# Explicit Rate-Optimal Streaming Codes with Smaller Field Size

Myna Vajha<sup>\*</sup>, Vinayak Ramkumar<sup>\*</sup>, M. Nikhil Krishnan<sup>†</sup>, P. Vijay Kumar<sup>\*</sup> \*Department of Electrical Communication Engineering, IISc Bangalore <sup>†</sup>Department of Electrical and Computer Engineering, University of Toronto {mynaramana, vinram93, nikhilkrishnan.m, pvk1729}@gmail.com

#### Abstract

Streaming codes are a class of packet-level erasure codes that ensure packet recovery over a sliding window channel which allows either a burst erasure of size b or a random erasures within any window of size  $(\tau + 1)$  time units, under a strict decoding-delay constraint  $\tau$ . The field size over which streaming codes are constructed is an important factor determining the complexity of implementation. The best known explicit rate-optimal streaming code requires a field size of  $q^2$  where  $q \ge \tau + b - a$  is a prime power. In this work, we present an explicit rate-optimal streaming code, for all possible  $\{a, b, \tau\}$  parameters, over a field of size  $q^2$  for prime power  $q \ge \tau$ . This is the smallest-known field size of a general explicit rate-optimal construction that covers all  $\{a, b, \tau\}$  parameter sets. We achieve this by modifying the non-explicit code construction due to Krishnan et al. to make it explicit, without change in field size.

#### **Index Terms**

Low-latency communication, streaming codes, packet-level FEC, random or burst erasure

#### I. INTRODUCTION

Enabling low-latency reliable communication for applications such as telesurgery, industrial automation, augmented reality and vehicular communication is a key target of 5G communication systems. For instance, telesurgery camera flow requires packet-loss rate less than  $10^{-3}$  and end-to-end latency below 150 ms [1]. To combat the packet drops that are an inevitable part of any communication network, one approach is to employ feedback-based methods such as ARQ. But such feedback-based schemes incur round-trip propagation delay, making it difficult to meet low-latency requirements. Blind re-transmission of packets is another option, but is inefficient as it amounts to using a repetition code. Streaming codes are a class of packet-level erasure codes and represent a natural way of achieving reliable, low-latency communication at the packet level.

A packet-expansion encoding framework for streaming codes was introduced in [2]. Given message packet at time t denoted by  $\underline{u}(t) \in \mathbb{F}_q^k$ , the coded packet  $\underline{x}(t) \in \mathbb{F}_q^n$ , at time t, is generated by appending parity packet  $\underline{p}(t) \in \mathbb{F}_q^{n-k}$  to  $\underline{x}(t)$ . More formally,  $\underline{x}(t) = [\underline{u}(t)^T \underline{p}(t)^T]^T$ . The encoder is causal and hence  $\underline{p}(t)$  depends only on  $\underline{u}(t)$  and prior message packets. In [2], [3], streaming codes that can handle burst erasure of size b under a decoding-delay constraint  $\tau$  are presented. The decoding-delay constraint  $\tau$  means that for recovery of message packet  $\underline{u}(t)$  only packets with index  $\leq t + \tau$  can be accessed. In [4], Badr et al. presented a delay constrained sliding window (DCSW) channel model that allows burst or random erasures. This channel can be viewed as a deterministic approximation of the Gilbert-Elliot channel [5]. In the DCSW channel model, within any sliding window of size w time units, either a burst erasure of length  $\leq b$  or else, at most a random erasures can occur. Additionally there is a decoding-delay constraint  $\tau$ . This model is non-trivial only if  $0 < a \leq b \leq \tau$ . As it turns out, we can, without loss in generality, set  $w = \tau + 1$  (see [6]). Thus the DCSW channel is parameterized by the three-parameter set  $\{a, b, \tau\}$ . An  $(a, b, \tau)$  streaming code is a packet-level code that can recover from all the permissible erasure patterns of an  $\{a, b, \tau\}$  DCSW channel, within decoding-delay  $\tau$ . Some other models of erasure codes for streaming can be found in [7]–[10].

In [4] an upper bound on the rate of an  $(a, b, \tau)$  streaming code was presented and it was later shown in [11], [12] that this rate is achievable for all possible  $\{a, b, \tau\}$  parameters. It follows from these results that the optimal

rate of  $(a, b, \tau)$  streaming code is given by  $R_{opt} = \frac{\tau+1-a}{\tau+1-a+b}$ . The rate-optimal codes presented in [11], [12] required a finite field alphabet that is exponential in  $\tau$ . A non-explicit rate-optimal streaming code, which requires a field  $\mathbb{F}_{q^2}$ with prime power  $q \ge \tau$ , is presented in [6]. Subsequently, an explicit construction was presented in [13] requiring field size  $q^2$ , for  $q \ge \tau + b - a$ , a prime power. Streaming codes for variable packet sizes are explored in [14]. Explicit rate-optimal constructions having linear field size for some  $\{a, b, \tau\}$  parameter ranges are presented in [6], [15], [16]. However, the construction in [13] remains the smallest field size explicit rate-optimal streaming code construction that exists for all possible  $\{a, b, \tau\}$ . Note that the field size required for the explicit code construction in [13] is larger than the field size  $q^2 \ge \tau^2$  requirement of the code in [6]. In the present paper, we present an explicit rate-optimal code having the same field-size requirement  $q^2$ , with prime power  $q \ge \tau$ , as that of the non-explicit code in [6]. Smaller field size constructions simplify implementation and are hence of significant, practical interest.

The principal contribution of the paper is thus an explicit rate-optimal streaming code construction for all possible  $\{a, b, \tau\}$  parameters. The construction is motivated by the structure of the non-explicit code in [6] and has smallest known field size of an explicit rate-optimal streaming code construction that holds for all  $\{a, b, \tau\}$  parameters.

Section II presents the diagonal-embedding framework for embedding a scalar code within the packet stream. The explicit construction of the scalar code having field size  $q^2 \ge \tau^2$  is presented in Section III. Proof that this construction, in conjunction with diagonal embedding, results in a rate-optimal streaming code is presented in Section IV.

We use [a : b] to denote the set  $\{a, a + 1, ..., b - 1, b\}$ . Given a  $(k \times n)$  matrix  $M, I \subseteq [0 : k - 1]$  and  $J \subseteq [0 : n - 1], M(I, J)$  will denote the sub-matrix of M comprised of rows with row-index in I and columns with column-index in J. We use |M| to denote the determinant of  $M, I_u$  denotes  $(u \times u)$  identity matrix and  $\underbrace{\mathbf{0}}_{(u \times u)}$ 

will denote the  $(u \times v)$  all-zero matrix.

#### **II.** PRELIMINARIES

#### A. Diagonal Embedding

Diagonal embedding, introduced in [3], can be viewed as a framework for deriving a packet-level code from a scalar code. This technique has been consistently used in the streaming-code literature. Let C be an [n, k] scalar code in systematic form, with first k code symbols being message symbols. Consider a packet-level code with coded packet at time t denoted by  $\underline{x}(t) = [x_0(t) \ x_1(t) \dots x_{n-1}(t)]^T$ . We will say that the packet-level code is obtained by diagonal embedding of the scalar code C if for all t, each n-tuple  $(x_0(t), x_1(t+1), \dots, x_{n-1}(t+n-1))$  is a codeword in the scalar code C. The packet-level code shares the rate  $\frac{k}{n}$  of the underlying scalar code. Diagonal embedding is illustrated in Fig.1.



Fig. 1: Packet-level code constructed by diagonal embedding of a scalar code of block length 6. Here each column indicates a coded packet.

## B. Properties Required of the Scalar Code

Let  $\delta = b - a$ . In order to show that the packet-level code constructed through diagonal embedding of an  $[n = \tau + 1 + \delta, k = n - b]$  scalar code C is a rate-optimal  $(a, b, \tau)$  streaming code it suffices to show that the following erasure recovery properties hold for codeword  $(c_0, c_1, \ldots, c_{n-1}) \in C$  that occupies the time indices  $0 \le t \le (n-1)$ . Analogous, time-shifted versions of these below conditions apply to the other embedded codewords (see [6] for details).

- B1 Any code symbol  $c_t$  with  $t \in [0 : \delta 1]$  should be recoverable from the erasure of a burst of b packets, corresponding to time indices in [t: t+b-1], by accessing non-erased code symbols in the set  $\{c_i \mid t < b_i\}$  $i \le \tau + t \ge \{c_i \mid i < t\}$ . The latter set represents previously-decoded code symbols.
- R1 Any code symbol  $c_t$  with  $t \in [0 : \delta 1]$  should be recoverable from any a random packet erasures, corresponding to time indices t and (a-1) indices in  $[t+1:\tau+t]$ , by accessing non-erased code symbols in the set  $\{c_i \mid t < i \le \tau + t\} \cup \{c_i \mid i < t\}.$
- B2 For any  $t \in [\delta : \tau + 1 a]$ , code symbols  $\{c_i \mid i \in [t : t + b 1]\}$  should be recoverable by accessing remaining code symbols  $\{c_i \mid i \notin [t:t+b-1]\}$ .
- R2 For any set  $A \subseteq [\delta : \tau + \delta]$  with |A| = a, code symbols  $\{c_i \mid i \in A\}$  should be recoverable by accessing the remaining code symbols  $\{c_i \mid i \notin A\}$ .

It follows from B2 property for  $t = \tau + 1 - a$  that the last n - k = b code symbols  $\{c_k, \ldots, c_{n-1}\}$  can be computed from the first k code symbols  $\{c_0, \ldots, c_{k-1}\}$ , thereby guaranteeing systematic encoding with  $\{c_0, \ldots, c_{k-1}\}$  as message symbols.

## **III. SCALAR CODE CONSTRUCTION**

Our explicit, rate-optimal streaming code construction will employ diagonal embedding as well as an n = n $\tau + 1 + \delta, k = n - b$  scalar code satisfying the four erasure recovery properties listed above in Section II. We begin by recursively defining a matrix that will be used to specify the parity-check matrix of our scalar code. This recursive definition can be viewed as an extension of the recursive matrix definition in [17] that was used to construct rate-optimal binary streaming codes for the situation when only burst erasures are present.

**Definition 1.** For any positive integers u, v and a, we recursively define the  $(u \times v)$  matrix  $\mathbf{P}_{u,v}^a$  as shown below:

$$\mathbf{P}_{u,v}^{a} = \begin{cases} \begin{bmatrix} I_{u} & \mathbf{0} & \mathbf{P}_{u,v-u-a}^{a} \\ & (u \times a) & \end{bmatrix} & u + a < v \\ \begin{bmatrix} I_{u} & \mathbf{0} \\ & (u \times (v-u)) \end{bmatrix} & u \le v \le u + a \\ \begin{bmatrix} I_{v} \\ \mathbf{P}_{u-v,v}^{a} \end{bmatrix} & v < u \end{cases}$$

For example,  $\mathbf{P}_{3,7}^2 = [I_3 \ \underbrace{\mathbf{0}}_{(3\times 2)} \ \mathbf{P}_{3,2}^2], \ \mathbf{P}_{3,2}^2 = \begin{bmatrix} I_2 \\ \mathbf{P}_{1,2}^2 \end{bmatrix}$  and  $\mathbf{P}_{1,2}^2 = [1 \ 0]$ . Therefore we have

$$\mathbf{P}_{3,7}^2 = \left[ \begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right].$$

**Construction 1.** Let  $\delta = b - a$ ,  $q \ge \tau$  be a prime power and  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Let the  $(a \times (\tau + 1 - a))$  matrix C over  $\mathbb{F}_q$  be such that any square sub-matrix of it is non-singular. We define an  $[n = \tau + 1 + \delta, k = n - b]$  scalar code having a parity check matrix H that is defined in step-by-step fashion below:

- initialize H to be the  $(b \times (\tau + 1 + \delta))$  all-zero matrix,
- set  $H([0:\delta-1], [0:\delta-1]) = \alpha I_{\delta}$ ,
- set  $H([0:\delta-1], [b:\tau-1]) = \mathbf{P}^{a}_{\delta,\tau-b}$ , set  $H([\delta:b-1], [0:a-1]) = I_{a}$ ,
- set  $H([\delta : b 1], [a : \tau]) = \mathbf{C}$ ,
- set  $H(0,\tau) = \alpha$  and  $H([1:\delta], [\tau+1:\tau+\delta]) = I_{\delta}$ .

The first  $\delta$  rows of the parity check matrix H are given by:

$$H([0:\delta-1],[0: au+\delta])$$

$$= \begin{bmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ (\delta \times a) \\ (\delta \times (\tau - b)) \\ (\delta \times (\tau - b)) \\ (\delta \times (\tau - b)) \end{bmatrix} \begin{pmatrix} \alpha & & & \mathbf{0} \\ & \mathbf{1} & & \mathbf{0} \\ & & \ddots & \vdots \\ & & & \mathbf{1} & \mathbf{0} \end{bmatrix},$$

and the last  $(b - \delta) = a$  rows of H by:

$$H([\delta:b-1], [0:\tau+\delta])$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{pmatrix} \mathbf{C} & \mathbf{0} & 1 \\ (a \times (\tau + 1 - a)) & (a \times (\delta - 1)) & \vdots \\ 0 \end{bmatrix}.$$

As defined above,  $H([\delta : b - 1], [0 : \tau])$  is the parity check matrix of a  $[\tau + 1, \tau + 1 - a]$  MDS code. A finite field of size  $q \ge \tau$  suffices to explicitly construct the matrix **C**. It can be verified that the last *a* rows of *H* are the same as that of the non-explicit code presented in [6].

#### A. Example Constructions

1)  $(a = 2, b = 5, \tau = 12)$ : Here  $\delta = 3, \tau - b = 7, \tau + 1 - a = 11$  and  $\tau + 1 + \delta = 16$ . The parity check matrix H of [n = 16, k = 11] scalar code is given in this case by:

. .

2)  $(a = 3, b = 6, \tau = 8)$ : Here  $\delta = 3, \tau - b = 2, \tau + 1 - a = 6$  and  $\tau + 1 + \delta = 12$ . The parity check matrix H of [n = 12, k = 6] scalar code is given in this case by:

$\alpha$	0	0	0	0	0	1	0	$\alpha$	0	0	0
0	$\alpha$	0	0	0	0	0	1	0	1	0	0
0	0	$\alpha$	0	0	0	1	0	0	0	1	0
1	0	0	C				0	0	1		
0	1	0	$\sim$				0	0	0		
0	0	1			(3)	×6)			0	0	0

## **IV. PROOF OF ERASURE RECOVERY PROPERTIES**

In this section we show that the  $[n = \tau + 1 + \delta, k = n - b]$  scalar code defined in Section III satisfies all the four erasure recovery conditions. This will in turn prove that the packet-level code obtained by diagonal embedding of this scalar code is a rate-optimal  $(a, b, \tau)$  streaming code. This  $(a, b, \tau)$  streaming code can be explicitly constructed over a finite field of size  $q^2 \ge \tau^2$ .

It can be proved that the scalar code satisfies R1 and R2 properties using arguments similar to that presented in [6]. Nevertheless for the sake of completeness, we provide proof of all the four properties here.

#### A. Proof of B1 Property

Property B1 is verified by showing that for every  $t \in [0 : \delta - 1]$ , there exists a parity check equation having support at t and zeros at indices  $[t + 1 : t + b - 1] \cup [t + \tau + 1 : \tau + \delta]$ . Using this parity-check equation, code symbol  $c_t$  can be recovered from a burst erasure confined to [t : t + b - 1] by accessing only the non-erased code symbols having index  $\leq t + \tau$ .

$$(\alpha \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \alpha \ 0 \ 0 \ )$$

It follows that  $\alpha c_0 + c_5 + c_{10} + \alpha c_{12} = 0$ . Hence  $c_0$  can be recovered by accessing symbols till  $c_{12}$ . Now for the  $(a = 3, b = 6, \tau = 8)$  example, consider a burst erasure such that  $\{c_2, c_3, c_4, c_5, c_6, c_7\}$  are lost. It follows from row 2 of H that  $\alpha c_2 + c_6 + c_{10} = 0$ , but since  $c_6$  is erased this equation alone is not sufficient to recover  $c_2$ . We get  $\alpha c_0 + c_6 + \alpha c_8 = 0$  from the 0-th row of H. From these two parity-check equations we obtain  $\alpha c_2 + c_{10} - \alpha c_0 - \alpha c_8 = 0$ , using which  $c_2$  can be recovered by accessing code symbols till  $c_{10}$ . We now prove the B1 property for general  $\{a, b, \tau\}$ . We use the notation h(t) to denote the t-th row of H. The symbol 'X' is used as a don't care symbol in the arguments below.

1)  $\tau - b \ge \delta$ : From the definition of the matrix  $P^a_{\delta,\tau-b}$ , the first  $\delta$  columns of  $P^a_{\delta,\tau-b}$  form  $I_{\delta}$ . The *t*-th row of H looks as shown below:

Therefore, using the parity check equation given by h(t) we can recover code symbol  $c_t$  from a burst erasures at [t:t+b-1] by accessing available code symbols with index  $\leq \tau + t$ .

2)  $\tau - b < \delta$ : Let  $\ell = \tau - b$ ,  $\delta = v\ell + x$  where  $0 \le x < \ell$ . Then by definition  $P_{\delta,\ell}^a = \left[I_\ell \cdots I_\ell P_{x,\ell}^a\right]^T$ . We further divide the proof of this case into three sub-cases.

a)  $t \in [0: \ell - 1]$ : In this case, the *t*-th row of *H* satisfies the requirement as shown below and hence can be used for recovery of  $c_t$ .

b)  $t \in [\ell : v\ell - 1]$ : Let  $t = v'\ell + x'$  where  $1 \le v' \le v - 1$  and  $0 \le x' < \ell$ . The t-th row h(t) is of the form:

This parity check equation does not have (b-1) zeros following the index t. Therefore we look at the equation given by  $(t - \ell)$ -th row of H:

Thus we get a parity check  $h(t) - h(t - \ell)$  as shown below:

Note that  $\tau + t - \ell = (\tau + t) - (\tau - b) = t + b$ . Therefore there are (b - 1) zeros following index t in the parity check equation shown above and this parity check equation can be used to recover code symbol  $c_t$ .

c)  $t \in [v\ell : v\ell + x - 1]$ : Let  $t = v\ell + x'$ , where  $x' \leq x - 1$ . The t-th parity check equation h(t) is of the form:

as the first x columns of  $P_{x,\ell}^a$  are given by  $I_x$ . Let  $y_i = (v-1)\ell + i - b$ . For any  $i \in [b + x' : \tau - 1]$ ,  $h(y_i)$  is as shown below:

Let  $S \subseteq [0: \tau + \delta]$  be the support of  $h(t) \cap [b + x': \tau - 1]$ . We now look at the parity check equation given by  $h(t) - \sum_{i \in S} h(y_i)$ . Clearly  $y_i < t$  for all  $i \in [b + x': \tau - 1]$ . It can be seen that the entries at indices  $[b + x': \tau - 1]$ 

of h(t) are either 0 or 1. Hence  $h(t) - \sum_{i \in S} h(y_i)$  takes the following form:

Note that  $\tau + (v-1)\ell + x' = \tau + t - \ell = t + b$ . Thus there are (b-1) zeros following index t in the above equation. Therefore code symbol  $c_t$  can be recovered by accessing code symbols with index  $\leq \tau + t$ .

## B. Proof of Property R1

Let  $H^{(t)}$  be the parity check matrix of the punctured code obtained by deleting coordinates  $[t + \tau + 1 : n - 1]$ from the scalar code and let  $\underline{h}_i^{(t)}$  denote the *i*-th column of  $H^{(t)}$ , for  $i \in [0 : \tau + t]$ . To prove R1 property, it is enough to show that  $\underline{h}_t^{(t)} \notin span(\{\underline{h}_i^{(t)} \mid i \in A\})$  for any  $A \subset [t+1 : \tau+t]$  with |A| = a-1, for all  $t \in [0 : \delta - 1]$ . 1) t = 0: In this case we need to look at  $H^{(0)}$ .

$$H^{(0)} = \begin{bmatrix} \alpha \mid X \cdots X & \alpha \\ 0 & \mathbf{C} \\ \vdots & I_{(a-1)} & \mathbf{C} \\ 0 & & ((a-1) \times (\tau+1-a)) \end{bmatrix}.$$

Note that the last (a - 1) rows of  $\underline{h}_0^{(0)}$  is all-zero. We also note that  $H^{(0)}([1 : a - 1], [1 : \tau])$  is the parity check matrix of an  $[\tau, \tau + 1 - a]$  MDS code. Hence, it is not possible for any other (a - 1) columns of  $H^{(0)}$  to linearly combine to obtain these (a - 1) zero entires, thus proving recoverability of  $c_0$ .

2)  $t \in [1 : \delta - 1]$ : Here look at the  $(a \times (\tau + 1))$  matrix  $\hat{H}_t = H(\{t\} \cup [\delta + 1 : b - 1], [t : \tau + t])$  which is a sub-matrix of  $H^{(t)}$ . Consider any  $A \subseteq [1 : \tau]$  with |A| = a - 1. To prove R1 property it is sufficient to show that 0-th column of  $\hat{H}_t$  doesn't lie in the linear span of columns of  $\hat{H}_t$  indexed by A.

$$\hat{H}_t = \begin{bmatrix} \alpha & X & \cdots & X & 1 \\ & & \\ & H([\delta+1:b-1], [t:\tau]) & \underbrace{\mathbf{0}}_{((a-1)\times t)} \end{bmatrix},$$

where X is either 0 or 1. If  $A \cap [\tau - t + 1 : \tau] \neq \phi$ , then by MDS property the last (a - 1) entires of 0-th column of  $\hat{H}_t$  can not be obtained by linear combination of  $\{\hat{H}_t([1:a-1],j) \mid j \in A\}$ . Now consider  $A \subseteq [1:\tau - t]$ . Suppose  $\hat{H}_t([0:a-1],0) = \sum_{j \in A} \beta_j \hat{H}_t([0:a-1],j)$  where  $\beta_j \in \mathbb{F}_{q^2}$ . It can be argued using MDS parity property of C matrix that there exists a unique linear combination of  $\{\hat{H}_t([1:a-1],j) \mid j \in A\}$  with all coefficients in  $\mathbb{F}_q \setminus \{0\}$  that result in  $\hat{H}_t([1:a-1],0)$ . Hence,  $\beta_j \in \mathbb{F}_q \setminus \{0\}$ , for all  $j \in A$ . Note that  $\hat{H}_t(0,j) \in \mathbb{F}_q$  for all  $j \in [1:\tau]$ , whereas  $\hat{H}_t(0,0) = \alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . So,  $\hat{H}_t(0,0) \neq \sum_{j \in A} \beta_j \hat{H}_t(0,j)$ , which is a contradiction. Thus, 0-th column of  $\hat{H}_t$  doesn't lie in span of columns in A.

#### C. Proof of Property B2

It is clear from the definition of the B2 property that in order to prove B2 property it suffices to show that the sub-matrix H([0:b-1], [t:t+b-1]) is invertible for any  $t \in [\delta: \tau + 1 - a]$ . Before describing the general proof for invertibility of H([0:b-1], [t:t+b-1]), we first present some examples which illustrate our arguments. For  $(a = 2, b = 5, \tau = 12)$ , the sub-matrix H([0:4], [3:7]) is as shown below:

Γ	0	0	1	0	0 -	1
	0	0	0	1	0	
	0	0	0	0	1	,
	$c_{0,1}$	$c_{0,2}$	$c_{0,3}$	$c_{0,4}$	$c_{0,5}$	
L	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{0,5}$ _	

1	0	$\alpha$	0	0	1
0	1	0	1	0	
1	0	0	0	1	.
$c_{0,8}$	$c_{0,9}$	$c_{0,10}$	0	0	
$c_{1,8}$	$c_{1,9}$	$c_{1,10}$	0	0	

The above matrix is non-singular if  $M = \begin{bmatrix} 1 & 0 & \alpha \\ c_{0,8} & c_{0,9} & c_{0,10} \\ c_{1,8} & c_{1,9} & c_{1,10} \end{bmatrix}$  has non-zero determinant. Clearly,  $|M| = \begin{vmatrix} c_{0,9} & c_{0,10} \\ c_{1,9} & c_{1,10} \end{vmatrix} + \alpha \begin{vmatrix} c_{0,8} & c_{0,9} \\ c_{1,8} & c_{1,9} & c_{1,10} \end{vmatrix}$  has non-zero determinant. Clearly,  $|M| = \begin{vmatrix} c_{0,9} & c_{0,10} \\ c_{1,9} & c_{1,10} \end{vmatrix} + \alpha \begin{vmatrix} c_{0,8} & c_{0,9} \\ c_{1,8} & c_{1,9} & c_{1,10} \end{vmatrix} \in \mathbb{F}_q$ ,  $\begin{vmatrix} c_{0,8} & c_{0,9} \\ c_{1,8} & c_{1,9} & c_{1,10} \end{vmatrix} \neq 0$  and  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ , we have  $|M| \neq 0$ . Now we move to the general proof of invertibility of H([0:b-1], [t:t+b-1]) for any  $t \in [\delta:\tau+1-a]$ .

Since  $H([\delta:b-1], [0:\tau])$  is the parity-check matrix of a  $[\tau+1, \tau+1-a]$  MDS Code, any  $(a \times a)$  sub-matrix of it is invertible. We will use this fact repeatedly in the proof given below. Let

$$\tau = vb + \ell$$
 where  $0 \le \ell < b$ .

Then  $\tau + 1 - a = vb + \ell - a + 1$ . We divide the proof into multiple cases based on the range of value of t. In the below proof, we often ignore the sign of the determinant as it is irrelevant to invertibility of a matrix.

*Case i*)  $t \in [\delta : (v-1)b]$ 

In this case  $t + b - 1 \le vb - 1 = \tau - \ell - 1$ . We note that

$$H([0:\delta-1],[\delta:vb-1]) = \left[\begin{array}{c|c} \mathbf{0}\\ \overline{\delta\times a} \end{array} \middle| I_{\delta} \left| \begin{array}{c} \mathbf{0}\\ \overline{\delta\times a} \end{array} \right| \cdots \left| I_{\delta} \left| \begin{array}{c} \mathbf{0}\\ \overline{\delta\times a} \end{array} \right].$$

From the structure of  $H([0:\delta-1], [\delta:vb-1])$  given above it can be observed that  $H([0:\delta-1], [t:t+b-1])$ , for  $t \in [\delta : (v-1)b]$ , is composed of a all-zero columns and  $\delta$  columns from identity matrix. Let  $t = xb + \theta$  where  $\theta < b$ .

Suppose 
$$\theta \leq \delta - 1$$
, then the sub-matrix  $H([0:b-1], [t:t+b-1])$  is of the following form:  

$$\begin{bmatrix} \mathbf{0} \\ (\theta \times (\delta - \theta)) \\ \hline I_{(\delta - \theta)} \\ \hline I_{(\delta - \theta)} \\ \hline H([\delta:b-1], [t:t+b-1]) \end{bmatrix}.$$

The determinant of the above matrix is equal to the determinant of  $H([\delta:b-1], [xb+\delta:(x+1)b-1])$  which is an  $(a \times a)$  sub-matrix of  $H([\delta : b - 1], [0 : \tau])$  and is hence non-zero.

For the case when  $\theta \ge \delta$ , the sub-matrix H([0:b-1], [t:t+b-1]) looks as shown below:

$$\underbrace{ \begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline (\delta \times (b-\theta)) & I_{\delta} & \underbrace{\mathbf{0}} \\ H([\delta:b-1], [t:t+b-1]) \end{array} } .$$

Let  $A = [t: (x+1)b - 1] \cup [(x+1)b + \delta: t+b - 1]$ . The determinant of the sub-matrix shown above is equal to the determinant of the matrix  $H([\delta : b - 1], A)$ . This determinant is non-zero since  $H([\delta : b - 1], A)$  is an  $(a \times a)$  sub-matrix of parity check matrix of a  $[\tau + 1, \tau + 1 - a]$  MDS code. Thus we have completed the proof of invertibility of H([0:b-1], [t:t+b-1]) for all  $t \in [\delta:(v-1)b]$ .

*Case ii)*  $t \in [(v-1)b+1: (v-1)b+\ell]$ 

Let  $t = (v-1)b + \theta$ . Therefore  $1 \le \theta \le \ell$  and  $t+b-1 = vb + \theta - 1 \le vb + \ell - 1$ . Note that

$$H([0:\delta-1], [(v-1)b:(vb+\ell-1)]) = \left[ \begin{array}{c|c} I_{\delta} & \mathbf{0} \\ \delta \times a & \delta \end{array} \right]$$

In this case the first  $\theta$  columns of  $\mathbf{P}^a_{\delta,\ell}$  are part of  $H([0:\delta-1], [t:t+b-1])$ . We first consider the case when  $\theta \leq \delta - 1$ . By definition of  $\mathbf{P}^a_{\delta,\ell}$ , we have  $\mathbf{P}^a_{\delta,\ell}([0:\theta-1][0:\theta-1]) = I_{\theta}$  for any  $\theta \leq \min\{\ell, \delta - 1\}$ . Hence the sub-matrix H([0:b-1], [t:t+b-1]) for the present case is as shown below:

$$\begin{bmatrix} \underbrace{\mathbf{0}}_{(\theta \times (\delta - \theta))} & \underbrace{\mathbf{0}}_{(\delta \times a)} & I_{\theta} \\ \hline I_{(\delta - \theta)} & \underbrace{\mathbf{0}}_{(\delta \times a)} & \mathbf{P}^{a}_{\delta, \ell}([\theta : \delta - 1], [0 : \theta - 1]) \\ \hline H([\delta : b - 1], [t : t + b - 1]) & \end{bmatrix}$$

The determinant of this matrix is equal to the determinant of  $H([\delta : b - 1], [vb + \delta : vb - 1])$  which is an  $(a \times a)$  sub-matrix of the parity check matrix of an  $[\tau + 1, \tau + 1 - a]$  MDS code. Hence H([0 : b - 1], [t : t + b - 1]) is non-singular.

Now suppose  $\ell \geq \theta \geq \delta$ . Then,

$$\mathbf{P}^{a}_{\delta,\ell}([0:\delta-1],[0:\theta-1]) = \begin{bmatrix} I_{\delta} & \mathbf{0} \\ (\delta \times (\theta - \delta)) \end{bmatrix}.$$

Therefore the sub-matrix H([0:b-1], [t:t+b-1]) for this case has the following form:

$$\begin{bmatrix} \mathbf{0} & I_{\delta} & \mathbf{0} \\ \underline{(\delta \times (b-\theta))} & I_{\delta} & \underline{\mathbf{0}} \\ \overline{H([\delta:b-1], [t:t+b-1])} \end{bmatrix} .$$

Let  $A = [t: vb-1] \cup [vb+\delta: t+b-1]$ . The determinant of H([0:b-1], [t:t+b-1]) is equal to determinant of  $H([\delta:b-1], A)$ , which is non-zero since A is an a- element subset of  $[0, \tau]$ . Thus we have argued that H([0:b-1], [t:t+b-1]) is non-singular for all  $t \in [(v-1)b+1: (v-1)b+\ell]$ .

Before proving the next case, we present a property of  $P^a_{\delta,\ell}$  which will be helpful in the proof.

**Lemma 1.** The sub-matrix formed by any  $\ell$  consecutive rows of  $P_{\delta,\ell}^a$  is invertible if  $\ell \leq \delta$ .

*Proof:* Let  $\delta = x\ell + u$  where  $0 \le u < \ell$  and  $x \ge 1$ . Then  $\mathbf{P}^a_{\delta\ell}$  is of the following form:

$$\left[\begin{array}{cc} I_{\ell} \\ \vdots \\ I_{\ell} \\ I_{u} \quad P_{u,\ell}^{a}([0:u-1],[u:\ell-1]) \end{array}\right]$$

It is easy to verify that any  $\ell$  consecutive rows are linearly independent in the above matrix.

*Case iii)*  $t \in [(v-1)b + \ell + 1 : (v-1)b + \delta]$ 

This case is possible only when  $\ell < \delta$ . Let  $t = (v - 1)b + \ell + \theta$ . Hence  $1 \le \theta \le \delta - \ell - 1$  and  $t + b - 1 = vb + \ell + \theta - 1 = \tau + \theta - 1$ . Here the entire  $\mathbf{P}^a_{\delta,\ell}$  is part of  $H([0:\delta-1], [t:t+b-1])$ .

The sub-matrix H([0:b-1], [t:t+b-1]) has the form:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline I_{(\delta-\ell-\theta)} & \mathbf{0} \\ \hline H([\delta:b-1], [t:t+b-1]) \end{bmatrix}, \text{ where } B = \underbrace{\begin{bmatrix} \alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{(\theta \times \theta)}.$$

Note that  $H([\delta:b-1], [(v-1)b+\delta:vb-1])$  is an  $(a \times a)$  sub-matrix of  $H([\delta:b-1], [0:\tau])$  and hence non-singular. Since *B* is invertible, it follows that H([0:b-1], [t:t+b-1]) is invertible if  $P^a_{\delta,\ell}([\theta:\theta+\ell-1], [0:\ell-1])$  is invertible. Since  $\ell \leq \delta - 1$ , it follows from Lemma 1 that  $P^a_{\delta,\ell}([\theta:\theta+\ell-1], [0:\ell-1])$  is invertible, thereby completing the proof for  $t \in [(v-1)b+\ell+1:(v-1)b+\delta]$ .

*Case iv)*  $t \in \{vb + \ell - a, vb + \ell - a + 1\}$ 

If  $t = vb + \ell - a$ , then  $t + b - 1 = \tau + \delta - 1$  and the sub-matrix H([0:b-1], [t:t+b-1]) has the following structure:

$$\begin{bmatrix} H([0:\delta-1],[t:\tau-1]) & \frac{\alpha & 0 & \cdots & 0 \\ 0 & & \\ \vdots & I_{\delta-1} & \\ 0 & & \\ H([\delta:b-1],[t:\tau]) & \frac{\mathbf{0}}{(a \times (\delta-1))} \end{bmatrix}$$

The determinant of H([0:b-1], [t:t+b-1]) can be expanded along 0-th row as  $\pm \alpha * |H([\delta:b-1], [\tau-a:\tau-1])| + z$ , where  $z \in \mathbb{F}_q$ . Note that  $H([\delta:b-1], [\tau-a:\tau-1])$  has non-zero determinant as it is an  $(a \times a)$  sub-matrix of MDS parity check matrix. Now since  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ , we have  $|H([0:b-1], [t:t+b-1])| \neq 0$ .

Now consider  $t = vb + \ell - a + 1$ . In this case  $t + b - 1 = \tau + \delta$  and the sub-matrix  $H([\delta : b - 1], [t : t + b - 1])$  looks like:

$$\begin{bmatrix} & \alpha & 0 & \cdots & 0 \\ & 0 & & \\ & \vdots & & I_{\delta} \\ & & & \\ \hline & & H([\delta+1:b-1], [t:\tau]) & \mathbf{0} \\ & & & \\ & & & \\ \hline \end{bmatrix}$$

The determinant of H([0:b-1], [t:t+b-1]) can be written as  $\pm \alpha * |H([\delta+1:b-1], [\tau-a+1:\tau-1])| + z$ , where  $z \in \mathbb{F}_q$ . Due to the MDS property, the  $(a-1) \times (a-1)$  sub-matrix  $H([\delta+1:b-1], [\tau-a+1:\tau-1])$  is invertible. Hence  $|H([0:b-1], [t:t+b-1])| \neq 0$  as  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ .

*Case* v)  $t \in [(v-1)b + \delta + 1 : vb]$ 

1)  $\ell < \delta$ : Let  $t = (v - 1)b + \ell + \theta + 1$ . Then  $\delta - \ell \le \theta \le b - \ell - 1$  and  $t + b - 1 = vb + \ell + \theta = \tau + \theta$ . Since  $n = \tau + \delta$  only  $\theta \le \delta$  is possible. The cases  $\theta = \delta$  and  $\theta = \delta - 1$  are already covered in case iv). Hence only  $\theta < \delta - 1$  is to be considered here. We also have  $0 \le b - \theta - \ell - 1 \le b - \delta - 1 < a$ . The sub-matrix H([0:b-1], [t:t+b-1]) is thus of the form:

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ (\delta \times (b-\theta-\ell-1)) \end{bmatrix} \mathbf{P}^{a}_{\delta,\ell} \begin{bmatrix} \alpha & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \\ \vdots & I_{\theta} \\ \vdots & \mathbf{0} \\ 0 & ((\delta-\theta-1)\times\theta) \\ H([\delta:b-1], [t:\tau]) & \mathbf{0} \\ (a \times \theta) \end{bmatrix}$$

10

The determinant of this matrix is equal to the determinant of the  $(b - \theta) \times (b - \theta)$  matrix R given below:

$$R = \begin{bmatrix} \mathbf{0} & | & \alpha \\ ((\delta - \theta) \times (b - \theta - 1 - \ell)) & \mathbf{P'} & [ & \alpha \\ \hline & & 0 \\ \hline & & & 0 \\ \hline & H([\delta : b - 1], [t : \tau]) & \end{bmatrix},$$

where  $\mathbf{P}' = \mathbf{P}^{a}_{\delta,\ell}(\{0\} \cup [\theta + 1 : \delta - 1], [0 : \ell - 1])$ . Let  $y = \delta - \theta - 1$  and  $\delta = x\ell + u$  where  $0 \le u < \ell$ . Note that  $(b - \theta - 1 - \ell) = (a - (\ell - y))$ .

a)  $y \le u$ : Notice that  $\ell - u \le \ell - y \le a$  here and therefore the reduced  $(b - \theta) \times (b - \theta)$  matrix R is of the form shown below:

The determinant of R is given by  $|R| = \pm |H([\delta : b - 1], A \setminus \{vb\})| \pm \alpha |H([\delta : b - 1], A \setminus \{vb + \ell\})|$ , where  $A = [t : vb + u - y - 1] \cup [vb + u : vb + \ell]$  is a set of a + 1 columns. The determinant |R| is clearly non zero as  $|H([\delta : b - 1], A \setminus \{vb\})|$  and  $|H([\delta : b - 1], A \setminus \{vb + \ell\})|$  are both non zero in  $\mathbb{F}_q$  and  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ .

b) y > u: The reduced matrix R has the following form:

$$\begin{bmatrix} \underbrace{\mathbf{0}}_{(y+1)\times(a-(\ell-y))} & \boxed{\begin{array}{c} 1 & 0 & \cdots & \cdots & 0 & \alpha \\ \hline \underbrace{\mathbf{0}}_{(y-u)\times u} & \underbrace{\mathbf{0}}_{y\times(\ell-y)} & I_{y-u} & 0 \\ \hline I_{u} & \underbrace{\mathbf{0}}_{y\times(\ell-y)} & \underbrace{I_{y-u} & 0 \\ \vdots \\ \hline H([\delta:b-1], [t:\tau]) & \end{bmatrix}}$$

,

where  $M = P^a_{\delta,\ell}([\delta - u : \delta - 1], [\ell - y - u : \ell - 1])$ . Let  $A = [t : t + a - \ell + y - 1] \cup [vb + u : vb + u + \ell - y - 1]$ be a set of size a. It can be seen that the determinant of R takes the form  $|R| = \pm \alpha |H([\delta : b - 1], A)| + z$ , where  $z \in \mathbb{F}_q$ . Now from  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  and invertibility of  $H([\delta : b - 1], A)$  it follows that  $|R| \neq 0$ .

2)  $\ell \ge \delta$ : The proof for  $t \in [(v-1)b + \delta + 1 : (v-1)b + \ell]$  is part of case ii). Therefore here we need to only consider  $t \in [(v-1)b + \ell + 1 : vb]$ . Let  $t = (v-1)b + \ell + 1 + \theta$ , where  $0 \le \theta \le b - \ell - 1$ . Then  $t+b-1 = vb + \ell + \theta = \tau + \theta$  and  $0 \le b - \ell - \theta - 1 < a$ . Here we consider only  $\theta < \delta - 1$  as the other possible cases are handled in case iv). The sub-matrix H([0:b-1], [t:t+b-1]) is of the form:

$$\begin{bmatrix} \mathbf{0} \\ (\delta \times (b-\theta-\ell-1)) \\ H([\delta:b-1], [t:\tau]) \end{bmatrix} \mathbf{I}_{\delta} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ ((\delta-\theta-1)\times\theta) \\ \mathbf{0} \\ \mathbf{0} \\ ((\delta-\theta-1)\times\theta) \\ \mathbf{0} \\ \mathbf{0} \\ ((\delta-\theta-1)\times\theta) \end{bmatrix}$$

Since  $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  it can be argued that determinant of H([0:b-1], [t:t+b-1]) is non-zero if the determinant of  $(b-\theta-1) \times (b-\theta-1)$  matrix

$$\underbrace{ \begin{bmatrix} \mathbf{0} \\ (\delta-\theta-1)\times(b-\ell-\theta-1) \\ \hline \\ H([\delta:b-1], [t:\tau-1]) \end{bmatrix} I_{\delta-\theta-1} \left[ \mathbf{0} \\ (\delta-\theta-1)\times(\ell-\delta) \\ \hline \\ H([\delta:b-1], [t:\tau-1]) \\ \hline \\ \end{bmatrix} \right] }_{ H([\delta:b-1], [t:\tau-1]) }$$

is non-zero. The determinant of this matrix is equal to  $|H([\delta : b-1], A)|$  where  $A = [t : t + (b-\ell) - 1] \cup [vb + \delta : vb + \ell - 1]$  is a set of size a. Therefore the determinant is non-zero.

*Case vi*)  $t \in [vb + 1 : vb + \ell - a - 1]$ 

This case is possible only if  $\ell \ge a + 2$ . Let  $t + b - 1 = \tau + \theta = vb + \ell + \theta$ , then  $t = (v - 1)b + \ell + \theta + 1$  and  $b - \ell \le \theta \le \delta - 2$ . The sub-matrix H([0:b-1], [t:t+b-1]) is of the form:

$$\begin{bmatrix} P^{a}_{\delta,\ell}([0:\delta-1], [\ell+\theta+1-b:\ell-1]) & \boxed{\begin{matrix} \alpha & 0 & \cdots & 0 \\ 0 & & \\ \vdots & & I_{\theta} \\ \vdots & & \\ 0 & ((\delta-\theta-1)\times\theta) \\ \hline H([\delta:b-1], [t:\tau]) & & \underbrace{\begin{matrix} 0 \\ (a\times\theta) \\ \end{array} \end{bmatrix}$$

As  $\ell + \theta + 1 - b \ge 1$ , the first row of above matrix has zeros in first  $(b - \theta - 1)$  columns. Therefore the matrix is invertible as long as the  $(b - \theta - 1) \times (b - \theta - 1)$  reduced matrix R shown below is invertible.

$$R = \left[\frac{P_{\delta,\ell}^{a}([\theta+1:\delta-1], [\ell+\theta+1-b:\ell-1])}{H([\delta:b-1], [t:\tau-1])}\right]$$
(1)

1)  $\ell \geq \delta$ : For this case  $P_{\delta,\ell}^a = [I_{\delta} \mathbf{0}]$ . Therefore matrix R is of the form:

$$\frac{\underbrace{\mathbf{0}}_{((\delta-\theta-1)\times(b-\ell))} \left| I_{\delta-\theta-1} \right| \underbrace{\mathbf{0}}_{((\delta-\theta-1)\times(\ell-\delta))} }{H([\delta:b-1],[t:\tau-1])} \right]$$

The determinant of R is equal to  $|H([\delta:b-1], A)|$  where  $A = [t:t+(b-\ell)-1] \cup [vb+\delta:vb+\ell-1]$  is set of a columns. This is clearly non zero.

2)  $\ell < \delta$ : Let  $y = \delta - \theta - 1$ . Note that  $\ell \ge b - \theta = \delta - \theta + a$  and hence we have  $0 < y < \ell$ . We will first examine the structure of the  $y \times (a + y)$  sub-matrix of  $P^a_{\delta,\ell}$  corresponding to last y rows and last (a + y) columns that appears in the reduced matrix R. To do that we define the variables  $v_0, v_1, v_2, \cdots$  where  $v_0 = \delta$ ,  $v_1 = \ell$  and

$$v_i = \begin{cases} v_{i-2} \mod v_{i-1} & i \text{ even,} \\ v_{i-2} \mod (v_{i-1}+a) & \text{otherwise} \end{cases}$$

If  $v_i = 0$ , we set D = i. For i odd if  $v_i \ge v_{i-1}$  also, we set D = i and  $v_D = 0$ . By this definition:

$$\delta = v_0 > \ell = v_1 > v_2 > v_3 \dots > v_D = 0.$$

We note that  $D \ge 2$  always.

An Example-  $\delta = 15, \ell = 4, a = 2$ : Here  $v_0 = \delta = 15$  and  $v_1 = \ell = 4$ . Now  $v_2 = v_0 \mod v_1 = 15 \mod 4 = 3$ . Then  $v_3 = v_1 \mod (v_2 + a) = 4 \mod (3 + 2) = 4$ . This means  $v_3 > v_2$ , therefore D = 3 and  $v_3 = 0$ . Thus for this example case  $v_0 = 15 > v_1 = 4 > v_2 = 3 > v_3 = 0$ . The indices  $v_0, \dots, v_D$  help describe the structure of  $P^a_{\delta, \ell}$ . For *i* even,

$$P_{v_{i-2},v_{i-1}}^{a} = \begin{cases} \begin{bmatrix} I_{v_{i-1}} \\ \vdots \\ I_{v_{i-1}} \\ P_{v_{i},v_{i-1}}^{a} \end{bmatrix} & i \neq D \\ \begin{bmatrix} I_{v_{i-1}} \\ \vdots \\ I_{v_{i-1}} \end{bmatrix} & i = D \end{cases}$$

For i odd and  $i \neq D$ ,

$$P_{v_{i-1},v_{i-2}}^{a} = \left[ I_{v_{i-1}} \underbrace{\mathbf{0}}_{v_{i-1} \times a} \cdots I_{v_{i-1}} \underbrace{\mathbf{0}}_{v_{i-1} \times a} P_{v_{i-1},v_{i}} \right]$$

For odd D,

$$P^{a}_{v_{D-1},v_{D-2}} = \left[ I_{v_{D-1}} \underbrace{\mathbf{0}}_{v_{D-1}\times a} \cdots I_{v_{D-1}} \underbrace{\mathbf{0}}_{v_{D-1}\times \mu} \right]$$

where

$$\mu = \begin{cases} a & \text{if } v_{D-2} \mod (v_{D-1} + a) = 0, \\ v_{D-2} \mod (v_{D-1} + a) - v_{D-1} & \text{otherwise.} \end{cases}$$

Look at  $y \times (a + y)$  sub-matrix of  $P_{\delta,\ell}^a$  appearing in R. Since  $0 < y < \ell$ , there exists some  $i \in [2 : D]$  such that  $v_{i-1} > y \ge v_i$ . We prove the *i* even and odd cases separately.

a) i even: For even  $i \ge 4$ , note that  $v_{i-2} > v_{i-1} > y$  and  $v_{i-3} \ge v_{i-1} + v_{i-2} + a > a + y$ . Hence, the  $y \times (a + y)$  sub-matrix of  $P^a_{\delta,\ell}$  under consideration is  $y \times (a + y)$  sub-matrix of  $P^a_{v_{i-2},v_{i-3}}$  corresponding to last (a + y) columns and last y rows. For even  $i \ge 4$ ,

$$P_{v_{i-2},v_{i-3}}^{a} = \left[ I_{v_{i-2}} \quad \underbrace{\mathbf{0}}_{v_{i-2} \times a} \quad \cdots \quad I_{v_{i-2}} \quad \underbrace{\mathbf{0}}_{v_{i-2} \times a} \quad P_{v_{i-2},v_{i-1}}^{a} \right].$$

For the case when even  $i = D \ge 4$ , this matrix looks as:

$$P^{a}_{v_{D-2},v_{D-3}} = \left[ I_{v_{D-2}} \quad \underbrace{\mathbf{0}}_{v_{D-2} \times a} \quad \cdots \quad I_{v_{D-2}} \quad \underbrace{\mathbf{0}}_{v_{D-2} \times a} \quad \frac{I_{v_{D-1}}}{I_{v_{D-1}}} \right].$$

If i = D = 2, then

$$P^a_{v_0,v_1} = \begin{bmatrix} I_{v_1} \\ \vdots \\ I_{v_1} \end{bmatrix}.$$

Note that when i = D we have  $V_{D-1} > y > 0$ . Hence, for even i = D the reduced sub-matrix R shown in equation (1) has the following form:

$$R = \begin{bmatrix} 0 & I_y \\ (y \times a) & I_y \\ \hline H([\delta:b-1], [t:t+a+y-1]) \end{bmatrix}$$

The determinant of R is same as determinant of  $H([\delta : b - 1], [t : t + a - 1])$ . This is clearly non-zero by the definition of H.

For the case when i is even with  $4 \le i < D$ , the matrix  $P^a_{v_{i-2},v_{i-3}}$  is of the following form:

$$P_{v_{i-2},v_{i-3}}^{a} = \begin{bmatrix} I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & \cdots & I_{v_{i-2}} & \underbrace{\mathbf{0}}_{v_{i-2} \times a} & \vdots \\ \hline I_{v_{i-1}} \\ \hline I_{v_{i-1}} \\ \hline P_{v_{i},v_{i-1}}^{a} \end{bmatrix}$$

If *i* is even and  $v_{i-1} \le a+y$ , then  $i \ge 4$  since  $v_1 = \ell > a+y$ . For the case when i < D is even with  $v_{i-1} \le a+y$ , the  $(y \times (a+y))$  sub-matrix of interest has the form:

$$\left[\begin{array}{c} \underbrace{0}_{y\times(a+y-v_{i-1})} \left| \begin{array}{c} \underbrace{0}_{(y-v_i)\times(v_{i-1}-(y-v_i))} \\ P^a_{v_i,v_{i-1}} \end{array} \right| I_{y-v_i} \\ \end{array} \right].$$

In this case the reduced sub-matrix R shown in equation (1) is as shown below:

$$R = \left[ \underbrace{\begin{array}{c|c} 0 \\ y \times (a+y-v_{i-1}) \\ \hline \\ \hline \\ H([\delta:b-1], [t:t+a+y-1]) \end{array}}^{0} I_{y-v_i} \\ I_{y-v_i} \\ \hline I_{y-v_i} \\ \hline \\ I_{y-v_i} \\ \hline \\ I_{y-v_i} \\ \hline \\ I_{y-v_i} \\ \hline \\ I_{$$

The determinant of R is same as the determinant of the matrix shown below:

$$R' = \left[ \underbrace{\begin{array}{c} 0 \\ v_i \times (a+y-v_{i-1}) \end{array}}_{H([\delta:b-1], [t:t+a+v_i-1])} \right],$$

Since  $v_{i-1} - y \leq a$  in this case,

$$P^{a}_{v_{i},v_{i-1}}([0:v_{i}-1],[0:v_{i}+(v_{i-1}-y-1)]) = \left[ \begin{array}{cc} I_{v_{i}} & \underbrace{\mathbf{0}}_{v_{i}\times(v_{i-1}-y)} \end{array} \right]$$

Therefore the determinant of R is equal to  $|H([\delta:b-1], A)|$  where  $A = [t:t+a+y-v_{i-1}-1] \cup [t+a+y-v_{i-1}+v_i:t+a+v_i-1]$  is a set of a columns. It is clear to see that it is hence non-zero.

Now consider the case when i < D is even with  $v_{i-1} > a + y$ . Then the  $(y \times (a + y))$  sub-matrix of  $P^a_{\delta,\ell}$  appearing in R is of the following form:

$$\begin{bmatrix} \mathbf{0} & & & & \\ I_{y-v_i} \\ \hline P^a_{v_i,v_{i-1}}([0:v_i-1], [v_{i-1}-a-y:v_{i-1}-1]) \end{bmatrix}$$

In this case the reduced sub-matrix R is of the form:

$$R = \left[ \frac{\underbrace{0}_{(y-v_i)\times(a+v_i)} | I_{y-v_i}}{\frac{P_{v_i,v_{i-1}}^a([0:v_i-1], [v_{i-1}-a-y:v_{i-1}-1])}{H([\delta:b-1], [t:t+a+y-1])}} \right].$$

The determinant of R is same as the determinant of matrix R' defined as:

$$R' = \left[\frac{P_{v_i,v_{i-1}}^a([0:v_i-1],[v_{i-1}-a-y:v_{i-1}+v_i-y-1])}{H([\delta:b-1],[t:t+a+v_i-1])}\right].$$

It can be seen that  $(a + v_i)$  columns of  $P^a_{v_i,v_{i-1}}$  appear in matrix R' and

$$P_{v_i,v_{i-1}}^a = \left[ \begin{array}{cccc} I_{v_i} & \underbrace{\mathbf{0}}_{(v_i \times a)} & \cdots & I_{v_i} & \underbrace{\mathbf{0}}_{(v_i \times a)} & P_{v_i,v_{i+1}}^a \end{array} \right].$$

• If  $y - v_i \ge v_{i+1}$  the  $(v_i \times (a + v_i))$  sub-matrix of  $P^a_{v_i,v_{i-1}}([0:v_i-1], [v_{i-1} - a - y:v_{i-1} + v_i - y - 1])$ 

$$\left[\begin{array}{ccccc}I_{v_i} & \underbrace{\mathbf{0}}_{(v_i \times a)} & \cdots & I_{v_i} & \underbrace{\mathbf{0}}_{(v_i \times a)}\end{array}\right]$$

There will be a all-zero columns among the  $a + v_i$  consecutive columns of the above matrix appearing in R'. Hence the determinant of R' is equal to determinant of sub-matrix composed of a columns of  $H([\delta : b-1], [t : t + a + v_i - 1])$ . Therefore  $|R'| \neq 0$ .

• Otherwise i.e.,  $y - v_i < v_{i+1}$ , then the  $a + v_i$  columns of  $P^a_{v_i, y_{i-1}}$  that are part of R' also include elements from  $P^a_{v_i, v_{i+1}}$ . Let  $y_1 = v_{i+1} + v_i - y$ . Then the matrix R' is of the form shown below:

$$\frac{0}{\frac{1}{|v_i-y_1|}} \frac{0}{|v_i\times a|} \left| \frac{P^a_{v_i,v_{i+1}}([0:v_i-1],[0:y_1-1])}{H([\delta:b-1],[t:t+a+v_i-1])} \right|$$

This matrix is invertible as  $P_{v_i,v_{i+1}}^a([0:y_1-1],[0:y_1-1]) = I_{y_1}$  and any a columns of  $H([\delta:b-1],[t:t+a+v_i-1])$  are linearly independent.

b) i odd: Since  $v_1 = \ell > y$  we have  $i \ge 3$  if i is odd. For odd i < D, the  $y \times (a + y)$  sub-matrix of  $P^a_{\delta,\ell}$  appearing in R is  $y \times (a + y)$  sub-matrix of  $P^a_{v_{i-1},v_{i-2}}$  corresponding to last (a + y) columns and last y rows. This is because  $v_{i-2} \ge v_{i-1} + a > a + y$  and  $v_{i-1} > y$ . For odd i < D.

$$P_{v_{i-1},v_{i-2}}^{a} = \left[ \begin{array}{cccc} I_{v_{i-1}} & \underbrace{\mathbf{0}}_{v_{i-1}\times a} & \cdots & I_{v_{i-1}} & \underbrace{\mathbf{0}}_{v_{i-1}\times a} & P_{v_{i-1},v_{i}} \end{array} \right].$$

Therefore the sub-matrix R appearing in equation (1) can be written as:

$$\left[\frac{0}{|I_{y-v_i}|} \underbrace{0}_{y \times a} \left| \begin{array}{c} P^a_{v_{i-1},v_i}([v_{i-1}-y:v_{i-1}-1],[0:v_i-1]) \\ H([\delta:b-1],[t:\tau-1]) \end{array} \right]\right]$$

The determinant of this matrix is equal to:

$$|H([\delta:b-1], [t+y-v_i:t+a+y-v_i])| * |P^a_{v_{i-1},v_i}([v_{i-1}-y:v_{i-1}-y+v_i-1], [0:v_i-1])|.$$

The first term in this determinant is non-zero by MDS property. The second term corresponds to determinant of sub-matrix of  $P_{v_{i-1},v_i}^a$  that is formed by picking  $v_i$  consecutive rows. By the property of  $P_{v_{i-1},v_i}^a$  given in Lemma 1 this is non-zero.

For the case odd i = D with  $v_{D-2} \mod (v_{D-1} + a) = 0$ , we have  $v_{D-1} > y$  and  $v_{D-2} \ge v_{D-1} + a > a + y$ and hence the  $y \times (a + y)$  sub-matrix appearing in R is contained in

$$P^a_{v_{D-1},v_{D-2}} = \left[ I_{v_{D-1}} \quad \underbrace{\mathbf{0}}_{v_{D-1}\times a} \quad \cdots \quad I_{v_{D-1}} \quad \underbrace{\mathbf{0}}_{v_{D-1}\times a} \right].$$

The reduced matrix R defined in (1) takes the form

$$R = \begin{bmatrix} I_y & \underbrace{0}_{(y \times a)} \\ \hline H([\delta:b-1], [t:t+a+y-1]) \end{bmatrix}.$$

Hence, the determinant of R is equal to determinant of  $H([\delta : b - 1], [t + y : t + a + y - 1])$ , which is clearly non-zero.

Now consider the case when odd i = D with  $v_{D-2} \mod (v_{D-1} + a) \ge v_{D-1}$ .

• For the case when  $v_{D-2} \ge a+y$ , the  $(y \times (a+y))$  sub-matrix we are interested in is a sub-matrix of  $P^a_{v_{D-1},v_{D-2}}$ and

$$P^a_{v_{D-1},v_{D-2}} = \left[ \begin{array}{cccc} I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1}\times a} & \cdots & I_{v_{D-1}} & \underbrace{\mathbf{0}}_{v_{D-1}\times p} \end{array} \right],$$

where  $p = v_{D-2} \mod (a + v_{D-1}) - v_{D-1}$ . The  $y \times (a + y)$  sub-matrix comprised of last y rows, last (a + y) columns is of the form shown below:

$$\left[\begin{array}{ccc} \mathbf{0} & I_y & \mathbf{0} \\ y \times (a-p) & & y \times p \end{array}\right]$$

Therefore the sub-matrix appearing in equation (1) can be written as:

$$R = \begin{bmatrix} \mathbf{0} & I_y & \mathbf{0} \\ y \times (a-p) & y \times p \\ H([\delta:b-1], [t:\tau-1]) \end{bmatrix}.$$

The determinant of this matrix is equal to  $|H([\delta:b-1], A)|$  where  $A = [t:t+a-p-1] \cup [\tau - p:\tau - 1]$ , which is non-zero since A is of size a.

• For the case when  $v_{D-2} < a + y$ ,  $P^a_{v_{D-1},v_{D-2}}$  is as shown below:

$$P^a_{v_{D-1},v_{D-2}} = \left[ I_{v_{D-1}} \quad \underbrace{\mathbf{0}}_{v_{D-1} \times p} \right],$$

where  $p = v_{D-2} - v_{D-1}$ . Since  $v_1 = \ell > a+y$ , this case occurs only when  $D \ge 5$ . In this case, the  $(y \times (a+y))$  sub-matrix of  $P^a_{\delta,\ell}$  appearing in R is the sub-matrix of  $P^a_{v_{D-3},v_{D-4}}$  corresponding to last (a+y) columns and last y rows.

$$P_{v_{D-3},v_{D-4}}^{a} = \left[ I_{v_{D-3}} \underbrace{\begin{array}{c} \mathbf{0} \\ v_{D-3} \times a \end{array}}_{v_{D-3} \times a} \cdots I_{v_{D-3}} \underbrace{\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ v_{D-3} \times a \end{array}}_{v_{D-3} \times a} \frac{I_{v_{D-2}}}{I_{v_{D-2}}} \right].$$

Thus the  $y \times (a + y)$  sub-matrix of interest has the form shown below:

$$\underbrace{\mathbf{0}}_{y\times(a-p)} \quad I_y \quad \underbrace{\mathbf{0}}_{y\times p} \ \right].$$

Therefore the sub-matrix appearing in equation (1) can be written as:

$$R = \begin{bmatrix} \mathbf{0} & I_y & \mathbf{0} \\ \underline{y \times (a-p)} & y \times p \\ \hline H([\delta:b-1], [t:\tau-1]) \end{bmatrix}$$

The determinant of this matrix is equal to  $|H([\delta:b-1], A)|$  where  $A = [t:t+a-p-1] \cup [\tau - p:\tau - 1]$ , which is non-zero.

## D. Proof of R2 property

If we prove that columns  $\{H([0:b-1],j) \mid j \in A\}$  are linearly independent for any set  $A \subseteq [\delta:\tau+\delta]$  with |A| = a, then R2 property follows. For  $\delta = 0$ , the scalar code reduces to a  $[\tau + 1, \tau + 1 - a]$  MDS code and proof is straightforward. Hence we need to consider only  $\delta > 0$ , for which have  $0 \notin A$ . If  $A \cap [1:\tau] \neq \phi$ , observe that columns of  $H([\delta:b-1], A)$  are either all-zero columns or distinct columns from an MDS parity check matrix. Hence the *a* columns  $\{H([\delta:b-1],j) \mid j \in A\}$  are not linearly dependent. If  $A \cap [0:\tau]$  is empty, then  $A \subseteq [\tau + 1:\tau + \delta]$ . By B2 property for  $t = \tau + 1 - a$ , it follows that the columns of H([0:b-1], A) are linearly independent. This completes the proof of R2 property.

#### REFERENCES

- [1] "5G Services Innovation," 5G-Americas, 2019.
- [2] E. Martinian and C. W. Sundberg, "Burst Erasure Correction Codes with Low Decoding Delay," *IEEE Trans. Inf. Theory*, vol. 50, no. 10, pp. 2494–2502, 2004.
- [3] E. Martinian and M. Trott, "Delay-Optimal Burst Erasure Code Construction," in *Proc. IEEE Int. Symp. Inf. Theory, Nice, France, June* 24-29, 2007, pp. 1006–1010.
- [4] A. Badr, P. Patil, A. Khisti, W. Tan, and J. G. Apostolopoulos, "Layered Constructions for Low-Delay Streaming Codes," *IEEE Trans. Inf. Theory*, vol. 63, no. 1, pp. 111–141, 2017.

- [5] M. Vajha, V. Ramkumar, M. Jhamtani, and P. V. Kumar, "On Sliding Window Approximation of Gilbert-Elliott Channel for Delay Constrained Setting," *CoRR*, vol. abs/2005.06921, 2020.
- [6] M. N. Krishnan, D. Shukla, and P. V. Kumar, "Low Field-size, Rate-Optimal Streaming Codes for Channels With Burst and Random Erasures," *IEEE Trans. Inf. Theory*, vol. 66, no. 8, pp. 4869–4891, 2020.
- [7] N. Adler and Y. Cassuto, "Burst-Erasure Correcting Codes With Optimal Average Delay," *IEEE Trans. Inf. Theory*, vol. 63, no. 5, pp. 2848–2865, 2017.
- [8] D. Leong and T. Ho, "Erasure Coding for Real-Time Streaming," in Proc. IEEE Int. Symp. Inf. Theory, Cambridge, MA, USA, July 1-6, 2012, pp. 289–293.
- [9] D. Leong, A. Qureshi, and T. Ho, "On Coding for Real-Time Streaming under Packet Erasures," in Proc. Int. Symp. Inf. Theory, Istanbul, Turkey, July 7-12, 2013, pp. 1012–1016.
- [10] Ö. F. Tekin, T. Ho, H. Yao, and S. Jaggi, "On erasure correction coding for streaming," in Proc. Inf. Theory and Applications Workshop, San Diego, CA, USA, February 5-10, 2012, pp. 221–226.
- [11] S. L. Fong, A. Khisti, B. Li, W. Tan, X. Zhu, and J. G. Apostolopoulos, "Optimal Streaming Codes for Channels With Burst and Arbitrary Erasures," *IEEE Trans. Inf. Theory*, vol. 65, no. 7, pp. 4274–4292, 2019.
- [12] M. N. Krishnan and P. V. Kumar, "Rate-Optimal Streaming Codes for Channels with Burst and Isolated Erasures," in Proc. IEEE Int. Symp. Inf. Theory, Vail, CO, USA, June 17-22, 2018, pp. 1809–1813.
- [13] E. Domanovitz, S. L. Fong, and A. Khisti, "An Explicit Rate-Optimal Streaming Code for Channels with Burst and Arbitrary Erasures," in Proc. IEEE Inf. Theory Workshop, Visby, Sweden, August 25-28, 2019, pp. 1–5.
- [14] M. Rudow and K. V. Rashmi, "Streaming Codes For Variable-Size Arrivals," in Proc. 56th Annual Allerton Conference on Communication, Control, and Computing, Monticello, IL, USA, October 2-5, 2018, pp. 733–740.
- [15] M. N. Krishnan, V. Ramkumar, M. Vajha, and P. V. Kumar, "Simple Streaming Codes for Reliable, Low-Latency Communication," *IEEE Commun. Lett.*, vol. 24, no. 2, pp. 249–253, 2020.
- [16] V. Ramkumar, M. Vajha, M. N. Krishnan, and P. V. Kumar, "Staggered Diagonal Embedding Based Linear Field Size Streaming Codes," in Proc. IEEE Int. Symp. Inf. Theory, Los Angeles, CA, USA, June 21-26, 2020, pp. 503–508.
- [17] H. D. Hollmann and L. M. Tolhuizen, "Optimal Codes for Correcting a Single (wrap-around) Burst of Erasures," *IEEE Trans. Inf. Theory*, vol. 54, no. 9, pp. 4361–4364, 2008.