^{\dagger}$. Hence, we do not need to care the phase factor $e^{i\theta(g,g')}$ when we focus on the states $\{W_q\}_{q\in\mathcal{G}}$.

This channel with the commutative group was discussed in the reference [11, Section VII-A-2]. The paper [12] studied such a channel model in the context of resource theory of asymmetry in the pure state case. Recently, the papers [13], [14] addressed this type of channels in the context of dense coding and private dense coding. This class of cq-symmetric channels is a quantum generalization of a regular channel [15], which is a useful class of channels in classical information theory. This class of classical channels is often called generalized additive [16, Section V] or conditional additive [16, Section 4] and contains a class of additive channels as a subclass. Such a channel appears even in wireless communication by considering binary phase-shift keying (BPSK) modulations [17, Section 4.3]. Its most simple example is the binary symmetric channel (BSC).

In the above symmetric channel, we define the stabilizer $\mathcal{K} \subset \mathcal{G}$ as

$$\mathcal{K} := \{ g \in \mathcal{G} | W_g = W_e \}, \tag{24}$$

where $e \in \mathcal{G}$ expresses the unit element of the group \mathcal{G} .

Since an element of K output the same state as the unit element e, we consider the channel with the input system $\mathcal{X} := \mathcal{G}/K$ as $W_{[q]} := W_q$. We call this type of channel an induced symmetric channel.

Lemma 1: Any induced symmetric channel satisfies the Condition (NR).

Example 1: We consider the case when $\mathcal{G}=\mathbb{Z}_d$ and \mathcal{H} is spanned by $\{|j\rangle\}_{j=0}^{d-1}$. We define the representation $U_g:=\mathsf{Z}^g$, where $\mathsf{Z}:=\sum_{j=0}^{d-1}e^{2\pi ji/d}|j\rangle\langle j|$. We define $|\phi\rangle:=\sum_{j=0}^{d-1}a_j|j\rangle$ with $a_j\neq 0$ for $j=0,\ldots,d-1$. When we choose the state ρ to be $|\phi\rangle\langle\phi|$, the vectors $\{U_g|\phi\rangle\}_{g=0,\ldots,d-1}$ are linearly independent. Hence, we have $\mathcal{K}=\{0\}$.

Example 2: Next, we consider the case with d=pq in Example 1. We choose $|\psi\rangle:=\sum_{j=0}^{p-1}b_j|qj\rangle$ with $b_j\neq 0$ for $j=0,\ldots,p-1$. When we choose the state ρ to be $|\psi\rangle\langle\psi|$, the vectors $\{U_j|\psi\rangle\}_{j=0}^{p-1}$ are linearly independent and $U_j|\psi\rangle=U_{j+pk}|\psi\rangle$ for $j=0,\ldots,p-1$ and $k=0,\ldots,q-1$. Hence, we have $\mathcal{K}=\{pk\}_{k=0}^{q-1}$.

B. Calculation of commitment capacity

To calculate the commitment capacity $\sup_{P \in P(\mathcal{X})} H(X|Y)_P$ of the induced channel, we prepare the following lemma.

Lemma 2: The function $P_X \mapsto H(X|Y)_{P_X}$ is concave.

In addition, for the calculation of the quantity $\sup_{P \in \mathcal{P}(\mathcal{X})} H(X|Y)_P$, we prepare the following things. A projective unitary representation $\{U_g\}_{g \in \mathcal{G}}$ on the Hilbert space \mathcal{H} is called irreducible when the following condition holds; When a subspace \mathcal{H}' of \mathcal{H} satisfies the condition $U_g\mathcal{H}' = \mathcal{H}'$ for $g \in \mathcal{G}$, \mathcal{H}' is \mathcal{H} or 0. An example of irreducible projective representation is given in Example 5. When \mathcal{G} is a commutative group and $\{U_g\}_{g \in \mathcal{G}}$ is an irreducible unitary representation of \mathcal{G} on \mathcal{H} , the dimension of \mathcal{H} is 1.

We denote the set of the irreducible projective unitary representations by \mathcal{G} . For an element $\lambda \in \mathcal{G}$, we denote the corresponding representation space and the corresponding unitary representation by \mathcal{H}_{λ} and U^{λ} , respectively. We denote the dimension of \mathcal{H}_{λ} by d_{λ} . Generally, the representation space \mathcal{H}_{Y} can be written as

$$\mathcal{H}_Y = \bigoplus_{\lambda \in \hat{\mathcal{G}}} \mathcal{H}_\lambda \otimes \mathbb{C}^{n_\lambda},\tag{25}$$

where n_{λ} expresses the multiplicity of the irreducible unitary presentation U^{λ} .

Example 3: To see the multiplicity in the most simple example, we consider the case of commutative group $\mathcal{G} = \mathbb{Z}_d$ with \mathcal{H} spanned by $\{|j\rangle\}_{j=1}^n$. Assume that U_g is given as $\sum_{j=1}^n e^{i2\pi g/d} |j\rangle\langle j|$ for $g \in \mathbb{Z}_d$. In this case, $U_{g,j} := e^{i2\pi g/d} |j\rangle\langle j|$ is an irreducible representation on the one-dimensional space spanned by $|j\rangle$. The representation $U_{g,j}$ has the same structure as $U_{g,0}$ for any $j=1,\ldots,n$, where $U_{g,0}:=e^{i2\pi g/d}|0\rangle\langle 0|$. Hence, the representation $U_{g,j}$ is equivalent to the representation $U_{g,0}$. The representation U_g contains n irreducible representations equivalent to the representation $U_{g,0}$. Now, we consider the space spanned by $\{|0,j\rangle\}_{j=1}^n$, which equals $\mathcal{H}_0 \otimes \mathbb{C}^n$, where \mathcal{H}_0 is spanned by $|0\rangle$ and \mathbb{C}^n is spanned by $\{|j\rangle\}_{j=1}^n$. When the representation on $\mathcal{H}_0 \otimes \mathbb{C}^n$ is given as $U_{g,0} \otimes I$, this representation has the same structure as the above representation U_g . In this case, the dimension of the second space expresses the number of the same representation n, which is considered as the multiplicity. In addition, U_g is a constant times of the identity I, $\mathcal{K} = \mathcal{G}$.

When the representation space \mathcal{H}_Y does not contain a subspace that equivalent to the irreducible representation space \mathcal{H}_{λ} , n_{λ} is zero. We denote the projection to the space $\mathcal{H}_{\lambda} \otimes \mathbb{C}^{n_{\lambda}}$ by P_{λ} . We define the state ρ_{λ} on $\mathbb{C}^{n_{\lambda}}$ and the probability $p(\lambda)$ as

$$p(\lambda) := \operatorname{Tr} P_{\lambda} \rho, \quad \rho_{\lambda} := \frac{1}{p(\lambda)} \operatorname{Tr}_{\mathcal{H}_{\lambda}} P_{\lambda} \rho P_{\lambda}.$$
 (26)

Since the average state $\sum_{g \in \mathcal{G}} \frac{1}{|\mathcal{G}|} W_g$ is commutative with U_g for any $g \in \mathcal{G}$, Schur's lemma [10, Lemma 2.4] guarantees that it has the form $\bigoplus_{\lambda \in \hat{\mathcal{G}}} \frac{p(\lambda)}{d_{\lambda}} I_{\lambda} \otimes \sigma_{\lambda}$. Since σ_{λ} coincides with ρ_{λ} , we have

$$\sum_{g \in \mathcal{G}} \frac{1}{|\mathcal{G}|} W_g = \bigoplus_{\lambda \in \hat{\mathcal{G}}} \frac{p(\lambda)}{d_{\lambda}} I_{\lambda} \otimes \rho_{\lambda}, \tag{27}$$

which implies that

$$S\left(\sum_{g\in\mathcal{G}}\frac{1}{|\mathcal{G}|}W_g\right) = \sum_{\lambda\in\hat{\mathcal{G}}}p(\lambda)\left(S(\rho_\lambda) + \log\frac{d_\lambda}{p(\lambda)}\right). \tag{28}$$

Using (28) and Lemma 2, we can show the following lemma.

Lemma 3: For an induced symmetric channel, the uniform distribution achieves the commitment capacity. \Box

In the above lemma, we need to address the induced symmetric channel instead of the symmetric channel because the symmetric channel does not satisfy Condition (NR) unless $\mathcal{K} = \{e\}$.

We denote the uniform distribution on the set \mathcal{X} of inputs of the induced symmetric channel by $P_{\mathrm{uni},\mathcal{X}}$. Then, due to Lemma 3, the commitment capacity is calculated as

$$\sup_{P \in \boldsymbol{P}(\mathcal{X})} H(X|Y)_P = H(X|Y)_{P_{\text{uni},\mathcal{X}}} = \log|\mathcal{X}| + S(\rho) - \sum_{\lambda \in \hat{\mathcal{G}}} p(\lambda) \left(S(\rho_\lambda) + \log\frac{d_\lambda}{p(\lambda)}\right),\tag{29}$$

where the second equation follows from (28). In particular, when $\mathcal{K} = \{e\}$, the symmetric channel satisfies

$$\sup_{P \in \mathcal{P}(\mathcal{G})} H(X|Y)_P = \log |\mathcal{G}| + S(\rho) - \sum_{\lambda \in \hat{\mathcal{G}}} p(\lambda) \Big(S(\rho_\lambda) + \log \frac{d_\lambda}{p(\lambda)} \Big). \tag{30}$$

In the following, we consider several typical cases. When the projective representation U is an irreducible representation U^{λ} , we have

$$\sup_{P \in P(\mathcal{G})} H(X|Y)_P = \log |\mathcal{X}| + S(\rho) - \log d_{\lambda}. \tag{31}$$

When \mathcal{G} is a commutative group and U has no multiplicity, we have

$$\sup_{P \in \mathbf{P}(\mathcal{G})} H(X|Y)_P = \log|\mathcal{X}| + S(\rho) + \sum_{\lambda \in \hat{\mathcal{G}}} p(\lambda) \log p(\lambda)$$
(32)

because the dimension of irreducible representation is 1.

Example 4: Eq. (32) guarantees that the capacity of the model in Example 1 is $\log d + \sum_{j=0}^{d-1} |a_j|^2 \log |a_j|^2$. Also, due to (32), the capacity of the model in Example 2 is calculated to $\log p + \sum_{j=0}^{p-1} |b_j|^2 \log |b_j|^2$. \square Example 5: We apply our result to dense coding with a general state [18], [19]. For this aim, we consider \mathbb{Z}_d , \mathbb{Z}_d , and \mathbb{Z}_d in the same way as Example 1. We define the operator \mathbb{Z}_d : $\sup_{j=0}^{d-1} |j+1\rangle\langle j|$, where $|d\rangle = |0\rangle$. For $\mathcal{G} = \mathbb{Z}_d^2$, we define $U_{j,k} := \mathsf{X}^j \mathsf{Z}^k$. Since the relation $U_{j,k}U_{j',k'} = (e^{2\pi/d})^{j'k}U_{j+j',k+k'}$ holds, $\{U_{j,k}\}_{(j,k)\in\mathbb{Z}_d^2}$ forms a projective irreducible representation [10, Section 8.1.1]. The receiver has the system \mathcal{H}_B with the same dimension as the sender's system \mathcal{H} . Assume that the sender and the receiver share the n copies of a state ρ on the composite system $\mathcal{H} \otimes \mathcal{H}_B$. Then, the sender is allowed to apply one of $\{U_{j,k}\}_{(j,k)\in\mathbb{Z}_d^2}$ on the system \mathcal{H} and send it to the receiver as one use of the channel. In this situation, \mathcal{H} is the irreducible representation space [10, Chapter 8], and the space \mathcal{H}_B shows the multiplicity. We assume that ρ is not commutative with $U_{j,k}$ unless (j,k)=(0,0). Then, we find that $\mathcal{K}=\{(0,0)\}$. By using (30), the capacity is calculated to $\log d + S(\rho) - S(\rho_B)$.

C. Proofs

This subsection proves the lemmas stated in this section.

1) Proof of Lemma 1: We show Condition (NR) by contradiction. We define the set $\mathcal{X}_0 \subset \mathcal{X} = \mathcal{G}/\mathcal{K}$ as

$$\mathcal{X}_0 := \left\{ [g] \in \mathcal{X} \middle| W_g \text{ cannot be written as } \sum_{[g'] \in \mathcal{X} \setminus \{[g]\}} P([g']) W_{g'} \right\}. \tag{33}$$

We assume that Condition (NR) does not hold, i.e., $\mathcal{X}_0 \neq \mathcal{X}$. We choose an element $[g] \in \mathcal{X} \setminus \mathcal{X}_0$ and a distribution P on $\mathcal{X} \setminus \{[g]\}$ such that $W_g = \sum_{[g'] \in \mathcal{X}} P([g']) W_{g'}$. For an element $[g_0] \in \mathcal{X}_0$, we have

$$W_{g_0} = U_{g_0g^{-1}} W_g U_{g_0g^{-1}}^{\dagger} = \sum_{[g'] \in \mathcal{X}} P([g']) U_{g_0g^{-1}} W_{g'} U_{g_0g^{-1}}^{\dagger}$$

$$= \sum_{[g'] \in \mathcal{X}} P([g']) W_{g_0g^{-1}g'} = \sum_{[g'] \in \mathcal{X}} P([gg_0^{-1}g']) W_{g'}, \tag{34}$$

which implies the contradiction to the condition $[g_0] \in \mathcal{X}_0$. Hence, Condition (NR) holds.

2) Proof of Lemma 2: We consider the state $\rho = \lambda |0\rangle \langle 0|_Z \otimes \sum_{x \in \mathcal{X}} P_0(x) \otimes W_x + (1-\lambda)|1\rangle \langle 1|_Z \otimes \sum_{x \in \mathcal{X}} P_1(x) \otimes W_x$ on the system Z, X, Y. Then, we have

$$\lambda H(X|Y)_{P_0} + (1-\lambda)H(X|Y)_{P_1} = H(X|YZ)[\rho] \le H(X|Y)[\rho] = H(X|Y)_{\lambda P_0 + (1-\lambda)P_1}.$$
 (35)

3) Proof of Lemma 3: Since a unitary operation does not change the information quantity $H(X|Y)_P$, we have $H(X|Y)_{P_g} = H(X|Y)_P$, where $P_g(x) := P(gx)$ for $x \in \mathcal{X}$. Hence, Lemma 2 implies that

$$H(X|Y)_P \le H(X|Y)_{\sum_g P_g} = H(X|Y)_{P_{\text{uni}},\mathcal{X}}.$$
(36)

V. Converse Part

A. Proof of (23)

The converse part (23) follows from the following theorem.

Theorem 2: Given an invertible protocol $\mathcal{P}_n = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}, \mathcal{T})$, there exists a distribution P_X on \mathcal{X} such that

$$(1 - \varepsilon - 3\sqrt[3]{\delta})R \le H(X \mid Y)_{P_X} + \frac{1 + \eta(\varepsilon)}{n},\tag{37}$$

where $R = R(\mathcal{P}_n)$, $\delta = \delta_p(\mathcal{P}_n)$, $\varepsilon = \varepsilon_p(\mathcal{P}_n)$, and $\eta(\varepsilon) := (\varepsilon + 1) \log(\varepsilon + 1) - \varepsilon \log(\varepsilon)$.

To show Theorem 2, we prepare the following proposition and the following lemma. In these statements, we consider various information quantities on the state ρ_2 after Bob's operation of Phase 2. The state ρ_2 has the random variables M, X^n, U^n, V^n, Z and the quantum system Y'_n . Hence, omit the symbol added to information quantities, $[\rho_2]$. That is, $I(M; U^n V^n Y'_n)[\rho_2]$ is simplified to $I(M; U^n V^n Y'_n)$.

Proposition 1: We consider a protocol $\mathcal{P}_n = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}, \mathcal{T})$, with $R = R(\mathcal{P}_n)$ and $\delta = \delta_p(\mathcal{P}_n)$, and assume that M is chosen uniformly from $[1:2^{nR}]$. There exists a function $h(X^nU^nV^n)$ such that the relation

$$\Pr\{M \neq h(X^n U^n V^n)\} < 3\sqrt[3]{\delta} \tag{38}$$

holds under A_1 , B. Notice that the random variables M, X^n, U^n, V^n are defined at the end of Phase 1 and they are defined with the state ρ_2 .

Lemma 4: Suppose Alice and Bob follow the protocol mentioned in section III-A such that two different messages $m \neq m'$ satisfy

$$\frac{1}{2} \left\| W_{\mathcal{B}, A_1(m)}^{U^n V^n Y_n'} - W_{\mathcal{B}, A_1(m')}^{U^n V^n Y_n'} \right\|_{1} \le \varepsilon. \tag{39}$$

Then,

$$I(M; U^n V^n Y_n') \le n\varepsilon R + \eta(\varepsilon), \tag{40}$$

where
$$\eta(\varepsilon) := (\varepsilon + 1) \log(\varepsilon + 1) - \varepsilon \log(\varepsilon)$$
.

Proof of Lemma 4: Since $W_{\mathcal{B},\mathcal{A}_1}^{U^nV^nY_n'} = \sum_{m'=1}^{2^{nR}} \frac{1}{2^{nR}} W_{\mathcal{B},A_1(m')}^{U^nV^nY_n'}$, the application of the triangle inequality implies

$$\frac{1}{2} \|W_{\mathcal{B}, A_1(m)}^{U^n V^n Y_n'} - W_{\mathcal{B}, A_1}^{U^n V^n Y_n'}\|_1 \le \sum_{m'=1}^{2^{nR}} \frac{1}{2 \cdot 2^{nR}} \|W_{\mathcal{B}, A_1(m)}^{U^n V^n Y_n'} - W_{\mathcal{B}, A_1(m')}^{U^n V^n Y_n'}\|_1 \le \varepsilon, \tag{41}$$

where the second inequality follows from (39). Then, since

$$I(M; U^{n}V^{n}Y'_{n}) = S(W_{\mathcal{B}, \mathcal{A}_{1}}^{U^{n}V^{n}Y'_{n}}) - \sum_{m=1}^{2^{nR}} \frac{1}{2^{nR}} S(W_{\mathcal{B}, \mathcal{A}_{1}(m)}^{U^{n}V^{n}Y'_{n}}),$$

(40) follows from the continuity property of von Neumann entropy, i.e., Fannes inequality [22], [23, Theorem 5.12].

Now, we prove Theorem 2 by using Proposition 1 and Lemmas 4 and 2.

Proof of Theorem 2: Applying Fano's inequality to Proposition 1, we have the relations

$$H(M \mid X^n U^n V^n Y_n') \le H(M \mid X^n U^n V^n) \le 1 + 3n\sqrt[3]{\delta}R.$$
 (42)

Then, we have

$$H(X^{n} \mid U^{n}V^{n}Y'_{n})$$

$$= H(MX^{n} \mid U^{n}V^{n}Y'_{n}) - H(M \mid X^{n}U^{n}V^{n}Y'_{n})$$

$$\geq H(M \mid U^{n}V^{n}Y'_{n}) - H(M \mid X^{n}U^{n}V^{n}Y'_{n})$$

$$\stackrel{(a)}{\geq} H(M \mid U^{n}V^{n}Y'_{n}) - 1 - 3n\sqrt[3]{\delta}R$$

$$= H(M) - I(M; U^{n}V^{n}Y'_{n}) - 1 - 3n\sqrt[3]{\delta}R$$

$$\stackrel{(b)}{\geq} H(M) - n\varepsilon R - \eta(\varepsilon) - 1 - 3n\sqrt[3]{\delta}R$$

$$= nR(1 - \varepsilon - 3\sqrt[3]{\delta}) - (1 + \eta(\varepsilon)), \tag{43}$$

where (a) follows from (42), and (b) follows from Lemma 4.

Next, we consider the following virtual operation on the state ρ_2 . We consider the state $\rho_3 := \Lambda_{Y_1'V_1 \to U_1Y_1} \circ \Lambda_{Y_2'V_2 \to Y_1'U_2Y_2} \circ \ldots \circ \Lambda_{Y_n'V_n \to Y_{n-1}'U_nY_n}(\rho_2)$. Under the state ρ_3 , Bob's system is composed of quantum system Y^n and the classical variable U^n . Then, we have

$$H(X^n \mid Y^n)[\rho_3] \ge H(X^n \mid Y^n U^n)[\rho_3].$$
 (44)

Also, we have

$$H(X^{n} | Y^{n})[\rho_{3}]$$

$$= \sum_{i=1}^{n} H(X_{i}|X^{i-1}Y^{n})[\rho_{3}]$$

$$\leq \sum_{i=1}^{n} H(X_{i} | Y_{i})[\rho_{3}]$$

$$= \sum_{i=1}^{n} H(X|Y)_{P_{X_{i}}}$$

$$\leq nH(X|Y)_{\sum_{i=1}^{n} \frac{1}{n}P_{X_{i}}}.$$
(45)

where the last inequality follows from Lemma 2. Since the operation $\Lambda_{Y_1'V_1 \to U_1Y_1} \circ \Lambda_{Y_2'V_2 \to Y_1'U_2Y_2} \circ \ldots \circ \Lambda_{Y_n'V_n \to Y_{n-1}'U_nY_n}$ is invertible, Bob's information under the state ρ_3 is equivalent to Bob's information under the state ρ_2 . Hence, we have

$$H(X^n \mid Y^n U^n)[\rho_3] = H(X^n \mid U^n V^n Y_n')[\rho_2]. \tag{46}$$

Therefore, we have

$$nH(X|Y)_{\sum_{i=1}^{n} \frac{1}{n} P_{X_{i}}}$$

$$\stackrel{(a)}{\geq} H(X^{n} \mid Y^{n})[\rho_{3}]$$

$$\stackrel{(b)}{\geq} H(X^{n} \mid Y^{n}U^{n})[\rho_{3}]$$

$$\stackrel{(c)}{=} H(X^{n} \mid U^{n}V^{n}Y'_{n})$$

$$\stackrel{(d)}{\geq} nR(1 - \varepsilon - 3\sqrt[3]{\delta}) - (1 + \eta(\varepsilon)),$$

$$(47)$$

where Steps (a), (b), (c), and (d) follow from (45), (44), (46), and (43), respectively. Thus, we obtain (37).

B. Proof of Proposition 1

Outline: In this proof, we discuss $\Pr\{M \neq h(X^nU^nV^n)\}$ when Alice behaves honestly in Phase 1 as \mathcal{A}_1 and Bob behaves honestly in Phase 1 as \mathcal{B} . Hence, we omit the symbol \mathcal{B} in the state description. We construct the required function $h(X^nU^nV^n)$ by following a sequence of steps. Here we give a brief outline. For every m, we first show the existence of a set which we call as $\operatorname{Good}(m) \subseteq \mathcal{Z}$, where for every $z \in \operatorname{Good}(m)$, the pair (m,z) has the property that the test $\mathcal{T} = \{T_{mz}, I - T_{mz}\}$ accepts the state $W_{\mathcal{A}_1(m),Z=z}^{U^nV^nY'_n}$ with high probability. We then define functions $F(\cdot \mid m)$ and $\overline{F}(\cdot \mid m)$. Using these functions we define the function $h(\cdot)$. We then invoke the properties of the set $\operatorname{Good}(m)$ and the functions $F(\cdot \mid m)$ and $\overline{F}(\cdot \mid m)$ to arrive at (38).

Step 1: The aim of this step is to define the set Good(m) and deriving its properties. Towards this, let $f(m,z) := \operatorname{Tr} W_{\mathcal{A}_1(m),Z=z}^{U^nV^nY_n'} T_{mz}$. The correctness condition implies

$$\sum_{z} P_{Z|M=m}(z) f(m,z)$$
=Pr $\{m \text{ is accepted } | \text{ Alice performs } A_1(m), A_2(m), \text{ Bob performs } \mathcal{B}, \mathcal{T}\}$
>1 - δ . (48)

For each m, define the set Good(m) as follows:

Good
$$(m) := \{z | f(m, z) > 1 - \sqrt[3]{\delta} \}.$$
 (49)

The existence of Good(m) follows from the following set of inequalities:

$$\Pr \{Good(m)\}
= \Pr \{f(Z,m) > 1 - \sqrt[3]{\delta}\}
= 1 - \Pr \{f(Z,m) \le 1 - \sqrt[3]{\delta}\}
= 1 - \Pr \{1 - f(Z,m) \ge \sqrt[3]{\delta}\}
\stackrel{(a)}{\ge} 1 - \frac{\mathbb{E}_{Z|M=m}[1 - f(Z,m)]}{\sqrt[3]{\delta}}
\stackrel{(b)}{\ge} 1 - \sqrt[3]{\delta^2},$$
(50)

where (a) follows from Markov inequality and (b) follows from (48). The relation (50) guarantees that Good(m) is a non-empty set.

Step 2: The aim of Step 2 is introducing the functions $F(x^n, u^n, v^n|mz)$, $F(x^n, u^n, v^n|m)$ and deriving their properties. Since the state on U^n, V^n, Y'_n is determined with $X^n = x^n, Z = z$, we use the notation, $W_{X^n=x^n,Z=z}^{U^n=u^n,V^n=v^n,Y_n'}$. Using the operator;

$$W_{U^{n}=u^{n},V^{n}=v^{n},X^{n}=x^{n},Z=z}^{Y''} := \frac{1}{\operatorname{Tr} W_{X^{n}=x^{n},Z=z}^{U^{n}=u^{n},V^{n}=v^{n},Y'_{n}}} W_{X^{n}=x^{n},Z=z}^{U^{n}=u^{n},V^{n}=v^{n},Y'_{n}},$$
(51)

we define the functions

$$F(x^{n}, u^{n}, v^{n}|mz) := \text{Tr}(W_{U^{n}=u^{n}, V^{n}=v^{n}, X^{n}=x^{n}, Z=z}^{Y'_{n}} \otimes |u^{n}, v^{n}\rangle\langle u^{n}, v^{n}|)T_{mz}$$
(52)

$$F(x^{n}, u^{n}, v^{n}|mz) := \text{Tr}(W_{U^{n}=u^{n}, V^{n}=v^{n}, X^{n}=x^{n}, Z=z}^{Y'_{n}} \otimes |u^{n}, v^{n}\rangle\langle u^{n}, v^{n}|)T_{mz}$$

$$F(x^{n}, u^{n}, v^{n}|m) := \max_{z \in \text{Good}(m)} F(x^{n}, u^{n}, v^{n}|mz).$$
(52)

As shown in Step 6, we have

$$\mathbb{E}_{X^n U^n V^n | M = m, Z = z} F(X^n U^n V^n | m'z') = \text{Tr } W_{A_1(m), Z = z}^{U^n, V^n, Y_n'} T_{m'z'}$$
(54)

for m, m', z, z'. Then, we have

$$\mathbb{E}_{X^n U^n V^n | M = m, Z = z} F(X^n U^n V^n | mz) = f(m, z).$$
(55)

Therefore, we have

$$\mathbb{E}_{X^{n}U^{n}V^{n}|M=m}F(X^{n}U^{n}V^{n}|m)$$

$$\geq \sum_{z\in Good(m)} P_{Z|M=m}(z)\mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}F(X^{n}U^{n}V^{n}|m)$$

$$\stackrel{(a)}{\geq} \sum_{z\in Good(m)} P_{Z|M=m}(z)\mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}F(X^{n}U^{n}V^{n}|mz)$$

$$\stackrel{(b)}{\equiv} \sum_{z\in Good(m)} P_{Z|M=m}(z)f(m,z)$$

$$\stackrel{(c)}{\geq} \sum_{z\in Good(m)} P_{Z|M=m}(z)(1-\sqrt[3]{\delta})$$

$$\stackrel{(d)}{\geq} (1-\sqrt[3]{\delta^{2}})(1-\sqrt[3]{\delta})$$

$$> 1-2\sqrt[3]{\delta}, \qquad (56)$$

where Steps (a), (b), (c), and (d) follow from (53), (55), (49), and (50), respectively.

Step 3: The aim of Step 3 is introducing the function $\bar{F}(x^n, u^n, v^n|m)$, and deriving its property. We define $\bar{F}(x^n, u^n, v^n|m) := \max_{m' \neq m} F(x^n, u^n, v^n|m')$. Then, as shown below, we have

$$\mathbb{E}_{X^n U^n V^n | M = m} \bar{F}(X^n, U^n, V^n | m) < \delta.$$
(57)

To show (57), we choose m' and $z' \in \text{Good}(m)$ as $\bar{F}(X^nU^nV^n|m) = F(X^nU^nV^n|m')$ and $F(X^nU^nV^n|m') = F(X^nU^nV^n|m'z')$. We define A_2' as Alice's dishonest operation in Phase 2 to send m'z' instead of mz. Hence, we have

$$\mathbb{E}_{X^{n}U^{n}V^{n}|M=m}\bar{F}(X^{n}U^{n}V^{n}|m) \stackrel{(a)}{=} \mathbb{E}_{X^{n}U^{n}V^{n}|M=m}F(X^{n}U^{n}V^{n}|m')$$

$$= \sum_{z} P_{Z|M=m}(z)\mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}F(X^{n}U^{n}V^{n}|m')$$

$$\stackrel{(b)}{=} \sum_{z} P_{Z|M=m}(z)\mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}F(X^{n}U^{n}V^{n}|m'z')$$

$$\stackrel{(c)}{=} \sum_{z} P_{Z|M=m}(z)\operatorname{Tr} W_{\mathcal{A}_{1}(m),Z=z}^{U^{n},V^{n},Y_{n}'}T_{m'z'}$$

$$\stackrel{(d)}{=} \Pr\{m' \text{ is accepted} \mid \text{Alice performs } A_1(m), A_2', \text{ Bob performs } \mathcal{B}, \mathcal{T}\} \stackrel{(e)}{\leq} \delta, \tag{58}$$

where Steps (a), (b), (c), (d), and (e), follow from the choice of m', the choice of z', (54), the definition of A'_2 , and (16), respectively.

Step 4: The aim of Step 4 is introducing the function $\hat{M}(x^n, u^n, v^n)$, and deriving its property. We define a function $\hat{M}(x^n, u^n, v^n)$ as follows.

$$h(x^n, u^n, v^n) := \underset{m}{\operatorname{argmax}} F(x^n, u^n, v^n | m).$$
 (59)

When m, x^n, u^n, v^n satisfies the condition, $m \neq \hat{M}(x^n, u^n, v^n)$, (59) guarantees that

$$F(x^{n}, u^{n}, v^{n}|m) < \max_{m'} F(x^{n}, u^{n}, v^{n}|m'),$$
(60)

which implies that

$$\bar{F}(x^n, u^n, v^n | m) = \max_{m'} F(x^n, u^n, v^n | m').$$
(61)

Therefore, we have $F(x^n, u^n, v^n|m) \leq \bar{F}(x^n, u^n, v^n|m)$, which implies that

$$1 \le \bar{F}(x^n, u^n, v^n | m) + (1 - F(x^n, u^n, v^n | m)).$$

Hence, we have

$$I[h(X^n, U^n, V^n) \neq m] \leq \bar{F}(x^n, u^n, v^n|m) + (1 - F(x^n, u^n, v^n|m)), \tag{62}$$

where the function $I[h(X^n, U^n, V^n) \neq m]$ is defined as

$$I[h(X^n,U^n,V^n)\neq m]:=\left\{\begin{array}{ll} 1 & \text{ when } h(X^n,U^n,V^n)\neq m;\\ 0 & \text{ when } h(X^n,U^n,V^n)=m. \end{array}\right.$$

Step 5: The aim of Step 5 is showing the desired relation (38). We have

$$\Pr\{h(X^{n}, U^{n}, V^{n}) \neq m\}
= \mathbb{E}_{X^{n}U^{n}V^{n}|M=m}I[h(X^{n}, U^{n}, V^{n}) \neq m]
\leq \mathbb{E}_{X^{n}U^{n}V^{n}|M=m}(\bar{F}(X^{n}, U^{n}, V^{n}|m) + (1 - F(X^{n}, U^{n}, V^{n}|m)))
= \mathbb{E}_{X^{n}U^{n}V^{n}|M=m}(\bar{F}(X^{n}, U^{n}, V^{n}|m)) + \mathbb{E}_{X^{n}U^{n}V^{n}|M=m}(1 - F(X^{n}, U^{n}, V^{n}|m)))
\stackrel{(b)}{<} \delta + 2\sqrt[3]{\delta},$$
(63)

where Steps (a) and (b) follow from (62) and the combination of (56) and (57), respectively.

Hence, we have

$$\Pr(M \neq h(X^n U^n V^n)) = \mathbb{E}_M \Pr\{h(X^n U^n V^n) \neq m\} < \delta + 2\sqrt[3]{\delta} \le 3\sqrt[3]{\delta},\tag{64}$$

which proves the desired statement (38).

Step 6: The claim in (54) can be shown as follows.

$$\mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}F(X^{n}U^{n}V^{n}|m'z') \\
= \mathbb{E}_{X^{n}U^{n}V^{n}|M=m,Z=z}\operatorname{Tr}(W_{U^{n},V^{n},X^{n},Z=z}^{Y_{n}}\otimes|U^{n},V^{n}\rangle\langle U^{n},V^{n}|)T_{m'z'} \\
= \mathbb{E}_{X^{n}|M=m,Z=z}\sum_{u^{n},v^{n}}\operatorname{Tr}W_{X^{n},Z=z}^{U^{n}=u^{n},V^{n}=v^{n},Y_{n}'}\operatorname{Tr}(W_{u^{n},v^{n},X^{n},Z=z}^{Y_{n}'}\otimes|u^{n},v^{n}\rangle\langle u^{n},v^{n}|)T_{m'z'} \\
= \mathbb{E}_{X^{n}|M=m,Z=z}\sum_{u^{n},v^{n}}\operatorname{Tr}(W_{X^{n},Z=z}^{U^{n}=u^{n},V^{n}=v^{n},Y_{n}'}\otimes|u^{n},v^{n}\rangle\langle u^{n},v^{n}|)T_{m'z'} \\
= \mathbb{E}_{X^{n}|M=m,Z=z}\operatorname{Tr}W_{A_{1}(m),X^{n},Z=z}^{U^{n},V^{n},Y_{n}'}T_{m'z'} \\
= \operatorname{Tr}W_{A_{1}(m),Z=z}^{U^{n},V^{n},Y_{n}'}T_{m'z'}.$$
(65)

C. Relation to existing converse part analyses

The proof partially follows techniques similar to those used in the papers[5], [6], [7] because these studies used a statement similar to Proposition 1. However, our Proposition 1 is different from the the corresponding statement in the papers [5], [6], [7]. In Proposition 1, the estimate of the message M is given as the function h of $X^nU^nV^n$. That is, the channel outputs (Y_1, \ldots, Y_n) are not the input variables of our function h because they do not exist in Phase 2 (Reveal phase) in our quantum setting. In contrast, the papers [5], [6], [7] used the variables U^nV^n and (Y_1, \ldots, Y_n) as the inputs of the estimate of the message M because the channel outputs (Y_1, \ldots, Y_n) exists even in Phase 2 (Reveal phase) in the classical setting. Due to the above reason, we need to invent an estimation function h different from their method.

The paper [8] also considered the converse part of in the classical setting only with non-interactive protocols. However, to derive the converse part, the paper [8] assumes that Bob can recover the original message M only with the received information via noisy channel and Z. That is, the paper [8] did not prove a statement corresponding to Proposition 1. In fact, if we show Proposition 1, this method works for the converse part of non-interactive protocols, i.e, $C_{p,non}(\mathbf{W})$. In this case, the converse part can be shown by the application of wiretap channel to the case when the main channel is the noiseless communication from Alice to Bob and the wiretap channel is the channel to the output of which is accessible to Bob in Phase 1. In addition, even when Proposition 1 is employed, the simple wiretap scenario does not work in interactive setting because the side information V^n, U^n cannot be handled in the simple wiretap scenario.

VI. DIRECT PART

A. Coding-theoretic formulation for non-interactive protocol

To study the performance of non-interactive protocol, we formulate a code for a cq-channel W. A map ϕ from $\mathcal{M} \times \mathcal{L}$ to \mathcal{X} is called a encoder, where $\mathcal{M} := \{1, \ldots, M\}$ and $\mathcal{L} := \{1, \ldots, L\}$. When Alice's message is $M \in \mathcal{M}$, she selects $L \in \mathcal{L}$ according to the uniform distribution and sends $\phi(M, L)$ via a cq channel W. Bob's verifier is $D = \{\mathcal{D}_{m,l}\}_{(m,l)\in\mathcal{M}\times\mathcal{L}}$, where $0 \leq \mathcal{D}_{m,l} \leq I$. A pair (ϕ, D) of an encoder and a verifier is called a code.

We introduce the parameters (A) – (C) for an encoder ϕ and a verifier $D = \{\mathcal{D}_{m,l}\}_{(m,l) \in \mathcal{M} \times \mathcal{L}}$ as follows. (A) Verifiable condition.

$$\varepsilon_A(\phi, D) := \max_{(m,l) \in \mathcal{M} \times \mathcal{L}} \varepsilon_{A,m,l}(\phi(m,l), D)$$
(66)

$$\varepsilon_{A,m,l}(x,D) := 1 - \text{Tr}[W_x \mathcal{D}_{m,l}]. \tag{67}$$

(B) Concealing condition

$$\delta_B(\phi) := \max_{m,m' \in \mathcal{M}} \left\| \sum_{l=1}^{L} \frac{1}{L} W_{\phi(m,l)} - \sum_{l'=1}^{L} \frac{1}{L} W_{\phi(m',l')} \right\|_1.$$
 (68)

(C) Binding condition. For $x \in \mathcal{X}$, we define the quantity $\delta_{C,x}(D)$ as the second largest value among $\{(1-\varepsilon_{A,m,l}(x,D))\}_{(m,l)\in M\times L}$. Then, we define

$$\delta_C(D) := \max_{x \in \mathcal{X}} \delta_{C,x}(D). \tag{69}$$

For a code (ϕ, D) , we define two numbers $|(\phi, D)|_1 := M$ and $|(\phi, D)|_2 := L$.

To construct a non-interactive protocol, we consider n use of the cq-channel, which is written as a cq-channel $\mathbf{W}^n := \{W_{x^n}^{(n)}\}_{x^n \in \mathcal{X}^n}$, where

$$W_{x^n}^{(n)} := W_{x_1} \otimes \dots \otimes W_{x_n} \tag{70}$$

with $x^n = (x_1, \dots, x_n)$. Given a code (ϕ_n, D_n) for the cq-channel W^n , we construct a non-interactive protocol with n use of the channel W as follows. In Phase 1, Alice chooses the random variable Z as the uniform random variable $L \in \mathcal{L}$. Given the message M, Alice chooses X^n to be $\phi_n(M, L)$, and sends it to Bob via the cq-channel W^n . Bob receives the state $W_{X^n}^n$. In Phase 2, Alice sends M and L to Bob. Bob applies the measurement $\{\mathcal{D}_{M,L}, I - \mathcal{D}_{M,L}\}$. When Bob's outcome corresponds to $\mathcal{D}_{M,L}$, he accepts the message M. Otherwise, he rejects it. This protocol accomplishes commitment instead of secrecy. We denote the above non-interactive protocol by $\mathcal{P}(\phi_n, D_n)$. Remember that the active concealing parameter $\varepsilon_a(\mathcal{P})$ and the active binding parameter $\delta_a(\mathcal{P})$ are defined for a protocol \mathcal{P} in the end of Section III-A. Then, we have the following lemma.

Lemma 5: The relations

$$\varepsilon_a(\mathcal{P}(\phi_n, D_n)) = \delta_B(\phi_n) \tag{71}$$

$$\delta_a(\mathcal{P}(\phi_n, D_n)) = \max(\varepsilon_A(\phi_n, D_n), \delta_C(D_n)$$
(72)

hold.

Proof: Since the quantity $\delta_B(\phi_n)$ is defined by (68), the condition (14) holds by replacing ε by $\delta_B(\phi_n)$. Hence, we have (71).

Since the quantity $\varepsilon_A(\phi_n, D_n)$ is defined by (66), the condition (17) holds by replacing δ by $\varepsilon_A(\phi_n, D_n)$. Since the quantity $\delta_C(D_n)$ is defined by (69), the condition (18) holds by replacing δ by $\delta_C(D_n)$. Hence, we have (72).

Therefore, to make a non-interactive protocol, it is sufficient to make the above type of code.

To construct a code, we introduce a pre-encoder and a pre-verifier, which are useful for this construction. A map ϕ from $\tilde{\mathcal{M}} := \{1, \dots, \tilde{\mathsf{M}}\}$ to \mathcal{X} is called a pre-encoder. We define $|\phi| := \tilde{\mathsf{M}}$. Bob's verifier is $D = \{\mathcal{D}_m\}_{m \in \tilde{\mathcal{M}}}$, where $0 \leq \mathcal{D}_m \leq I$. We introduce the conditions (a) ,(b α), and (c) for an pre-encoder ϕ and a pre-verifier $D = \{\mathcal{D}_m\}_{m \in \tilde{\mathcal{M}}}$ as follows.

(a) Verifiable condition.

$$\varepsilon_{A}(\phi, D) := \max_{m \in \tilde{\mathcal{M}}} \varepsilon_{A,m}(\phi(m), D) \le \varepsilon_{A}$$

$$\varepsilon_{A,m}(x, D) := 1 - \text{Tr}[W_{x}\mathcal{D}_{m}].$$
(73)

$$\varepsilon_{A,m}(x,D) := 1 - \text{Tr}[W_x \mathcal{D}_m]. \tag{74}$$

(b α) Rényi equivocation type of concealing condition of order $\alpha > 1$.

$$E_{\alpha}(\phi) := \log \tilde{\mathsf{M}} - \min_{\sigma \in \mathcal{S}(\mathcal{H}_Y)} \frac{1}{\alpha - 1} \log \sum_{m=1}^{\tilde{\mathsf{M}}} \frac{1}{\tilde{\mathsf{M}}} 2^{(\alpha - 1)\tilde{D}_{\alpha}(W_{\phi(m)} \| \sigma)}. \tag{75}$$

(c) Binding condition. For $x \in \mathcal{X}$, we define the quantity $\delta_{C,x}(D)$ as the second largest value among $\{(1 - \varepsilon_{A,m}(x,C))\}_{m \in \mathcal{M}}$. We define

$$\delta_C(D) := \max_{x \in \mathcal{X}} \delta_{C,x}(D). \tag{76}$$

We can easily show that

$$\tilde{H}_{\alpha}(\tilde{M}|Y) = E_{\alpha}(\phi). \tag{77}$$

B. Asymptotic analysis for non-interactive protocol

Now, we show the inequality (22), i.e., the existence of the non-interactive protocol to achieve the rate $\sup_{P_X} H(X|Y)_{P_X}$. For this aim, we discuss a sequence of codes $\{(\phi_n, D_n)\}$. We say that a sequence of codes $\{(\phi_n, D_n)\}$ is secure when $\varepsilon_A(\phi_n, D_n) \to 0$, $\delta_B(\phi_n) \to 0$, and $\delta_C(D_n) \to 0$. Then, we have the following theorem.

Theorem 3: Assume Condition (NR). For any distribution $P \in \mathbf{P}(\mathcal{X})$, there exists a secure sequence codes $\{(\phi_n, D_n)\}$ with $\mathsf{M}_n := |(\phi_n, D_n)|_1 = 2^{nR_1}$ and $\mathsf{L}_n := |(\phi_n, D_n)|_2 = 2^{nR_2}$ when there exists a distribution P on \mathcal{X} such that

$$R_1 + R_2 < H(X)_P, \ R_2 > I(X;Y)_P.$$
 (78)

Therefore, there exists the above type of a code when there exists a distribution P on \mathcal{X} such that $R_1 < H(X|Y)_P$. The combination of this fact and Lemma 5 yields (22). That is, for the direct part (22), it is sufficient to show Theorem 3.

To show Theorem 3, we discuss a sequence of pre-codes $\{(\phi_n, D_n)\}$. We say that a sequence of pre-codes $\{(\phi_n, D_n)\}$ is (α, r_α) -secure when $\varepsilon_A(\phi_n, D_n) \to 0$, $\delta_C(D_n) \to 0$, and $\lim_{n \to \infty} \frac{1}{n} E_\alpha(\phi_n) \ge r_\alpha$ for $\alpha > 1$.

Theorem 4: Assume Condition (NR). For any distribution $P \in \mathcal{P}(\mathcal{X})$, there exists a (α, r_{α}) -secure sequence of pre-codes $\{(\phi_n, D_n)\}$ with $\tilde{\mathsf{M}}_n := |(\phi_n, D_n)| = 2^{\lfloor nR_1 \rfloor + \lfloor nR_2 \rfloor}$ when there exists a distribution P on \mathcal{X} such that

$$R_1 + R_2 < H(X)_P, \ r_\alpha = R_1 + R_2 - \tilde{I}_\alpha(X;Y)_P.$$
 (79)

C. Proof of Theorem 3

Here, we show Theorem 3 by using Theorem 4. Given R_1, R_2 that satisfies the condition (78), we define $\mathcal{M}_n := \mathbb{F}_2^{\lfloor nR_1 \rfloor} \mathcal{L}_n := \mathbb{F}_2^{\lfloor nR_2 \rfloor}$. Using Theorem 4, we choose a pre-code $(\tilde{\phi}_n, D_n)$, and the set $\tilde{\mathcal{M}}_n$ is identified with $\mathbb{F}_2^{\lfloor nR_1 \rfloor + \lfloor nR_2 \rfloor}$.

We denote the projection from $\mathcal{M}_n \oplus \mathcal{L}_n$ to \mathcal{M}_n by P. We randomly choose an invertible linear map F from $\tilde{\mathcal{M}}_n$ to $\mathcal{M}_n \oplus \mathcal{L}_n$ such that $P \circ F$ satisfies the universal2 hash condition (see [24], [25] for more details on universal2 hash functions).

Then, there exists a liner invertible function f from $\tilde{\mathcal{M}}_n$ to $\mathcal{M}_n \oplus \mathcal{L}_n$ such that

$$\|\rho_{P \circ f(\tilde{M}),Y} - \rho_{mix,M} \otimes \rho_E\|_1 \le 2^{\frac{2}{\alpha} - 1 + \frac{\alpha - 1}{\alpha} (\log |\mathcal{B}| - E_\alpha(\tilde{\phi}))}$$

$$\tag{80}$$

for $\alpha \in (1,2]$, where $\rho_{\tilde{M},Y} := \sum_{\tilde{m} \in \tilde{\mathcal{M}}_n} \frac{1}{|\tilde{\mathcal{M}}_n|} |\tilde{m}\rangle \langle \tilde{m}| \otimes W_{\tilde{\phi}_n(\tilde{m})}^{(n)}$. The inequality in (80) follows because of the Proposition 2 mentioned below at the end of this subsection. We define $\phi_n(m,l) := \tilde{\phi}_n(f^{-1}(m,l))$. We have

$$\rho_{P \circ f(\tilde{M}), Y} = \sum_{m \in \mathcal{M}_n} \frac{1}{|\mathcal{M}_n|} |m\rangle \langle m| \otimes \sum_{l \in \mathcal{L}} \frac{1}{|\mathcal{L}|} W_{\phi_n(m, l)}^{(n)}.$$
(81)

Hence,

$$\delta_B(\phi_n) = \|\rho_{P \circ f(\tilde{M}),Y} - \rho_{mix,M} \otimes \rho_Y\|_1. \tag{82}$$

Since r_a satisfies the second condition in (79), when α is close to 1, we have

$$\lim_{n \to \infty} \frac{\log |\mathcal{B}| - E_{\alpha}(\tilde{\phi})}{n} = R_1 - r_{\alpha} = R_1 - (R_1 + R_2) + \tilde{I}_{\alpha}(X;Y)_Y = \tilde{I}_{\alpha}(X;Y)_Y - R_2 < 0.$$
 (83)

The combination of (83), (82), and (80) shows that $\delta_B(\phi_n) \to 0$. Other two conditions $\varepsilon_A(\phi_n, D_n) \to 0$ and $\delta_C(D_n) \to 0$ follow from Theorem 4.

Proposition 2 ([26][27]): Let G be a universal hash function from A to B Then, we have

$$\mathbb{E}_{G} \| \rho_{G(A)E} - \rho_{mix,B} \otimes \rho_{E} \|_{1} \leq 2^{\frac{2}{\alpha} - 1 + \frac{\alpha - 1}{\alpha} (\log |\mathcal{B}| - \tilde{H}_{\alpha}(A|E))}$$
(84)

for
$$\alpha \in (1,2]$$
.

D. Outline of proof of Theorem 4

Here, we present the outline of Theorem 4. To realize Binding condition (c), we need to exclude the existence of $x^n \in \mathcal{X}^n$ and $m \neq m' \in \tilde{\mathcal{M}}_n$ such that $1 - \varepsilon_{A,m}(x^n, D)$ and $1 - \varepsilon_{A,m'}(x^n, D)$ are far from 0. For this aim, we focus on Hamming distance $d_H(x^n, x^{n'})$ between $x^n = (x_1^n, \dots, x_n^n), x^{n'} = (x_1^{n'}, \dots, x_n^{n'}) \in \mathcal{X}^n$ as

$$d_H(x^n, x^{n'}) := |\{k | x_k^n \neq x_k^{n'}\}|. \tag{85}$$

and Hermitian matricess $\{\Xi_x\}_{x\in\mathcal{X}}$ to satisfy the following conditions;

$$Tr[W_x \Xi_x] = 0, (86)$$

$$\zeta_1 := \min_{x \neq x' \in \mathcal{X}} -(\operatorname{Tr}[W_{x'}\Xi_x]) > 0, \tag{87}$$

$$\zeta_2 := \max_{x,x' \in \mathcal{X}} \text{Tr}[W_{x'}(\Xi_x - \text{Tr}[W_{x'}\Xi_x])^2] < \infty.$$
(88)

For $x^n = (x_1^n, \dots, x_n^n) \in \mathcal{X}^n$, we define

$$\Xi_{x^n}^{(n)} := \sum_{i=1}^n I^{\otimes (i-1)} \otimes \Xi_{x_i^n} \otimes I^{\otimes (n-i)}. \tag{89}$$

Then, given an encoder ϕ_n mapping $\tilde{\mathcal{M}}_n$ to \mathcal{X}^n , we employ the following projection to Bob's decoder to include the message m in his decoded list;

$$\{\Xi_{\phi_n(m)}^{(n)} \ge -\varepsilon_1 n\}. \tag{90}$$

The projection (90) performs $1 - \varepsilon_{A,m}(x^n, D)$ small when $d_H(x^n, \phi_n(m))$ is larger than a certain threshold. Indeed, we have the following lemma.

Lemma 6: When Condition (NR) holds, there exist functions $\{\Xi_x\}_{x\in\mathcal{X}}$ that satisfies the conditions (86), (87), and (88).

Proof: We show the desired statement for each $x \in \mathcal{X}$. If any a self-adjoint operator A_x satisfies that $\operatorname{Tr} W_x A_x$ belongs to the convex hull of $\{\operatorname{Tr} W_{x'} A_x\}_{x' \in \mathcal{X} \setminus \{x\}}, W_x$ belongs to the set $\{\sum_{x' \in \mathcal{X} \setminus \{x\}} P(x') W_{x'} | P \in \mathcal{P}(\mathcal{X} \setminus \{x\})\}$. Due to Condition (NR), W_x does not belong to the set $\{\sum_{x' \in \mathcal{X} \setminus \{x\}} P(x') W_{x'} | P \in \mathcal{P}(\mathcal{X} \setminus \{x\})\}$. Considering the contraposition of the above statement, we have the following; there exists a self-adjoint operator A_x such that $\operatorname{Tr} W_x A_x > \operatorname{Tr} W_{x'} A_x$ for $x' \in \mathcal{X} \setminus \{x\}$. We choose a basis $\{|e_{j,x}\rangle\}_j$ to diagonal A_x , and define $P_x(j) := \langle e_{j,x} | W_x | e_{j,x} \rangle$ and $P_{x'}(j) := \langle e_{j,x} | W_{x'} | e_{j,x} \rangle$. Then, P_x does not belong to the convex hull of $\{P_{x'}\}_{x' \neq x}$. Hence, applying Lemma 1 of [9], we obtained the desired statement for $x \in \mathcal{X}$.

VII. PROOF OF THEOREM 4

Step 0: We set $\overline{\mathsf{M}}_n := \frac{3}{2} \cdot 2^{\lfloor nR_1 \rfloor + \lfloor nR_2 \rfloor}$. Hence, $\widetilde{\mathsf{M}}_n = \frac{2}{3} \cdot \overline{\mathsf{M}}_n$. We prepare the verifier used in this proof as follows.

Definition 1 (Verifier D_{ϕ_n}): Given a distribution P on \mathcal{X} , we define the verifier D_{ϕ_n} for a given encoder ϕ_n (a map from $\overline{\mathcal{M}}_n := \{1, \ldots, \overline{\mathcal{M}}_n\}$ to \mathcal{X}^n) in the following way. Using the condition (90), we define the projection $\Pi_{x^n} := \{\Xi_{x^n}^{(n)} \ge -n\varepsilon_1\}$. We define the verifier $D_{\phi_n} = \{\Pi_{\phi_n(m)}\}_m$. \square Remember that, for $x^n = (x_1^n, \ldots, x_n^n), x^{n'} = (x_1^{n'}, \ldots, x_n^{n'}) \in \mathcal{X}^n$, Hamming distance $d_H(x^n, x^{n'})$ is

Remember that, for $x^n = (x_1^n, \dots, x_n^n), x^{n'} = (x_1^{n'}, \dots, x_n^{n'}) \in \mathcal{X}^n$, Hamming distance $d_H(x^n, x^{n'})$ is defined to be the number of k such that $x_k^n \neq x_k^{n'}$ as (85) in Subsection VI-D. In the proof of Theorem 4, we need to extract an encoder ϕ_n and elements $m \in \mathcal{M}_n$ that satisfies the following Hamming distance condition;

$$d_H(\phi_n(m), \phi_n(j)) > n\varepsilon_2 \text{ for } \forall j \neq m.$$
 (91)

For this aim, given a code ϕ_n and a real number $\varepsilon_2 > 0$, we define the function $\eta_{\phi_n,\varepsilon_2}^C$ from $\overline{\mathcal{M}}_n$ to $\{0,1\}$

$$\eta_{\phi_n,\varepsilon_2}^C(m) := \begin{cases} 0 & \text{when (91) holds} \\ 1 & \text{otherwise.} \end{cases}$$
 (92)

As shown in Appendix A, we have the following lemma.

Lemma 7: When a code $\tilde{\phi}_n$ defined in a subset $\tilde{\mathcal{M}}_n \subset \overline{\mathcal{M}}_n$ satisfies

$$d_H(\tilde{\phi}_n(m), \tilde{\phi}_n(m')) > n\varepsilon_2 \tag{93}$$

for two distinct elements $m \neq m' \in \tilde{\mathcal{M}}_n$, the verifier $D_{\tilde{\phi}_n}$ defined in Definition 1 satisfies

$$\delta_D(D_{\tilde{\phi}_n}) \le \frac{\zeta_2}{n[\zeta_1 \frac{\varepsilon_2}{2} - \varepsilon_1]_+^2}.$$
(94)

Step 1: The aim of this step is preparation of lemmas related to random coding.

To show Theorem 4, we assume that the variable $\Phi_n(m)$ for $m \in \overline{\mathcal{M}}_n$ is subject to the distribution P^n independently. Then, we have the following four lemmas, which are shown later. In this proof, we treat the code Φ_n as a random variable. Hence, the expectation and the probability for this variable are denoted by \mathbb{E}_{Φ_n} and \Pr_{Φ_n} , respectively. We prepare the following lemmas whose proofs are given in Appendices.

Lemma 8: We have the average version of Verifiable condition (a), i.e.,

$$\lim_{n \to \infty} \mathbb{E}_{\Phi_n} \sum_{m=1}^{\overline{M}_n} \frac{1}{\overline{M}_n} \varepsilon_{A,m}(\Phi_n, D_{\Phi_n}) = 0.$$
 (95)

Lemma 9 ([9, Lemma 12]): When $R_1 + R_2 < H(X)_P$, for $\varepsilon_2 > 0$, we have

$$\lim_{n \to \infty} \mathbb{E}_{\Phi_n} \sum_{m=1}^{\overline{\mathsf{M}}_n} \frac{1}{\overline{\mathsf{M}}_n} \eta_{\Phi_n, \varepsilon_2}^C(m) = 0.$$
 (96)

Lemma 10: We choose $\sigma_{P,\alpha} \in \mathcal{S}(\mathcal{H}_Y)$ as

$$\sigma_{P,\alpha} := \underset{\sigma \in \mathcal{S}(\mathcal{H}_Y)}{\operatorname{argmin}} \tilde{D}_{\alpha}(\boldsymbol{W} \times P \| \sigma \otimes P). \tag{97}$$

We have

$$\mathbb{E}_{\Phi_n} \sum_{i=1}^{\overline{M}_n} \frac{1}{\overline{M}_n} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\Phi_n(i)} \| \sigma_{P,\alpha}^n)} = 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_P}.$$
(98)

Step 2: The aim of this step is the extraction of an encoder ϕ_n and messages m with a small decoding error probability that satisfies the condition (91).

We define $\varepsilon_{3,n}$ as

$$\varepsilon_{3,n} := 9\mathbb{E}_{\Phi_n} \sum_{m=1}^{\overline{M}_n} \frac{1}{\overline{M}_n} \Big(\Big(\varepsilon_{A,m}(\phi_n, D_{\Phi_n}) + \eta_{\Phi_n, \varepsilon_2}^C(m) \Big) \Big). \tag{99}$$

Here the function $\eta_{\Phi_n,\varepsilon_2}^C$ reflects the Hamming distance condition (91). Lemmas 8 and 9 guarantees that $\varepsilon_{3,n}\to 0$. Then, there exists a sequence of codes ϕ_n such that

$$\sum_{m=1}^{\overline{M}_n} \frac{1}{\overline{M}_n} \left(\varepsilon_{A,m}(\phi_n, D_{\phi_n}) + \eta_{\phi_n, \varepsilon_2}^C(m) \right) \le \frac{\varepsilon_{3,n}}{3}$$
(100)

$$\sum_{m=1}^{\overline{M}_n} \frac{1}{\overline{M}_n} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\phi_n(m)} \parallel \sigma_{P,\alpha}^n)} \le 3 \cdot 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_P}. \tag{101}$$

Due to Eq. (100), Markov inequality guarantees that there exist $\frac{2}{3} \cdot \overline{\mathsf{M}}_n$ elements $\tilde{\mathcal{M}}_n := \{m_1, \dots, m_{\frac{2}{3} \cdot \overline{\mathsf{M}}_n}\}$ such that every element $m \in \tilde{\mathcal{M}}_n$ satisfies

$$\varepsilon_{A,m}(\phi_n, D_{\phi_n}) + \eta_{\phi_n, \varepsilon_2}^C(m) \le \varepsilon_{3,n},$$
 (102)

which implies that

$$\varepsilon_{A,m}(\phi_n, D_{\phi_n}) \le \varepsilon_{3,n} \tag{103}$$

$$\eta_{\phi_n,\varepsilon_2}^C(m) = 0 \tag{104}$$

because $\eta_{\phi_n,\varepsilon_2}^C$ takes value 0 or 1. Then, we define a code $\tilde{\phi}_n$ on $\tilde{\mathcal{M}}_n$ as $\tilde{\phi}_n(m) := \phi_n(m)$ for $m \in \tilde{\mathcal{M}}_n$. Eq. (103) guarantees Verifiable condition (a). For m, m', Eq. (101) guarantees that

$$\sum_{m \in \tilde{\mathcal{M}}_n} \frac{1}{|\tilde{\mathcal{M}}_n|} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\tilde{\phi}_n(m)} \| \sigma_{P,\alpha}^{\otimes n})} = \sum_{m \in \tilde{\mathcal{M}}_n} \frac{3}{2\overline{\mathsf{M}}_n} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\phi_n(m)} \| \sigma_{P,\alpha}^{\otimes n})} \le \frac{9}{2} \cdot 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_P}. \tag{105}$$

Step 3: The aim of this step is the evaluation of the parameter $\delta_C(D_{\tilde{\phi}_n,3})$. The relation (104) guarantees the condition

$$d_H(\tilde{\phi}_n(m), \tilde{\phi}_n(m')) > n\varepsilon_2 \tag{106}$$

for $m \neq m' \in \tilde{\mathcal{M}}_n$. Therefore, Lemma 7 guarantees Binding condition (c), i.e.,

$$\delta_C(D_{\tilde{\phi}_n}) \le \frac{\zeta_2}{n[\zeta_1 \frac{\varepsilon_2}{2} - \varepsilon_1]_+^2} \to 0. \tag{107}$$

Step 4: The aim of this step is the evaluation of the parameter $E_{\alpha}(\tilde{\phi}_n)$. Eq. (105) guarantees that

$$\min_{\sigma_{n} \in \mathcal{S}(\mathcal{H}_{Y}^{\otimes n})} \sum_{m \in \tilde{\mathcal{M}}_{n}} \frac{1}{|\tilde{\mathcal{M}}_{n}|} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\tilde{\phi}_{n}(m)} \| \sigma_{n})}$$

$$\leq \sum_{m \in \tilde{\mathcal{M}}_{n}} \frac{1}{|\tilde{\mathcal{M}}_{n}|} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\tilde{\phi}_{n}(m)} \| \sigma_{P,\alpha}^{\otimes n})}$$

$$\stackrel{(a)}{\leq} \frac{9}{2} \cdot 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_{P}}, \tag{108}$$

where (a) follows from (105). Hence, we obtain Condition $(b\alpha)$, i.e., the relation $\lim_{n\to\infty} \frac{1}{n} E_{\alpha}(\phi_n) \ge r_{\alpha}$ with (79) as

$$\lim_{n \to \infty} \frac{1}{n} E_{\alpha}(\tilde{\phi}_n) \ge R_1 + R_2 - \tilde{I}_{\alpha}(X; Y)_P. \tag{109}$$

VIII. CONCLUSION

We have calculated various types of commitment capacities. To show the direct part, we have extended the method by [9] to the quantum setting. To show the converse part, we have shown Proposition 1, which constructs a function to estimate the message from the random variables X^n, U^n, V^n . This function has been constructed from an invertible protocol, and satisfies the required property (38) due to the security parameters of the original invertible protocol. This part was omitted in the preceding papers [5], [6], [7]. Since any interactive protocol in the classical setting satisfies the invertible condition, our converse proof covers the classical setting without any condition.

However, we could not prove the converse part for a general interactive protocol in the cq-channel setting. When the invertible condition does not hold, there exists no inverse TP-CP map $\Lambda_{Y_1'V_1 \to U_1Y_1}$, $\Lambda_{Y_2'V_2 \to Y_1'U_2Y_2}, \ldots, \Lambda_{Y_n'V_n \to Y_{n-1}'U_nY_n}$ to satisfy the condition (19). Hence, the relation (46) does not hold in general. We need to find another method to avoid this problem for a general interactive protocol. Therefore, it is a interesting future problem to calculate the capacities $C_p(W)$ and $C_a(W)$.

In the direct part, we have constructed a specific code to satisfy Conditions (A), (B), and (C), and have converted it to a non-interactive protocol to achieve the commitment capacity. For this construction, we have constructed a pre-code to satisfy Conditions (a), $(b\alpha)$, and (c) by using Hamming distance as Theorem 4. However, we have not constructed a special type of list decoding unlike the reference [9] due to the following reason. If we apply the same list decoder, we need to apply a measurement, which might destroy the received quantum state. Therefore, we can expect that this approach does not work well for cq-channels. It is another interesting future direction to construct secure list decoding for a cq-channel that has a similar performance as that in the reference [9].

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APPENDIX A PROOF OF LEMMA 7

Step 1: The aim of this step is the evaluation of $W_{x^n}^n(\Pi_{x^{n'},3})$. The conditions (86) and (87) imply that

$$Tr[W_{x^{n'}}^{(n)}\Xi_{x^n}] \le -\zeta_1 d(x^n, x^{n'}). \tag{110}$$

By using the method by [28], the condition (88) implies that

$$Tr[W_{r^{n'}}^{(n)}(\Xi_{x^n}^{(n)} - Tr[W_{r^{n'}}^{(n)}\Xi_{x^n}^{(n)}])^2] \le n\zeta_2.$$
(111)

Hence, applying Chebyshev inequality to the variable $\xi_{x^n}(Y^n)$, we have

$$W_{x^{n'}}^{n}(\Pi_{x^{n},2}) = \text{Tr}[W_{x^{n'}}^{n}\{\Xi_{x^{n}}^{(n)} \ge -n\varepsilon_{1}\}]$$

$$\le \frac{n\zeta_{2}}{[\zeta_{1}d(x^{n}, x^{n'}) - n\varepsilon_{1}]_{+}^{2}}.$$
(112)

Step 2: The aim of this step is the evaluation of smaller value of $\text{Tr}[W_{x^n}^n\Pi_{\tilde{\phi}_n(m),3}]$ and $\text{Tr}[W_{x^n}^n\Pi_{\tilde{\phi}_n(m'),3}]$. Since Eq. (93) implies

$$n\varepsilon_2 < d(\tilde{\phi}_n(m), \tilde{\phi}_n(m')) \le d_H(x^n, \tilde{\phi}_n(m)) + d_H(x^n, \tilde{\phi}_n(m')),$$
 (113)

we have

$$\max(\left[\zeta_1 d_H(x^n, \tilde{\phi}_n(m)) - n\varepsilon_1\right]_+, \left[\zeta_1 d_H(x^n, \tilde{\phi}_n(m')) - n\varepsilon_1\right]_+) \ge \left[n\left(\zeta_1 \frac{\varepsilon_2}{2} - \varepsilon_1\right)\right]_+^2. \tag{114}$$

Hence, (112) guarantees that

$$\min(\operatorname{Tr}[W_{x^{n}}^{n}\Pi_{\tilde{\phi}_{n}(m),3}], \operatorname{Tr}[W_{x^{n}}^{n}\Pi_{\tilde{\phi}_{n}(m'),3}]) \\
\leq \frac{n\zeta_{2}}{\max([\zeta_{1}d(x^{n},\tilde{\phi}_{n}(m)) - n\varepsilon_{1}]_{+}^{2}, [\zeta_{1}d(x^{n},\tilde{\phi}_{n}(m')) - n\varepsilon_{1}]_{+}^{2})} \\
\leq \frac{n\zeta_{2}}{[n(\zeta_{1}\frac{\varepsilon_{2}}{2} - \varepsilon_{1})]_{+}^{2}} = \frac{\zeta_{2}}{n[\zeta_{1}\frac{\varepsilon_{2}}{2} - \varepsilon_{1}]_{+}^{2}}, \tag{115}$$

which implies the desired statement.

APPENDIX B PROOF OF LEMMA 8

To evaluate the value

$$\mathbb{E}_{X^n} \operatorname{Tr} W_{X^n}^{(n)}(I - \Pi_{X^n}) = \mathbb{E}_{X^n} \operatorname{Tr} W_{X^n}^{(n)} \{\Xi_{x^n}^{(n)} < -n\varepsilon_1\},$$
(116)

we denote the eigenvalue of Ξ_x with eigenvector $|e_{j,x}\rangle$ by $\xi(j,x)$. We define the random variable J,X whose joint distribution is $P_{JX}(jx) = P(x)\langle e_{j,x}|W_x|e_{j,x}\rangle$. We consider their n independent variables $J^n = (J_1,\ldots,J_n)$ and $X^n = (X_1,\ldots,X_n)$. Hence, we define $\xi^n(J^n,X^n) := \sum_{i=1}^n \xi(J_i,X_i)$. The value (116) equals the probability $\Pr(\xi^n(J^n,X^n) \leq -n\varepsilon_1)$. Since the expectation of $\xi(J_i,X_i)$ is zero and the variance of $\xi(J_i,X_i)$ is upper bounded by ζ_2 , this value goes to zero. Hence, we obtain Lemma 8.

APPENDIX C PROOF OF LEMMA 10

Eq. (98) can be shown as follows.

$$\mathbb{E}_{\Phi} \sum_{i=1}^{\overline{M}_{n}} \frac{1}{\overline{M}_{n}} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\Phi_{n}(i)} \| \sigma_{P,\alpha}^{\otimes n})} = \mathbb{E}_{\Phi} \sum_{i=1}^{\overline{M}_{n}} \frac{1}{\overline{M}_{n}} \prod_{j=1}^{n} 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{\Phi_{n}(i)_{j}} \| \sigma_{P,\alpha})}$$

$$= \sum_{i=1}^{\overline{M}_{n}} \frac{1}{\overline{M}_{n}} \prod_{j=1}^{n} \sum_{x \in \mathcal{X}} P(x) 2^{(\alpha-1)\tilde{D}_{\alpha}(W_{x} \| \sigma_{P,\alpha})}$$

$$= \sum_{i=1}^{\overline{M}_{n}} \frac{1}{\overline{M}_{n}} \prod_{j=1}^{n} 2^{(\alpha-1)\tilde{D}_{\alpha}(W \times P \| \sigma_{P,\alpha} \otimes P)}$$

$$\stackrel{(a)}{=} \sum_{i=1}^{M_{n}} \frac{1}{\overline{M}_{n}} \prod_{j=1}^{n} 2^{(\alpha-1)\tilde{I}_{\alpha}(X;Y)_{P}} = \sum_{i=1}^{\overline{M}_{n}} \frac{1}{\overline{M}_{n}} 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_{P}} = 2^{n(\alpha-1)\tilde{I}_{\alpha}(X;Y)_{P}}, \tag{117}$$

where (a) follows from (97).

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