

A VARIATIONAL PRINCIPLE FOR THE METRIC MEAN DIMENSION OF LEVEL SETS

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ABSTRACT. We prove a variational principle for the upper and lower metric mean dimension of level sets

$$\left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}$$

associated to continuous potentials $\varphi : X \rightarrow \mathbb{R}$ and continuous dynamics $f : X \rightarrow X$ defined on compact metric spaces and exhibiting the specification property. This result relates the upper and lower metric mean dimension of the above mentioned sets with growth rates of measure-theoretic entropy of partitions decreasing in diameter associated to some special measures. Moreover, we present several examples to which our result may be applied to. Similar results were previously known for the topological entropy and for the topological pressure.

1. INTRODUCTION

One of the most important notions in Dynamical Systems is that of *topological entropy*. It is a topological invariant and, roughly speaking, measures how chaotic a system is. In particular, it is an effective tool to decide whether two systems are conjugated or not. Nevertheless, there are plenty of systems with infinite topological entropy (for instance, they form a C^0 -generic set in the space of homeomorphisms of a compact manifold [39] with dimension greater than one) and thus, in this context, the entropy is not useful anymore. Therefore, in order to study these types of systems, new dynamical quantities are required and an example of such a quantity is the *metric mean dimension*.

The notion of metric mean dimension was introduced by Lindenstrauss and Weiss in [24] as metric-dependent analog of the *mean dimension*, a topological invariant associated to a dynamical system which was introduced by Gromov [12]. This last notion has several applications, like in the study of embedding problems [18], and the metric mean dimension presents an upper bound to it. But more than that, the metric mean dimension turned out to be useful in several contexts like in the study of compression [15, 16].

In the present paper we give a modest contribution to the study of ergodic theoretical aspects of the metric mean dimension by presenting a variational principle. Previous connections between ergodic theory and metric mean dimension were presented, for instance, by Lindenstrauss and Tsukamoto [23], Velozo and Velozo [35], Tsukamoto [34], Shi [29], Gutman and Śpiewak [17] and Yang, Chen and Zhou [36]. For more on these works, see Section 2.8. The main novelty of our work with respect to the previously mentioned ones is that our variational principle holds for special subsets and not only for the whole phase space. More precisely, we consider

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$$K_\alpha = \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}$$

associated to continuous potentials $\varphi : X \rightarrow \mathbb{R}$ and continuous dynamics $f : X \rightarrow X$ defined on compact metric spaces exhibiting the specification property and present a relation between the *upper and lower metric mean dimension* of the above mentioned sets and growth rates of measure-theoretic entropy of partitions decreasing in diameter associated to some special measures. This is the content of Theorem A. In Section 4 we present several examples to which our result is applicable.

1.1. Multifractal analysis. The general idea of multifractal analysis consists in decomposing the phase space into subsets of points with similar dynamical behavior, for instance, in sets of points with the same Birkhoff average, the same Lyapunov exponents or the same local entropies, and to describe the size of each of such subsets from a geometrical or topological viewpoint. The information (collection of numbers) obtained via this procedure for one such decomposition of the phase space is called a *multifractal spectrum*. Then, in the best-case scenario the idea is that if one knows some of these spectra one could fully recover the dynamics (see for instance [2, 3]). This phenomenon is sometimes called *multifractal rigidity*. But even when we are not in such a nice world, we still can get useful information about the dynamics from these various spectra (see for instance [11, 25, 30, 31]). Our main result, Theorem A, may be seen as a small contribution to the study of one such spectra, namely, the one obtained by measuring the size of level sets of Birkhoff averages with respect to the metric mean dimension. In particular, as a consequence of our result we get that the map $\alpha \mapsto \underline{\text{mdim}}_{\text{M}}(K_\alpha, f, d)$ is concave when restricted to the set of parameters $\alpha \in \mathbb{R}$ for which $K_\alpha \neq \emptyset$. As far as we know, this is the first time this spectrum was considered and we hope that our results may be of some more help in the study of multifractal analysis of systems with infinite topological entropy.

2. DEFINITIONS AND STATEMENTS

Let (X, d) be a compact metric space and $f : X \rightarrow X$ be a continuous map. Given $n \in \mathbb{N}$, we define the dynamical metric $d_n : X \times X \rightarrow [0, \infty)$ by

$$d_n(x, z) = \max \left\{ d(x, z), d(f(x), f(z)), \dots, d(f^{n-1}(x), f^{n-1}(z)) \right\}.$$

It is easy to see that d_n is indeed a metric and, moreover, generates the same topology as d . Furthermore, given $\varepsilon > 0$, $n \in \mathbb{N}$ and a point $x \in X$, we define the open (n, ε) -ball around x by

$$B_n(x, \varepsilon) = \{y \in X; d_n(x, y) < \varepsilon\}.$$

We sometimes call these (n, ε) -balls *dynamical balls* of radius ε and length n . We say that a set $E \subset X$ is (n, ε) -separated by f if $d_n(x, z) > \varepsilon$ for every $x, z \in E$.

2.1. The metric mean dimension. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, let us denote by $s(f, n, \varepsilon)$ the maximal cardinality of all (n, ε) -separated subsets of X by f which, due to the compactness of X , is finite.

The *upper metric mean dimension* of f with respect to d is given by

$$\overline{\text{mdim}}_{\text{M}}(X, f, d) = \limsup_{\varepsilon \rightarrow 0} \frac{h(f, \varepsilon)}{|\log \varepsilon|}$$

where

$$h(f, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(f, n, \varepsilon).$$

Similarly, the *lower metric mean dimension* of f with respect to d is given by

$$\underline{\text{mdim}}_M(X, f, d) = \liminf_{\varepsilon \rightarrow 0} \frac{h(f, \varepsilon)}{|\log \varepsilon|}.$$

In the case when $\underline{\text{mdim}}_M(X, f, d) = \overline{\text{mdim}}_M(X, f, d)$ this common value is called the *metric mean dimension* of f with respect to d and is denoted simply by $\text{mdim}_M(X, f, d)$.

Recall that the *topological entropy* of the map f is given by

$$h_{\text{top}}(f) = \lim_{\varepsilon \rightarrow 0} h(f, \varepsilon).$$

Consequently, $\overline{\text{mdim}}_M(X, f, d) = \underline{\text{mdim}}_M(X, f, d) = 0$ whenever the topological entropy of f is finite. In particular, the metric mean dimension is a suitable quantity to study systems with infinite topological entropy. For more on these quantities see [24, 23, 34] and references therein.

2.2. The metric mean dimension for non-compact subset. We now present the notion of metric mean dimension on non-compact sets introduced in [10]. Given a set $Z \subset X$, let us consider

$$m(Z, s, N, \varepsilon) = \inf_{\Gamma} \left\{ \sum_{i \in I} \exp(-sn_i) \right\},$$

where the infimum is taken over all covers $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_{i \in I}$ of Z with $n_i \geq N$. We also consider

$$m(Z, s, \varepsilon) = \lim_{N \rightarrow \infty} m(Z, s, N, \varepsilon).$$

One can show (see for instance [26]) that there exists a certain number $s_0 \in [0, +\infty)$ such that $m(Z, s, \varepsilon) = 0$ for every $s > s_0$ and $m(Z, s, \varepsilon) = +\infty$ for every $s < s_0$. In particular, we may consider

$$h(Z, f, \varepsilon) = \inf\{s : m(Z, s, \varepsilon) = 0\} = \sup\{s : m(Z, s, \varepsilon) = +\infty\}.$$

The *upper metric mean dimension of f on Z* is then defined as the following limit

$$\overline{\text{mdim}}_M(Z, f, d) = \limsup_{\varepsilon \rightarrow 0} \frac{h(Z, f, \varepsilon)}{|\log \varepsilon|}.$$

Similarly, the *lower metric mean dimension of f on Z* is defined as

$$\underline{\text{mdim}}_M(Z, f, d) = \liminf_{\varepsilon \rightarrow 0} \frac{h(Z, f, \varepsilon)}{|\log \varepsilon|}.$$

In the case when $Z = X$ one can check that the two definitions of upper/lower metric mean dimension given above actually coincide.

2.3. Level sets of a continuous map. Let $C(X, \mathbb{R})$ denote the set of all continuous maps $\varphi : X \rightarrow \mathbb{R}$ and take $\varphi \in C(X, \mathbb{R})$. For $\alpha \in \mathbb{R}$, let

$$K_\alpha = \left\{ x \in X : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \alpha \right\}. \quad (1)$$

We also consider the set

$$\mathcal{L}_\varphi = \{\alpha \in \mathbb{R} : K_\alpha \neq \emptyset\}.$$

It is easy to see that \mathcal{L}_φ is a bounded and non-empty set [32, Lemma 2.1]. Moreover, if f satisfies the so called specification property (see Section 2.6) then \mathcal{L}_φ is an

interval of \mathbb{R} and, moreover, $\mathcal{L}_\varphi = \{\int \varphi d\mu; \mu \in \mathcal{M}_f(X)\}$ where $\mathcal{M}_f(X)$ stands for the set of all invariant measures (see [33, Lemma 2.5]).

2.4. The auxiliary quantities $\Lambda_\varphi \overline{\text{mdim}}_M(f, \alpha, d)$ and $\Lambda_\varphi \underline{\text{mdim}}_M(f, \alpha, d)$. Fix $\alpha \in \mathbb{R}$ and $\varphi \in C(X, \mathbb{R})$. For $\delta > 0$ and $n \in \mathbb{N}$ define the set

$$P(\alpha, \delta, n) = \left\{ x \in X : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \alpha \right| < \delta \right\}.$$

Let $N(\alpha, \delta, n, \varepsilon)$ denote the minimal number of (n, ε) -balls needed to cover $P(\alpha, \delta, n)$. Define

$$\Lambda_\varphi(\alpha, \varepsilon) = \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon)$$

and

$$\begin{aligned} \Lambda_\varphi \overline{\text{mdim}}_M(f, \alpha, d) &= \limsup_{\varepsilon \rightarrow 0} \frac{\Lambda_\varphi(\alpha, \varepsilon)}{|\log \varepsilon|}, \\ \Lambda_\varphi \underline{\text{mdim}}_M(f, \alpha, d) &= \liminf_{\varepsilon \rightarrow 0} \frac{\Lambda_\varphi(\alpha, \varepsilon)}{|\log \varepsilon|}. \end{aligned} \quad (2)$$

Remark 2.1. Observe that, if $M(\alpha, \delta, n, \varepsilon)$ denotes the maximal cardinality of a (n, ε) -separated set contained in $P(\alpha, \delta, n)$, then we have that

$$N(\alpha, \delta, n, \varepsilon) \leq M(\alpha, \delta, n, \varepsilon) \leq N(\alpha, \delta, n, \varepsilon/2).$$

In particular,

$$\Lambda_\varphi(\alpha, \varepsilon) = \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log M(\alpha, \delta, n, \varepsilon). \quad (3)$$

2.5. The main quantities $H_\varphi \overline{\text{mdim}}_M(f, \alpha, d)$ and $H_\varphi \underline{\text{mdim}}_M(f, \alpha, d)$. Given $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$, let us consider

$$\mathcal{M}_f(X, \varphi, \alpha) = \left\{ \mu \text{ is } f\text{-invariant and } \int \varphi d\mu = \alpha \right\}.$$

A simple observation is that $\mathcal{M}_f(X, \varphi, \alpha) \neq \emptyset$ for every $\alpha \in \mathcal{L}_\varphi$ (see [32, Lemma 4.1]).

Let $\mu \in \mathcal{M}_f(X)$. We say that $\xi = \{C_1, \dots, C_k\}$ is a measurable partition of X if every C_i is a measurable set, $\mu(X \setminus \cup_{i=1}^k C_i) = 0$ and $\mu(C_i \cap C_j) = 0$ for every $i \neq j$. The *entropy* of ξ with respect to μ is given by

$$H_\mu(\xi) = - \sum_{i=1}^k \mu(C_i) \log(\mu(C_i)).$$

Given a measurable partition ξ , we consider $\xi^n = \bigvee_{j=0}^{n-1} f^{-j}\xi$. Then, the *metric entropy of (f, μ) with respect to ξ* is given by

$$h_\mu(f, \xi) = \lim_{n \rightarrow +\infty} \frac{1}{n} H_\mu(\xi^n).$$

Using this quantity we define

$$H_\varphi \overline{\text{mdim}}_M(f, \alpha, d) = \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|\xi| < \varepsilon} h_\mu(f, \xi) \quad (4)$$

and

$$H_\varphi \underline{\text{mdim}}_M(f, \alpha, d) = \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|\xi| < \varepsilon} h_\mu(f, \xi)$$

where $|\xi|$ denotes the diameter of the partition ξ and the infimum is taken over all finite measurable partitions of X satisfying $|\xi| < \varepsilon$.

We also recall that the *metric entropy* of (f, μ) is given by

$$h_\mu(f) = \sup_{\xi} h_\mu(f, \xi)$$

where the supremum is taken over all finite measurable partitions ξ of X .

2.6. Specification property. We say that f satisfies the *specification property* if for every $\varepsilon > 0$, there exists an integer $m = m(\varepsilon)$ such that for any collection of finite intervals $I_j = [a_j, b_j] \subset \mathbb{N}$, $j = 1, \dots, k$, satisfying $a_{j+1} - b_j \geq m(\varepsilon)$ for every $j = 1, \dots, k-1$ and any x_1, \dots, x_k in X , there exists a point $x \in X$ such that

$$d(f^{p+a_j}x, f^p x_j) < \varepsilon \text{ for all } p = 0, \dots, b_j - a_j \text{ and every } j = 1, \dots, k.$$

The specification property is present in many interesting examples. For instance, every topologically mixing locally maximal hyperbolic set has the specification property and factors of systems with specification have specification (see for instance [22]). Other examples of systems satisfying this property which are more adapted to our purposes will appear in Section 4.

2.7. Main result. Our main result may be seen as an extension of [32, Theorem 5.1] to the infinite entropy setting.

Theorem A. *Suppose $f : X \rightarrow X$ is a continuous transformation with the specification property. Let $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$ be such that $K_\alpha \neq \emptyset$. Then*

$$\overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d) = \Lambda_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) = \text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d).$$

Similarly,

$$\underline{\text{mdim}}_{\text{M}}(K_\alpha, f, d) = \Lambda_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d) = \text{H}_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d).$$

We consider the equalities between $\overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d)$ and $\text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ and between $\underline{\text{mdim}}_{\text{M}}(K_\alpha, f, d)$ and $\text{H}_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ to be the most important part of our result because it relates a topological quantity with one that has an ergodic-theoretical flavor. Moreover, in some cases it allow us to obtain some interesting properties about the multifractal spectrum. For instance, will show below that

Proposition 2.2. *Under the assumptions of Theorem A, the map*

$$\mathcal{L}_\varphi \ni \alpha \mapsto \text{H}_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d)$$

is concave.

Consequently, combining this result with Theorem A we get that

Corollary 2.3. *Under the assumptions of Theorem A, the map*

$$\mathcal{L}_\varphi \ni \alpha \mapsto \underline{\text{mdim}}_{\text{M}}(K_\alpha, f, d)$$

is concave.

An interesting question is whether we can change the order between the limit and the supremum in the definition of $\text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ and $\text{H}_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d)$. This would allow, for instance, to talk about the existence of “maximizing measures”: measures that realize the supremum. Such a measure would capture the complexity of the system over all scales $\varepsilon > 0$. It was observed in [23, Section VIII] that a similar question involving different ergodic quantities is, in general, false. Nevertheless, under the additional assumption that f has the marker property, one can do such a

change (in the setting of [23]) as observed by Yang, Chen and Zhou [36]. As for our hypothesis that f satisfies the specification property, we do not know whether it is actually required for Theorem A to hold or if it is just an artefact of the technique.

2.8. Related results. As already mentioned, for the topological entropy a result similar to Theorem A was obtained in [32]. In fact, our result was inspired by that one. Moreover, [32] was extended to the framework of topological pressure in [33].

As for variational results involving the upper metric mean dimension, there are several works dealing with this problem. For instance, [23] presented a variational principle relating the metric mean dimension with the supremum of certain rate distortion functions over invariant measures of the system. This was further explored in [35]. More recently, [29] obtained variational principles for the metric mean dimension in terms of Brin-Katok local entropy and Shapira's entropy of an open cover. One result that is more connected to ours is the one obtained in [17] which says that

$$\overline{\text{mdim}}_{\text{M}}(X, f, d) = \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \sup_{\mu \in \mathcal{M}_f(X)} \inf_{|\xi| < \varepsilon} h_{\mu}(f, \xi) \quad (5)$$

and

$$\underline{\text{mdim}}_{\text{M}}(X, f, d) = \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \sup_{\mu \in \mathcal{M}_f(X)} \inf_{|\xi| < \varepsilon} h_{\mu}(f, \xi). \quad (6)$$

These are variational results for the upper/lower metric mean dimension of the entire space X while Theorem A applies also to level sets of continuous maps φ . Observe that in the case when φ is a constant map equal to α , the α -level set of it coincides with X . In particular, whenever f has the specification property, (5) and (6) may be seen as particular cases of our result. We stress however that the results in [17] do not assume such property.

3. PROOFS OF THEOREM A AND PROPOSITION 2.2

In this section we present the proofs of Theorem A and Proposition 2.2 starting with the latter one which is much simpler.

Proof of Proposition 2.2. Given measures $\mu_1, \mu_2 \in \mathcal{M}_f(X)$, using that the map $\mu \rightarrow H_{\mu}(\xi)$ is concave for any finite and measurable partition ξ (see [8, Lemma 9.5.1]), it follows that for any $t \in [0, 1]$,

$$t h_{\mu_1}(f, \xi) + (1-t) h_{\mu_2}(f, \xi) \leq h_{t\mu_1 + (1-t)\mu_2}(f, \xi).$$

In particular,

$$t \inf_{|\xi| < \varepsilon} h_{\mu_1}(f, \xi) + (1-t) \inf_{|\xi| < \varepsilon} h_{\mu_2}(f, \xi) \leq \inf_{|\xi| < \varepsilon} h_{t\mu_1 + (1-t)\mu_2}(f, \xi). \quad (7)$$

Now, given $\alpha_1, \alpha_2 \in \mathcal{L}_{\varphi}$, by the comments in Section 2.3 there exist invariant measures $\mu_1, \mu_2 \in \mathcal{M}_f(X)$ such $\alpha_i = \int \varphi d\mu_i$, $i = 1, 2$. For any $t \in [0, 1]$, consider $\mu = t\mu_1 + (1-t)\mu_2$ and $\alpha = t\alpha_1 + (1-t)\alpha_2$. Then, $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$. Combining this observation with (7) we get that

$$\begin{aligned} & t \sup_{\mu_1 \in \mathcal{M}_f(X, \varphi, \alpha_1)} \inf_{|\xi| < \varepsilon} h_{\mu_1}(f, \xi) + (1-t) \sup_{\mu_2 \in \mathcal{M}_f(X, \varphi, \alpha_2)} \inf_{|\xi| < \varepsilon} h_{\mu_2}(f, \xi) \\ & \leq \sup_{\mu_1 \in \mathcal{M}_f(X, \varphi, \alpha_1), \mu_2 \in \mathcal{M}_f(X, \varphi, \alpha_2)} \inf_{|\xi| < \varepsilon} h_{t\mu_1 + (1-t)\mu_2}(f, \xi) \\ & \leq \sup_{\mu \in \mathcal{M}_f(X, \varphi, \alpha)} \inf_{|\xi| < \varepsilon} h_{\mu}(f, \xi). \end{aligned}$$

Then, dividing everything by $|\log \varepsilon|$, taking “ $\liminf_{\varepsilon \rightarrow 0}$ ” and using that $\liminf(a) + \liminf(b) \leq \liminf(a + b)$ we get that

$$tH_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha_1, d) + (1-t)H_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha_2, d) \leq H_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, t\alpha_1 + (1-t)\alpha_2, d).$$

Consequently, the map $\mathcal{L}_{\alpha} \ni \alpha \mapsto H_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d)$ is concave concluding the proof of the proposition. \square

Remark 3.1. It is not clear to us whether a version of Proposition 2.2 holds for the map $\mathcal{L}_{\alpha} \ni \alpha \mapsto H_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d)$. In fact, in order to get the desired conclusion in the aforementioned proposition we have used the property that $\liminf(a) + \liminf(b) \leq \liminf(a + b)$ which obviously does not hold for the \limsup .

We now present the the poof of Theorem A. We start considering the first claim of the theorem and, for the sake of clarity of the presentation, we split it into three main propositions. We emphasize that this proof is an adaptation of the proof of Theorem 5.1 of [32] to our setting. Fix $\varphi \in C(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$ such that $K_{\alpha} \neq \emptyset$. Moreover, assume initially that all the quantities $\overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, f, d)$, $\Lambda_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d)$ and $H_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d)$ are finite.

Proposition 3.2. *Under the hypotheses of Theorem A we have that*

$$\overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, f, d) \leq \Lambda_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d).$$

Proof. Let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers converging to zero such that

$$\overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, f, d) = \lim_{j \rightarrow \infty} \frac{h(K_{\alpha}, f, \varepsilon_j)}{|\log \varepsilon_j|}.$$

In particular we have that

$$\limsup_{j \rightarrow \infty} \frac{\Lambda_{\varphi}(\alpha, \varepsilon_j)}{|\log \varepsilon_j|} \leq \limsup_{\varepsilon \rightarrow 0} \frac{\Lambda_{\varphi}(\alpha, \varepsilon)}{|\log \varepsilon|} = \Lambda_{\varphi} \overline{\text{mdim}}_{\mathbb{M}}(f, \alpha, d).$$

Given $\delta > 0$ and $k \in \mathbb{N}$, let us consider the set

$$\begin{aligned} G(\alpha, \delta, k) &= \bigcap_{n=k}^{\infty} P(\alpha, \delta, n) \\ &= \bigcap_{n=k}^{\infty} \left\{ x \in X : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \alpha \right| < \delta \right\}. \end{aligned}$$

As a consequence of the definition we have that $K_{\alpha} \subset \bigcup_{k \in \mathbb{N}} G(\alpha, \delta, k)$.

Now, given $k \in \mathbb{N}$, since $G(\alpha, \delta, k) \subset P(\alpha, \delta, n)$ for $n \geq k$, it follows that $G(\alpha, \delta, k)$ may be covered by $N(\alpha, \delta, n, \varepsilon_j)$ dynamical balls of radius ε_j and length n . Thus, for every $s \geq 0$ and $n \geq k$ we have

$$m(G(\alpha, \delta, k), s, \varepsilon_j) \leq N(\alpha, \delta, n, \varepsilon_j) \exp(-ns).$$

Let $s = s(\varepsilon_j) > \Lambda_{\varphi}(\alpha, \varepsilon_j)$ and $\gamma(\varepsilon_j) = (s - \Lambda_{\varphi}(\alpha, \varepsilon_j))/2$. Then, if $\delta_j > 0$ is small enough, there exists an increasing sequence $\{n_{\ell}\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$ such that

$$N(\alpha, \delta_j, n_{\ell}, \varepsilon_j) \leq \exp(n_{\ell}(\Lambda_{\varphi}(\alpha, \varepsilon_j) + \gamma(\varepsilon_j))).$$

Thus, assuming without lost of generality that $n_1 \geq k$ and combining the previous observations we conclude that

$$m(G(\alpha, k, \delta_j), s(\varepsilon_j), \varepsilon_j) \leq \exp(-n_{\ell} \gamma(\varepsilon_j)).$$

In particular, as $\gamma(\varepsilon_j) > 0$, letting $n_{\ell} \rightarrow \infty$ we obtain $m(G(\alpha, k, \delta_j), s(\varepsilon_j), \varepsilon_j) = 0$. Consequently,

$$h(G(\alpha, k, \delta_j), f, \varepsilon_j) \leq s(\varepsilon_j)$$

which implies that

$$h(K_\alpha, f, \varepsilon_j) \leq \sup_k h(G(\alpha, k, \delta_j), f, \varepsilon_j) \leq s(\varepsilon_j).$$

Hence,

$$\begin{aligned} \overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d) &= \limsup_{j \rightarrow \infty} \frac{h(K_\alpha, f, \varepsilon_j)}{|\log \varepsilon_j|} \\ &\leq \limsup_{j \rightarrow \infty} \frac{s(\varepsilon_j)}{|\log \varepsilon_j|} \\ &\leq \limsup_{j \rightarrow \infty} \frac{2\gamma(\varepsilon_j)}{|\log \varepsilon_j|} + \limsup_{j \rightarrow \infty} \frac{\Lambda_\varphi(\alpha, \varepsilon_j)}{|\log \varepsilon_j|} \\ &\leq \limsup_{j \rightarrow \infty} \frac{2\gamma(\varepsilon_j)}{|\log \varepsilon_j|} + \Lambda_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d). \end{aligned}$$

Therefore, as we can choose $s(\varepsilon_j)$ arbitrarily close to $\Lambda_\varphi(\alpha, \varepsilon_j)$, the limsup in the last step is zero for an adequate choice of $s(\varepsilon_j)$. Then, $\overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d) \leq \Lambda_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ completing the proof of the proposition. \square

Proposition 3.3. *Under the hypotheses of Theorem A we have that*

$$\text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) \leq \overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d).$$

The strategy of the proof consists in constructing a fractal set F contained in K_α and a special probability measure η supported on F that satisfies the hypothesis of the so called Entropy Distribution Principle (see Lemma 3.10). This will be enough to get the desired inequality. As a step towards the definition of F , we introduce a family of finite sets \mathcal{S}_k which play a major role in the construction.

In order to prove Proposition 3.3 we will need the following auxiliary quantity. For $\mu \in \mathcal{M}_f(X)$, $\delta > 0$ and $n \in \mathbb{N}$, let us denote by $N_\mu(\delta, \varepsilon, n)$ the minimal number (n, ε) -balls needed to cover a set of μ -measure bigger than $1 - \delta$. Then, we define

$$h_\mu(f, \varepsilon, \delta) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log N_\mu(\delta, \varepsilon, n). \quad (8)$$

Proof of Proposition 3.3. Fix $\gamma > 0$ and let $\{\delta_k\}_{k \in \mathbb{N}}$ be a decreasing sequence converging to 0. Take $\varepsilon = \varepsilon(\gamma) > 0$ and $\mu \in \mathcal{M}_f(X, \varphi, \alpha)$ so that

$$\frac{\inf_{|\xi| < 5\varepsilon} h_\mu(f, \xi)}{|\log 5\varepsilon|} \geq \text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{\gamma}{2}$$

and

$$\frac{h(K_\alpha, f, \varepsilon/2)}{|\log \varepsilon/2|} \leq \overline{\text{mdim}}_{\text{M}}(K_\alpha, f, d) + \gamma. \quad (9)$$

Let \mathcal{U} be a finite open cover of X with diameter $\text{diam}(\mathcal{U}) \leq 5\varepsilon$ and Lebesgue number $\text{Leb}(\mathcal{U}) \geq \frac{5\varepsilon}{4}$. We now construct an auxiliary measure which is a finite combination of ergodic measures and ‘‘approximates’’ μ . To prove this lemma we follow the idea from [37, p. 535]. In what follows, $\partial\xi$ will denote the boundary of the partition ξ which is just the union of the boundaries of all the elements of the partition and $\xi \succ \mathcal{U}$ means that ξ refines \mathcal{U} , that is, each element of ξ is contained in an element of \mathcal{U} .

Lemma 3.4. *For each $k \in \mathbb{N}$, there exists a measure $\nu_k \in \mathcal{M}_f(X)$ satisfying*

- (a) $\nu_k = \sum_{i=1}^{j(k)} \lambda_i \nu_i^k$, where $\lambda_i > 0$, $\sum_{i=1}^{j(k)} \lambda_i = 1$ and $\nu_i^k \in \mathcal{M}_f^{\text{erg}}(X)$;
- (b) $\inf_{\xi \succ \mathcal{U}} h_\mu(f, \xi) \leq \inf_{\xi \succ \mathcal{U}} h_{\nu_k}(f, \xi) + \delta_k/2$;

$$(c) \left| \int_X \varphi d\nu_k - \int_X \varphi d\mu \right| < \delta_k.$$

Proof of Lemma 3.4. Given $k \in \mathbb{N}$, let $\beta_k > 0$ be such that for every $\tau_1, \tau_2 \in \mathcal{M}_f(X)$,

$$d_{\mathcal{M}_f(X)}(\tau_1, \tau_2) < \beta_k \implies \left| \int \varphi d\tau_1 - \int \varphi d\tau_2 \right| < \delta_k$$

where $d_{\mathcal{M}_f(X)}$ is a metric in $\mathcal{M}_f(X)$. Let $\mathcal{P} = \{P_1, \dots, P_{j(k)}\}$ be a partition of $\mathcal{M}_f(X)$ whose diameter with respect to $d_{\mathcal{M}_f(X)}$ is smaller than β_k . By the Ergodic Decomposition Theorem there exists a measure $\hat{\mu}$ on $\mathcal{M}_f(X)$ satisfying $\hat{\mu}(\mathcal{M}_f^{\text{erg}}(X)) = 1$ such that

$$\int \psi(x) d\mu(x) = \int_{\mathcal{M}_f(X)} \left(\int_X \psi(x) d\tau(x) \right) d\hat{\mu}(\tau) \text{ for every } \psi \in C(X, \mathbb{R}).$$

Let us consider now $\lambda_i = \hat{\mu}(P_i)$ and take $\nu_i^k \in P_i \cap \mathcal{M}_f^{\text{erg}}(X)$ such that $\inf_{\xi \succ \mathcal{U}} h_{\nu_i^k}(f, \xi) \geq \inf_{\xi \succ \mathcal{U}} h_\tau(f, \xi) - \delta_k/2$ for $\hat{\mu}$ -almost every $\tau \in P_i \cap \mathcal{M}_f^{\text{erg}}(X)$. Observe that such a measure ν_i^k exists because $\sup_{\tau \in \mathcal{M}_f^{\text{erg}}(X)} \inf_{\xi \succ \mathcal{U}} h_\tau(f, \xi) < +\infty$. This latter fact follows from Lemma 3 and Theorem 5 of [29] and the fact that the upper metric mean dimension is finite. Finally, define $\nu_k = \sum_{i=1}^{j(k)} \lambda_i \nu_i^k$. It is easy to see that ν_k satisfies properties a) and c) from the statement. Let us now check that it also satisfies b). By [19, Proposition 5] we know that

$$\inf_{\xi \succ \mathcal{U}} h_\mu(f, \xi) = \int_{\mathcal{M}_f(X)} \inf_{\xi \succ \mathcal{U}} h_\tau(f, \xi) d\hat{\mu}(\tau).$$

Thus, by our choice of the measures ν_i^k it follows that

$$\begin{aligned} \inf_{\xi \succ \mathcal{U}} h_\mu(f, \xi) &= \int_{\mathcal{M}_f(X)} \inf_{\xi \succ \mathcal{U}} h_\tau(f, \xi) d\hat{\mu}(\tau) \\ &\leq \sum_{i=1}^{j(k)} \lambda_i \inf_{\xi \succ \mathcal{U}} h_{\nu_i^k}(f, \xi) + \delta_k/2 \\ &\leq \inf_{\xi \succ \mathcal{U}} h_{\nu_k}(f, \xi) + \delta_k/2 \end{aligned}$$

completing the proof of the lemma. \square

Let ν_k be as in the previous lemma. Using the fact that each measure ν_i^k is ergodic, by the proof of [29, Theorem 9] there exists a finite Borel measurable partition ξ_k which refines \mathcal{U} so that

$$h_{\nu_i^k}(f, 5\varepsilon, \gamma) \leq h_{\nu_i^k}(f, \xi_k) \leq h_{\nu_i^k}(f, 5\varepsilon/4, \gamma) + \delta_k. \quad (10)$$

Now, take a finite Borel partition ξ refining \mathcal{U} with $\mu(\partial\xi) = 0$ such that

$$h_\mu(f, \xi) - \delta_k \leq \inf_{\zeta \succ \mathcal{U}} h_{\nu_k}(f, \zeta).$$

In particular, since $\xi_k \succ \mathcal{U}$,

$$h_\mu(f, \xi) - \delta_k \leq h_{\nu_k}(f, \xi_k). \quad (11)$$

Moreover, since $\xi \succ \mathcal{U}$ it follows that $|\xi| < 5\varepsilon$ and thus

$$\frac{h_\mu(f, \xi)}{|\log 5\varepsilon|} \geq H_\varphi \overline{\text{mdim}}_M(f, \alpha, d) - \gamma. \quad (12)$$

Again, since each ν_i^k is ergodic, there exists $\ell_k \in \mathbb{N}$ large enough for which the set

$$Y_i(k) = \left\{ x \in X : \left| \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) - \int_X \varphi d\nu_i^k \right| < \delta_k \quad \forall n \geq \ell_k \right\}$$

has ν_i^k -measure bigger than $1 - \gamma$ for every $k \in \mathbb{N}$ and $i \in \{1, \dots, j(k)\}$.

By [33, Lemma 3.6], there exists $\hat{n}_k \rightarrow \infty$ with $[\lambda_i \hat{n}_k] \geq \ell_k$ so that the maximal cardinality of an $([\lambda_i \hat{n}_k], 5\varepsilon/4)$ -separated set in $Y_i(k)$, denoted by $M_{k,i}$, satisfies

$$M_{k,i} \geq \exp \left([\lambda_i \hat{n}_k] \left(h_{\nu_i^k}(f, 5\varepsilon/4, \gamma) - \frac{4\gamma}{j(k)} \right) \right). \quad (13)$$

Furthermore, the sequence \hat{n}_k can be chosen such that $\hat{n}_k \geq 2^{m_k}$ where $m_k = m(\varepsilon/2^{k+2})$ is as in the definition of the specification property. Let $n_k := m_k(j(k) - 1) + \sum_i [\lambda_i \hat{n}_k]$. Observe that $n_k/\hat{n}_k \rightarrow 1$.

Denote by $E_{i,k}([\lambda_i \hat{n}_k], 5\varepsilon/4)$ a maximal $([\lambda_i \hat{n}_k], 5\varepsilon/4)$ -separated set in $Y_i(k)$. By the specification property, for each

$$x_1 \in E_{1,k}(n_1, 5\varepsilon/4), x_2 \in E_{2,k}(n_2, 5\varepsilon/4), \dots, x_{j(k)} \in E_{j(k),k}(n_{j(k)}, 5\varepsilon/4),$$

there exists $y = y(x_1, \dots, x_{j(k)}) \in X$ so that the pieces of orbits

$$\{x_i, f(x_i), \dots, f^{[\lambda_i \hat{n}_k]-1}(x_i); i = 1, \dots, j(k)\}$$

are $\varepsilon/2^k$ -shadowed by y with gap m_k . We claim that if $(x_1, \dots, x_{j(k)}) \neq (x'_1, \dots, x'_{j(k)})$ then $y(x_1, \dots, x_{j(k)}) \neq y'(x'_1, \dots, x'_{j(k)})$. Indeed, if $x_i \neq x'_i$,

$$\begin{aligned} \frac{5\varepsilon}{4} &< d_{[\lambda_i \hat{n}_k]}(x_i, x'_i) \\ &\leq d_{[\lambda_i \hat{n}_k]}(x_i, f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y)) \\ &\quad + d_{[\lambda_i \hat{n}_k]}(x'_i, f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y')) \\ &\quad + d_{[\lambda_i \hat{n}_k]}(f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y), f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y')) \\ &< 2 \frac{\varepsilon}{2^{k+2}} + d_{[\lambda_i \hat{n}_k]}(f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y), f^{[\lambda_1 \hat{n}_k] + \dots + [\lambda_{i-1} \hat{n}_k] + (i-1)m_k}(y')), \end{aligned}$$

which implies that $d_{n_k}(y, y') > 9\varepsilon/8$ proving our claim. Moreover, as a by-product of this observation we get that

$$S_k = \{y(x_1, \dots, x_{j(k)}) : x_i \in E_{i,k}([\lambda_i \hat{n}_k], 5\varepsilon/4) \text{ for } i = 1, \dots, j(k)\}$$

is a $(n_k, 9\varepsilon/8)$ -separated set with cardinality $M_k := \prod_{i=1}^{j(k)} M_{k,i}$. Combining (10), (11) and (13) with the choices of ε , γ and n_k and recalling that $n_k/\hat{n}_k \rightarrow 1$ we

get that for k sufficiently large

$$\begin{aligned}
 M_k &= \prod_{i=i}^{j(k)} \#E_{i,k}([\lambda_i \hat{n}_k], 9\varepsilon/8) \geq \exp \left(\sum_{i=1}^{j(k)} [\lambda_i \hat{n}_k] \left(h_{\nu_i^k}(f, 5\varepsilon/4, \gamma) - \frac{4\gamma}{j(k)} \right) \right) \quad (14) \\
 &\geq \exp \left(\hat{n}_k \sum_{i=1}^{j(k)} \lambda_i h_{\nu_i^k}(f, 5\varepsilon/4, \gamma) - 4\hat{n}_k \gamma \right) \\
 &\geq \exp \left(\hat{n}_k \sum_{i=1}^{j(k)} \lambda_i h_{\nu_i^k}(f, \xi_k) - 4\hat{n}_k \gamma - \hat{n}_k \delta_k \right) \\
 &\geq \exp(\hat{n}_k (h_{\nu_k}(f, \xi_k) - 4\gamma - \delta_k)) \\
 &\geq \exp(R_k n_k (h_{\nu_k}(f, \xi_k, \gamma) - 4\gamma - \delta_k)) \\
 &\geq \exp(R_k n_k (h_{\mu}(f, \xi) - 4\gamma - 2\delta_k)) \\
 &\geq \exp(R_k n_k (h_{\mu}(f, \xi) - 5\gamma))
 \end{aligned}$$

for some $R_k \in (0, 1)$.

Let $y = y(x_1, \dots, x_k) \in \mathcal{S}_k$. Then,

$$\begin{aligned}
 |S_{n_k} \varphi(y) - n_k \alpha| &\leq \left| S_{n_k} \varphi(y) - n_k \left(\int \varphi d\nu_k - \delta_k \right) \right| \\
 &\leq \sum_{i=1}^{j(k)-1} \left| S_{[\lambda_i \hat{n}_k]} \varphi(f^{\sum_{i=1}^{i-1} [\lambda_i \hat{n}_k] + (i-1)m_k}(y)) - n_k \lambda_i \int \varphi d\nu_i^k \right| \\
 &\quad + n_k \delta_k + m_k (j(k) - 1) \|\varphi\| \\
 &\leq \sum_{i=1}^{j(k)-1} \left| S_{[\lambda_i \hat{n}_k]} \varphi(x_i) - [\lambda_i n_k] \int \varphi d\nu_i^k \right| \\
 &\quad + n_k \delta_k + m_k j(k) \|\varphi\| + n_k \text{Var}(\varphi, \varepsilon/2^k) \\
 &< \delta_k \sum_{i=1}^{j(k)-1} [\lambda_i \hat{n}_k] + m_k j(k) \|\varphi\| + n_k \delta_k + n_k \text{Var}(\varphi, \varepsilon/2^k).
 \end{aligned}$$

Thus, for sufficiently large k ,

$$\left| \frac{1}{n_k} S_{n_k} \varphi(y) - \alpha \right| \leq \delta_k + \text{Var}(\varphi, \varepsilon/2^k) + \frac{1}{k}. \quad (15)$$

We now choose a sequence $\{N_k\}_{k \in \mathbb{N}}$ of positive integers such that $N_1 = 1$ and

- (1) $[R_k N_k] \geq 2^{n_{k+1} + m_{k+1}}$, for $k \geq 2$;
- (2) $[R_{k+1} N_{k+1}] \geq 2^{[R_1 N_1 n_1] + \dots + [R_k N_k (n_k + m_k)]}$, for $k \geq 1$.

Observe that this sequence $\{N_k\}_{k \in \mathbb{N}}$ grows very fast and

$$\lim_{k \rightarrow \infty} \frac{n_{k+1} + m_{k+1}}{R_k N_k} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{R_1 N_1 n_1 + \dots + R_k N_k (n_k + m_k)}{R_{k+1} N_{k+1}} = 0. \quad (16)$$

Moreover, we enumerate the points in \mathcal{S}_k as

$$\mathcal{S}_k = \{x_i^k : i = 1, \dots, M_k\}.$$

For any $(i_1, \dots, i_{N_k}) \in \{1, 2, \dots, M_k\}^{[R_k N_k]}$, let $y(i_1, \dots, i_{[R_k N_k]}) \in X$ be given by the specification property so that its orbit $\varepsilon/2^k$ -shadows, with gap m_k , the pieces of orbits $\{x_{i_j}^k, f(x_{i_j}^k), \dots, f^{n_k-1}(x_{i_j}^k)\}$, $j = 1, 2, \dots, [R_k N_k]$. Then, define

$$\mathcal{C}_k = \{y(i_1, \dots, i_{[R_k N_k]}) \in X : (i_1, \dots, i_{[R_k N_k]}) \in \{1, 2, \dots, M_k\}^{[R_k N_k]}\}.$$

Moreover, consider

$$c_k = [R_k N_k] n_k + ([R_k N_k] - 1) m_k.$$

We now observe that different sequences in $\{1, 2, \dots, M_k\}^{[R_k N_k]}$ give rise to different points in \mathcal{C}_k and that such points are uniformly separated with respect to d_{c_k} .

Lemma 3.5 (Lemma 5.1 of [32]). *If $(i_1, \dots, i_{[R_k N_k]}) \neq (j_1, \dots, j_{[R_k N_k]})$, then*

$$d_{c_k}(y(i_1, \dots, i_{[R_k N_k]}), y(j_1, \dots, j_{[R_k N_k]})) > \varepsilon.$$

In particular $\#\mathcal{C}_k = M_k^{[R_k N_k]}$.

Our next step is to construct inductively an auxiliary sequence of finite sets \mathcal{T}_k . Let $\mathcal{T}_1 = \mathcal{C}_1$ and $t_1 = c_1$. Now, suppose that we have already constructed the set \mathcal{T}_k and we will describe how to construct \mathcal{T}_{k+1} . Consider

$$\begin{aligned} t_{k+1} &= t_k + m_{k+1} + c_{k+1} \\ &= [R_1 N_1] n_1 + [R_2 N_2] (n_2 + m_2) + \dots + [R_{k+1} N_{k+1}] (n_{k+1} + m_{k+1}). \end{aligned} \quad (17)$$

For $x \in \mathcal{T}_k$ and $y \in \mathcal{C}_{k+1}$, let $z = z(x, y)$ be some point such that

$$d_{t_k}(x, z) < \frac{\varepsilon}{2^{k+1}} \text{ and } d_{c_{k+1}}(y, f^{t_k+m_{k+1}}(z)) < \frac{\varepsilon}{2^{k+1}}. \quad (18)$$

Observe that the existence of such a point is guaranteed by the specification property of f . Then, let us consider

$$\mathcal{T}_{k+1} = \{z(x, y) : x \in \mathcal{T}_k, y \in \mathcal{C}_{k+1}\}.$$

By proceeding as in the proof of the Lemma 3.5 we can see that different pairs (x, y) , $x \in \mathcal{T}_k$, $y \in \mathcal{C}_{k+1}$, produce different points $z = z(x, y)$. In particular, $\#\mathcal{T}_{k+1} = \#\mathcal{T}_k \cdot \#\mathcal{C}_{k+1}$. Therefore, proceeding inductively,

$$\#\mathcal{T}_k = \#\mathcal{C}_1 \dots \#\mathcal{C}_k = M_1^{[R_1 N_1]} \dots M_k^{[R_k N_k]}.$$

In particular, by Lemma 3.5 and (18) we have that for every $x \in \mathcal{T}_k$ and $y, y' \in \mathcal{C}_{k+1}$ with $y \neq y'$,

$$d_{t_k}(z(x, y), z(x, y')) < \frac{\varepsilon}{2^{k+2}} \text{ and } d_{t_{k+1}}(z(x, y), z(x, y')) > \frac{3\varepsilon}{4}. \quad (19)$$

For every $k \in \mathbb{N}$ let us consider

$$F_k := \bigcup_{x \in \mathcal{T}_k} \overline{B}_{t_k}(x, \varepsilon/2^{k+1}),$$

where $\overline{B}_{t_k}(x, \varepsilon/2^{k+1})$ denotes the closure of the open ball $B_{t_k}(x, \varepsilon/2^{k+1})$. As a simple consequence of (19) we have the following observation.

Lemma 3.6 (Lemma 5.2 of [32]). *For every k the following is satisfied:*

- (1) *for any $x, x' \in \mathcal{T}_k$, $x \neq x'$, the sets $\overline{B}_{t_k}(x, \varepsilon/2^{k+1})$ and $\overline{B}_{t_k}(x', \varepsilon/2^{k+1})$ are disjoint;*
- (2) *if $z \in \mathcal{T}_{k+1}$ is such that $z = z(x, y)$ for some $x \in \mathcal{T}_k$ and $y \in \mathcal{C}_{k+1}$, then*

$$\overline{B}_{t_{k+1}}\left(z, \frac{\varepsilon}{2^{k+2}}\right) \subset \overline{B}_{t_k}\left(x, \frac{\varepsilon}{2^{k+1}}\right).$$

Hence, $F_{k+1} \subset F_k$.

Consider

$$F := \bigcap_{k \in \mathbb{N}} F_k.$$

Observe that, since each F_k is a closed and non-empty set and, moreover, $F_{k+1} \subset F_k$, the set F is a non-empty and closed set too. Furthermore, using (15) we may prove that

Lemma 3.7 (Lemma 5.3 of [32]). *Under the above conditions,*

$$F \subset K_\alpha.$$

Now, for every $k \geq 1$, let us consider the probability measure η_k given by

$$\eta_k = \frac{1}{\#\mathcal{T}_k} \sum_{z \in \mathcal{T}_k} \delta_z.$$

Observe that, as $\mathcal{T}_k \subset F_k$, $\eta_k(F_k) = 1$. Moreover,

Lemma 3.8 (Lemma 5.4 of [32]). *The sequence of probability measures $(\eta_k)_{k \in \mathbb{N}}$ converges in the weak*-topology to some probability measure η . Furthermore, the limiting measure η satisfies $\eta(F) = 1$.*

An important feature of the measure η that can be obtained by exploring its definition and (14) is that the η -measure of some appropriate dynamical balls decay exponentially fast. More precisely,

Lemma 3.9 (Lemma 5.5 of [32]). *For every n sufficiently large and $q \in X$ so that $B_n(q, \frac{\varepsilon}{2}) \cap F \neq \emptyset$ one has*

$$\eta\left(B_n\left(q, \frac{\varepsilon}{2}\right)\right) \leq \exp(-n(h_\mu(f, \xi) - 8\gamma)).$$

In order to conclude our proof we need a simple yet interesting fact whose proof we include for the sake of completeness. This is a version of the *Entropy Distribution Principle* of [32] (see [32, Theorem 3.6]). Observe that for this result, the measure involved does not need to be invariant, as it is the case of the measure η obtained in the previous lemmas.

Lemma 3.10. *Let $f : X \rightarrow X$ be a continuous transformation and $\varepsilon > 0$. Given a set $Z \subset X$ and a constant $s \geq 0$, suppose there exist a constant $C > 0$ and a Borel probability measure η satisfying:*

- (i) $\eta(Z) > 0$;
- (ii) $\eta(B_n(x, \varepsilon)) \leq Ce^{-ns}$ for every ball $B_n(x, \varepsilon)$ such that $B_n(x, \varepsilon) \cap Z \neq \emptyset$.

Then $h(Z, f, \varepsilon) \geq s$.

Proof of Lemma 3.10. Let $\Gamma = \{B_{n_i}(x_i, \varepsilon)\}_i$ be some cover of Z . Without loss of generality we may assume that $B_{n_i}(x_i, \varepsilon) \cap Z \neq \emptyset$ for every i . In such case we have that

$$\begin{aligned} \sum_i \exp(-sn_i) &\geq C^{-1} \sum_i \eta(B_{n_i}(x_i, \varepsilon)) \geq C^{-1} \eta\left(\bigcup_i B_{n_i}(x_i, \varepsilon)\right) \\ &\geq C^{-1} \eta(Z) > 0. \end{aligned}$$

Therefore, $m(Z, s, \varepsilon) > 0$ and hence $h(Z, f, \varepsilon) \geq s$. \square

By Lemma 3.7 we have that $h(K_\alpha, f, \varepsilon/2) \geq h(F, f, \varepsilon/2)$. Lemmas 3.9 and 3.10 gives us that $h(F, f, \varepsilon/2) \geq h_\mu(f, \xi) - 8\gamma$. Consequently,

$$h(K_\alpha, f, \varepsilon/2) \geq h_\mu(f, \xi) - 8\gamma.$$

Thus, combining this observation with (9) and (12) we get that

$$\begin{aligned} \overline{\text{H}_\varphi \text{mdim}_M}(f, \alpha, d) - 9\gamma &\leq \frac{h_\mu(f, \xi) - 8\gamma}{|\log 5\varepsilon|} \\ &\leq \frac{h(K_\alpha, f, \varepsilon/2)}{|\log \varepsilon/2| + \log 10} \\ &\leq \overline{\text{mdim}_M}(K_\alpha, f, d) + \gamma. \end{aligned}$$

Thus, since $\gamma > 0$ is arbitrary, the proof of the proposition is complete. \square

Proposition 3.11. *Under the hypotheses of Theorem A we have that*

$$H_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) \geq \Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d).$$

Proof. Fix $\gamma > 0$. Let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers which converges to zero and satisfies

$$\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) = \lim_{j \rightarrow \infty} \frac{\Lambda_{\varphi}(\alpha, \varepsilon_j)}{|\log \varepsilon_j|}.$$

Then, there exists $\varepsilon_0 > 0$ so that for all $\varepsilon_j \in (0, \varepsilon_0]$ we have

$$\frac{\Lambda_{\varphi}(\alpha, \varepsilon_j)}{|\log \varepsilon_j|} > \Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma.$$

In particular, for every $\varepsilon_j \in (0, \varepsilon_0]$,

$$\Lambda_{\varphi}(\alpha, \varepsilon_j) > \left(\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j|.$$

Fix $j \in \mathbb{N}$ such that $\varepsilon_j \in (0, \varepsilon_0]$. By the alternative expression of $\Lambda_{\varphi}(\alpha, \varepsilon_j)$ given in (3) it follows that there exists a sequence of positive numbers $(\delta_{j,k})_{k \in \mathbb{N}}$ converging to zero and such that for every $k \in \mathbb{N}$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log M(\alpha, \delta_{j,k}, n, \varepsilon_j) &> \Lambda_{\varphi}(\alpha, \varepsilon_j) - \frac{2}{3}\gamma \\ &> \left(\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j| - \frac{2}{3}\gamma. \end{aligned}$$

Similarly, there exists a sequence $(n_{j,k})_{k \in \mathbb{N}}$ in \mathbb{N} satisfying $\lim_{k \rightarrow \infty} n_{j,k} = +\infty$ and

$$\begin{aligned} M_{j,k} &:= M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j) \\ &> \exp \left(n_{j,k} \left(\left(\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j| - \gamma \right) \right). \end{aligned} \tag{20}$$

Consider a maximal $(n_{j,k}, \varepsilon_j)$ -separated set $C_{j,k}$ of $P(\alpha, \delta_{j,k}, n_{j,k})$. For each $j, k \in \mathbb{N}$ consider

$$\sigma_k^{(j)} = \frac{1}{M_{j,k}} \sum_{x \in C_{j,k}} \delta_x,$$

and

$$\mu_k^{(j)} = \frac{1}{n_{j,k}} \sum_{i=0}^{n_{j,k}-1} (f^i)_* (\sigma_k^{(j)}) = \frac{1}{M_{j,k}} \sum_{x \in C_{j,k}} \frac{1}{n_{j,k}} \sum_{i=0}^{n_{j,k}-1} \delta_{f^i(x)}.$$

It is not difficult to see that any accumulation point of $\{\mu_k^{(j)}\}_{k \in \mathbb{N}}$, say $\mu^{(j)}$, is f -invariant (see [38, Theorem 6.9]). Moreover, $\int_X \varphi d\mu^{(j)} = \alpha$ for every $j \in \mathbb{N}$.

Indeed, we may assume without loss of generality that $\lim_{k \rightarrow \infty} \mu_k^{(j)} = \mu^{(j)}$. Then, for every j and k in \mathbb{N} we have

$$\left| \int_X \varphi d\mu_k^{(j)} - \alpha \right| \leq \frac{1}{M_{j,k}} \sum_{x \in C_{j,k}} \left| \frac{1}{n_{j,k}} \sum_{i=0}^{n_{j,k}-1} \varphi(f^i(x)) - \alpha \right| \leq \delta_{j,k}.$$

Thus,

$$\begin{aligned} \left| \int_X \varphi d\mu^{(j)} - \alpha \right| &\leq \left| \int_X \varphi d\mu^{(j)} - \int_X \varphi d\mu_k^{(j)} \right| + \left| \int_X \varphi d\mu_k^{(j)} - \alpha \right| \\ &\leq \left| \int_X \varphi d\mu^{(j)} - \int_X \varphi d\mu_k^{(j)} \right| + \delta_{j,k}. \end{aligned}$$

Consequently, making $k \rightarrow +\infty$ we conclude that $\int_X \varphi d\mu^{(j)} = \alpha$ for every $j \in \mathbb{N}$ as claimed.

For every $j \in \mathbb{N}$ choose a Borel partition $\xi(j) = \{A_1, \dots, A_\ell\}$ of X so that $\text{diam}(\xi(j)) < \varepsilon_j$ and $\mu^{(j)}(\partial A_i) = 0$ for $0 \leq i \leq \ell$ (see [38, Lemma 8.5(ii)]). Then,

$$H_{\sigma_k^{(j)}} \left(\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j) \right) = \log M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j).$$

Indeed, observe that if x and y belong to the same element of $\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j)$ then $d_{n_{j,k}}(x, y) < \varepsilon_j$. In particular, no element of $\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j)$ can contain more than one point of a maximal $(n_{j,k}, \varepsilon_j)$ -separated set. Thus, exactly $M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j)$ elements of $\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j)$ have $\sigma_k^{(j)}$ -measure equal to $\frac{1}{M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j)}$. All others have zero $\sigma_k^{(j)}$ -measure.

Fix natural numbers q and $n_{j,k}$ with $1 < q < n_{j,k}$ and define, for $0 \leq s \leq q-1$, $a(s) = [(n_{j,k} - s)/q]$ where $[p]$ denotes the integer part of p . Fix $0 \leq s \leq q-1$. Then, by [38, Remark 2(ii), p. 188] we have that

$$\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j) = \bigvee_{r=0}^{a(s)-1} f^{-(rq+s)} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) \vee \bigvee_{t \in L} f^{-t}\xi(j)$$

where L is a set with cardinality at most $2q$. Therefore, using [38, Theorem 4.3(viii)] and [38, Corollary 4.2.1],

$$\begin{aligned} \log M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j) &= H_{\sigma_k^{(j)}} \left(\bigvee_{i=0}^{n_{j,k}-1} f^{-i}\xi(j) \right) \\ &\leq \sum_{i=0}^{a(s)-1} H_{\sigma_k^{(j)}} f^{-(rq+s)} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) + \sum_{t \in L} H_{\sigma_k^{(j)}}(f^{-t}\xi(j)) \\ &\leq \sum_{i=0}^{a(s)-1} H_{\sigma_k^{(j)} \circ f^{-(rq+s)}} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) + 2q \log(\ell). \end{aligned}$$

Summing the previous inequality over s from 0 to $q-1$ and using [38, Remark 2(iii), p. 188], we get that that

$$q \log M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j) \leq \sum_{p=0}^{n_{j,k}-1} H_{\sigma_k^{(j)} \circ f^{-p}} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) + 2q^2 \log(\ell).$$

Thus, dividing everything by $n_{j,k}$ in the above inequality and using (20) and the concavity of the map $\mu \rightarrow H_\mu(\xi)$ we obtain

$$\begin{aligned} & q \left(\left(\Lambda_\varphi \overline{\text{mdim}}_M(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j| - \gamma \right) \\ & < \frac{q}{n_{j,k}} \log M(\alpha, \delta_{j,k}, n_{j,k}, \varepsilon_j) \\ & \leq H_{\mu_k^{(j)}} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) + \frac{2q^2 \log(\ell)}{n_{j,k}}. \end{aligned} \tag{21}$$

Now, since the elements of $\bigvee_{i=0}^{q-1} f^{-i}\xi(j)$ have boundaries of $\mu^{(j)}$ -measure zero, it follows from the weak convergence of the measures $\mu_k^{(j)}$ to $\mu^{(j)}$ that $\lim_{k \rightarrow \infty} \mu_k^{(j)}(B) =$

$\mu^{(j)}(B)$ for each element B of $\bigvee_{i=0}^{q-1} f^{-i}\xi(j)$ and, therefore,

$$\lim_{k \rightarrow \infty} H_{\mu_k} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right) = H_{\mu^{(j)}} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right).$$

Thus, by (21) we have that

$$q \left(\left(\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j| - \frac{2}{3}\gamma \right) \leq H_{\mu^{(j)}} \left(\bigvee_{i=0}^{q-1} f^{-i}\xi(j) \right).$$

Dividing both sides of the previous inequality by q and letting q go to $+\infty$ we obtain

$$\left(\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \right) \cdot |\log \varepsilon_j| - \frac{2}{3}\gamma \leq h_{\mu^{(j)}}(f, \xi(j)), \text{ for all } j \in \mathbb{N},$$

which implies that

$$\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \leq \frac{h_{\mu^{(j)}}(f, \xi(j)) + \frac{2}{3}\gamma}{|\log \varepsilon_j|}, \text{ for all } j \in \mathbb{N}.$$

Therefore,

$$\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma \leq \frac{\inf_{|\xi| < \varepsilon_j} h_{\mu^{(j)}}(f, \xi) + \gamma}{|\log \varepsilon_j|} \text{ for all } j \in \mathbb{N}$$

and consequently,

$$\begin{aligned} \Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) - \frac{1}{3}\gamma &\leq \limsup_{j \rightarrow \infty} \frac{\sup_{\nu \in \mathcal{M}_f(X, \alpha, d)} \inf_{|\xi| < \varepsilon_j} h_{\nu}(f, \xi)}{|\log \varepsilon_j|} \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{\sup_{\nu \in \mathcal{M}_f(X, \alpha, d)} \inf_{|\xi| < \varepsilon} h_{\nu}(f, \xi)}{|\log \varepsilon|} \\ &= H_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d) \end{aligned}$$

completing the proof of the proposition. \square

Finally, the first claim of Theorem A follows directly by combining Propositions 3.2, 3.3 and 3.11. For the general case, that is, without the assumption that the quantities $\overline{\text{mdim}}_{\text{M}}(K_{\alpha}, f, d)$, $\Lambda_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ and $H_{\varphi} \overline{\text{mdim}}_{\text{M}}(f, \alpha, d)$ are finite, we observe that a simple modification of our proof show us that if one of the quantities is infinite then the other two must also be infinite and, therefore, the first claim of Theorem A is still true. As for the second claim in Theorem A, again one can easily see that simple adaptations of the previous proof yield the desired conclusion. The proof of Theorem A is complete.

4. EXAMPLES

In this section we present some examples of settings with positive upper/lower metric mean dimension where our results may be applied. Moreover, we also present a simple application of Theorem A to calculate $\text{mdim}_{\text{M}}(K_{\alpha}, f, d)$.

Example 4.1. Let (Z, D) be a compact metric space with upper box-counting dimension $\overline{\text{dim}}_{\text{B}} Z < \infty$. Let us consider $X = Z^{\mathbb{N}}$ endowed with the metric

$$d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} \frac{1}{2^n} D(x_n, y_n)$$

and let $\sigma : X \rightarrow X$ be the shift map. It is well known that σ has the specification property and $\overline{\text{mdim}}_{\text{M}}(X, \sigma, d) = \overline{\text{dim}}_{\text{B}} Z$ and $\underline{\text{mdim}}_{\text{M}}(X, \sigma, d) = \underline{\text{dim}}_{\text{B}} Z$ (see for

instance [28]). In particular, we may apply Theorem A to it getting, for instance, that for any $\varphi \in C^0(X, \mathbb{R})$ and $\alpha \in \mathbb{R}$,

$$\overline{\text{mdim}}_{\text{M}}(K_\alpha, \sigma, d) = \Lambda_\varphi \overline{\text{mdim}}_{\text{M}}(\sigma, \alpha, d) = \text{H}_\varphi \overline{\text{mdim}}_{\text{M}}(\sigma, \alpha, d).$$

Example 4.2. Let $X = [0, 1]^\mathbb{N}$ be endowed with the metric induced by the Euclidian distance in $[0, 1]$ as in the previous example and consider the set

$$E = \left\{ \{x^{(i,j)}\}_{i,j \in \mathbb{N}} \in X : x_n^{(i,j)} = \frac{1}{2^j} \text{ if } i = n \text{ and } x_n^{(i,j)} = 0 \text{ if } i \neq n \right\} \cup \{e\},$$

where $e = (0, 0, \dots)$, which is closed and shift invariant. If 2^E denotes the space of subsets of X endowed with the Hausdorff distance d_H , by [20, Proposition 3.6] we have that

$$\overline{\text{mdim}}_{\text{M}}(E, \sigma, d) = 0 \text{ and } \text{mdim}_{\text{M}}(2^E, \sigma_\sharp, d_H) = 1,$$

where σ_\sharp is the induced map by σ on the hyperspace 2^E . By [6, Proposition 4] we have that σ_\sharp has the specification property and then Theorem A may be applied.

Example 4.3. It was proved in [1, 9] that for C^0 -generic homeomorphisms acting on a compact and smooth manifold X with dimension greater than one, the upper metric mean dimension with respect to the smooth metric coincides with the dimension of the manifold. Moreover, they also proved that the set of homeomorphisms with positive lower metric mean dimension is C^0 dense in the set of homeomorphisms of X . Now, in order to be able to apply Theorem A to elements of those sets, we need to guarantee that they have the specification property. For this purpose we restrict ourselves to the set of conservative homeomorphisms, where the specification property holds C^0 -generically.

We fix a good Borel probability measure $\mu \in \mathcal{M}(X)$, i.e., a probability measure that satisfies the following conditions:

- (C₁) [Non-atomic] For every $x \in X$ one has $\mu(\{x\}) = 0$;
- (C₂) [Full support] For every nonempty open set $U \subset X$ one has $\mu(U) > 0$;
- (C₃) [Boundary with zero measure] $\mu(\partial X) = 0$.

In a forthcoming paper by S. Rom ana and G. Lacerda it is proved that there exists a Baire generic subset of $\text{Homeo}_\mu(X, d)$ (the set of conservative homeomorphisms on X) with metric mean dimension equal to the dimension of X . Consequently, since according to [14] the specification property is a Baire generic property in $\text{Homeo}_\mu(X, d)$, there exists a C^0 -open and dense subset of $\text{Homeo}_\mu(X, d)$ whose elements have positive upper metric mean dimension and the specification property and, in particular, Theorem A may be applied to those elements.

In the next two examples we consider the specification property for linear operators acting on Banach spaces and we start by recalling the appropriate definition for this setting. Let B be a Banach space over \mathbb{K} ($= \mathbb{R}$ or \mathbb{C}) and $T : B \rightarrow B$ be a linear operator. We say that T has the *operator specification property* if there exists a sequence of T -invariant sets $\{K_m\}_{m \in \mathbb{N}}$ with $B = \bigcup_{m \in \mathbb{N}} K_m$ for which $T|_{K_m} : K_m \rightarrow K_m$ satisfies the specification property. We emphasize that the sets K_m do not need to be compact, although in the all known examples we have compactness for such sets.

Example 4.4. Fix $\nu = (\nu_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ so that $\nu_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \nu_n < \infty$. Given $1 \leq p < +\infty$, consider

$$\ell^p(\nu) = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \|(x_n)_{n \in \mathbb{N}}\|_{\ell^p(\nu)} := \left(\sum_{n=1}^{\infty} |x_n|^p \nu_n \right)^{\frac{1}{p}} < \infty \right\},$$

which is a Banach space, and the shift map $\sigma : \ell^p(\nu) \rightarrow \ell^p(\nu)$. By [5, Theorem 2.1] we have that $\sigma : \ell^p(\nu) \rightarrow \ell^p(\nu)$ has the operator specification property with $K_m = mK$, $m \in \mathbb{N}$, where K is the compact set $K = \{(x_n)_{n \in \mathbb{N}} \in \ell^p(\nu) : |x_n| \leq 1 \text{ for all } n \in \mathbb{N}\}$. We now observe that $T|_K : K \rightarrow K$ has positive metric mean dimension. More precisely,

Lemma 4.5.

$$\overline{\text{mdim}}_{\text{M}}(K, \sigma, \|\cdot\|_{\ell^p(\nu)}) = \underline{\text{mdim}}_{\text{M}}(K, \sigma, \|\cdot\|_{\ell^p(\nu)}) = 1.$$

Proof. Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we observe that

$$\left\{ x \in K : x_i \in \left\{ 0, \frac{\varepsilon}{\sqrt[p]{\nu_1}}, \frac{2\varepsilon}{\sqrt[p]{\nu_1}}, \dots, \left\lfloor 1/\frac{\varepsilon}{\sqrt[p]{\nu_1}} \right\rfloor \frac{\varepsilon}{\sqrt[p]{\nu_1}} \right\} \text{ for all } 1 \leq i \leq n \right\}$$

is a (n, ε) -separated set in K . In particular,

$$\begin{aligned} \underline{\text{mdim}}_{\text{M}}(K, \sigma, \|\cdot\|_{\ell^p(\nu)}) &= \liminf_{\varepsilon \rightarrow 0} \frac{h(\sigma, \varepsilon)}{|\log \varepsilon|} \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{\limsup_{n \rightarrow \infty} \frac{1}{n} \left| \log \left(\left\lfloor 1/\frac{\varepsilon}{\sqrt[p]{\nu_1}} \right\rfloor \right)^n \right|}{|\log \varepsilon|} = 1. \end{aligned} \quad (22)$$

In order to get the reverse inequality, let $\ell \in \mathbb{N}$ be so that $\sum_{n \geq \ell} \nu_n < \frac{\varepsilon}{2}$ and define

$$M = \left(\sum_{k \in \mathbb{N}} \nu_k \right)^{1/p} > 0. \text{ We consider an open cover of } [-1, 1] \text{ by}$$

$$I_k = \left(\frac{(k-1)\varepsilon}{12M}, \frac{(k+1)\varepsilon}{12M} \right), \text{ for } -\lfloor 12M/\varepsilon \rfloor \leq k \leq \lfloor 12M/\varepsilon \rfloor.$$

Note that each I_k has length $\frac{\varepsilon}{6M}$. Given $n \geq 1$, let us consider the following open cover of $K^{\mathbb{N}}$:

$$\{x : x_1 \in I_{k_1}, x_2 \in I_{k_2}, \dots, x_{n+\ell} \in I_{k_{n+\ell}}\},$$

where $-\lfloor 12M/\varepsilon \rfloor \leq k_1, \dots, k_{n+\ell} \leq \lfloor 12M/\varepsilon \rfloor$. Observe that each element of this open cover has diameter less than ε with respect to the metric d_n (induced by $\|\cdot\|_{\ell^p(\nu)}$). So,

$$\limsup_{\varepsilon \rightarrow 0} \frac{h(\sigma, \varepsilon)}{|\log \varepsilon|} \leq \limsup_{\varepsilon \rightarrow 0} \frac{\limsup_{n \rightarrow \infty} \frac{1}{n} \log (2 \lfloor 12M/\varepsilon \rfloor)^{n+\ell+1}}{|\log \varepsilon|} = 1.$$

Hence,

$$\overline{\text{mdim}}_{\text{M}}(K, \sigma, \|\cdot\|_{\ell^p(\nu)}) \leq 1. \quad (23)$$

Finally, combining (22) and (23) we get the desired result. \square

As a consequence of the previous proof we also get that

$$\text{mdim}_{\text{M}}(K_m, \sigma, \|\cdot\|_{\ell^p(\nu)}) = 1$$

for all $m \in \mathbb{N}$. In particular, we may apply Theorem A to $\sigma|_{K_m} : K_m \rightarrow K_m$ for every $m \in \mathbb{N}$.

Example 4.6. Another class of examples is given by the weighted shifts. Let $\nu = (\nu_n)_{n \in \mathbb{N}} = (1)_{n \in \mathbb{N}}$ and consider $\ell^p(\nu)$ as in the previous example. Observe that in this case $\ell^p(\nu) = \ell^p$. Let $w = (w_n)_{n \in \mathbb{N}}$ be a weight sequence and define the *weighted shift* on ℓ^p as $B_w((x_n)_{n \in \mathbb{N}}) = (w_{n+1}x_{n+1})_{n \in \mathbb{N}}$. It was observed in [5, p. 602] that if one considers $a = (a_n)_{n \in \mathbb{N}}$ given by

$$a_1 = 1 \text{ and } a_n := w_2 \dots w_n, \text{ for all } n > 1,$$

and $\bar{\nu} = (\bar{\nu}_n)_{n \in \mathbb{N}}$ given by

$$\bar{\nu}_n = \frac{1}{\prod_{j=2}^n |w_j|^p}, \text{ for all } n \in \mathbb{N},$$

then

$$\phi_a : (x_n)_{n \in \mathbb{N}} \in \ell^p \mapsto \phi_a((x_n)_{n \in \mathbb{N}}) = (a_1 x_1, a_2 x_2, \dots) \in \ell^p(\bar{\nu})$$

defines a topological conjugacy between the weighted shift and the backward shift given in the previous example. Moreover, they observed that this topological conjugacy is also an isometry, which implies that

$$\overline{\text{mdim}}_{\text{M}} \left(\phi_a^{-1}(K_m), B_w, \|\cdot\|_{\ell^p} \right) = \underline{\text{mdim}}_{\text{M}} \left(\phi_a^{-1}(K_m), B_w, \|\cdot\|_{\ell^p} \right) = 1$$

for all $m \in \mathbb{N}$. Furthermore, if $\sum_{n=1}^{\infty} \bar{\nu}_n < \infty$ we have that B_w has the operator specification property (see [5, Theorem 2.3]) and then we are in the context of Theorem A.

Example 4.7. Let us consider $X = [0, 1]^{\mathbb{Z}}$ endowed with the metric

$$d((x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}}) = \sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} |x_n - y_n|$$

and let $\sigma : X \rightarrow X$ be the left shift map. Similarly to Example 4.1, σ has the specification property and, moreover, $\text{mdim}_{\text{M}}(X, \sigma, d) = 1$. In particular, Theorem A may be applied in this setting. Let λ be the Lebesgue measure on $[0, 1]$ and consider $\mu = \lambda^{\mathbb{Z}}$. Then, it is a well known fact that μ is ergodic. Given $\varphi \in C^0(X, \mathbb{R})$, take $\alpha = \int \varphi d\mu$. We will show that $\text{mdim}_{\text{M}}(K_\alpha, \sigma, d) = 1$. In order to do it we recall the definition of *Brin-Katok local entropy*: for an ergodic measure $\mu \in \mathcal{M}_\sigma(X)$, $\varepsilon > 0$ and a point $x \in X$, let us consider

$$h_\mu^{BK}(\varepsilon, x) = \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)),$$

where $B_n(x, \varepsilon)$ is defined as in Section 2. Since μ is ergodic, the map $x \mapsto h_\mu^{BK}(\varepsilon, x)$ is constant μ -almost everywhere. Denote this constant by $h_\mu^{BK}(\varepsilon)$. Then, we have the following observation.

Lemma 4.8 (See [17].). *For any ergodic measures $\mu \in \mathcal{M}_\sigma(X)$ and any $\varepsilon > 0$,*

$$h_\mu^{BK}(\varepsilon) \leq \inf_{|\xi| < \varepsilon} h_\mu(\sigma, \xi),$$

where the infimum is taken over all finite measurable partitions of X with diameter smaller than ε .

Therefore, considering the measure $\mu = \lambda^{\mathbb{Z}}$ given above and using Theorem A we get that

$$\begin{aligned} \underline{\text{mdim}}_{\text{M}} \left(K_\alpha, \sigma, d \right) &= H_\varphi \underline{\text{mdim}}_{\text{M}}(f, \alpha, d) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} \inf_{|\xi| < \varepsilon} h_\mu(f, \xi) \\ &\geq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} h_\mu^{BK}(\varepsilon). \end{aligned}$$

Now, in [29, Example 12] it is proved that $\liminf_{\varepsilon \rightarrow 0} \frac{1}{|\log \varepsilon|} h_{\mu}^{BK}(\varepsilon) \geq 1$. Consequently,

$$\begin{aligned} 1 &= \text{mdim}_{\mathbb{M}}(X, \sigma, d) = \overline{\text{mdim}}_{\mathbb{M}}(X, \sigma, d) \\ &\geq \overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d) \geq \underline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d) \geq 1 \end{aligned}$$

and thus, $\text{mdim}_{\mathbb{M}}(K_{\alpha}, \sigma, d) = \overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d) = \underline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d) = 1$ as claimed. We observe that our Theorem A combined with Lemma 4.8 may be very useful for giving lower bounds for $\overline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d)$ and $\underline{\text{mdim}}_{\mathbb{M}}(K_{\alpha}, \sigma, d)$. In fact, it is enough to take an ergodic measure μ satisfying $\alpha = \int \varphi d\mu$ and estimate $h_{\mu}^{BK}(\varepsilon)$ which, in general, may be easier than estimating $\inf_{|\xi| < \varepsilon} h_{\mu}(\sigma, \xi)$.

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REFERENCES

- [1] J. M. Acevedo, S. Romaña and R. Arias. *Density of the level sets of the metric mean dimension for homeomorphisms*, Preprint <https://arxiv.org/abs/2207.11873>, 2022.
- [2] L. Barreira, Y. Pesin and J. Schmeling. *On a general concept of multifractality: multifractal spectra for dimensions, entropies, and Lyapunov exponents*. *Multifractal rigidity*, *Chaos* **7** (1997), 27–38.
- [3] L. Barreira, Y. Pesin and J. Schmeling. *Multifractal spectra and multifractal rigidity for horseshoes*. *J. Dyn. Control Syst.* **3** (1997), 33–49.
- [4] L. Barreira and B. Saussol. *Variational principle and multifractal spectra*, *Transactions of the American Mathematical Society* **353** (2001), 3919–3944.
- [5] S. Bartoll, F. Martínez-Giménez and A. Peris. *The specification property for backward shifts*, *Journal of Difference Equations and Applications* (2012), **18**, 599–605.
- [6] W. Bauer and K. Sigmund. *Topological Dynamics of Transformations Induced on the Space of Probability Measures*, *Monatshefte für Mathematik* (1975), **79**, 81–92.
- [7] N. C. Bernardes, P. R. Cirilo, U. B. Darji, A. Messaoudi and E. R. Pujals. *Expansivity and shadowing in linear dynamics*, *J. Math. Anal. and App.* **461** (2018), 796–816.
- [8] M. Brin and G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 2004.
- [9] M. Carvalho, F. B. Rodrigues and P. Varandas, *Generic homeomorphisms have full metric mean dimension*, *Ergod. Th. and Dynam. Sys.* **42** (2022), 40–64.
- [10] D. Cheng, Z. Liand B. Selmi. *Upper metric mean dimensions with potential on subsets*. *Nonlinearity* **34** (2021), 852–867.
- [11] V. Climenhaga, *The thermodynamic approach to multifractal analysis*. *Ergodic Theory and Dynamical Systems*, **34** (2014), 1409–1450.
- [12] M. Gromov *Topological invariants of dynamical systems and spaces of holomorphic maps I*, *Math. Phys. Anal. Geom.* **2** (1999), 323–415.
- [13] P.-A. Guihéneuf. *Propriétés dynamiques génériques des homéomorphismes conservatifs*. *Ensaio Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro (2012).
- [14] P.-A. Guihéneuf and T. Lefeuvre. *On the genericity of the shadowing property for conservative homeomorphisms*. *Proc. Amer. Math. Soc.* **146** (2018), 4225–4237.
- [15] Y. Gutman, A. Śpiewak, *New uniform bounds for almost lossless analog compression*, In 2019 IEEE International Symposium on Information Theory (ISIT), pages 1702–1706, 2019.
- [16] Y. Gutman, A. Śpiewak, *Metric mean dimension and analog compression*, *IEEE Transactions on Information Theory*, **66** (2020), 6977–6998.
- [17] Y. Gutman, A. Śpiewak, *Around the variational principle for metric mean dimension*, *Studia Mathematica* **261** (2021), 345–360.
- [18] Y. Gutman and M. Tsukamoto, *Embedding minimal dynamical systems into Hilbert cubes*, *Invent. Math.*, **221** (2020), 113–166.
- [19] W. Huang, A. Maass, P. Romagnoly and X. Ye, *Entropy pairs and local Abramov formula for measure-theoretic entropy for a cover*, *Ergodic Theory and Dynamical Systems* **24** (2004), 1127–1153.

- [20] X. Huang and X. Wang, *The metric mean dimension of hyperspace induced by symbolic dynamical systems*. International Journal of General Systems. DOI: [03081079.2022.2052060](https://doi.org/10.1080/03081079.2022.2052060).
- [21] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Etudes Sci. Publ. Math. **51** (1980), 137–173.
- [22] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge, 1995.
- [23] E. Lindenstrauss and M. Tsukamoto, *From rate distortion theory to metric mean dimension: variational principle*, IEEE Trans. Inform. Theory, **64** (2018), 3590–3609.
- [24] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math., **115** (2000), 1–24.
- [25] L. Olsen, *A multifractal formalism*, Adv. Math. **116** (1995), 82–196.
- [26] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications* Chicago Lectures in Mathematics. Chicago, 1997.
- [27] M. Pollicott and M. Yuri, *Dynamical Systems and Ergodic Theory*, Cambridge University Press, Cambridge, 1998.
- [28] F.B. Rodrigues and J.M. Acevedo. *Mean Dimension and Metric Mean Dimension for Non-autonomous Dynamical Systems*, J Dyn Control Syst (2021). DOI: [s10883-021-09541-6](https://doi.org/10.1088/1088-3601/202101010101).
- [29] R. Shi, *On variational principles for metric mean dimension*, IEEE Transactions on Information Theory. **68** (2022), 4282–4288.
- [30] Y. Shi, X. Tian, P. Varandas, X. Wang, *On multifractal analysis and large deviations of singular-hyperbolic attractors*, Preprint <https://arxiv.org/abs/2111.05477>, 2021.
- [31] F. Takens and E. Verbitski, *Multifractal analysis of local entropies for expansive homeomorphisms with specification*, Comm. Math. Phys. **203** (1999), 593–612.
- [32] F. Takens and E. Verbitskiy. *On the variational principle for the topological entropy of certain non-compact sets*, Ergod. Th. and Dynam. Sys. **23** (2003), 317–348.
- [33] D. Thompson. *A variational principle for topological pressure for certain non-compact sets*, Journal of the London Mathematical Society **80** (2009), 585–602.
- [34] M. Tsukamoto. *Double variational principle for mean dimension with potential*. Advances in Mathematics **361** (2020).
- [35] A. Velozo and R. Velozo, *Rate distortion theory, metric mean dimension and measure theoretic entropy*, Preprint <https://arxiv.org/abs/1707.05762>, 2017.
- [36] R. Yang, E. Chen and X. Zhou. *Some notes on variational principle for metric mean dimension*. To appear in IEEE Transactions on Information Theory. DOI: [10.1109/TIT.2022.3229058](https://doi.org/10.1109/TIT.2022.3229058).
- [37] L. S. Young. *Large deviations in dynamical systems*, Trans. Amer. Math. Soc. **318** (1990), 525–543
- [38] P. Walters, *An Introduction to Ergodic Theory* (Graduate Texts in Mathematics, 79). Springer, New York, 1982.
- [39] K. Yano, *A remark on the topological entropy of homeomorphisms*, Invent. Math. **59** (1980) 215–220.

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