# Explicit Rate-Optimal Streaming Codes with Smaller Field Size 

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#### Abstract

Streaming codes are a class of packet-level erasure codes that ensure packet recovery over a sliding window channel which allows either a burst erasure of size $b$ or $a$ random erasures within any window of size $(\tau+1)$ time units, under a strict decoding-delay constraint $\tau$. The field size over which streaming codes are constructed is an important factor determining the complexity of implementation. The best known explicit rate-optimal streaming code requires a field size of $q^{2}$ where $q \geq \tau+b-a$ is a prime power. In this work, we present an explicit rate-optimal streaming code, for all possible $\{a, b, \tau\}$ parameters, over a field of size $q^{2}$ for prime power $q \geq \tau$. This is the smallest-known field size of a general explicit rate-optimal construction that covers all $\{a, b, \tau\}$ parameter sets. We achieve this by modifying the non-explicit code construction due to Krishnan et al. to make it explicit, without change in field size.


## Index Terms

Low-latency communication, streaming codes, packet-level FEC, random or burst erasure

## I. Introduction

Enabling low-latency reliable communication for applications such as telesurgery, industrial automation, augmented reality and vehicular communication is a key target of 5 G communication systems. For instance, telesurgery camera flow requires packet-loss rate less than $10^{-3}$ and end-to-end latency below 150 ms [1]. To combat the packet drops that are an inevitable part of any communication network, one approach is to employ feedback-based methods such as ARQ. But such feedback-based schemes incur round-trip propagation delay, making it difficult to meet low-latency requirements. Blind re-transmission of packets is another option, but is inefficient as it amounts to using a repetition code. Streaming codes are a class of packet-level erasure codes and represent a natural way of achieving reliable, low-latency communication at the packet level.

A packet-expansion encoding framework for streaming codes was introduced in [2]. Given message packet at time $t$ denoted by $\underline{u}(t) \in \mathbb{F}_{q}^{k}$, the coded packet $\underline{x}(t) \in \mathbb{F}_{q}^{n}$, at time $t$, is generated by appending parity packet $\underline{p}(t) \in \mathbb{F}_{q}^{n-k}$ to $\underline{x}(t)$. More formally, $\underline{x}(t)=\left[\underline{u}(t)^{T} \underline{p}(t)^{T}\right]^{T}$. The encoder is causal and hence $\underline{p}(t)$ depends only on $\underline{u}(t)$ and prior message packets. In [2], [3], streaming codes that can handle burst erasure of size $b$ under a decoding-delay constraint $\tau$ are presented. The decoding-delay constraint $\tau$ means that for recovery of message packet $\underline{u}(t)$ only packets with index $\leq t+\tau$ can be accessed. In [4], Badr et al. presented a delay constrained sliding window (DCSW) channel model that allows burst or random erasures. This channel can be viewed as a deterministic approximation of the Gilbert-Elliot channel [5]. In the DCSW channel model, within any sliding window of size $w$ time units, either a burst erasure of length $\leq b$ or else, at most $a$ random erasures can occur. Additionally there is a decoding-delay constraint $\tau$. This model is non-trivial only if $0<a \leq b \leq \tau$. As it turns out, we can, without loss in generality, set $w=\tau+1$ (see [6]). Thus the DCSW channel is parameterized by the three-parameter set $\{a, b, \tau\}$. An $(a, b, \tau)$ streaming code is a packet-level code that can recover from all the permissible erasure patterns of an $\{a, b, \tau\}$ DCSW channel, within decoding-delay $\tau$. Some other models of erasure codes for streaming can be found in [7]-[10].

In [4] an upper bound on the rate of an $(a, b, \tau)$ streaming code was presented and it was later shown in [11], [12] that this rate is achievable for all possible $\{a, b, \tau\}$ parameters. It follows from these results that the optimal
rate of $(a, b, \tau)$ streaming code is given by $R_{\mathrm{opt}}=\frac{\tau+1-a}{\tau+1-a+b}$. The rate-optimal codes presented in [11], [12] required a finite field alphabet that is exponential in $\tau$. A non-explicit rate-optimal streaming code, which requires a field $\mathbb{F}_{q^{2}}$ with prime power $q \geq \tau$, is presented in [6]. Subsequently, an explicit construction was presented in [13] requiring field size $q^{2}$, for $q \geq \tau+b-a$, a prime power. Streaming codes for variable packet sizes are explored in [14]. Explicit rate-optimal constructions having linear field size for some $\{a, b, \tau\}$ parameter ranges are presented in [6], [15], [16]. However, the construction in [13] remains the smallest field size explicit rate-optimal streaming code construction that exists for all possible $\{a, b, \tau\}$. Note that the field size required for the explicit code construction in [13] is larger than the field size $q^{2} \geq \tau^{2}$ requirement of the code in [6]. In the present paper, we present an explicit rate-optimal code having the same field-size requirement $q^{2}$, with prime power $q \geq \tau$, as that of the non-explicit code in [6]. Smaller field size constructions simplify implementation and are hence of significant, practical interest.

The principal contribution of the paper is thus an explicit rate-optimal streaming code construction for all possible $\{a, b, \tau\}$ parameters. The construction is motivated by the structure of the non-explicit code in [6] and has smallest known field size of an explicit rate-optimal streaming code construction that holds for all $\{a, b, \tau\}$ parameters.

Section $\Pi$ presents the diagonal-embedding framework for embedding a scalar code within the packet stream. The explicit construction of the scalar code having field size $q^{2} \geq \tau^{2}$ is presented in Section III. Proof that this construction, in conjunction with diagonal embedding, results in a rate-optimal streaming code is presented in Section IV.

We use $[a: b]$ to denote the set $\{a, a+1, \ldots, b-1, b\}$. Given a $(k \times n)$ matrix $M, I \subseteq[0: k-1]$ and $J \subseteq[0: n-1], M(I, J)$ will denote the sub-matrix of $M$ comprised of rows with row-index in $I$ and columns with column-index in $J$. We use $|M|$ to denote the determinant of $M, I_{u}$ denotes $(u \times u)$ identity matrix and $\underbrace{0}$
will denote the $(u \times v)$ all-zero matrix.

## II. Preliminaries

## A. Diagonal Embedding

Diagonal embedding, introduced in [3], can be viewed as a framework for deriving a packet-level code from a scalar code. This technique has been consistently used in the streaming-code literature. Let $\mathcal{C}$ be an $[n, k]$ scalar code in systematic form, with first $k$ code symbols being message symbols. Consider a packet-level code with coded packet at time $t$ denoted by $\underline{x}(t)=\left[x_{0}(t) x_{1}(t) \ldots x_{n-1}(t)\right]^{T}$. We will say that the packet-level code is obtained by diagonal embedding of the scalar code $\mathcal{C}$ if for all $t$, each $n$-tuple $\left(x_{0}(t), x_{1}(t+1), \ldots, x_{n-1}(t+n-1)\right)$ is a codeword in the scalar code $\mathcal{C}$. The packet-level code shares the rate $\frac{k}{n}$ of the underlying scalar code. Diagonal embedding is illustrated in Fig. 1 .


Fig. 1: Packet-level code constructed by diagonal embedding of a scalar code of block length 6. Here each column indicates a coded packet.

## B. Properties Required of the Scalar Code

Let $\delta=b-a$. In order to show that the packet-level code constructed through diagonal embedding of an $[n=\tau+1+\delta, k=n-b]$ scalar code $\mathcal{C}$ is a rate-optimal $(a, b, \tau)$ streaming code it suffices to show that the following erasure recovery properties hold for codeword $\left(c_{0}, c_{1}, \ldots, c_{n-1}\right) \in \mathcal{C}$ that occupies the time indices $0 \leq t \leq(n-1)$. Analogous, time-shifted versions of these below conditions apply to the other embedded codewords (see [6] for details).

B1 Any code symbol $c_{t}$ with $t \in[0: \delta-1]$ should be recoverable from the erasure of a burst of $b$ packets, corresponding to time indices in $[t: t+b-1]$, by accessing non-erased code symbols in the set $\left\{c_{i} \mid t<\right.$ $i \leq \tau+t\} \cup\left\{c_{i} \mid i<t\right\}$. The latter set represents previously-decoded code symbols.
R1 Any code symbol $c_{t}$ with $t \in[0: \delta-1]$ should be recoverable from any $a$ random packet erasures, corresponding to time indices $t$ and $(a-1)$ indices in $[t+1: \tau+t]$, by accessing non-erased code symbols in the set $\left\{c_{i} \mid t<i \leq \tau+t\right\} \cup\left\{c_{i} \mid i<t\right\}$.
B2 For any $t \in[\delta: \tau+1-a]$, code symbols $\left\{c_{i} \mid i \in[t: t+b-1]\right\}$ should be recoverable by accessing remaining code symbols $\left\{c_{i} \mid i \notin[t: t+b-1]\right\}$.
R2 For any set $A \subseteq[\delta: \tau+\delta]$ with $|A|=a$, code symbols $\left\{c_{i} \mid i \in A\right\}$ should be recoverable by accessing the remaining code symbols $\left\{c_{i} \mid i \notin A\right\}$.
It follows from B2 property for $t=\tau+1-a$ that the last $n-k=b$ code symbols $\left\{c_{k}, \ldots, c_{n-1}\right\}$ can be computed from the first $k$ code symbols $\left\{c_{0}, \ldots, c_{k-1}\right\}$, thereby guaranteeing systematic encoding with $\left\{c_{0}, \ldots, c_{k-1}\right\}$ as message symbols.

## III. Scalar Code Construction

Our explicit, rate-optimal streaming code construction will employ diagonal embedding as well as an $n n=$ $\tau+1+\delta, k=n-b]$ scalar code satisfying the four erasure recovery properties listed above in Section We begin by recursively defining a matrix that will be used to specify the parity-check matrix of our scalar code. This recursive definition can be viewed as an extension of the recursive matrix definition in [17] that was used to construct rate-optimal binary streaming codes for the situation when only burst erasures are present.

Definition 1. For any positive integers $u, v$ and $a$, we recursively define the $(u \times v)$ matrix $\mathbf{P}_{u, v}^{a}$ as shown below:

$$
\mathbf{P}_{u, v}^{a}= \begin{cases}{\left[\begin{array}{lll}
I_{u} & \underbrace{\mathbf{0}}_{(u \times a)} & \mathbf{P}_{u, v-u-a}^{a}
\end{array}\right]} & u+a<v \\
{\left[\begin{array}{ll}
I_{u} & \underbrace{\mathbf{0}}_{(u \times(v-u))}
\end{array}\right]} & u \leq v \leq u+a \\
& \\
{\left[\begin{array}{c}
I_{v} \\
\mathbf{P}_{u-v, v}^{a}
\end{array}\right]} & v<u\end{cases}
$$

For example, $\mathbf{P}_{3,7}^{2}=[I_{3} \underbrace{\mathbf{0}}_{(3 \times 2)} \mathbf{P}_{3,2}^{2}], \mathbf{P}_{3,2}^{2}=\left[\begin{array}{c}I_{2} \\ \mathbf{P}_{1,2}^{2}\end{array}\right]$ and $\mathbf{P}_{1,2}^{2}=\left[\begin{array}{ll}1 & 0\end{array}\right]$. Therefore we have

$$
\mathbf{P}_{3,7}^{2}=\left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}\right]
$$

Construction 1. Let $\delta=b-a, q \geq \tau$ be a prime power and $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$. Let the $(a \times(\tau+1-a))$ matrix $\mathbf{C}$ over $\mathbb{F}_{q}$ be such that any square sub-matrix of it is non-singular. We define an $[n=\tau+1+\delta, k=n-b]$ scalar code having a parity check matrix $H$ that is defined in step-by-step fashion below:

- initialize $H$ to be the $(b \times(\tau+1+\delta))$ all-zero matrix,
- set $H([0: \delta-1],[0: \delta-1])=\alpha I_{\delta}$,
- set $H([0: \delta-1],[b: \tau-1])=\mathbf{P}_{\delta, \tau-b}^{a}$,
- set $H([\delta: b-1],[0: a-1])=I_{a}$,
- set $H([\delta: b-1],[a: \tau])=\mathbf{C}$,
- set $H(0, \tau)=\alpha$ and $H([1: \delta],[\tau+1: \tau+\delta])=I_{\delta}$.

The first $\delta$ rows of the parity check matrix $H$ are given by:

$$
H([0: \delta-1],[0: \tau+\delta])
$$

$$
=[\begin{array}{cccc}
\alpha & & & \\
& \alpha & & \\
& & \ddots & \\
& & & \alpha
\end{array} \underbrace{\mathbf{0}}_{(\delta \times a)}|\underbrace{\mathbf{P}_{\delta, \tau-b}^{a}}_{(\delta \times(\tau-b))}| \begin{array}{ccccc}
\alpha & & & & 0 \\
& 1 & & & 0 \\
& & \ddots & & \vdots \\
& & & 1 & 0
\end{array}]
$$

and the last $(b-\delta)=a$ rows of $H$ by:

$$
\begin{gathered}
H([\delta: b-1],[0: \tau+\delta]) \\
=\left[\left.\begin{array}{cccc|c|c}
1 & & & \\
& 1 & & \\
& & \ddots & & \\
& & & 1 & \underbrace{\mathbf{C}}_{(a \times(\tau+1-a))} & \underbrace{\mathbf{0}}_{(a \times(\delta-1))}
\end{array} \right\rvert\, \begin{array}{c}
1 \\
0 \\
\vdots \\
\end{array}\right.
\end{gathered}
$$

As defined above, $H([\delta: b-1],[0: \tau])$ is the parity check matrix of a $[\tau+1, \tau+1-a]$ MDS code. A finite field of size $q \geq \tau$ suffices to explicitly construct the matrix $\mathbf{C}$. It can be verified that the last $a$ rows of $H$ are the same as that of the non-explicit code presented in [6].

## A. Example Constructions

1) $(a=2, b=5, \tau=12)$ : Here $\delta=3, \tau-b=7, \tau+1-a=11$ and $\tau+1+\delta=16$. The parity check matrix $H$ of $[n=16, k=11]$ scalar code is given in this case by:

$$
\left[\right]
$$

2) $(a=3, b=6, \tau=8)$ : Here $\delta=3, \tau-b=2, \tau+1-a=6$ and $\tau+1+\delta=12$. The parity check matrix $H$ of $[n=12, k=6]$ scalar code is given in this case by:

$$
\left[\right]
$$

## IV. Proof of Erasure Recovery Properties

In this section we show that the $[n=\tau+1+\delta, k=n-b]$ scalar code defined in Section III satisfies all the four erasure recovery conditions. This will in turn prove that the packet-level code obtained by diagonal embedding of this scalar code is a rate-optimal $(a, b, \tau)$ streaming code. This $(a, b, \tau)$ streaming code can be explicitly constructed over a finite field of size $q^{2} \geq \tau^{2}$.

It can be proved that the scalar code satisfies R1 and R2 properties using arguments similar to that presented in [6]. Nevertheless for the sake of completeness, we provide proof of all the four properties here.

## A. Proof of B1 Property

Property B 1 is verified by showing that for every $t \in[0: \delta-1]$, there exists a parity check equation having support at $t$ and zeros at indices $[t+1: t+b-1] \cup[t+\tau+1: \tau+\delta]$. Using this parity-check equation, code symbol $c_{t}$ can be recovered from a burst erasure confined to $[t: t+b-1]$ by accessing only the non-erased code symbols having index $\leq t+\tau$.

For the $(a=2, b=5, \tau=12)$ example, suppose symbols $\left\{c_{0}, c_{1}, c_{2}, c_{3}, c_{4}\right\}$ are erased. Row 0 of the parity check matrix $H$ for this example is as shown below:

$$
\left(\begin{array}{llllllllllllllll}
\alpha & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha & 0 & 0 & 0
\end{array}\right)
$$

It follows that $\alpha c_{0}+c_{5}+c_{10}+\alpha c_{12}=0$. Hence $c_{0}$ can be recovered by accessing symbols till $c_{12}$. Now for the $(a=3, b=6, \tau=8)$ example, consider a burst erasure such that $\left\{c_{2}, c_{3}, c_{4}, c_{5}, c_{6}, c_{7}\right\}$ are lost. It follows from row 2 of $H$ that $\alpha c_{2}+c_{6}+c_{10}=0$, but since $c_{6}$ is erased this equation alone is not sufficient to recover $c_{2}$. We get $\alpha c_{0}+c_{6}+\alpha c_{8}=0$ from the 0 -th row of $H$. From these two parity-check equations we obtain $\alpha c_{2}+c_{10}-\alpha c_{0}-\alpha c_{8}=0$, using which $c_{2}$ can be recovered by accessing code symbols till $c_{10}$. We now prove the B1 property for general $\{a, b, \tau\}$. We use the notation $h(t)$ to denote the $t$-th row of $H$. The symbol ' X ' is used as a don't care symbol in the arguments below.

1) $\tau-b \geq \delta$ : From the definition of the matrix $P_{\delta, \tau-b}^{a}$, the first $\delta$ columns of $P_{\delta, \tau-b}^{a}$ form $I_{\delta}$. The $t$-th row of $H$ looks as shown below:

Therefore, using the parity check equation given by $h(t)$ we can recover code symbol $c_{t}$ from a burst erasures at $[t: t+b-1]$ by accessing available code symbols with index $\leq \tau+t$.
2) $\tau-b<\delta$ : Let $\ell=\tau-b, \delta=v \ell+x$ where $0 \leq x<\ell$. Then by definition $P_{\delta, \ell}^{a}=\left[\begin{array}{llll}I_{\ell} & \cdots & I_{\ell} P_{x, \ell}^{a}\end{array}\right]^{T}$. We further divide the proof of this case into three sub-cases.
a) $t \in[0: \ell-1]$ : In this case, the $t$-th row of $H$ satisfies the requirement as shown below and hence can be used for recovery of $c_{t}$.
b) $t \in[\ell: v \ell-1]$ : Let $t=v^{\prime} \ell+x^{\prime}$ where $1 \leq v^{\prime} \leq v-1$ and $0 \leq x^{\prime}<\ell$. The $t$-th row $h(t)$ is of the form:

$$
\left.\right)
$$

This parity check equation does not have $(b-1)$ zeros following the index $t$. Therefore we look at the equation given by $(t-\ell)$-th row of $H$ :

Thus we get a parity check $h(t)-h(t-\ell)$ as shown below:

$$
\left.\right)
$$

Note that $\tau+t-\ell=(\tau+t)-(\tau-b)=t+b$. Therefore there are $(b-1)$ zeros following index $t$ in the parity check equation shown above and this parity check equation can be used to recover code symbol $c_{t}$.
c) $t \in[v \ell: v \ell+x-1]$ : Let $t=v \ell+x^{\prime}$, where $x^{\prime} \leq x-1$. The $t$-th parity check equation $h(t)$ is of the form:

$$
\left.\right)
$$

as the first $x$ columns of $P_{x, \ell}^{a}$ are given by $I_{x}$. Let $y_{i}=(v-1) \ell+i-b$. For any $i \in\left[b+x^{\prime}: \tau-1\right], h\left(y_{i}\right)$ is as shown below:

Let $S \subseteq[0: \tau+\delta]$ be the support of $h(t) \cap\left[b+x^{\prime}: \tau-1\right]$. We now look at the parity check equation given by $h(t)-\sum_{i \in S} h\left(y_{i}\right)$. Clearly $y_{i}<t$ for all $i \in\left[b+x^{\prime}: \tau-1\right]$. It can be seen that the entries at indices $\left[b+x^{\prime}: \tau-1\right]$
of $h(t)$ are either 0 or 1 . Hence $h(t)-\sum_{i \in S} h\left(y_{i}\right)$ takes the following form:

$$
\left.\begin{array}{cccccccccccccc}
0 & & & t & & & \tau+(v-1) \ell+x^{\prime} & & & \tau+t & & & \tau+\delta \\
\left(\begin{array}{lllllllllll} 
& \cdots & X & \alpha & 0 & \cdots & 0 & 1 & X & \cdots & X
\end{array}\right. & 0 & \cdots & 0
\end{array}\right)
$$

Note that $\tau+(v-1) \ell+x^{\prime}=\tau+t-\ell=t+b$. Thus there are $(b-1)$ zeros following index $t$ in the above equation. Therefore code symbol $c_{t}$ can be recovered by accessing code symbols with index $\leq \tau+t$.

## B. Proof of Property R1

Let $H^{(t)}$ be the parity check matrix of the punctured code obtained by deleting coordinates $[t+\tau+1: n-1]$ from the scalar code and let $\underline{h}_{i}^{(t)}$ denote the $i$-th column of $H^{(t)}$, for $i \in[0: \tau+t]$. To prove $R 1$ property, it is enough to show that $\underline{h}_{t}^{(t)} \notin \operatorname{span}\left\langle\left\{\underline{h}_{i}^{(t)} \mid i \in A\right\}\right\rangle$ for any $A \subset[t+1: \tau+t]$ with $|A|=a-1$, for all $t \in[0: \delta-1]$.

1) $t=0$ : In this case we need to look at $H^{(0)}$.

$$
H^{(0)}=\left[\begin{array}{c|cccc}
\alpha & X & \cdots & \cdots & X
\end{array} \quad \alpha\right.
$$

Note that the last $(a-1)$ rows of $\underline{h}_{0}^{(0)}$ is all-zero. We also note that $H^{(0)}([1: a-1],[1: \tau])$ is the parity check matrix of an $[\tau, \tau+1-a]$ MDS code. Hence, it is not possible for any other $(a-1)$ columns of $H^{(0)}$ to linearly combine to obtain these $(a-1)$ zero entires, thus proving recoverability of $c_{0}$.
2) $t \in[1: \delta-1]:$ Here look at the $(a \times(\tau+1))$ matrix $\hat{H}_{t}=H(\{t\} \cup[\delta+1: b-1],[t: \tau+t])$ which is a sub-matrix of $H^{(t)}$. Consider any $A \subseteq[1: \tau]$ with $|A|=a-1$. To prove $R 1$ property it is sufficient to show that $0-$ th column of $\hat{H}_{t}$ doesn't lie in the linear span of columns of $\hat{H}_{t}$ indexed by $A$.

$$
\hat{H}_{t}=\left[\begin{array}{cccc}
\alpha & X & \cdots & \cdots
\end{array}\right] \quad X \quad 1
$$

where $X$ is either 0 or 1 . If $A \cap[\tau-t+1: \tau] \neq \phi$, then by MDS property the last $(a-1)$ entires of $0-$ th column of $\hat{H}_{t}$ can not be obtained by linear combination of $\left\{\hat{H}_{t}([1: a-1], j) \mid j \in A\right\}$. Now consider $A \subseteq[1: \tau-t]$. Suppose $\hat{H}_{t}([0: a-1], 0)=\sum_{j \in A} \beta_{j} \hat{H}_{t}([0: a-1], j)$ where $\beta_{j} \in \mathbb{F}_{q^{2}}$. It can be argued using MDS parity property of $C$ matrix that there exists a unique linear combination of $\left\{\hat{H}_{t}([1: a-1], j) \mid j \in A\right\}$ with all coefficients in $\mathbb{F}_{q} \backslash\{0\}$ that result in $\hat{H}_{t}([1: a-1], 0)$. Hence, $\beta_{j} \in \mathbb{F}_{q} \backslash\{0\}$, for all $j \in A$. Note that $\hat{H}_{t}(0, j) \in \mathbb{F}_{q}$ for all $j \in[1: \tau]$, whereas $\hat{H}_{t}(0,0)=\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$. So, $\hat{H}_{t}(0,0) \neq \sum_{j \in A} \beta_{j} \hat{H}_{t}(0, j)$, which is a contradiction. Thus, $0-$ th column of $\hat{H}_{t}$ doesn't lie in span of columns in $A$.

## C. Proof of Property B2

It is clear from the definition of the B 2 property that in order to prove B 2 property it suffices to show that the sub-matrix $H([0: b-1],[t: t+b-1])$ is invertible for any $t \in[\delta: \tau+1-a]$. Before describing the general proof for invertibility of $H([0: b-1],[t: t+b-1])$, we first present some examples which illustrate our arguments. For $(a=2, b=5, \tau=12)$, the sub-matrix $H([0: 4],[3: 7])$ is as shown below:

$$
\left[\begin{array}{cc|ccc}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\hline c_{0,1} & c_{0,2} & c_{0,3} & c_{0,4} & c_{0,5} \\
c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} & c_{0,5}
\end{array}\right]
$$

where $c_{i, j}=H(\delta+i, a+j)$. This matrix is invertible since $\left[\begin{array}{cc}c_{0,1} & c_{0,2} \\ c_{1,1} & c_{1,2}\end{array}\right]$ is a $(2 \times 2)$ sub-matrix of the parity check matrix of a $[13,11]$ MDS code. Now consider $H([0: 4],[10: 14])$ which has the following structure:
$\left[\begin{array}{ccc|cc}1 & 0 & \alpha & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline c_{0,8} & c_{0,9} & c_{0,10} & 0 & 0 \\ c_{1,8} & c_{1,9} & c_{1,10} & 0 & 0\end{array}\right]$.

The above matrix is non-singular if $M=\left[\begin{array}{ccc}1 & 0 & \alpha \\ c_{0,8} & c_{0,9} & c_{0,10} \\ c_{1,8} & c_{1,9} & c_{1,10}\end{array}\right]$ has non-zero determinant. Clearly, $|M|=\left|\begin{array}{ll}c_{0,9} & c_{0,10} \\ c_{1,9} & c_{1,10}\end{array}\right|+$ $\alpha\left|\begin{array}{cc}c_{0,8} & c_{0,9} \\ c_{1,8} & c_{1,9}\end{array}\right|$. Since $\left|\begin{array}{ll}c_{0,9} & c_{0,10} \\ c_{1,9} & c_{1,10}\end{array}\right| \in \mathbb{F}_{q},\left|\begin{array}{cc}c_{0,8} & c_{0,9} \\ c_{1,8} & c_{1,9}\end{array}\right| \neq 0$ and $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$, we have $|M| \neq 0$.

Now we move to the general proof of invertibility of $H([0: b-1],[t: t+b-1])$ for any $t \in[\delta: \tau+1-a]$. Since $H([\delta: b-1],[0: \tau])$ is the parity-check matrix of a $[\tau+1, \tau+1-a]$ MDS Code, any $(a \times a)$ sub-matrix of it is invertible. We will use this fact repeatedly in the proof given below. Let

$$
\tau=v b+\ell \quad \text { where } \quad 0 \leq \ell<b
$$

Then $\tau+1-a=v b+\ell-a+1$. We divide the proof into multiple cases based on the range of value of $t$. In the below proof, we often ignore the sign of the determinant as it is irrelevant to invertibility of a matrix.

Case i) $t \in[\delta:(v-1) b]$
In this case $t+b-1 \leq v b-1=\tau-\ell-1$. We note that

$$
H([0: \delta-1],[\delta: v b-1])=[\underbrace{\mathbf{0}}_{\delta \times a}\left|I_{\delta}\right| \underbrace{\mathbf{0}}_{\delta \times a}|\cdots| I_{\delta} \mid \underbrace{\mathbf{0}}_{\delta \times a}]
$$

From the structure of $H([0: \delta-1],[\delta: v b-1])$ given above it can be observed that $H([0: \delta-1],[t: t+b-1])$, for $t \in[\delta:(v-1) b]$, is composed of $a$ all-zero columns and $\delta$ columns from identity matrix. Let $t=x b+\theta$ where $\theta<b$.

Suppose $\theta \leq \delta-1$, then the sub-matrix $H([0: b-1],[t: t+b-1])$ is of the following form:

$$
\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{(\theta \times(\delta-\theta))} & \underbrace{\mathbf{0}}_{(\delta \times a)} & I_{\theta} \\
\cline { 1 - 1 } I_{(\delta-\theta)} & \underbrace{\mathbf{0}}_{((\delta-\theta) \times \theta)} \\
\hline H([\delta: b-1],[t: t+b-1])
\end{array}\right]
$$

The determinant of the above matrix is equal to the determinant of $H([\delta: b-1],[x b+\delta:(x+1) b-1])$ which is an $(a \times a)$ sub-matrix of $H([\delta: b-1],[0: \tau])$ and is hence non-zero.
For the case when $\theta \geq \delta$, the sub-matrix $H([0: b-1],[t: t+b-1])$ looks as shown below:

$$
\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{(\delta \times(b-\theta))} & I_{\delta} & \underbrace{\mathbf{0}}_{(\delta \times(\theta-\delta))} \\
\hline H([\delta: b-1],[t: t+b-1])
\end{array}\right] .
$$

Let $A=[t:(x+1) b-1] \cup[(x+1) b+\delta: t+b-1]$. The determinant of the sub-matrix shown above is equal to the determinant of the matrix $H([\delta: b-1], A)$. This determinant is non-zero since $H([\delta: b-1], A)$ is an $(a \times a)$ sub-matrix of parity check matrix of a $[\tau+1, \tau+1-a]$ MDS code. Thus we have completed the proof of invertibility of $H([0: b-1],[t: t+b-1])$ for all $t \in[\delta:(v-1) b]$.

Case ii) $t \in[(v-1) b+1:(v-1) b+\ell]$
Let $t=(v-1) b+\theta$. Therefore $1 \leq \theta \leq \ell$ and $t+b-1=v b+\theta-1 \leq v b+\ell-1$. Note that

$$
H([0: \delta-1],[(v-1) b:(v b+\ell-1)])=[I_{\delta}|\underbrace{0}_{\delta \times a}| \mathbf{P}_{\delta, \ell}^{a}] .
$$

In this case the first $\theta$ columns of $\mathbf{P}_{\delta, \ell}^{a}$ are part of $H([0: \delta-1],[t: t+b-1])$. We first consider the case when $\theta \leq \delta-1$. By definition of $\mathbf{P}_{\delta, \ell}^{a}$, we have $\mathbf{P}_{\delta, \ell}^{a}([0: \theta-1][0: \theta-1])=I_{\theta}$ for any $\theta \leq \min \{\ell, \delta-1\}$. Hence the sub-matrix $H([0: b-1],[t: t+b-1])$ for the present case is as shown below:

$$
\left[\right]
$$

The determinant of this matrix is equal to the determinant of $H([\delta: b-1],[v b+\delta: v b-1])$ which is an $(a \times a)$ sub-matrix of the parity check matrix of an $[\tau+1, \tau+1-a]$ MDS code. Hence $H([0: b-1],[t: t+b-1])$ is non-singular.

Now suppose $\ell \geq \theta \geq \delta$. Then,

$$
\mathbf{P}_{\delta, \ell}^{a}([0: \delta-1],[0: \theta-1])=\left[\begin{array}{ll}
I_{\delta} & \underbrace{\mathbf{0}}_{(\delta \times(\theta-\delta))}
\end{array}\right] .
$$

Therefore the sub-matrix $H([0: b-1],[t: t+b-1])$ for this case has the following form:

$$
\left[\begin{array}{l|c|c}
\underbrace{\mathbf{0}}_{(\delta \times(b-\theta))} & I_{\delta} & \underbrace{\mathbf{0}}_{(\delta \times(\theta-\delta))} \\
\hline H([\delta: b-1],[t: t+b-1])
\end{array}\right]
$$

Let $A=[t: v b-1] \cup[v b+\delta: t+b-1]$. The determinant of $H([0: b-1],[t: t+b-1])$ is equal to determinant of $H([\delta: b-1], A)$, which is non-zero since $A$ is an $a-$ element subset of $[0, \tau]$. Thus we have argued that $H([0: b-1],[t: t+b-1])$ is non-singular for all $t \in[(v-1) b+1:(v-1) b+\ell]$.

Before proving the next case, we present a property of $P_{\delta, \ell}^{a}$ which will be helpful in the proof.
Lemma 1. The sub-matrix formed by any $\ell$ consecutive rows of $P_{\delta, \ell}^{a}$ is invertible if $\ell \leq \delta$.
Proof: Let $\delta=x \ell+u$ where $0 \leq u<\ell$ and $x \geq 1$. Then $\mathbf{P}_{\delta, \ell}^{a}$ is of the following form:

$$
\left[\begin{array}{cc} 
& I_{\ell} \\
& \vdots \\
& I_{\ell} \\
I_{u} & P_{u, \ell}^{a}([0: u-1],[u: \ell-1])
\end{array}\right]
$$

It is easy to verify that any $\ell$ consecutive rows are linearly independent in the above matrix.
Case iii) $t \in[(v-1) b+\ell+1:(v-1) b+\delta]$
This case is possible only when $\ell<\delta$. Let $t=(v-1) b+\ell+\theta$. Hence $1 \leq \theta \leq \delta-\ell-1$ and $t+b-1=$ $v b+\ell+\theta-1=\tau+\theta-1$. Here the entire $\mathbf{P}_{\delta, \ell}^{a}$ is part of $H([0: \delta-1],[t: t+b-1])$.

The sub-matrix $H([0: b-1],[t: t+b-1])$ has the form:

$$
\left[\right], \text { where } B=\underbrace{\left[\begin{array}{cccc}
\alpha & & & \\
& 1 & & \\
& & \ddots & \\
& & & 1
\end{array}\right]}_{(\theta \times \theta)}
$$

Note that $H([\delta: b-1],[(v-1) b+\delta: v b-1])$ is an $(a \times a)$ sub-matrix of $H([\delta: b-1],[0: \tau])$ and hence non-singular. Since $B$ is invertible, it follows that $H([0: b-1],[t: t+b-1])$ is invertible if $P_{\delta, \ell}^{a}([\theta: \theta+\ell-1],[0: \ell-1])$ is invertible. Since $\ell \leq \delta-1$, it follows from Lemma 1 that $P_{\delta, \ell}^{a}([\theta: \theta+\ell-1],[0: \ell-1])$ is invertible, thereby completing the proof for $t \in[(v-1) b+\ell+1:(v-1) b+\delta]$.

Case iv) $t \in\{v b+\ell-a, v b+\ell-a+1\}$
If $t=v b+\ell-a$, then $t+b-1=\tau+\delta-1$ and the sub-matrix $H([0: b-1],[t: t+b-1])$ has the following structure:

$$
\left[\begin{array}{c|c|ccc} 
& \alpha & 0 & \cdots & 0 \\
\cline { 2 - 4 } & 0 & & I_{\delta-1} \\
& \vdots & \\
\hline H([\delta: \delta-1],[t: \tau-1])
\end{array}\right]
$$

The determinant of $H([0: b-1],[t: t+b-1])$ can be expanded along 0-th row as $\pm \alpha * \mid H([\delta: b-1],[\tau-a$ : $\tau-1]) \mid+z$, where $z \in \mathbb{F}_{q}$. Note that $H([\delta: b-1],[\tau-a: \tau-1])$ has non-zero determinant as it is an $(a \times a)$ sub-matrix of MDS parity check matrix. Now since $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$, we have $|H([0: b-1],[t: t+b-1])| \neq 0$.

Now consider $t=v b+\ell-a+1$. In this case $t+b-1=\tau+\delta$ and the sub-matrix $H([\delta: b-1],[t: t+b-1])$ looks like:

$$
\left[\right]
$$

The determinant of $H([0: b-1],[t: t+b-1])$ can be written as $\pm \alpha *|H([\delta+1: b-1],[\tau-a+1: \tau-1])|+z$, where $z \in \mathbb{F}_{q}$. Due to the MDS property, the $(a-1) \times(a-1)$ sub-matrix $H([\delta+1: b-1],[\tau-a+1: \tau-1])$ is invertible. Hence $|H([0: b-1],[t: t+b-1])| \neq 0$ as $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$.

Case v) $t \in[(v-1) b+\delta+1: v b]$

1) $\ell<\delta$ : Let $t=(v-1) b+\ell+\theta+1$. Then $\delta-\ell \leq \theta \leq b-\ell-1$ and $t+b-1=v b+\ell+\theta=\tau+\theta$. Since $n=\tau+\delta$ only $\theta \leq \delta$ is possible. The cases $\theta=\delta$ and $\theta=\delta-1$ are already covered in case iv). Hence only $\theta<\delta-1$ is to be considered here. We also have $0 \leq b-\theta-\ell-1 \leq b-\delta-1<a$. The sub-matrix $H([0: b-1],[t: t+b-1])$ is thus of the form:

$$
\left[\begin{array}{c|c|c|cc}
\underbrace{\mathbf{0}}_{(\delta \times(b-\theta-\ell-1))} & \mathbf{P}_{\delta, \ell}^{a} & \vdots & 0 & \cdots \cdots \\
\cline { 3 - 4 } & & 0 & \\
\cline { 3 - 4 } & & \vdots & \underbrace{}_{\theta} \\
\hline H([\delta: b-1],[t: \tau]) & \underbrace{\mathbf{0}}_{(a \times \theta)}
\end{array}\right]
$$

The determinant of this matrix is equal to the determinant of the $(b-\theta) \times(b-\theta)$ matrix $R$ given below:

$$
R=\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{((\delta-\theta) \times(b-\theta-1-\ell))} & & \mathbf{P}^{\prime} \\
& & 0 \\
& & \vdots \\
H([\delta: b-1],[t: \tau])
\end{array}\right],
$$

where $\mathbf{P}^{\prime}=\mathbf{P}_{\delta, \ell}^{a}(\{0\} \cup[\theta+1: \delta-1],[0: \ell-1])$. Let $y=\delta-\theta-1$ and $\delta=x \ell+u$ where $0 \leq u<\ell$. Note that $(b-\theta-1-\ell)=(a-(\ell-y))$.
a) $y \leq u$ : Notice that $\ell-u \leq \ell-y \leq a$ here and therefore the reduced $(b-\theta) \times(b-\theta)$ matrix $R$ is of the form shown below:

$$
\left[\begin{array}{c|c|c|c|c}
\underbrace{\mathbf{0}}_{(y+1) \times(a-(\ell-y))} & \begin{array}{ccccc}
1 & 0 & \cdots \cdots \cdots \cdots \cdots & 0 \\
\underbrace{\mathbf{0}}_{y \times(u-y)} & I_{y} & \underbrace{\mathbf{0}}_{y \times(\ell-u)} & 0 \\
\vdots \\
0
\end{array} & H([\delta: b-1],[t: \tau])
\end{array}\right]
$$

The determinant of $R$ is given by $|R|= \pm|H([\delta: b-1], A \backslash\{v b\})| \pm \alpha|H([\delta: b-1], A \backslash\{v b+\ell\})|$, where $A=[t: v b+u-y-1] \cup[v b+u: v b+\ell]$ is a set of $a+1$ columns. The determinant $|R|$ is clearly non zero as $|H([\delta: b-1], A \backslash\{v b\})|$ and $|H([\delta: b-1], A \backslash\{v b+\ell\})|$ are both non zero in $\mathbb{F}_{q}$ and $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$.
b) $y>u$ : The reduced matrix $R$ has the following form:

where $M=P_{\delta, \ell}^{a}([\delta-u: \delta-1],[\ell-y-u: \ell-1])$. Let $A=[t: t+a-\ell+y-1] \cup[v b+u: v b+u+\ell-y-1]$ be a set of size $a$. It can be seen that the determinant of R takes the form $|R|= \pm \alpha|H([\delta: b-1], A)|+z$, where $z \in \mathbb{F}_{q}$. Now from $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$ and invertibility of $H([\delta: b-1], A)$ it follows that $|R| \neq 0$.
2) $\ell \geq \delta$ : The proof for $t \in[(v-1) b+\delta+1:(v-1) b+\ell]$ is part of case ii). Therefore here we need to only consider $t \in[(v-1) b+\ell+1: v b]$. Let $t=(v-1) b+\ell+1+\theta$, where $0 \leq \theta \leq b-\ell-1$. Then $t+b-1=v b+\ell+\theta=\tau+\theta$ and $0 \leq b-\ell-\theta-1<a$. Here we consider only $\theta<\delta-1$ as the other possible cases are handled in case iv). The sub-matrix $H([0: b-1],[t: t+b-1])$ is of the form:


Since $\alpha \in \mathbb{F}_{q^{2}} \backslash \mathbb{F}_{q}$ it can be argued that determinant of $H([0: b-1],[t: t+b-1])$ is non-zero if the determinant of $(b-\theta-1) \times(b-\theta-1)$ matrix
$\left[\begin{array}{c|c|c|c}\underbrace{\boldsymbol{0}}_{(\delta-\theta-1) \times(b-\ell-\theta-1)} & \underbrace{\mathbf{0}}_{((\delta-\theta-1) \times(\theta+1))} & I_{\delta-\theta-1} & \underbrace{\mathbf{0}}_{(\delta-\theta-1) \times(\ell-\delta)} \\ H([\delta: b-1],[t: \tau-1])\end{array}\right]$
is non-zero. The determinant of this matrix is equal to $|H([\delta: b-1], A)|$ where $A=[t: t+(b-\ell)-1] \cup[v b+\delta$ : $v b+\ell-1]$ is a set of size $a$. Therefore the determinant is non-zero.

Case vi) $t \in[v b+1: v b+\ell-a-1]$
This case is possible only if $\ell \geq a+2$. Let $t+b-1=\tau+\theta=v b+\ell+\theta$, then $t=(v-1) b+\ell+\theta+1$ and $b-\ell \leq \theta \leq \delta-2$. The sub-matrix $H([0: b-1],[t: t+b-1])$ is of the form:

$$
\left[\begin{array}{c|c|c} 
& \alpha & 0 \cdots \cdots \\
\cline { 2 - 4 } P_{\delta, \ell}^{a}([0: \delta-1],[\ell+\theta+1-b: \ell-1]) & 0 & I_{\theta} \\
\cline { 3 - 4 } & \underbrace{\mathbf{0}}_{((\delta-\theta-1) \times \theta)} \\
\hline H([\delta: b-1],[t: \tau]) & \underbrace{\mathbf{0}}_{(a \times \theta)}
\end{array}\right]
$$

As $\ell+\theta+1-b \geq 1$, the first row of above matrix has zeros in first $(b-\theta-1)$ columns. Therefore the matrix is invertible as long as the $(b-\theta-1) \times(b-\theta-1)$ reduced matrix $R$ shown below is invertible.

$$
\begin{equation*}
R=\left[\frac{P_{\delta, \ell}^{a}([\theta+1: \delta-1],[\ell+\theta+1-b: \ell-1])}{H([\delta: b-1],[t: \tau-1])}\right] \tag{1}
\end{equation*}
$$

1) $\ell \geq \delta:$ For this case $P_{\delta, \ell}^{a}=\left[\begin{array}{ll}I_{\delta} \mathbf{0}\end{array}\right]$. Therefore matrix $R$ is of the form:

$$
\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{((\delta-\theta-1) \times(b-\ell))} & I_{\delta-\theta-1} & \underbrace{\mathbf{0}}_{((\delta-\theta-1) \times(\ell-\delta))} \\
\hline H([\delta: b-1],[t: \tau-1])
\end{array}\right]
$$

The determinant of $R$ is equal to $|H([\delta: b-1], A)|$ where $A=[t: t+(b-\ell)-1] \cup[v b+\delta: v b+\ell-1]$ is set of $a$ columns. This is clearly non zero.
2) $\ell<\delta$ : Let $y=\delta-\theta-1$. Note that $\ell \geq b-\theta=\delta-\theta+a$ and hence we have $0<y<\ell$. We will first examine the structure of the $y \times(a+y)$ sub-matrix of $P_{\delta, \ell}^{a}$ corresponding to last $y$ rows and last $(a+y)$ columns that appears in the reduced matrix $R$. To do that we define the variables $v_{0}, v_{1}, v_{2}, \cdots$ where $v_{0}=\delta, v_{1}=\ell$ and

$$
v_{i}=\left\{\begin{array}{lll}
v_{i-2} & \bmod v_{i-1} & i \text { even } \\
v_{i-2} & \bmod \left(v_{i-1}+a\right) & \text { otherwise }
\end{array}\right.
$$

If $v_{i}=0$, we set $D=i$. For $i$ odd if $v_{i} \geq v_{i-1}$ also, we set $D=i$ and $v_{D}=0$. By this definition:

$$
\delta=v_{0}>\ell=v_{1}>v_{2}>v_{3} \cdots>v_{D}=0
$$

We note that $D \geq 2$ always.
An Example $-\delta=15, \ell=4, a=2$ : Here $v_{0}=\delta=15$ and $v_{1}=\ell=4$. Now $v_{2}=v_{0} \bmod v_{1}=15 \bmod 4=3$. Then $v_{3}=v_{1} \bmod \left(v_{2}+a\right)=4 \bmod (3+2)=4$. This means $v_{3}>v_{2}$, therefore $D=3$ and $v_{3}=0$. Thus for this example case $v_{0}=15>v_{1}=4>v_{2}=3>v_{3}=0$.

The indices $v_{0}, \cdots, v_{D}$ help describe the structure of $P_{\delta, \ell}^{a}$. For $i$ even,

$$
P_{v_{i-2}, v_{i-1}}^{a}=\left\{\begin{array}{l}
{\left[\begin{array}{c}
I_{v_{i-1}} \\
\vdots \\
I_{v_{i-1}} \\
P_{v_{i}, v_{i-1}}^{a}
\end{array}\right]} \\
i \neq D \\
{\left[\begin{array}{c}
I_{v_{i-1}} \\
\vdots \\
I_{v_{i-1}}
\end{array}\right]}
\end{array} \quad i=D\right.
$$

For $i$ odd and $i \neq D$,

$$
P_{v_{i-1}, v_{i-2}}^{a}=[I_{v_{i-1}} \underbrace{\mathbf{0}}_{v_{i-1} \times a} \cdots I_{v_{i-1}} \underbrace{\mathbf{0}}_{v_{i-1} \times a} P_{v_{i-1}, v_{i}}] .
$$

For odd $D$,

$$
P_{v_{D-1}, v_{D-2}}^{a}=\left[\begin{array}{lllll}
I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times a} & \cdots & I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times \mu}
\end{array}\right]
$$

where

$$
\mu=\left\{\begin{array}{lll}
a & \text { if } & v_{D-2} \quad \bmod \left(v_{D-1}+a\right)=0, \\
v_{D-2} & \bmod \left(v_{D-1}+a\right)-v_{D-1} & \text { otherwise } .
\end{array}\right.
$$

Look at $y \times(a+y)$ sub-matrix of $P_{\delta, \ell}^{a}$ appearing in $R$. Since $0<y<\ell$, there exists some $i \in[2: D]$ such that $v_{i-1}>y \geq v_{i}$. We prove the $i$ even and odd cases separately.
a) $i$ even: For even $i \geq 4$, note that $v_{i-2}>v_{i-1}>y$ and $v_{i-3} \geq v_{i-1}+v_{i-2}+a>a+y$. Hence, the $y \times(a+y)$ sub-matrix of $P_{\delta, \ell}^{a}$ under consideration is $y \times(a+y)$ sub-matrix of $P_{v_{i-2}, v_{i-3}}^{a}$ corresponding to last $(a+y)$ columns and last $y$ rows. For even $i \geq 4$,

$$
P_{v_{i-2}, v_{i-3}}^{a}=\left[\begin{array}{llllll}
I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & \cdots & I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & P_{v_{i-2}, v_{i-1}}^{a}
\end{array}\right] .
$$

For the case when even $i=D \geq 4$, this matrix looks as:

$$
P_{v_{D-2}, v_{D-3}}^{a}=\left[\begin{array}{lllll|l}
I_{v_{D-2}} & \underbrace{\mathbf{0}}_{v_{D-2} \times a} & \cdots & I_{v_{D-2}} & \underbrace{\mathbf{0}}_{v_{D-2} \times a} & \frac{I_{v_{D-1}}}{\vdots} \\
\hline I_{v_{D-1}}
\end{array}\right] .
$$

If $i=D=2$, then

$$
P_{v_{0}, v_{1}}^{a}=\left[\begin{array}{c}
I_{v_{1}} \\
\vdots \\
I_{v_{1}}
\end{array}\right] .
$$

Note that when $i=D$ we have $V_{D-1}>y>0$. Hence, for even $i=D$ the reduced sub-matrix $R$ shown in equation (1) has the following form:

$$
R=\left[\begin{array}{c|c}
\underbrace{0}_{(y \times a)} & I_{y} \\
\hline H([\delta: b-1],[t: t+a+y-1])
\end{array}\right] .
$$

The determinant of $R$ is same as determinant of $H([\delta: b-1],[t: t+a-1])$. This is clearly non-zero by the definition of $H$.

For the case when $i$ is even with $4 \leq i<D$, the matrix $P_{v_{i-2}, v_{i-3}}^{a}$ is of the following form:

$$
P_{v_{i-2}, v_{i-3}}^{a}=\left[\begin{array}{lllll|l}
I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & \cdots & I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & \frac{I_{v_{i-1}}}{\vdots} \\
& \frac{I_{v_{i-1}}}{\vdots} \\
\hline
\end{array}\right] .
$$

If $i$ is even and $v_{i-1} \leq a+y$, then $i \geq 4$ since $v_{1}=\ell>a+y$. For the case when $i<D$ is even with $v_{i-1} \leq a+y$, the $(y \times(a+y))$ sub-matrix of interest has the form:

$$
\left[\begin{array}{l|l|l}
\underbrace{0}_{y \times\left(a+y-v_{i-1}\right)} & \underbrace{\mathbf{0}}_{\left(y-v_{i}\right) \times\left(v_{i-1}-\left(y-v_{i}\right)\right)} & I_{y-v_{i}} \\
P_{v_{i}, v_{i-1}}^{a}
\end{array}\right] .
$$

In this case the reduced sub-matrix $R$ shown in equation (1) is as shown below:

$$
R=\left[\begin{array}{c|c|c}
\underbrace{0}_{y \times\left(a+y-v_{i-1}\right)} & \underbrace{\mathbf{0}}_{\left(y-v_{i}\right) \times\left(v_{i-1}-\left(y-v_{i}\right)\right)} & I_{y-v_{i}} \\
\left.\left.\cline { 2 - 3 } P_{v_{i}, v_{i-1}}^{a}-1\right]\right)
\end{array}\right] .
$$

The determinant of $R$ is same as the determinant of the matrix shown below:

$$
R^{\prime}=[\frac{\underbrace{0}_{v_{i} \times\left(a+y-v_{i-1}\right)} \mid P_{v_{i}, v_{i-1}}^{a}\left(\left[0: v_{i}-1\right],\left[0: v_{i-1}-y+v_{i}-1\right]\right)}{H\left([\delta: b-1],\left[t: t+a+v_{i}-1\right]\right)}],
$$

Since $v_{i-1}-y \leq a$ in this case,

$$
P_{v_{i}, v_{i-1}}^{a}\left(\left[0: v_{i}-1\right],\left[0: v_{i}+\left(v_{i-1}-y-1\right)\right]\right)=\left[\begin{array}{ll}
I_{v_{i}} & \underbrace{\mathbf{0}}_{v_{i} \times\left(v_{i-1}-y\right)}
\end{array}\right] .
$$

Therefore the determinant of $R$ is equal to $|H([\delta: b-1], A)|$ where $A=\left[t: t+a+y-v_{i-1}-1\right] \cup\left[t+a+y-v_{i-1}+v_{i}\right.$ : $\left.t+a+v_{i}-1\right]$ is a set of $a$ columns. It is clear to see that it is hence non-zero.

Now consider the case when $i<D$ is even with $v_{i-1}>a+y$. Then the $(y \times(a+y))$ sub-matrix of $P_{\delta, \ell}^{a}$ appearing in $R$ is of the following form:

$$
\left[\begin{array}{c|c}
\underbrace{\mathbf{0}}_{\left(y-v_{i}\right) \times\left(a+v_{i}\right)} & I_{y-v_{i}} \\
\hline P_{v_{i}, v_{i-1}}^{a}\left(\left[0: v_{i}-1\right],\left[v_{i-1}-a-y: v_{i-1}-1\right]\right)
\end{array}\right]
$$

In this case the reduced sub-matrix $R$ is of the form:

$$
R=\left[\begin{array}{c}
\underbrace{\mathbf{0}}_{\left(y-v_{i}\right) \times\left(a+v_{i}\right)} \\
\frac{I_{y-v_{i}}^{a}}{P_{v_{i}, v_{i-1}}\left(\left[0: v_{i}-1\right],\left[v_{i-1}-a-y: v_{i-1}-1\right]\right)} \\
H([\delta: b-1],[t: t+a+y-1])
\end{array}\right]
$$

The determinant of $R$ is same as the determinant of matrix $R^{\prime}$ defined as:

$$
R^{\prime}=\left[\frac{P_{v_{i}, v_{i-1}}^{a}\left(\left[0: v_{i}-1\right],\left[v_{i-1}-a-y: v_{i-1}+v_{i}-y-1\right]\right)}{H\left([\delta: b-1],\left[t: t+a+v_{i}-1\right]\right)}\right] .
$$

It can be seen that $\left(a+v_{i}\right)$ columns of $P_{v_{i}, v_{i-1}}^{a}$ appear in matrix $R^{\prime}$ and

$$
P_{v_{i}, v_{i-1}}^{a}=\left[\begin{array}{lllll}
I_{v_{i}} & \underbrace{\mathbf{0}}_{\left(v_{i} \times a\right)} & \cdots & I_{v_{i}} & \underbrace{\mathbf{0}}_{\left(v_{i} \times a\right)}
\end{array} P_{v_{i}, v_{i+1}}^{a}\right] .
$$

- If $y-v_{i} \geq v_{i+1}$ the $\left(v_{i} \times\left(a+v_{i}\right)\right)$ sub-matrix of $P_{v_{i}, v_{i-1}}^{a}\left(\left[0: v_{i}-1\right],\left[v_{i-1}-a-y: v_{i-1}+v_{i}-y-1\right]\right)$
contained in $R^{\prime}$ is a sub-matrix that is composed of $\left(a+v_{i}\right)$ consecutive columns from the matrix:

$$
\left[\begin{array}{lllll}
I_{v_{i}} & \underbrace{\mathbf{0}}_{\left(v_{i} \times a\right)} & \cdots & I_{v_{i}} & \underbrace{\mathbf{0}}_{\left(v_{i} \times a\right)}
\end{array}\right]
$$

There will be $a$ all-zero columns among the $a+v_{i}$ consecutive columns of the above matrix appearing in $R^{\prime}$. Hence the determinant of $R^{\prime}$ is equal to determinant of sub-matrix composed of $a$ columns of $H([\delta: b-1],[t$ : $\left.\left.t+a+v_{i}-1\right]\right)$. Therefore $\left|R^{\prime}\right| \neq 0$.

- Otherwise ie., $y-v_{i}<v_{i+1}$, then the $a+v_{i}$ columns of $P_{v_{i}, y_{i-1}}^{a}$ that are part of $R^{\prime}$ also include elements from $P_{v_{i}, v_{i+1}}^{a}$. Let $y_{1}=v_{i+1}+v_{i}-y$. Then the matrix $R^{\prime}$ is of the form shown below:

$$
\left[\begin{array}{c|c|c}
\frac{\mathbf{0}}{I_{v_{i}-y_{1}}} & \underbrace{}_{v_{i} \times a} & P_{v_{i}, v_{i+1}}^{a}\left(\left[0: v_{i}-1\right],\left[0: y_{1}-1\right]\right) \\
H\left([\delta: b-1],\left[t: t+a+v_{i}-1\right]\right)
\end{array}\right]
$$

This matrix is invertible as $P_{v_{i}, v_{i+1}}^{a}\left(\left[0: y_{1}-1\right],\left[0: y_{1}-1\right]\right)=I_{y_{1}}$ and any $a$ columns of $H([\delta: b-1],[t:$ $\left.t+a+v_{i}-1\right]$ ) are linearly independent.
b) $i$ odd: Since $v_{1}=\ell>y$ we have $i \geq 3$ if $i$ is odd. For odd $i<D$, the $y \times(a+y)$ sub-matrix of $P_{\delta, \ell}^{a}$ appearing in $R$ is $y \times(a+y)$ sub-matrix of $P_{v_{i-1}, v_{i-2}}^{a}$ corresponding to last $(a+y)$ columns and last $y$ rows. This is because $v_{i-2} \geq v_{i-1}+a>a+y$ and $v_{i-1}>y$. For odd $i<D$.

$$
P_{v_{i-1}, v_{i-2}}^{a}=\left[\begin{array}{lllll}
I_{v_{i-1}} & \underbrace{\mathbf{0}}_{v_{i-1} \times a} & \cdots & I_{v_{i-1}} & \underbrace{\mathbf{0}}_{v_{i-1} \times a}
\end{array} P_{v_{i-1}, v_{i}}\right] .
$$

Therefore the sub-matrix $R$ appearing in equation (1) can be written as:

$$
\left[\begin{array}{c|c|c}
{\underset{I}{I_{y-v_{i}}}}_{\mathbf{0}}^{\underbrace{\mathbf{0}}_{y \times a}} \mid P_{v_{i-1}, v_{i}}^{a}\left(\left[v_{i-1}-y: v_{i-1}-1\right],\left[0: v_{i}-1\right]\right) \\
H([\delta: b-1],[t: \tau-1])
\end{array}\right] .
$$

The determinant of this matrix is equal to:

$$
\begin{array}{r}
\left|H\left([\delta: b-1],\left[t+y-v_{i}: t+a+y-v_{i}\right]\right)\right| * \\
\left|P_{v_{i-1}, v_{i}}^{a}\left(\left[v_{i-1}-y: v_{i-1}-y+v_{i}-1\right],\left[0: v_{i}-1\right]\right)\right| .
\end{array}
$$

The first term in this determinant is non-zero by MDS property. The second term corresponds to determinant of sub-matrix of $P_{v_{i-1}, v_{i}}^{a}$ that is formed by picking $v_{i}$ consecutive rows. By the property of $P_{v_{i-1}, v_{i}}^{a}$ given in Lemma 1 this is non-zero.

For the case odd $i=D$ with $v_{D-2} \bmod \left(v_{D-1}+a\right)=0$, we have $v_{D-1}>y$ and $v_{D-2} \geq v_{D-1}+a>a+y$ and hence the $y \times(a+y)$ sub-matrix appearing in $R$ is contained in

$$
P_{v_{D-1}, v_{D-2}}^{a}=\left[\begin{array}{lllll}
I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times a} & \cdots & I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times a}
\end{array}\right] .
$$

The reduced matrix $R$ defined in (1) takes the form

$$
R=\left[\begin{array}{c|c}
I_{y} & \underbrace{0}_{(y \times a)} \\
\hline H([\delta: b-1],[t: t+a+y-1])
\end{array}\right] .
$$

Hence, the determinant of $R$ is equal to determinant of $H([\delta: b-1],[t+y: t+a+y-1])$, which is clearly non-zero.

Now consider the case when odd $i=D$ with $v_{D-2} \bmod \left(v_{D-1}+a\right) \geq v_{D-1}$.

- For the case when $v_{D-2} \geq a+y$, the $(y \times(a+y))$ sub-matrix we are interested in is a sub-matrix of $P_{v_{D-1}, v_{D-2}}^{a}$ and

$$
P_{v_{D-1}, v_{D-2}}^{a}=\left[\begin{array}{lllll}
I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times a} & \cdots & I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times p}
\end{array}\right]
$$

where $p=v_{D-2} \bmod \left(a+v_{D-1}\right)-v_{D-1}$. The $y \times(a+y)$ sub-matrix comprised of last $y$ rows, last $(a+y)$ columns is of the form shown below:

$$
[\underbrace{\mathbf{0}}_{y \times(a-p)} I_{y} \underbrace{\mathbf{0}}_{y \times p}]
$$

Therefore the sub-matrix appearing in equation (1) can be written as:

$$
R=\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{y \times(a-p)} & I_{y} & \underbrace{\mathbf{0}}_{y \times p} \\
\hline H([\delta: b-1],[t: \tau-1])
\end{array}\right] .
$$

The determinant of this matrix is equal to $|H([\delta: b-1], A)|$ where $A=[t: t+a-p-1] \cup[\tau-p: \tau-1]$, which is non-zero since $A$ is of size $a$.

- For the case when $v_{D-2}<a+y, P_{v_{D-1}, v_{D-2}}^{a}$ is as shown below:

$$
P_{v_{D-1}, v_{D-2}}^{a}=\left[\begin{array}{ll}
I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1} \times p}
\end{array}\right],
$$

where $p=v_{D-2}-v_{D-1}$. Since $v_{1}=\ell>a+y$, this case occurs only when $D \geq 5$. In this case, the $(y \times(a+y))$ sub-matrix of $P_{\delta, \ell}^{a}$ appearing in $R$ is the sub-matrix of $P_{v_{D-3}, v_{D-4}}^{a}$ corresponding to last ( $a+y$ ) columns and last $y$ rows.

$$
P_{v_{D-3}, v_{D-4}}^{a}=\left[\begin{array}{cccc|c}
I_{v_{D-3}} & \underbrace{\mathbf{0}}_{v_{D-3} \times a} & \cdots & I_{v_{D-3}} & \underbrace{\mathbf{0}}_{v_{D-3} \times a} \\
& \frac{I_{v_{D-2}}}{} \\
\frac{I_{v_{D-2}}}{} \\
\hline P_{v_{D-1}, v_{D-2}}^{a}
\end{array}\right]
$$

Thus the $y \times(a+y)$ sub-matrix of interest has the form shown below:

$$
[\underbrace{\mathbf{0}}_{y \times(a-p)} I_{y} \underbrace{\mathbf{0}}_{y \times p}] .
$$

Therefore the sub-matrix appearing in equation (1) can be written as:

$$
R=\left[\begin{array}{c|c|c}
\underbrace{\mathbf{0}}_{y \times(a-p)} & I_{y} & \underbrace{\mathbf{0}}_{y \times p} \\
\hline H([\delta: b-1],[t: \tau-1])
\end{array}\right] .
$$

The determinant of this matrix is equal to $|H([\delta: b-1], A)|$ where $A=[t: t+a-p-1] \cup[\tau-p: \tau-1]$, which is non-zero.

## D. Proof of R2 property

If we prove that columns $\{H([0: b-1], j) \mid j \in A\}$ are linearly independent for any set $A \subseteq[\delta: \tau+\delta]$ with $|A|=a$, then R2 property follows. For $\delta=0$, the scalar code reduces to a $[\tau+1, \tau+1-a]$ MDS code and proof is straightforward. Hence we need to consider only $\delta>0$, for which have $0 \notin A$. If $A \cap[1: \tau] \neq \phi$, observe that columns of $H([\delta: b-1], A)$ are either all-zero columns or distinct columns from an MDS parity check matrix. Hence the $a$ columns $\{H([\delta: b-1], j) \mid j \in A\}$ are not linearly dependent. If $A \cap[0: \tau]$ is empty, then $A \subseteq[\tau+1: \tau+\delta]$. By B2 property for $t=\tau+1-a$, it follows that the columns of $H([0: b-1], A)$ are linearly independent. This completes the proof of R2 property.

## References

[1] "5G Services Innovation," 5G-Americas, 2019.
[2] E. Martinian and C. W. Sundberg, "Burst Erasure Correction Codes with Low Decoding Delay," IEEE Trans. Inf. Theory, vol. 50, no. 10, pp. 2494-2502, 2004.
[3] E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," in Proc. IEEE Int. Symp. Inf. Theory, Nice, France, June 24-29, 2007, pp. 1006-1010.
[4] A. Badr, P. Patil, A. Khisti, W. Tan, and J. G. Apostolopoulos, "Layered Constructions for Low-Delay Streaming Codes," IEEE Trans. Inf. Theory, vol. 63, no. 1, pp. 111-141, 2017.
[5] M. Vajha, V. Ramkumar, M. Jhamtani, and P. V. Kumar, "On Sliding Window Approximation of Gilbert-Elliott Channel for Delay Constrained Setting," CoRR, vol. abs/2005.06921, 2020.
[6] M. N. Krishnan, D. Shukla, and P. V. Kumar, "Low Field-size, Rate-Optimal Streaming Codes for Channels With Burst and Random Erasures," IEEE Trans. Inf. Theory, vol. 66, no. 8, pp. 4869-4891, 2020.
[7] N. Adler and Y. Cassuto, "Burst-Erasure Correcting Codes With Optimal Average Delay," IEEE Trans. Inf. Theory, vol. 63, no. 5, pp. 2848-2865, 2017.
[8] D. Leong and T. Ho, "Erasure Coding for Real-Time Streaming," in Proc. IEEE Int. Symp. Inf. Theory, Cambridge, MA, USA, July 1-6, 2012, pp. 289-293.
[9] D. Leong, A. Qureshi, and T. Ho, "On Coding for Real-Time Streaming under Packet Erasures," in Proc. Int. Symp. Inf. Theory, Istanbul, Turkey, July 7-12, 2013, pp. 1012-1016.
[10] Ö. F. Tekin, T. Ho, H. Yao, and S. Jaggi, "On erasure correction coding for streaming," in Proc. Inf. Theory and Applications Workshop, San Diego, CA, USA, February 5-10, 2012, pp. 221-226.
[11] S. L. Fong, A. Khisti, B. Li, W. Tan, X. Zhu, and J. G. Apostolopoulos, "Optimal Streaming Codes for Channels With Burst and Arbitrary Erasures," IEEE Trans. Inf. Theory, vol. 65, no. 7, pp. 4274-4292, 2019.
[12] M. N. Krishnan and P. V. Kumar, "Rate-Optimal Streaming Codes for Channels with Burst and Isolated Erasures," in Proc. IEEE Int. Symp. Inf. Theory, Vail, CO, USA, June 17-22, 2018, pp. 1809-1813.
[13] E. Domanovitz, S. L. Fong, and A. Khisti, "An Explicit Rate-Optimal Streaming Code for Channels with Burst and Arbitrary Erasures," in Proc. IEEE Inf. Theory Workshop, Visby, Sweden, August 25-28, 2019, pp. 1-5.
[14] M. Rudow and K. V. Rashmi, "Streaming Codes For Variable-Size Arrivals," in Proc. 56th Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, USA, October 2-5, 2018, pp. 733-740.
[15] M. N. Krishnan, V. Ramkumar, M. Vajha, and P. V. Kumar, "Simple Streaming Codes for Reliable, Low-Latency Communication," IEEE Commun. Lett., vol. 24, no. 2, pp. 249-253, 2020.
[16] V. Ramkumar, M. Vajha, M. N. Krishnan, and P. V. Kumar, "Staggered Diagonal Embedding Based Linear Field Size Streaming Codes," in Proc. IEEE Int. Symp. Inf. Theory, Los Angeles, CA, USA, June 21-26, 2020, pp. 503-508.
[17] H. D. Hollmann and L. M. Tolhuizen, "Optimal Codes for Correcting a Single (wrap-around) Burst of Erasures," IEEE Trans. Inf. Theory, vol. 54, no. 9, pp. 4361-4364, 2008.

